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LOCAL-GLOBAL PRINCIPLES FOR ZERO-CYCLES ON HOMOGENEOUS SPACES OVER ARITHMETIC FUNCTION FIELDS

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Abstract. We study the existence of zero-cycles of degree one on varieties that are defined over a function field of a curve over a complete discretely valued field. We show that local-global principles hold for such zero-cycles provided that local-global principles hold for the existence of rational points over extensions of the function field. This assertion is analogous to a known result concerning varieties over number fields. Many of our results are shown to hold more generally in the henselian case.

1. Introduction

The study of rational points on varieties is a fundamental subject of arithmetic geometry. Local-global principles (and their obstructions) are one of the main tools in understanding whether a rational point exists. A related object of study is the index of an algebraic variety, and one may ask whether it is equal to 1, i.e., whether the variety admits a zero-cycle of degree one. Equivalently, for every prime ℓ, does there exist a point defined over a finite field extension of degree prime to ℓ?

In this paper, we consider varieties over certain two-dimensional fields—in particular, one-variable function fields over complete discretely valued fields (so-called semiglobal fields). These fields are amenable to patching methods. Local-global principles for rational points on homogeneous spaces over such fields were studied, for example, in [CTPS12], [CTPS16], [HHK15a]. Here we exhibit several situations where such local-global principles imply corresponding local-global statements for zero-cycles. Parallel results over number fields were first obtained by Liang ([Lia13, Prop. 3.2.3]; see also [CT15, Section 8.2]). However, our situation involves substantial new difficulties to overcome.

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A semiglobal field admits several natural collections of overfields with respect to which local-global principles can be studied. After a choice of normal projective model of the semiglobal field, one may consider two distinct collections of overfields, the first associated with *patches* on such a model (described at the beginning of Section 2), and the second consisting of overfields which are fraction fields of complete local rings at points of the closed fiber of the model. Finally, as in the number field case, one may also work with the set of completions with respect to discrete valuations.

Consider a scheme Z of finite type over a semiglobal field F. We show (Theorem 3.4) that in the case of overfields coming from patching, a local-global principle for rational points for the base change Z_E to all finite separable field extensions E/F implies a local-global principle for separable zero-cycles of degree one over F (or analogously, of degree prime to some given prime ℓ). This also gives a local-global principle for the (separable) index of Z (Corollary 3.5). The analogous results hold when the collection of overfields under consideration comes from points of the closed fiber of a model as described above (Theorem 3.9). Moreover, the latter results extend to the case of function fields of curves over excellent henselian discrete valuation rings. In particular, we obtain local-global principles for zero-cycles in that situation; see Proposition 3.12. The required local-global hypothesis for rational points over a semiglobal field holds, for instance, if Z is a torsor under a linear algebraic group that is connected and rational (see Corollary 3.10).

In the situation where Z is a principal or projective homogeneous space under a linear algebraic group, we also obtain local-global principles with respect to discrete valuations (Theorem 3.16) under additional hypotheses on F (e.g., when F is the function field of a curve over a complete discretely valued field with algebraically closed residue field of characteristic 0). In certain cases, the existence of local zero-cycles of degree one already implies the existence of a global rational point (Theorems 3.23 and 3.27).

The results are obtained using a combination of methods. Many of the local-global statements rely on descent results that we prove for finite field extensions and for the existence of rational points, in the context of a pair of fields $L \subseteq L'$ and a finite separable field extension E'/L'. In the former type of descent result (e.g., Proposition 2.3), we find a finite separable field extension E/L such that $E \otimes_L L' \cong E'$. In the latter type (e.g., Proposition 2.10), given an L-scheme Z with an E'-point, we find an appropriate finite separable field extension E/L such that Z has a point over $E \otimes_L L'$. Descent of extensions of fields arising in the context of patching is also studied in [HHKPS17], which builds on the results here. For our local-global principles with respect to discrete valuations, we also use structural properties of linear algebraic groups, as well as results about nonabelian cohomology in degree two.

The manuscript is organized as follows. Section 2 contains descent results. The first two subsections concern fields occurring in patching. We show that finite separable field extensions of some of these overfields descend to the semiglobal field F (Subsection 2.1); in other cases it is still possible to descend the existence of points (Subsection 2.2; in particular, see Proposition 2.10). Subsection 2.3 contains local descent results. Local-global principles for zero-cycles are proven in Section 3; this is first done with respect to patches and points on arithmetic curves over complete discrete valuation rings (Subsection 3.1), then generalized to excellent henselian valuation rings (Subsection 3.2). In certain cases, we obtain local-global principles, with respect to discrete valuations (Subsections 3.3 and 3.4), for principal and projective homogeneous spaces over certain two-dimensional fields, including semiglobal fields.

2. Descent results

This section contains descent results which will be essential in proving the local-global principles in the following section. We first consider collections of fields coming from patching. We recall the following notation, which was established in [HH10], [HHK09], [HHK15b].

Definition 2.1. Let K be a discretely valued field with valuation ring T, uniformizer t, and residue field k. Let F be a one-variable function field over K; i.e., a finitely generated field extension of transcendence degree one in which K is algebraically closed. A *normal model* of F is an integral T-scheme $\mathcal{X}$ with function field F that is flat and projective over T of relative dimension 1, and that is normal as a scheme. If in addition the scheme $\mathcal{X}$ is regular, we say that $\mathcal{X}$ is a *regular model*. The *closed fiber* of $\mathcal{X}$ is $\mathcal{X}_k := \mathcal{X} \times_T k$.

In the case in which the discrete valuation ring T is complete, we call F a *semiglobal field*. (We will also often consider the more general case in which T is excellent and henselian.)

The following notation will be used throughout this manuscript.

Notation 2.2. In the context of Definition 2.1, let $\mathcal{X}$ be a normal model for F. If P is a (not necessarily closed) point of the closed fiber X of $\mathcal{X}$, let R_P be the local ring of $\mathcal{X}$ at P, let $\widehat{R}_P$ be its completion with respect to the maximal ideal $\mathfrak{m}_P$, and let F_P be the fraction field of $\widehat{R}_P$. In the case that P is a closed point of X, the *branches* of X at P are the height 1 prime ideals of $\widehat{R}_P$ that contain t. We write $R_\wp$ for the local ring of $\widehat{R}_P$ at a branch $\wp$. This is a discrete valuation ring. We write $\widehat{R}_\wp$ for its completion, and $F_\wp$ for the fraction field of $\widehat{R}_\wp$.

If U is a nonempty connected affine open subset of X, then we write R_U for the subring of F consisting of rational functions that are regular at each point of U. We let $\widehat{R}_U$ be the t-adic completion of R_U. This is an integral domain by [HHK15b, Proposition 3.4], and we let F_U be the fraction field of $\widehat{R}_U$. If $P \in U \subseteq U'$, then $\widehat{R}_{U'} \subseteq \widehat{R}_U \subset \widehat{R}_P$ and $F_{U'} \subseteq F_U \subset F_P$.

The spectra of the rings $\widehat{R}_P$ and $\widehat{R}_U$ above are thought of as patches on $\mathcal{X}$. For the sake of readers who are familiar with rigid geometry, we remark that our fields F_P and F_U are the same as the rings of meromorphic functions on the corresponding rigid open and rigid closed affinoid sets obtained by deleting the closed fiber from the patches; see [Ray74] concerning the relationship between formal schemes and rigid analytic spaces.

2.1. Descent of field extensions.

The following two statements generalize [HHK15b, Proposition 3.5]. By the *trivial étale algebra* (*of degree* n) over a field L we mean the direct product of n copies of L.

Proposition 2.3. *Let $\mathcal{X}$ be a normal model of a semiglobal field F, let P be a closed point of $\mathcal{X}$, let $\wp$ be a branch of the closed fiber X at P, and let $E_\wp$ be a finite separable field extension of $F_\wp$. Then there exists a finite separable field* *extension E_P of F_P such that $E_P \otimes_{F_P} F_\wp \cong E_\wp$ as extensions of F_P, and such that E_P induces the trivial étale algebra over $F_{\wp'}$ for every other branch $\wp'$ at P.*

Proof. Let $f_\wp(x) \in F_\wp[x]$ be the minimal polynomial of a primitive element of $E_\wp$. For each other branch $\wp'$ at P, let $f_{\wp'}(x) \in F_{\wp'}[x]$ be a separable polynomial of the same degree that splits completely over $F_{\wp'}$ and thus defines the trivial étale algebra over $F_{\wp'}$. The field F_P is dense in $\prod F_{\wp'}$ by [Bou72, Theorem VI.7.2.1], where the product ranges over all the branches at P (including $\wp$). So applying Krasner's lemma (e.g., [Lan70, Prop. II.2.4]) to the above polynomials yields the desired extension of F_P, which is a field since $E_\wp$ is. □

Proposition 2.4. *Let $\mathcal{X}$ be a normal model of a semiglobal field F, let U be a nonempty connected affine open subset of the closed fiber of $\mathcal{X}$, and let E_U be a finite separable field extension of F_U. Then there is a finite separable field extension E of F such that $E \otimes_F F_U \cong E_U$ as extensions of F_U.*

Proof. Let $\bar{U}$ be the closure of U in the closed fiber X of $\mathcal{X}$. After blowing up $\mathcal{X}$ at the points of $\mathcal{P}_U := \bar{U} \smallsetminus U$ (which changes the model $\mathcal{X}$ but does not change F, U, or F_U), we may assume that $\bar{U}$ is unibranched at each point of $\mathcal{P}_U$ (see [Lip75, Lecture 1]). Let $\mathcal{P}$ be a finite set of closed points of X that satisfies $\mathcal{P} \cap \bar{U} = \mathcal{P}_U$ and which contains at least one point on each irreducible component of X. Let $\mathcal{U}$ be the set of connected components of $X \smallsetminus \mathcal{P}$. Then $U \in \mathcal{U}$, and each element of $\mathcal{U}$ is affine.

For each point $P \in \mathcal{P}_U$, consider the unique branch $\wp$ at P on U. Then $E_\wp := E_U \otimes_{F_U} F_\wp$ is a finite direct product of finite separable field extensions $E_{\wp,i}$ of $F_\wp$. By Proposition 2.3, for each i there is a finite separable field extension $E_{P,i}$ of F_P such that $E_{P,i} \otimes_{F_P} F_\wp \cong E_{\wp,i}$ and such that $E_{P,i} \otimes_{F_P} F_{\wp'}$ is a trivial étale algebra over $F_{\wp'}$ for every other branch $\wp'$ at P. So the direct product of the fields $E_{P,i}$ (ranging over i) is a finite étale F_P-algebra E_P satisfying $E_P \otimes_{F_P} F_\wp \cong E_\wp$ and such that $E_P \otimes_{F_P} F_{\wp'}$ is the trivial étale algebra of degree $n := [E_U : F_U]$ over $F_{\wp'}$. Here E_P is well defined for each $P \in \mathcal{P}_U$ since $\wp$ is unique given P.

For every $P \in \mathcal{P}$ that is not in $\mathcal{P}_U$, let E_P be the trivial étale algebra of degree n over F_P. Similarly, for every $U' \in \mathcal{U}$ other than U, let $E_{U'}$ be the trivial étale algebra of degree n over $F_{U'}$, and for every branch $\wp$ at a point that is not in $\bar{U}$, let $E_\wp$ be the trivial étale algebra of degree n over $F_\wp$. Thus for *every* branch $\wp$ at a point $P \in \mathcal{P}$ lying on some $U' \in \mathcal{U}$ (including the case $U' = U$), we have isomorphisms $E_P \otimes_{F_P} F_\wp \cong E_\wp \cong E_{U'} \otimes_{F_{U'}} F_\wp$. But patching holds for finite separable algebras in this context; see [HHK15b, Proposition 3.7 and Example 2.7]. So there is a finite étale F-algebra E that compatibly induces all of the algebras E_P, $E_{U'}$, $E_\wp$. Since E_U is a field, so is E. □

The above two propositions suggest analogous statements in which the roles of P and U are interchanged: The analogue of Proposition 2.3 would assert that if $\wp$ is a branch at a point P in $\bar{U} \smallsetminus U$, then every finite separable field extension of $F_\wp$ would be induced by a finite separable field extension of F_U. The analogue of Proposition 2.4 would say that every finite separable field extension of F_P is induced by a finite separable field extension of F. This does not hold in general, as the following example shows.

Example 2.5. Let T denote the complete discrete valuation ring $k[[t]]$, where k is a field of characteristic $p > 0$. Let $\mathcal{X}$ be the projective x-line over T, and let

P be the origin on the projective k-line X. Then F_P equals $k((t,x))$, the fraction field of $k[[t,x]]$. Consider the field extension E_P/F_P generated by the solutions of $y^p - y = \frac{\alpha}{t}$, where α is a transcendental power series (i.e., a power series in x transcendental over $F = k((t))(x)$). Then one can show that E_P is not induced by an extension of F in the above sense. In fact, this example is a special instance of [HHKPS17, Proposition 2.15], to which we refer the reader for a proof.

In [HHKPS17, Section 2], it is shown that versions of descent for field extensions with the roles of P and U interchanged do hold when the residue field k of K has characteristic 0, and that as a consequence there are local-global principles in that situation. To treat the more general case, we consider a different type of descent in the next subsection.

2.2. **Descent of existence of points.** In order to prove a local-global principle (with respect to patches) for zero-cycles in arbitrary characteristic, we prove a descent result for the existence of points instead of field extensions (Proposition 2.10). Specifically, let T be an excellent henselian (e.g., complete) discrete valuation ring, and choose a normal model $\mathcal{X}$ of a one-variable function field F over the fraction field K of T. In the context of Notation 2.2, we show that if E_P/F_P is a separable field extension whose degree is not divisible by some prime number ℓ, and if Z is an F-scheme of finite type which has an E_P-point, then there is a finite separable field extension E/F of degree prime to ℓ such that Z has a point over $E \otimes_F F_P$. First, some preparation is needed.

For P a point on the closed fiber of $\mathcal{X}$ as above, the henselization R_P^{h} of R_P is the same as the henselization at P of the coordinate ring of an affine open subset of $\mathcal{X}$ that contains P. Since that coordinate ring is of finite type over the excellent henselian discrete valuation ring T, Artin's approximation theorem [Art69, Theorem 1.10] applies to a system of polynomial equations over R_P^{h} and asserts that if there is a solution over $\widehat{R}_P$, then there is a solution over R_P^{h}. By clearing denominators, the same assertion holds with $\widehat{R}_P$ and R_P^{h} replaced by F_P and F_P^{h}, where F_P^{h} is the fraction field of R_P^{h} (this being a separable field extension of F). We use this in the proof of the next proposition.

Note that we may pick a fixed algebraic closure $\bar{F}_P$ of F_P, and let $\bar{F}$ be the algebraic closure of F in $\bar{F}_P$. Thus $\bar{F}$ is an algebraic closure of F that contains F_P^{h}.

Proposition 2.6. *Let T be an excellent henselian discrete valuation ring with fraction field K. Let F be a one-variable function field over K (e.g., a semiglobal field if K is complete), and let $\mathcal{X}$ be a normal model of F. Let P be a (not necessarily closed) point of the closed fiber of $\mathcal{X}$, and let Z be an F-scheme of finite type. Let ℓ be a prime number, and suppose that there is a finite separable field extension E_P/F_P of degree prime to ℓ such that $Z(E_P)$ is nonempty. Then there is a finite separable field extension E'_P/F_P^{h} of degree prime to ℓ such that $Z(E'_P)$ is nonempty.*

Proof. Let d be the smallest positive integer that is prime to ℓ such that there is a finite separable field extension E_P/F_P of degree d for which Z has an E_P-point ξ. Since E_P is separable over F_P, there are exactly d distinct F_P-embeddings $\sigma_1, \dots, \sigma_d$ of E_P into an algebraic closure $\bar{F}_P$ of F_P. By the minimality of d and E_P, the point $(\sigma_1(\xi), \dots, \sigma_d(\xi)) \in Z^d(\bar{F}_P)$ does not lie on the closed subset $\Delta \subset Z^d$, where two or more of the entries are equal. Consider the image $\mathcal{D} \subset S^d(Z)$ of Δ in the dth symmetric power of Z; i.e., $S^d(Z) \smallsetminus \mathcal{D} = (Z^d \smallsetminus \Delta)/S_d$. The image

$\zeta \in S^d(Z) \smallsetminus \mathcal{D}$ of $(\sigma_1(\xi), \ldots, \sigma_d(\xi))$ is an F_P-point on this F-scheme, corresponding to a morphism $\operatorname{Spec}(F_P) \to S^d(Z) \smallsetminus \mathcal{D}$. The image of this morphism is a point of (the underlying topological space of) $S^d(Z) \smallsetminus \mathcal{D}$, and this lies in some affine open subset $\operatorname{Spec}(A) \subseteq S^d(Z) \smallsetminus \mathcal{D}$. This point corresponds to a solution over F_P to a system of polynomial equations over F that defines A.

By Artin's approximation theorem, there is a solution to this system of equations over the field F_P^{h}. This corresponds to an F_P^{h}-point ζ' on $S^d(Z) \smallsetminus \mathcal{D}$. Pick a point on $Z^d \smallsetminus \Delta$ that maps to ζ'. Each entry lies in an algebraic closure of F_P^{h} or, equivalently, of F. The d entries are distinct, and this set of entries is stable under the absolute Galois group of F_P^{h} since ζ' is defined over F_P^{h}. Thus the entries form a disjoint union of orbits under this absolute Galois group, say, of orders $d_1, \ldots, d_r$, with $\sum_i d_i = d$. Since d is prime to ℓ, so is some d_i. Let ξ' be an entry lying in the ith orbit; this defines a point of Z, say, with field of definition E'_P. Then the field E'_P is separable over F_P^{h} by the distinctness of the entries; the degree of E_P over F_P^{h} is d_i, which is prime to ℓ; and ξ' is an E'_P-point of Z. □

The above proof actually shows more—viz., that if $[E_P : F_P]$ is minimal for the given property, then E'_P can be chosen so that $[E'_P : F_P^{\mathrm{h}}] = [E_P : F_P]$. This follows from the fact that, at the end of the proof, d_i must equal d (i.e., there is just one orbit) by minimality of d and because ξ' induces a point of Z over an extension of F_P of degree at most d_i.

Lemma 2.7. *Let $L \subseteq L' \subseteq E'$ be separable algebraic field extensions, where $[E' : L']$ is finite. Let Z be an L-scheme of finite type such that $Z(E')$ is nonempty. Then there are finite separable field extensions $L \subseteq \widetilde{L} \subseteq \widetilde{E}$ such that $\widetilde{L} \subseteq L'$ and $\widetilde{E} \subseteq E'$, that $[\widetilde{E} : \widetilde{L}] = [E' : L']$; $Z(\widetilde{E})$ is nonempty, and that E' is the compositum of its subfields $\widetilde{E}$ and L'.*

Proof. Let $\xi \in Z(E')$. Since Z is an L-scheme of finite type, there is an affine open subset of Z that contains ξ and is L-isomorphic to a Zariski closed subset Y of $\mathbb{A}^n_L$ for some n. Let $y_1, \ldots, y_n \in E'$ be the coordinates of the image of ξ in Y. Let z be a primitive element of the finite separable field extension E'/L', say, with minimal monic polynomial g over L' of degree $d = [E' : L']$. Thus each y_i is of the form $\sum_{j=0}^{d-1} c_{ij} z^j$ with $c_{ij} \in L'$. Let $\widetilde{L}$ be the subfield of L' generated over L by the coefficients of g and by the elements c_{ij}; this is finite over L, and it is separable over L since L' is. The polynomial g is irreducible over $\widetilde{L}$ because it is irreducible over L'. Thus $\widetilde{E} := \widetilde{L}(z) \subseteq E'$ is separable and of degree d over $\widetilde{L}$, and $\widetilde{E} \subseteq \widetilde{E}L' = L'(z) = E'$. Also, each y_i lies in $\widetilde{E}$ since $z \in \widetilde{E}$ and $c_{ij} \in \widetilde{L} \subseteq \widetilde{E}$. So $(y_1, \ldots, y_r) \in Y(\widetilde{E})$, and thus $\xi \in Z(\widetilde{E})$. □

Lemma 2.8. *Let ℓ be a prime number, and let $L \subseteq \widetilde{L} \subseteq \widetilde{E}$ be finite separable field extensions such that $[\widetilde{E} : \widetilde{L}]$ is prime to ℓ. Let $\widehat{E}$ be the Galois closure of $\widetilde{E}/L$. Then for every Sylow ℓ-subgroup S of $\operatorname{Gal}(\widehat{E}/L)$, there is some $\sigma \in \operatorname{Gal}(\widehat{E}/L)$ such that the compositum $\widetilde{L}\widehat{E}^S \subseteq \widehat{E}$ contains $\sigma(\widetilde{E})$.*

Proof. The intersection $\operatorname{Gal}(\widehat{E}/\widetilde{L}) \cap S \subseteq \operatorname{Gal}(\widehat{E}/L)$ is an ℓ-subgroup of $\operatorname{Gal}(\widehat{E}/\widetilde{L})$, so it is contained in a Sylow ℓ-subgroup S^* of $\operatorname{Gal}(\widehat{E}/\widetilde{L})$. Let S' be a Sylow ℓ-subgroup of $\operatorname{Gal}(\widehat{E}/\widetilde{E})$. Since $[\widetilde{E} : \widetilde{L}]$ is prime to ℓ, the group S' is also a Sylow ℓ-subgroup of $\operatorname{Gal}(\widehat{E}/\widetilde{L})$. Thus S^*, S' are conjugate subgroups of $\operatorname{Gal}(\widehat{E}/\widetilde{L})$, say, by an element $\sigma \in \mathrm{Gal}(\widehat{E}/\widetilde{L}) \subseteq \mathrm{Gal}(\widehat{E}/L)$. Since $S' \subseteq \mathrm{Gal}(\widehat{E}/\widetilde{E})$, its conjugate $S^* = (S')^\sigma$ is contained in $\mathrm{Gal}(\widehat{E}/\sigma(\widetilde{E}))$. Thus $\widehat{E}^{S^*}$ contains $\sigma(\widetilde{E})$. So $\widetilde{L}\widehat{E}^S = \widehat{E}^{\mathrm{Gal}(\widehat{E}/\widetilde{L})}\widehat{E}^S = \widehat{E}^{\mathrm{Gal}(\widehat{E}/\widetilde{L})\cap S}$, which contains $\widehat{E}^{S^*}$ and hence contains $\sigma(\widetilde{E})$. □

Recall that if L is a field, Z is an L-scheme of finite type, and A/L is a finite direct product of field extensions L_i/L, an A-point on Z is a collection of points in $Z(L_i)$ for each i. In particular, $Z(A)$ is nonempty if and only if $Z(L_i)$ is nonempty for all i.

Proposition 2.9. *Let ℓ be a prime number, and let $L \subseteq L' \subseteq E'$ be separable algebraic field extensions, where $[E' : L']$ is finite and prime to ℓ. Let Z be an L-scheme of finite type such that $Z(E')$ is nonempty. Then there is a finite separable field extension E/L of degree prime to ℓ such that $Z(E \otimes_L L')$ is nonempty.*

Proof. Let $\widetilde{L}$ and $\widetilde{E}$ be as in Lemma 2.7; in particular, $[\widetilde{E} : \widetilde{L}] = [E' : L']$ is prime to ℓ. Let $\widehat{E}$ be the Galois closure of $\widetilde{E}$ over L, let S be a Sylow ℓ-subgroup of $\mathrm{Gal}(\widehat{E}/L)$, and let E be the fixed field $\widehat{E}^S$. We will show that E has the desired properties.

Since S is a Sylow ℓ-subgroup of $\mathrm{Gal}(\widehat{E}/L)$, the degree of E over L is prime to ℓ. In order to show that $Z(E \otimes_L L')$ is nonempty, it is sufficient to show that $Z(E \otimes_L \widetilde{L})$ is nonempty, because $L \subseteq \widetilde{L} \subseteq L'$. Since E/L is a separable field extension of finite degree, $E \otimes_L \widetilde{L}$ is a finite separable algebra over $\widetilde{L}$, and hence is a finite direct product $\prod \widetilde{E}_i$ of finite separable field extensions of $\widetilde{L}$. It therefore suffices to show that $Z(\widetilde{E}_i)$ is nonempty for all i. Here each $\widetilde{E}_i$ is isomorphic to a compositum of $\widetilde{L}$ and E with respect to some L-embeddings of those two fields into the common overfield $\widehat{E}$ (since $\widehat{E}/L$ is Galois).

After conjugating, we may assume that the above L-embedding of $\widetilde{L}$ in $\widehat{E}$ is the given one (i.e., the original composition $\widetilde{L} \hookrightarrow \widetilde{E} \hookrightarrow \widehat{E}$), while allowing the L-embedding of E into $\widehat{E}$ to vary. The images of E in $\widehat{E}$ under the various L-algebra embeddings are just the Galois conjugates $\sigma(E)$ for $\sigma \in \mathrm{Gal}(\widehat{E}/L)$. Since E is the fixed field $\widehat{E}^S$, its Galois conjugates are the fixed fields of the conjugates of S—viz., the fields $\widehat{E}^{S'}$ where S' varies over the Sylow ℓ-subgroups of $\mathrm{Gal}(\widehat{E}/L)$. So it suffices to show that, for each S', there is a point of Z defined over the compositum $\widetilde{L}\widehat{E}^{S'} \subseteq \widehat{E}$.

So consider any S'. By Lemma 2.8 (which applies since $[\widetilde{E} : \widetilde{L}]$ is prime to ℓ), $\widetilde{L}\widehat{E}^{S'}$ contains $\tau(\widetilde{E})$ for some $\tau \in \mathrm{Gal}(\widehat{E}/L)$. But Z is an L-variety that has an $\widetilde{E}$-point ξ, by Lemma 2.7. Hence Z also has a $\tau(\widetilde{E})$-point—viz., $\tau(\xi)$. But $Z(\tau(\widetilde{E})) \subseteq Z(\widetilde{L}\widehat{E}^{S'})$ since $\tau(\widetilde{E}) \subseteq \widetilde{L}\widehat{E}^{S'}$. So, indeed, Z has a point defined over $\widetilde{L}\widehat{E}^{S'}$. □

Proposition 2.10. *Let T be an excellent henselian discrete valuation ring with fraction field K. Let F be a one-variable function field over K (e.g., a semiglobal field, if K is complete), and let $\mathcal{X}$ be a normal model of F. Let P be a (not necessarily closed) point of the closed fiber of $\mathcal{X}$, and let Z be an F-scheme of finite type. Let ℓ be a prime number, and suppose that there is a finite separable field extension E_P/F_P of degree prime to ℓ such that $Z(E_P)$ is nonempty. Then there is a finite separable field extension E/F of degree prime to ℓ such that $Z(E \otimes_F F_P)$ is nonempty.*

Proof. By Proposition 2.6, there is a finite separable field extension E'_P/F^{h}_P of degree prime to ℓ such that $Z(E'_P)$ is nonempty. Applying Proposition 2.9 with $L = F$, $L' = F^{\mathrm{h}}_P$, and $E' = E'_P$, we obtain a finite separable field extension E/F of degree prime to ℓ such that $Z(E \otimes_F F^{\mathrm{h}}_P)$ is nonempty. But F^{h}_P is contained in F_P, so $Z(E \otimes_F F_P)$ is nonempty. □

2.3. **Local descent results.** In this subsection, we establish local descent results which will be used to prove local-global principles with respect to discrete valuations in Subsection 3.3.

Let A be a complete regular local ring of dimension 2 with field of fractions F and residue field k. For any prime π of A, let F_π denote the completion of F with respect to the discrete valuation associated with π, and let $k(\pi)$ denote the residue field.

The following lemma is proved in [PPS16, Lemma 5.1] for Galois extensions, and a similar proof gives the general case.

Lemma 2.11. *Let A be a complete regular local ring of dimension 2, let F be its field of fractions, and let k be its residue field. Let $\pi, \delta \in A$ generate the maximal ideal, and let E_π/F_π be a finite separable unramified field extension. If* $\mathrm{char}(k)$ *does not divide $[E_\pi : F_\pi]$, then there exists a finite separable field extension E/F such that $E \otimes_F F_\pi \simeq E_\pi$, and the integral closure of A in E is a complete regular local ring with fraction field E and maximal ideal (π', δ'), where π' and δ' generate the unique primes lying over π and δ, respectively.*

Proof. The residue field $F(\pi)$ of F_π is the field of fractions of $A/(\pi)$. Since A is a complete regular local ring, $A/(\pi)$ is a complete discrete valuation ring, k is the residue field of $A/(\pi)$, and the image $\bar{\delta}$ of δ is a uniformizer in $A/(\pi)$.

Let $E(\pi)$ be the residue field of E_π; this is a complete discretely valued field. Let $L(\pi)$ be the maximal unramified field extension of $F(\pi)$ contained in $E(\pi)$, and let L_π be the subextension of E_π/F_π whose residue field is $L(\pi)$. Let κ be the residue field of $E(\pi)$ (or, equivalently, of $L(\pi)$) at its discrete valuation. Since $[E_\pi : F_\pi]$ is coprime to $\mathrm{char}(k)$, so is $[\kappa : k]$, and thus κ is a finite separable field extension of k. Write $\kappa = k[t]/(f(t))$ for some monic separable polynomial $f(t) \in k[t]$. By lifting the polynomial $f(t)$ to a monic polynomial over A, we obtain a finite étale A-algebra B; this is a complete regular local ring with maximal ideal (π, δ) at which the residue field is κ. The residue field of L (i.e., of B) at π is $L(\pi)$. Since the same is true for L_π, it follows that the complete discretely valued fields $L \otimes_F F_\pi$ and L_π are isomorphic over F_π. Note that the image $\bar{\delta} \in B/(\pi) \subset L(\pi)$ of δ is a uniformizer for $L(\pi)$.

Now $E(\pi)/L(\pi)$ is totally ramified of degree d prime to $\mathrm{char}(k)$. So by [Lan70, Proposition II.5.12], $E(\pi) = L(\pi)(\sqrt[d]{v\bar{\delta}})$ for some unit $v \in B/(\pi)$, the valuation ring of $L(\pi)$. Let $u \in B$ be a lift of $v \in B/(\pi)$, and let $E = L(\sqrt[d]{u\delta})$. Then E is a finite separable field extension of F, and $E \otimes_F F_\pi \simeq E \otimes_L L \otimes_F F_\pi \simeq E \otimes_L L_\pi \simeq L_\pi(\sqrt[d]{u\delta}) \simeq E_\pi$. The integral closure of B in E (or, equivalently, of A in E) is a two-dimensional complete local ring with maximal ideal $(\pi, \sqrt[d]{u\delta})$, and hence it is regular. Its fraction field is E, and the ideals in this ring that are generated by π and $\sqrt[d]{u\delta}$ are the unique prime ideals lying over the ideals (π) and (δ) in A. So E is as asserted. □

The above lemma can be used to obtain a descent statement for not necessarily unramified extensions.

Lemma 2.12. *Let A be a complete regular local ring of dimension 2, let F be its field of fractions, and let k be its residue field. Let $\pi, \delta \in A$ generate the maximal ideal, and let E_π/F_π be a finite separable field extension. Suppose that* $\mathrm{char}(k)$ *does not divide $[E_\pi : F_\pi]$. Let ℓ be a prime number. If $[E_\pi : F_\pi]$ is prime to ℓ, then there exists a finite separable field extension E/F such that the following hold:*

- *$[E : F]$ is prime to ℓ.*
- *$E \otimes_F F_\pi$ is a field.*
- *E_π is isomorphic to a subfield of $E \otimes_F F_\pi$.*
- *The integral closure of A in E is a complete regular local ring with fraction field E and maximal ideal (π', δ'), where π' and δ' generate the unique primes lying over π and δ, respectively.*

Proof. Let L_π/F_π be the maximal unramified extension contained in E_π. By Lemma 2.11, there is a finite separable extension L/E such that $L \otimes F_\pi \simeq L_\pi$, that the integral closure of A in L is regular with maximal ideal (π', δ'), and that (π') and (δ') are the unique primes lying over the primes (π) and (δ) of A. Thus, replacing F_π with L_π and F with L, we may assume that E_π/F_π is totally ramified. Since $\mathrm{char}(k)$ does not divide $n := [E_\pi : F_\pi]$, neither does the characteristic of the residue field $k(\pi)$ of F_π. Hence $E_\pi = F_\pi(\sqrt[n]{u\pi})$ for some $u \in F_\pi$ which is a unit at the discrete valuation of F_π [Lan70, Section II.5, Proposition 12]. Let $\bar{u}$ be the image of u in $k(\pi)$. Since $k(\pi)$ is the field of fractions of $A/(\pi)$, we have $\bar{u} = \bar{v}\bar{\delta}^i$ for some i and some unit $\bar{v} \in A/(\pi)$ with preimage $v \in A$. Thus $u^{-1}v\delta^i$ lies in the valuation ring of F_π and is congruent to 1 mod π. By Hensel's lemma, this element has an nth root in F_π, so $E_\pi = F_\pi(\sqrt[n]{v\delta^i\pi})$. Let $E = F(\sqrt[n]{\delta}, \sqrt[n]{v\pi})$. Since n is not divisible by either $\mathrm{char}(k)$ or ℓ, the field extension E/F is separable and of degree prime to ℓ. Moreover, $E \otimes_F F_\pi$ is isomorphic to the field $F_\pi(\sqrt[n]{\delta}, \sqrt[n]{v\pi})$, which contains E_π as a subfield. Since E/F is finite and since A is a complete local ring with fraction field F, the integral closure B of A in E is a complete local ring with fraction field E. By [PS14, Lemma 3.2], B is a regular local ring with maximal ideal $(\sqrt[n]{\delta}, \sqrt[n]{v\pi})$. Since $(\sqrt[n]{\delta})$ and $(\sqrt[n]{v\pi})$ are the unique primes of B lying over the primes (δ) and (π) of A, the result follows. □

Finally, we require a simultaneous descent result.

Lemma 2.13. *Let A be a complete regular local ring of dimension 2, let F be its field of fractions, let (π, δ) be its maximal ideal, and let k be its residue field. Suppose that* $\mathrm{char}(k) = 0$. *Let E_π/F_π and E_δ/F_δ be finite field extensions. Let ℓ be a prime. Suppose that the degrees of E_π/F_π and E_δ/F_δ are prime to ℓ. Then there exists a finite* (*separable*) *field extension E/F such that the following hold:*

- *$[E : F]$ is prime to ℓ.*
- *The integral closure of A in E is a complete regular local ring.*
- *$E \otimes_F F_\pi$ and $E \otimes_F F_\delta$ are fields.*
- *E_π is isomorphic to a subfield of $E \otimes_F F_\pi$.*
- *E_δ is isomorphic to a subfield of $E \otimes_F F_\delta$.*

Proof. By Lemma 2.12, there exists a finite (separable) field extension $\tilde{E}/F$ such that $[\tilde{E} : F]$ is prime to ℓ, that $\tilde{E} \otimes_F F_\pi$ is a field which contains an isomorphic copy of E_π as a subfield, and that the integral closure $\tilde{B}$ of A in $\tilde{E}$ is a regular local ring with maximal ideal (π', δ'), with π' and δ' lying over π and δ, respectively. Moreover, the ideals (π') and (δ') are uniquely determined by π and δ. Since δ' is a prime lying over δ, $F_\delta \subseteq \tilde{E}_{\delta'}$. Since E_δ/F_δ is a separable field extension, $E_\delta \otimes_{F_\delta} \tilde{E}_{\delta'}$ is a product of field extensions $E_i/\tilde{E}_{\delta'}$. Since $[E_\delta : F_\delta]$ is prime to ℓ, $[E_i : \tilde{E}_{\delta'}]$ is prime to ℓ for some i.

Again by Lemma 2.12 (this time applied to the complete regular local ring $\widetilde{B}$ and $E_i/\widetilde{E}_{\delta'}$), there exists a finite field extension $E/\widetilde{E}$ of degree prime to ℓ with the following properties: $E \otimes_{\widetilde{E}} \widetilde{E}_{\delta'}$ is a field containing E_i as a subfield, the integral closure B of $\widetilde{B}$ in E is a complete regular local ring having fraction field E and maximal ideal of the form (π'', δ''), and (π'') and (δ'') are the unique primes of B that lie over the primes (π') and (δ') of $\widetilde{B}$. Note that the uniqueness of (δ') implies that $\tilde{E}_{\delta'} = \tilde{E} \otimes_F F_\delta$. Since E_δ is isomorphic to a subfield of E_i, E_δ is isomorphic to a subfield of $E \otimes_{\tilde{E}} \tilde{E}_{\delta'} = E \otimes_{\tilde{E}} (\tilde{E} \otimes_F F_\delta) = E \otimes_F F_\delta$ as claimed. Similarly, E_π is a subfield of $\tilde{E} \otimes_F F_\pi$, which is a subfield of $E \otimes_F F_\pi$ by base change. The uniqueness of (π'') implies that the latter is a field. Since $[E : \tilde{E}]$ and $[\tilde{E} : F]$ are prime to ℓ, $[E : F]$ is prime to ℓ. Since the integral closure of A in E is B, the assertion follows. □

3. Local-global principles for zero-cycles

Let F be a field, and let Z be an F-scheme. A *zero-cycle* on Z is a finite $\mathbb{Z}$-linear combination $\sum n_i P_i$ of closed points P_i of Z. Its *degree* is $\sum n_i \deg(P_i)$, where $\deg(P_i)$ is the degree of the residue field of P_i over F. We say that a closed point P of Z is *separable* if its residue field is separable over F. A zero-cycle $\sum n_i P_i$ is called separable if each P_i is.

We will prove local-global principles for zero-cycles in different settings. First, we consider collections of overfields coming from patching and from points on the closed fiber of a model. Second, we will use this to obtain local-global principles with respect to discrete valuations. Many of the statements in this section rely on the descent results in the previous section. In our results, we have to assume a local-global principle for points (in the respective setting) over the function field F as well as over all of its finite separable field extensions. This is analogous to results in the number field case; see [Lia13, Prop. 3.2.3], [CT15, Section 8.2]. Example 3.7 shows that this hypothesis is actually necessary. Corollary 3.10 exhibits situations in which the assumption is satisfied.

3.1. **Local-global principles with respect to patches and points.** In this subsection, we prove that certain local-global principles for the existence of rational points imply analogous local-global principles for zero-cycles, for varieties over semiglobal fields. This will be proven in two contexts, one where the overfields come from patching, and one where they correspond to points on the closed fiber of a model.

Notation 3.1. Let $\mathcal{X}$ be a normal model of a one-variable function field F over a discretely valued field K, and let X denote the closed fiber. Let $\mathcal{P}$ be a finite nonempty set of closed points of X that meets each irreducible component of X, and let $\mathcal{U}$ be the set of connected components of the complement of $\mathcal{P}$ in X. Let $\mathcal{B}$ be the set of branches of X at points of $\mathcal{P}$. We let $\Omega_{\mathcal{X}}$ be the collection of field extensions F_P/F where P is a (not necessarily closed) point of X, and let $\Omega_{\mathcal{X},\mathcal{P}}$ denote the collection of field extensions F_ξ/F where $\xi \in \mathcal{P} \cup \mathcal{U}$. (See Notation 2.2.)

Notation 3.2. Let $\mathcal{X}$ be a normal model for F as above, and let E/F be a finite field extension. We let $\mathcal{X}_E$ denote the normalization of $\mathcal{X}$ in E. (This is a normal model for E.) If $\mathcal{P}$ is a finite set of closed points of $\mathcal{X}$, we let $\mathcal{P}_E$ denote its preimage under the natural map $\mathcal{X}_E \to \mathcal{X}$.

Definition 3.3. Let F be a field, let Z be an F-scheme of finite type, and let Ω be a collection of overfields of F. We use the following terminology:

- The pair (Z, Ω) *satisfies a local-global principle for rational points* if it has the following property: $Z(F) \neq \varnothing$ if and only if $Z(L) \neq \varnothing$ for every $L \in \Omega$.
- The pair (Z, Ω) *satisfies a local-global principle for closed points of degree prime to* ℓ if it has the following property: Z has a closed point of degree prime to ℓ if and only if the base change Z_L has a closed point of degree prime to ℓ for every $L \in \Omega$.
- The pair (Z, Ω) *satisfies a local-global principle for zero-cycles of degree one* if and only if it has the following property: Z has a zero-cycle of degree one if and only if Z_L has a zero-cycle of degree one for all $L \in \Omega$.

Similarly, one can speak of a local-global principle for separable closed points or separable zero-cycles.

Below we consider the case where F is a semiglobal field (i.e., K is complete). Note that a finite separable field extension E/F is again a semiglobal field.

Theorem 3.4. *Let F be a semiglobal field with normal model $\mathcal{X}$. Let $\mathcal{P}$ be a finite nonempty set of closed points that meets every irreducible component of the closed fiber X of $\mathcal{X}$. Let Z be an F-scheme of finite type. Assume that, for all finite separable field extensions E/F, $(Z_E, \Omega_{\mathcal{X}_E,\mathcal{P}_E})$ satisfies a local-global principle for rational points. Then, for every prime number ℓ, $(Z, \Omega_{\mathcal{X},\mathcal{P}})$ satisfies a local-global principle for separable closed points of degree prime to ℓ.*

Proof. Let $\mathcal{U}$ be as in Notation 3.1. We need to prove that if Z_{F_ξ} has a separable closed point z_ξ of degree prime to ℓ for every $\xi \in \mathcal{P} \cup \mathcal{U}$, then Z has a separable closed point z of degree prime to ℓ. For each ξ, let E_ξ/F_ξ be the residue field extension at z_ξ.

By Proposition 2.4, for each $U \in \mathcal{U}$, there is a finite separable field extension A_U of F that induces E_U (i.e., $A_U \otimes_F F_U$ is isomorphic to E_U) and hence has degree prime to ℓ. In particular, each $Z(A_U \otimes_F F_U)$ is nonempty. By Proposition 2.10, for each $P \in \mathcal{P}$, there is a finite separable field extension A_P of F of degree prime to ℓ such that $Z(A_P \otimes_F F_P)$ is nonempty. Let A be the tensor product over F of both of the fields A_P and A_U. This is an étale F-algebra of degree prime to ℓ, and it is the direct product of finite separable field extensions A_i of F, each of which is a compositum of the fields A_P and A_U. Since the degree of A is the sum of the degrees of the fields A_i over F, at least one of those fields A_i has degree prime to ℓ. Write E for this field. So E/F is separable of degree prime to ℓ, and, for each $\xi \in \mathcal{P} \cup \mathcal{U}$, $Z(E \otimes_F F_\xi)$ is nonempty.

Let $\mathcal{X}_E$, $\mathcal{P}_E$ be as in Notation 3.2, let X_E be the closed fiber, and let $\mathcal{U}_E$ be the set of connected components of the complement of $\mathcal{P}_E$ in X_E. For each $P \in \mathcal{P}$, $E \otimes_F F_P$ is the direct product of the fields $E_{P'}$, where P' runs over the points of $\mathcal{P}_E$ that lie over P; and the corresponding assertion holds for each $U \in \mathcal{U}$. Hence for each $\xi' \in \mathcal{P}_E \cup \mathcal{U}_E$, $Z(E_{\xi'})$ is nonempty. By assumption, this implies that $Z(E)$ is nonempty; i.e., Z has a point defined over a finite separable field extension of F of degree prime to ℓ. □

The hypothesis in the above theorem cannot be weakened to consider merely F instead of all finite separable field extensions; see Example 3.7.

Given a variety V over a field k, the *index* (resp., *separable index*) of V is the greatest common divisor of the degrees of the finite (resp., finite separable) field extensions of k over which V has a rational point. This is the same as the smallest positive degree of a zero-cycle (resp., separable zero-cycle) on V. In this terminology, Theorem 3.4 says that if the separable index of each Z_{F_ξ} is prime to ℓ, then so is the separable index of Z; i.e., if each Z_{F_ξ} has a zero-cycle of degree prime to ℓ, then so does Z.

Corollary 3.5. *Let F be a semiglobal field with normal model $\mathcal{X}$, and let X denote the closed fiber. Let $\mathcal{P}$ be a finite nonempty set of closed points of X which meets every irreducible component of X. Let Z be an F-scheme of finite type such that, for every finite separable extension E/F, $(Z_E, \Omega_{\mathcal{X}_E,\mathcal{P}_E})$ satisfies a local-global principle for rational points. Then the prime numbers that divide the separable index of Z are precisely those that divide the separable index of some Z_L for $L \in \Omega_{\mathcal{X},\mathcal{P}}$. In particular, the separable index of Z is equal to* 1 *if and only if the separable index of each Z_L is equal to* 1.

Corollary 3.6. *In Corollary* 3.5*, if* $\operatorname{char} F = 0$*, or if Z is regular and generically smooth, then the assertion remains true if the separable index is replaced by the index. In particular, $(Z, \Omega_{\mathcal{X},\mathcal{P}})$ satisfies a local-global principle for zero-cycles of degree one.*

Proof. This is trivial in the former case. In the latter case, it follows from the fact that, under those hypotheses, the index is equal to the separable index [GLL13, Theorem 9.2]. □

The above results (and their proofs) show that if $(Z_E, \Omega_{\mathcal{X}_E,\mathcal{P}_E})$ satisfies a local-global principle for rational points for all finite separable E/F, the index (resp., separable index) of Z divides some power of the least common multiple of the indices (resp., separable indices) of the fields Z_L for $L \in \Omega_{\mathcal{X},\mathcal{P}}$. It would be interesting to obtain a bound on the exponent, in terms of F and Z. (See [HHKPS17, Section 2.2] for a partial result in this direction.)

Example 3.7. This example shows that, in Theorem 3.4 and in Corollaries 3.5 and 3.6, we cannot simply assume that Z satisfies a local-global principle over F, but we must also consider finite field extensions of F. Let k be a field of characteristic 0, let $T = k[[t]]$, let $K = k((t))$, let $F = K(x)$, and let $\mathcal{X}$ be the projective x-line over T. On the closed fiber of $\mathcal{X}$, take $\mathcal{P}$ to be the set $\{P_0, P_1\}$ consisting of the two points on the closed fiber where $x = 0$ and where $x = 1$. The complement of $\mathcal{P}$ in the closed fiber is a connected affine open set U; take $\mathcal{U} = \{U\}$.

Let $E = F[y]/(y^2 - x(x-t)(x-1)(x-1-t))$, a degree two separable field extension of F that splits over F_U but whose base change to F_{P_i} is a degree two field extension for $i = 0, 1$. So the (separable) index of $Y := \operatorname{Spec}(E)$ is 2 over F and over each F_{P_i}, but it is 1 over F_U. The normalization $\mathcal{Y}$ of $\mathcal{X}$ in E is a degree two branched cover of $\mathcal{X}$ whose fiber over the generic point of $\mathcal{X}$ is Y, and whose fiber over the closed point of $\operatorname{Spec}(T)$ consists of two copies of the projective k-line that meet at two points. Its reduction graph contains a loop, so by [HHK15a, Proposition 6.2] there is a degree three connected *split cover* $\mathcal{Z} \to \mathcal{Y}$ (i.e., $\mathcal{Z} \times_{\mathcal{Y}} Q$ consists of three copies of Q for every point $Q \in \mathcal{Y}$ other than the generic point). The fiber Z of $\mathcal{Z}$ over the generic point of $\mathcal{X}$ is the spectrum of a degree six separable field extension L of F, and hence the index of Z over F is 6. But since $\mathcal{Z} \to \mathcal{Y}$ is a split cover, the index of Z over each F_{P_i} is 2 and over F_U is 1 (the same as for Y).

Since the index of Z over F (resp., over F_{P_i}, F_U) is 6 (resp., 2, 1), it follows that $Z(F_{P_i}) = \emptyset$, and hence $(Z, \Omega_{\mathcal{X},\mathcal{P}})$ trivially satisfies a local-global principle for rational points, but the conclusions of Theorem 3.4 and its corollaries fail here if we take $\ell = 3$. The explanation is that $(Z_E, \Omega_{\mathcal{Y},\mathcal{P}_E})$ does not satisfy a local-global principle for rational points due to the above index computations for Z.

We next prove analogous results in which *all* of the points on the closed fiber X of $\mathcal{X}$ are used. That is, instead of considering a collection of overfields of the form $\Omega_{\mathcal{X},\mathcal{P}}$ for some finite set $\mathcal{P}$ as before, we consider the collection $\Omega_{\mathcal{X}}$ of all overfields of F of the form F_P, where P ranges over all (not necessarily closed) points of X.

Proposition 3.8. *Let F be a semiglobal field with normal model $\mathcal{X}$, and let $\mathcal{P}$ be a finite nonempty set of closed points of $\mathcal{X}$ which meets every irreducible component of the closed fiber. Let Z be an F-scheme of finite type. Then the following are equivalent:*

1. $(Z, \Omega_{\mathcal{X}})$ *satisfies a local-global principle for rational points.*
2. *For every choice of $\mathcal{P}$ as in Notation* 3.1*, $(Z, \Omega_{\mathcal{X},\mathcal{P}})$ satisfies a local-global principle for rational points.*

Proof. First, suppose that $(Z, \Omega_{\mathcal{X}})$ satisfies a local-global principle for rational points. Let $\mathcal{P}$ and $\mathcal{U}$ be as in Notation 3.1. Assume that Z has a rational point over F_ξ for each $\xi \in \mathcal{P} \cup \mathcal{U}$. Then every point P of X that is not contained in $\mathcal{P}$ lies in some $U \in \mathcal{U}$, and hence $F_U \subseteq F_P$. Thus Z has a rational point over F_P for every point $P \in X$ (including the generic points of irreducible components). Hence by hypothesis, Z has an F-point. Thus $(Z, \Omega_{\mathcal{X},\mathcal{P}})$ satisfies a local-global principle for rational points.

Conversely, suppose that $(Z, \Omega_{\mathcal{X},\mathcal{P}})$ satisfies a local-global principle for rational points for every choice of $\mathcal{P}$ as above. Assume that $Z(F_P)$ is nonempty for every (not necessarily closed) point $P \in X$. Given an irreducible component X_i of X, consider its generic point η_i. By [HHK15a, Proposition 5.8], since $Z(F_{\eta_i})$ is nonempty, it follows that $Z(F_{U_i})$ is also nonempty for some nonempty affine open subset $U_i \subset X_i$ that does not meet any other irreducible component of X. Let $\mathcal{U}$ be the collection of these disjoint open sets U_i of X, and let $\mathcal{P}$ be the complement of their union in X; note that this is a nonempty finite set of points. By hypothesis, Z has an F-point (note that $Z(F_P)$ is nonempty for each $P \in \mathcal{P}$). So $(Z, \Omega_{\mathcal{X}})$ satisfies a local-global principle for rational points. □

Using Proposition 3.8, we obtain the following result, which parallels Theorem 3.4 and Corollaries 3.5 and 3.6.

Theorem 3.9. *Let F be a semiglobal field with normal model $\mathcal{X}$. Let Z be an F-scheme of finite type such that, for every finite separable field extension E/F,*

$(Z_E, \Omega_{\mathcal{X}_E})$ satisfies a local-global principle for rational points. Then the following hold:

(a) *For every prime number ℓ, $(Z, \Omega_{\mathcal{X}})$ satisfies a local-global principle for separable closed points of degree prime to ℓ.*

(b) *The prime numbers that divide the separable index of Z are precisely those that divide the separable index of Z_L for some $L \in \Omega_{\mathcal{X}}$. In particular, the separable index of Z is equal to 1 if and only if the separable index of each Z_L is equal to 1.*

(c) *If $\mathrm{char}(F) = 0$ or Z is regular and generically smooth, the previous assertion remains true if the separable index is replaced by the index. In particular, $(Z, \Omega_{\mathcal{X}})$ satisfies a local-global principle for zero-cycles of degree one.*

Proof. We begin by proving (a). Suppose that, for every P in the closed fiber X of $\mathcal{X}$, Z has a point over a finite separable extension E_P of F_P having degree prime to ℓ. For each irreducible component X_i of X with generic point η_i, Proposition 2.10 yields a finite separable field extension E_i of F of degree prime to ℓ such that $Z(E_i \otimes_F F_{\eta_i})$ is nonempty. By [HHK15a, Proposition 5.8], there is a nonempty affine open subset $U_i \subset X_i$ that meets no other irreducible component of X such that $Z(E_i \otimes_F F_{U_i})$ is nonempty. Since E_i/F is a separable field extension of degree prime to ℓ, so is E_{U_i}/F_{U_i} for some direct factor E_{U_i} of $E_i \otimes_F F_{U_i}$. Here $Z(E_{U_i})$ is nonempty since $Z(E_i \otimes_F F_{U_i})$ is nonempty.

As in the proof of Proposition 3.8, we let $\mathcal{U}$ be the collection of open sets U_i, and we let $\mathcal{P}$ consist of the (finitely many) closed points of X that do not lie in any U_i. For any finite separable field extension E/F, $(Z_E, \Omega_{\mathcal{X}_E, \mathcal{P}_E})$ satisfies a local-global principle for rational points, by the hypothesis of the theorem together with Proposition 3.8 applied to Z_E. So by Theorem 3.4, $(Z, \Omega_{\mathcal{X},\mathcal{P}})$ satisfies a local-global principle for separable closed points of degree prime to ℓ. Since Z has a separable point of degree prime to ℓ over F_ξ for each $\xi \in \mathcal{P} \cup \mathcal{U}$, it follows that Z has a separable point of degree prime to ℓ.

The other two statements are then immediate, as in the analogous case for $\Omega_{\mathcal{X},\mathcal{P}}$ in Corollaries 3.5 and 3.6. □

The following corollary exhibits some cases in which our theorem above applies, for homogeneous spaces under linear algebraic groups that are connected and (retract) rational. Here, by a *homogeneous space* Z under a linear algebraic group G defined over a field F, we mean an F-scheme Z together with a group scheme action of G on Z over F such that the action of the group $G(\bar{F})$ on the set $Z(\bar{F})$ is simply transitive.

Corollary 3.10. *Let F be a semiglobal field with normal model $\mathcal{X}$. Let G be a linear algebraic group over F (i.e., a smooth affine group scheme over F), and let Z be a homogeneous space for G. In each of the following cases, the prime numbers that divide the separable index of Z are exactly those that divide the separable index of some Z_L, where $L \in \Omega_{\mathcal{X}}$, and moreover, $(Z, \Omega_{\mathcal{X}})$ satisfies a local-global principle for separable zero-cycles of degree one.*

(1) *G is connected and retract rational, and Z is a torsor under G.*

(2) *G is connected and rational, and $G(E)$ acts transitively on $Z(E)$ for every field extension E/F.*

The same conclusion holds without separability (i.e., for the index and for zero-cycles) if Z is smooth; e.g., this holds in case (1).

Proof. In both cases, $(Z, \Omega_{\mathcal{X}})$ satisfies a local-global principle for rational points. In the first case, this was shown in [Kra10] (see also [HHK15a, Corollary 6.5] for the case in which G is rational); in the second, this follows from [HHK15a, Corollary 2.8]. These local-global principles also hold for Z_E if E/F is a finite separable field extension of F, since the above assumptions on F are preserved under such a base change. Thus Theorem 3.9(b) applies. If Z is smooth (e.g., in case (1)), Theorem 3.9(c) also applies. □

Theorem 3.9 also applies in the case of certain other linear algebraic groups that need not be rational; see [HHK14, Section 4.3] for examples.

3.2. **Local-global principles over excellent henselian discrete valuation rings.** In this subsection, T is assumed to be excellent henselian instead of complete. We now extend the results from the previous subsection to that case via Artin approximation.

Let T be an excellent henselian discrete valuation ring with residue field k and fraction field K. Let $\mathcal{X}$ be a normal flat projective T-curve with closed fiber X and function field F; i.e., $\mathcal{X}$ is a normal model of F. Every finite separable field extension of F is also the function field of a normal flat projective T-curve. The completion $\widehat{T}$ of T is a complete discretely valued field with the same residue field k; its fraction field $\widehat{K}$ is the completion of K. Because T is henselian, the base change $\widehat{\mathcal{X}} := \mathcal{X} \times_T \widehat{T}$ is a normal connected projective $\widehat{T}$-curve with closed fiber X and function field $\widehat{F} := \mathrm{frac}(F \otimes_K \widehat{K})$. The complete local rings of $\widehat{\mathcal{X}}$ and $\mathcal{X}$ at a point $P \in X$ are naturally isomorphic, so we may write $\widehat{R}_P$ and F_P without ambiguity for those rings and their fraction fields. Similarly, for U as an affine open subset of the closed fiber, the notations $\widehat{R}_U$ and F_U are unambiguous, being the same for $\widehat{\mathcal{X}}$ and $\mathcal{X}$. If E/F is a finite separable field extension, we write $\widehat{E} = E \otimes_F \widehat{F} = \mathrm{frac}(E \otimes_K \widehat{K})$. This is the function field of the normalization $\mathcal{X}_E$ of $\mathcal{X}$ in E, whose closed fiber is denoted by X_E.

Proposition 3.11. *In the above situation, let Z be an F-scheme of finite type, and let E/F be a finite separable field extension. If $Z(\widehat{E})$ is nonempty, then so is $Z(E)$.*

Proof. By hypothesis, Z_E has a point over $\widehat{E}$. By Artin's approximation theorem [Art69, Theorems 1.10 and 1.12], Z_E also has an E-point. Equivalently, Z has an E-point. □

Let F be as above with normal model $\mathcal{X}$, and let Z be an F-scheme of finite type. As in Definition 3.3, $\Omega_{\mathcal{X}}$ denotes the set of all overfields of the form F_P. By Proposition 3.11, $(Z, \Omega_{\mathcal{X}})$ satisfies a local-global principle for rational points if and only if $(\widehat{Z}, \Omega_{\widehat{\mathcal{X}}})$ satisfies such a local-global principle, where $\widehat{Z} := Z \times_F \widehat{F}$.

Preserving the notation given just before Lemma 3.11, we have as follows.

Proposition 3.12. *The assertions of Theorem* 3.9 *remain true if F is the function field of a curve over an excellent henselian discrete valuation ring T as above.*

In particular, assume that Z is an F-scheme of finite type which is regular and generically smooth such that, for every finite separable field extension E/F, $(Z_E, \Omega_{\mathcal{X}_E})$ satisfies a local-global principle for rational points. Then $(Z, \Omega_{\mathcal{X}})$ satisfies a local-global principle for zero-cycles of degree one.

Proof. Let ℓ be prime, and assume that, for every $P \in X$, there is a finite separable extension E_P/F_P of degree prime to ℓ such that $Z(E_P)$ is nonempty. By Proposition 2.10, for each P, there is a finite separable field extension A_P/F of degree prime to ℓ such that $Z(A_P \otimes_F F_P)$ is nonempty. If η is the generic point of an irreducible component of X, then by applying [HHK15a, Proposition 5.8] to $\widehat{Z}_{A_\eta}$, we deduce that $Z(A_\eta \otimes_F F_U) = \widehat{Z}(A_\eta \otimes_F F_U)$ is nonempty for some dense open subset U of that component that meets no other component. Write $A_U = A_\eta$. Let $\mathcal{U}$ be the collection of these (finitely many) sets U, and let $\mathcal{P}$ be the complement in X of the union of these sets U.

Proceeding as in the second half of the proof of Theorem 3.4 using the field extensions A_P, A_U of F (for $P \in \mathcal{P}$ and $U \in \mathcal{U}$), we obtain a finite separable extension E/F of degree prime to ℓ such that $Z(E \times_F F_\xi)$ is nonempty for each $\xi \in \mathcal{P} \cup \mathcal{U}$. Now every point $P \in X$ outside of $\mathcal{P}$ lies on some $U \in U$, and hence F_P contains F_U; so $Z(E \times_F F_P)$ is nonempty for every point P on the closed fiber of $\mathcal{X}$. Thus by hypothesis, $Z(E)$ is nonempty. This proves the analogue of Theorem 3.9(a) in the henselian case.

The henselian analogues of the remaining assertions are then immediate, as before. □

As a consequence, Corollary 3.10 remains valid if F is the function field of a curve over an excellent henselian discrete valuation ring.

In the above situation, with T being an excellent henselian discrete valuation ring and P a point of $\mathcal{X}$, let R_P^{h} be the henselization of the local ring R_P at its maximal ideal, and let F_P^{h} be its fraction field. Also, for every discrete valuation v on F, let $R_v \subset F$ be the associated valuation ring, with henselization R_v^{h} and completion $\widehat{R}_v$, having fraction fields F_v^{h} and F_v. Recall that a point Q of $\mathcal{X}$ is called the *center* of v if R_v contains the local ring $\mathcal{O}_{Z,Q}$ of Q on Z, and if the maximal ideal $\mathfrak{m}_Q$ of $\mathcal{O}_{Z,Q}$ is the contraction of the maximal ideal $\mathfrak{m}_v$ of R_v.

Proposition 3.13. *Let T be an excellent henselian discrete valuation ring, and let F and $\mathcal{X}$ be as above. For every discrete valuation v on F, there is a point P on the closed fiber X of $\mathcal{X}$ such that $F_P^{\mathrm{h}} \subseteq F_v^{\mathrm{h}}$.*

Proof. We proceed as in the proof of [HHK15a, Proposition 7.4]. First, note that v has a center Q on $\mathcal{X}$ that is not the generic point of $\mathcal{X}$. In the case in which T is complete, this was shown in [HHK15a, Lemma 7.3]; but the proof used the completeness hypothesis only to cite Hensel's lemma in the proof of Lemma 7.1 there, and that holds by definition for henselian rings. If Q lies on X, then we may take $P = Q$. Otherwise, Q is a closed point of the generic fiber of $\mathcal{X}$, of codimension 1, and v is defined by Q; so $F_Q = F_v$. Since T is henselian, the closure of Q in $\mathcal{X}$ meets X at a single closed point P. Every étale neighborhood of P in $\mathcal{X}$ also defines an étale neighborhood of Q, since the residue field of Q is a henselian discretely valued field whose residue field corresponds to P. Thus $R_P^{\mathrm{h}} \subseteq R_Q^{\mathrm{h}} = R_v^{\mathrm{h}}$, and hence $F_P^{\mathrm{h}} \subseteq F_v^{\mathrm{h}}$. □

3.3. **Local-global principles with respect to discrete valuations in characteristic** 0**.** In the previous subsections, we considered local-global principles with respect to a collection of overfields arising from patching. More classically, local-global principles are stated with respect to overfields that are completions at discrete valuations. The aim of this subsection is to discuss when such local-global principles for rational points imply analogous principles for zero-cycles. Moreover, we show that in some cases, the existence of local zero-cycles of degree one even implies the existence of a global rational point.

Notation 3.14. For a field F, let Ω_F be the collection of overfields of F that are completions of F at discrete valuations.

Below we consider the function field F of a curve over an excellent henselian discrete valuation ring. In this situation, we begin by explaining how local-global principles with respect to discrete valuations relate to local-global principles as studied in the previous subsections.

Proposition 3.15. *Let T be an excellent henselian discrete valuation ring, and let F be the function field of a normal flat projective T-curve $\mathcal{X}$ with closed fiber X. Let Z be an F-scheme of finite type. If (Z, Ω_F) satisfies a local-global principle for rational points, then $(Z, \Omega_{\mathcal{X}})$ satisfies a local-global principle for rational points.*

Proof. Suppose that Z has an F_P-point for every point $P \in X$. We want to show that Z has an F-point. By Proposition 3.13, for every discrete valuation v on F, there is a point $P \in X$ such that $F_P^{\mathrm{h}} \subseteq F_v^{\mathrm{h}}$. By Artin's approximation theorem [Art69, Theorem 1.10] applied to the coordinate ring of an affine neighborhood of P in $\mathcal{X}$, Z has an F_P^{h}-point. Hence Z has an F_v^{h}-point, and thus also an F_v-point. By the local-global hypothesis, Z has an F-point. □

We next consider the case of torsors under a linear algebraic group G over a function field F. These are classified up to isomorphism by the first Galois cohomology set $H^1(F, G)$. We define $Ш(F, G)$ to be the kernel of the local-global map $H^1(F, G) \to \prod_v H^1(F_v, G)$, where v runs over all discrete valuations on F. Hence $Ш(F, G)$ is trivial if and only if (Z, Ω_F) satisfies a local-global principle for rational points for every G-torsor Z.

Theorem 3.16. *Let T be an excellent henselian discrete valuation ring, let K be its field of fractions, and let k be its residue field. Suppose that* $\mathrm{char}(k) = 0$. *Let F be a one-variable function field over K, and let $\mathcal{X}$ be a regular model of F over T. Let G be a connected linear algebraic group over F which is the generic fiber of a reductive smooth group scheme over $\mathcal{X}$. Let Z be a G-torsor over F. Suppose that, for all finite separable field extensions E/F, (Z_E, Ω_E) satisfies a local-global principle for rational points. Then (Z, Ω_F) satisfies a local-global principle for zero-cycles of degree one. In particular, this applies if* $Ш(E, G_E)$ *is trivial for all finite separable field extensions E/F.*

Proof. Let Z be a G-torsor over F. Let $\mathcal{Y}$ be a sequence of blowups of $\mathcal{X}$ such that Z is unramified except along a union of regular curves with normal crossings; i.e., Z extends to a torsor on the complement of the union of these curves. Since the underlying group scheme of G is smooth over $\mathcal{X}$, it is also smooth over $\mathcal{Y}$. Let Y be the closed fiber of $\mathcal{Y}$.

Let $P \in \mathcal{Y}$ be a closed point. Let R_P be the regular local ring at P, and let $\widehat{R}_P$ be the completion of R_P at its maximal ideal. Since the ramification locus of Z is a union of regular curves with normal crossings, there exist $\pi, \delta \in R_P$ such that the maximal ideal at P is (π, δ) and that Z is unramified on R_P, except possibly at π and δ.

Suppose that Z has a zero-cycle of degree one over F_v for all discrete valuations v of F. Let ℓ be a prime. Then there exist field extensions $E_\pi/F_{P,\pi}$ and $E_\delta/F_{P,\delta}$ of degree prime to ℓ such that $Z(E_\pi) \neq \emptyset$ and $Z(E_\delta) \neq \emptyset$. Here $F_{P,\pi} := (F_P)_\pi$ and $F_{P,\delta} := (F_P)_\delta$ are associated with F_P as in the beginning of Subsection 2.3. By Lemma 2.13, there exists a field extension E_P/F_P of degree prime to ℓ such that the integral closure B_P of $\widehat{R}_P$ in E_P is a complete regular local ring and that E_π (resp., E_δ) is isomorphic to a subfield of the field $E_P \otimes_{F_P} F_{P,\pi}$ (resp., $E_P \otimes_{F_P} F_{P,\delta}$). Moreover, the maximal ideal of B_P is of the form (π', δ') for unique primes π' and δ' lying over π and δ, respectively.

Since $Z(E_\pi)$ and $Z(E_\delta)$ are nonempty, $Z(E_P \otimes F_{P,\pi})$ and $Z(E_P \otimes F_{P,\delta})$ are nonempty. Hence $Z_{E_P \otimes F_{P,\pi}} \simeq G_{E_P \otimes F_{P,\pi}}$ and $Z_{E_P \otimes F_{P,\delta}} \simeq G_{E_P \otimes F_{P,\delta}}$. In particular, Z is unramified at π' and δ'. But Z is unramified on R_P, except possibly at π and δ; thus it is indeed unramified at every height 1 prime ideal of B_P. Hence the class of Z_{E_P} comes from a class ζ in $H^1(B_P, G)$ [CTS79, Corollary 6.14]. Since ζ is trivial over the completion at π', the image of ζ in $H^1(k(\pi'), G)$ is trivial, where $k(\pi')$ is the residue field of E_π. Since $\zeta \in H^1(B_P, G)$, the image of ζ in $H^1(k(\pi'), G)$ comes from the image $\zeta(\pi')$ of ζ in $H^1(B_P/(\pi'), G)$. Since $B_P/(\pi')$ is a discrete valuation ring with field of fractions $k(\pi')$ and the image of $\zeta(\pi')$ in $H^1(k(\pi'), G)$ is trivial, $\zeta(\pi')$ is trivial [Nis84]. Hence the image of ζ in $H^1(\kappa, G)$ is trivial, where κ is the residue field of B_P. Since B_P is a complete regular local ring, Hensel's lemma implies that ζ is trivial in $H^1(B_P, G)$, and hence its image is trivial in $H^1(E_P, G)$. Thus $Z(E_P) \neq \emptyset$. Since $\gcd(\ell, [E_P : F_P]) = 1$ and since the prime ℓ is arbitrary, Z has a zero-cycle of degree one over F_P.

Since (Z, Ω_F) satisfies a local-global principle for rational points, $(Z, \Omega_{\mathcal{Y}})$ satisfies a local-global principle for rational points by Proposition 3.15, and the same holds for finite separable field extensions E/F. Since Z has a zero-cycle of degree one over F_P for all $P \in Y$, Z has a zero-cycle of degree one over F by Proposition 3.12. $\square$

Remark 3.17. As the proof shows, the above theorem holds under the weaker assumption that for every blowup (i.e., birational projective morphism) $\mathcal{Y} \to \mathcal{X}$, and for every finite separable field extension E/F, $(Z_E, \Omega_{\mathcal{Y}_E})$ satisfies a local-global principle for rational points. This hypothesis is equivalent to the one stated in Theorem 3.16 in important special cases, e.g., when K is complete, or when k is algebraically closed and G is stably rational. The former case follows from [HHK15a, Theorem 8.10(ii)]. In the latter case, the henselization F_P^{h} of F at a point P on the closed fiber is algebraic over F, so every discrete valuation on F_P^{h} restricts to a discrete valuation on F. Thus if $Z(F_v) \neq \emptyset$ for every $v \in \Omega_F$, then $Z((F_P^{\mathrm{h}})_v) \neq \emptyset$ for every $v \in \Omega_{F_P^{\mathrm{h}}}$. By [BKG04, Cor. 7.7], it follows that $Z(F_P^{\mathrm{h}}) \neq \emptyset$, and hence $Z(F_P) \neq \emptyset$ as required.

Corollary 3.18. *Let T, F, $\mathcal{X}$, and G be as in Theorem* 3.16*. Assume that the linear algebraic group G is retract rational over F, and let Z be a G-torsor over F. Then (Z, Ω_F) satisfies a local-global principle for zero-cycles of degree one.*

Proof. By Remark 3.17, it suffices to check that $(Z_E, \Omega_{\mathcal{Y}_E})$ satisfies a local-global principle for rational points, for all blowups $\mathcal{Y}$ of a regular model $\mathcal{X}$ and all finite separable field extensions E/F. That condition holds by [Kra10] (see also Corollary 3.10). $\square$

3.4. **Local-global principles with respect to discrete valuations: Case of an algebraically closed residue field.** The remainder of this section is devoted to results about torsors and projective homogeneous spaces over certain two-dimensional fields over an algebraically closed field k of characteristic 0, including semiglobal fields. More precisely, we have the following.

Hypothesis 3.19. *Let k be an algebraically closed field of characteristic* 0*. Let F be one of the following:*

(a) *The fraction field of a normal, two-dimensional, excellent henselian local ring with residue field k.*
(b) *A one-variable function field over the fraction field of an excellent henselian discrete valuation ring with residue field k.*

Note that if F is a field as in Hypothesis 3.19, then finite field extensions of F are also of the same type.

Proposition 3.20. *Let F be a field as in Hypothesis* 3.19*. Then F has the following properties:*

(1) $\mathrm{cd}(F) \leq 2$.
(2) *For central simple algebras over finite field extensions of F, period and index coincide.*
(3) *For any semisimple simply connected group G over F, $H^1(F,G)=1$.*
(4) *For any quasi-trivial torus P over F, the diagonal map*

$$H^2(F,P) \to \prod_v H^2(F_v,P)$$

is injective. (*Here v runs through all discrete valuations on F.*)

Proof. We first assume that F is as in Hypothesis 3.19(a). For the first three properties, see [CTGP04, Thm. 1.4], whereas property (4) is an immediate consequence of [CTOP00, Cor. 1.10].

Next assume F is as in Hypothesis 3.19(b). For property (1), the fraction field K of an excellent henselian discrete valuation ring with algebraically closed residue field is C_1 by [Ser00, II, Section 3.3(c)], and hence a one-variable function field F over K is C_2 by [Ser00, II, Section 4.5, Example (b)]. Therefore $\mathrm{cd}(F) \leq 2$ by [Ser00, end of II, Section 4.5], giving property (1). Property (2) is a special case of [HHK09, Corollary 5.6]. Property (4) follows from [CTOP00, Corollary 1.10(b)]. It remains to show property (3). The absolute Galois group of the fraction field K of an excellent henselian discrete valuation ring with residue field k is procyclic and in particular abelian, because the finite extensions of K are all obtained by taking roots of the uniformizer. Hence $\bar{K}\otimes_K F/F$ is a procyclic field extension, of cohomological dimension at most 1 by Tsen's theorem. But F^{ab} contains $\bar{K}\otimes_K F$ since the absolute Galois group of K is abelian, and moreover this field extension is algebraic. Hence $\mathrm{cd}(F^{\mathrm{ab}}) \leq 1$ by [Ser00, II, Section 4.1, Proposition 10]. The statement now follows from [CTGP04, Thm. 1.2(v)] (see also [Par10, end of Section 6]). □

Remark 3.21. As pointed out by Starr, Prop. 3.20(1)–(3) for fields of type (b) in Hypothesis 3.19 can also be deduced by a localization process from global results established using the theory of rationally simply connected varieties (see [Sta17, Prop. 4.4]).

Let G be a connected reductive linear algebraic group over a field K of characteristic 0. By [BK00, Lemma 1.4.1] and [CT08, Prop. 4.1, Cor. 5.3], there exists a central extension

$$1 \to P \to H \to G \to 1 \qquad (*)$$

such that H is a connected reductive group, its derived group H^{ss} is a (semisimple) simply connected group, the quotient H/H^{ss} is a torus Q, and the kernel P is a quasi-trivial torus. If moreover G is rational over K, there exists such a presentation of G for which the torus Q is a quasi-trivial torus.

Proposition 3.22. *Let F be any field of characteristic 0, and let G be a connected reductive linear algebraic group over F. Suppose that $H^1(F, M) = 1$ for all semisimple simply connected groups M over F. Let Z be a G-torsor over F.*

(a) *If Z has a zero-cycle of degree one, then Z has a rational point.*
(b) *If G is rational, then Z has a rational point if and only if the image of the class of Z under the boundary map $H^1(F, G) \to H^2(F, P)$ vanishes. (Here P is as in the above short exact sequence.)*

Proof. For the proof of the first statement, let $1 \to P \to H \to G \to 1$ be as in $(*)$ above. Since P is a quasi-trivial torus, $H^1(F, P)$ is trivial, and hence we have an exact sequence of pointed sets

$$1 \to H^1(F, H) \to H^1(F, G) \to H^2(F, P).$$

Let $[Z] \in H^1(F, G)$ denote the class of Z, and let ζ be the image of $[Z]$ in $H^2(F, P)$. Suppose that Z has a zero-cycle of degree one. Then since P is a torus, using restriction-corestriction, it follows that ζ is the trivial element. Hence there exists an H-torsor $\widetilde{Z}$ such that its class $[\widetilde{Z}] \in H^1(F, H)$ maps to $[Z]$ in $H^1(F, G)$. Since, for any finite field extension L/F, the map $H^1(L, H) \to H^1(L, G)$ has trivial kernel and Z has a zero-cycle of degree one, $\widetilde{Z}$ has a zero-cycle of degree one.

Consider the exact sequence $1 \to H^{ss} \to H \to Q \to 1$. Since H^{ss} is semisimple simply connected, by the hypothesis $H^1(F, H^{ss})$ is trivial. Hence the map $H^1(F, H) \to H^1(F, Q)$ has trivial kernel.

Let $\bar{Z}$ be a torsor representing the image of $[\widetilde{Z}]$ in $H^1(F, Q)$. Since $\widetilde{Z}$ has a zero-cycle of degree one, $\bar{Z}$ has a zero-cycle of degree one. Since Q is a torus, once again the restriction-corestriction argument gives that $\bar{Z}$ has a rational point. Since the map $H^1(F, H) \to H^1(F, Q)$ has trivial kernel, $\widetilde{Z}$ is trivial. In particular, Z is trivial and hence has a rational point.

To see the second assertion, assume that G is rational over F. Then, by the discussion preceding Proposition 3.22, we may assume that in the exact sequence

$$1 \to H^{ss} \to H \to Q \to 1,$$

with H^{ss} semisimple simply connected, Q is a quasi-trivial torus. We thus have $H^1(F, H) = 1$. Cohomology of the exact sequence

$$1 \to P \to H \to G \to 1$$

then gives the result. □

Theorem 3.23. *Let F be a field as in Hypothesis 3.19. Let G be a connected and rational linear algebraic group over F. Let Z be a G-torsor over F. If Z has a zero-cycle of degree one over F_v for all discrete valuations v of F, then Z has a rational point over F.*

Proof. Since F has characteristic 0, the kernel $H^1(F, R_u(G))$ of the map $H^1(F,G) \to H^1(F, G/R_u(G))$ is trivial by [Ser00, III, Section 2.1, Prop. 6]; here $R_u(G)$ denotes the unipotent radical of G. The analogous statement holds for all field extensions E/F. Hence we may assume without loss of generality that G is reductive. The hypotheses of Proposition 3.22 then hold by Proposition 3.20(3).

Let

$$1 \to P \to H \to G \to 1$$

be a central extension as in the discussion preceding Proposition 3.22. As in the proof of that proposition, this gives rise to an exact sequence of pointed sets

$$1 \to H^1(F,H) \to H^1(F,G) \to H^2(F,P),$$

and similarly,

$$1 \to H^1(F_v,H) \to H^1(F_v,G) \to H^2(F_v,P)$$

for each discrete valuation v of F. By assumption, Z_{F_v} has a zero-cycle of degree one for each such v. By restriction-corestriction (as in the proof of Proposition 3.22), the class of Z_{F_v} maps to the trivial element in $H^2(F_v,P)$ for each v. According to Proposition 3.20(4), this implies that the class of Z maps to the trivial element in $H^2(F,P)$. Hence Proposition 3.22(b) applies and implies the result. □

Remark 3.24. For an alternative proof after reducing to the reductive case, first note that $Z(F_v) \neq \emptyset$ by Proposition 3.22(a). The local case (when F is as in Hypothesis 3.19(a)) now follows from [BKG04, Corollary 7.7]. In the case of a function field of a normal curve $\mathcal{X}$ over an excellent henselian discrete valuation ring T as in Hypothesis 3.19(b), the local case implies that Z has a point over the fraction field of the henselization of the local ring of $\mathcal{X}$ at any closed point P. Hence Z also has a point over the fraction field of the complete local ring at P. It also has a point over F_η, for each generic point η of the closed fiber. Thus it has a point over the function field of $\mathcal{X} \times_T \widehat{T}$ by [HHK15a, Theorem 5.10]. This case of the theorem then follows from Lemma 3.11.

Below we study local-global principles for zero-cycles on projective homogeneous spaces. We use a criterion for the existence of rational points from [CTGP04]. We begin by recalling some notation and facts (see [CTGP04, pp. 333–335], which closely follows the work of Borovoi [Bor93]).

Let F be a field of characteristic 0. Let G be a connected reductive linear algebraic group over F, and let Z be a projective homogeneous G-space. Let $\bar{H}$ be the isotropy group of an $\bar{F}$-point of Z; note that since Z is projective, $\bar{H}$ is parabolic and hence connected. As in [CTGP04], one can define an associated F-torus H^{tor} (this is an F-form of the maximal torus quotient of $\bar{H}$). Because Z is projective, the F-torus H^{tor} is a quasi-trivial torus by [CTGP04, Lemma 5.6].

As in [CTGP04], one may further define an F-kernel $L = (\bar{H}, \kappa)$, a cohomology set $H^2(F,L)$, and a class $\eta(Z) \in H^2(F,L)$ associated with Z. This class is a *neutral class* if and only if Z comes from a G-torsor; i.e., if and only if there exists a G-torsor Y and a G-equivariant morphism $Y \to Z$. There is a natural map $t_* : H^2(F,L) \to H^2(F, H^{\mathrm{tor}})$ which is functorial in the field F [CTGP04], and which sends neutral classes in $H^2(F,L)$ to the trivial element in $H^2(F, H^{\mathrm{tor}})$.

The following proposition is an immediate consequence of [CTGP04, Prop. 5.4] and will be the key ingredient in the proof of the theorem below.

Proposition 3.25. *Suppose that F is a field which satisfies properties* (1), (2), *and* (3) *in Proposition* 3.20. *Let G be a semisimple simply connected linear algebraic group over F, and let Z be a projective homogeneous space under G. Then $Z(F) \neq \emptyset$ if and only if $t_*(\eta(Z)) = 1 \in H^2(F, H^{\text{tor}})$.*

Proof. If $Z(F)$ contains a rational point x, then the map $G \to Z$ given by the action of G on x shows that $\eta(Z)$ is neutral, and thus $t_*(\eta(Z)) = 1 \in H^2(F, H^{\text{tor}})$. Conversely, if $t_*(\eta(Z)) = 1$, then $\eta(Z)$ is neutral by [CTGP04, Prop. 5.4]; i.e., it comes from a G-torsor. By assumption, $H^1(F, G)$ consists of a single element; hence that torsor has an F-rational point which maps to a point on Z, showing that $Z(F) \neq \emptyset$. □

Theorem 3.26. *Let F be a field of characteristic* 0 *which satisfies properties* (1), (2), *and* (3) *in Proposition* 3.20. *Let G be a connected linear algebraic group over F, and let Z be a projective homogeneous G-space. If Z has a zero-cycle of degree one, then Z has a rational point. In particular, this assertion holds whenever F is a field as in Hypothesis* 3.19.

Proof. Let $R(G)$ be the radical of G, i.e., the maximal connected solvable subgroup of G. This is contained in any parabolic subgroup of G. Since Z is projective, the action of G on Z factors through $G^{ss} = G/R(G)$. Let $G^{sc} \to G^{ss}$ be the simply connected cover of G^{ss}. Thus we may view Z as a projective homogenous G^{sc}-space. Replacing G with G^{sc}, we may therefore assume that G is semisimple and simply connected (see [CTGP04, proof of Corollary 5.7]).

If Z has a zero-cycle of degree one, there exist finite field extensions F_i/F such that $Z(F_i) \neq \emptyset$ for all i and the gcd of the degrees of F_i/F is 1. Since $Z(F_i) \neq \emptyset$, the image of $t_*(\eta(Z))$ in $H^2(F_i, H^{\text{tor}})$ is trivial by Proposition 3.25. Using a restriction-corestriction argument, we conclude that $t_*(\eta(Z))$ is trivial in $H^2(F, H^{\text{tor}})$. One then concludes that $Z(F) \neq \emptyset$ by a second application of Proposition 3.25. The last statement follows from Proposition 3.20. □

If F is the function field of a p-adic curve, there are projective homogeneous spaces under simply connected groups which admit zero-cycles of degree one, but with no rational points. See [Par05].

Theorem 3.27. *Let F be a field as in Hypothesis* 3.19, *let G be a connected linear algebraic group over F, and let Z be a projective homogeneous G-space. If Z has a zero-cycle of degree one over F_v for each discrete valuation v of F, then Z has a rational point over F.*

Proof. As in the first paragraph of the proof of Theorem 3.26, we may reduce to the case that G is semisimple simply connected. If Z has a zero-cycle of degree one over each F_v, then a corestriction-restriction argument shows that the class $t^*(\eta(Z)) = 1 \in H^2(F, H^{\text{tor}})$ has trivial image in each $H^2(F_v, H^{\text{tor}})$. Since H^{tor} is a quasi-trivial torus, Proposition 3.20(4) implies that $t^*(\eta(Z)) = 1 \in H^2(F, H^{\text{tor}})$. An application of Proposition 3.25 then yields $Z(F) \neq \emptyset$. □

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