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Boundary Layer of Transport Equation with In-Flow Boundary

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Abstract

Consider the steady neutron transport equation in two dimensional convex domains with an in-flow boundary condition. We establish the diffusive limit while the boundary layers are present. Our contribution relies on a delicate decomposition of boundary data to separate the regular and singular boundary layers, novel weighted $W^{1,\infty}$ estimates for the Milne problem with geometric correction in convex domains, and an $L^{2m} - L^{\infty}$ framework which yields stronger remainder estimates.

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1. Introduction

1.1. Problem Formulation

We consider the steady neutron transport equation in a two-dimensional bounded convex domain with an in-flow boundary. In the spacial domain $\vec{x} = (x_1, x_2) \in \Omega$ where $\partial\Omega \in C^4$ and the velocity domain $\vec{w} = (w_1, w_2) \in \mathbb{S}^1$, the neutron density $u^\epsilon(\vec{x}, \vec{w})$ satisfies

$$\begin{cases} \epsilon\vec{w}\cdot\nabla_x u^\epsilon + u^\epsilon - \bar{u}^\epsilon = 0 \ \text{ in } \ \Omega\times\mathbb{S}^1, \\ u^\epsilon(\vec{x}_0, \vec{w}) = g(\vec{x}_0, \vec{w}) \ \text{ for } \ \vec{w}\cdot\vec{\nu} < 0 \ \text{ and } \ \vec{x}_0 \in \partial\Omega, \end{cases} \tag{1.1}$$

where

$$\bar{u}^\epsilon(\vec{x}) = \frac{1}{2\pi}\int_{\mathbb{S}^1} u^\epsilon(\vec{x}, \vec{w})\,\mathrm{d}\vec{w}; \tag{1.2}$$

$\vec{\nu}$ is the outward unit normal vector, with the Knudsen number $0 < \epsilon << 1$. We intend to study the behavior of u^ϵ as $\epsilon \to 0$.

Based on the flow direction, we can divide the boundary $\Gamma = \{(\vec{x}, \vec{w}) : \vec{x} \in \partial\Omega\}$ into the in-flow boundary Γ^-, the out-flow boundary Γ^+ and the grazing set Γ^0

$$\Gamma^- = \{(\vec{x}, \vec{w}) : \vec{x} \in \partial\Omega,\ \vec{w}\cdot\vec{\nu} < 0\}, \tag{1.3}$$
$$\Gamma^+ = \{(\vec{x}, \vec{w}) : \vec{x} \in \partial\Omega,\ \vec{w}\cdot\vec{\nu} > 0\}, \tag{1.4}$$
$$\Gamma^0 = \{(\vec{x}, \vec{w}) : \vec{x} \in \partial\Omega,\ \vec{w}\cdot\vec{\nu} = 0\}. \tag{1.5}$$

It is easy to see that $\Gamma = \Gamma^+ \cup \Gamma^- \cup \Gamma^0$. In particular, the boundary condition is only given for Γ^-.

1.2. Background and Method

1.2.1. Asymptotic Analysis Diffusive limits, or more general hydrodynamic limits, are central to connecting kinetic theory and fluid mechanics. The basic idea is to consider the asymptotic behaviors of the solutions to Boltzmann equation, transport equation, or Vlasov systems. Since the early 20th century, this type of problem has been extensively studied in many different settings: steady or unsteady, linear or nonlinear, strong solution or weak solution, etc..

Among all these variations, one of the simplest but most important models—neutron transport equation in bounded domains-has attracted a lot of attention since the dawn of the atomic age. The neutron transport equation is usually regarded as a linear prototype of the more complicated nonlinear Boltzmann equation, and thus is an ideal starting point to develop new theories and techniques. We refer to [10–20] for more details.

For the steady neutron transport equation, the exact solution can be approximated by the sum of an interior solution U and a boundary layer $\mathcal{U}$. The interior solution satisfies certain fluid equations or thermodynamic equations, and the boundary layer satisfies a half-space kinetic equation, which decays rapidly when it is away from the boundary.

The justification of the diffusive limit usually involves two steps:

(1) Expanding $U = \sum_{k=0}^{\infty} \epsilon^k U_k$ and $\mathcal{U} = \sum_{k=0}^{\infty} \epsilon^k \mathcal{U}_k$ as power series of ϵ and proving the coefficients U_k and $\mathcal{U}_k$ are well-defined. Traditionally, the estimates of the interior solutions U_k are relatively straightforward. On the other hand, boundary layers $\mathcal{U}_k$ satisfy one-dimensional half-space problems which lose some key structures of the original equations. The well-posedness of boundary layer equations are sometimes extremely difficult and it is possible that they are actually ill-posed (for example certain type of Prandtl layers [6]).

(2) Proving that $R = u^{\epsilon} - U_0 - \mathcal{U}_0 = o(1)$ as $\epsilon \to 0$. Ideally, this should be done just by expanding to the leading-order level U_0 and $\mathcal{U}_0$. However, in singular perturbation problems, the estimates of the remainder R usually involve negative powers of ϵ, which requires an expansion to higher-order terms U_N and $\mathcal{U}_N$ for $N \geqq 1$ such that we have a sufficient power of ϵ. In other words, we define $R = u^{\epsilon} - \sum_{k=0}^{N} \epsilon^k U_k - \sum_{k=0}^{N} \epsilon^k \mathcal{U}_k$ for $N \geqq 1$ instead of $R = u^{\epsilon} - U_0 - \mathcal{U}_0$ to get better estimates of R.

1.2.2. Classical Approach The construction of kinetic boundary layers has long been believed to be satisfactorily solved since Bensoussan, Lions and Papanicolaou published their remarkable paper [1] in 1979. Their formulation, based on the flat Milne problem, was later extended to treat the nonlinear Boltzmann equation (see [22,23]).

In detail, in Ω, let $\eta \in [0, \infty)$ denote the rescaled normal variable with respect to the boundary, $\tau \in [-\pi, \pi)$ the tangential variable, and $\phi \in [-\pi, \pi)$ the velocity variable defined in (2.21), (2.25), and (2.31). The boundary layer $\mathcal{U}_0$ satisfies the flat Milne problem,

$$\sin\phi \frac{\partial \mathcal{U}_0}{\partial \eta} + \mathcal{U}_0 - \bar{\mathcal{U}}_0 = 0. \tag{1.6}$$

Unfortunately, in [24], we demonstrated that both the proof and result of this formulation are invalid due to a lack of regularity in estimating $\dfrac{\partial \mathcal{U}_0}{\partial \tau}$. Also, this glitch was further captured by numerical tests in [21]. This pulls the whole research back to the starting point, and any later results based on this type of boundary layer should be reexamined.

To be more specific, the remainder estimates require $\mathscr{U}_1 \in L^\infty$, which needs $\dfrac{\partial \mathscr{U}_0}{\partial \tau} \in L^\infty$. However, though [1] shows that $\mathscr{U}_0 \in L^\infty$, it does not necessarily mean that $\dfrac{\partial \mathscr{U}_0}{\partial \eta} \in L^\infty$. Furthermore, this singularity $\dfrac{\partial \mathscr{U}_0}{\partial \eta} \notin L^\infty$ will be transferred to $\dfrac{\partial \mathscr{U}_0}{\partial \tau} \notin L^\infty$. A careful construction of boundary data justifies this invalidity, that is the chain of estimates

$$R = o(1) \Leftarrow \mathscr{U}_1 \in L^\infty \Leftarrow \frac{\partial \mathscr{U}_0}{\partial \tau} \in L^\infty \Leftarrow \frac{\partial \mathscr{U}_0}{\partial \eta} \in L^\infty, \tag{1.7}$$

is broken since the rightmost estimate is wrong.

Note that the difficulty of above classical approach is purely due to the geometry of the curved boundary $\partial\Omega$. When $\partial\Omega$ is flat, that is when Ω is the half space $\mathbb{R}\times\mathbb{R}^+$, the flat Milne problem (1.6) provides the correct description of the kinetic boundary layer.

1.2.3. Geometric Correction While the classical method fails, a new approach with geometric correction to the boundary layer construction has been developed to ensure regularity in the cases of disk and annulus in [24] and [25]. The new boundary layer $\mathscr{U}_0$ satisfies the ϵ-Milne problem with geometric correction,

$$\sin\phi \frac{\partial \mathscr{U}_0}{\partial \eta} + \frac{\epsilon}{R_\kappa - \epsilon\eta} \cos\phi \frac{\partial \mathscr{U}_0}{\partial \phi} + \mathscr{U}_0 - \bar{\mathscr{U}}_0 = 0, \tag{1.8}$$

where R_κ is the radius of curvature of the boundary. We proved that the solution recovers the well-posedness and exponential decay as in the flat Milne problem, and the regularity in τ is indeed improved, that is $\dfrac{\partial \mathscr{U}_0}{\partial \tau} \in L^\infty$. A similar formulation was first introduced by Chandrasekhar in [3] to describe the transfer of radiations with spherical symmetry, and our analysis provides a rigorous justification of its implementation in the construction of kinetic boundary layers.

However, this new method fails to treat more general domains. Roughly speaking, we have two contradictory goals to achieve:

- To prove diffusive limits, the remainder estimates require higher-order regularity estimates of the boundary layer.
- The geometric correction $\dfrac{\epsilon}{R_\kappa - \epsilon\eta} \cos\phi \dfrac{\partial \mathscr{U}_0}{\partial \phi}$ in the boundary layer equation is related to the curvature of the boundary curve, which prevents higher-order regularity estimates.

In other words, the improvement of regularity is still not enough to close the proof. We may analyze the effects of different domains and formulations as follows:

- In the absence of the geometric correction $\dfrac{\epsilon}{R_\kappa - \epsilon\eta} \cos\phi \dfrac{\partial \mathscr{U}_0}{\partial \phi}$, which is the flat Milne problem as in [1], the key tangential derivative $\dfrac{\partial \mathscr{U}_0}{\partial \tau}$ is not bounded. Therefore, the expansion breaks down.

- In the domain of disk or annulus, when R_κ is constant, as in [24] and [25], $\dfrac{\partial \mathcal{U}_0}{\partial \tau}$ is bounded, since the tangential derivative $\dfrac{\partial}{\partial \tau}$ commutes with the equation, and thus we do not even need to estimate $\dfrac{\partial \mathcal{U}_0}{\partial \eta}$.
- For general smooth convex domains, when R_κ is a function of τ, $\dfrac{\partial \mathcal{U}_0}{\partial \tau}$ relates to the normal derivative $\dfrac{\partial \mathcal{U}_0}{\partial \eta}$, which has been shown to be possibly unbounded in [24]. Therefore, we get stuck again at the regularity estimates.

1.2.4. Diffusive Boundary In [7] and [8], for the case of diffusive boundary, we pushed the above argument from both sides, that is improvements in remainder estimates and boundary layer regularity.

In detail, consider the boundary layer expansion

$$\mathcal{U}(\eta, \tau, \vec{w}) \sim \mathcal{U}_0(\eta, \tau, \vec{w}) + \epsilon \mathcal{U}_1(\eta, \tau, \vec{w}). \tag{1.9}$$

The diffusive boundary condition

$$u^\epsilon(\vec{x}_0, \vec{w}) = \frac{1}{2} \int_{\vec{w}\cdot\vec{\nu}>0} u^\epsilon(\vec{x}_0, \vec{w})(\vec{w} \cdot \vec{\nu})\, \mathrm{d}\vec{w} + \epsilon g(\vec{x}_0, \vec{w}) \tag{1.10}$$

leads to an important simplification: $\mathcal{U}_0 = 0$. As stated in [24], the next-order boundary layer $\mathcal{U}_1$ must formally satisfy

$$\sin\phi \frac{\partial \mathcal{U}_1}{\partial \eta} + \frac{\epsilon}{R_\kappa - \epsilon\eta} \cos\phi \frac{\partial \mathcal{U}_1}{\partial \phi} + \mathcal{U}_1 - \bar{\mathcal{U}}_1 = 0. \tag{1.11}$$

Naturally, the diffusive limit requires an estimate of $\dfrac{\partial \mathcal{U}_1}{\partial \tau}$. Here, a key observation is that $W = \dfrac{\partial \mathcal{U}_1}{\partial \tau}$ satisfies

$$\begin{aligned} &\sin\phi \frac{\partial W}{\partial \eta} + \frac{\epsilon}{R_\kappa - \epsilon\eta} \cos\phi \frac{\partial W}{\partial \phi} + W - \bar{W} \\ &\quad = -\frac{\partial_\tau R_\kappa}{R_\kappa - \epsilon\eta} \left(\frac{\epsilon}{R_\kappa - \epsilon\eta} \cos\phi \frac{\partial \mathcal{U}_1}{\partial \phi} \right). \end{aligned} \tag{1.12}$$

Note that the right-hand side is part of the $\mathcal{U}_1$ equation and its estimate depends on $\sin\phi \dfrac{\partial \mathcal{U}_1}{\partial \eta}$. In other words, the estimate of $\dfrac{\partial \mathcal{U}_1}{\partial \tau}$ depends on $\sin\phi \dfrac{\partial \mathcal{U}_1}{\partial \eta}$, not just $\dfrac{\partial \mathcal{U}_1}{\partial \eta}$ which is possibly unbounded. The $\sin\phi$ is crucial to eliminate the singularity. This forms the major proof in [7] and [8], that is the weighted regularity of $\mathcal{U}_1$.

Our main idea is to delicately track $\mathcal{U}_1$ along the characteristics in the mild formulation, and prove the weighted $W^{1,\infty}$ estimates of the boundary layer. In particular, we showed that $\dfrac{\partial \mathcal{U}_1}{\partial \tau}$ is bounded even when R_κ is not constant for general convex domains. Furthermore, with a novel $L^{2m} - L^\infty$ framework, we prove a new remainder estimate, which does not require any higher regularity estimates of the boundary layer.

In summary, in [7] and [8], we proved the diffusive limit: u^ϵ converges to the solution of a Laplace's equation with Neumann boundary condition.

1.2.5. In-Flow Boundary and Basic Ideas It is notable that, for the case of in-flow boundary as equation (1.1), the situation is much worse. The leading-order boundary layer $\mathcal{U}_0$ is no longer zero:

$$\sin\phi\frac{\partial\mathcal{U}_0}{\partial\eta}+\frac{\epsilon}{R_\kappa-\epsilon\eta}\cos\phi\frac{\partial\mathcal{U}_0}{\partial\phi}+\mathcal{U}_0-\bar{\mathcal{U}}_0=0, \tag{1.13}$$

$$\sin\phi\frac{\partial\mathcal{U}_1}{\partial\eta}+\frac{\epsilon}{R_\kappa-\epsilon\eta}\cos\phi\frac{\partial\mathcal{U}_1}{\partial\phi}+\mathcal{U}_1-\bar{\mathcal{U}}_1=-\cos\phi\frac{\partial\mathcal{U}_0}{\partial\tau}. \tag{1.14}$$

The remainder contains the term $\dfrac{\partial\mathcal{U}_1}{\partial\tau}$, which depends on the estimate of $\dfrac{\partial^2\mathcal{U}_0}{\partial\tau^2}$. Then we must prove $W^{2,\infty}$ estimates in the boundary layer equation. In principle, this is impossible for general kinetic equations as [5] pointed out.

On the other hand, we have a key observation that actually the singularity that prevents higher-order regularity concentrates in the neighborhood of the grazing set, so it is natural to isolate the singular part from the whole solution and tackle them in different approaches.

Inspired by [21], we introduce a new regularization argument. Instead of trying different weighted norms, we may also modify the boundary data and smoothen the boundary layer in this modified problem.

To be precise, we decompose the boundary data $g=\mathscr{G}+\mathfrak{G}$, such that

- the boundary layer $\mathscr{U}$ with data $\mathscr{G}$, which we call regular boundary layer, attains second-order regularity in the tangential direction, that is $\dfrac{\partial^2\mathscr{U}}{\partial\tau^2}\in L^\infty$; $\mathscr{G}=g$ in most of the region except a small neighborhood of the grazing set;
- the boundary layer $\mathfrak{U}$ with data $\mathfrak{G}$, which we call singular boundary layer, attains only first-order regularity in the tangential direction that is $\dfrac{\partial\mathfrak{U}}{\partial\tau}\in L^\infty$, but the support of $\mathfrak{G}$ is restricted to a very small neighborhood of the grazing set with diameter ϵ^α for some $0<\alpha<1$.

In other words, for the remainder estimates, the extra power of ϵ comes from two sources: $\mathscr{U}$ gains power by expanding to the higher order, and $\mathfrak{U}$ gains power through a small support ϵ^α.

Definitely, this decomposition comes with a price. Even if we assume $\dfrac{\partial g}{\partial\phi}=O(1)$, after the decomposition, we can at most have $\dfrac{\partial\mathscr{G}}{\partial\phi}=O(\epsilon^{-\alpha})$ and $\dfrac{\partial\mathfrak{G}}{\partial\phi}=O(\epsilon^{-\alpha})$. We have to prove a much stronger weighted $W^{1,\infty}$ estimates to suppress such loss of power in ϵ. Moreover, this decomposition introduces two contradictory goals in the estimates:

- to obtain $W^{2,\infty}$ estimate of $\mathscr{U}$ with data $\mathscr{G}$, we want α to be as small as possible; the smaller α is (better smoothness of $\mathscr{G}$), the better estimates we get;
- to obtain $W^{1,\infty}$ estimate of $\mathfrak{U}$ with data $\mathfrak{G}$, we want α to be as large as possible; the larger α is (smaller support of $\mathfrak{G}$), the better estimates we get.

This balance is quite delicate and the estimates for the ϵ-Milne problem with geometric correction in [7,24,25] and [8] are not sufficient. We have to start from scratch and prove the stronger version.

1.2.6. Main Methods To fully solve such a problem, we need an intricate synthesis of previously developed methods, and the fresh regularization argument stated above.

We inherit and modify the following ideas and techniques, which can be considered the minor contribution:

- **Geometric Correction:**
  The ϵ-Milne problem with geometric correction for $f = \mathscr{U}$ or $\mathfrak{U}$,

$$\sin\phi\frac{\partial f}{\partial\eta} + \frac{\epsilon}{R_\kappa - \epsilon\eta}\cos\phi\frac{\partial f}{\partial\phi} + f - \bar{f} = S, \tag{1.15}$$

  has been shown to be the correct formulation to describe kinetic boundary layers (see [24]). In this paper, we start from scratch and justify the detailed dependence of f on the source term S. In particular, we isolate the contribution of $\bar{S}$ and $S-\bar{S}$.
- **Canonical Weighted $W^{1,\infty}$ Estimates of Boundary Layers:**
  The weighted $W^{1,\infty}$ estimates in ϵ-Milne problem with geometric correction is the key to estimate $\dfrac{\partial f}{\partial\tau}$ (see [7]). In this paper, we highlight the dependence of $W^{1,\infty}$ norm on the characteristic curves and the boundary data. The convexity and the kinetic distance

$$\zeta(\eta,\phi) = \left(1 - \left(\frac{R_\kappa - \epsilon\eta}{R_\kappa}\cos\phi\right)^2\right)^{\frac{1}{2}}, \tag{1.16}$$

  is key to this proof.
- **Remainder Estimates:**
  This is the key step to reduce the regularity requirement in boundary layers. It was originally developed in [24] and later strengthened in [7]. In the remainder equation for $R(\vec{x},\vec{w}) = u^\epsilon - U - \mathcal{U}$,

$$\epsilon\vec{w}\cdot\nabla_x R + R - \bar{R} = S, \tag{1.17}$$

  the estimate justified in [24] using the $L^2 - L^\infty$ framework is

$$\|R\|_{L^\infty} \lesssim \frac{1}{\epsilon^3}\|S\|_{L^2} + \text{higher-order terms}. \tag{1.18}$$

  We intend to show that $\|R\|_{L^\infty} = o(1)$ as $\epsilon \to 0$. Since we cannot expand to higher-order boundary layers to further improve S, the coefficient ϵ^{-3} is too singularity. A key improvement in [7] for the diffusive boundary case is to develop the $L^{2m}-L^{\infty}$ framework to prove a stronger estimate for $m\geqq 2$,

$$\|R\|_{L^{\infty}}\lesssim\frac{1}{\epsilon^{2+\frac{1}{m}}}\|S\|_{L^{\frac{2m}{2m-1}}}+\text{higher-order terms.} \tag{1.19}$$

In this paper, we adapt it to treat the in-flow boundary case with a modified $L^{2m}-L^{\infty}$ framework. The main idea is to introduce a special test function in the weak formulation to treat $\bar{R}$ and $R-\bar{R}$ separately, and further to bootstrap in order to improve the L^{∞} estimate by a modified double Duhamel's principle. The proof relies on a delicate analysis using interpolation and Young's inequality.

The key novelty of this paper lies in the innovative regularization argument and the corresponding regularity estimates, which constitute the major contribution:

- **Improved Weighted $W^{1,\infty}$ Estimates of Boundary Layers:**
We combine several different formulations to track the characteristics and justify that the solution of (1.15) satisfies

$$\begin{aligned}&\left\|\zeta\frac{\partial f}{\partial\eta}\right\|_{L^{\infty}L^{\infty}}+\left\|\frac{\epsilon}{R_{\kappa}-\epsilon\eta}\cos\phi\frac{\partial f}{\partial\phi}\right\|_{L^{\infty}L^{\infty}}\\&\quad\leqq C\left|\ln(\epsilon)\right|^{8}\left(\|p\|_{L^{\infty}_{-}}+\left\|(\epsilon+\zeta)\frac{\partial p}{\partial\phi}\right\|_{L^{\infty}_{-}}+\|S\|_{L^{\infty}L^{\infty}}\right.\\&\qquad\left.+\left\|\zeta\frac{\partial S}{\partial\eta}\right\|_{L^{\infty}L^{\infty}}+\|f\|_{L^{\infty}L^{\infty}}\right),\end{aligned} \tag{1.20}$$

where the boundary data $p=\mathscr{G}$ or $\mathfrak{G}$. The extra weight $\epsilon+\zeta$ suppresses the singularity in $\dfrac{\partial\mathscr{G}}{\partial\phi}$ and $\dfrac{\partial\mathfrak{G}}{\partial\phi}$. In particular, the estimate does not depend on $\dfrac{\partial S}{\partial\phi}$. This is the key step to isolate the contributions of $\sin\dfrac{\partial f}{\partial\eta}$ and $\dfrac{\epsilon}{R_{\kappa}-\epsilon\eta}\cos\phi\dfrac{\partial f}{\partial\phi}$, which is crucial, later on, for the $W^{2,\infty}$ estimates.

The estimate is obtained through a delicate absorbing argument and a novel characteristic analysis of half-space kinetic equations.

- $\boldsymbol{\dfrac{\partial^2}{\partial\tau^2}}$ **Estimate of Regular Boundary Layer**:
As pointed out in [5], weighted $W^{2,\infty}$ estimates of general kinetic equations are not available. This is true even for $\mathscr{U}$ with modified boundary data. In principle, we cannot bound $\dfrac{\partial^2\mathscr{U}_0}{\partial\eta^2}$ and $\dfrac{\partial^2\mathscr{U}_0}{\partial\phi^2}$. Instead, we propose a delicate analysis to show that we can estimate $\dfrac{\partial^2\mathscr{U}_0}{\partial\tau^2}$ without referring to the other second-order derivatives. This is quite unusual and cannot be done in a direct fashion.

Roughly speaking, we need a chain of estimates

$$
\begin{aligned}
\left\|\frac{\partial \mathscr{U}_1}{\partial \tau}\right\|_{L^\infty L^\infty} &\Leftarrow \left\|\frac{\partial}{\partial \tau}\left(\frac{\partial \mathscr{U}_0}{\partial \tau}\right)\right\|_{L^\infty L^\infty}\\
&\Leftarrow \left\|\zeta \frac{\partial}{\partial \eta}\left(\frac{\partial \mathscr{U}_0}{\partial \tau}\right)\right\|_{L^\infty L^\infty} + \left\|\frac{\epsilon}{R_\kappa - \epsilon\eta}\cos\phi \frac{\partial}{\partial \phi}\left(\frac{\partial \mathscr{U}_0}{\partial \tau}\right)\right\|_{L^\infty L^\infty}\\
&\Leftarrow \left\|\frac{\epsilon}{R_\kappa - \epsilon\eta}\cos\phi \frac{\partial}{\partial \phi}\left(\frac{\partial \mathscr{U}_0}{\partial \eta}\right)\right\|_{L^\infty L^\infty}\\
&\Leftarrow \left\|\frac{\epsilon}{R_\kappa - \epsilon\eta}\cos\phi \frac{\partial \mathscr{U}_0}{\partial \phi}\right\|_{L^\infty L^\infty}. \qquad (1.21)
\end{aligned}
$$

Here, none of these steps is a direct application of the above improved weighted $W^{1,\infty}$ estimates. Instead, we need a careful arrangement of these terms and utilize absorbing arguments in a delicate way. Eventually, we can justify that

$$
\left\|\epsilon^2 \frac{\partial \mathscr{U}_1}{\partial \tau}\right\|_{L^{\frac{2m}{2m-1}}} \sim \epsilon^{3-\frac{1}{2m}-\alpha}. \qquad (1.22)
$$

- $\dfrac{\partial}{\partial \tau}$ **Estimate with Smallness of Singular Boundary Layer**:
  Here, the major difficulty is to preserve the smallness of boundary data. The key observation is that in our proof of well-posedness and $W^{1,\infty}$ estimates, we only use two types of quantities: the integral in ϕ and the value along the characteristics. Therefore, we introduce a domain decomposition as $\chi_1 : \zeta \leqq \epsilon^\alpha$ and $\chi_2 : \zeta \geqq \epsilon^\alpha$, and estimate $\mathfrak{U}$ in each domain separately.
  (1) χ_1: since $\mathfrak{G} = O(1)$, we know that $\mathfrak{U} = O(1)$ and its major contribution is from the boundary data, so it is relatively large but restricted to a small domain for $\alpha > 0$.
  (2) χ_2: since $\mathfrak{G} = 0$, we know that $\mathfrak{U} = O(\epsilon^\alpha)$ and its major contribution is from the non-local operator $\bar{\mathfrak{U}}$, so it is relatively small and spread over most of the domain.
  In the remainder estimate, the estimate of $\mathfrak{U}$ is in $L^{\frac{2m}{2m-1}}$, so we can combine these two contributions in an integral to obtain smallness

$$
\left\|\epsilon \frac{\partial \mathfrak{U}_0}{\partial \tau}\right\|_{L^{\frac{2m}{2m-1}}} \sim \epsilon^{2-\frac{1}{2m}+\alpha}. \qquad (1.23)
$$

Applying these new techniques, we successfully obtain the diffusive limit: u^ϵ converges to the solution of a Laplace's equation with Dirichlet boundary condition.

1.3. Main Theorem

Theorem 1.1. *Assume $g(\vec{x}_0, \vec{w}) \in C^4(\Gamma^-)$. Then for the steady neutron transport equation (1.1), there exists a unique solution $u^\epsilon(\vec{x}, \vec{w}) \in L^\infty(\Omega \times \mathbb{S}^1)$. Moreover, for any $0 < \delta << 1$, the solution obeys the estimate*

$$
\left\|u^\epsilon - U - \mathcal{U}\right\|_{L^\infty(\Omega\times\mathbb{S}^1)} \leqq C(\delta)\epsilon^{\frac{1}{2}-\delta}, \qquad (1.24)
$$

where $U(\vec{x})$ satisfies the Laplace equation with Dirichlet boundary condition

$$\begin{cases} \Delta_x U(\vec{x}) = 0 \ \ in \ \ \Omega, \\ U(\vec{x}_0) = D(\vec{x}_0) \ \ on \ \ \partial\Omega, \end{cases} \tag{1.25}$$

and $\mathcal{U}(\eta, \tau, \phi)$ satisfies the ϵ-Milne problem with geometric correction

$$\begin{cases} \sin\phi \dfrac{\partial \mathcal{U}}{\partial \eta} - \dfrac{\epsilon}{R_\kappa(\tau) - \epsilon\eta} \cos\phi \dfrac{\partial \mathcal{U}}{\partial \phi} + \mathcal{U} - \bar{\mathcal{U}} = 0, \\ \mathcal{U}(0, \tau, \phi) = g(\tau, \phi) - D(\tau) \ \ for \ \ \sin\phi > 0, \\ \mathcal{U}(L, \tau, \phi) = \mathcal{U}(L, \tau, \mathscr{R}[\phi]) \end{cases} \tag{1.26}$$

for $L = \epsilon^{-\frac{1}{2}}$, $\mathscr{R}[\phi] = -\phi$, η the rescaled normal variable, τ the tangential variable, and ϕ the velocity variable.

Remark 1.2. The implicitly defined function D is determined through the study of the ϵ-Milne problem with geometric correction

$$\begin{cases} \sin\phi \dfrac{\partial \mathcal{F}}{\partial \eta} - \dfrac{\epsilon}{R_\kappa(\tau) - \epsilon\eta} \cos\phi \dfrac{\partial \mathcal{F}}{\partial \phi} + \mathcal{F} - \bar{\mathcal{F}} = 0, \\ \mathcal{F}(0, \tau, \phi) = g(\tau, \phi) \ \ \text{for} \ \ \sin\phi > 0, \\ \mathcal{F}(L, \tau, \phi) = \mathcal{F}(L, \tau, \mathscr{R}[\phi]). \end{cases} \tag{1.27}$$

Theorems 4.3 and 4.8 confirm that there exists $\mathcal{F}_L(\tau) \in \mathbb{R}$ such that $\mathcal{F} - \mathcal{F}_L$ satisfies desired L^2 and L^∞ estimates. The proof of these theorems shows that the mapping $g \to \mathcal{F}_L$ is one-to-one and linear. Here we simply take $D(\tau) = \mathcal{F}_L(\tau)$ and it can also be rewritten as $D(\vec{x}_0)$.

Remark 1.3. The boundary layer $\mathcal{U}$ is defined through the equation (1.26). Based on the analysis in Theorems 4.3, 4.8, and 4.9, we know that $\mathcal{U} \in L^\infty$ is uniquely determined and decays exponentially fast to zero as $\eta \to \infty$. The mapping $g \to \mathcal{U}$ is linear and it provides the boundary data D for the interior solution.

Remark 1.4. Note that the effects of the boundary layer decay very fast away from the boundary. Roughly speaking, this theorem states that for $\vec{x}$ not very close to the boundary, $u^\epsilon(\vec{x}, \vec{w})$ can be approximated by the solution of a Laplace equation with Dirichlet boundary condition.

1.4. Notation and Paper Structure

Throughout this paper, $C > 0$ denotes a constant that only depends on the domain Ω, but does not depend on the data or ϵ. It is referred as universal and can change from one inequality to another. When we write $C(z)$, it means a certain positive constant depending on the quantity z. We write $a \lesssim b$ to denote $a \leqq Cb$.

This paper is organized as follows: in Section 2, we present the asymptotic analysis of the equation (1.1) and introduce the decomposition of boundary layers; in Section 3, we establish the L^∞ well-posedness of the remainder equation; in Section 4, we prove the well-posedness and decay of the ϵ-Milne problem with geometric correction; in Section 5, we study the weighted regularity of the ϵ-Milne problem with geometric correction; finally, in Section 6, we give a detailed analysis of the asymptotic expansion and prove the main theorem.

Remark 1.5. The general structure of this paper is very similar to that of [7] and [8]. In particular, Section 3, 4 and 5 seem to be an obvious adaption of the corresponding theorems there. However, we introduce new techniques to delicately improve the results of [7], so it needs a careful handling and a fresh start from scratch.

2. Asymptotic Analysis

In this section, we will present the asymptotic expansions of the neutron transport equation (1.1).

2.1. Interior Expansion

We define the interior expansion as follows:

$$U(\vec{x},\vec{w}) \sim U_0(\vec{x},\vec{w}) + \epsilon U_1(\vec{x},\vec{w}) + \epsilon^2 U_2(\vec{x},\vec{w}), \tag{2.1}$$

where U_k can be determined by comparing the order of ϵ by plugging (2.1) into the equation (1.1). Thus we have

$$U_0 - \bar{U}_0 = 0, \tag{2.2}$$
$$U_1 - \bar{U}_1 = -\vec{w}\cdot\nabla_x U_0, \tag{2.3}$$
$$U_2 - \bar{U}_2 = -\vec{w}\cdot\nabla_x U_1. \tag{2.4}$$

Plugging (2.2) into (2.3), we obtain

$$U_1 = \bar{U}_1 - \vec{w}\cdot\nabla_x \bar{U}_0. \tag{2.5}$$

Plugging (2.5) into (2.4), we get

$$\begin{aligned} U_2 - \bar{U}_2 &= -\vec{w}\cdot\nabla_x(\bar{U}_1 - \vec{w}\cdot\nabla_x\bar{U}_0) \\ &= -\vec{w}\cdot\nabla_x\bar{U}_1 + w_1^2\partial^2_{x_1x_1}\bar{U}_0 + w_2^2\partial^2_{x_2x_2}\bar{U}_0 + 2w_1w_2\partial^2_{x_1x_2}\bar{U}_0. \end{aligned} \tag{2.6}$$

Integrating (2.6) over $\vec{w}\in\mathbb{S}^1$, we achieve the final form

$$\Delta_x\bar{U}_0 = 0, \tag{2.7}$$

which further implies $U_0(\vec{x},\vec{w})$ satisfies the equation

$$\begin{cases} U_0 &= \bar{U}_0, \\ \Delta_x\bar{U}_0 &= 0. \end{cases} \tag{2.8}$$

In a similar fashion, for $k = 1, 2$, U_k satisfies

$$\begin{cases} U_k &= \bar{U}_k - \vec{w}\cdot\nabla_x U_{k-1}, \\ \Delta_x\bar{U}_k &= -\displaystyle\int_{\mathbb{S}^1}\vec{w}\cdot\nabla_x U_{k-1}\,\mathrm{d}\vec{w}. \end{cases} \tag{2.9}$$

It is easy to see that $\bar{U}_k$ satisfies an elliptic equation. However, the boundary condition of $\bar{U}_k$ is unknown at this stage, since generally U_k does not necessarily satisfy the in-flow boundary condition of (1.1). Therefore, we have to resort to boundary layers.

2.2. Boundary Layer Expansion

Besides the Cartesian coordinate system for interior solutions, we need a local coordinate system in a neighborhood of the boundary to describe boundary layers.

Assume the Cartesian coordinate system is $\vec{x} = (x_1, x_2)$. Using polar coordinates system $(r, \theta) \in [0, \infty) \times [-\pi, \pi)$ and choosing pole in Ω, we assume $\vec{x}_0 \in \partial\Omega$ is

$$\begin{cases} x_{1,0} = r(\theta)\cos\theta, \\ x_{2,0} = r(\theta)\sin\theta, \end{cases} \tag{2.10}$$

where $r(\theta) > 0$ is a given function describing the boundary $\partial\Omega$. Our local coordinate system is similar to the polar coordinate system, but varies to satisfy the specific requirements.

In a neighborhood of the boundary, for each θ, we have the outward unit normal vector

$$\vec{\nu} = \left(\frac{r(\theta)\cos\theta + r'(\theta)\sin\theta}{\sqrt{r(\theta)^2 + r'(\theta)^2}}, \frac{r(\theta)\sin\theta - r'(\theta)\cos\theta}{\sqrt{r(\theta)^2 + r'(\theta)^2}} \right). \tag{2.11}$$

We can determine each point $\vec{x} \in \bar{\Omega}$ as $\vec{x} = \vec{x}_0 - \mu\vec{\nu}$ where μ is the normal distance to a boundary point $\vec{x}_0$. In detail, this means

$$\begin{cases} x_1 = r(\theta)\cos\theta - \mu\dfrac{r(\theta)\cos\theta + r'(\theta)\sin\theta}{\sqrt{r(\theta)^2 + r'(\theta)^2}}, \\ x_2 = r(\theta)\sin\theta - \mu\dfrac{r(\theta)\sin\theta - r'(\theta)\cos\theta}{\sqrt{r(\theta)^2 + r'(\theta)^2}}, \end{cases} \tag{2.12}$$

where $r'(\theta) = \dfrac{\mathrm{d}r}{\mathrm{d}\theta}$. It is easy to see that $\mu = 0$ denotes the boundary $\partial\Omega$ and $\mu > 0$ denotes the interior of Ω (before touching the other side of the domain boundary). (μ, θ) is the desired local coordinate system.

By chain rule (see [7]), we may deduce that

$$\frac{\partial\theta}{\partial x_1} = \frac{MP}{P^3 + Q\mu}, \quad \frac{\partial\mu}{\partial x_1} = -\frac{N}{P}, \quad \frac{\partial\theta}{\partial x_2} = \frac{NP}{P^3 + Q\mu}, \quad \frac{\partial\mu}{\partial x_2} = \frac{M}{P}, \tag{2.13}$$

where

$$\begin{aligned} &P = (r^2 + r'^2)^{\frac{1}{2}}, \quad Q = rr'' - r^2 - 2r'^2, \quad M = -r\sin\theta + r'\cos\theta, \\ &N = r\cos\theta + r'\sin\theta. \end{aligned} \tag{2.14}$$

Therefore, note the fact that for C^2 convex domains, the curvature is

$$\kappa(\theta) = \frac{r^2 + 2r'^2 - rr''}{(r^2 + r'^2)^{\frac{3}{2}}}, \tag{2.15}$$

and the radius of curvature is

$$R_\kappa(\theta) = \frac{1}{\kappa(\theta)} = \frac{(r^2 + r'^2)^{\frac{3}{2}}}{r^2 + 2r'^2 - rr''}. \tag{2.16}$$

Substitution 1:
Let $(x_1, x_2) \rightarrow (\mu, \theta)$ with $(\mu, \theta) \in [0, R_{\min}) \times [-\pi, \pi)$ for $R_{\min} = \min_\theta R_\kappa$ as

$$\begin{cases} x_1 = r(\theta)\cos\theta - \mu \dfrac{r(\theta)\cos\theta + r'(\theta)\sin\theta}{\sqrt{r(\theta)^2 + r'(\theta)^2}}, \\ x_2 = r(\theta)\sin\theta - \mu \dfrac{r(\theta)\sin\theta - r'(\theta)\cos\theta}{\sqrt{r(\theta)^2 + r'(\theta)^2}}, \end{cases} \tag{2.17}$$

and then the equation (1.1) is transformed into

$$\begin{cases} \epsilon \left(w_1 \dfrac{-r\cos\theta - r'\sin\theta}{(r^2 + r'^2)^{\frac{1}{2}}} + w_2 \dfrac{-r\sin\theta + r'\cos\theta}{(r^2 + r'^2)^{\frac{1}{2}}} \right) \dfrac{\partial u^\epsilon}{\partial \mu} \\ + \epsilon \left(w_1 \dfrac{-r\sin\theta + r'\cos\theta}{(r^2 + r'^2)(1 - \kappa\mu)} + w_2 \dfrac{r\cos\theta + r'\sin\theta}{(r^2 + r'^2)(1 - \kappa\mu)} \right) \dfrac{\partial u^\epsilon}{\partial \theta} + u^\epsilon - \bar{u}^\epsilon = 0, \\ u^\epsilon(0, \theta, \vec{w}) = g(\theta, \vec{w}) \ \text{ for } \ \vec{w} \cdot \vec{\nu} < 0, \end{cases} \tag{2.18}$$

where

$$\vec{w} \cdot \vec{\nu} = w_1 \frac{r\cos\theta + r'\sin\theta}{(r^2 + r'^2)^{\frac{1}{2}}} + w_2 \frac{r\sin\theta - r'\cos\theta}{(r^2 + r'^2)^{\frac{1}{2}}}. \tag{2.19}$$

Noting the fact that

$$\left(\frac{M}{P}\right)^2 + \left(\frac{N}{P}\right)^2 = \left(\frac{-r\cos\theta - r'\sin\theta}{(r^2 + r'^2)^{\frac{1}{2}}}\right)^2 + \left(\frac{-r\sin\theta + r'\cos\theta}{(r^2 + r'^2)^{\frac{1}{2}}}\right)^2 = 1, \tag{2.20}$$

we can further simplify (2.18).
Substitution 2:
Let $\theta \rightarrow \tau$ with $\tau \in [-\pi, \pi)$ as

$$\begin{cases} \sin\tau = \dfrac{r\sin\theta - r'\cos\theta}{(r^2 + r'^2)^{\frac{1}{2}}}, \\ \cos\tau = \dfrac{r\cos\theta + r'\sin\theta}{(r^2 + r'^2)^{\frac{1}{2}}}, \end{cases} \tag{2.21}$$

which implies

$$\frac{\mathrm{d}\tau}{\mathrm{d}\theta} = \kappa (r^2 + r'^2)^{\frac{1}{2}} > 0. \tag{2.22}$$

Then the equation (1.1) is transformed into

$$\begin{cases} -\epsilon\,(w_1\cos\tau + w_2\sin\tau)\dfrac{\partial u^\epsilon}{\partial \mu} - \dfrac{\epsilon}{R_\kappa - \mu}(w_1\sin\tau - w_2\cos\tau)\dfrac{\partial u^\epsilon}{\partial \tau} \\ \quad + u^\epsilon - \bar{u}^\epsilon = 0, \\ u^\epsilon(0, \tau, \vec{w}) = g(\tau, \vec{w}) \ \text{ for } \ \vec{w} \cdot \vec{\nu} < 0, \end{cases} \tag{2.23}$$

where

$$\vec{w}\cdot\vec{\nu} = w_1\cos\tau + w_2\sin\tau. \tag{2.24}$$

Substitution 3:
We further make the scaling transform for $\mu \to \eta$ with $\eta \in \left[0, \dfrac{R_{\min}}{\epsilon}\right)$ as

$$\eta = \frac{\mu}{\epsilon}, \tag{2.25}$$

which implies

$$\frac{\partial u^\epsilon}{\partial \mu} = \frac{1}{\epsilon}\frac{\partial u^\epsilon}{\partial \eta}. \tag{2.26}$$

Then the equation (1.1) is transformed into

$$\begin{cases} -\left(w_1\cos\tau + w_2\sin\tau\right)\dfrac{\partial u^\epsilon}{\partial \eta} - \dfrac{\epsilon}{R_\kappa - \epsilon\eta}\left(w_1\sin\tau - w_2\cos\tau\right)\dfrac{\partial u^\epsilon}{\partial \tau} \\ \qquad + u^\epsilon - \bar{u}^\epsilon = 0, \\ u^\epsilon(0,\tau,\vec{w}) = g(\tau,\vec{w}) \ \text{ for } \ \vec{w}\cdot\vec{\nu} < 0, \end{cases} \tag{2.27}$$

where

$$\vec{w}\cdot\vec{\nu} = w_1\cos\tau + w_2\sin\tau. \tag{2.28}$$

Substitution 4:
Define the velocity substitution for $(w_1, w_2) \to \xi$ with $\xi \in [-\pi, \pi)$ as

$$\begin{cases} w_1 = -\sin\xi \\ w_2 = -\cos\xi. \end{cases} \tag{2.29}$$

We have the succinct form of the equation (1.1) as

$$\begin{cases} \sin(\tau+\xi)\dfrac{\partial u^\epsilon}{\partial \eta} - \dfrac{\epsilon}{R_\kappa - \epsilon\xi}\cos(\tau+\xi)\dfrac{\partial u^\epsilon}{\partial \tau} + u^\epsilon - \bar{u}^\epsilon = 0, \\ u^\epsilon(0,\tau,\xi) = g(\tau,\xi) \ \text{ for } \ \sin(\tau+\xi) > 0. \end{cases} \tag{2.30}$$

Substitution 5:
As [24] and [7] reveal, we need a further rotational substitution for $\xi \to \phi$ with $\phi \in [-\pi, \pi)$ as

$$\phi = \tau + \xi \tag{2.31}$$

and achieve the form

$$\begin{cases} \sin\phi\dfrac{\partial u^\epsilon}{\partial \eta} - \dfrac{\epsilon}{R_\kappa - \epsilon\eta}\cos\phi\left(\dfrac{\partial u^\epsilon}{\partial \phi} + \dfrac{\partial u^\epsilon}{\partial \tau}\right) + u^\epsilon - \bar{u}^\epsilon = 0, \\ u^\epsilon(0,\tau,\phi) = g(\tau,\phi) \ \text{ for } \ \sin\phi > 0. \end{cases} \tag{2.32}$$

This step is trying to compensate the variations of the normal vector ν along the boundary. A bi-product of such substitution is that we decompose the tangential derivative and introduce a new velocity derivative.

We define the boundary layer expansion as follows:

$$\mathscr{U}(\eta,\tau,\phi)\sim\mathscr{U}_0(\eta,\tau,\phi)+\epsilon\mathscr{U}_1(\eta,\tau,\phi), \tag{2.33}$$

where $\mathscr{U}_k$ can be determined by comparing the order of ϵ via plugging (2.33) into the equation (2.32). Thus, in a neighborhood of the boundary, we have

$$\sin\phi\frac{\partial\mathscr{U}_0}{\partial\eta}-\frac{\epsilon}{R_\kappa-\epsilon\eta}\cos\phi\frac{\partial\mathscr{U}_0}{\partial\phi}+\mathscr{U}_0-\bar{\mathscr{U}}_0=0, \tag{2.34}$$

$$\sin\phi\frac{\partial\mathscr{U}_1}{\partial\eta}-\frac{\epsilon}{R_\kappa-\epsilon\eta}\cos\phi\frac{\partial\mathscr{U}_1}{\partial\phi}+\mathscr{U}_1-\bar{\mathscr{U}}_1=\frac{1}{R_\kappa-\epsilon\eta}\cos\phi\frac{\partial\mathscr{U}_0}{\partial\tau}, \tag{2.35}$$

where

$$\bar{\mathscr{U}}_k(\eta,\tau)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathscr{U}_k(\eta,\tau,\phi)\,\mathrm{d}\phi. \tag{2.36}$$

We call this type of equations the ϵ-Milne problem with geometric correction.

2.3. Decomposition and Modification

In this section, we introduce the important decomposition of boundary data, which can greatly improve the regularity. The idea is adapted from [21] for the flat Milne problem.

Consider the ϵ-Milne problem with geometric correction with $L=\epsilon^{-\frac{1}{2}}$ and $\mathscr{R}[\phi]=-\phi$,

$$\begin{cases}\sin\phi\dfrac{\partial f}{\partial\eta}-\dfrac{\epsilon}{R_\kappa-\epsilon\eta}\cos\phi\dfrac{\partial f}{\partial\phi}+f-\bar{f}=0,\\ f(0,\phi)=g(\phi)\ \text{ for }\ \sin\phi>0,\\ f(L,\phi)=f(L,\mathscr{R}[\phi]).\end{cases} \tag{2.37}$$

We assume that $g(\phi)$ is not a constant and $0\leqq g(\phi)\leqq 1$. This is always achievable and we do not lose the generality since the equation is linear. For some $\alpha>0$ which will be determined later, define two C^∞ auxiliary functions

$$g_1(\phi)=\begin{cases}0 & \text{for }\ \phi\in(0,\epsilon^\alpha]\cup[\pi-\epsilon^\alpha,\pi),\\ g(\phi) & \text{for }\ \phi\in[2\epsilon^\alpha,\pi-2\epsilon^\alpha],\end{cases} \tag{2.38}$$

and

$$g_2(\phi)=\begin{cases}1 & \text{for }\ \phi\in(0,\epsilon^\alpha]\cup[\pi-\epsilon^\alpha,\pi),\\ g(\phi) & \text{for }\ \phi\in[2\epsilon^\alpha,\pi-2\epsilon^\alpha].\end{cases} \tag{2.39}$$

A standard construction using mollifier justifies the existence of g_i for $i=1,2$. Also, we can easily obtain $\left|\dfrac{\partial g_i}{\partial\phi}\right|\leqq C\epsilon^{-\alpha}$ and $\left|\dfrac{\partial^2 g_i}{\partial\phi^2}\right|\leqq C\epsilon^{-2\alpha}$. Let $f_1(\eta,\phi)$ and $f_2(\eta,\phi)$ be the solutions to the equation (2.37) with in-flow data $g_1(\phi)$ and $g_2(\phi)$

respectively. Then by Theorem 4.8, we know f_1 and f_2 are well-defined in L^∞. By Theorem 4.10, they satisfy the maximum principle, which means

$$f_1(0,0^+)-\bar{f}_1(0)=f_1(0,\pi^-)-\bar{f}_1(0)=-\bar{f}_1(0)<0, \tag{2.40}$$
$$f_2(0,0^+)-\bar{f}_2(0)=f_2(0,\pi^-)-\bar{f}_2(0)=1-\bar{f}_2(0)>0. \tag{2.41}$$

Therefore, there exists a constant $0<\lambda<1$ such that

$$\lambda\Big(f_1(0,0^+)-\bar{f}_1(0)\Big)+(1-\lambda)\Big(f_2(0,0^+)-\bar{f}_2(0)\Big)=0, \tag{2.42}$$
$$\lambda\Big(f_1(0,\pi^-)-\bar{f}_1(0)\Big)+(1-\lambda)\Big(f_2(0,\pi^-)-\bar{f}_2(0)\Big)=0. \tag{2.43}$$

Let $g_\lambda(\phi)=\lambda g_1(\phi)+(1-\lambda)g_2(\phi)$ and the corresponding solution to the equation (2.37) is $f_\lambda(\eta,\phi)$. We have

$$f_\lambda(0,0^+)-\bar{f}_\lambda(0)=f_\lambda(0,\pi^-)-\bar{f}_\lambda(0)=0. \tag{2.44}$$

Since for $\phi\in(0,\epsilon^\alpha]\cup[\pi-\epsilon^\alpha,\pi)$, $g_\lambda=1-\lambda$ is a constant, we naturally have $\dfrac{\partial g_\lambda}{\partial\phi}=0$. We may formally solve from equation (2.37) that

$$\begin{aligned}\frac{\partial f_\lambda}{\partial\eta}\bigg|_{\eta=0,\phi\in(0,\epsilon^\alpha]\cup[\pi-\epsilon^\alpha,\pi)}&=\frac{1}{\sin\phi}\Bigg(\frac{\epsilon}{R_\kappa-\epsilon\eta}\cos\phi\frac{\partial g_\lambda}{\partial\phi}\bigg|_{\phi\in(0,\epsilon^\alpha]\cup[\pi-\epsilon^\alpha,\pi)}\\&\quad-\Big(f_\lambda-\bar{f}_\lambda\Big)\bigg|_{\eta=0,\phi\in(0,\epsilon^\alpha]\cup[\pi-\epsilon^\alpha,\pi)}\Bigg)=0.\end{aligned} \tag{2.45}$$

Note that $g_\lambda(\phi)=g(\phi)$ for $\phi\in[2\epsilon^\alpha,\pi-2\epsilon^\alpha]$, so our modification is restricted to a small region near the grazing set and we can smoothen the normal derivative at the boundary.

This method can be easily generalized to treat other $g(\phi)$. In principle, for $g(\phi)\in C^1$, we can define a decomposition

$$g(\phi)=\mathscr{G}(\phi)+\mathfrak{G}(\phi), \tag{2.46}$$

such that $\mathfrak{G}(\phi)=0$ for $\sin\phi\geqq 2\epsilon^\alpha$, and the solution to the equation (2.37) with in-flow data $\mathscr{G}(\phi)$ has L^∞ normal derivative at η=0. Such a decomposition comes with a price. Originally, we have $\left\|\dfrac{\partial g}{\partial\phi}\right\|_{L^\infty}\leqq C$. However, now we only have $\left\|\dfrac{\partial\mathscr{G}}{\partial\phi}\right\|_{L^\infty}\leqq C\epsilon^{-\alpha}$ and $\left\|\dfrac{\partial\mathfrak{G}}{\partial\phi}\right\|_{L^\infty}\leqq C\epsilon^{-\alpha}$ due to the short-ranged cut-off function.

2.4. Matching Procedure

The bridge between the interior solution and boundary layer is the boundary condition of (1.1), so we first consider the boundary expansion

$$U_0 + \mathscr{U}_0 + \mathfrak{U}_0 = g, \tag{2.47}$$

$$U_1 + \mathscr{U}_1 = 0. \tag{2.48}$$

Here $\mathscr{U}_0$ and $\mathfrak{U}_0$ are boundary layers corresponding to the decomposed boundary data $\mathscr{G}$ and $\mathfrak{G}$ respectively. We call $\mathscr{U}$ the regular boundary layer and $\mathfrak{U}$ the singular boundary layer. They should both satisfy the ϵ-Milne problem with geometric correction.
Step 0: Preliminaries.
Define the weight function

$$\zeta(\eta, \phi) = \left(1 - \left(\frac{R_\kappa - \epsilon\eta}{R_\kappa}\cos\phi\right)^2\right)^{\frac{1}{2}}. \tag{2.49}$$

Let

$$F(\epsilon; \eta, \tau) = -\frac{\epsilon}{R_\kappa(\tau) - \epsilon\eta}, \tag{2.50}$$

and the length for ϵ-Milne problem as $L = \epsilon^{-\frac{1}{2}}$. For $\phi \in [-\pi, \pi]$, denote $\mathscr{R}[\phi] = -\phi$.
Step 1: Construction of $\mathscr{U}_0$, $\mathfrak{U}_0$ and U_0.
Define the zeroth-order boundary layer as

$$\begin{cases} \mathscr{U}_0(\eta, \tau, \phi) = \mathscr{F}_0(\eta, \tau, \phi) - \mathscr{F}_{0,L}(\tau), \\ \sin\phi \dfrac{\partial \mathscr{F}_0}{\partial \eta} + F(\epsilon; \eta, \tau)\cos\phi \dfrac{\partial \mathscr{F}_0}{\partial \phi} + \mathscr{F}_0 - \bar{\mathscr{F}}_0 = 0, \\ \mathscr{F}_0(0, \tau, \phi) = \mathscr{G}(\tau, \phi) \ \text{ for } \ \sin\phi > 0, \\ \mathscr{F}_0(L, \tau, \phi) = \mathscr{F}_0(L, \tau, \mathscr{R}[\phi]), \end{cases} \tag{2.51}$$

with $\mathscr{F}_{0,L}(\tau)$ is defined in Theorems 4.3 and 4.8, and

$$\begin{cases} \mathfrak{U}_0(\eta, \tau, \phi) = \mathfrak{F}_0(\eta, \tau, \phi) - \mathfrak{F}_{0,L}(\tau), \\ \sin\phi \dfrac{\partial \mathfrak{F}_0}{\partial \eta} + F(\epsilon; \eta, \tau)\cos\phi \dfrac{\partial \mathfrak{F}_0}{\partial \phi} + \mathfrak{F}_0 - \bar{\mathfrak{F}}_0 = 0, \\ \mathfrak{F}_0(0, \tau, \phi) = \mathfrak{G}(\tau, \phi) \ \text{ for } \ \sin\phi > 0, \\ \mathfrak{F}_0(L, \tau, \phi) = \mathfrak{F}_0(L, \tau, \mathscr{R}[\phi]), \end{cases} \tag{2.52}$$

with $\mathfrak{F}_{0,L}(\tau)$ is defined in Theorems 4.3 and 4.8. Also, we define the zeroth-order interior solution $U_0(\vec{x}, \vec{w})$ as

$$\begin{cases} U_0(\vec{x}, \vec{w}) = \bar{U}_0(\vec{x}), \\ \Delta_x \bar{U}_0(\vec{x}) = 0 \ \text{ in } \ \Omega, \\ \bar{U}_0(\vec{x}_0) = \mathscr{F}_{0,L}(\tau) + \mathfrak{F}_{0,L}(\tau) \ \text{ on } \ \partial\Omega. \end{cases} \tag{2.53}$$

Roughly speaking $\mathscr{F}_{0,L}(\tau)$ and $\mathfrak{F}_{0,L}(\tau)$ represent the value of $\mathscr{F}_0$ and $\mathfrak{F}_0$ at infinity (since $L \to \infty$ as $\epsilon \to 0$).
Step 2: Construction of $\mathscr{U}_1$ and U_1.
Define the first-order boundary layer as

$$
\begin{cases}
\mathscr{U}_1(\eta,\tau,\phi) = \mathscr{F}_1(\eta,\tau,\phi) - \mathscr{F}_{1,L}(\tau),\\
\sin\phi\dfrac{\partial\mathscr{F}_1}{\partial\eta} + F(\epsilon;\eta,\tau)\cos\phi\dfrac{\partial\mathscr{F}_1}{\partial\phi} + \mathscr{F}_1 - \bar{\mathscr{F}}_1 = \dfrac{1}{R_\kappa - \epsilon\eta}\cos\phi\dfrac{\partial\mathscr{U}_0}{\partial\tau},\\
\mathscr{F}_1(0,\tau,\phi) = \vec{w}\cdot\nabla_x U_0(0,\tau,\vec{w}) \ \text{ for } \ \sin\phi > 0,\\
\mathscr{F}_1(L,\tau,\phi) = \mathscr{F}_1(L,\tau,\mathscr{R}[\phi]),
\end{cases}
\tag{2.54}
$$

with $\mathscr{F}_{1,L}(\tau)$ is defined in Theorem 4.3 and Theorem 4.8. Then we define the first-order interior solution $U_1(\vec{x},\vec{w})$ as

$$
\begin{cases}
U_1(\vec{x},\vec{w}) = \bar{U}_1(\vec{x}) - \vec{w}\cdot\nabla_x U_0(\vec{x},\vec{w}),\\
\Delta_x\bar{U}_1(\vec{x}) = -\displaystyle\int_{\mathbb{S}^1}\Big(\vec{w}\cdot\nabla_x U_0(\vec{x},\vec{w})\Big)\mathrm{d}\vec{w} \ \text{ in } \ \Omega,\\
\quad\bar{U}_1(\vec{x}_0) = f_{1,L}(\tau) \ \text{ on } \ \partial\Omega.
\end{cases}
\tag{2.55}
$$

Note that we do not define $\mathfrak{U}_1$ here.
Step 3: Construction of U_2.
Since we do not expand to $\mathscr{U}_2$ and $\mathfrak{U}_2$, we define the second-order interior solution as

$$
\begin{cases}
U_2(\vec{x},\vec{w}) = \bar{U}_2(\vec{x}) - \vec{w}\cdot\nabla_x U_1(\vec{x},\vec{w}),\\
\Delta_x\bar{U}_2(\vec{x}) = -\displaystyle\int_{\mathbb{S}^1}\Big(\vec{w}\cdot\nabla_x U_1(\vec{x},\vec{w})\Big)\mathrm{d}\vec{w} \ \text{ in } \ \Omega,\\
\quad\bar{U}_2(\vec{x}_0) = 0 \ \text{ on } \ \partial\Omega.
\end{cases}
\tag{2.56}
$$

Here, we might have $O(\epsilon^3)$ error in this step due to the trivial boundary data. Thanks to the remainder estimate, it will not affect the diffusive limit.

3. Remainder Estimate

In this section, we consider the remainder equation for $u(\vec{x},\vec{w})$:

$$
\begin{cases}
\epsilon\vec{w}\cdot\nabla_x u + u - \bar{u} = f(\vec{x},\vec{w}) \ \text{ in } \ \Omega,\\
u(\vec{x}_0,\vec{w}) = h(\vec{x}_0,\vec{w}) \ \text{ for } \ \vec{w}\cdot\vec{\nu} < 0 \ \text{ and } \ \vec{x}_0 \in \partial\Omega.
\end{cases}
\tag{3.1}
$$

We define the L^p norm with $1 \leqq p < \infty$ and L^∞ norms in $\Omega\times\mathbb{S}^1$ as usual:

$$
\|f\|_{L^p(\Omega\times\mathbb{S}^1)} = \left(\int_\Omega\int_{\mathbb{S}^1}|f(\vec{x},\vec{w})|^p\,\mathrm{d}\vec{w}\,\mathrm{d}\vec{x}\right)^{\frac{1}{p}},
\tag{3.2}
$$

$$
\|f\|_{L^\infty(\Omega\times\mathbb{S}^1)} = \mathrm{esssup}_{(\vec{x},\vec{w})\in\Omega\times\mathbb{S}^1}\,|f(\vec{x},\vec{w})|\,.
\tag{3.3}
$$

Define the L^p norm with $1 \leqq p < \infty$ and L^∞ norms on the boundary as follows:

$$\|f\|_{L^p(\Gamma)} = \left(\iint_\Gamma |f(\vec{x},\vec{w})|^p \,|\vec{w}\cdot\vec{\nu}|\,\mathrm{d}\vec{w}\,\mathrm{d}\vec{x}\right)^{\frac{1}{p}}, \tag{3.4}$$

$$\|f\|_{L^p(\Gamma^\pm)} = \left(\iint_{\Gamma^\pm} |f(\vec{x},\vec{w})|^p \,|\vec{w}\cdot\vec{\nu}|\,\mathrm{d}\vec{w}\,\mathrm{d}\vec{x}\right)^{\frac{1}{p}}, \tag{3.5}$$

$$\|f\|_{L^\infty(\Gamma)} = \mathrm{esssup}_{(\vec{x},\vec{w})\in\Gamma} |f(\vec{x},\vec{w})|, \tag{3.6}$$

$$\|f\|_{L^\infty(\Gamma^\pm)} = \mathrm{esssup}_{(\vec{x},\vec{w})\in\Gamma^\pm} |f(\vec{x},\vec{w})|. \tag{3.7}$$

From now on, we denote $\mathrm{d}\gamma = (\vec{w}\cdot\vec{\nu})\,\mathrm{d}\vec{w}\,\mathrm{d}\vec{x}$ on the boundary.

In the following, we always assume that Ω is convex and bounded. The proof can be decomposed into two major steps:

- L^{2m} estimates:

Directly energy estimates can bound $\|u-\bar{u}\|_{L^2(\Omega\times\mathbb{S}^1)}$, so the key is how to control $\bar{u}$. Here, we utilize the spectral gap of the transport operator $\vec{w}\cdot\nabla_x$ in bounded domains to construct special test functions such that $\|\bar{u}\|_{L^{2m}(\Omega\times\mathbb{S}^1)}$ shows up explicitly in the weak formulation. In particular, in the L^{2m} estimates, we apply Young's inequality and interpolation estimates to tackle the other terms in the weak formulation.

- L^∞ estimates:

This is based on the mild formulation. We use Duhamel's principle to rewrite the solution along the characteristics. Moreover, we may expand $\bar{u}$ into the velocity integral and apply Duhamel's principle again to get multiple integrals. Here, convexity and boundedness guarantee that the mild formulation will not produce singularities inside the domain (see [9]). Then a delicate substitution will transform pointwise estimates into the control of space integrals, which is provided by the above L^{2m} estimates.

This general method constitutes the so-called $L^{2m} - L^\infty$ framework.

The remainder estimates for the neutron transport equation with diffusive boundary was proved in [7] and [8]. The case with in-flow boundary was first shown in [24] based on $L^2 - L^\infty$ framework. The main results in [24] are as follows:

Theorem 3.1. *The unique solution $u(\vec{x},\vec{w})$ to the equation (3.1) satisfies*

$$\frac{1}{\epsilon^{\frac{1}{2}}}\|u\|_{L^2(\Gamma^+)} + \|u\|_{L^2(\Omega\times\mathbb{S}^1)} \leqq C\left(\frac{1}{\epsilon^2}\|f\|_{L^2(\Omega\times\mathbb{S}^1)} + \frac{1}{\epsilon^{\frac{1}{2}}}\|h\|_{L^2(\Gamma^-)}\right). \tag{3.8}$$

Theorem 3.2. *The unique solution $u(\vec{x},\vec{w})$ to the equation (3.1) satisfies*

$$\|u\|_{L^\infty(\Gamma^+)} + \|u\|_{L^\infty(\Omega\times\mathbb{S}^1)} \leqq C\bigg(\frac{1}{\epsilon^3}\|f\|_{L^2(\Omega\times\mathbb{S}^1)} + \|f\|_{L^\infty(\Omega\times\mathbb{S}^1)} + \frac{1}{\epsilon^{\frac{3}{2}}}\|h\|_{L^2(\Gamma^-)} + \|h\|_{L^\infty(\Gamma^-)}\bigg). \tag{3.9}$$

The existence and uniqueness of solution $u(\vec{x},\vec{w})$ was justified in [24]. Here we will focus on the a priori estimates and prove an improved version.

3.1. L^{2m} Estimate

In the following, we assume $m > 2$ is an integer and let $o(1)$ denote a sufficiently small constant.

Lemma 3.3. (Green's identity) *Assume* $u(\vec{x}, \vec{w}),\ v(\vec{x}, \vec{w}) \in L^2(\Omega \times \mathbb{S}^1)$ *and* $\vec{w} \cdot \nabla_x u,\ \vec{w} \cdot \nabla_x v \in L^2(\Omega \times \mathbb{S}^1)$ *with* $u,\ v \in L^2(\Gamma)$. *Then*

$$\iint_{\Omega\times\mathbb{S}^1} \Big((\vec{w} \cdot \nabla_x u)v + (\vec{w} \cdot \nabla_x u)v\Big) \mathrm{d}\vec{x}\, \mathrm{d}\vec{w} = \int_\Gamma uv\, \mathrm{d}\gamma. \tag{3.10}$$

Proof. See [2, Chapter 9] and [4]. □

Theorem 3.4. *The unique solution* $u(\vec{x}, \vec{w})$ *to the equation* (3.1) *satisfies*

$$\begin{aligned}
&\frac{1}{\epsilon^{\frac{1}{2}}} \|u\|_{L^2(\Gamma^+)} + \|\bar{u}\|_{L^{2m}(\Omega\times\mathbb{S}^1)} + \frac{1}{\epsilon} \|u - \bar{u}\|_{L^2(\Omega\times\mathbb{S}^1)} \\
&\quad\leqq C\bigg(o(1)\epsilon^{\frac{1}{m}}\Big(\|u\|_{L^\infty(\Omega\times\mathbb{S}^1)} + \|u\|_{L^\infty(\Gamma^+)}\Big) \\
&\qquad + \frac{1}{\epsilon} \|f\|_{L^2(\Omega\times\mathbb{S}^1)} + \frac{1}{\epsilon^2} \|f\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathbb{S}^1)} \\
&\qquad + \frac{1}{\epsilon^{\frac{1}{2}}} \|h\|_{L^2(\Gamma^-)} + \|h\|_{L^m(\Gamma^-)}\bigg).
\end{aligned} \tag{3.11}$$

Proof. Step 1: Kernel Estimate.
Applying Green's identity to the equation (3.1). Then for any $\phi \in L^2(\Omega \times \mathbb{S}^1)$ satisfying $\vec{w} \cdot \nabla_x\phi \in L^2(\Omega \times \mathbb{S}^1)$ and $\phi \in L^2(\Gamma)$, we have

$$\epsilon \int_\Gamma u\phi\, \mathrm{d}\gamma - \epsilon \iint_{\Omega\times\mathbb{S}^1} (\vec{w} \cdot \nabla_x\phi)u + \iint_{\Omega\times\mathbb{S}^1} (u - \bar{u})\phi = \iint_{\Omega\times\mathbb{S}^1} f\phi. \tag{3.12}$$

Our goal is to choose a particular test function ϕ. We first construct an auxiliary function ξ. Naturally $u \in L^\infty(\Omega \times \mathbb{S}^1)$ implies $\bar{u} \in L^{2m}(\Omega)$ which further leads to $(\bar{u})^{2m-1} \in L^{\frac{2m}{2m-1}}(\Omega)$. We define $\xi(\vec{x})$ on Ω satisfying

$$\begin{cases} \Delta\xi = (\bar{u})^{2m-1} & \text{in } \Omega, \\ \quad\xi = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.13}$$

In the bounded domain Ω, based on the standard elliptic estimates, we have a unique ξ satisfying

$$\|\xi\|_{W^{2,\frac{2m}{2m-1}}(\Omega)} \leqq C \left\|(\bar{u})^{2m-1}\right\|_{L^{\frac{2m}{2m-1}}(\Omega)} = C\, \|\bar{u}\|^{2m-1}_{L^{2m}(\Omega)}. \tag{3.14}$$

We plug the test function

$$\phi = -\vec{w} \cdot \nabla_x\xi \tag{3.15}$$

into the weak formulation (3.12) and estimate each term there. By Sobolev embedding theorem, we have

$$\|\phi\|_{L^2(\Omega)} \leqq C\,\|\xi\|_{H^1(\Omega)} \leqq C\,\|\xi\|_{W^{2,\frac{2m}{2m-1}}(\Omega)} \leqq C\,\|\bar{u}\|_{L^{2m}(\Omega)}^{2m-1}, \quad (3.16)$$

$$\|\phi\|_{L^{\frac{2m}{2m-1}}(\Omega)} \leqq C\,\|\xi\|_{W^{1,\frac{2m}{2m-1}}(\Omega)} \leqq C\,\|\bar{u}\|_{L^{2m}(\Omega)}^{2m-1}. \quad (3.17)$$

Then we can decompose

$$-\epsilon \iint_{\Omega\times\mathbb{S}^1} (\vec{w}\cdot\nabla_x\phi)u = -\epsilon \iint_{\Omega\times\mathbb{S}^1} (\vec{w}\cdot\nabla_x\phi)\bar{u} -\epsilon \iint_{\Omega\times\mathbb{S}^1} (\vec{w}\cdot\nabla_x\phi)(u-\bar{u}). \quad (3.18)$$

We estimate the two term on the right-hand side of (3.18) separately. By (3.13) and (3.15), we have

$$\begin{aligned} &-\epsilon \iint_{\Omega\times\mathbb{S}^1} (\vec{w}\cdot\nabla_x\phi)\bar{u} \\ &= \epsilon \iint_{\Omega\times\mathbb{S}^1} \bar{u}\Big(w_1(w_1\partial_{11}\xi + w_2\partial_{12}\xi) + w_2(w_1\partial_{12}\xi + w_2\partial_{22}\xi)\Big) \\ &= \epsilon \iint_{\Omega\times\mathbb{S}^1} \bar{u}\Big(w_1^2\partial_{11}\xi + w_2^2\partial_{22}\xi\Big) = 2\epsilon\pi \int_\Omega \bar{u}(\partial_{11}\xi + \partial_{22}\xi) \\ &= \epsilon\,\|\bar{u}\|_{L^{2m}(\Omega)}^{2m}. \end{aligned} \quad (3.19)$$

In the second equality, the cross terms vanish due to the symmetry of the integral over $\mathbb{S}^1$. On the other hand, for the second term in (3.18), Hölder's inequality and the elliptic estimate imply

$$\begin{aligned} -\epsilon \iint_{\Omega\times\mathbb{S}^1} (\vec{w}\cdot\nabla_x\phi)(u-\bar{u}) &\leqq C\epsilon\,\|u-\bar{u}\|_{L^{2m}(\Omega\times\mathbb{S}^1)}\,\|\nabla_x\phi\|_{L^{\frac{2m}{2m-1}}(\Omega)} \\ &\leqq C\epsilon\,\|u-\bar{u}\|_{L^{2m}(\Omega\times\mathbb{S}^1)}\,\|\xi\|_{W^{2,\frac{2m}{2m-1}}(\Omega)} \\ &\leqq C\epsilon\,\|u-\bar{u}\|_{L^{2m}(\Omega\times\mathbb{S}^1)}\,\|\bar{u}\|_{L^{2m}(\Omega)}^{2m-1}. \end{aligned} \quad (3.20)$$

Based on (3.14), (3.16), (3.17), Sobolev embedding theorem and the trace theorem, we have

$$\begin{aligned} \|\nabla_x\xi\|_{L^{\frac{m}{m-1}}(\Gamma)} &\leqq C\,\|\nabla_x\xi\|_{W^{\frac{1}{2m},\frac{2m}{2m-1}}(\Gamma)} \leqq C\,\|\nabla_x\xi\|_{W^{1,\frac{2m}{2m-1}}(\Omega)} \\ &\leqq C\,\|\xi\|_{W^{2,\frac{2m}{2m-1}}(\Omega)} \leqq C\,\|\bar{u}\|_{L^{2m}(\Omega)}^{2m-1}. \end{aligned} \quad (3.21)$$

Based on (3.14), (3.17) and Hölder's inequality, we have

$$\begin{aligned} \epsilon \int_\Gamma u\phi\,\mathrm{d}\gamma &= \epsilon \int_{\Gamma^+} u\phi\,\mathrm{d}\gamma + \epsilon \int_{\Gamma^-} u\phi\,\mathrm{d}\gamma \\ &\leqq C\epsilon\,\|\nabla_x\xi\|_{L^{\frac{m}{m-1}}(\Gamma)}\Big(\|u\|_{L^m(\Gamma^+)} + \|h\|_{L^m(\Gamma^-)}\Big) \\ &\leqq C\epsilon\,\|\bar{u}\|_{L^{2m}(\Omega\times\mathbb{S}^1)}^{2m-1}\Big(\|u\|_{L^m(\Gamma^+)} + \|h\|_{L^m(\Gamma^-)}\Big). \end{aligned} \quad (3.22)$$

Also, we have

$$\iint_{\Omega\times\mathbb{S}^1}(u-\bar{u})\phi \leqq C\left\|\phi\right\|_{L^2(\Omega\times\mathbb{S}^1)}\left\|u-\bar{u}\right\|_{L^2(\Omega\times\mathbb{S}^1)} \leqq C\left\|\bar{u}\right\|_{L^{2m}(\Omega)}^{2m-1}\left\|u-\bar{u}\right\|_{L^2(\Omega\times\mathbb{S}^1)}, \tag{3.23}$$

$$\iint_{\Omega\times\mathbb{S}^1} f\phi \leqq C\left\|\phi\right\|_{L^2(\Omega\times\mathbb{S}^1)}\left\|f\right\|_{L^2(\Omega\times\mathbb{S}^1)} \leqq C\left\|\bar{u}\right\|_{L^{2m}(\Omega)}^{2m-1}\left\|f\right\|_{L^2(\Omega\times\mathbb{S}^1)}. \tag{3.24}$$

Collecting terms in (3.19), (3.20), (3.22), (3.23) and (3.24), we obtain

$$\epsilon\left\|\bar{u}\right\|_{L^{2m}(\Omega\times\mathbb{S}^1)} \leqq C\bigg(\epsilon\left\|u-\bar{u}\right\|_{L^{2m}(\Omega\times\mathbb{S}^1)}+\left\|u-\bar{u}\right\|_{L^2(\Omega\times\mathbb{S}^1)}+\epsilon\left\|u\right\|_{L^m(\Gamma^+)} + \left\|f\right\|_{L^2(\Omega\times\mathbb{S}^1)}+\epsilon\left\|h\right\|_{L^m(\Gamma^-)}\bigg), \tag{3.25}$$

Step 2: Energy Estimate.
In the weak formulation (3.12), we may take the test function $\phi = u$ to get the energy estimate

$$\frac{1}{2}\epsilon\int_{\Gamma}|u|^2\,\mathrm{d}\gamma+\left\|u-\bar{u}\right\|_{L^2(\Omega\times\mathbb{S}^1)}^2=\iint_{\Omega\times\mathbb{S}^1} fu. \tag{3.26}$$

This naturally implies

$$\epsilon\left\|u\right\|_{L^2(\Gamma^+)}^2+\left\|u-\bar{u}\right\|_{L^2(\Omega\times\mathbb{S}^1)}^2=\iint_{\Omega\times\mathbb{S}^1} fu+\epsilon\left\|h\right\|_{L^2(\Gamma^-)}^2. \tag{3.27}$$

On the other hand, we can square on both sides of (3.25) to obtain

$$\epsilon^2\left\|\bar{u}\right\|_{L^{2m}(\Omega\times\mathbb{S}^1)}^2 \leqq C\bigg(\epsilon^2\left\|u-\bar{u}\right\|_{L^{2m}(\Omega\times\mathbb{S}^1)}^2+\left\|u-\bar{u}\right\|_{L^2(\Omega\times\mathbb{S}^1)}^2+\epsilon^2\left\|u\right\|_{L^m(\Gamma^+)} + \left\|f\right\|_{L^2(\Omega\times\mathbb{S}^1)}^2+\epsilon^2\left\|h\right\|_{L^m(\Gamma^-)}^2\bigg), \tag{3.28}$$

Multiplying (3.28) by a sufficiently small constant and adding it to (3.27) to absorb $\left\|u-\bar{u}\right\|_{L^2(\Omega\times\mathbb{S}^1)}^2$, we deduce

$$\begin{aligned}&\epsilon\left\|u\right\|_{L^2(\Gamma^+)}^2+\epsilon^2\left\|\bar{u}\right\|_{L^{2m}(\Omega\times\mathbb{S}^1)}^2+\left\|u-\bar{u}\right\|_{L^2(\Omega\times\mathbb{S}^1)}^2\\ &\leqq C\bigg(\epsilon^2\left\|u-\bar{u}\right\|_{L^{2m}(\Omega\times\mathbb{S}^1)}^2+\epsilon^2\left\|u\right\|_{L^m(\Gamma^+)}+\left\|f\right\|_{L^2(\Omega\times\mathbb{S}^1)}^2\\ &\quad+\iint_{\Omega\times\mathbb{S}^1} fu+\epsilon\left\|h\right\|_{L^2(\Gamma^-)}^2+\epsilon^2\left\|h\right\|_{L^m(\Gamma^-)}^2\bigg).\end{aligned} \tag{3.29}$$

By interpolation estimate and Young's inequality, we have

$$\left\|u\right\|_{L^m(\Gamma^+)} \leqq \left\|u\right\|_{L^2(\Gamma^+)}^{\frac{2}{m}}\left\|u\right\|_{L^\infty(\Gamma^+)}^{\frac{m-2}{m}}=\left(\frac{1}{\epsilon^{\frac{m-2}{m^2}}}\left\|u\right\|_{L^2(\Gamma^+)}^{\frac{2}{m}}\right)\left(\epsilon^{\frac{m-2}{m^2}}\left\|u\right\|_{L^\infty(\Gamma^+)}^{\frac{m-2}{m}}\right)$$

$$
\begin{aligned}
&\leqq C\left(\frac{1}{\epsilon^{\frac{m-2}{m^2}}}\|u\|_{L^2(\Gamma^+)}^{\frac{2}{m}}\right)^{\frac{m}{2}}+o(1)\left(\epsilon^{\frac{m-2}{m^2}}\|u\|_{L^\infty(\Gamma^+)}^{\frac{m-2}{m}}\right)^{\frac{m}{m-2}}\\
&\leqq \frac{C}{\epsilon^{\frac{m-2}{2m}}}\|u\|_{L^2(\Gamma^+)}+o(1)\epsilon^{\frac{1}{m}}\|u\|_{L^\infty(\Gamma^+)}.
\end{aligned}
\tag{3.30}
$$

Similarly, we have

$$
\begin{aligned}
\|u-\bar{u}\|_{L^{2m}(\Omega\times\mathbb{S}^1)} \leqq\ & \|u-\bar{u}\|_{L^2(\Omega\times\mathbb{S}^1)}^{\frac{1}{m}}\|u-\bar{u}\|_{L^\infty(\Omega\times\mathbb{S}^1)}^{\frac{m-1}{m}}\\
=\ & \left(\frac{1}{\epsilon^{\frac{m-1}{m^2}}}\|u-\bar{u}\|_{L^2(\Omega\times\mathbb{S}^1)}^{\frac{1}{m}}\right)\left(\epsilon^{\frac{m-1}{m^2}}\|u-\bar{u}\|_{L^\infty(\Omega\times\mathbb{S}^1)}^{\frac{m-1}{m}}\right)\\
\leqq\ & C\left(\frac{1}{\epsilon^{\frac{m-1}{m^2}}}\|u-\bar{u}\|_{L^2(\Omega\times\mathbb{S}^1)}^{\frac{1}{m}}\right)^{m}\\
& +o(1)\left(\epsilon^{\frac{m-1}{m^2}}\|u-\bar{u}\|_{L^\infty(\Omega\times\mathbb{S}^1)}^{\frac{m-1}{m}}\right)^{\frac{m}{m-1}}\\
\leqq\ & \frac{C}{\epsilon^{\frac{m-1}{m}}}\|u-\bar{u}\|_{L^2(\Omega\times\mathbb{S}^1)}+o(1)\epsilon^{\frac{1}{m}}\|u-\bar{u}\|_{L^\infty(\Omega\times\mathbb{S}^1)}.
\end{aligned}
\tag{3.31}
$$

We need this extra $\epsilon^{\frac{1}{m}}$ for the convenience of L^∞ estimate. Then we know for sufficiently small ϵ,

$$
\begin{aligned}
\epsilon^2\|u\|_{L^m(\Gamma^+)}^2 &\leqq C\epsilon^{2-\frac{m-2}{m}}\|u\|_{L^2(\Gamma^+)}^2+o(1)\epsilon^{2+\frac{2}{m}}\|u\|_{L^\infty(\Gamma^+)}^2\\
&\leqq o(1)\epsilon\|u\|_{L^2(\Gamma^+)}^2+o(1)\epsilon^{2+\frac{2}{m}}\|u\|_{L^\infty(\Gamma^+)}^2.
\end{aligned}
\tag{3.32}
$$

Similarly, we have

$$
\begin{aligned}
\epsilon^2\|u-\bar{u}\|_{L^{2m}(\Omega\times\mathbb{S}^1)}^2 &\leqq \epsilon^{2-\frac{2m-2}{m}}\|u-\bar{u}\|_{L^2(\Omega\times\mathbb{S}^1)}^2+o(1)\epsilon^{2+\frac{2}{m}}\|u\|_{L^\infty(\Omega\times\mathbb{S}^1)}^2\\
&\leqq o(1)\|u-\bar{u}\|_{L^2(\Omega\times\mathbb{S}^1)}^2+o(1)\epsilon^{2+\frac{2}{m}}\|u\|_{L^\infty(\Omega\times\mathbb{S}^1)}^2.
\end{aligned}
\tag{3.33}
$$

Inserting (3.32) and (3.33) into (3.29), we can absorb $\|u-\bar{u}\|_{L^2(\Omega\times\mathbb{S}^1)}$ and $\epsilon\|u\|_{L^2(\Gamma^+)}^2$ into left-hand side to obtain

$$
\begin{aligned}
&\epsilon\|u\|_{L^2(\Gamma^+)}^2+\epsilon^2\|\bar{u}\|_{L^{2m}(\Omega\times\mathbb{S}^1)}^2+\|u-\bar{u}\|_{L^2(\Omega\times\mathbb{S}^1)}^2\\
&\quad\leqq C\bigg(o(1)\epsilon^{2+\frac{2}{m}}\left(\|u\|_{L^\infty(\Omega\times\mathbb{S}^1)}^2+\|u\|_{L^\infty(\Gamma^+)}^2\right)+\|f\|_{L^2(\Omega\times\mathbb{S}^1)}^2\\
&\qquad+\iint_{\Omega\times\mathbb{S}^1}fu+\epsilon\|h\|_{L^2(\Gamma^-)}^2+\epsilon^2\|h\|_{L^m(\Gamma^-)}^2\bigg).
\end{aligned}
\tag{3.34}
$$

We can decompose

$$
\iint_{\Omega\times\mathbb{S}^1}fu=\iint_{\Omega\times\mathbb{S}^1}f\bar{u}+\iint_{\Omega\times\mathbb{S}^1}f(u-\bar{u}).
\tag{3.35}
$$

Hölder's inequality and Cauchy's inequality imply

$$\iint_{\Omega\times\mathbb{S}^1} f\bar{u} \leqq \|f\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathbb{S}^1)} \|\bar{u}\|_{L^{2m}(\Omega\times\mathbb{S}^1)} \leqq \frac{C}{\epsilon^2}\|f\|^2_{L^{\frac{2m}{2m-1}}(\Omega\times\mathbb{S}^1)} + o(1)\epsilon^2\|\bar{u}\|^2_{L^{2m}(\Omega\times\mathbb{S}^1)}, \tag{3.36}$$

and

$$\iint_{\Omega\times\mathbb{S}^1} f(u-\bar{u}) \leqq C\|f\|^2_{L^2(\Omega\times\mathbb{S}^1)} + o(1)\|u-\bar{u}\|^2_{L^2(\Omega\times\mathbb{S}^1)}. \tag{3.37}$$

Hence, absorbing $\epsilon^2\|\bar{u}\|^2_{L^{2m}(\Omega\times\mathbb{S}^1)}$ and $\|u-\bar{u}\|^2_{L^2(\Omega\times\mathbb{S}^1)}$ into left-hand side of (3.34), we get

$$\begin{aligned}&\epsilon\|u\|^2_{L^2(\Gamma^+)} + \epsilon^2\|\bar{u}\|^2_{L^{2m}(\Omega\times\mathbb{S}^1)} + \|u-\bar{u}\|^2_{L^2(\Omega\times\mathbb{S}^1)}\\ &\leqq C\bigg(o(1)\epsilon^{2+\frac{2}{m}}\Big(\|u\|^2_{L^\infty(\Omega\times\mathbb{S}^1)} + \|u\|^2_{L^\infty(\Gamma^+)}\Big)\\ &\quad + \|f\|^2_{L^2(\Omega\times\mathbb{S}^1)} + \frac{1}{\epsilon^2}\|f\|^2_{L^{\frac{2m}{2m-1}}(\Omega\times\mathbb{S}^1)} + \epsilon\|h\|^2_{L^2(\Gamma^-)}\\ &\quad + \epsilon^2\|h\|^2_{L^m(\Gamma^-)}\bigg),\end{aligned} \tag{3.38}$$

which implies

$$\begin{aligned}&\frac{1}{\epsilon^{\frac{1}{2}}}\|u\|_{L^2(\Gamma^+)} + \|\bar{u}\|_{L^{2m}(\Omega\times\mathbb{S}^1)} + \frac{1}{\epsilon}\|u-\bar{u}\|_{L^2(\Omega\times\mathbb{S}^1)}\\ &\leqq C\bigg(o(1)\epsilon^{\frac{1}{m}}\Big(\|u\|_{L^\infty(\Omega\times\mathbb{S}^1)} + \|u\|_{L^\infty(\Gamma^+)}\Big)\\ &\quad + \frac{1}{\epsilon}\|f\|_{L^2(\Omega\times\mathbb{S}^1)} + \frac{1}{\epsilon^2}\|f\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathbb{S}^1)} + \frac{1}{\epsilon^{\frac{1}{2}}}\|h\|_{L^2(\Gamma^-)}\\ &\quad + \|h\|_{L^m(\Gamma^-)}\bigg).\end{aligned} \tag{3.39}$$

□

3.2. L^∞ Estimate

Theorem 3.5. *The unique solution $u(\vec{x},\vec{w})$ to the equation (3.1) satisfies*

$$\begin{aligned}\|u\|_{L^\infty(\Omega\times\mathbb{S}^1)} \leqq C\bigg(&\frac{1}{\epsilon^{1+\frac{1}{m}}}\|f\|_{L^2(\Omega\times\mathbb{S}^1)} + \frac{1}{\epsilon^{2+\frac{1}{m}}}\|f\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathbb{S}^1)} + \|f\|_{L^\infty(\Omega\times\mathbb{S}^1)}\\ &+ \frac{1}{\epsilon^{\frac{1}{2}+\frac{1}{m}}}\|h\|_{L^2(\Gamma^-)} + \frac{1}{\epsilon^{\frac{1}{m}}}\|h\|_{L^m(\Gamma^-)} + \|h\|_{L^\infty(\Gamma^-)}\bigg).\end{aligned} \tag{3.40}$$

Proof. Step 1: Double Duhamel iterations.
We can rewrite the equation (3.1) along the characteristics as

$$u(\vec{x},\vec{w}) = h(\vec{x}-\epsilon t_b\vec{w},\vec{w})\mathrm{e}^{-t_b} + \int_0^{t_b} f\Big(\vec{x}-\epsilon(t_b-s)\vec{w},\vec{w}\Big)\mathrm{e}^{-(t_b-s)}\,\mathrm{d}s + \frac{1}{2\pi}\int_0^{t_b}\bigg(\int_{\mathbb{S}^1} u\Big(\vec{x}-\epsilon(t_b-s)\vec{w},\vec{w}_t\Big)\,\mathrm{d}\vec{w}_t\bigg)\mathrm{e}^{-(t_b-s)}\,\mathrm{d}s, \quad (3.41)$$

where the backward exit time t_b is defined as

$$t_b(\vec{x},\vec{w}) = \inf\{t \geqq 0 : (\vec{x}-\epsilon t\vec{w},\vec{w}) \in \Gamma^-\}, \quad (3.42)$$

which represents the first time that the characteristics track back and hit the in-flow boundary. Note we have replaced $\bar{u}$ by the integral of u over the dummy velocity variable $\vec{w}_t$. For the last term in this formulation, we apply the Duhamel's principle again to $u\Big(\vec{x}-\epsilon(t_b-s)\vec{w},\vec{w}_t\Big)$ and obtain

$$\begin{aligned}
u(\vec{x},\vec{w}) &= h(\vec{x}-\epsilon t_b\vec{w},\vec{w})\mathrm{e}^{-t_b} + \int_0^{t_b} f\Big(\vec{x}-\epsilon(t_b-s)\vec{w},\vec{w}\Big)\mathrm{e}^{-(t_b-s)}\,\mathrm{d}s \\
&\quad + \frac{1}{2\pi}\int_0^{t_b}\bigg(\int_{\mathbb{S}^1} h\Big(\vec{x}-\epsilon(t_b-s)\vec{w}-\epsilon s_b\vec{w}_t,\vec{w}_t\Big)\mathrm{e}^{-s_b}\,\mathrm{d}\vec{w}_t\bigg)\mathrm{e}^{-(t_b-s)}\,\mathrm{d}s \\
&\quad + \frac{1}{2\pi}\int_0^{t_b}\bigg(\int_{\mathbb{S}^1}\bigg(\int_0^{s_b} f\Big(\vec{x}-\epsilon(t_b-s)\vec{w} \\
&\quad -\epsilon(s_b-r)\vec{w}_t,\vec{w}_t\Big)\mathrm{e}^{-(s_b-r)}\,\mathrm{d}r\bigg)\,\mathrm{d}\vec{w}_t\bigg)\mathrm{e}^{-(t_b-s)}\,\mathrm{d}s \\
&\quad + \frac{1}{2\pi}\int_0^{t_b}\bigg(\int_{\mathbb{S}^1}\bigg(\int_0^{s_b} \bar{u}\Big(\vec{x}-\epsilon(t_b-s)\vec{w} \\
&\quad -\epsilon(s_b-r)\vec{w}_t\Big)\mathrm{e}^{-(s_b-r)}\,\mathrm{d}r\bigg)\,\mathrm{d}\vec{w}_t\bigg)\mathrm{e}^{-(t_b-s)}\,\mathrm{d}s, \qquad (3.43)
\end{aligned}$$

where the exiting time from $\Big(\vec{x}-\epsilon(t_b-s)\vec{w},\vec{w}_t\Big)$ is defined as

$$s_b(\vec{x},\vec{w},s,\vec{w}_t) = \inf\{r \geqq 0 : \Big(\vec{x}-\epsilon(t_b-s)\vec{w}-\epsilon r\vec{w}_t,\vec{w}_t\Big) \in \Gamma^-\}. \quad (3.44)$$

Step 2: Estimates of all but the last term in (3.43).
We can directly estimate as follows:

$$\left|h(\vec{x}-\epsilon t_b\vec{w},\vec{w})\mathrm{e}^{-t_b}\right| \leqq \|h\|_{L^\infty(\Gamma^-)}, \quad (3.45)$$

$$\left|\frac{1}{2\pi}\int_0^{t_b}\bigg(\int_{\mathbb{S}^1} h\Big(\vec{x}-\epsilon(t_b-s)\vec{w}-\epsilon s_b\vec{w}_t,\vec{w}_t\Big)\mathrm{e}^{-s_b}\,\mathrm{d}\vec{w}_t\bigg)\mathrm{e}^{-(t_b-s)}\,\mathrm{d}s\right| \leqq \|h\|_{L^\infty(\Gamma^-)}, \quad (3.46)$$

$$\left|\int_0^{t_b} f\Big(\vec{x}-\epsilon(t_b-s)\vec{w},\vec{w}\Big)\mathrm{e}^{-(t_b-s)}\,\mathrm{d}s\right| \leqq \|f\|_{L^\infty(\Omega\times\mathbb{S}^1)}, \quad (3.47)$$

$$\left|\frac{1}{2\pi}\int_0^{t_b}\left(\int_{\mathbb{S}^1}\left(\int_0^{s_b} f(\vec{x}-\epsilon(t_b-s)\vec{w}\right.\right.\right.$$
$$\left.\left.\left.-\epsilon(s_b-r)\vec{w}_t,\vec{w}_t)\mathrm{e}^{-(s_b-r)}\,\mathrm{d}r\right)\mathrm{d}\vec{w}_t\right)\mathrm{e}^{-(t_b-s)}\,\mathrm{d}s\right| \leqq \|f\|_{L^\infty(\Omega\times\mathbb{S}^1)}. \quad (3.48)$$

Step 3: Estimates of the last term in (3.43).
Now we decompose the last term in (3.43) as

$$\int_0^{t_b}\int_{\mathbb{S}^1}\int_0^{s_b} = \int_0^{t_b}\int_{\mathbb{S}^1}\int_{s_b-r\leqq\delta} + \int_0^{t_b}\int_{\mathbb{S}^1}\int_{s_b-r\geqq\delta} = I_1 + I_2, \quad (3.49)$$

for some small $\delta > 0$ to be determined later. We can estimate I_1 directly as

$$I_1 \leqq \int_0^{t_b}\mathrm{e}^{-(t_b-s)}\left(\int_{\max(0,s_b-\delta)}^{s_b}\|u\|_{L^\infty(\Omega\times\mathbb{S}^1)}\,\mathrm{d}r\right)\mathrm{d}s \leqq C\delta\,\|u\|_{L^\infty(\Omega\times\mathbb{S}^1)}. \quad (3.50)$$

Then we can bound I_2 as

$$I_2 \leqq C\int_0^{t_b}\int_{\mathbb{S}^1}\int_0^{\max(0,s_b-\delta)}\left|\bar{u}\Big(\vec{x}-\epsilon(t_b-s)\vec{w}\right.$$
$$\left.-\epsilon(s_b-r)\vec{w}_t\Big)\right|\mathrm{e}^{-(t_b-s)}\,\mathrm{d}r\,\mathrm{d}\vec{w}_t\,\mathrm{d}s. \quad (3.51)$$

By the definition of t_b and s_b, we always have $\vec{x}-\epsilon(t_b-s)\vec{w}-\epsilon(s_b-r)\vec{w}_t \in \bar{\Omega}$. Hence, we may interchange the order of integration and apply Hölder's inequality to obtain

$$I_2 \leqq C\int_0^{t_b}\left(\left(\int_{\mathbb{S}^1}\int_0^{\max(0,s_b-\delta)}\mathbf{1}_\Omega\Big(\vec{x}-\epsilon(t_b-s)\vec{w}-\epsilon(s_b-r)\vec{w}_t\Big)\right.\right.$$
$$\left.\left|\bar{u}\Big(\vec{x}-\epsilon(t_b-s)\vec{w}-\epsilon(s_b-r)\vec{w}_t\Big)\right|^{2m}\mathrm{d}\vec{w}_t\,\mathrm{d}r\right)^{\frac{1}{2m}}$$
$$\times\left(\int_{\mathbb{S}^1}\int_0^{\max(0,s_b-\delta)}\mathbf{1}_\Omega\Big(\vec{x}-\epsilon(t_b-s)\vec{w}\right.$$
$$\left.\left.-\epsilon(s_b-r)\vec{w}_t\Big)\,\mathrm{d}\vec{w}_t\,\mathrm{d}r\right)^{\frac{2m-1}{2m}}\right)\mathrm{e}^{-(t_b-s)}\,\mathrm{d}s. \quad (3.52)$$

Note that $\vec{w}_t \in \mathbb{S}^1$, which is essentially a one-dimensional variable. Thus, we may write it in a new variable ψ as $\vec{w}_t = (\cos\psi, \sin\psi)$. Then we define the change of variable $[-\pi,\pi)\times\mathbb{R} \to \Omega : (\psi, r) \to (y_1, y_2) = \vec{y} = \vec{x}-\epsilon(t_b-s)\vec{w}-\epsilon(s_b-r)\vec{w}_t$, that is

$$\begin{cases} y_1 = x_1 - \epsilon(t_b-s)w_1 - \epsilon(s_b-r)\cos\psi, \\ y_2 = x_2 - \epsilon(t_b-s)w_2 - \epsilon(s_b-r)\sin\psi. \end{cases} \quad (3.53)$$

Therefore, for $s_b - r \geqq \delta$, we can directly compute the Jacobian

$$\left|\frac{\partial(y_1,y_2)}{\partial(\psi,r)}\right| = \left\|\begin{matrix} -\epsilon(s_b-r)\sin\psi & \epsilon\cos\psi \\ \epsilon(s_b-r)\cos\psi & \epsilon\sin\psi \end{matrix}\right\| = \epsilon^2(s_b-r) \geqq \epsilon^2\delta. \quad (3.54)$$

Hence, we may simplify (3.52) as

$$
\begin{aligned}
I_2 &\leqq C\int_0^{t_b}\left(\int_\Omega \frac{1}{\epsilon^2\delta}\left|\bar{u}(\vec{y})\right|^{2m}\,\mathrm{d}\vec{y}\right)^{\frac{1}{2m}}\mathrm{e}^{-(t_b-s)}\,\mathrm{d}s\\
&\leqq \frac{C}{\epsilon^{\frac{1}{m}}\delta^{\frac{1}{2m}}}\int_0^{t_b}\left(\int_\Omega \left|\bar{u}(\vec{y})\right|^{2m}\,\mathrm{d}\vec{y}\right)^{\frac{1}{2m}}\mathrm{e}^{-(t_b-s)}\,\mathrm{d}s\\
&\leqq \frac{C}{\epsilon^{\frac{1}{m}}\delta^{\frac{1}{2m}}}\left\|\bar{u}\right\|_{L^{2m}(\Omega\times\mathbb{S}^1)}.
\end{aligned}
\tag{3.55}
$$

Step 4: Synthesis.
In summary, collecting (3.45), (3.46), (3.47), (3.48), (3.50) and (3.55), for any $(\vec{x},\vec{w})\in\bar{\Omega}\times\mathbb{S}^1$, we have

$$
\begin{aligned}
\left|u(\vec{x},\vec{w})\right| \leqq{}& C\delta\left\|u\right\|_{L^\infty(\Omega\times\mathbb{S}^1)}+\frac{C}{\epsilon^{\frac{1}{m}}\delta^{\frac{1}{2m}}}\left\|\bar{u}\right\|_{L^{2m}(\Omega\times\mathbb{S}^1)}\\
&+C\bigg(\left\|f\right\|_{L^\infty(\Omega\times\mathbb{S}^1)}+\left\|h\right\|_{L^\infty(\Gamma^-)}\bigg).
\end{aligned}
\tag{3.56}
$$

Let δ be sufficiently small such that $C\delta\leqq\dfrac{1}{2}$. Taking supremum over $(\vec{x},\vec{w})\in\Gamma^+$ in (3.56) and using Theorem 3.4, we have

$$
\begin{aligned}
\left\|u\right\|_{L^\infty(\Gamma^+)} \leqq{}& \frac{1}{2}\left\|u\right\|_{L^\infty(\Omega\times\mathbb{S}^1)}+C\bigg(o(1)\Big(\left\|u\right\|_{L^\infty(\Omega\times\mathbb{S}^1)}+\left\|u\right\|_{L^\infty(\Gamma^+)}\Big)\\
&+\frac{1}{\epsilon^{1+\frac{1}{m}}}\left\|f\right\|_{L^2(\Omega\times\mathbb{S}^1)}+\frac{1}{\epsilon^{2+\frac{1}{m}}}\left\|f\right\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathbb{S}^1)}+\left\|f\right\|_{L^\infty(\Omega\times\mathbb{S}^1)}\\
&+\frac{1}{\epsilon^{\frac{1}{2}+\frac{1}{m}}}\left\|h\right\|_{L^2(\Gamma^-)}+\frac{1}{\epsilon^{\frac{1}{m}}}\left\|h\right\|_{L^m(\Gamma^-)}+\left\|h\right\|_{L^\infty(\Gamma^-)}\bigg).
\end{aligned}
\tag{3.57}
$$

Absorbing $o(1)\left\|u\right\|_{L^\infty(\Gamma^+)}$ into the left-hand side, we obtain

$$
\begin{aligned}
\left\|u\right\|_{L^\infty(\Gamma^+)} \leqq{}& \frac{1}{2}\left\|u\right\|_{L^\infty(\Omega\times\mathbb{S}^1)}+C\bigg(o(1)\left\|u\right\|_{L^\infty(\Omega\times\mathbb{S}^1)}\\
&+\frac{1}{\epsilon^{1+\frac{1}{m}}}\left\|f\right\|_{L^2(\Omega\times\mathbb{S}^1)}+\frac{1}{\epsilon^{2+\frac{1}{m}}}\left\|f\right\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathbb{S}^1)}+\left\|f\right\|_{L^\infty(\Omega\times\mathbb{S}^1)}\\
&+\frac{1}{\epsilon^{\frac{1}{2}+\frac{1}{m}}}\left\|h\right\|_{L^2(\Gamma^-)}+\frac{1}{\epsilon^{\frac{1}{m}}}\left\|h\right\|_{L^m(\Gamma^-)}+\left\|h\right\|_{L^\infty(\Gamma^-)}\bigg).
\end{aligned}
\tag{3.58}
$$

Taking supremum over $(\vec{x},\vec{w})\in\Omega\times\mathbb{S}^1$ in (3.56) and using Theorem 3.4, we have

$$
\begin{aligned}
\left\|u\right\|_{L^\infty(\Omega\times\mathbb{S}^1)} \leqq{}& \frac{1}{2}\left\|u\right\|_{L^\infty(\Omega\times\mathbb{S}^1)}+C\bigg(o(1)\Big(\left\|u\right\|_{L^\infty(\Omega\times\mathbb{S}^1)}+\left\|u\right\|_{L^\infty(\Gamma^+)}\Big)\\
&+\frac{1}{\epsilon^{1+\frac{1}{m}}}\left\|f\right\|_{L^2(\Omega\times\mathbb{S}^1)}+\frac{1}{\epsilon^{2+\frac{1}{m}}}\left\|f\right\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathbb{S}^1)}+\left\|f\right\|_{L^\infty(\Omega\times\mathbb{S}^1)}\\
&+\frac{1}{\epsilon^{\frac{1}{2}+\frac{1}{m}}}\left\|h\right\|_{L^2(\Gamma^-)}+\frac{1}{\epsilon^{\frac{1}{m}}}\left\|h\right\|_{L^m(\Gamma^-)}+\left\|h\right\|_{L^\infty(\Gamma^-)}\bigg).
\end{aligned}
\tag{3.59}
$$

Inserting (3.58) into (3.59), we obtain

$$\|u\|_{L^\infty(\Omega\times\mathbb{S}^1)} \leqq \frac{1}{2}\|u\|_{L^\infty(\Omega\times\mathbb{S}^1)} + C\bigg(o(1)\|u\|_{L^\infty(\Omega\times\mathbb{S}^1)} + \frac{1}{\epsilon^{1+\frac{1}{m}}}\|f\|_{L^2(\Omega\times\mathbb{S}^1)} + \frac{1}{\epsilon^{2+\frac{1}{m}}}\|f\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathbb{S}^1)} + \|f\|_{L^\infty(\Omega\times\mathbb{S}^1)} + \frac{1}{\epsilon^{\frac{1}{2}+\frac{1}{m}}}\|h\|_{L^2(\Gamma^-)} + \frac{1}{\epsilon^{\frac{1}{m}}}\|h\|_{L^m(\Gamma^-)} + \|h\|_{L^\infty(\Gamma^-)}\bigg). \tag{3.60}$$

Absorb $\frac{1}{2}\|u\|_{L^\infty(\Omega\times\mathbb{S}^1)}$ and $o(1)\|u\|_{L^\infty(\Omega\times\mathbb{S}^1)}$ into the left-hand side, we obtain

$$\|u\|_{L^\infty(\Omega\times\mathbb{S}^1)} \leqq C\bigg(\frac{1}{\epsilon^{1+\frac{1}{m}}}\|f\|_{L^2(\Omega\times\mathbb{S}^1)} + \frac{1}{\epsilon^{2+\frac{1}{m}}}\|f\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathbb{S}^1)} + \|f\|_{L^\infty(\Omega\times\mathbb{S}^1)} + \frac{1}{\epsilon^{\frac{1}{2}+\frac{1}{m}}}\|h\|_{L^2(\Gamma^-)} + \frac{1}{\epsilon^{\frac{1}{m}}}\|h\|_{L^m(\Gamma^-)} + \|h\|_{L^\infty(\Gamma^-)}\bigg). \tag{3.61}$$

□

4. Well-Posedness of ϵ-Milne Problem with Geometric Correction

We consider the ϵ-Milne problem with geometric correction

$$\begin{cases} \sin\phi\dfrac{\partial f}{\partial\eta} + F(\eta)\cos\phi\dfrac{\partial f}{\partial\phi} + f - \bar{f} = S(\eta,\phi), \\ f(0,\phi) = h(\phi) \ \text{ for } \ \sin\phi > 0, \\ f(L,\phi) = f(L,\mathscr{R}[\phi]). \end{cases} \tag{4.1}$$

for $f(\eta,\phi)$ in the domain $(\eta,\phi)\in[0,L]\times[-\pi,\pi)$ where $L=\epsilon^{-\frac{1}{2}}$, $\mathscr{R}[\phi]=-\phi$ and

$$F(\eta) = -\frac{\epsilon}{R_\kappa - \epsilon\eta}, \tag{4.2}$$

for the radius of curvature R_κ. Here, for convenience, we temporarily ignore the superscript on ϵ and τ. Define a potential function $V(\eta)$ satisfying $V(0)=0$ and $\dfrac{\partial V}{\partial\eta} = -F(\eta)$. Then we can direct compute

$$V(\eta) = \ln\left(\frac{R_\kappa}{R_\kappa - \epsilon\eta}\right). \tag{4.3}$$

We define the norms in the space $(\eta,\phi)\in[0,L]\times[-\pi,\pi)$ as follows:

$$\|f\|_{L^2L^2} = \left(\int_0^L\int_{-\pi}^{\pi}|f(\eta,\phi)|^2\,\mathrm{d}\phi\,\mathrm{d}\eta\right)^{\frac{1}{2}}, \tag{4.4}$$

$$\|f\|_{L^\infty L^\infty} = \operatorname{esssup}_{(\eta,\phi)\in[0,L]\times[-\pi,\pi)}|f(\eta,\phi)|, \tag{4.5}$$

$$\|f\|_{L^\infty L^2} = \operatorname{esssup}_{\eta\in[0,L]}\left(\int_{-\pi}^{\pi}|f(\eta,\phi)|^2\,\mathrm{d}\phi\right)^{\frac{1}{2}}. \tag{4.6}$$

Similarly,

$$\|f(\eta)\|_{L^2} = \left(\int_{-\pi}^{\pi} |f(\eta, \phi)|^2 \, \mathrm{d}\phi \right)^{\frac{1}{2}}, \tag{4.7}$$

$$\|f(\eta)\|_{L^\infty} = \operatorname{esssup}_{\phi \in [-\pi, \pi)} |f(\eta, \phi)| . \tag{4.8}$$

Also, we define the weighted norms at in-flow boundary as

$$\|h\|_{L^2_-} = \left(\int_{\sin\phi > 0} |h(\phi)|^2 \sin\phi \, \mathrm{d}\phi \right)^{\frac{1}{2}}, \tag{4.9}$$

$$\|h\|_{L^\infty_-} = \operatorname{esssup}_{\sin\phi > 0} |h(\phi)| . \tag{4.10}$$

Also define

$$\langle f, g \rangle_\phi (\eta) = \int_{-\pi}^{\pi} f(\eta, \phi) g(\eta, \phi) \, \mathrm{d}\phi \tag{4.11}$$

as the L^2 inner product in ϕ.

In the following, we will always assume that for some $C, K > 0$ uniform in ϵ,

$$\|h\|_{L^\infty_-} + \left\| \mathrm{e}^{K\eta} S \right\|_{L^\infty L^\infty} \leqq C. \tag{4.12}$$

The well-posedness, exponential decay and maximum principle of the equation (4.1) has been well studied in [24]. Here we will focus on the a priori estimates and present detail analysis for the dependence of f on the boundary data h and the source term S.

4.1. L^2 Estimates

4.1.1. $\bar{S} = 0$ Case Assume that S satisfies $\bar{S}(\eta) = 0$ for any η. We may decompose the solution

$$f(\eta, \phi) = q_f(\eta) + r_f(\eta, \phi), \tag{4.13}$$

where the hydrodynamical part q_f is in the null space of the operator $f - \bar{f}$, and the microscopic part r_f is the orthogonal complement, that is

$$q_f(\eta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\eta, \phi) \, \mathrm{d}\phi = \bar{f}, \quad r_f(\eta, \phi) = f(\eta, \phi) - q_f(\eta). \tag{4.14}$$

In what follows, when there is no confusion, we simply write $f = q + r$.

Lemma 4.1. *Assume* (4.12) *holds and* $\bar{S}(\eta) = 0$ *for any* $\eta \in [0, L]$. *Then the unique solution* $f(\eta, \phi)$ *to the equation* (4.1) *satisfies*

$$\|r\|_{L^2 L^2} \leqq C \Big(\|h\|_{L^2_-} + \|S\|_{L^2 L^2} \Big), \tag{4.15}$$

and there exists $q_L \in \mathbb{R}$ *such that*

$$|q_L| \leqq C\bigg(\|h\|_{L^2_-} + \|S\|_{L^2L^2}\bigg) + C\left|\int_0^L \langle \sin\phi, S\rangle_\phi(y)\,\mathrm{d}y\right|, \tag{4.16}$$

$$\|q - q_L\|_{L^2L^2} \leqq C\bigg(\|h\|_{L^2_-} + \|S\|_{L^2L^2}\bigg) + C\left(\int_0^L \left(\int_\eta^L \langle \sin\phi, S\rangle_\phi(y)\,\mathrm{d}y\right)^2 \mathrm{d}\eta\right)^{\frac{1}{2}}. \tag{4.17}$$

Also, for any $\eta \in [0, L]$,

$$\langle \sin\phi, r\rangle_\phi(\eta) = 0. \tag{4.18}$$

Proof. Step 1: Estimate of r.
Multiplying f on both sides of (4.1) and integrating over $\phi \in [-\pi, \pi)$, we get the energy estimate

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\eta}\langle f, f\sin\phi\rangle_\phi(\eta) + F(\eta)\left\langle \frac{\partial f}{\partial\phi}, f\cos\phi\right\rangle_\phi(\eta) + \|r(\eta)\|_{L^2}^2 = \langle S, f\rangle_\phi(\eta). \tag{4.19}$$

An integration by parts reveals

$$F(\eta)\left\langle \frac{\partial f}{\partial\phi}, f\cos\phi\right\rangle_\phi(\eta) = \frac{1}{2}F(\eta)\langle f, f\sin\phi\rangle_\phi(\eta). \tag{4.20}$$

Also, the assumption $\bar{S}(\eta) = 0$ leads to

$$\langle S, f\rangle_\phi(\eta) = \langle S, q\rangle_\phi(\eta) + \langle S, r\rangle_\phi(\eta) = \langle S, r\rangle_\phi(\eta). \tag{4.21}$$

Hence, we have the simplified form of (4.19) as follows:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\eta}\langle f, f\sin\phi\rangle_\phi(\eta) + \frac{1}{2}F(\eta)\langle f, f\sin\phi\rangle_\phi(\eta) + \|r(\eta)\|_{L^2}^2 = \langle S, r\rangle_\phi(\eta). \tag{4.22}$$

Define

$$\alpha(\eta) = \frac{1}{2}\langle f, f\sin\phi\rangle_\phi(\eta). \tag{4.23}$$

Then (4.22) can be rewritten as follows:

$$\frac{\mathrm{d}\alpha}{\mathrm{d}\eta} + F(\eta)\alpha(\eta) + \|r(\eta)\|_{L^2}^2 = \langle S, r\rangle_\phi(\eta). \tag{4.24}$$

We can solve this differential equation for α on $[\eta, L]$ and $[0, \eta]$ respectively to obtain

$$\alpha(\eta) = \alpha(L)\exp\left(\int_\eta^L F(y)\,\mathrm{d}(y)\right) + \int_\eta^L \exp\left(\int_\eta^y F(z)\,\mathrm{d}z\right)\left(\|r(y)\|_{L^2}^2 - \langle S, r\rangle_\phi(y)\right)\mathrm{d}y, \tag{4.25}$$

$$\alpha(\eta) = \alpha(0)\exp\left(-\int_0^\eta F(y)\,\mathrm{d}(y)\right) + \int_0^\eta \exp\left(-\int_y^\eta F(z)\,\mathrm{d}z\right)\left(-\|r(y)\|_{L^2}^2 + \langle S, r\rangle_\phi(y)\right)\mathrm{d}y. \tag{4.26}$$

The specular reflexive boundary $f(L,\phi) = f(L,\mathscr{R}[\phi])$ ensures $\alpha(L) = 0$. Hence, based on (4.25), we have

$$\begin{aligned}\alpha(\eta) &\geqq \int_\eta^L \exp\left(\int_\eta^y F(z)\,\mathrm{d}z\right)\left(-\langle S, r\rangle_\phi(y)\right)\mathrm{d}y\\ &\geqq -C\int_\eta^L \langle S, r\rangle_\phi(y)\,\mathrm{d}y.\end{aligned} \tag{4.27}$$

Also, (4.26) implies

$$\begin{aligned}\alpha(\eta) &\leqq \alpha(0)\exp\left(-\int_0^\eta F(y)\,\mathrm{d}(y)\right)\\ &\quad + \int_0^\eta \exp\left(-\int_y^\eta F(z)\,\mathrm{d}z\right)\left(\langle S, r\rangle_\phi(y)\right)\mathrm{d}y\\ &\leqq C\|h\|_{L^2_-}^2 + C\int_0^\eta\left(\langle S, r\rangle_\phi(y)\right)\mathrm{d}y,\end{aligned} \tag{4.28}$$

due to the fact

$$\alpha(0) = \frac{1}{2}\langle \sin\phi f, f\rangle_\phi(0) \leqq \frac{1}{2}\left(\int_{\sin\phi>0} h^2(\phi)\sin\phi\,\mathrm{d}\phi\right) \leqq C\|h\|_{L^2_-}^2. \tag{4.29}$$

Then in (4.26) taking $\eta = L$, from $\alpha(L) = 0$, we have

$$\begin{aligned}&\int_0^L \exp\left(\int_0^y F(z)\,\mathrm{d}z\right)\|r(y)\|_{L^2}^2\,\mathrm{d}y\\ &\leqq \alpha(0) + \int_0^L \exp\left(\int_0^y F(z)\,\mathrm{d}z\right)\langle S, r\rangle_\phi(y)\,\mathrm{d}y\\ &\leqq C\|h\|_{L^2_-}^2 + C\int_0^L \langle S, r\rangle_\phi(y)\,\mathrm{d}y.\end{aligned} \tag{4.30}$$

On the other hand, we can directly estimate as follows:

$$\int_0^L \exp\left(\int_0^y F(z)\,\mathrm{d}z\right)\|r(y)\|_{L^2}^2\,\mathrm{d}y \geqq C\int_0^L \|r(y)\|_{L^2}^2\,\mathrm{d}y. \tag{4.31}$$

Combining (4.30) and (4.31) yields

$$\int_0^L \|r(\eta)\|_{L^2}^2\,\mathrm{d}\eta \leqq C\|h\|_{L^2_-}^2 + C\int_0^L \langle S, r\rangle_\phi(y)\,\mathrm{d}y. \tag{4.32}$$

By Cauchy's inequality, we have

$$\left|\int_0^L \langle S, r\rangle_\phi (y)\, \mathrm{d}y\right| \leqq C_0 \int_0^L \|r(\eta)\|_{L^2}^2 \, \mathrm{d}\eta + \frac{4}{C_0}\int_0^L \|S(\eta)\|_{L^2}^2 \, \mathrm{d}\eta, \quad (4.33)$$

for $C_0 > 0$ small. Therefore, absorbing $\int_0^L \|r(\eta)\|_{L^2}^2 \, \mathrm{d}\eta$ and summarizing (4.32) and (4.33), we deduce

$$\int_0^L \|r(\eta)\|_{L^2}^2 \, \mathrm{d}\eta \leqq C\left(\|h\|_{L^2_-}^2 + \int_0^L \|S(\eta)\|_{L^2}^2 \, \mathrm{d}\eta\right). \quad (4.34)$$

Step 2: Orthogonality relation.
A direct integration over $\phi \in [-\pi, \pi)$ in (4.1) implies

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\langle \sin\phi, f\rangle_\phi (\eta) = -F\left\langle \cos\phi, \frac{\mathrm{d}f}{\mathrm{d}\phi}\right\rangle_\phi (\eta) + \bar{S}(\eta) = -F\,\langle \sin\phi, f\rangle_\phi (\eta), \quad (4.35)$$

due to $\bar{S} = 0$. The specular reflexive boundary $f(L,\phi) = f(L,\mathscr{R}[\phi])$ implies $\langle \sin\phi, f\rangle_\phi (L) = 0$. Then we have

$$\langle \sin\phi, f\rangle_\phi (\eta) = 0. \quad (4.36)$$

It is easy to see

$$\langle \sin\phi, q\rangle_\phi (\eta) = 0. \quad (4.37)$$

Hence, we may derive

$$\langle \sin\phi, r\rangle_\phi (\eta) = 0. \quad (4.38)$$

This leads to orthogonal relation (4.18).
Step 3: Estimate of q.
Multiplying $\sin\phi$ on both sides of (4.1) and integrating over $\phi \in [-\pi, \pi)$ lead to

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left\langle \sin^2\phi, f\right\rangle_\phi (\eta) = -\langle \sin\phi, r\rangle_\phi (\eta) - F(\eta)\left\langle \sin\phi\cos\phi, \frac{\partial f}{\partial\phi}\right\rangle_\phi (\eta) + \langle \sin\phi, S\rangle_\phi (\eta). \quad (4.39)$$

We can further integrate by parts as follows:

$$-F(\eta)\left\langle \sin\phi\cos\phi, \frac{\partial f}{\partial\phi}\right\rangle_\phi (\eta) = F(\eta)\,\langle \cos(2\phi), f\rangle_\phi (\eta) = F(\eta)\,\langle \cos(2\phi), r\rangle_\phi (\eta). \quad (4.40)$$

Using the orthogonal relation (4.18), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left\langle \sin^2\phi, f\right\rangle_\phi (\eta) = F(\eta)\,\langle \cos(2\phi), r\rangle_\phi (\eta) + \langle \sin\phi, S\rangle_\phi (\eta).$$

Define

$$\beta(\eta) = \left\langle \sin^2\phi, f \right\rangle_\phi (\eta). \tag{4.41}$$

Hence, we can integrate (4.41) over $[0, \eta]$ to get that

$$\beta(\eta) - \beta(0) = \int_0^\eta F(y) \left\langle \cos(2\phi), r \right\rangle_\phi (y)\, \mathrm{d}y + \int_0^\eta \left\langle \sin\phi, S \right\rangle_\phi (y)\, \mathrm{d}y. \tag{4.42}$$

Then the initial data

$$\begin{aligned}\beta(0) = \left\langle \sin^2\phi, f \right\rangle_\phi (0) &\leqq \Big(\left\langle f, f\,|\sin\phi| \right\rangle_\phi (0) \Big)^{\frac{1}{2}} \|\sin\phi\|_{L^2}^{\frac{3}{2}} \\ &\leqq C\Big(\left\langle f, f\,|\sin\phi| \right\rangle_\phi (0) \Big)^{\frac{1}{2}}.\end{aligned} \tag{4.43}$$

Obviously, we have

$$\begin{aligned}\left\langle f, f\,|\sin\phi| \right\rangle_\phi (0) = &\int_{\sin\phi>0} h^2(\phi)\sin\phi\, \mathrm{d}\phi \\ &- \int_{\sin\phi<0} \Big(f(0,\phi) \Big)^2 \sin\phi\, \mathrm{d}\phi.\end{aligned} \tag{4.44}$$

However, based on the definition of $\alpha(\eta)$ and (4.27), we can obtain

$$\begin{aligned}&\int_{\sin\phi>0} h^2(\phi)\sin\phi\, \mathrm{d}\phi + \int_{\sin\phi<0} \Big(f(0,\phi) \Big)^2 \sin\phi\, \mathrm{d}\phi = 2\alpha(0) \\ &\geqq -C \int_0^L \left\langle S, r \right\rangle_\phi (y)\, \mathrm{d}y.\end{aligned}$$

Hence, we can deduce

$$\begin{aligned}-\int_{\sin\phi<0} \Big(f(0,\phi) \Big)^2 \sin\phi\, \mathrm{d}\phi &\leqq \int_{\sin\phi>0} h^2(\phi)\sin\phi\, \mathrm{d}\phi + C\int_0^L \left\langle S, r \right\rangle_\phi (y)\, \mathrm{d}y \\ &\leqq C\Big(\|h\|_{L^2_-}^2 + \int_0^L \|S(\eta)\|_{L^2}^2\, \mathrm{d}\eta \Big).\end{aligned} \tag{4.45}$$

From (4.34), we can deduce

$$\beta(0) \leqq C\Big(\|h\|_{L^2_-} + \|S\|_{L^2L^2} \Big). \tag{4.46}$$

Since $F \in L^1[0, L] \cap L^2[0, L]$, $r \in L^2([0, L] \times [-\pi, \pi))$, by (4.46) and (4.15), we have

$$|\beta(L)| \leqq |\beta(0)| + \left| \int_0^L F(y) \left\langle \cos(2\phi), r \right\rangle_\phi (y)\, \mathrm{d}y \right| + \left| \int_0^L \left\langle \sin\phi, S \right\rangle_\phi (y)\, \mathrm{d}y \right|$$

$$\leqq C\Big(\|h\|_{L^2_-} + \|S\|_{L^2L^2}\Big) + C\|F\|_{L^2L^2}\|r\|_{L^2L^2} + \left|\int_0^L \langle \sin\phi, S\rangle_\phi (y)\,\mathrm{d}y\right|$$

$$\leqq C\Big(\|h\|_{L^2_-} + \|S\|_{L^2L^2}\Big) + \left|\int_0^L \langle \sin\phi, S\rangle_\phi (y)\,\mathrm{d}y\right|. \tag{4.47}$$

We define

$$q_L = \frac{\beta(L)}{\|\sin\phi\|_{L^2}^2}. \tag{4.48}$$

Naturally, we have

$$|q_L| \leqq C\Big(\|h\|_{L^2_-} + \|S\|_{L^2L^2}\Big) + C\left|\int_0^L \langle \sin\phi, S\rangle_\phi (y)\,\mathrm{d}y\right|. \tag{4.49}$$

Note that q_L is not necessarily $q(L)$. Moreover,

$$\beta(L) - \beta(\eta) = \int_\eta^L F(y)\,\langle\cos(2\phi), r\rangle_\phi (y)\,\mathrm{d}y + \int_\eta^L \langle \sin\phi, S\rangle_\phi (y)\,\mathrm{d}y. \tag{4.50}$$

Note

$$\begin{aligned}\beta(\eta) &= \left\langle \sin^2\phi, f\right\rangle_\phi (\eta) = \left\langle \sin^2\phi, q\right\rangle_\phi (\eta) + \left\langle \sin^2\phi, r\right\rangle_\phi (\eta)\\ &= q(\eta)\,\|\sin\phi\|_{L^2}^2 + \left\langle \sin^2\phi, r\right\rangle_\phi (\eta).\end{aligned} \tag{4.51}$$

Thus we can estimate

$$\begin{aligned}&\|\sin\phi\|_{L^2}^2\,\|q(\eta) - q_L\|_{L^2}\\ &\quad= \beta(L) - \beta(\eta) + \left\langle \sin^2\phi, r\right\rangle_\phi (\eta)\\ &\quad\leqq C\bigg(\int_\eta^L \left|F(y)\,\langle\cos(2\phi), r(y)\rangle_\phi\,\mathrm{d}y\right|\,\mathrm{d}\eta + \left|\int_\eta^L \langle\sin\phi, S\rangle_\phi (y)\,\mathrm{d}y\right|\\ &\qquad + \left|\left\langle \sin^2\phi, r\right\rangle_\phi (\eta)\right|\bigg)\\ &\quad\leqq C\bigg(\|r(\eta)\|_{L^2} + \int_\eta^L |F(y)|\,\|r(y)\|_{L^2}\,\mathrm{d}y\\ &\qquad + \left|\int_\eta^L \langle\sin\phi, S\rangle_\phi (y)\,\mathrm{d}y\right|\bigg).\end{aligned} \tag{4.52}$$

Then we integrate (4.52) over $\eta \in [0, L]$. Cauchy's inequality implies

$$\begin{aligned}\int_0^L \left(\int_\eta^L |F(y)|\,\|r(y)\|_{L^2}\,\mathrm{d}y\right)^2 \mathrm{d}\eta &\leqq \|r\|_{L^2L^2}^2 \int_0^L \int_\eta^L |F(y)|^2\,\mathrm{d}y\,\mathrm{d}\eta\\ &\leqq C\|r\|_{L^2L^2}^2.\end{aligned} \tag{4.53}$$

Hence, we have

$$\|q - q_L\|_{L^2L^2} \leqq C\Big(\|h\|_{L^2_-} + \|S\|_{L^2L^2}\Big) + C\bigg(\int_0^L \bigg(\int_\eta^L \langle \sin\phi, S\rangle_\phi (y)\,\mathrm{d}y\bigg)^2 \mathrm{d}\eta\bigg)^{\frac{1}{2}}. \quad (4.54)$$

□

4.1.2. $\bar{S} \neq 0$ Case For general S, we define $S = \bar{S} + (S - \bar{S}) = S_Q + S_R$.

Lemma 4.2. *Assume* (4.12) *holds. The unique solution* $f(\eta, \phi)$ *to the equation* (4.1) *satisfies*

$$\|r\|_{L^2L^2} \leqq C\Big(\|h\|_{L^2_-} + \|S\|_{L^2L^2}\Big) + C\bigg(\int_0^L \bigg(\int_\eta^L \big|S_Q(y)\big|\,\mathrm{d}y\bigg)^2 \mathrm{d}\eta\bigg)^{\frac{1}{2}}, \quad (4.55)$$

and there exists $q_L \in \mathbb{R}$ *such that*

$$|q_L| \leqq C\Big(\|h\|_{L^2_-} + \|S\|_{L^2L^2}\Big) + C\left|\int_0^L \langle \sin\phi, S_R\rangle_\phi (y)\,\mathrm{d}y\right| + C\left|\int_0^L \int_\eta^L \big|S_Q(y)\big|\,\mathrm{d}y\,\mathrm{d}\eta\right|, \quad (4.56)$$

$$\|q - q_L\|_{L^2L^2} \leqq C\Big(\|h\|_{L^2_-} + \|S\|_{L^2L^2}\Big) + C\bigg(\int_0^L \bigg(\int_\eta^L \langle \sin\phi, S_R\rangle_\phi (y)\,\mathrm{d}y\bigg)^2 \mathrm{d}\eta\bigg)^{\frac{1}{2}} + C\bigg(\int_0^L \bigg(\int_\eta^L \int_y^L \big|S_Q(z)\big|\,\mathrm{d}z\,\mathrm{d}y\bigg)^2 \mathrm{d}\eta\bigg)^{\frac{1}{2}}. \quad (4.57)$$

Also, for any $\eta \in [0, L]$,

$$\langle \sin\phi, r\rangle_\phi (\eta) = -\int_\eta^L \mathrm{e}^{V(\eta)-V(y)} S_Q(y)\,\mathrm{d}y. \quad (4.58)$$

Proof. We can apply superposition property for this linear problem. For simplicity, we just call the above estimates as the L^2 estimates.
Step 1: Construction of auxiliary function f^1.
We first solve f^1 as the solution to

$$\begin{cases} \sin\phi \dfrac{\partial f^1}{\partial\eta} + F(\eta)\cos\phi \dfrac{\partial f^1}{\partial\phi} + f^1 - \bar{f}^1 = S_R(\eta,\phi), \\ f^1(0,\phi) = h(\phi) \ \text{ for } \ \sin\phi > 0, \\ f^1(L,\phi) = f^1(L, \mathscr{R}[\phi]). \end{cases} \quad (4.59)$$

Since $\bar{S}_R = 0$, by Lemma 4.1, we know there exists a unique solution f^1 satisfying the L^2 estimate.
Step 2: Construction of auxiliary function f^2.
We seek a function f^2 satisfying

$$-\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sin\phi\frac{\partial f^2}{\partial\eta}+F(\eta)\cos\phi\frac{\partial f^2}{\partial\phi}\right)\mathrm{d}\phi+S_Q=0. \tag{4.60}$$

The following analysis shows this type of function can always be found. An integration by parts transforms the equation (4.60) into

$$-\int_{-\pi}^{\pi}\sin\phi\frac{\partial f^2}{\partial\eta}\,\mathrm{d}\phi-\int_{-\pi}^{\pi}F(\eta)\sin\phi f^2\,\mathrm{d}\phi+2\pi S_Q=0. \tag{4.61}$$

Setting

$$f^2(\phi,\eta)=a(\eta)\sin\phi. \tag{4.62}$$

and plugging this ansatz into (4.61), we have

$$-\frac{\mathrm{d}a}{\mathrm{d}\eta}\int_{-\pi}^{\pi}\sin^2\phi\,\mathrm{d}\phi-F(\eta)a(\eta)\int_{-\pi}^{\pi}\sin^2\phi\,\mathrm{d}\phi+2\pi S_Q=0. \tag{4.63}$$

Hence, we have

$$-\frac{\mathrm{d}a}{\mathrm{d}\eta}-F(\eta)a(\eta)+2S_Q=0. \tag{4.64}$$

This is a first order linear ordinary differential equation, which possesses infinite solutions. We can directly solve it to obtain

$$a(\eta)=\exp\left(-\int_0^{\eta}F(y)\,\mathrm{d}y\right)\left(a(0)+\int_0^{\eta}\exp\left(\int_0^{y}F(z)\,\mathrm{d}z\right)2S_Q(y)\,\mathrm{d}y\right). \tag{4.65}$$

We may take

$$a(0)=-\int_0^{L}\exp\left(\int_0^{y}F(z)\,\mathrm{d}z\right)2S_Q(y)\,\mathrm{d}y. \tag{4.66}$$

Then, we can directly verify

$$|a(\eta)|\leqq C\int_{\eta}^{L}\left|S_Q(y)\right|\,\mathrm{d}y, \tag{4.67}$$

and f^2 satisfies the L^2 estimate.
Step 3: Construction of auxiliary function f^3.
Based on above construction, we can directly verify that

$$\int_{-\pi}^{\pi}\left(-\sin\phi\frac{\partial f^2}{\partial\eta}-F(\eta)\cos\phi\frac{\partial f^2}{\partial\phi}-f^2+\bar{f}^2+S_Q\right)\mathrm{d}\phi=0. \tag{4.68}$$

Then we can solve f^3 as the solution to

$$\begin{cases} \sin\phi\dfrac{\partial f^3}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial f^3}{\partial\phi}+f^3-\bar{f}^3 \\ \quad =-\sin\phi\dfrac{\partial f^2}{\partial\eta}-F(\eta)\cos\phi\dfrac{\partial f^2}{\partial\phi}-f^2+\bar{f}^2+S_Q, \\ f^3(0,\phi)=-a(0)\sin\phi \ \text{ for } \ \sin\phi>0, \\ f^3(L,\phi)=f^3(L,\mathscr{R}[\phi]). \end{cases} \tag{4.69}$$

By (4.68), we can apply Lemma 4.1 to obtain a unique solution f^3 satisfying the L^2 estimate.

Step 4: Construction of auxiliary function f^4.

We now define $f^4=f^2+f^3$ and an explicit verification shows

$$\begin{cases} \sin\phi\dfrac{\partial f^4}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial f^4}{\partial\phi}+f^4-\bar{f}^4=S_Q(\eta,\phi), \\ f^4(0,\phi)=0 \ \text{ for } \ \sin\phi>0, \\ f^4(L,\phi)=f^4(L,\mathscr{R}[\phi]), \end{cases} \tag{4.70}$$

and f^4 satisfies the L^2 estimate.

In summary, we deduce that f^1+f^4 is the solution of (4.1) and satisfies the L^2 estimate. □

Combining all above, letting $f_L=q_L$, we have the following theorem:

Theorem 4.3. *Assume* (4.12) *holds. There exists* $f_L\in\mathbb{R}$ *satisfying*

$$\begin{aligned} |f_L| \leqq\; & C\Big(\|h\|_{L^2_-}+\|S\|_{L^2L^2}\Big)+C\left|\int_0^L \langle\sin\phi,S_R\rangle_\phi(y)\,\mathrm{d}y\right| \\ &+C\left|\int_0^L\int_\eta^L \big|S_Q(y)\big|\,\mathrm{d}y\,\mathrm{d}\eta\right|, \end{aligned} \tag{4.71}$$

such that the unique solution $f(\eta,\phi)$ *to the equation* (4.1) *satisfies*

$$\begin{aligned} \|f-f_L\|_{L^2L^2} \leqq\; & C\Big(\|h\|_{L^2_-}+\|S\|_{L^2L^2}\Big) \\ &+C\bigg(\int_0^L\bigg(\int_\eta^L \langle\sin\phi,S_R\rangle_\phi(y)\,\mathrm{d}y\bigg)^2\mathrm{d}\eta\bigg)^{\frac12} \\ &+C\bigg(\int_0^L\bigg(\int_\eta^L \big|S_Q(y)\big|\,\mathrm{d}y\bigg)^2\mathrm{d}\eta\bigg)^{\frac12} \\ &+C\bigg(\int_0^L\bigg(\int_\eta^L\int_y^L \big|S_Q(z)\big|\,\mathrm{d}z\,\mathrm{d}y\bigg)^2\mathrm{d}\eta\bigg)^{\frac12}. \end{aligned} \tag{4.72}$$

Here C *represents some constant uniform in* ϵ.

4.2. L^∞ Estimates

4.2.1. Formulation Consider the ϵ-transport problem for $f(\eta,\phi)$ in $(\eta,\phi)\in[0,L]\times[-\pi,\pi)$

$$\begin{cases} \sin\phi\dfrac{\partial f}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial f}{\partial\phi}+f=H(\eta,\phi),\\ f(0,\phi)=h(\phi)\ \text{ for }\ \sin\phi>0,\\ f(L,\phi)=f(L,\mathscr{R}[\phi]).\end{cases} \tag{4.73}$$

Here, we assume $H=S+\bar{f}\in L^\infty$. Define the energy as follows:

$$E(\eta,\phi)=\mathrm{e}^{-V(\eta)}\cos\phi. \tag{4.74}$$

Along the characteristics, this energy is conserved and the equation can be simplified as follows:

$$\sin\phi\frac{\mathrm{d}f}{\mathrm{d}\eta}+f=H. \tag{4.75}$$

Since V is increasing, an implicit function $\eta^+(\eta,\phi)$ can be determined through

$$|E(\eta,\phi)|=\mathrm{e}^{-V(\eta^+)}, \tag{4.76}$$

which means (η^+,ϕ_0) with $\sin\phi_0=0$ is on the same characteristics as (η,ϕ). Define the quantities for $0\leqq\eta'\leqq\eta^+$ as follows:

$$\phi'(\eta,\phi;\eta')=\cos^{-1}\left(\mathrm{e}^{V(\eta')-V(\eta)}\cos\phi\right), \tag{4.77}$$

$$\mathscr{R}[\phi'(\eta,\phi;\eta')]=-\cos^{-1}\left(\mathrm{e}^{V(\eta')-V(\eta)}\cos\phi\right)=-\phi'(\eta,\phi;\eta'), \tag{4.78}$$

where the inverse trigonometric function can be defined single-valued in the domain $[0,\pi)$ and the quantities are always well-defined due to the monotonicity of V. Finally we put

$$G_{\eta,\eta'}(\phi)=\int_{\eta'}^{\eta}\frac{1}{\sin\left(\phi'(\eta,\phi;\xi)\right)}\,\mathrm{d}\xi. \tag{4.79}$$

We can rewrite the solution to the equation (4.73) along the characteristics as

$$f(\eta,\phi)=\mathcal{K}[h](\eta,\phi)+\mathcal{T}[H](\eta,\phi), \tag{4.80}$$

where
Region I:
For $\sin\phi>0$,

$$\mathcal{K}[h](\eta,\phi)=h\Big(\phi'(\eta,\phi;0)\Big)\exp(-G_{\eta,0}), \tag{4.81}$$

$$\mathcal{T}[H](\eta,\phi)=\int_0^{\eta}\frac{H\Big(\eta',\phi'(\eta,\phi;\eta')\Big)}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\exp(-G_{\eta,\eta'})\,\mathrm{d}\eta'. \tag{4.82}$$

Region II:
For $\sin\phi < 0$ and $|E(\eta,\phi)| \leqq e^{-V(L)}$,

$$\mathcal{K}[h](\eta,\phi) = h(\phi'(\eta,\phi;0))\exp(-G_{L,0} - G_{L,\eta}) \tag{4.83}$$

$$\mathcal{T}[H](\eta,\phi) = \int_0^L \frac{H\Big(\eta',\phi'(\eta,\phi;\eta')\Big)}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\exp(-G_{L,\eta'} - G_{L,\eta})\,d\eta' + \int_\eta^L \frac{H\Big(\eta',\mathscr{R}[\phi'(\eta,\phi;\eta')]\Big)}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\exp(G_{\eta,\eta'})\,d\eta'. \tag{4.84}$$

Region III:
For $\sin\phi < 0$ and $|E(\eta,\phi)| \geqq e^{-V(L)}$,

$$\mathcal{K}[h](\eta,\phi) = h\Big(\phi'(\eta,\phi;0)\Big)\exp(-G_{\eta^+,0} - G_{\eta^+,\eta}) \tag{4.85}$$

$$\mathcal{T}[H](\eta,\phi) = \int_0^{\eta^+} \frac{H\Big(\eta',\phi'(\eta,\phi;\eta')\Big)}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\exp(-G_{\eta^+,\eta'} - G_{\eta^+,\eta})\,d\eta' + \int_\eta^{\eta^+} \frac{H\Big(\eta',\mathscr{R}[\phi'(\eta,\phi;\eta')]\Big)}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\exp(G_{\eta,\eta'})\,d\eta'. \tag{4.86}$$

Here, the decomposition of regions is based on whether the characteristics touches $\eta = L$ and $\sin\phi = 0$. In order to achieve the estimate of f, we need to control $\mathcal{K}[h]$ and $\mathcal{T}[H]$.

4.2.2. Preliminaries We first give several technical lemmas to be used for proving L^∞ estimates of f. The proofs are given in [24, Lemma 4.7-4.9], so we omit them here.

Lemma 4.4. *For any* $0 \leqq \beta \leqq 1$, *we have*

$$\left\| e^{\beta\eta}\mathcal{K}[h] \right\|_{L^\infty L^\infty} \leqq \|h\|_{L^\infty_-}. \tag{4.87}$$

In particular,

$$\|\mathcal{K}[h]\|_{L^\infty L^\infty} \leqq \|h\|_{L^\infty_-}. \tag{4.88}$$

Lemma 4.5. *The integral operator* $\mathcal{T}$ *satisfies*

$$\|\mathcal{T}[H]\|_{L^\infty L^\infty} \leqq \|H\|_{L^\infty L^\infty}, \tag{4.89}$$

and for any $0 \leqq \beta \leqq \dfrac{1}{2}$

$$\left\| e^{\beta\eta}\mathcal{T}[H] \right\|_{L^\infty L^\infty} \leqq \left\| e^{\beta\eta} H \right\|_{L^\infty L^\infty}. \tag{4.90}$$

Lemma 4.6. *For any* $\delta > 0$ *there is a constant* $C(\delta) > 0$ *independent of data such that*

$$\|\mathcal{T}[H]\|_{L^\infty L^2} \leqq C(\delta)\|H\|_{L^2 L^2} + \delta\|H\|_{L^\infty L^\infty}. \tag{4.91}$$

4.2.3. Estimates of ϵ-Milne Equation with Geometric Correction Consider the equation satisfied by $\mathscr{V} = f - f_L$ as follows:

$$\begin{cases} \sin\phi \dfrac{\partial \mathscr{V}}{\partial \eta} + F(\eta)\cos\phi \dfrac{\partial \mathscr{V}}{\partial \phi} + \mathscr{V} = \bar{\mathscr{V}} + S, \\ \mathscr{V}(0,\phi) = p(\phi) := h(\phi) - f_L \quad \text{for} \quad \sin\phi > 0, \\ \mathscr{V}(L,\phi) = \mathscr{V}(L, \mathscr{R}[\phi]). \end{cases} \tag{4.92}$$

Theorem 4.7. *Assume* (4.12) *holds. The unique solution* $f(\eta,\phi)$ *to the equation* (4.1) *satisfies*

$$\|f - f_L\|_{L^\infty L^\infty} \leqq C\Big(|f_L| + \|h\|_{L^\infty_-} + \|S\|_{L^\infty L^\infty} + \|f - f_L\|_{L^2L^2}\Big). \tag{4.93}$$

Proof. We first show the following important facts:

$$\big\|\bar{\mathscr{V}}\big\|_{L^2L^2} \leqq \|\mathscr{V}\|_{L^2L^2}, \tag{4.94}$$

$$\big\|\bar{\mathscr{V}}\big\|_{L^\infty L^\infty} \leqq \|\mathscr{V}\|_{L^\infty L^2}. \tag{4.95}$$

We can directly derive them by Cauchy's inequality as follows:

$$\begin{aligned} \|\bar{\mathscr{V}}\|^2_{L^2L^2} &= \int_0^L \int_{-\pi}^{\pi} \left(\frac{1}{2\pi}\right)^2 \left(\int_{-\pi}^{\pi} \mathscr{V}(\eta,\phi')\,\mathrm{d}\phi'\right)^2 \mathrm{d}\phi\,\mathrm{d}\eta \\ &\leqq \int_0^L \int_{-\pi}^{\pi} \left(\frac{1}{2\pi}\right)\left(\int_{-\pi}^{\pi} \mathscr{V}^2(\eta,\phi')\,\mathrm{d}\phi'\right) \mathrm{d}\phi\,\mathrm{d}\eta \\ &= \int_0^L \left(\int_{-\pi}^{\pi} \mathscr{V}^2(\eta,\phi')\,\mathrm{d}\phi'\right) \mathrm{d}\eta = \|\mathscr{V}\|^2_{L^2L^2}. \end{aligned} \tag{4.96}$$

$$\begin{aligned} \|\bar{\mathscr{V}}\|^2_{L^\infty L^\infty} &= \sup_\eta \bar{\mathscr{V}}^2(\eta) = \sup_\eta \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathscr{V}(\eta,\phi)\,\mathrm{d}\phi\right)^2 \\ &\leqq \sup_\eta \left(\frac{1}{2\pi}\right)^2 \left(\int_{-\pi}^{\pi} \mathscr{V}^2(\eta,\phi)\,\mathrm{d}\phi\right)\left(\int_{-\pi}^{\pi} 1^2\,\mathrm{d}\phi\right) \\ &= \sup_\eta \left(\int_{-\pi}^{\pi} \mathscr{V}^2(\eta,\phi)\,\mathrm{d}\phi\right) = \|\mathscr{V}\|^2_{L^\infty L^2}. \end{aligned} \tag{4.97}$$

Then by Lemma 4.6, (4.94) and (4.95), we can show

$$\begin{aligned} \big\|\mathcal{T}[\bar{\mathscr{V}}]\big\|_{L^\infty L^2} &\leqq C(\delta)\big\|\bar{\mathscr{V}}\big\|_{L^2L^2} + \delta\big\|\bar{\mathscr{V}}\big\|_{L^\infty L^\infty} \\ &\leqq C(\delta)\|\mathscr{V}\|_{L^2L^2} + \delta\|\mathscr{V}\|_{L^\infty L^2}. \end{aligned} \tag{4.98}$$

By (4.92),

$$\mathscr{V} = \mathcal{K}[p] + \mathcal{T}[\bar{\mathscr{V}}] + \mathcal{T}[S]. \tag{4.99}$$

Therefore, based on Lemma 4.4, Lemma 4.5 and (4.98), we can directly estimate

$$\begin{aligned} \|\mathscr{V}\|_{L^\infty L^2} &\leqq \|\mathcal{K}[p]\|_{L^\infty L^2} + \|\mathcal{T}[S]\|_{L^\infty L^2} + C(\delta)\|\mathscr{V}\|_{L^2L^2} + \delta\|\mathscr{V}\|_{L^\infty L^2} \\ &\leqq \|p\|_{L^\infty_-} + \|S\|_{L^\infty L^\infty} + C(\delta)\|\mathscr{V}\|_{L^2L^2} + \delta\|\mathscr{V}\|_{L^\infty L^2}. \end{aligned} \tag{4.100}$$

We can take $\delta = \dfrac{1}{2}$ to obtain

$$\|\mathscr{V}\|_{L^\infty L^2} \leqq C\bigg(\|\mathscr{V}\|_{L^2L^2} + \|p\|_{L^\infty_-} + \|S\|_{L^\infty L^\infty}\bigg). \tag{4.101}$$

Therefore, based on Lemma 4.5, (4.101) and (4.95), we can achieve

$$\begin{aligned}\|\mathscr{V}\|_{L^\infty L^\infty} &\leqq \|\mathcal{K}[p]\|_{L^\infty L^\infty} + \|\mathcal{T}[S]\|_{L^\infty L^\infty} + \left\|\mathcal{T}[\bar{\mathscr{V}}]\right\|_{L^\infty L^\infty}\\ &\leqq C\bigg(\|p\|_{L^\infty_-} + \|S\|_{L^\infty L^\infty} + \left\|\bar{\mathscr{V}}\right\|_{L^\infty L^\infty}\bigg)\\ &\leqq C\bigg(\|p\|_{L^\infty_-} + \|S\|_{L^\infty L^\infty} + \|\mathscr{V}\|_{L^\infty L^2}\bigg)\\ &\leqq C\bigg(\|p\|_{L^\infty_-} + \|S\|_{L^\infty L^\infty} + \|\mathscr{V}\|_{L^2 L^2}\bigg).\end{aligned} \tag{4.102}$$

□

Combining Theorem 4.7 and Theorem 4.3, we deduce the main theorem.

Theorem 4.8. *Assume* (4.12) *holds. There exists* $f_L \in \mathbb{R}$ *satisfying*

$$\begin{aligned}|f_L| \leqq\ & C\bigg(\|h\|_{L^2_-} + \|S\|_{L^2L^2}\bigg) + C\left|\int_0^L \langle\sin\phi, S_R\rangle_\phi(y)\,\mathrm{d}y\right|\\ &+C\left|\int_0^L\int_\eta^L \left|S_Q(y)\right|\,\mathrm{d}y\,\mathrm{d}\eta\right|,\end{aligned} \tag{4.103}$$

such that the unique solution $f(\eta,\phi)$ *to the equation* (4.1) *satisfies*

$$\begin{aligned}\|f - f_L\|_{L^\infty L^\infty} \leqq\ & C\bigg(\|h\|_{L^2_-} + \|S\|_{L^2L^2} + \|h\|_{L^\infty_-} + \|S\|_{L^\infty L^\infty}\bigg)\\ &+C\left(\int_0^L\left(\int_\eta^L \langle\sin\phi, S_R\rangle_\phi(y)\,\mathrm{d}y\right)^2\mathrm{d}\eta\right)^{\frac{1}{2}}\\ &+C\left|\int_0^L \langle\sin\phi, S_R\rangle_\phi(y)\,\mathrm{d}y\right|\\ &+C\left|\int_0^L\int_\eta^L \left|S_Q(y)\right|\,\mathrm{d}y\,\mathrm{d}\eta\right|\\ &+C\left(\int_0^L\left(\int_\eta^L \left|S_Q(y)\right|\,\mathrm{d}y\right)^2\mathrm{d}\eta\right)^{\frac{1}{2}}\\ &+C\left(\int_0^L\left(\int_\eta^L\int_y^L \left|S_Q(z)\right|\,\mathrm{d}z\,\mathrm{d}y\right)^2\mathrm{d}\eta\right)^{\frac{1}{2}}.\end{aligned} \tag{4.104}$$

Here C *represents some constant uniform in* ϵ.

4.3. Exponential Decay

In this section, we prove the spatial decay of the solution to the ϵ-Milne problem with geometric correction.

Theorem 4.9. *Assume (4.12) holds. For $K_0 > 0$ sufficiently small, there exists $f_L \in \mathbb{R}$ satisfying*

$$|f_L| \leqq C\bigg(\|h\|_{L^2_-} + \|S\|_{L^2L^2}\bigg) + C\left|\int_0^L \langle \sin\phi, S_R\rangle_\phi(y)\,\mathrm{d}y\right| + C\left|\int_0^L\int_\eta^L \big|S_Q(y)\big|\,\mathrm{d}y\,\mathrm{d}\eta\right|, \tag{4.105}$$

such that the unique solution $f(\eta,\phi)$ to the equation (4.1) satisfies

$$\begin{aligned}\|f-f_L\|_{L^\infty L^\infty} \leqq{}& C\bigg(\|h\|_{L^2_-} + \left\|\mathrm{e}^{K_0\eta}S\right\|_{L^2L^2} + \|h\|_{L^\infty_-} + \left\|\mathrm{e}^{K_0\eta}S\right\|_{L^\infty L^\infty}\bigg)\\ &+C\bigg(\int_0^L \mathrm{e}^{2K_0\eta}\bigg(\int_\eta^L \langle \sin\phi, S_R\rangle_\phi(y)\,\mathrm{d}y\bigg)^2\mathrm{d}\eta\bigg)^{\frac12}\\ &+C\left|\int_0^L \langle \sin\phi, S_R\rangle_\phi(y)\,\mathrm{d}y\right|\\ &+C\left|\int_0^L\int_\eta^L \big|S_Q(y)\big|\,\mathrm{d}y\,\mathrm{d}\eta\right|\\ &+C\bigg(\int_0^L \mathrm{e}^{2K_0\eta}\bigg(\int_\eta^L \big|S_Q(y)\big|\,\mathrm{d}y\bigg)^2\mathrm{d}\eta\bigg)^{\frac12}\\ &+C\bigg(\int_0^L \mathrm{e}^{2K_0\eta}\bigg(\int_\eta^L\int_y^L \big|S_Q(z)\big|\,\mathrm{d}z\,\mathrm{d}y\bigg)^2\mathrm{d}\eta\bigg)^{\frac12}.\end{aligned} \tag{4.106}$$

Here C represents some constant uniform in ϵ.

Proof. Define $Z = \mathrm{e}^{K_0\eta}\mathscr{V}$ for $\mathscr{V} = f - f_L$.
Step 1: L^2 estimates.
We use the decomposition in (4.14). The orthogonal property reveals

$$\langle f, f\sin\phi\rangle_\phi(\eta) = \langle r, r\sin\phi\rangle_\phi(\eta). \tag{4.107}$$

Multiplying $\mathrm{e}^{2K_0\eta}f$ on both sides of equation (4.1) and integrating over $\phi \in [-\pi,\pi)$, we obtain

$$\begin{aligned}&\frac12\frac{\mathrm{d}}{\mathrm{d}\eta}\bigg(\mathrm{e}^{2K_0\eta}\langle r, r\sin\phi\rangle_\phi(\eta)\bigg) + \frac12 F(\eta)\bigg(\mathrm{e}^{2K_0\eta}\langle r, r\sin\phi\rangle_\phi(\eta)\bigg)\\ &\quad-\mathrm{e}^{2K_0\eta}\bigg(K_0\langle r, r\sin\phi\rangle_\phi(\eta) - \langle r, r\rangle_\phi(\eta)\bigg)\\ &= \mathrm{e}^{2K_0\eta}\langle S, f\rangle_\phi(\eta).\end{aligned} \tag{4.108}$$

For $K_0 < \min\left\{\frac{1}{2}, K\right\}$ for K in (4.12), we have

$$\frac{3}{2}\left\|r(\eta)\right\|_{L^2}^2 \geqq -K_0\left\langle r, r\sin\phi\right\rangle_\phi(\eta) + \left\langle r, r\right\rangle_\phi(\eta) \geqq \frac{1}{2}\left\|r(\eta)\right\|_{L^2}^2. \quad (4.109)$$

Similar to the proof of Lemma 4.1, formula as (4.108) and (4.109) imply

$$\left\|\mathrm{e}^{K_0\eta}r\right\|_{L^2L^2}^2 = \int_0^L \mathrm{e}^{2K_0\eta}\left\langle r, r\right\rangle_\phi(\eta)\,\mathrm{d}\eta \leqq C\left(\|h\|_{L^2_-}^2 + \left\|\mathrm{e}^{K_0\eta}S\right\|_{L^2L^2}^2\right). \quad (4.110)$$

From the proof of Lemma 4.1 and Cauchy's inequality, we can deduce

$$\begin{aligned}
&\int_0^L \mathrm{e}^{2K_0\eta}\left(\int_{-\pi}^{\pi}\left(f(\eta,\phi)-f_L\right)^2\mathrm{d}\phi\right)\mathrm{d}\eta\\
&\leqq \int_0^L \mathrm{e}^{2K_0\eta}\left(\int_{-\pi}^{\pi} r^2(\eta,\phi)\,\mathrm{d}\phi\right)\mathrm{d}\eta + \int_0^L \mathrm{e}^{2K_0\eta}\left(\int_{-\pi}^{\pi}\Big(q(\eta)-q_L\Big)^2\mathrm{d}\phi\right)\mathrm{d}\eta\\
&\leqq \int_0^L \mathrm{e}^{2K_0\eta}\left\|r(\eta)\right\|_{L^2}^2\,\mathrm{d}\eta\\
&\quad+\int_0^L \mathrm{e}^{2K_0\eta}\left(\int_\eta^L |F(y)|\,\|r(y)\|_{L^2}\,\mathrm{d}y\right)^2\mathrm{d}\eta\\
&\quad+\int_0^L \mathrm{e}^{2K_0\eta}\left(\int_\eta^L \left\langle\sin\phi, S\right\rangle_\phi(y)\,\mathrm{d}y\right)^2\mathrm{d}\eta\\
&\leqq C\left(\|h\|_{L^2_-}^2 + \left\|\mathrm{e}^{K_0\eta}S\right\|_{L^2L^2}^2\right)\\
&\quad+C\left(\int_0^L \mathrm{e}^{2K_0\eta}\left\|r(\eta)\right\|_{L^2}^2\,\mathrm{d}\eta\right)\left(\int_0^L\int_\eta^L \mathrm{e}^{2K_0(\eta-y)}F^2(y)\,\mathrm{d}y\,\mathrm{d}\eta\right)\\
&\quad+\int_0^L \mathrm{e}^{2K_0\eta}\left(\int_\eta^L \left\langle\sin\phi, S\right\rangle_\phi(y)\,\mathrm{d}y\right)^2\mathrm{d}\eta\\
&\leqq C\left(\|h\|_{L^2_-}^2 + \left\|\mathrm{e}^{K_0\eta}S\right\|_{L^2L^2}^2\right)\\
&\quad+C\left(\int_0^L \mathrm{e}^{2K_0\eta}\left\|r(\eta)\right\|_{L^2}^2\,\mathrm{d}\eta\right)\left(\int_0^L\int_\eta^L F^2(y)\,\mathrm{d}y\,\mathrm{d}\eta\right)\\
&\quad+\int_0^L \mathrm{e}^{2K_0\eta}\left(\int_\eta^L \left\langle\sin\phi, S\right\rangle_\phi(y)\,\mathrm{d}y\right)^2\mathrm{d}\eta\\
&\leqq C\left(\|h\|_{L^2_-}^2 + \left\|\mathrm{e}^{K_0\eta}S\right\|_{L^2L^2}^2\right) + \int_0^L \mathrm{e}^{2K_0\eta}\\
&\quad\left(\int_\eta^L \left\langle\sin\phi, S\right\rangle_\phi(y)\,\mathrm{d}y\right)^2\mathrm{d}\eta. \qquad (4.111)
\end{aligned}$$

This completes the proof of the L^2 estimate when $\bar{S}=0$. By the method introduced in Lemma 4.2, we can extend the above L^2 estimates to the general S case. Note all the auxiliary functions constructed in Lemma 4.2 satisfy the estimates. We have

$$\begin{aligned}\|Z\|_{L^2L^2} \leqq C\bigg(\|h\|_{L^2_-} + \|S\|_{L^2L^2}\bigg) \\ +C\bigg(\int_0^L \mathrm{e}^{2K_0\eta}\bigg(\int_\eta^L \langle\sin\phi, S_R\rangle_\phi(y)\,\mathrm{d}y\bigg)^2\mathrm{d}\eta\bigg)^{\frac{1}{2}} \\ +C\bigg(\int_0^L \mathrm{e}^{2K_0\eta}\bigg(\int_\eta^L \left|S_Q(y)\right|\,\mathrm{d}y\bigg)^2\mathrm{d}\eta\bigg)^{\frac{1}{2}} \\ +C\bigg(\int_0^L \mathrm{e}^{2K_0\eta}\bigg(\int_\eta^L\int_y^L \left|S_Q(z)\right|\,\mathrm{d}z\,\mathrm{d}y\bigg)^2\mathrm{d}\eta\bigg)^{\frac{1}{2}}, \end{aligned} \tag{4.112}$$

Step 2: L^∞ estimates.
Z satisfies the equation

$$\begin{cases} \sin\phi\dfrac{\partial Z}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial Z}{\partial\phi}+Z=\bar{Z}+\mathrm{e}^{K_0\eta}S+K_0\sin\phi Z, \\ Z(0,\phi)=p(\phi)=h(\phi)-f_L \quad\text{for}\quad \sin\phi>0 \\ Z(L,\phi)=Z(L,\mathscr{R}[\phi]). \end{cases} \tag{4.113}$$

Then by Lemma 4.6, (4.94) and (4.95), we can show

$$\begin{aligned}\left\|\bar{Z}\right\|_{L^\infty L^2} &\leqq C(\delta)\left\|\bar{Z}\right\|_{L^2L^2}+\delta\left\|\bar{Z}\right\|_{L^\infty L^\infty} \\ &\leqq C(\delta)\|Z\|_{L^2L^2}+\delta\|Z\|_{L^\infty L^2}.\end{aligned} \tag{4.114}$$

We know

$$Z=\mathcal{K}[p]+\mathcal{T}[\bar{Z}+\mathrm{e}^{K_0\eta}S+K_0\sin\phi Z]. \tag{4.115}$$

Therefore, based on Lemma 4.4 and (4.98), we can directly estimate

$$\begin{aligned}\|Z\|_{L^\infty L^2} &\leqq \|\mathcal{K}[p]\|_{L^\infty L^2}+\left\|\mathcal{T}[\mathrm{e}^{K_0\eta}S]\right\|_{L^\infty L^2} \\ &\quad+\|\mathcal{T}[K_0\sin\phi Z]\|_{L^\infty L^2}+C(\delta)\|Z\|_{L^2L^2}+\delta\|Z\|_{L^\infty L^2} \\ &\leqq \|p\|_{L^\infty_-}+\left\|\mathrm{e}^{K_0\eta}S\right\|_{L^\infty L^\infty}+K_0\|Z\|_{L^\infty L^\infty}+C(\delta)\|Z\|_{L^2L^2} \\ &\quad+\delta\|Z\|_{L^\infty L^2}.\end{aligned} \tag{4.116}$$

We can take $\delta=\dfrac{1}{2}$ to obtain

$$\|Z\|_{L^\infty L^2} \leqq C\bigg(\|p\|_{L^\infty_-}+\left\|\mathrm{e}^{K_0\eta}S\right\|_{L^\infty L^\infty}+K_0\|Z\|_{L^\infty L^\infty}+\|Z\|_{L^2L^2}\bigg). \tag{4.117}$$

Then based on Lemma 4.4, Lemma 4.5, Lemma 4.6 and (4.117), we can deduce

$$\|Z\|_{L^\infty L^\infty} \leqq \|\mathcal{K}[p]\|_{L^\infty L^\infty}+\left\|\mathcal{T}[\mathrm{e}^{K_0\eta}S]\right\|_{L^\infty L^\infty}+\left\|\bar{Z}\right\|_{L^\infty L^\infty}$$

$$
\begin{aligned}
&+\|K_0 \sin\phi Z\|_{L^\infty L^\infty}\\
&\leqq \|p\|_{L^\infty_-} + \left\|\mathrm{e}^{K_0\eta} S\right\|_{L^\infty L^\infty} + \|\bar{Z}\|_{L^\infty L^\infty} + K_0\|Z\|_{L^\infty L^\infty}\\
&\leqq \|p\|_{L^\infty_-} + \left\|\mathrm{e}^{K_0\eta} S\right\|_{L^\infty L^\infty} + \|Z\|_{L^\infty L^2} + K_0\|Z\|_{L^\infty L^\infty}\\
&\leqq C\bigg(\|Z\|_{L^2L^2} + \left\|\mathrm{e}^{K_0\eta} S\right\|_{L^2L^2}\\
&\quad + \left\|\mathrm{e}^{K_0\eta} S\right\|_{L^\infty L^\infty} + \|p\|_{L^\infty_-} + K_0\|Z\|_{L^\infty L^\infty}\bigg). \qquad (4.118)
\end{aligned}
$$

Taking K_0 sufficiently small, we absorb $K_0\|Z\|_{L^\infty L^\infty}$ to the left-hand side and obtain

$$
\|Z\|_{L^\infty L^\infty} \leqq C\left(\|Z\|_{L^2L^2} + \left\|\mathrm{e}^{K_0\eta} S\right\|_{L^2L^2} + \left\|\mathrm{e}^{K_0\eta} S\right\|_{L^\infty L^\infty} + \|p\|_{L^\infty_-}\right). \qquad (4.119)
$$

Then the final result is obvious. □

4.4. Maximum Principle

In [24], the author proved the maximum principle.

Theorem 4.10. *The unique solution $f(\eta,\phi)$ to the equation with $S = 0$ satisfies the maximum principle, that is*

$$
\inf_{\sin\phi>0} h(\phi) \leqq f(\eta,\phi) \leqq \sup_{\sin\phi>0} h(\phi). \qquad (4.120)
$$

5. Regularity of ϵ-Milne Problem with Geometric Correction

In this section, we study the regularity of the ϵ-Milne problem with geometric correction (4.1). Define the weight function

$$
\zeta(\eta,\phi) = \left(1 - \left(\frac{R_\kappa - \epsilon\eta}{R_\kappa}\cos\phi\right)^2\right)^{\frac{1}{2}}. \qquad (5.1)
$$

For $\eta = 0$, ζ reduces to $\sin\phi$ and it is zero only at the grazing set. The farther (η,ϕ) is away from the grazing set, the larger ζ is. Also, we can easily show that

$$
\sin\phi\frac{\partial\zeta}{\partial\eta} + F(\eta)\cos\phi\frac{\partial\zeta}{\partial\phi} = 0. \qquad (5.2)
$$

Along the characteristics, ζ is a constant.

We use the notation from previous section, that is $\mathscr{V}(\eta,\phi)=f(\eta,\phi)-f_L$ satisfies the difference equation

$$\begin{cases}\sin\phi\dfrac{\partial\mathscr{V}}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial\mathscr{V}}{\partial\phi}+\mathscr{V}=\bar{\mathscr{V}}+S,\\ \mathscr{V}(0,\phi)=p(\phi)=h(\phi)-f_L\ \text{ for }\ \sin\phi>0,\\ \mathscr{V}(L,\phi)=\mathscr{V}(L,\mathscr{R}[\phi]).\end{cases} \tag{5.3}$$

The regularity has been thoroughly studied in [7]. Hence, here we will focus on the a priori estimates and prove an improved version of the regularity theorem. For simplicity, we always assume the quantities discussed are well-defined. The major upshot is that we can avoid using the information of $\dfrac{\partial S}{\partial\phi}$.

5.1. Mild Formulation

Taking η derivative in (5.3) and multiplying ζ, we obtain the ϵ-transport problem for $\mathscr{A}=\zeta\dfrac{\partial\mathscr{V}}{\partial\eta}$ as

$$\begin{cases}\sin\phi\dfrac{\partial\mathscr{A}}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial\mathscr{A}}{\partial\phi}+\mathscr{A}=\tilde{\mathscr{A}}+S_{\mathscr{A}},\\ \mathscr{A}(0,\phi)=p_{\mathscr{A}}(\phi)\ \text{ for }\ \sin\phi>0,\\ \mathscr{A}(L,\phi)=\mathscr{A}(L,R\phi),\end{cases} \tag{5.4}$$

where $p_{\mathscr{A}}$ and $S_{\mathscr{A}}$ will be specified later with

$$\tilde{\mathscr{A}}(\eta,\phi)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\zeta(\eta,\phi)}{\zeta(\eta,\phi_*)}\mathscr{A}(\eta,\phi_*)\,\mathrm{d}\phi_*. \tag{5.5}$$

Here for clarity, we use dummy variable ϕ_*. Define the energy as before

$$E(\eta,\phi)=\mathrm{e}^{-V(\eta)}\cos\phi=\cos\phi\frac{R_\kappa-\epsilon\eta}{R_\kappa}. \tag{5.6}$$

Along the characteristics, where this energy is conserved and ζ is a constant, the equation (5.4) can be simplified as follows:

$$\sin\phi\frac{\mathrm{d}\mathscr{A}}{\mathrm{d}\eta}+\mathscr{A}=\tilde{\mathscr{A}}+S_{\mathscr{A}}. \tag{5.7}$$

Also, we recall the notation to describe the characteristics in Section 4.2. Similar to ϵ-Milne problem with geometric correction, we can define the solution along the characteristics as follows:

$$\mathscr{A}(\eta,\phi)=\mathcal{K}[p_{\mathscr{A}}]+\mathcal{T}[\tilde{\mathscr{A}}+S_{\mathscr{A}}], \tag{5.8}$$

where
Region I:

For $\sin\phi > 0$,

$$\mathcal{K}[p_{\mathscr{A}}] = p_{\mathscr{A}}\Big(\phi'(\eta,\phi;0)\Big)\exp(-G_{\eta,0}) \tag{5.9}$$

$$\mathcal{T}[\tilde{\mathscr{A}} + S_{\mathscr{A}}] = \int_0^{\eta} \frac{(\tilde{\mathscr{A}} + S_{\mathscr{A}})\Big(\eta',\phi'(\eta,\phi;\eta')\Big)}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)} \exp(-G_{\eta,\eta'})\,\mathrm{d}\eta'. \tag{5.10}$$

Region II:
For $\sin\phi < 0$ and $|E(\eta,\phi)| \leqq \mathrm{e}^{-V(L)}$,

$$\mathcal{K}[p_{\mathscr{A}}] = p_{\mathscr{A}}\Big(\phi'(\eta,\phi;0)\Big)\exp(-G_{L,0} - G_{L,\eta}) \tag{5.11}$$

$$\begin{aligned}\mathcal{T}[\tilde{\mathscr{A}} + S_{\mathscr{A}}] = &\int_0^{L} \frac{(\tilde{\mathscr{A}} + S)\Big(\eta',\phi'(\eta,\phi;\eta')\Big)}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)} \exp(-G_{L,\eta'} - G_{L,\eta})\,\mathrm{d}\eta' \\ &+ \int_{\eta}^{L} \frac{(\tilde{\mathscr{A}} + S)\Big(\eta',\mathscr{R}[\phi'(\eta,\phi;\eta')]\Big)}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)} \exp(-G_{\eta',\eta})\,\mathrm{d}\eta'.\end{aligned} \tag{5.12}$$

Region III:
For $\sin\phi < 0$ and $|E(\eta,\phi)| \geqq \mathrm{e}^{-V(L)}$,

$$\mathcal{K}[p_{\mathscr{A}}] = p_{\mathscr{A}}\Big(\phi'(\eta,\phi;0)\Big)\exp(-G_{\eta^+,0} - G_{\eta^+,\eta}) \tag{5.13}$$

$$\begin{aligned}\mathcal{T}[\tilde{\mathscr{A}} + S_{\mathscr{A}}] = &\int_0^{\eta^+} \frac{(\tilde{\mathscr{A}} + S_{\mathscr{A}})\Big(\eta',\phi'(\eta,\phi;\eta')\Big)}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)} \exp(-G_{\eta^+,\eta'} - G_{\eta^+,\eta})\,\mathrm{d}\eta' \\ &+ \int_{\eta}^{\eta^+} \frac{(\tilde{\mathscr{A}} + S_{\mathscr{A}})\Big(\eta',\mathscr{R}[\phi'(\eta,\phi;\eta')]\Big)}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)} \exp(-G_{\eta',\eta})\,\mathrm{d}\eta'.\end{aligned} \tag{5.14}$$

Then we need to estimate $\mathcal{K}[p_{\mathscr{A}}]$ and $\mathcal{T}[\tilde{\mathscr{A}} + S_{\mathscr{A}}]$ in each region. We assume $0 < \delta << 1$ and $0 < \delta_0 << 1$ are small quantities which will be determined later. Since we always assume that (η,ϕ) and (η',ϕ') are on the same characteristics, when there is no confusion, we simply write ϕ' or $\phi'(\eta')$ instead of $\phi'(\eta,\phi;\eta')$.

5.2. *Region I:* $\sin\phi > 0$

We consider

$$\mathcal{K}[p_{\mathscr{A}}] = p_{\mathscr{A}}\Big(\phi'(\eta,\phi;0)\Big)\exp(-G_{\eta,0}) \tag{5.15}$$

$$\mathcal{T}[\tilde{\mathscr{A}} + S_{\mathscr{A}}] = \int_0^{\eta} \frac{(\tilde{\mathscr{A}} + S_{\mathscr{A}})\Big(\eta',\phi'(\eta,\phi;\eta')\Big)}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\exp(-G_{\eta,\eta'})\,\mathrm{d}\eta'. \tag{5.16}$$

Based on Lemma 4.7, Lemma 4.8, we can directly obtain

$$\|\mathcal{K}[p_{\mathscr{A}}]\|_{L^\infty L^\infty} \leqq \|p_{\mathscr{A}}\|_{L^\infty_-}, \tag{5.17}$$

$$\|\mathcal{T}[S_{\mathscr{A}}]\|_{L^\infty L^\infty} \leqq \|S_{\mathscr{A}}\|_{L^\infty L^\infty}. \tag{5.18}$$

Hence, we only need to estimate

$$I = \mathcal{T}[\tilde{\mathscr{A}}] = \int_0^{\eta} \frac{\tilde{\mathscr{A}}\Big(\eta',\phi'(\eta,\phi;\eta')\Big)}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\exp(-G_{\eta,\eta'})\,\mathrm{d}\eta'. \tag{5.19}$$

We divide it into several steps as follows:
Step 0: Preliminaries.
We have

$$E(\eta',\phi') = \frac{R_\kappa - \epsilon\eta'}{R_\kappa}\cos\phi'. \tag{5.20}$$

We can directly obtain

$$\begin{aligned}\zeta(\eta',\phi') &= \frac{1}{R_\kappa}\sqrt{R_\kappa^2 - \Big((R_\kappa - \epsilon\eta')\cos\phi'\Big)^2}\\ &= \frac{1}{R_\kappa}\sqrt{R_\kappa^2 - (R_\kappa - \epsilon\eta')^2 + (R_\kappa - \epsilon\eta')^2\sin^2\phi'},\\ &\leqq \frac{1}{R_\kappa}\Big(\sqrt{R_\kappa^2 - (R_\kappa - \epsilon\eta')^2} + \sqrt{(R_\kappa - \epsilon\eta')^2\sin^2\phi'}\Big)\\ &\leqq C\Big(\sqrt{\epsilon\eta'} + \sin\phi'\Big),\end{aligned} \tag{5.21}$$

and

$$\zeta(\eta',\phi') \geqq \frac{1}{R_\kappa}\sqrt{R_\kappa^2 - (R_\kappa - \epsilon\eta')^2} \geqq C\sqrt{\epsilon\eta'}. \tag{5.22}$$

Also, we know for $0 \leqq \eta' \leqq \eta$ that,

$$\sin\phi' = \sqrt{1 - \cos^2\phi'} = \sqrt{1 - \left(\frac{R_\kappa - \epsilon\eta}{R_\kappa - \epsilon\eta'}\right)^2\cos^2\phi} \tag{5.23}$$

$$= \frac{\sqrt{(R_\kappa - \epsilon\eta')^2 \sin^2\phi + (2R_\kappa - \epsilon\eta - \epsilon\eta')(\epsilon\eta - \epsilon\eta')\cos^2\phi}}{R_\kappa - \epsilon\eta'}. \quad (5.24)$$

Since

$$0 \leqq (2R_\kappa - \epsilon\eta - \epsilon\eta')(\epsilon\eta - \epsilon\eta')\cos^2\phi \leqq 2R_\kappa\epsilon(\eta - \eta'), \quad (5.25)$$

we have

$$\sin\phi \leqq \sin\phi' \leqq 2\sqrt{\sin^2\phi + \epsilon(\eta - \eta')}, \quad (5.26)$$

which means

$$\frac{1}{2\sqrt{\sin^2\phi + \epsilon(\eta - \eta')}} \leqq \frac{1}{\sin\phi'} \leqq \frac{1}{\sin\phi}. \quad (5.27)$$

Therefore,

$$\begin{aligned} -\int_{\eta'}^{\eta} \frac{1}{\sin\phi'(y)}\,\mathrm{d}y &\leqq -\int_{\eta'}^{\eta} \frac{1}{2\sqrt{\sin^2\phi + \epsilon(\eta - y)}}\,\mathrm{d}y \\ &= \frac{1}{\epsilon}\left(\sin\phi - \sqrt{\sin^2\phi + \epsilon(\eta - \eta')}\right) \\ &= -\frac{\eta - \eta'}{\sin\phi + \sqrt{\sin^2\phi + \epsilon(\eta - \eta')}} \\ &\leqq -\frac{\eta - \eta'}{2\sqrt{\sin^2\phi + \epsilon(\eta - \eta')}}. \end{aligned} \quad (5.28)$$

Define a cut-off function $\chi \in C^\infty[-\pi, \pi]$ satisfying

$$\chi(\phi) = \begin{cases} 1 \text{ for } & |\sin\phi| \leqq \delta, \\ 0 \text{ for } & |\sin\phi| \geqq 2\delta, \end{cases} \quad (5.29)$$

In the following, we will divide the estimate of I into several cases based on the value of $\sin\phi$, $|\cos\phi|$, $\sin\phi'$, $\epsilon\eta'$ and $\epsilon(\eta - \eta')$. Let $\mathbf{1}$ denote the indicator function. We write

$$\begin{aligned} I = &\int_0^\eta \mathbf{1}_{\{\sin\phi \geqq \delta_0\}} \mathbf{1}_{\{|\cos\phi| \geqq \delta_0\}} + \int_0^\eta \mathbf{1}_{\{0 \leqq \sin\phi \leqq \delta_0\}} \mathbf{1}_{\{\chi(\phi_*) < 1\}} \\ &+ \int_0^\eta \mathbf{1}_{\{0 \leqq \sin\phi \leqq \delta_0\}} \mathbf{1}_{\{\chi(\phi_*) = 1\}} \mathbf{1}_{\{\sqrt{\epsilon\eta'} \geqq \sin\phi'\}} \\ &+ \int_0^\eta \mathbf{1}_{\{0 \leqq \sin\phi \leqq \delta_0\}} \mathbf{1}_{\{\chi(\phi_*) = 1\}} \mathbf{1}_{\{\sqrt{\epsilon\eta'} \leqq \sin\phi'\}} \mathbf{1}_{\{\sin^2\phi \leqq \epsilon(\eta - \eta')\}} \\ &+ \int_0^\eta \mathbf{1}_{\{0 \leqq \sin\phi \leqq \delta_0\}} \mathbf{1}_{\{\chi(\phi_*) = 1\}} \mathbf{1}_{\{\sqrt{\epsilon\eta'} \leqq \sin\phi'\}} \mathbf{1}_{\{\sin^2\phi \geqq \epsilon(\eta - \eta')\}} \\ &+ \int_0^\eta \mathbf{1}_{\{|\cos\phi| \leqq \delta_0\}} \\ = &\, I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \end{aligned} \quad (5.30)$$

Step 1: Estimate of I_1 for $\sin\phi \geqq \delta_0$ and $|\cos\phi| \geqq \delta_0$.
For $\sin\phi \geqq \delta_0$ and $|\cos\phi| \geqq \delta_0$, we do not need the mild formulation of $\mathscr{A}$. Instead, we directly estimate

$$|\mathscr{A}| \leqq \left|\frac{\partial \mathscr{V}}{\partial \eta}\right|. \tag{5.31}$$

We will estimate I_1 based on the characteristics of $\mathscr{V}$ itself instead of the derivative. Here, we will use two formulations of the equation (5.3):

- Formulation I: η is the principal variable, $\phi = \phi(\eta)$, and the equation can be rewritten as

$$\sin\phi \frac{\mathrm{d}\mathscr{V}}{\mathrm{d}\eta} + \mathscr{V} = \bar{\mathscr{V}} + S. \tag{5.32}$$

- Formulation II: ϕ is the principal variable, $\eta = \eta(\phi)$ and the equation can be rewritten as

$$F(\eta)\cos\phi \frac{\mathrm{d}\mathscr{V}}{\mathrm{d}\phi} + \mathscr{V} = \bar{\mathscr{V}} + S. \tag{5.33}$$

These two formulations are equivalent and can be applied to different regions of the domain.

We may decompose $\mathscr{V} = \mathscr{V}_1 + \mathscr{V}_2$ where $\mathscr{V}_1$ satisfies

$$\begin{cases} \sin\phi \dfrac{\partial \mathscr{V}_1}{\partial \eta} + F(\eta)\cos\phi \dfrac{\partial \mathscr{V}_1}{\partial \phi} + \mathscr{V}_1 = \bar{\mathscr{V}}, \\ \mathscr{V}_1(0,\phi) = p(\phi) \ \text{ for } \ \sin\phi > 0, \\ \mathscr{V}_1(L,\phi) = \mathscr{V}_1(L,\mathscr{R}[\phi]), \end{cases} \tag{5.34}$$

and $\mathscr{V}_2$ satisfies

$$\begin{cases} \sin\phi \dfrac{\partial \mathscr{V}_2}{\partial \eta} + F(\eta)\cos\phi \dfrac{\partial \mathscr{V}_2}{\partial \phi} + \mathscr{V}_2 = S, \\ \mathscr{V}_2(0,\phi) = 0 \ \text{ for } \ \sin\phi > 0, \\ \mathscr{V}_2(L,\phi) = \mathscr{V}_2(L,\mathscr{R}[\phi]). \end{cases} \tag{5.35}$$

Assume $\mathscr{V}$ is well-defined in L^∞. Then by tracking along the characteristics, we can easily see that $\mathscr{V}_1$ and $\mathscr{V}_2$ are well-defined.

Using Formulation I, we rewrite the equation (5.34) along the characteristics as

$$\mathscr{V}_1(\eta,\phi) = \exp\left(-G_{\eta,0}\right)\left(p\Big(\phi'(0)\Big) + \int_0^\eta \frac{\bar{\mathscr{V}}(\eta')}{\sin\Big(\phi'(\eta')\Big)} \exp\left(G_{\eta',0}\right) \mathrm{d}\eta'\right) \tag{5.36}$$

where (η',ϕ'), $(0,\phi'(0))$ and (η,ϕ) are on the same characteristic with $\sin\phi' \geqq 0$, and

$$G_{t,s} = \int_s^t \frac{1}{\sin\Big(\phi'(\xi)\Big)} \mathrm{d}\xi. \tag{5.37}$$

Taking the η derivative on both sides of (5.36), we have

$$\frac{\partial \mathscr{V}_1}{\partial \eta} = X_1 + X_2 + X_3 + X_4 + X_5, \tag{5.38}$$

where

$$X_1 = -\exp\left(-G_{\eta,0}\right) \frac{\partial G_{\eta,0}}{\partial \eta}\left(p\Big(\phi'(0)\Big) + \int_0^\eta \frac{\bar{\mathscr{V}}(\eta')}{\sin\Big(\phi'(\eta')\Big)} \exp\left(G_{\eta',0}\right) \mathrm{d}\eta'\right), \tag{5.39}$$

$$X_2 = \exp\left(-G_{\eta,0}\right) \frac{\partial p\Big(\phi'(0)\Big)}{\partial \eta}, \tag{5.40}$$

$$X_3 = \frac{\bar{\mathscr{V}}(\eta)}{\sin\phi}, \tag{5.41}$$

$$X_4 = -\exp\left(-G_{\eta,0}\right) \int_0^\eta \bar{\mathscr{V}}(\eta') \exp\left(G_{\eta',0}\right) \frac{\cos\Big(\phi'(\eta')\Big)}{\sin^2\Big(\phi'(\eta')\Big)} \frac{\partial \phi'(\eta')}{\partial \eta} \mathrm{d}\eta', \tag{5.42}$$

$$X_5 = \exp\left(-G_{\eta,0}\right) \int_0^\eta \frac{\bar{\mathscr{V}}(\eta')}{\sin\Big(\phi'(\eta')\Big)} \exp\left(G_{\eta',0}\right) \frac{\partial G_{\eta',0}}{\partial \eta} \mathrm{d}\eta'. \tag{5.43}$$

Then we need to estimate each term. This procedure is standard, so we omit the details. Note the fact that for $0 \leqq \eta' \leqq \eta$, we have $\sin\phi' \geqq \sin\phi \geqq \delta_0$ and

$$\int_0^\eta \frac{1}{\sin\Big(\phi'(\eta')\Big)} \exp\left(-G_{\eta,\eta'}\right) \mathrm{d}\eta' \leqq \int_0^\infty \mathrm{e}^{-y} \,\mathrm{d}y = 1, \tag{5.44}$$

with the substitution $y = G_{\eta,\eta'}$. The estimates can be listed as follows:

$$|X_1| \leqq \frac{C}{\delta_0}\|\mathscr{V}\|_{L^\infty L^\infty}, \quad |X_2| \leqq \frac{C}{\delta_0}\left\|\frac{\partial p}{\partial \phi}\right\|_{L^\infty_-}, \quad |X_3| \leqq \frac{C}{\delta_0}\|\mathscr{V}\|_{L^\infty L^\infty}, \tag{5.45}$$

$$|X_4| \leqq \frac{C}{\delta_0}\|\mathscr{V}\|_{L^\infty L^\infty}, \quad |X_5| \leqq \frac{C}{\delta_0}\|\mathscr{V}\|_{L^\infty L^\infty}. \tag{5.46}$$

In total, we have

$$\left|\frac{\partial \mathscr{V}_1}{\partial \eta}\right| \leqq \frac{C}{\delta_0}\left(\left\|\frac{\partial p}{\partial \phi}\right\|_{L^\infty_-} + \|\mathscr{V}\|_{L^\infty L^\infty}\right). \tag{5.47}$$

Using Formulation II, we rewrite the equation (5.35) along the characteristics as

$$\mathscr{V}_2(\eta, \phi) = \exp\left(-H_{\phi,\phi_*}\right) \int_{\phi_*}^\phi \frac{S\Big(\eta'(\phi'), \phi'\Big)}{F\Big(\eta'(\phi')\Big)\cos\phi'} \exp\left(H_{\phi',\phi_*}\right) \mathrm{d}\phi', \tag{5.48}$$

where (η', ϕ'), $(0, \phi_*)$ and (η, ϕ) are on the same characteristic with $\sin\phi' \geqq 0$, and

$$H_{t,s} = \int_s^t \frac{1}{F\Big(\eta'(\xi)\Big)\cos\xi'}\,\mathrm{d}\xi. \tag{5.49}$$

Taking η derivative on both sides of (5.48), we have

$$\frac{\partial \mathscr{V}_2}{\partial\eta} = Y_1 + Y_2 + Y_3 + Y_4 + Y_5, \tag{5.50}$$

where

$$Y_1 = -\exp\left(-H_{\phi,\phi_*}\right)\frac{\partial H_{\phi,\phi_*}}{\partial\eta}\int_{\phi_*}^{\phi}\frac{S\Big(\eta'(\phi'),\phi'\Big)}{F\Big(\eta'(\phi')\Big)\cos\phi'}\exp\left(H_{\phi',\phi_*}\right)\,\mathrm{d}\phi', \tag{5.51}$$

$$Y_2 = \frac{S(0,\phi_*)}{F(0)\cos\phi_*}\frac{\partial\phi_*}{\partial\eta}, \tag{5.52}$$

$$Y_3 = -\exp\left(-H_{\phi,\phi_*}\right)\int_{\phi_*}^{\phi} S\Big(\eta'(\phi'),\phi'\Big)$$

$$\frac{1}{F^2\Big(\eta'(\phi')\Big)\cos\phi'}\frac{\partial F\Big(\eta'(\phi')\Big)}{\partial\eta}\exp\left(H_{\phi',\phi_*}\right)\,\mathrm{d}\phi', \tag{5.53}$$

$$Y_4 = \exp\left(-H_{\phi,\phi_*}\right)\int_{\phi_*}^{\phi}\frac{S\Big(\eta'(\phi'),\phi'\Big)}{F\Big(\eta'(\phi')\Big)\cos\phi'}\exp\left(H_{\phi',\phi_*}\right)\frac{\partial H_{\phi',\phi_*}}{\partial\eta}\,\mathrm{d}\phi', \tag{5.54}$$

$$Y_5 = \exp\left(-H_{\phi,\phi_*}\right)\int_{\phi_*}^{\phi}\frac{\partial_{\eta'} S\Big(\eta'(\phi'),\phi'\Big)}{F\Big(\eta'(\phi')\Big)\cos\phi'}\frac{\partial\eta'(\phi')}{\partial\eta}\exp\left(H_{\phi',\phi_*}\right)\,\mathrm{d}\phi'. \tag{5.55}$$

Then we just need to estimate each term. Along the characteristics, we know

$$\mathrm{e}^{-V(\eta')}\cos\phi' = \mathrm{e}^{-V(\eta)}\cos\phi, \tag{5.56}$$

which implies

$$\cos\phi' = \mathrm{e}^{V(\eta')-V(\eta)}\cos\phi \geqq \mathrm{e}^{V(0)-V(L)}\cos\phi \geqq \mathrm{e}^{V(0)-V(L)}\delta_0. \tag{5.57}$$

We can further deduce that

$$\cos\phi' \geqq \left(1 - \frac{\epsilon^{\frac{1}{2}}}{R_\kappa}\right)\delta_0 \geqq \frac{\delta_0}{2}, \tag{5.58}$$

when ϵ is sufficiently small. Also, we have

$$\int_{\phi_*}^{\phi}\frac{1}{F\Big(\eta'(\phi')\Big)\cos\phi'}\exp\left(H_{\phi,\phi'}\right)\,\mathrm{d}\phi' \leqq \int_0^\infty \mathrm{e}^{-y}\,\mathrm{d}y = 1, \tag{5.59}$$

with the substitution $y = H_{\phi,\phi'}$. Similarly to X_i estimates, we may directly obtain

$$|Y_1| \leqq \frac{C}{\delta_0}\|S\|_{L^\infty L^\infty}, \quad |Y_2| \leqq \frac{C}{\delta_0}\|S\|_{L^\infty L^\infty}, \quad |Y_3| \leqq \frac{C}{\delta_0}\|S\|_{L^\infty L^\infty}, \tag{5.60}$$

$$|Y_4| \leqq \frac{C}{\delta_0}\|S\|_{L^\infty L^\infty}, \quad |Y_5| \leqq \frac{C}{\delta_0}\left\|\frac{\partial S}{\partial \eta}\right\|_{L^\infty L^\infty}. \tag{5.61}$$

In total, we have

$$\left|\frac{\partial \mathscr{V}_2}{\partial \eta}\right| \leqq \frac{C}{\delta_0}\left(\|S\|_{L^\infty L^\infty} + \left\|\frac{\partial S}{\partial \eta}\right\|_{L^\infty L^\infty}\right). \tag{5.62}$$

Combining (5.47) and (5.62), we have

$$\begin{aligned}\left|\frac{\partial \mathscr{V}}{\partial \eta}\right| &\leqq \left|\frac{\partial \mathscr{V}_1}{\partial \eta}\right| + \left|\frac{\partial \mathscr{V}_2}{\partial \eta}\right| \\ &\leqq \frac{C}{\delta_0}\left(\left\|\frac{\partial p}{\partial \phi}\right\|_{L^\infty_-} + \|S\|_{L^\infty L^\infty} + \left\|\frac{\partial S}{\partial \eta}\right\|_{L^\infty L^\infty} + \|\mathscr{V}\|_{L^\infty L^\infty}\right).\end{aligned} \tag{5.63}$$

Hence, noting that $\zeta \geqq \sin\phi \geqq \delta_0$, we know

$$I_1 \leqq \frac{C}{\delta_0^2}\left(\left\|\zeta\frac{\partial p}{\partial \phi}\right\|_{L^\infty_-} + \|S\|_{L^\infty L^\infty} + \left\|\zeta\frac{\partial S}{\partial \eta}\right\|_{L^\infty L^\infty} + \|\mathscr{V}\|_{L^\infty L^\infty}\right). \tag{5.64}$$

Step 2: Estimate of I_2 for $0 \leqq \sin\phi \leqq \delta_0$ and $\chi(\phi_*) < 1$.
We have

$$\begin{aligned}I_2 &= \frac{1}{2\pi}\int_0^\eta \left(\int_{-\pi}^{\pi} \frac{\zeta(\eta',\phi')}{\zeta(\eta',\phi_*)}\Big(1-\chi(\phi_*)\Big)\mathscr{A}(\eta',\phi_*)\,\mathrm{d}\phi_*\right) \\ &\quad \frac{1}{\sin\phi'}\exp(-G_{\eta,\eta'})\,\mathrm{d}\eta' \\ &= \frac{1}{2\pi}\int_0^\eta \left(\int_{-\pi}^{\pi} \zeta(\eta',\phi')\Big(1-\chi(\phi_*)\Big)\frac{\partial \mathscr{V}(\eta',\phi_*)}{\partial \eta'}\,\mathrm{d}\phi_*\right) \\ &\quad \frac{1}{\sin\phi'}\exp(-G_{\eta,\eta'})\,\mathrm{d}\eta'.\end{aligned} \tag{5.65}$$

Based on the ϵ-Milne problem of $\mathscr{V}$ as

$$\begin{aligned}&\sin\phi_*\frac{\partial \mathscr{V}(\eta',\phi_*)}{\partial \eta'} + F(\eta')\cos\phi_*\frac{\partial \mathscr{V}(\eta',\phi_*)}{\partial \phi_*} + \mathscr{V}(\eta',\phi_*) - \bar{\mathscr{V}}(\eta') \\ &= S(\eta',\phi_*),\end{aligned} \tag{5.66}$$

we have

$$\begin{aligned}\frac{\partial \mathscr{V}(\eta',\phi_*)}{\partial \eta'} = -\frac{1}{\sin\phi_*}\bigg(&F(\eta')\cos\phi_*\frac{\partial \mathscr{V}(\eta',\phi_*)}{\partial \phi_*} + \mathscr{V}(\eta',\phi_*) \\ &-\bar{\mathscr{V}}(\eta') - S(\eta',\phi_*)\bigg)\end{aligned} \tag{5.67}$$

Hence, inserting (5.67) into (5.65), we have the term in the large paranthesis

$$\begin{aligned}\tilde{\mathscr{A}} :&= \int_{-\pi}^{\pi} \zeta(\eta', \phi')\Big(1 - \chi(\phi_*)v\Big)\frac{\partial \mathscr{V}(\eta', \phi_*)}{\partial \eta'}\,\mathrm{d}\phi_* \\ &= -\int_{-\pi}^{\pi} \zeta(\eta', \phi')\Big(1 - \chi(\phi_*)\Big)\frac{1}{\sin\phi_*}\Big(\mathscr{V}(\eta', \phi_*) - \bar{\mathscr{V}}(\eta') - S(\eta', \phi_*)\Big)\,\mathrm{d}\phi_* \\ &\quad - \int_{-\pi}^{\pi} \zeta(\eta', \phi')\Big(1 - \chi(\phi_*)\Big)\frac{1}{\sin\phi_*}F(\eta')\cos\phi_*\frac{\partial \mathscr{V}(\eta', \phi_*)}{\partial \phi_*}\,\mathrm{d}\phi_* \\ :&= \tilde{\mathscr{A}}_1 + \tilde{\mathscr{A}}_2. \end{aligned} \tag{5.68}$$

Using the definition of χ, we may directly obtain

$$\begin{aligned}\left|\tilde{\mathscr{A}}_1\right| &= \left|\int_{-\pi}^{\pi} \zeta(\eta', \phi')\Big(1 - \chi(\phi_*)\Big)\frac{1}{\sin\phi_*}\Big(\mathscr{V}(\eta', \phi_*) - \bar{\mathscr{V}}(\eta') - S(\eta', \phi_*)\Big)\,\mathrm{d}\phi_*\right| \\ &\leqq \frac{C}{\delta}\left|\int_{-\pi}^{\pi}\Big(\mathscr{V}(\eta', \phi_*) - \bar{\mathscr{V}}(\eta') - S(\eta', \phi_*)\Big)\,\mathrm{d}\phi_*\right| \\ &\leqq \frac{C}{\delta}\Big(\|\mathscr{V}\|_{L^\infty L^\infty} + \|S\|_{L^\infty L^\infty}\Big). \end{aligned} \tag{5.69}$$

On the other hand, an integration by parts yields

$$\tilde{\mathscr{A}}_2 = \int_{-\pi}^{\pi} \frac{\partial}{\partial \phi_*}\bigg(\zeta(\eta', \phi')\Big(1 - \chi(\phi_*)\Big)\frac{1}{\sin\phi_*}F(\eta')\cos\phi_*\bigg)\mathscr{V}(\eta', \phi_*)\,\mathrm{d}\phi_*, \tag{5.70}$$

which further implies that

$$\left|\tilde{\mathscr{A}}_2\right| \leqq \frac{C\epsilon}{\delta^2}\|\mathscr{V}\|_{L^\infty L^\infty}. \tag{5.71}$$

Since we can use substitution to show

$$\int_0^{\eta} \frac{1}{\sin\phi'}\exp(-G_{\eta,\eta'})\,\mathrm{d}\eta' \leqq 1, \tag{5.72}$$

we have

$$\begin{aligned}|I_2| &\leqq C\left(\frac{1}{\delta} + \frac{\epsilon}{\delta^2}\right)\Big(\|\mathscr{V}\|_{L^\infty L^\infty} + \|S\|_{L^\infty L^\infty}\Big)\int_0^{\eta}\frac{1}{\sin\phi'}\exp(-G_{\eta,\eta'})\,\mathrm{d}\eta' \\ &\leqq C\left(\frac{1}{\delta} + \frac{\epsilon}{\delta^2}\right)\Big(\|\mathscr{V}\|_{L^\infty L^\infty} + \|S\|_{L^\infty L^\infty}\Big). \end{aligned} \tag{5.73}$$

Step 3: Estimate of I_3 for $0 \leqq \sin\phi \leqq \delta_0$, $\chi(\phi_*) = 1$ and $\sqrt{\epsilon\eta'} \geqq \sin\phi'$.
Based on (5.21), this implies

$$\zeta(\eta', \phi') \leqq C\sqrt{\epsilon\eta'}.$$

Then combining this with (5.22), we can directly obtain

$$\int_{-\pi}^{\pi} \frac{\zeta(\eta', \phi')}{\zeta(\eta', \phi_*)} \chi(\phi_*) \mathscr{A}(\eta', \phi_*) \, d\phi_* \\ \leqq C \int_{-\delta}^{\delta} \mathscr{A}(\eta', \phi_*) \, d\phi_* \leqq C\delta \|\mathscr{A}\|_{L^\infty L^\infty}. \tag{5.74}$$

Hence, we have

$$|I_3| \leqq C\delta \|\mathscr{A}\|_{L^\infty L^\infty} \int_0^\eta \frac{1}{\sin\phi'} \exp(-G_{\eta,\eta'}) \, d\eta' \leqq C\delta \|\mathscr{A}\|_{L^\infty L^\infty}. \tag{5.75}$$

Step 4: Estimate of I_4 for $0 \leqq \sin\phi \leqq \delta_0$, $\chi(\phi_*) = 1$, $\sqrt{\epsilon\eta'} \leqq \sin\phi'$ and $\sin^2\phi \leqq \epsilon(\eta - \eta')$.

Based on (5.21), this implies

$$\zeta(\eta', \phi') \leqq C \sin\phi'. \tag{5.76}$$

Based on (5.28), we have

$$-G_{\eta,\eta'} = -\int_{\eta'}^{\eta} \frac{1}{\sin\phi'(y)} \, dy \leqq -\frac{\eta - \eta'}{2\sqrt{\epsilon(\eta - \eta')}} \leqq -C\sqrt{\frac{\eta - \eta'}{\epsilon}}. \tag{5.77}$$

Hence, considering $\zeta(\eta', \phi_*) \geqq \sqrt{\epsilon\eta'}$, we know

$$\begin{aligned} |I_4| &\leqq C \int_0^\eta \left(\int_{-\pi}^{\pi} \frac{\zeta(\eta', \phi')}{\zeta(\eta', \phi_*)} \chi(\phi_*) \mathscr{A}(\eta', \phi_*) \, d\phi_* \right) \frac{1}{\sin\phi'} \exp(-G_{\eta,\eta'}) \, d\eta' \\ &\leqq C \int_0^\eta \left(\int_{-\delta}^{\delta} \frac{1}{\zeta(\eta', \phi_*)} \mathscr{A}(\eta', \phi_*) \, d\phi_* \right) \frac{\zeta(\eta', \phi')}{\sin\phi'} \exp(-G_{\eta,\eta'}) \, d\eta' \\ &\leqq C \|\mathscr{A}\|_{L^\infty L^\infty} \int_0^\eta \left(\int_{-\delta}^{\delta} \frac{1}{\zeta(\eta', \phi_*)} \, d\phi_* \right) \frac{\sin\phi'}{\sin\phi'} \exp(-G_{\eta,\eta'}) \, d\eta' \\ &\leqq C\delta \|\mathscr{A}\|_{L^\infty L^\infty} \int_0^\eta \frac{1}{\sqrt{\epsilon\eta'}} \exp(-G_{\eta,\eta'}) \, d\eta' \\ &\leqq C\delta \|\mathscr{A}\|_{L^\infty L^\infty} \int_0^\eta \frac{1}{\sqrt{\epsilon\eta'}} \exp\left(-C\sqrt{\frac{\eta - \eta'}{\epsilon}} \right) d\eta'. \end{aligned} \tag{5.78}$$

Define $z = \dfrac{\eta'}{\epsilon}$, which implies $d\eta' = \epsilon \, dz$. Substituting this into above integral, we have

$$\begin{aligned} |I_4| &\leqq C\delta \|\mathscr{A}\|_{L^\infty L^\infty} \int_0^{\frac{\eta}{\epsilon}} \frac{1}{\sqrt{z}} \exp\left(-C\sqrt{\frac{\eta}{\epsilon} - z} \right) dz \\ &= C\delta \|\mathscr{A}\|_{L^\infty L^\infty} \left(\int_0^1 \frac{1}{\sqrt{z}} \exp\left(-C\sqrt{\frac{\eta}{\epsilon} - z} \right) dz \right. \\ &\quad \left. + \int_1^{\frac{\eta}{\epsilon}} \frac{1}{\sqrt{z}} \exp\left(-C\sqrt{\frac{\eta}{\epsilon} - z} \right) dz \right). \end{aligned} \tag{5.79}$$

We can estimate these two terms separately:

$$\int_0^1 \frac{1}{\sqrt{z}} \exp\left(- C\sqrt{\frac{\eta}{\epsilon} - z}\right) \mathrm{d}z \leqq \int_0^1 \frac{1}{\sqrt{z}} \mathrm{d}z = 2. \tag{5.80}$$

$$\begin{aligned} \int_1^{\frac{\eta}{\epsilon}} \frac{1}{\sqrt{z}} \exp\left(- C\sqrt{\frac{\eta}{\epsilon} - z}\right) \mathrm{d}z &\leqq \int_1^{\frac{\eta}{\epsilon}} \exp\left(- C\sqrt{\frac{\eta}{\epsilon} - z}\right) \mathrm{d}z \\ &\overset{t^2=\frac{\eta}{\epsilon}-z}{\leqq} 2\int_0^\infty t\mathrm{e}^{-Ct}\, \mathrm{d}t < \infty. \end{aligned} \tag{5.81}$$

Hence, we know

$$|I_4| \leqq C\delta \|\mathscr{A}\|_{L^\infty L^\infty}. \tag{5.82}$$

Step 5: Estimate of I_5 for $0 \leqq \sin\phi \leqq \delta_0$, $\chi(\phi_*) = 1$, $\sqrt{\epsilon\eta'} \leqq \sin\phi'$ and $\sin^2\phi \geqq \epsilon(\eta - \eta')$.

Based on (5.21), this implies

$$\zeta(\eta', \phi') \leqq C \sin\phi'. \tag{5.83}$$

Based on (5.28), we have

$$- G_{\eta,\eta'} = - \int_{\eta'}^{\eta} \frac{1}{\sin\phi'(y)} \mathrm{d}y \leqq -\frac{C(\eta - \eta')}{\sin\phi}. \tag{5.84}$$

Hence, we have

$$|I_5| \leqq C\|\mathscr{A}\|_{L^\infty L^\infty} \int_0^\eta \left(\int_{-\delta}^{\delta} \frac{1}{\zeta(\eta', \phi_*)} \mathrm{d}\phi_* \right) \exp\left(-\frac{C(\eta - \eta')}{\sin\phi}\right) \mathrm{d}\eta'. \tag{5.85}$$

Here, we use a different way to estimate the inner integral. We use substitution to find

$$\begin{aligned} \int_{-\delta}^{\delta} \frac{1}{\zeta(\eta', \phi_*)} \mathrm{d}\phi_* &= \int_{-\delta}^{\delta} \frac{1}{\sqrt{R_\kappa^2 - (R_\kappa - \epsilon\eta')^2 \cos\phi_*^2}} \mathrm{d}\phi_* \\ &\overset{\sin\phi_* \; small}{\leqq} C\int_{-\delta}^{\delta} \frac{\cos\phi_*}{\sqrt{R_\kappa^2 - (R_\kappa - \epsilon\eta')^2 \cos\phi_*^2}} \mathrm{d}\phi_* \\ &= C\int_{-\delta}^{\delta} \frac{\cos\phi_*}{\sqrt{R_\kappa^2 - (R_\kappa - \epsilon\eta')^2 + (R_\kappa - \epsilon\eta')^2 \sin\phi_*^2}} \mathrm{d}\phi_* \\ &\overset{y=\sin\phi_*}{=} C\int_{-\delta}^{\delta} \frac{1}{\sqrt{R_\kappa^2 - (R_\kappa - \epsilon\eta')^2 + (R_\kappa - \epsilon\eta')^2 y^2}} \mathrm{d}y. \end{aligned} \tag{5.86}$$

Define

$$\begin{aligned} &p = \sqrt{R_\kappa^2 - (R_\kappa - \epsilon\eta')^2} = \sqrt{2R_\kappa\epsilon\eta' - \epsilon^2\eta'^2} \leqq C\sqrt{\epsilon\eta'}, \quad q = R_\kappa - \epsilon\eta' \geqq C, \\ &r = \frac{p}{q} \leqq C\sqrt{\epsilon\eta'}. \end{aligned} \tag{5.87}$$

Then we have

$$\begin{aligned}\int_{-\delta}^{\delta}\frac{1}{\zeta(\eta',\phi_*)}\,\mathrm{d}\phi_* &\leqq C\int_{-\delta}^{\delta}\frac{1}{\sqrt{p^2+q^2y^2}}\,\mathrm{d}y\\ &\leqq C\int_{-2}^{2}\frac{1}{\sqrt{p^2+q^2y^2}}\,\mathrm{d}y \leqq C\int_{-2}^{2}\frac{1}{\sqrt{r^2+y^2}}\,\mathrm{d}y\\ &\leqq C\int_{0}^{2}\frac{1}{\sqrt{r^2+y^2}}\,\mathrm{d}y = \bigg(\ln(y+\sqrt{r^2+y^2})-\ln(r)\bigg)\bigg|_0^2\\ &\leqq C\bigg(\ln(2+\sqrt{r^2+4})-\ln(r)\bigg) \leqq C\bigg(1+\ln(r)\bigg)\\ &\leqq C\bigg(1+|\ln(\epsilon)|+\big|\ln(\eta')\big|\bigg). \end{aligned} \tag{5.88}$$

Hence, we know

$$|I_5| \leqq C\|\mathscr{A}\|_{L^\infty L^\infty}\int_0^{\eta}\bigg(1+|\ln(\epsilon)|+\big|\ln(\eta')\big|\bigg) \exp\bigg(-\frac{C(\eta-\eta')}{\sin\phi}\bigg)\,\mathrm{d}\eta'. \tag{5.89}$$

We may directly compute

$$\left|\int_0^{\eta}\bigg(1+|\ln(\epsilon)|\bigg)\exp\bigg(-\frac{C(\eta-\eta')}{\sin\phi}\bigg)\,\mathrm{d}\eta'\right| \leqq C\sin\phi(1+|\ln(\epsilon)|). \tag{5.90}$$

Hence, we only need to estimate

$$\left|\int_0^{\eta}\big|\ln(\eta')\big|\exp\bigg(-\frac{C(\eta-\eta')}{\sin\phi}\bigg)\,\mathrm{d}\eta'\right|. \tag{5.91}$$

If $\eta \leqq 2$, using Cauchy's inequality, we have

$$\begin{aligned}&\left|\int_0^{\eta}\big|\ln(\eta')\big|\exp\bigg(-\frac{C(\eta-\eta')}{\sin\phi}\bigg)\,\mathrm{d}\eta'\right|\\ &\leqq \bigg(\int_0^{\eta}\ln^2(\eta')\,\mathrm{d}\eta'\bigg)^{\frac12}\bigg(\int_0^{\eta}\exp\bigg(-\frac{2C(\eta-\eta')}{\sin\phi}\bigg)\,\mathrm{d}\eta'\bigg)^{\frac12}\\ &\leqq \bigg(\int_0^{2}\ln^2(\eta')\,\mathrm{d}\eta'\bigg)^{\frac12}\bigg(\int_0^{\eta}\exp\bigg(-\frac{2C(\eta-\eta')}{\sin\phi}\bigg)\,\mathrm{d}\eta'\bigg)^{\frac12}\\ &\leqq \sqrt{\sin\phi}. \end{aligned} \tag{5.92}$$

If $\eta \geqq 2$, we decompose and apply Cauchy's inequality to obtain

$$\begin{aligned}&\left|\int_0^{\eta}\big|\ln(\eta')\big|\exp\bigg(-\frac{C(\eta-\eta')}{\sin\phi}\bigg)\,\mathrm{d}\eta'\right|\\ &\leqq \left|\int_0^{2}\big|\ln(\eta')\big|\exp\bigg(-\frac{C(\eta-\eta')}{\sin\phi}\bigg)\,\mathrm{d}\eta'\right| + \left|\int_2^{\eta}\ln(\eta')\exp\bigg(-\frac{C(\eta-\eta')}{\sin\phi}\bigg)\,\mathrm{d}\eta'\right|\end{aligned}$$

$$
\begin{aligned}
&\leqq \left(\int_0^2 \ln^2(\eta')\,\mathrm{d}\eta'\right)^{\frac{1}{2}}\left(\int_0^2 \exp\left(-\frac{2C(\eta-\eta')}{\sin\phi}\right)\,\mathrm{d}\eta'\right)^{\frac{1}{2}}\\
&\quad + \ln(L)\left|\int_2^\eta \exp\left(-\frac{C(\eta-\eta')}{\sin\phi}\right)\,\mathrm{d}\eta'\right|\\
&\leqq C\bigg(\sqrt{\sin\phi}+|\ln(\epsilon)|\sin\phi\bigg) \leqq C\Big(1+|\ln(\epsilon)|\Big)\sqrt{\sin\phi}. \qquad (5.93)
\end{aligned}
$$

Hence, we have

$$
|I_5| \leqq C\Big(1+|\ln(\epsilon)|\Big)\sqrt{\delta_0}\|\mathscr{A}\|_{L^\infty L^\infty}. \qquad (5.94)
$$

Step 6: Estimate of I_6 for $|\cos\phi| < \delta_0$.
We have

$$
\begin{aligned}
I_6 &= \frac{1}{2\pi}\int_0^\eta\left(\int_{-\pi}^{\pi}\frac{\zeta(\eta',\phi')}{\zeta(\eta',\phi_*)}\mathscr{A}(\eta',\phi_*)\,\mathrm{d}\phi_*\right)\frac{1}{\sin\phi'}\exp(-G_{\eta,\eta'})\,\mathrm{d}\eta'\\
&= \frac{1}{2\pi}\int_0^\eta\left(\int_{-\pi}^{\pi}\zeta(\eta',\phi')\Big(1-\chi(\phi_*)\Big)\frac{\partial\mathscr{V}(\eta',\phi_*)}{\partial\eta'}\,\mathrm{d}\phi_*\right)\frac{1}{\sin\phi'}\exp(-G_{\eta,\eta'})\,\mathrm{d}\eta'\\
&\quad +\frac{1}{2\pi}\int_0^\eta\left(\int_{-\pi}^{\pi}\chi(\phi_*)\frac{\zeta(\eta',\phi')}{\zeta(\eta',\phi_*)}\mathscr{A}(\eta',\phi_*)\,\mathrm{d}\phi_*\right)\frac{1}{\sin\phi'}\exp(-G_{\eta,\eta'})\,\mathrm{d}\eta'. \qquad (5.95)
\end{aligned}
$$

The first term can be estimated as I_2:

$$
\begin{aligned}
&\frac{1}{2\pi}\int_0^\eta\left(\int_{-\pi}^{\pi}\zeta(\eta',\phi')\Big(1-\chi(\phi_*)\Big)\frac{\mathscr{V}(\eta',\phi_*)}{\partial\eta'}\,\mathrm{d}\phi_*\right)\frac{1}{\sin\phi'}\exp(-G_{\eta,\eta'})\,\mathrm{d}\eta'\\
&\quad\leqq C\left(\frac{1}{\delta}+\frac{\epsilon}{\delta^2}\right)\bigg(\|\mathscr{V}\|_{L^\infty L^\infty}+\|S\|_{L^\infty L^\infty}\bigg). \qquad (5.96)
\end{aligned}
$$

It is easy to check that $\sqrt{\epsilon\eta'} \leqq \sin\phi \leqq \sin\phi'$ and $\sin^2\phi \geqq \epsilon(\eta-\eta')$, so the second term can be estimated as I_5.

$$
\begin{aligned}
&\frac{1}{2\pi}\int_0^\eta\left(\int_{-\pi}^{\pi}\chi(\phi_*)\frac{\zeta(\eta',\phi')}{\zeta(\eta',\phi_*)}\mathscr{A}(\eta',\phi_*)\,\mathrm{d}\phi_*\right)\frac{1}{\sin\phi'}\exp(-G_{\eta,\eta'})\,\mathrm{d}\eta'\\
&\quad\leqq C\Big(1+|\ln(\epsilon)|\Big)\sqrt{\sin\phi}\sup_{|\sin\phi_*|\leqq\delta}|\mathscr{A}(\eta,\phi_*)|\\
&\quad\leqq C\Big(1+|\ln(\epsilon)|\Big)\sup_{|\sin\phi_*|\leqq\delta}|\mathscr{A}(\eta,\phi_*)|. \qquad (5.97)
\end{aligned}
$$

Note that now we lose the smallness since $\sin\phi \geqq \dfrac{1}{2}$, so we need a more detailed analysis. Actually, the value of $|\mathscr{A}|$ for $|\sin\phi| \leqq \delta$, is covered in I_2, I_3, I_4, I_5 and the following II_2, II_3, II_4, III. Therefore, in fact, we get the estimate

$$
\begin{aligned}
&\frac{1}{2\pi}\int_0^\eta\left(\int_{-\pi}^{\pi}\chi(\phi_*)\frac{\zeta(\eta',\phi')}{\zeta(\eta',\phi_*)}\mathscr{A}(\eta',\phi_*)\,\mathrm{d}\phi_*\right)\frac{1}{\sin\phi'}\exp(-G_{\eta,\eta'})\,\mathrm{d}\eta'\\
&\quad\leqq C\Big(1+|\ln(\epsilon)|\Big)\bigg(\|p_{\mathscr{A}}\|_{L^\infty_-}+\|S_{\mathscr{A}}\|_{L^\infty L^\infty}\bigg)
\end{aligned}
$$

$$+C\Big(1+|\ln(\epsilon)|\Big)\bigg(\frac{1}{\delta}+\frac{\epsilon}{\delta^2}\bigg)\bigg(\|\mathscr{V}\|_{L^\infty L^\infty}+\|S\|_{L^\infty L^\infty}\bigg)$$
$$+C\Big(1+|\ln(\epsilon)|\Big)\bigg(\delta+\Big(1+|\ln(\epsilon)|\Big)\sqrt{\delta_0}\bigg)\|\mathscr{A}\|_{L^\infty L^\infty}. \quad (5.98)$$

Therefore, we have

$$\begin{aligned}|I_6| \leqq\ & C\Big(1+|\ln(\epsilon)|\Big)\bigg(\|p_{\mathscr{A}}\|_{L^\infty_-}+\|S_{\mathscr{A}}\|_{L^\infty L^\infty}\bigg)\\ &+C\Big(1+|\ln(\epsilon)|\Big)\bigg(\frac{1}{\delta}+\frac{\epsilon}{\delta^2}\bigg)\bigg(\|\mathscr{V}\|_{L^\infty L^\infty}+\|S\|_{L^\infty L^\infty}\bigg)\\ &+C\Big(1+|\ln(\epsilon)|\Big)\bigg(\delta+\Big(1+|\ln(\epsilon)|\Big)\sqrt{\delta_0}\bigg)\|\mathscr{A}\|_{L^\infty L^\infty}. \quad (5.99)\end{aligned}$$

Step 7: Synthesis.
Collecting all the terms in previous steps, we have proved

$$\begin{aligned}|I| \leqq\ & C\Big(1+|\ln(\epsilon)|\Big)\bigg(\|p_{\mathscr{A}}\|_{L^\infty_-}+\|S_{\mathscr{A}}\|_{L^\infty L^\infty}\bigg)\\ &+\frac{C}{\delta_0^2}\bigg(\left\|\zeta\frac{\partial p}{\partial\phi}\right\|_{L^\infty_-}+\|S\|_{L^\infty L^\infty}+\left\|\zeta\frac{\partial S}{\partial\eta}\right\|_{L^\infty L^\infty}+\|\mathscr{V}\|_{L^\infty L^\infty}\bigg)\\ &+C\Big(1+|\ln(\epsilon)|\Big)\bigg(\frac{1}{\delta}+\frac{\epsilon}{\delta^2}\bigg)\bigg(\|\mathscr{V}\|_{L^\infty L^\infty}+\|S\|_{L^\infty L^\infty}\bigg)\\ &+C\Big(1+|\ln(\epsilon)|\Big)\bigg(\delta+\Big(1+|\ln(\epsilon)|\Big)\sqrt{\delta_0}\bigg)\|\mathscr{A}\|_{L^\infty L^\infty}. \quad (5.100)\end{aligned}$$

5.3. Region II: $\sin\phi<0$ *and* $|E(\eta,\phi)|\leqq \mathrm{e}^{-V(L)}$

We consider

$$\mathcal{K}[p_{\mathscr{A}}]=p_{\mathscr{A}}\Big(\phi'(\eta,\phi;0)\Big)\exp(-G_{L,0}-G_{L,\eta}) \quad (5.101)$$

$$\begin{aligned}\mathcal{T}[\tilde{\mathscr{A}}+S_{\mathscr{A}}]=\ & \int_0^L\frac{(\tilde{\mathscr{A}}+S)\Big(\eta',\phi'(\eta,\phi;\eta')\Big)}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\exp(-G_{L,\eta'}-G_{L,\eta})\,\mathrm{d}\eta'\\ &+\int_\eta^L\frac{(\tilde{\mathscr{A}}+S)\Big(\eta',\mathscr{R}[\phi'(\eta,\phi;\eta')]\Big)}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\exp(-G_{\eta',\eta})\,\mathrm{d}\eta'. \quad (5.102)\end{aligned}$$

Based on Lemma 4.7, Lemma 4.8, we can directly obtain

$$|\mathcal{K}[p_{\mathscr{A}}]| \leqq \|p_{\mathscr{A}}\|_{L^\infty_-}, \quad (5.103)$$
$$|\mathcal{T}[S_{\mathscr{A}}]| \leqq \|S_{\mathscr{A}}\|_{L^\infty L^\infty}. \quad (5.104)$$

Hence, we only need to estimate

$$II = \mathcal{T}[\tilde{\mathscr{A}} + S_{\mathscr{A}}] = \int_0^L \frac{\tilde{\mathscr{A}}\Big(\eta', \phi'(\eta, \phi; \eta')\Big)}{\sin\Big(\phi'(\eta, \phi; \eta')\Big)} \exp(-G_{L,\eta'} - G_{L,\eta})\, \mathrm{d}\eta' + \int_\eta^L \frac{\tilde{\mathscr{A}}\Big(\eta', \mathscr{R}[\phi'(\eta, \phi; \eta')]\Big)}{\sin\Big(\phi'(\eta, \phi; \eta')\Big)} \exp(-G_{\eta',\eta})\, \mathrm{d}\eta'. \tag{5.105}$$

In particular, since the integral $\int_0^\eta \cdots$ can be estimated as in Region I, so we only need to estimate the integral $\int_\eta^L \cdots$. Also, noting that fact that

$$\exp(-G_{L,\eta'} - G_{L,\eta}) \leqq \exp(-G_{\eta',\eta}), \tag{5.106}$$

we only need to estimate

$$\int_\eta^L \frac{\tilde{\mathscr{A}}\Big(\eta', \mathscr{R}[\phi'(\eta, \phi; \eta')]\Big)}{\sin\Big(\phi'(\eta, \phi; \eta')\Big)} \exp(-G_{\eta',\eta})\, \mathrm{d}\eta'. \tag{5.107}$$

Here the proof is almost identical to that in Region I, so we only point out the key differences.
Step 0: Preliminaries.
We need to update one key result. For $0 \leqq \eta \leqq \eta'$,

$$\begin{aligned}\sin\phi' &= \sqrt{1-\cos^2\phi'} = \sqrt{1 - \left(\frac{R_\kappa - \epsilon\eta}{R_\kappa - \epsilon\eta'}\right)^2 \cos^2\phi} \\ &= \frac{\sqrt{(R_\kappa - \epsilon\eta')^2 \sin^2\phi + (2R_\kappa - \epsilon\eta - \epsilon\eta')(\epsilon\eta' - \epsilon\eta)\cos^2\phi}}{R_\kappa - \epsilon\eta'} \\ &\leqq |\sin\phi|\,. \end{aligned} \tag{5.108}$$

Then we have

$$-\int_\eta^{\eta'} \frac{1}{\sin\phi'(y)}\, \mathrm{d}y \leqq -\frac{\eta' - \eta}{|\sin\phi|}. \tag{5.109}$$

In the following, we will divide the estimate of II into several cases based on the value of $\sin\phi$, $|\cos\phi|$, $\sin\phi'$ and $\epsilon\eta'$. We write

$$\begin{aligned} II = &\int_\eta^L \mathbf{1}_{\{\sin\phi \leqq -\delta_0\}} \mathbf{1}_{\{|\cos\phi| \geqq \delta_0\}} + \int_\eta^L \mathbf{1}_{\{-\delta_0 \leqq \sin\phi \leqq 0\}} \mathbf{1}_{\{\chi(\phi_*) < 1\}} \\ &+ \int_\eta^L \mathbf{1}_{\{-\delta_0 \leqq \sin\phi \leqq 0\}} \mathbf{1}_{\{\chi(\phi_*) = 1\}} \mathbf{1}_{\{\sqrt{\epsilon\eta'} \geqq \sin\phi'\}} \end{aligned}$$

$$
\begin{aligned}
&+\int_{\eta}^{L} \mathbf{1}_{\{-\delta_0\leqq\sin\phi\leqq 0\}}\mathbf{1}_{\{\chi(\phi_*)=1\}}\mathbf{1}_{\{\sqrt{\epsilon\eta'}\leqq\sin\phi'\}}\\
&+\int_{\eta}^{L} \mathbf{1}_{\{|\cos\phi|\leqq\delta_0\}}\\
&= II_1+II_2+II_3+II_4+II_5.
\end{aligned} \tag{5.110}
$$

Step 1: Estimate of II_1 for $\sin\phi\leqq-\delta_0$.
We first estimate $\sin\phi'$. Along the characteristics, we know that

$$
\mathrm{e}^{-V(\eta')}\cos\phi'=\mathrm{e}^{-V(\eta)}\cos\phi, \tag{5.111}
$$

which implies that

$$
\begin{aligned}
\cos\phi' &= \mathrm{e}^{V(\eta')-V(\eta)}\cos\phi\leqq \mathrm{e}^{V(L)-V(0)}\cos\phi\\
&= \mathrm{e}^{V(L)-V(0)}\sqrt{1-\delta_0^2}.
\end{aligned} \tag{5.112}
$$

We can further deduce that

$$
\cos\phi'\leqq\left(1-\frac{\epsilon^{\frac{1}{2}}}{R_\kappa}\right)^{-1}\sqrt{1-\delta_0^2}. \tag{5.113}
$$

Then we have

$$
\sin\phi'\geqq\sqrt{1-\left(1-\frac{\epsilon^{\frac{1}{2}}}{R_\kappa}\right)^{-2}(1-\delta_0^2)}\geqq\delta_0-\epsilon^{\frac{1}{4}}>\frac{\delta_0}{2}, \tag{5.114}
$$

when ϵ is sufficiently small.

Similar to Region I, we will use two formulations to handle different terms and we will decompose $\mathscr{V}=\mathscr{V}_1+\mathscr{V}_2$.

Using Formulation I, we rewrite the $\mathscr{V}_1$ equation along the characteristics as

$$
\begin{aligned}
\mathscr{V}_1(\eta,\phi) &= p\Big(\phi'(0)\Big)\exp(-G_{L,0}-G_{L,\eta})\\
&\quad+\int_0^L\frac{\bar{\mathscr{V}}(\eta')}{\sin\Big(\phi'(\eta')\Big)}\exp(-G_{L,\eta'}-G_{L,\eta})\,\mathrm{d}\eta'\\
&\quad+\int_\eta^L\frac{\bar{\mathscr{V}}(\eta')}{\sin\Big(\phi'(\eta')\Big)}\exp(-G_{\eta',\eta})\,\mathrm{d}\eta',
\end{aligned} \tag{5.115}
$$

where (η',ϕ') and (η,ϕ) are on the same characteristic with $\sin\phi'\geqq 0$. Then taking η derivative on both sides of (5.115) yields

$$
\frac{\partial\mathscr{V}_1}{\partial\eta}=X_1+X_2+X_3+X_4+X_5+X_6+X_7, \tag{5.116}
$$

where

$$X_1 = \frac{\partial p\Big(\phi'(0)\Big)}{\partial \eta} \exp(-G_{L,0} - G_{L,\eta}), \tag{5.117}$$

$$X_2 = -p\Big(\phi'(0)\Big) \exp(-G_{L,0} - G_{L,\eta}) \bigg(\frac{\partial G_{L,0}}{\partial \eta} + \frac{\partial G_{L,\eta}}{\partial \eta} \bigg), \tag{5.118}$$

$$X_3 = -\int_0^L \bar{\mathscr{V}}(\eta') \frac{\cos\Big(\phi'(\eta')\Big)}{\sin^2\Big(\phi'(\eta')\Big)} \frac{\partial \phi'(\eta')}{\partial \eta} \exp(-G_{L,\eta'} - G_{L,\eta})\, \mathrm{d}\eta', \tag{5.119}$$

$$X_4 = -\int_0^L \frac{\bar{\mathscr{V}}(\eta')}{\sin\Big(\phi'(\eta')\Big)} \exp(-G_{L,\eta'} - G_{L,\eta}) \times \bigg(\frac{\partial G_{L,\eta'}}{\partial \eta} + \frac{\partial G_{L,\eta}}{\partial \eta} \bigg) \mathrm{d}\eta', \tag{5.120}$$

$$X_5 = -\int_\eta^L \bar{\mathscr{V}}(\eta') \frac{\cos\Big(\phi'(\eta')\Big)}{\sin^2\Big(\phi'(\eta')\Big)} \frac{\partial \phi'(\eta')}{\partial \eta} \exp(-G_{\eta',\eta})\, \mathrm{d}\eta', \tag{5.121}$$

$$X_6 = -\int_\eta^L \frac{\bar{\mathscr{V}}(\eta')}{\sin\Big(\phi'(\eta')\Big)} \exp(-G_{\eta',\eta}) \frac{\partial G_{\eta',\eta}}{\partial \eta}\, \mathrm{d}\eta', \tag{5.122}$$

$$X_7 = -\frac{\bar{\mathscr{V}}(\eta)}{\sin(\phi)}. \tag{5.123}$$

We need to estimate each term. The estimates are standard, so we only list the results:

$$|X_1| \leqq \frac{C}{\delta_0} \left\| \frac{\partial p}{\partial \phi} \right\|_{L^\infty_-}, \quad |X_2| \leqq \frac{C}{\delta_0} \|p\|_{L^\infty_-}, \quad |X_3| \leqq \frac{C}{\delta_0} \|\mathscr{V}\|_{L^\infty L^\infty}, \quad |X_4| \leqq \frac{C}{\delta_0} \|\mathscr{V}\|_{L^\infty L^\infty}, \tag{5.124}$$

$$|X_5| \leqq \frac{C}{\delta_0} \|\mathscr{V}\|_{L^\infty L^\infty}, \quad |X_6| \leqq \frac{C}{\delta_0} \|\mathscr{V}\|_{L^\infty L^\infty}, \quad |X_7| \leqq \frac{C}{\delta_0} \|\mathscr{V}\|_{L^\infty L^\infty}. \tag{5.125}$$

In total, we have

$$\left| \frac{\partial \mathscr{V}_1}{\partial \eta} \right| \leqq \frac{C}{\delta_0} \bigg(\|p\|_{L^\infty} + \left\| \frac{\partial p}{\partial \phi} \right\|_{L^\infty} + \|\mathscr{V}\|_{L^\infty L^\infty} \bigg). \tag{5.126}$$

Using Formulation II, we rewrite the $\mathscr{V}_2$ equation along the characteristics as

$$\mathscr{V}_2(\eta,\phi) = \int_{\phi_*}^{\phi^*} \frac{S\Big(\eta'(\phi'),\phi'\Big)}{F\Big(\eta'(\phi')\Big)\cos(\phi')} \exp(-H_{\phi^*,\phi'} - H_{-\phi^*,\phi})\,\mathrm{d}\phi' + \int_{\phi}^{-\phi^*} \frac{S\Big(\eta'(\phi'),\phi'\Big)}{F\Big(\eta'(\phi')\Big)\cos(\phi')} \exp(-H_{\phi',\phi})\,\mathrm{d}\phi', \tag{5.127}$$

where (η',ϕ'), $(0,\phi_*)$, (L,ϕ^*), $(L,-\phi^*)$ and (η,ϕ) are on the same characteristic with $\sin\phi' \geqq 0$ and $\phi^* \geqq 0$. Then taking η derivative on both sides of (5.127) yields

$$\frac{\partial \mathscr{V}_2}{\partial \eta} = Y_1 + Y_2 + Y_3 + Y_4 + Y_5 + Y_6 + Y_7 + Y_8, \tag{5.128}$$

where

$$Y_1 = \frac{S(L,\phi^*)}{F(L)\cos(\phi^*)} \exp(-H_{-\phi^*,\phi}) \frac{\partial \phi^*}{\partial \eta} - \frac{S(0,\phi_*)}{F(0)\cos(\phi_*)} \exp(-H_{\phi^*,\phi_*} - H_{-\phi^*,\phi}) \frac{\partial \phi_*}{\partial \eta}, \tag{5.129}$$

$$Y_2 = -\int_{\phi_*}^{\phi^*} S\Big(\eta'(\phi'),\phi'\Big) \frac{1}{F^2\Big(\eta'(\phi')\Big)\cos(\phi')} \frac{\partial F\Big(\eta'(\phi')\Big)}{\partial \eta} \exp(-H_{\phi^*,\phi'} - H_{-\phi^*,\phi})\,\mathrm{d}\phi', \tag{5.130}$$

$$Y_3 = -\int_{\phi_*}^{\phi^*} \frac{S\Big(\eta'(\phi'),\phi'\Big)}{F\Big(\eta'(\phi')\Big)\cos(\phi')} \exp(-H_{\phi^*,\phi'} - H_{-\phi^*,\phi}) \left(\frac{\partial H_{\phi^*,\phi'}}{\partial \eta} + \frac{\partial H_{-\phi^*,\phi}}{\partial \eta}\right) \mathrm{d}\phi', \tag{5.131}$$

$$Y_4 = \int_{\phi_*}^{\phi^*} \frac{\partial_{\eta'} S\Big(\eta'(\phi'),\phi'\Big)}{F\Big(\eta'(\phi')\Big)\cos(\phi')} \frac{\partial \eta'(\phi')}{\partial \eta} \exp(-H_{\phi^*,\phi'} - H_{-\phi^*,\phi})\,\mathrm{d}\phi', \tag{5.132}$$

$$Y_5 = -\frac{S(L,-\phi^*)}{F(L)\cos(-\phi^*)} \exp(-H_{-\phi^*,\phi}) \frac{\partial \phi^*}{\partial \eta}, \tag{5.133}$$

$$Y_6 = -\int_{\phi}^{-\phi^*} S\Big(\eta'(\phi'),\phi'\Big) \frac{1}{F^2\Big(\eta'(\phi')\Big)\cos(\phi')} \frac{\partial F\Big(\eta'(\phi')\Big)}{\partial \eta} \exp(-H_{\phi',\phi})\,\mathrm{d}\phi', \tag{5.134}$$

$$Y_7 = -\int_{\phi}^{-\phi^*} \frac{S\Big(\eta'(\phi'),\phi'\Big)}{F\Big(\eta'(\phi')\Big)\cos(\phi')} \exp(-H_{\phi',\phi}) \frac{\partial H_{\phi',\phi}}{\partial \eta} \,\mathrm{d}\phi', \tag{5.135}$$

$$Y_8 = \int_{\phi}^{-\phi^*} \frac{\partial_{\eta'} S\Big(\eta'(\phi'),\phi'\Big)}{F\Big(\eta'(\phi')\Big)\cos(\phi')} \frac{\partial \eta'(\phi')}{\partial \eta} \exp(-H_{\phi',\phi}) \,\mathrm{d}\phi'. \tag{5.136}$$

We need to estimate each term. The estimates are standard, so we only list the results:

$$|Y_1| \leqq \frac{C}{\delta_0}\|S\|_{L^\infty L^\infty}, \quad |Y_2| \leqq \frac{C}{\delta_0}\|S\|_{L^\infty L^\infty}, \quad |Y_3| \leqq \frac{C}{\delta_0}\|S\|_{L^\infty L^\infty},$$
$$|Y_4| \leqq \frac{C}{\delta_0}\left\|\frac{\partial S}{\partial \eta}\right\|_{L^\infty L^\infty}, \tag{5.137}$$
$$|Y_5| \leqq \frac{C}{\delta_0}\|S\|_{L^\infty L^\infty}, \quad |Y_6| \leqq \frac{C}{\delta_0}\|S\|_{L^\infty L^\infty}, \quad |Y_7| \leqq \frac{C}{\delta_0}\|S\|_{L^\infty L^\infty},$$
$$|Y_8| \leqq \frac{C}{\delta_0}\left\|\frac{\partial S}{\partial \eta}\right\|_{L^\infty L^\infty}. \tag{5.138}$$

In total, we have

$$\left|\frac{\partial \mathscr{V}_2}{\partial \eta}\right| \leqq \frac{C}{\delta_0}\left(\|S\|_{L^\infty L^\infty} + \left\|\frac{\partial S}{\partial \eta}\right\|_{L^\infty L^\infty}\right). \tag{5.139}$$

Combining (5.126) and (5.139), noting that $\zeta \geqq \sin\phi \geqq \delta_0$, we have

$$|II_1| \leqq \frac{C}{\delta_0^2}\Bigg(\|p\|_{L^\infty_-} + \left\|\zeta\frac{\partial p}{\partial \phi}\right\|_{L^\infty_-} + \|S\|_{L^\infty L^\infty}$$
$$+ \left\|\zeta\frac{\partial S}{\partial \eta}\right\|_{L^\infty L^\infty} + \|\mathscr{V}\|_{L^\infty L^\infty}\Bigg). \tag{5.140}$$

Step 2: Estimate of II_2 for $-\delta_0 \leqq \sin\phi \leqq 0$ and $\chi(\phi_*) < 1$.

This is similar to the estimate of I_2 based on the integral

$$\int_{\eta}^{L} \frac{1}{\sin\phi'} \exp(-G_{\eta',\eta}) \,\mathrm{d}\eta' \leqq 1. \tag{5.141}$$

Then we have

$$|II_2| \leqq \left(\frac{1}{\delta} + \frac{\epsilon}{\delta^2}\right)\left(\|\mathscr{V}\|_{L^\infty L^\infty} + \|S\|_{L^\infty L^\infty}\right). \tag{5.142}$$

Step 3: Estimate of II_3 for $-\delta_0 \leqq \sin\phi \leqq 0$, $\chi(\phi_*) = 1$ and $\sqrt{\epsilon\eta'} \geqq \sin\phi'$.

This is similar to the estimate of I_3, we have

$$|II_3| \leqq C\delta\|\mathscr{A}\|_{L^\infty L^\infty}. \tag{5.143}$$

Step 4: Estimate of II_4 for $-\delta_0 \leqq \sin\phi \leqq 0$, $\chi(\phi_*) = 1$ and $\sqrt{\epsilon\eta'} \leqq \sin\phi'$.

This step is different. We do not need to further decompose the cases. Based on (5.109), we have,

$$-G_{\eta,\eta'} \leqq -\frac{\eta'-\eta}{|\sin\phi|}. \tag{5.144}$$

Then following the same argument in estimating I_5, we obtain

$$|II_4| \leqq C\|\mathscr{A}\|_{L^\infty L^\infty}\int_\eta^L \Big(1+|\ln(\epsilon)|+\big|\ln(\eta')\big|\Big) \exp\left(-\frac{\eta'-\eta}{|\sin\phi|}\right)\mathrm{d}\eta'. \tag{5.145}$$

If $\eta \geqq 2$, we directly obtain

$$\begin{aligned}\left|\int_\eta^L \big|\ln(\eta')\big|\exp\left(-\frac{\eta'-\eta}{|\sin\phi|}\right)\mathrm{d}\eta'\right| &\leqq \left|\int_2^L \ln(\eta')\exp\left(-\frac{\eta'-\eta}{|\sin\phi|}\right)\mathrm{d}\eta'\right|\\ &\leqq \ln(2)\left|\int_2^L \exp\left(-\frac{\eta'-\eta}{|\sin\phi|}\right)\mathrm{d}\eta'\right|\\ &\leqq C\sqrt{|\sin\phi|}.\end{aligned} \tag{5.146}$$

If $\eta \leqq 2$, we decompose as

$$\begin{aligned}&\left|\int_\eta^L \big|\ln(\eta')\big|\exp\left(-\frac{\eta'-\eta}{|\sin\phi|}\right)\mathrm{d}\eta'\right|\\ &\quad\leqq \left|\int_\eta^2 \big|\ln(\eta')\big|\exp\left(-\frac{\eta'-\eta}{|\sin\phi|}\right)\mathrm{d}\eta'\right|\\ &\qquad+\left|\int_2^L \big|\ln(\eta')\big|\exp\left(-\frac{\eta'-\eta}{|\sin\phi|}\right)\mathrm{d}\eta'\right|.\end{aligned} \tag{5.147}$$

The second term is identical to the estimate in $\eta \geqq 2$. We apply Cauchy's inequality to the first term

$$\begin{aligned}&\left|\int_\eta^2 \big|\ln(\eta')\big|\exp\left(-\frac{\eta'-\eta}{|\sin\phi|}\right)\mathrm{d}\eta'\right|\\ &\leqq \left(\int_\eta^2 \ln^2(\eta')\,\mathrm{d}\eta'\right)^{\frac{1}{2}}\left(\int_\eta^2 \exp\left(-\frac{2(\eta'-\eta)}{|\sin\phi|}\right)\mathrm{d}\eta'\right)^{\frac{1}{2}}\\ &\leqq \left(\int_0^2 \ln^2(\eta')\,\mathrm{d}\eta'\right)^{\frac{1}{2}}\left(\int_\eta^2 \exp\left(-\frac{2(\eta'-\eta)}{|\sin\phi|}\right)\mathrm{d}\eta'\right)^{\frac{1}{2}}\\ &\leqq C\sqrt{|\sin\phi|}.\end{aligned} \tag{5.148}$$

Hence, we have

$$|II_4| \leqq C(1+|\ln(\epsilon)|)\sqrt{\delta_0}\|\mathscr{A}\|_{L^\infty L^\infty}. \tag{5.149}$$

Step 5: Estimate of II_5 for $|\cos\phi| < \delta_0$.
This is similar to the estimate of I_6, we have

$$
\begin{aligned}
|II_5| \leqq\ & C\Big(1+|\ln(\epsilon)|\Big)\Big(\,\|p_{\mathscr{A}}\|_{L^\infty_-} + \|S_{\mathscr{A}}\|_{L^\infty L^\infty}\Big)\\
&+C\Big(1+|\ln(\epsilon)|\Big)\Big(\frac{1}{\delta}+\frac{\epsilon}{\delta^2}\Big)\Big(\|\mathscr{V}\|_{L^\infty L^\infty}+\|S\|_{L^\infty L^\infty}\Big)\\
&+C\Big(1+|\ln(\epsilon)|\Big)\Big(\delta+\Big(1+|\ln(\epsilon)|\Big)\sqrt{\delta_0}\Big)\|\mathscr{A}\|_{L^\infty L^\infty}. \qquad (5.150)
\end{aligned}
$$

Step 6: Synthesis.
Collecting all the terms in previous steps, we have proved

$$
\begin{aligned}
|II| \leqq\ & C\Big(1+|\ln(\epsilon)|\Big)\Big(\,\|p_{\mathscr{A}}\|_{L^\infty_-} + \|S_{\mathscr{A}}\|_{L^\infty L^\infty}\Big)\\
&+\frac{C}{\delta_0^2}\Big(\,\|p\|_{L^\infty_-}+\Big\|\zeta\frac{\partial p}{\partial\phi}\Big\|_{L^\infty_-}+\|S\|_{L^\infty L^\infty}+\Big\|\zeta\frac{\partial S}{\partial\eta}\Big\|_{L^\infty L^\infty}+\|\mathscr{V}\|_{L^\infty L^\infty}\Big)\\
&+C\Big(1+|\ln(\epsilon)|\Big)\Big(\frac{1}{\delta}+\frac{\epsilon}{\delta^2}\Big)\Big(\|\mathscr{V}\|_{L^\infty L^\infty}+\|S\|_{L^\infty L^\infty}\Big)\\
&+C\Big(1+|\ln(\epsilon)|\Big)\Big(\delta+\Big(1+|\ln(\epsilon)|\Big)\sqrt{\delta_0}\Big)\|\mathscr{A}\|_{L^\infty L^\infty}. \qquad (5.151)
\end{aligned}
$$

5.4. Region III: $\sin\phi < 0$ *and* $|E(\eta,\phi)| \geqq \mathrm{e}^{-V(L)}$

We consider

$$
\mathcal{K}[p_{\mathscr{A}}] = p_{\mathscr{A}}\Big(\phi'(\eta,\phi;0)\Big)\exp(-G_{\eta^+,0}-G_{\eta^+,\eta}) \qquad (5.152)
$$

$$
\begin{aligned}
\mathcal{T}[\tilde{\mathscr{A}}+S_{\mathscr{A}}] =\ & \int_0^{\eta^+}\frac{(\tilde{\mathscr{A}}+S_{\mathscr{A}})\Big(\eta',\phi'(\eta,\phi;\eta')\Big)}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\exp(-G_{\eta^+,\eta'}-G_{\eta^+,\eta})\,\mathrm{d}\eta'\\
&+\int_\eta^{\eta^+}\frac{(\tilde{\mathscr{A}}+S_{\mathscr{A}})\Big(\eta',\mathscr{R}[\phi'(\eta,\phi;\eta')]\Big)}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\exp(-G_{\eta',\eta})\,\mathrm{d}\eta'. \qquad (5.153)
\end{aligned}
$$

Based on [24, Lemma 4.7, Lemma 4.8], we still have

$$
|\mathcal{K}[p_{\mathscr{A}}]| \leqq \|p_{\mathscr{A}}\|_{L^\infty_-}, \qquad (5.154)
$$

$$
|\mathcal{T}[S_{\mathscr{A}}]| \leqq \|S_{\mathscr{A}}\|_{L^\infty L^\infty}. \qquad (5.155)
$$

Hence, we only need to estimate

$$III = \mathcal{T}[\tilde{\mathscr{A}}] = \int_0^{\eta^+} \frac{\tilde{\mathscr{A}}\Big(\eta', \phi'(\eta, \phi; \eta')\Big)}{\sin\Big(\phi'(\eta, \phi; \eta')\Big)} \exp(-G_{\eta^+,\eta'} - G_{\eta^+,\eta})\,\mathrm{d}\eta' + \int_{\eta}^{\eta^+} \frac{\tilde{\mathscr{A}}\Big(\eta', \mathscr{R}[\phi'(\eta, \phi; \eta')]\Big)}{\sin\Big(\phi'(\eta, \phi; \eta')\Big)} \exp(-G_{\eta',\eta})\,\mathrm{d}\eta'. \quad (5.156)$$

Note that $|E(\eta, \phi)| \geqq \mathrm{e}^{-V(L)}$ implies

$$\mathrm{e}^{-V(\eta)} \cos\phi \geqq \mathrm{e}^{-V(L)}. \quad (5.157)$$

Hence, we can further deduce that

$$\cos\phi \geqq \mathrm{e}^{V(\eta)-V(L)} \geqq \mathrm{e}^{V(0)-V(L)} \geqq \left(1 - \frac{\epsilon^{\frac{1}{2}}}{R_\kappa}\right). \quad (5.158)$$

Hence, we know

$$|\sin\phi| \leqq \sqrt{1 - \left(1 - \frac{\epsilon^{\frac{1}{2}}}{R_\kappa}\right)^2} \leqq \epsilon^{\frac{1}{4}}. \quad (5.159)$$

Hence, when ϵ is sufficiently small, we always have

$$|\sin\phi| \leqq \epsilon^{\frac{1}{4}} \leqq \delta_0. \quad (5.160)$$

This means we do not need to bother with the estimate of $\sin\phi \leqq -\delta_0$ as Step 1 in estimating I and II. Also, it is not necessary to discuss the case $|\cos\phi| < \delta_0$.

Then the integral $\int_0^{\eta}(\cdots)$ is similar to the argument in Region I, and the integral $\int_{\eta}^{\eta^+}(\cdots)$ is similar to the argument in Region II. Hence, combining the methods in Region I and Region II, we can show the desired result, that is

$$\begin{aligned} |III| \leqq\ & C\Big(1 + |\ln(\epsilon)|\Big)\Big(\|p_{\mathscr{A}}\|_{L^\infty_-} + \|S_{\mathscr{A}}\|_{L^\infty L^\infty}\Big) \\ & + C\Big(1 + |\ln(\epsilon)|\Big)\left(\frac{1}{\delta} + \frac{\epsilon}{\delta^2}\right)\Big(\|\mathscr{V}\|_{L^\infty L^\infty} + \|S\|_{L^\infty L^\infty}\Big) \\ & + C\Big(1 + |\ln(\epsilon)|\Big)\Big(\delta + \Big(1 + |\ln(\epsilon)|\Big)\sqrt{\delta_0}\Big)\|\mathscr{A}\|_{L^\infty L^\infty}. \quad (5.161) \end{aligned}$$

5.5. Estimate of Normal Derivative

Theorem 5.1. *The solution $\mathscr{A}$ to the equation* (5.4) *satisfies*

$$\|\mathscr{A}\|_{L^\infty L^\infty} \leqq C\,|\ln(\epsilon)|\bigg(\|p_{\mathscr{A}}\|_{L^\infty_-} + \|S_{\mathscr{A}}\|_{L^\infty L^\infty}\bigg) + C\,|\ln(\epsilon)|^8\bigg(\|p\|_{L^\infty_-} + \left\|\zeta\frac{\partial p}{\partial\phi}\right\|_{L^\infty_-} + \|S\|_{L^\infty L^\infty} + \left\|\zeta\frac{\partial S}{\partial\eta}\right\|_{L^\infty L^\infty} + \|\mathscr{V}\|_{L^\infty L^\infty}\bigg). \tag{5.162}$$

Proof. Combining the analysis in above three regions and taking supremum over all (η, ϕ), we have

$$\begin{aligned}\|\mathscr{A}\|_{L^\infty L^\infty} \leqq{}& C\Big(1+|\ln(\epsilon)|\Big)\bigg(\|p_{\mathscr{A}}\|_{L^\infty_-} + \|S_{\mathscr{A}}\|_{L^\infty L^\infty}\bigg)\\ &+\frac{C}{\delta_0^2}\bigg(\|p\|_{L^\infty_-} + \left\|\zeta\frac{\partial p}{\partial\phi}\right\|_{L^\infty_-} + \|S\|_{L^\infty L^\infty}\\ &+\left\|\zeta\frac{\partial S}{\partial\eta}\right\|_{L^\infty L^\infty} + \|\mathscr{V}\|_{L^\infty L^\infty}\bigg)\\ &+C\Big(1+|\ln(\epsilon)|\Big)\bigg(\frac{1}{\delta}+\frac{\epsilon}{\delta^2}\bigg)\bigg(\|\mathscr{V}\|_{L^\infty L^\infty} + \|S\|_{L^\infty L^\infty}\bigg)\\ &+C\Big(1+|\ln(\epsilon)|\Big)\bigg(\delta + \Big(1+|\ln(\epsilon)|\Big)\sqrt{\delta_0}\bigg)\|\mathscr{A}\|_{L^\infty L^\infty}.\end{aligned} \tag{5.163}$$

Then we choose quantities δ and δ_0 to perform the absorbing argument. First we choose $\delta = C_0\Big(1+|\ln(\epsilon)|\Big)^{-1}$ for $C_0 > 0$ sufficiently small such that

$$C\delta \leqq \frac{1}{4}. \tag{5.164}$$

Then we take $\delta_0 = C_0\Big(1+|\ln(\epsilon)|\Big)^{-4}$ such that

$$C\Big(1+|\ln(\epsilon)|\Big)^2\sqrt{\delta_0} \leqq \frac{1}{4} \tag{5.165}$$

for ϵ sufficiently small. Note that this mild decay of δ_0 with respect to ϵ also justifies the assumption in Case III that

$$\epsilon^{\frac{1}{4}} \leqq \frac{\delta_0}{2}, \tag{5.166}$$

for ϵ sufficiently small. Hence, we can absorb all the term related to $\|\mathscr{A}\|_{L^\infty L^\infty}$ on the right-hand side of (5.163) to the left-hand side to obtain

$$\|\mathscr{A}\|_{L^\infty L^\infty} \leqq C\,|\ln(\epsilon)|\bigg(\|p_{\mathscr{A}}\|_{L^\infty_-} + \|S_{\mathscr{A}}\|_{L^\infty L^\infty}\bigg) + C\,|\ln(\epsilon)|^8\bigg(\|p\|_{L^\infty_-} + \left\|\zeta\frac{\partial p}{\partial\phi}\right\|_{L^\infty_-} + \|S\|_{L^\infty L^\infty} + \left\|\zeta\frac{\partial S}{\partial\eta}\right\|_{L^\infty L^\infty} + \|\mathscr{V}\|_{L^\infty L^\infty}\bigg). \tag{5.167}$$

□

5.6. A Priori Estimate of Derivatives

In this subsection, we further estimate the normal and velocity derivatives.

Theorem 5.2. *The solution $\mathscr{V}$ to the difference equation (5.3) satisfies*

$$\left\|\zeta\frac{\partial\mathscr{V}}{\partial\eta}\right\|_{L^\infty L^\infty}+\left\|F(\eta)\cos\phi\frac{\partial\mathscr{V}}{\partial\phi}\right\|_{L^\infty L^\infty} \leq C\,|\ln(\epsilon)|^8\Bigg(\|p\|_{L^\infty_-}+\left\|(\epsilon+\zeta)\frac{\partial p}{\partial\phi}\right\|_{L^\infty_-} +\|S\|_{L^\infty L^\infty}+\left\|\zeta\frac{\partial S}{\partial\eta}\right\|_{L^\infty L^\infty}+\|\mathscr{V}\|_{L^\infty L^\infty}\Bigg). \tag{5.168}$$

Proof. Based on Theorem 5.1, we have

$$\|\mathscr{A}\|_{L^\infty L^\infty}\leq C\,|\ln(\epsilon)|\Big(\|p_{\mathscr{A}}\|_{L^\infty_-}+\|S_{\mathscr{A}}\|_{L^\infty L^\infty}\Big) +C\,|\ln(\epsilon)|^8\Bigg(\|p\|_{L^\infty_-}+\left\|\zeta\frac{\partial p}{\partial\phi}\right\|_{L^\infty_-} +\|S\|_{L^\infty L^\infty}+\left\|\zeta\frac{\partial S}{\partial\eta}\right\|_{L^\infty L^\infty}+\|\mathscr{V}\|_{L^\infty L^\infty}\Bigg). \tag{5.169}$$

Taking derivatives on both sides of (5.3) and multiplying ζ, we have

$$p_{\mathscr{A}}=-\epsilon\cos\phi\frac{\partial p}{\partial\phi}-p+\bar{\mathscr{V}}(0)+S(0,\phi), \tag{5.170}$$

$$S_{\mathscr{A}}=\frac{\partial F}{\partial\eta}\zeta\cos\phi\frac{\partial\mathscr{V}}{\partial\phi}+\zeta\frac{\partial S}{\partial\eta}. \tag{5.171}$$

Since $|F(\eta)|\leq C\epsilon$ and $\left|\frac{\partial F}{\partial\eta}\right|\leq C\epsilon F$, we may directly estimate

$$\|p_{\mathscr{A}}\|_{L^\infty}\leq C\Bigg(\|p\|_{L^\infty_-}+\epsilon\left\|\frac{\partial p}{\partial\phi}\right\|_{L^\infty_-}+\|S\|_{L^\infty L^\infty}+\|\mathscr{V}\|_{L^\infty L^\infty}\Bigg), \tag{5.172}$$

$$\|S_{\mathscr{A}}\|_{L^\infty L^\infty}\leq C\Bigg(\epsilon\left\|F(\eta)\cos\phi\frac{\partial\mathscr{V}}{\partial\phi}\right\|_{L^\infty L^\infty}+\left\|\zeta\frac{\partial S}{\partial\eta}\right\|_{L^\infty L^\infty}\Bigg). \tag{5.173}$$

Then inserting (5.172) and (5.173) into (5.169), we derive

$$\|\mathscr{A}\|_{L^\infty L^\infty}\leq C\epsilon\left\|F(\eta)\cos\phi\frac{\partial\mathscr{V}}{\partial\phi}\right\|_{L^\infty L^\infty} +C\,|\ln(\epsilon)|^8\Bigg(\|p\|_{L^\infty_-}+\left\|(\epsilon+\zeta)\frac{\partial p}{\partial\phi}\right\|_{L^\infty_-} +\|S\|_{L^\infty L^\infty}+\left\|\zeta\frac{\partial S}{\partial\eta}\right\|_{L^\infty L^\infty}+\|\mathscr{V}\|_{L^\infty L^\infty}\Bigg). \tag{5.174}$$

Since

$$\|\mathscr{A}\|_{L^\infty L^\infty} = \left\|\zeta\frac{\partial\mathscr{V}}{\partial\eta}\right\|_{L^\infty L^\infty} \geqq \left\|\sin\phi\frac{\partial\mathscr{V}}{\partial\eta}\right\|_{L^\infty L^\infty}, \tag{5.175}$$

we know

$$\begin{aligned}\left\|\sin\phi\frac{\partial\mathscr{V}}{\partial\eta}\right\|_{L^\infty L^\infty} \leqq C\epsilon\left\|F(\eta)\cos\phi\frac{\partial\mathscr{V}}{\partial\phi}\right\|_{L^\infty L^\infty}&\\ +C\left|\ln(\epsilon)\right|^8\Bigg(\|p\|_{L^\infty_-}+\left\|(\epsilon+\zeta)\frac{\partial p}{\partial\phi}\right\|_{L^\infty_-}&\\ +\|S\|_{L^\infty L^\infty}+\left\|\zeta\frac{\partial S}{\partial\eta}\right\|_{L^\infty L^\infty}+\|\mathscr{V}\|_{L^\infty L^\infty}\Bigg).&\end{aligned} \tag{5.176}$$

Considering the equation (5.3), since $\zeta(\eta,\phi)\geqq|\sin\phi|$, we have

$$\begin{aligned}\left\|F(\eta)\cos\phi\frac{\partial\mathscr{V}}{\partial\phi}\right\|_{L^\infty L^\infty} &\leqq \left\|\sin\phi\frac{\partial\mathscr{V}}{\partial\eta}\right\|_{L^\infty L^\infty}\\ &\quad+\|\mathscr{V}\|_{L^\infty L^\infty}+\left\|\bar{\mathscr{V}}\right\|_{L^\infty L^\infty}+\|S\|_{L^\infty L^\infty}\\ &\leqq C\epsilon\left\|F(\eta)\cos\phi\frac{\partial\mathscr{V}}{\partial\phi}\right\|_{L^\infty L^\infty}\\ &\quad+C\left|\ln(\epsilon)\right|^8\Bigg(\|p\|_{L^\infty_-}+\left\|(\epsilon+\zeta)\frac{\partial p}{\partial\phi}\right\|_{L^\infty_-}\\ &\quad+\|S\|_{L^\infty L^\infty}+\left\|\zeta\frac{\partial S}{\partial\eta}\right\|_{L^\infty L^\infty}+\|\mathscr{V}\|_{L^\infty L^\infty}\Bigg).\end{aligned} \tag{5.177}$$

Absorbing $\left\|F(\eta)\cos\phi\frac{\partial\mathscr{V}}{\partial\phi}\right\|_{L^\infty L^\infty}$ into the left-hand side, we obtain

$$\begin{aligned}\left\|F(\eta)\cos\phi\frac{\partial\mathscr{V}}{\partial\phi}\right\|_{L^\infty L^\infty} &\leqq C\left|\ln(\epsilon)\right|^8\Bigg(\|p\|_{L^\infty_-}+\left\|(\epsilon+\zeta)\frac{\partial p}{\partial\phi}\right\|_{L^\infty_-}\\ &\quad+\|S\|_{L^\infty L^\infty}+\left\|\zeta\frac{\partial S}{\partial\eta}\right\|_{L^\infty L^\infty}+\|\mathscr{V}\|_{L^\infty L^\infty}\Bigg).\end{aligned} \tag{5.178}$$

Therefore, we further derive

$$\begin{aligned}\left\|\zeta\frac{\partial\mathscr{V}}{\partial\eta}\right\|_{L^\infty L^\infty} &\leqq C\left|\ln(\epsilon)\right|^8\Bigg(\|p\|_{L^\infty_-}+\left\|(\epsilon+\zeta)\frac{\partial p}{\partial\phi}\right\|_{L^\infty_-}\\ &\quad+\|S\|_{L^\infty L^\infty}+\left\|\zeta\frac{\partial S}{\partial\eta}\right\|_{L^\infty L^\infty}+\|\mathscr{V}\|_{L^\infty L^\infty}\Bigg).\end{aligned} \tag{5.179}$$

□

Theorem 5.3. *For $K_0 > 0$ sufficiently small, the solution $\mathscr{V}$ to the difference equation (5.3) satisfies*

$$\left\| \mathrm{e}^{K_0\eta}\zeta \frac{\partial \mathscr{V}}{\partial \eta} \right\|_{L^\infty L^\infty} + \left\| \mathrm{e}^{K_0\eta} F(\eta)\cos\phi \frac{\partial \mathscr{V}}{\partial \phi} \right\|_{L^\infty L^\infty}$$
$$\leqq C\left|\ln(\epsilon)\right|^8 \bigg(\|p\|_{L^\infty_-} + \left\| (\epsilon+\zeta)\frac{\partial p}{\partial \phi} \right\|_{L^\infty_-}$$
$$+ \left\| \mathrm{e}^{K_0\eta} S \right\|_{L^\infty L^\infty} + \left\| \mathrm{e}^{K_0\eta}\zeta \frac{\partial S}{\partial \eta} \right\|_{L^\infty L^\infty} + \left\| \mathrm{e}^{K_0\eta}\mathscr{V} \right\|_{L^\infty L^\infty} \bigg). \tag{5.180}$$

Proof. This proof is almost identical to Theorem 5.2. The only difference is that $S_{\mathscr{A}}$ is added by $K_0\mathscr{A}\sin\phi$. When K_0 is sufficiently small, we can also absorb them into the left-hand side. Hence, this is obvious. □

6. Diffusive Limit

6.1. Analysis of Regular Boundary Layer

In this subsection, we will justify that the regular boundary layers are all well-defined.

Step 1: Well-Posedness of $\mathscr{U}_0$.

$\mathscr{U}_0$ satisfies the ϵ-Milne problem with geometric correction

$$\begin{cases} \sin\phi \dfrac{\partial \mathscr{U}_0}{\partial \eta} + F(\eta)\cos\phi \dfrac{\partial \mathscr{U}_0}{\partial \phi} + \mathscr{U}_0 - \bar{\mathscr{U}}_0 = 0, \\ \mathscr{U}_0(0,\tau,\phi) = \mathscr{G}(\tau,\phi) - \mathscr{F}_0(\tau) \ \text{ for } \ \sin\phi > 0, \\ \mathscr{U}_0(L,\tau,\phi) = \mathscr{U}_0(L,\tau,\mathscr{R}[\phi]), \end{cases} \tag{6.1}$$

Therefore, since $\|\mathscr{G}\|_{L^\infty} \leqq C$, by Theorem 4.9, we know

$$\left\| \mathrm{e}^{K_0\eta}\mathscr{U}_0 \right\|_{L^\infty L^\infty} \leqq C. \tag{6.2}$$

Step 2: Tangential Derivatives of $\mathscr{U}_0$.

The τ derivative $W = \dfrac{\partial \mathscr{U}_0}{\partial \tau}$ satisfies

$$\begin{cases} \sin\phi \dfrac{\partial W}{\partial \eta} + F(\eta)\cos\phi \dfrac{\partial W}{\partial \phi} + W - \bar{W} = -\dfrac{R_\kappa'}{R_\kappa - \epsilon\eta} F(\eta)\cos\phi \dfrac{\partial \mathscr{U}_0}{\partial \phi}, \\ W(0,\tau,\phi) = \dfrac{\partial \mathscr{G}}{\partial \tau}(\tau,\phi) - \dfrac{\partial \mathscr{F}_0}{\partial \tau}(\tau) \ \text{ for } \ \sin\phi > 0, \\ W(L,\tau,\phi) = W(L,\tau,\mathscr{R}[\phi]), \end{cases} \tag{6.3}$$

where R_κ' represents the θ derivative of R_κ. Here we need the regularity estimates of $\mathscr{U}_0$.

Based on Theorem 5.3, we know

$$
\begin{aligned}
&\left\| \mathrm{e}^{K_0\eta} F(\eta)\cos\phi \frac{\partial \mathscr{U}_0}{\partial\phi} \right\|_{L^\infty L^\infty} \\
&\quad \leqq C\left|\ln(\epsilon)\right|^8 \left(\|\mathscr{G}\|_{L^\infty_-} + \left\|(\epsilon+\zeta)\frac{\partial \mathscr{G}}{\partial\phi}\right\|_{L^\infty_-} + \left\|\mathrm{e}^{K_0\eta}\mathscr{U}_0\right\|_{L^\infty L^\infty}\right) \\
&\quad \leqq C\left|\ln(\epsilon)\right|^8 .
\end{aligned} \tag{6.4}
$$

Note that here although $\left\|\frac{\partial \mathscr{G}}{\partial\phi}\right\|_{L^\infty} \leqq C\epsilon^{-\alpha}$, with the help of $\epsilon+\zeta$, we can get rid of this negative power. Therefore, by Theorem 4.9, we have

$$
\left\|\mathrm{e}^{K_0\eta} W\right\|_{L^\infty L^\infty} \leqq C\left|\ln(\epsilon)\right|^8 . \tag{6.5}
$$

Step 3: Well-Posedness of $\mathscr{U}_1$.
$\mathscr{U}_1$ satisfies the ϵ-Milne problem with geometric correction

$$
\begin{cases}
\sin\phi\dfrac{\partial \mathscr{U}_1}{\partial\eta} + F(\eta)\cos\phi\dfrac{\partial \mathscr{U}_1}{\partial\phi} + \mathscr{U}_1 - \bar{\mathscr{U}}_1 = \dfrac{W}{R_\kappa - \epsilon\eta}\cos\phi, \\
\mathscr{U}_1(0,\tau,\phi) = \vec{w}\cdot\nabla_x U_0(\vec{x}_0,\vec{w}) - \mathscr{F}_{1,L}(\tau) \ \text{ for } \ \sin\phi > 0, \\
\mathscr{U}_1(L,\tau,\phi) = \mathscr{U}_1(L,\tau,\mathscr{R}[\phi]).
\end{cases} \tag{6.6}
$$

Therefore, by Theorem 4.9, we know

$$
\left\|\mathrm{e}^{K_0\eta}\mathscr{U}_1\right\|_{L^\infty L^\infty} \leqq C\left\|\mathrm{e}^{K_0\eta} W\right\|_{L^\infty L^\infty} \leqq C\left|\ln(\epsilon)\right|^8 . \tag{6.7}
$$

Step 4: Tangential Derivatives of $\mathscr{U}_1$.
The τ derivative $V = \dfrac{\partial \mathscr{U}_1}{\partial\tau}$ satisfies

$$
\begin{cases}
\sin\phi\dfrac{\partial V}{\partial\eta} + F(\eta)\cos\phi\dfrac{\partial V}{\partial\phi} + V - \bar{V} = S_1 + S_2 + S_3, \\
V(0,\tau,\phi) = \dfrac{\partial}{\partial\tau}\left(\vec{w}\cdot\nabla_x U_0(\vec{x}_0,\vec{w}) - \mathscr{F}_{1,L}(\tau)\right) \ \text{ for } \ \sin\phi > 0, \\
V(L,\tau,\phi) = V(L,\tau,\mathscr{R}[\phi]),
\end{cases} \tag{6.8}
$$

where

$$
S_1 = -\frac{R_\kappa'}{R_\kappa - \epsilon\eta} F(\eta)\cos\phi\frac{\partial \mathscr{U}_1}{\partial\phi}, \tag{6.9}
$$

$$
S_2 = -\frac{R_\kappa'}{(R_\kappa - \epsilon\eta)^2} W\cos\phi, \tag{6.10}
$$

$$
S_3 = \frac{1}{R_\kappa - \epsilon\eta}\cos\phi\frac{\partial W}{\partial\tau}. \tag{6.11}
$$

Based on Theorem 5.3, we have

$$
\left\|\mathrm{e}^{K_0\eta} S_1\right\|_{L^\infty L^\infty} \leqq C\left\|\mathrm{e}^{K_0\eta} F(\eta)\cos\phi\frac{\partial \mathscr{U}_1}{\partial\phi}\right\|_{L^\infty L^\infty} \tag{6.12}
$$

$$
\begin{aligned}
&\leqq C\bigg(\left\|\mathrm{e}^{K_0\eta}\frac{W}{R_\kappa-\epsilon\eta}\cos\phi\right\|_{L^\infty L^\infty}\\
&\quad+\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial}{\partial\eta}\left(\frac{W}{R_\kappa-\epsilon\eta}\cos\phi\right)\right\|_{L^\infty L^\infty}\bigg)\\
&\leqq C\bigg(\left\|\mathrm{e}^{K_0\eta}W\right\|_{L^\infty L^\infty}+\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial W}{\partial\eta}\right\|_{L^\infty L^\infty}\bigg),
\end{aligned}
$$

$$
\begin{aligned}
\left\|\mathrm{e}^{K_0\eta}S_2\right\|_{L^\infty L^\infty}&\leqq C\left\|\mathrm{e}^{K_0\eta}\frac{R_\kappa'}{(R_\kappa-\epsilon\eta)^2}W\cos\phi\right\|_{L^\infty L^\infty}\\
&\leqq C\left\|\mathrm{e}^{K_0\eta}W\right\|_{L^\infty L^\infty},
\end{aligned}\tag{6.13}
$$

$$
\left\|\mathrm{e}^{K_0\eta}S_3\right\|_{L^\infty L^\infty}\leqq C\left\|\mathrm{e}^{K_0\eta}\frac{\partial W}{\partial\tau}\right\|_{L^\infty L^\infty}.\tag{6.14}
$$

Step 5: Tangential Derivatives of W.

The τ derivative $Z=\dfrac{\partial W}{\partial\tau}$ satisfies

$$
\begin{cases}
\sin\phi\dfrac{\partial Z}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial Z}{\partial\phi}+Z-\bar{Z}=T_1+T_2,\\
Z(0,\tau,\phi)=\dfrac{\partial^2\mathscr{G}}{\partial\tau^2}(\tau,\phi)-\dfrac{\partial^2\mathscr{F}_0}{\partial\tau^2}(\tau)\ \text{ for }\ \sin\phi>0,\\
Z(L,\tau,\phi)=Z(L,\tau,\mathscr{R}[\phi]),
\end{cases}\tag{6.15}
$$

where

$$
T_1=-\frac{R_\kappa'}{R_\kappa-\epsilon\eta}F(\eta)\cos\phi\frac{\partial W}{\partial\phi},\tag{6.16}
$$

$$
T_2=-\frac{\partial}{\partial\tau}\left(\frac{R_\kappa'}{R_\kappa-\epsilon\eta}\right)F(\eta)\cos\phi\frac{\partial\mathscr{U}_0}{\partial\phi}.\tag{6.17}
$$

Based on Theorem 5.3, we have

$$
\left\|\mathrm{e}^{K_0\eta}T_1\right\|_{L^\infty L^\infty}\leqq C\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial W}{\partial\phi}\right\|_{L^\infty L^\infty},\tag{6.18}
$$

$$
\left\|\mathrm{e}^{K_0\eta}T_2\right\|_{L^\infty L^\infty}\leqq C\left\|F(\eta)\cos\phi\frac{\partial\mathscr{U}_0}{\partial\phi}\right\|_{L^\infty L^\infty}\leqq C\left|\ln(\epsilon)\right|^8.\tag{6.19}
$$

Therefore, we have

$$
\begin{aligned}
\left\|\mathrm{e}^{K_0\eta}S_3\right\|_{L^\infty L^\infty}\leqq\left\|\mathrm{e}^{K_0\eta}Z\right\|_{L^\infty L^\infty}&\leqq C\left|\ln(\epsilon)\right|^8\\
&\quad+C\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial W}{\partial\phi}\right\|_{L^\infty L^\infty}.
\end{aligned}\tag{6.20}
$$

In total, we have

$$
\begin{aligned}
&\left\|e^{K_0\eta}S_1\right\|_{L^\infty L^\infty}+\left\|e^{K_0\eta}S_2\right\|_{L^\infty L^\infty}+\left\|e^{K_0\eta}S_3\right\|_{L^\infty L^\infty}\\
&\quad\leqq C\left|\ln(\epsilon)\right|^8+C\bigg(\left\|e^{K_0\eta}\zeta\frac{\partial W}{\partial\eta}\right\|_{L^\infty L^\infty}\\
&\qquad+\left\|e^{K_0\eta}F(\eta)\cos\phi\frac{\partial W}{\partial\phi}\right\|_{L^\infty L^\infty}\bigg).
\end{aligned}
\tag{6.21}
$$

Hence, we need the regularity estimate of W. However, this cannot be done directly. We will first study the normal derivative of $\mathscr{U}_0$.

Step 6: Regularity of Normal Derivative.

The normal derivative $A=\dfrac{\partial\mathscr{U}_0}{\partial\eta}$ satisfies

$$
\begin{cases}
\sin\phi\dfrac{\partial A}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial A}{\partial\phi}+A-\bar{A}=\dfrac{\epsilon}{R-\epsilon\eta}F(\eta)\cos\phi\dfrac{\partial\mathscr{U}_0}{\partial\phi},\\
A(0,\tau,\phi)=\dfrac{1}{\sin\phi}\bigg(F(\eta)\cos\phi\dfrac{\partial\mathscr{G}}{\partial\phi}(\tau,\phi)-\mathscr{G}(0,\tau,\phi)+\bar{\mathscr{U}}_0(0,\tau,\phi)\bigg)\ \text{for}\ \ \sin\phi>0,\\
A(L,\tau,\phi)=A(L,\tau,\mathscr{R}[\phi]),
\end{cases}
\tag{6.22}
$$

This is where the cut-off in $\mathscr{G}$ plays a role. Based on the construction of $\mathscr{G}$ and using $|F(\eta)|\leqq C\epsilon$, we know $\|A(0,\phi,\tau)\|_{L^\infty_-}\leqq C\epsilon^{-\alpha}$ and $\left\|(\epsilon+\zeta)\dfrac{\partial A}{\partial\phi}(0,\phi,\tau)\right\|_{L^\infty_-}\leqq C\epsilon^{-\alpha}$. Therefore, using Theorem 4.9, we have

$$
\begin{aligned}
\left\|e^{K_0\eta}A\right\|_{L^\infty L^\infty}&\leqq C\bigg(\|A(0,\phi,\tau)\|_{L^\infty_-}+\left\|e^{K_0\eta}F(\eta)\cos\phi\frac{\partial\mathscr{U}_0}{\partial\phi}\right\|_{L^\infty L^\infty}\bigg)\\
&\leqq C\epsilon^{-\alpha}.
\end{aligned}
\tag{6.23}
$$

By Theorem 5.3, we know that

$$
\begin{aligned}
&\left\|e^{K_0\eta}\zeta\frac{\partial A}{\partial\eta}\right\|_{L^\infty L^\infty}+\left\|e^{K_0\eta}F(\eta)\cos\phi\frac{\partial A}{\partial\phi}\right\|_{L^\infty L^\infty}\\
&\leqq C\left|\ln(\epsilon)\right|^8\bigg(\epsilon^{-\alpha}+\left\|e^{K_0\eta}\frac{\epsilon}{R-\epsilon\eta}F(\eta)\cos\phi\frac{\partial\mathscr{U}_0}{\partial\phi}\right\|_{L^\infty L^\infty}\\
&\quad+\left\|e^{K_0\eta}\zeta\frac{\partial}{\partial\eta}\bigg(\frac{\epsilon}{R-\epsilon\eta}F(\eta)\cos\phi\frac{\partial\mathscr{U}_0}{\partial\phi}\bigg)\right\|_{L^\infty L^\infty}\bigg)\\
&\leqq C\left|\ln(\epsilon)\right|^8\bigg(\epsilon^{-\alpha}+\epsilon\left\|e^{K_0\eta}F(\eta)\cos\phi\frac{\partial\mathscr{U}_0}{\partial\phi}\right\|_{L^\infty L^\infty}\\
&\quad+\epsilon\left\|e^{K_0\eta}F(\eta)\cos\phi\frac{\partial A}{\partial\phi}\right\|_{L^\infty L^\infty}\bigg)
\end{aligned}
\tag{6.24}
$$

Then we may absorb $\left\|e^{K_0\eta}F(\eta)\cos\phi\dfrac{\partial A}{\partial\phi}\right\|_{L^\infty L^\infty}$ into the left-hand side to obtain

$$\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial A}{\partial\eta}\right\|_{L^\infty L^\infty}+\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial A}{\partial\phi}\right\|_{L^\infty L^\infty}\leqq C\epsilon^{-\alpha}\left|\ln(\epsilon)\right|^8. \tag{6.25}$$

Step 7: Regularity of Tangential Derivative.
We turn to the regularity of W. Based on Theorem 5.3, we have

$$\begin{aligned}
&\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial W}{\partial\eta}\right\|_{L^\infty L^\infty}+\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial W}{\partial\phi}\right\|_{L^\infty L^\infty}\\
&\leqq C\left|\ln(\epsilon)\right|^8\left(1+\left\|\mathrm{e}^{K_0\eta}\frac{R_\kappa'}{R_\kappa-\epsilon\eta}F(\eta)\cos\phi\frac{\partial\mathscr{U}_0}{\partial\phi}\right\|_{L^\infty L^\infty}\right.\\
&\quad\left.+\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial}{\partial\eta}\left(\frac{R_\kappa'}{R_\kappa-\epsilon\eta}F(\eta)\cos\phi\frac{\partial\mathscr{U}_0}{\partial\phi}\right)\right\|_{L^\infty L^\infty}\right)\\
&\leqq C\left|\ln(\epsilon)\right|^8\left(1+\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial\mathscr{U}_0}{\partial\phi}\right\|_{L^\infty L^\infty}\right.\\
&\quad\left.+\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial A}{\partial\phi}\right\|_{L^\infty L^\infty}\right)\\
&\leqq C\epsilon^{-\alpha}\left|\ln(\epsilon)\right|^{16}.
\end{aligned} \tag{6.26}$$

Step 8: Synthesis.
Using above estimates, we actually have shown that

$$\left\|\mathrm{e}^{K_0\eta}V\right\|_{L^\infty L^\infty}\leqq C\epsilon^{-\alpha}\left|\ln(\epsilon)\right|^{16}. \tag{6.27}$$

Theorem 6.1. *For $K_0>0$ sufficiently small, the regular boundary layer satisfies*

$$\begin{aligned}
&\left\|\mathrm{e}^{K_0\eta}\mathscr{U}_0\right\|_{L^\infty L^\infty}\leqq C, &&\left\|\mathrm{e}^{K_0\eta}\mathscr{U}_1\right\|_{L^\infty L^\infty}\leqq C\left|\ln(\epsilon)\right|^8,\\
&\left\|\mathrm{e}^{K_0\eta}\frac{\partial\mathscr{U}_0}{\partial\tau}\right\|_{L^\infty L^\infty}\leqq C\left|\ln(\epsilon)\right|^8, &&\left\|\mathrm{e}^{K_0\eta}\frac{\partial\mathscr{U}_1}{\partial\tau}\right\|_{L^\infty L^\infty}\leqq C\epsilon^{-\alpha}\left|\ln(\epsilon)\right|^{16}.
\end{aligned} \tag{6.28}$$

6.2. Analysis of Singular Boundary Layer

In this subsection, we will justify that the singular boundary layers are all well-defined.
Step 1: Well-Posedness of $\mathfrak{U}_0$.
$\mathfrak{U}_0$ satisfies the ϵ-Milne problem with geometric correction

$$\begin{cases}
\sin\phi\dfrac{\partial\mathfrak{U}_0}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial\mathfrak{U}_0}{\partial\phi}+\mathfrak{U}_0-\bar{\mathfrak{U}}_0=0,\\
\mathfrak{U}_0(0,\tau,\phi)=\mathfrak{G}(\tau,\phi)-\mathfrak{F}_{0,L}(\tau)\ \text{ for }\ \sin\phi>0,\\
\mathfrak{U}_0(L,\tau,\phi)=\mathfrak{U}_0(L,\tau,\mathscr{R}[\phi]).
\end{cases} \tag{6.29}$$

Therefore, by Theorem 4.9, we know

$$\left\|\mathrm{e}^{K_0\eta}\mathfrak{U}_0\right\|_{L^\infty L^\infty}\leqq C. \tag{6.30}$$

However, this is not sufficient for future use and we need more detailed analysis. We will divide the domain $(\eta,\phi)\in[0,L]\times[-\pi,\pi)$ into two regions:

- Region I χ_1: $0 \leqq \zeta < 2\epsilon^{\alpha}$.
- Region II χ_2: $2\epsilon^{\alpha} \leqq \zeta \leqq 1$.

Here we use χ_i to represent either the corresponding region or the indicator function. It is easy to see that $\mathfrak{G} = 0$ in Region II. Similarly we decompose the solution $\mathfrak{U}_0 = \chi_1\mathfrak{U}_0 + \chi_2\mathfrak{U}_0 = f_1 + f_2$ in these two regions. In the following, the estimates for f_i will be restricted to the region χ_i for $i = 1, 2$. Using Theorem 4.3, we can easily show that

$$\left\|\mathrm{e}^{K_0\eta}\mathfrak{U}_0\right\|_{L^2L^2} \leqq C\epsilon^{\alpha}. \tag{6.31}$$

The key to L^{∞} estimates in Theorem 4.10 is Lemma 4.6 and Lemma 4.7. Their proofs are basically tracking along the characteristics. Hence, we know that

$$\begin{aligned}\left\|\mathrm{e}^{K_0\eta}\bar{\mathfrak{U}}_0\right\|_{L^\infty L^\infty} &\leqq C\bigg(\epsilon^{\alpha}\left\|\mathrm{e}^{K_0\eta}f_1\right\|_{L^\infty L^2} + \left\|\mathrm{e}^{K_0\eta}f_2\right\|_{L^\infty L^2}\bigg)\\ &\leqq C\bigg(\left\|\mathrm{e}^{K_0\eta}\mathfrak{U}_0\right\|_{L^2L^2} + \delta\epsilon^{\alpha}\left\|\mathrm{e}^{K_0\eta}f_1\right\|_{L^\infty L^\infty}\\ &\qquad +\delta\left\|\mathrm{e}^{K_0\eta}f_2\right\|_{L^\infty L^\infty}\bigg).\end{aligned} \tag{6.32}$$

Thus, considering $\chi_1\mathfrak{G} = \mathfrak{G}$ and $\chi_2\mathfrak{G} = 0$, we may directly obtain

$$\begin{aligned}\left\|\mathrm{e}^{K_0\eta}f_1\right\|_{L^\infty L^\infty} &\leqq C\bigg(\left\|\chi_1\mathfrak{G}\right\|_{L^\infty} + \left\|\mathrm{e}^{K_0\eta}\bar{\mathfrak{U}}_0\right\|_{L^\infty L^\infty}\bigg)\\ &\leqq C\bigg(\left\|\chi_1\mathfrak{G}\right\|_{L^\infty} + \left\|\mathrm{e}^{K_0\eta}\mathfrak{U}_0\right\|_{L^2L^2}\\ &\qquad +\delta\epsilon^{\alpha}\left\|\mathrm{e}^{K_0\eta}f_1\right\|_{L^\infty L^\infty} + \delta\left\|\mathrm{e}^{K_0\eta}f_2\right\|_{L^\infty L^\infty}\bigg)\\ &\leqq C\bigg(1 + \delta\epsilon^{\alpha}\left\|\mathrm{e}^{K_0\eta}f_1\right\|_{L^\infty L^\infty} + \delta\left\|\mathrm{e}^{K_0\eta}f_2\right\|_{L^\infty L^\infty}\bigg),\end{aligned} \tag{6.33}$$

$$\begin{aligned}\left\|\mathrm{e}^{K_0\eta}f_2\right\|_{L^\infty L^\infty} &\leqq C\bigg(\left\|\chi_2\mathfrak{G}\right\|_{L^\infty} + \left\|\mathrm{e}^{K_0\eta}\bar{\mathfrak{U}}_0\right\|_{L^\infty L^\infty}\bigg)\\ &\leqq C\bigg(\left\|\chi_2\mathfrak{G}\right\|_{L^\infty} + \left\|\mathrm{e}^{K_0\eta}\mathfrak{U}_0\right\|_{L^2L^2} + \delta\epsilon^{\alpha}\left\|\mathrm{e}^{K_0\eta}f_1\right\|_{L^\infty L^\infty}\\ &\qquad +\delta\left\|\mathrm{e}^{K_0\eta}f_2\right\|_{L^\infty L^\infty}\bigg)\\ &\leqq C\bigg(\epsilon^{\alpha} + \delta\epsilon^{\alpha}\left\|\mathrm{e}^{K_0\eta}f_1\right\|_{L^\infty L^\infty} + \delta\left\|\mathrm{e}^{K_0\eta}f_2\right\|_{L^\infty L^\infty}\bigg).\end{aligned} \tag{6.34}$$

Letting δ small, absorbing $\left\|\mathrm{e}^{K_0\eta}f_1\right\|_{L^\infty L^\infty}$ and $\left\|\mathrm{e}^{K_0\eta}f_2\right\|_{L^\infty L^\infty}$, we know

$$\left\|\mathrm{e}^{K_0\eta} f_1\right\|_{L^\infty L^\infty} \leqq C\left(1+\delta\left\|\mathrm{e}^{K_0\eta} f_2\right\|_{L^\infty L^\infty}\right), \tag{6.35}$$

$$\left\|\mathrm{e}^{K_0\eta} f_2\right\|_{L^\infty L^\infty} \leqq C\left(\epsilon^\alpha+\delta\epsilon^\alpha\left\|\mathrm{e}^{K_0\eta} f_1\right\|_{L^\infty L^\infty}\right). \tag{6.36}$$

Combining them together, we can easily see that

$$\left\|\mathrm{e}^{K_0\eta} f_1\right\|_{L^\infty L^\infty} \leqq C, \tag{6.37}$$

$$\left\|\mathrm{e}^{K_0\eta} f_2\right\|_{L^\infty L^\infty} \leqq C\epsilon^\alpha. \tag{6.38}$$

In total, we can derive

$$\left\|\mathrm{e}^{K_0\eta}\bar{\mathfrak{U}}_0\right\|_{L^\infty L^\infty} \leqq C\epsilon^\alpha. \tag{6.39}$$

Step 2: Regularity of $\mathfrak{U}_0$.
This is very similar to the well-posedness proof, we will also consider the regularity of $\mathfrak{U}_0$ in two regions. Note that in the proof of Theorem 5.3, the L^∞ estimates relies on two kinds of quantities:

- $\left|\zeta\dfrac{\partial\mathfrak{U}_0}{\partial\eta}\right|$ on the same characteristics.
- $\displaystyle\int_{-\pi}^{\pi}\zeta\frac{\partial\mathfrak{U}_0}{\partial\eta}\,\mathrm{d}\phi$ for some $\eta>0$.

Correspondingly, we may handle them separately: for the first case, since ζ is preserved along the characteristics, we can directly separate the estimate of f_1 and f_2; for the second case, we may use the simple domain decomposition

$$\begin{aligned}\int_{-\pi}^{\pi}\zeta\frac{\partial\mathfrak{U}_0}{\partial\eta}(\eta,\phi)\,\mathrm{d}\phi &= \int_{\chi_1}\zeta\frac{\partial f_1}{\partial\eta}\,\mathrm{d}\phi+\int_{\chi_2}\zeta\frac{\partial f_2}{\partial\eta}\,\mathrm{d}\phi\\ &\leqq C\left(\epsilon^\alpha\left\|\zeta\frac{\partial f_1}{\partial\eta}\right\|_{L^\infty L^2}+\left\|\zeta\frac{\partial f_2}{\partial\eta}\right\|_{L^\infty L^2}\right).\end{aligned} \tag{6.40}$$

Then following a similar absorbing argument as in above well-posedness proof, we have

$$\begin{aligned}&\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial f_1}{\partial\eta}\right\|_{L^\infty L^\infty}+\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial f_1}{\partial\phi}\right\|_{L^\infty L^\infty}\\ &\quad\leqq C\left|\ln(\epsilon)\right|^8\left(\|\mathfrak{G}\|_{L^\infty}+\left\|(\epsilon+\zeta)\frac{\partial\mathfrak{G}}{\partial\phi}\right\|_{L^\infty}+\left\|\mathrm{e}^{K_0\eta}\mathfrak{U}_0\right\|_{L^\infty L^\infty}\right)\\ &\quad\leqq C\left|\ln(\epsilon)\right|^8,\end{aligned} \tag{6.41}$$

$$\begin{aligned}&\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial f_2}{\partial\eta}\right\|_{L^\infty L^\infty}+\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial f_2}{\partial\phi}\right\|_{L^\infty L^\infty}\\ &\quad\leqq C\left|\ln(\epsilon)\right|^8\left(\left\|\mathrm{e}^{K_0\eta}f_2\right\|_{L^\infty L^\infty}+\epsilon^\alpha\left\|\mathrm{e}^{K_0\eta}f_1\right\|_{L^\infty L^\infty}\right)\leqq C\epsilon^\alpha\left|\ln(\epsilon)\right|^8.\end{aligned} \tag{6.42}$$

Note that although $\left\|\dfrac{\partial\mathfrak{G}}{\partial\phi}\right\|_{L^\infty} \leqq C\epsilon^{-\alpha}$, with the help of $\epsilon+\zeta$, we can get rid of this negative power.

Step 3: Tangential Derivatives of $\mathfrak{U}_0$.

The τ derivative $P=\dfrac{\partial\mathfrak{U}_0}{\partial\tau}$ satisfies

$$\begin{cases} \sin\phi\dfrac{\partial P}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial P}{\partial\phi}+P-\bar{P}=-\dfrac{R_\kappa'}{R_\kappa-\epsilon\eta}F(\eta)\cos\phi\dfrac{\partial\mathfrak{U}_0}{\partial\phi}, \\ P(0,\tau,\phi)=\dfrac{\partial\mathfrak{G}}{\partial\tau}(\tau,\phi)-\dfrac{\partial\mathfrak{F}_{0,L}}{\partial\tau}(\tau) \ \text{ for } \ \sin\phi>0, \\ P(L,\tau,\phi)=P(L,\tau,\mathscr{R}[\phi]). \end{cases} \tag{6.43}$$

It is easy to check that

$$\int_{-\pi}^{\pi}\cos\phi\frac{\partial\mathfrak{U}_0}{\partial\phi}\,\mathrm{d}\phi=\int_{-\pi}^{\pi}\mathfrak{U}_0\sin\phi\,\mathrm{d}\phi=0, \tag{6.44}$$

due to the orthogonal property. Hence, using Theorem 4.3 with $S_Q=0$, we have

$$\left\|\mathrm{e}^{K_0\eta}P\right\|_{L^2L^2}\leqq C\epsilon^{\alpha}\left|\ln(\epsilon)\right|^8, \tag{6.45}$$

which further implies that

$$\begin{aligned} \left\|\mathrm{e}^{K_0\eta}P_1\right\|_{L^\infty L^\infty} &\leqq C\Bigg(\left\|\frac{\partial\mathfrak{G}}{\partial\tau}\right\|_{L^\infty L^\infty}+\left\|\mathrm{e}^{K_0\eta}P\right\|_{L^2L^2} \\ &\qquad +\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial\mathfrak{U}_0}{\partial\phi}\right\|_{L^\infty L^\infty}\Bigg) \\ &\leqq C\left|\ln(\epsilon)\right|^8, \end{aligned} \tag{6.46}$$

$$\begin{aligned} \left\|\mathrm{e}^{K_0\eta}P_2\right\|_{L^\infty L^\infty} &\leqq C\Bigg(\mathrm{e}^{K_0\eta}\|P\|_{L^2L^2}+\epsilon^{\alpha}\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial f_1}{\partial\phi}\right\|_{L^\infty L^\infty} \\ &\qquad +\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial f_2}{\partial\phi}\right\|_{L^\infty L^\infty}\Bigg) \\ &\leqq C\epsilon^{\alpha}\left|\ln(\epsilon)\right|^8, \end{aligned} \tag{6.47}$$

where $P_1=\dfrac{\partial f_1}{\partial\tau}$ and $P_2=\dfrac{\partial f_2}{\partial\tau}$.

Theorem 6.2. *Let*

$$\begin{cases} \chi_1: 0\leqq\zeta<2\epsilon^{\alpha}, \\ \chi_2: 2\epsilon^{\alpha}\leqq\zeta\leqq 1. \end{cases} \tag{6.48}$$

For $K_0>0$ sufficiently small, the singular boundary layer satisfies

$$\begin{aligned} &\left\|\mathrm{e}^{K_0\eta}(\chi_1\mathfrak{U}_0)\right\|_{L^\infty L^\infty}\leqq C, && \left\|\mathrm{e}^{K_0\eta}(\chi_2\mathfrak{U}_0)\right\|_{L^\infty L^\infty}\leqq C\epsilon^{\alpha}, \\ &\left\|\mathrm{e}^{K_0\eta}\frac{\partial(\chi_1\mathfrak{U}_0)}{\partial\tau}\right\|_{L^\infty L^\infty}\leqq C\left|\ln(\epsilon)\right|^8, && \left\|\mathrm{e}^{K_0\eta}\frac{\partial(\chi_2\mathfrak{U}_0)}{\partial\tau}\right\|_{L^\infty L^\infty}\leqq C\epsilon^{\alpha}\left|\ln(\epsilon)\right|^8. \end{aligned} \tag{6.49}$$

6.3. Analysis of Interior Solution

In this subsection, we will justify that the interior solutions are all well-defined.
Step 1: Well-Posedness of U_0.
U_0 satisfies an elliptic equation

$$\begin{cases} U_0(\vec{x},\vec{w}) = \bar{U}_0(\vec{x}), \\ \Delta_x \bar{U}_0(\vec{x}) = 0 \ \text{ in } \ \Omega, \\ \bar{U}_0(\vec{x}_0) = \mathscr{F}_{0,L}(\tau) + \mathfrak{F}_{0,L}(\tau) \ \text{ on } \ \partial\Omega. \end{cases} \tag{6.50}$$

Based on standard elliptic theory, we have

$$\|U_0\|_{H^3(\Omega)} \leqq C\bigg(\left\|\mathscr{F}_{0,L}\right\|_{H^{\frac{5}{2}}(\partial\Omega)} + \left\|\mathfrak{F}_{0,L}\right\|_{H^{\frac{5}{2}}(\partial\Omega)} \bigg) \leqq C. \tag{6.51}$$

Step 2: Well-Posedness of U_1.
U_1 satisfies an elliptic equation

$$\begin{cases} U_1(\vec{x},\vec{w}) = \bar{U}_1(\vec{x}) - \vec{w}\cdot\nabla_x U_0(\vec{x},\vec{w}), \\ \Delta_x \bar{U}_1(\vec{x}) = -\displaystyle\int_{\mathbb{S}^1} \Big(\vec{w}\cdot\nabla_x U_0(\vec{x},\vec{w})\Big)\mathrm{d}\vec{w} \ \text{ in } \ \Omega, \\ \bar{U}_1(\vec{x}_0) \quad = f_{1,L}(\tau) \ \text{ on } \ \partial\Omega. \end{cases} \tag{6.52}$$

Based on standard elliptic theory, we have

$$\|U_1\|_{H^3(\Omega)} \leqq C\bigg(\left\|\mathscr{F}_{1,L}\right\|_{H^{\frac{5}{2}}(\partial\Omega)} + \|U_0\|_{H^2(\Omega)} \bigg) \leqq C\left|\ln(\epsilon)\right|^8. \tag{6.53}$$

Step 3: Well-Posedness of U_2.
U_2 satisfies an elliptic equation

$$\begin{cases} U_2(\vec{x},\vec{w}) = \bar{U}_2(\vec{x}) - \vec{w}\cdot\nabla_x U_1(\vec{x},\vec{w}), \\ \Delta_x \bar{U}_2(\vec{x}) = -\displaystyle\int_{\mathbb{S}^1} \Big(\vec{w}\cdot\nabla_x U_1(\vec{x},\vec{w})\Big)\mathrm{d}\vec{w} \ \text{ in } \ \Omega, \\ \bar{U}_2(\vec{x}_0) \quad = 0 \ \text{ on } \ \partial\Omega. \end{cases} \tag{6.54}$$

Based on standard elliptic theory, we have

$$\|U_2\|_{H^3(\Omega)} \leqq C\bigg(\left\|\bar{U}_0\right\|_{H^3(\Omega)} + \left\|\bar{U}_1\right\|_{H^2(\Omega)} \bigg) \leqq C\left|\ln(\epsilon)\right|^8. \tag{6.55}$$

Theorem 6.3. *The interior solution satisfies*

$$\|U_0\|_{H^3(\Omega)} \leqq C, \quad \|U_1\|_{H^3(\Omega)} \leqq C\left|\ln(\epsilon)\right|^8, \quad \|U_2\|_{H^3(\Omega)} \leqq C\left|\ln(\epsilon)\right|^8. \tag{6.56}$$

6.4. Proof of Main Theorem

Theorem 6.4. *Assume $g(\vec{x}_0,\vec{w})\in C^4(\Gamma^-)$. Then for the steady neutron transport equation (1.1), there exists a unique solution $u^\epsilon(\vec{x},\vec{w})\in L^\infty(\Omega\times\mathbb{S}^1)$. Moreover, for any $0<\delta<<1$, the solution obeys the estimate*

$$\left\|u^\epsilon-U-\mathcal{U}\right\|_{L^\infty(\Omega\times\mathbb{S}^1)}\leqq C(\delta)\epsilon^{\frac{1}{2}-\delta}, \tag{6.57}$$

where $U(\vec{x})$ satisfies the Laplace equation with Dirichlet boundary condition

$$\begin{cases}\Delta_x U(\vec{x})=0 \ \ in \ \ \Omega,\\ U(\vec{x}_0)=D(\vec{x}_0) \ \ on \ \ \partial\Omega,\end{cases} \tag{6.58}$$

and $\mathcal{U}(\eta,\tau,\phi)$ satisfies the ϵ-Milne problem with geometric correction

$$\begin{cases}\sin\phi\dfrac{\partial\mathcal{U}}{\partial\eta}-\dfrac{\epsilon}{R_\kappa(\tau)-\epsilon\eta}\cos\phi\dfrac{\partial\mathcal{U}}{\partial\phi}+\mathcal{U}-\bar{\mathcal{U}}=0,\\ \mathcal{U}(0,\tau,\phi)=g(\tau,\phi)-D(\tau) \ \ for \ \ \sin\phi>0,\\ \mathcal{U}(L,\tau,\phi)=\mathcal{U}(L,\tau,\mathscr{R}[\phi]),\end{cases} \tag{6.59}$$

for $L=\epsilon^{-\frac{1}{2}}$, $\mathscr{R}[\phi]=-\phi$, η the rescaled normal variable, τ the tangential variable, and ϕ the velocity variable.

Proof. Based on Theorem 3.5, we know there exists a unique $u^\epsilon(\vec{x},\vec{w})\in L^\infty(\Omega\times\mathbb{S}^1)$, so we focus on the diffusive limit. We divide the proof into several steps:
Step 1: Remainder definitions.
We define the remainder as

$$R=u^\epsilon-\sum_{k=0}^{2}\epsilon^k U_k-\sum_{k=0}^{1}\epsilon^k\mathscr{U}_k-\mathfrak{U}_0=u^\epsilon-Q-\mathscr{Q}-\mathfrak{Q}, \tag{6.60}$$

where

$$Q=U_0+\epsilon U_1+\epsilon^2U_2, \tag{6.61}$$
$$\mathscr{Q}=\mathscr{U}_0+\epsilon\mathscr{U}_1, \tag{6.62}$$
$$\mathfrak{Q}=\mathfrak{U}_0. \tag{6.63}$$

Noting the equation (2.32) is equivalent to the equation (1.1), we write $\mathcal{L}$ to denote the neutron transport operator as follows:

$$\begin{aligned}\mathcal{L}[u]&=\epsilon\vec{w}\cdot\nabla_x u+u-\bar{u}\\ &=\sin\phi\frac{\partial u}{\partial\eta}-\frac{\epsilon}{R_\kappa-\epsilon\eta}\cos\phi\left(\frac{\partial u}{\partial\phi}+\frac{\partial u}{\partial\tau}\right)+u-\bar{u}.\end{aligned} \tag{6.64}$$

Step 2: Estimates of $\mathcal{L}[Q]$.
The interior contribution can be estimated as

$$\mathcal{L}[Q]=\epsilon\vec{w}\cdot\nabla_x Q+Q-\bar{Q}=\epsilon^3\vec{w}\cdot\nabla_x U_2. \tag{6.65}$$

By Theorem 6.3, we have

$$\|\mathcal{L}[Q]\|_{L^\infty(\Omega\times\mathbb{S}^1)} \leqq \left\|\epsilon^3\vec{w}\cdot\nabla_x U_2\right\|_{L^\infty(\Omega\times\mathbb{S}^1)} \leqq C\epsilon^3\left\|\nabla_x U_2\right\|_{L^\infty(\Omega\times\mathbb{S}^1)} \leqq C\epsilon^3\left|\ln(\epsilon)\right|^8. \tag{6.66}$$

This implies

$$\|\mathcal{L}[Q]\|_{L^2(\Omega\times\mathbb{S}^1)} \leqq C\epsilon^3\left|\ln(\epsilon)\right|^8, \tag{6.67}$$

$$\|\mathcal{L}[Q]\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathbb{S}^1)} \leqq C\epsilon^3\left|\ln(\epsilon)\right|^8, \tag{6.68}$$

$$\|\mathcal{L}[Q]\|_{L^\infty(\Omega\times\mathbb{S}^1)} \leqq C\epsilon^3\left|\ln(\epsilon)\right|^8. \tag{6.69}$$

Step 3: Estimates of $\mathcal{L}\mathscr{Q}$.
We need to estimate $\mathscr{U}_0+\epsilon\mathscr{U}_1$. The boundary layer contribution can be estimated as

$$\begin{aligned}\mathcal{L}[\mathscr{U}_0+\epsilon\mathscr{U}_1] &= \sin\phi\frac{\partial(\mathscr{U}_0+\epsilon\mathscr{U}_1)}{\partial\eta}\\ &\quad-\frac{\epsilon}{R_\kappa-\epsilon\eta}\cos\phi\left(\frac{\partial(\mathscr{U}_0+\epsilon\mathscr{U}_1)}{\partial\phi}+\frac{\partial(\mathscr{U}_0+\epsilon\mathscr{U}_1)}{\partial\tau}\right)\\ &\quad+(\mathscr{U}_0+\epsilon\mathscr{U}_1)-(\bar{\mathscr{U}}_0+\epsilon\bar{\mathscr{U}}_1)\\ &= -\epsilon^2\frac{1}{R_\kappa-\epsilon\eta}\cos\phi\frac{\partial\mathscr{U}_1}{\partial\tau}.\end{aligned} \tag{6.70}$$

By Theorem 6.1, we have

$$\left\|-\epsilon^2\frac{1}{R_\kappa-\epsilon\eta}\cos\phi\frac{\partial\mathscr{U}_1}{\partial\tau}\right\|_{L^\infty(\Omega\times\mathbb{S}^1)} \leqq C\epsilon^2\left\|\frac{\partial\mathscr{U}_1}{\partial\tau}\right\|_{L^\infty(\Omega\times\mathbb{S}^1)} \leqq C\epsilon^{2-\alpha}\left|\ln(\epsilon)\right|^8. \tag{6.71}$$

Also, the exponential decay of $\dfrac{\partial\mathscr{U}_1}{\partial\tau}$ and the rescaling of $\eta=\dfrac{\mu}{\epsilon}$ implies that

$$\begin{aligned}&\left\|-\epsilon^2\frac{1}{R_\kappa-\epsilon\eta}\cos\phi\frac{\partial\mathscr{U}_1}{\partial\tau}\right\|_{L^2(\Omega\times\mathbb{S}^1)} \leqq \epsilon^2\left\|\frac{\partial\mathscr{U}_1}{\partial\tau}\right\|_{L^2(\Omega\times\mathbb{S}^1)}\\ &\leqq \epsilon^2\left(\int_{-\pi}^{\pi}\int_0^{R_{\min}}(R_{\min}-\mu)\left\|\frac{\partial\mathscr{U}_1}{\partial\tau}(\mu,\tau)\right\|_{L^\infty}^2\mathrm{d}\mu\,\mathrm{d}\tau\right)^{\frac{1}{2}}\\ &\leqq \epsilon^{\frac{5}{2}}\left(\int_{-\pi}^{\pi}\int_0^{\frac{R_{\min}}{\epsilon}}(R_{\min}-\epsilon\eta)\left\|\frac{\partial\mathscr{U}_1}{\partial\tau}(\eta,\tau)\right\|_{L^\infty}^2\mathrm{d}\eta\,\mathrm{d}\tau\right)^{\frac{1}{2}}\\ &\leqq C\epsilon^{\frac{5}{2}-\alpha}\left|\ln(\epsilon)\right|^8\left(\int_{-\pi}^{\pi}\int_0^{\frac{R_{\min}}{\epsilon}}\mathrm{e}^{-2K_0\eta}\,\mathrm{d}\eta\,\mathrm{d}\tau\right)^{\frac{1}{2}}\\ &\leqq C\epsilon^{\frac{5}{2}-\alpha}\left|\ln(\epsilon)\right|^8.\end{aligned} \tag{6.72}$$

Similarly, we have

$$\left\| -\epsilon^2 \frac{1}{R_\kappa - \epsilon\eta} \cos\phi \frac{\partial \mathscr{U}_1}{\partial \tau} \right\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathbb{S}^1)} \leqq C\epsilon^{3-\frac{1}{2m}-\alpha} \left|\ln(\epsilon)\right|^8. \tag{6.73}$$

In total, we have

$$\|\mathcal{L}[\mathscr{Q}]\|_{L^2(\Omega\times\mathbb{S}^1)} \leqq C\epsilon^{\frac{5}{2}-\alpha} \left|\ln(\epsilon)\right|^8, \tag{6.74}$$

$$\|\mathcal{L}[\mathscr{Q}]\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathbb{S}^1)} \leqq C\epsilon^{3-\frac{1}{2m}-\alpha} \left|\ln(\epsilon)\right|^8, \tag{6.75}$$

$$\|\mathcal{L}[\mathscr{Q}]\|_{L^\infty(\Omega\times\mathbb{S}^1)} \leqq C\epsilon^{2-\alpha} \left|\ln(\epsilon)\right|^8. \tag{6.76}$$

Step 4: Estimates of $\mathcal{L}\mathfrak{Q}$.
We need to estimate $\mathfrak{U}_0$. The boundary layer contribution can be estimated as

$$\begin{aligned}\mathcal{L}[\mathfrak{U}_0] &= \sin\phi \frac{\partial \mathfrak{U}_0}{\partial \eta} - \frac{\epsilon}{R_\kappa - \epsilon\eta} \cos\phi \left(\frac{\partial \mathfrak{U}_0}{\partial \phi} + \frac{\partial \mathfrak{U}_0}{\partial \tau} \right) + \mathfrak{U}_0 - \bar{\mathfrak{U}}_0 \\ &= -\epsilon \frac{1}{R_\kappa - \epsilon\eta} \cos\phi \frac{\partial \mathfrak{U}_0}{\partial \tau}.\end{aligned} \tag{6.77}$$

By Theorem 6.2, we have

$$\begin{aligned}\left\| -\epsilon \frac{1}{R_\kappa - \epsilon\eta} \cos\phi \frac{\partial \mathfrak{U}_0}{\partial \tau} \right\|_{L^\infty(\Omega\times\mathbb{S}^1)} &\leqq C\epsilon \left\| \frac{\partial \mathfrak{U}_0}{\partial \tau} \right\|_{L^\infty(\Omega\times\mathbb{S}^1)} \\ &\leqq C\epsilon \left|\ln(\epsilon)\right|^8.\end{aligned} \tag{6.78}$$

Also, the exponential decay of $\dfrac{\partial \mathfrak{U}_0}{\partial \tau}$ and the rescaling $\eta = \dfrac{\mu}{\epsilon}$ implies

$$\begin{aligned}&\left\| -\epsilon \frac{1}{R_\kappa - \epsilon\eta} \cos\phi \frac{\partial \mathfrak{U}_0}{\partial \tau} \right\|_{L^2(\Omega\times\mathbb{S}^1)} \leqq \epsilon \left\| \frac{\partial \mathfrak{U}_0}{\partial \tau} \right\|_{L^2(\Omega\times\mathbb{S}^1)} \\ &\leqq \epsilon \left(\int_{-\pi}^{\pi} \int_0^{R_{\min}} \int_{-\pi}^{\pi} \chi_1(R_{\min} - \mu) \left\| \frac{\partial P_1}{\partial \tau}(\mu, \tau) \right\|_{L^\infty}^2 \mathrm{d}\phi\, \mathrm{d}\mu\, \mathrm{d}\tau \right)^{\frac{1}{2}} \\ &\quad + \epsilon \left(\int_{-\pi}^{\pi} \int_0^{R_{\min}} \int_{-\pi}^{\pi} \chi_2(R_{\min} - \mu) \left\| \frac{\partial P_2}{\partial \tau} \right\|_{L^\infty}^2 \mathrm{d}\phi\, \mathrm{d}\mu\, \mathrm{d}\tau \right)^{\frac{1}{2}} \\ &\leqq \epsilon^{\frac{3}{2}} \left(\int_{-\pi}^{\pi} \int_0^{\frac{R_{\min}}{\epsilon}} \int_{-\pi}^{\pi} \chi_1(R_{\min} - \epsilon\eta) \left\| \frac{\partial P_1}{\partial \tau}(\eta, \tau) \right\|_{L^\infty}^2 \mathrm{d}\phi\, \mathrm{d}\eta\, \mathrm{d}\tau \right)^{\frac{1}{2}} \\ &\quad + \epsilon^{\frac{3}{2}} \left(\int_{-\pi}^{\pi} \int_0^{\frac{R_{\min}}{\epsilon}} \int_{-\pi}^{\pi} \chi_2(R_{\min} - \epsilon\eta) \left\| \frac{\partial P_2}{\partial \tau}(\eta, \tau) \right\|_{L^\infty}^2 \mathrm{d}\phi\, \mathrm{d}\eta\, \mathrm{d}\tau \right)^{\frac{1}{2}} \\ &\leqq C\left(\epsilon^{1+\frac{3}{2}\alpha} + \epsilon^{\frac{3}{2}+\alpha} \right) \left|\ln(\epsilon)\right|^8 \left(\int_{-\pi}^{\pi} \int_0^{\frac{R_{\min}}{\epsilon}} \mathrm{e}^{-2K_0\eta}\, \mathrm{d}\eta\, \mathrm{d}\tau \right)^{\frac{1}{2}} \\ &\leqq C\epsilon^{1+\frac{3}{2}\alpha} \left|\ln(\epsilon)\right|^8.\end{aligned} \tag{6.79}$$

Here the smallness of χ_1 quantity comes from the small domain $|\phi| \leqq \epsilon^{\alpha}$ and $|\eta| \leqq \epsilon^{2\alpha-1}$. The smallness of χ_2 quantity comes from the extra ϵ^{α} for $0 < \alpha < 1$. Similarly, we have

$$\left\| -\epsilon \frac{1}{R_\kappa - \epsilon\eta} \cos\phi \frac{\partial \mathfrak{U}_0}{\partial \tau} \right\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathbb{S}^1)} \leqq C\epsilon^{2-\frac{1}{2m}+\alpha} |\ln(\epsilon)|^8 . \tag{6.80}$$

In total, we have

$$\|\mathcal{L}[\mathscr{Q}]\|_{L^2(\Omega\times\mathbb{S}^1)} \leqq C\epsilon^{1+\frac{3}{2}\alpha} |\ln(\epsilon)|^8 , \tag{6.81}$$

$$\|\mathcal{L}[\mathscr{Q}]\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathbb{S}^1)} \leqq C\epsilon^{2-\frac{1}{2m}+\alpha} |\ln(\epsilon)|^8 , \tag{6.82}$$

$$\|\mathcal{L}[\mathscr{Q}]\|_{L^\infty(\Omega\times\mathbb{S}^1)} \leqq C\epsilon |\ln(\epsilon)|^8 . \tag{6.83}$$

Step 5: Source Term and Boundary Condition.
In summary, since $\mathcal{L}[u^\epsilon] = 0$, collecting estimates in Step 2 to Step 4, we can prove

$$\|\mathcal{L}[R]\|_{L^2(\Omega\times\mathbb{S}^1)} \leqq C\left(\epsilon^{\frac{5}{2}-\alpha} + \epsilon^{1+\frac{3}{2}\alpha}\right) |\ln(\epsilon)|^8 , \tag{6.84}$$

$$\|\mathcal{L}[R]\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathbb{S}^1)} \leqq C\left(\epsilon^{3-\frac{1}{2m}-\alpha} + \epsilon^{2-\frac{1}{2m}+\alpha}\right) |\ln(\epsilon)|^8 , \tag{6.85}$$

$$\|\mathcal{L}[R]\|_{L^\infty(\Omega\times\mathbb{S}^1)} \leqq C\left(\epsilon^{2-\alpha} + \epsilon\right) |\ln(\epsilon)|^8 . \tag{6.86}$$

We can directly obtain that the boundary data is satisfied up to $O(\epsilon)$, so we know that

$$\|R\|_{L^2(\Gamma^-)} \leqq C\epsilon^2, \tag{6.87}$$

$$\|R\|_{L^m(\Gamma^-)} \leqq C\epsilon^2, \tag{6.88}$$

$$\|R\|_{L^\infty(\Gamma^-)} \leqq C\epsilon^2 \tag{6.89}$$

Step 6: Diffusive Limit.
Hence, the remainder R satisfies the equation

$$\begin{cases} \epsilon\vec{w}\cdot\nabla_x R + R - \bar{R} = \mathcal{L}[R] \ \text{in} \ \Omega\times\mathbb{S}^1, \\ R = R \ \text{for} \ \vec{w}\cdot\vec{\nu} < 0 \ \text{and} \ \vec{x}_0 \in \partial\Omega. \end{cases} \tag{6.90}$$

By Theorem 3.5, we have, for m sufficiently large, that

$$\begin{aligned} \|R\|_{L^\infty(\Omega\times\mathbb{S}^1)} \leqq\ & C\Bigg(\frac{1}{\epsilon^{1+\frac{1}{m}}} \|\mathcal{L}[R]\|_{L^2(\Omega\times\mathbb{S}^1)} + \frac{1}{\epsilon^{2+\frac{1}{m}}} \|\mathcal{L}[R]\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathbb{S}^1)} \\ & + \|\mathcal{L}[R]\|_{L^\infty(\Omega\times\mathbb{S}^1)} \\ & + \frac{1}{\epsilon^{\frac{1}{2}+\frac{1}{m}}} \|R\|_{L^2(\Gamma^-)} + \frac{1}{\epsilon^{\frac{1}{m}}} \|R\|_{L^m(\Gamma^-)} + \|R\|_{L^\infty(\Gamma^-)} \Bigg), \\ \leqq\ & C\Bigg(\frac{1}{\epsilon^{1+\frac{1}{m}}} \left(\epsilon^{\frac{5}{2}-\alpha} + \epsilon^{1+\frac{3}{2}\alpha}\right) |\ln(\epsilon)|^8 \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\epsilon^{2+\frac{1}{m}}}\left(\epsilon^{3-\frac{1}{2m}-\alpha}+\epsilon^{2-\frac{1}{2m}+\alpha}\right)|\ln(\epsilon)|^8+(\epsilon)\,|\ln(\epsilon)|^8\\
&+\frac{1}{\epsilon^{\frac{1}{2}+\frac{1}{m}}}(\epsilon^2)+\frac{1}{\epsilon^{\frac{1}{m}}}(\epsilon^2)+(\epsilon^2)\Bigg)\\
\leqq{}& C\left(\epsilon^{1-\frac{3}{2m}-\alpha}+\epsilon^{\alpha-\frac{3}{2m}}\right)|\ln(\epsilon)|^8. \qquad (6.91)
\end{aligned}
$$

Here, we need

$$
1-\frac{3}{2m}-\alpha>0, \quad \alpha-\frac{3}{2m}>0, \tag{6.92}
$$

which means that

$$
\frac{3}{2m}<\alpha<1-\frac{3}{2m}. \tag{6.93}
$$

For $m>3$, this is always achievable. Also, we know that

$$
\min_{\alpha}\left\{\epsilon^{1-\frac{3}{2m}-\alpha}+\epsilon^{\alpha-\frac{3}{2m}}\right\}=2\epsilon^{\frac{1}{2}}. \tag{6.94}
$$

Since it is easy to see that

$$
\left\|\sum_{k=1}^{2}\epsilon^k U_k+\sum_{k=1}^{1}\epsilon^k \mathscr{U}_k\right\|_{L^\infty(\Omega\times\mathbb{S}^1)}\leqq C\epsilon, \tag{6.95}
$$

our result naturally follows. We simply take $U=U_0$ and $\mathcal{U}=\mathscr{U}_0+\mathfrak{U}_0$. It is obvious that $\mathcal{U}$ satisfies the ϵ-Milne problem with geometric correction with the full boundary data $g(\phi,\tau)-\mathscr{F}_{0,L}(\tau)-\mathfrak{F}_{0,L}(\tau)$. This completes the proof of main theorem. □

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