QUANTUM COHOMOLOGY AND TORIC MINIMAL MODEL PROGRAMS

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Abstract. We give a quantum version of the Danilov-Jurkiewicz presentation of the cohomology of a compact toric orbifold with projective coarse moduli space. More precisely, we construct a canonical isomorphism from a formal version of the Batyrev ring from [4] to the quantum orbifold cohomology at a canonical bulk deformation. This isomorphism generalizes results of Givental [23], Iritani [34] and Fukaya-Oh-Ohta-Ono [21] for toric manifolds and Coates-Lee-Corti-Tseng [11] for weighted projective spaces. The proof uses a quantum version of Kirwan surjectivity (Theorem 2.6 below) and an equality of dimensions (Theorem 4.19 below) deduced using a toric minimal model program (tmmp). We show that there is a natural decomposition of the quantum cohomology where summands correspond to singularities in the tmmp, each of which gives rise to a collection of Hamiltonian non-displaceable Lagrangian tori.

Contents

1. Introduction

According to results of Danilov and Jurkiewicz [16, 35, 36], the rational cohomology ring of a complete rationally-smooth toric variety is the quotient of a polynomial ring generated by prime invariant divisors by the Stanley-Reisner ideal. In addition to relations corresponding to linear equivalence of invariant divisors, there are higher degree relations corresponding to collections of divisors whose intersection is empty.

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One can reformulate this presentation of the cohomology ring in terms of equivariant cohomology as follows. Let G be a complex reductive group acting on a smooth polarized projective variety X. If the action on the semistable locus X^{ss} is locally free then the geometric invariant theory (git) quotient $X/\!\!/G = X^{\mathrm{ss}}/G$, by which we mean the stack-theoretic quotient of the semistable locus by the group action, is a smooth proper Deligne-Mumford stack with projective coarse moduli space. A result of Kirwan [38] says that the natural map $H_G(X, \mathbb{Q}) \to H(X/\!\!/G, \mathbb{Q})$ is surjective. Under suitable properness assumptions the same holds for quasi-projective X.

In particular, let G be a torus acting on a finite-dimensional vector space X with weights contained in an open half-space. The quotient $X/\!\!/G$ is a smooth proper Deligne-Mumford toric stack as in Borisov-Chen-Smith [7] and any such toric stack with projective coarse moduli space arises in this way. The equivariant cohomology $H_G(X)$ may be identified with the ring of polynomial functions on $\mathfrak{g}$ and each weight maps to a divisor class in $H(X/\!\!/G)$ under the Kirwan map. The Stanley-Reisner ideal SR_X^G is precisely the kernel of the Kirwan map. For example, if $G =^{\times}$ acts by scalar multiplication on $X =^k$, then $H_G(X) = \mathbb{Q}[\xi]$ is a polynomial ring in a single generator ξ, the git quotient is $X/\!\!/G = \mathbb{P}^{k-1}$, and the intersection of the k prime invariant divisors is empty. The Stanley-Reisner ideal is the ideal $\langle \xi^k \rangle$ generated by ξ^k. This gives the standard description of the cohomology ring of projective space $H(\mathbb{P}^{k-1}) = H_G(X)/SR_X^G = \mathbb{Q}[\xi]/\langle \xi^k \rangle$.

In this paper we give a similar presentation of the quantum cohomology of compact toric orbifolds with projective coarse moduli spaces, via the quantum version of the Kirwan map introduced in [50, 51, 52]. The results here generalize those of Batyrev [4], Givental [23], Iritani [32, 33, 34], and Fukaya-Oh-Ohta-Ono [21], who use results of McDuff-Tolman [43]. In particular, Iritani [34] computed the quantum cohomology of toric manifolds using localization arguments for toric varieties that appear as certain complete intersections, while Fukaya et al [21] gave a computation using open-closed Gromov-Witten invariants defined via Kuranishi structures. The orbifold quantum cohomology of weighted projective spaces is computed in Coates-Lee-Corti-Tseng [11]. After the first version of this manuscript appeared a mirror theorem for toric stacks was proved by Coates, Corti, Iritani, and Tseng [12] and applied to give a Batyrev-style presentation in [13, Theorem 5.13].

A novel feature of the approach here is the appearance of minimal model programs, which are used to prove injectivity of the quantum Kirwan map modulo the quantum Stanley-Reisner ideal. The critical values of the Givental-Hori-Vafa potential acquire a natural geometric meaning in our approach: their logarithms are the transition times in the minimal model program, see Theorem 5.5 below, and the dimension of the orbifold cohomology and the logarithm of the lowest eigenvalue of quantum multiplication by the first Chern class decrease under each transition. We also obtain a more conceptual understanding of the appearance of open families of non-displaceable Lagrangians in toric orbifolds, as a consequence of the existence of infinitely many minimal model programs, see Remark 5.3.

We introduce the following notations.

Notation 1.1. (a) (Novikov coefficients) Let Λ denote the *universal Novikov field* of formal power series of q with rational exponents

$$\Lambda = \left\{ \sum_{\rho} c_\rho q^\rho \;\middle|\; \begin{array}{l} c_\rho \in, \rho \in \mathbb{Q} \\ \forall e > 0, \#\{\rho | c_\rho < e\} < \infty \end{array} \right\}.$$

We denote by $\Lambda_0 \subset \Lambda$ the subring with only non-negative powers of q.

(b) (Equivariant quantum cohomology) Let

$$QH_G(X) := H_G(X,) \otimes_\Lambda$$

denote the (ungraded) equivariant quantum cohomology of X. We denote by $QH_G(X, \mathbb{Q}) := H_G(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \Lambda$ the subspace with rational coefficients. Equivariant enumeration of stable maps to X defines a family of products

$$\star_\alpha : T_\alpha QH_G(X, \mathbb{Q})^2 \to T_\alpha QH_G(X, \mathbb{Q})$$

forming (part of) the structure of a *Frobenius manifold* on $QH_G(X, \mathbb{Q})$ [23] for α in a formal neighborhood of a symplectic class $\omega \in H_2^G(X, \mathbb{Q})$. Explicitly the product $\beta \star_{\alpha+\omega} \gamma$ is defined by

$$\langle \beta \star_{\alpha+\omega} \gamma, \delta \rangle = \sum_{d \in H_2(X, \mathbb{Z}), n \geq 0} \frac{q^{\langle d, \omega \rangle}}{n!} \int_{[\overline{\mathcal{M}}_{0,n+3}(X,d)_G]} \mathrm{ev}^*(\alpha, \ldots, \alpha, \beta, \gamma, \delta^\vee) \tag{1}$$

where the integral denotes push-forward to BG using the equivariant virtual fundamental class described in [28].

(c) (Inertia stacks) The *inertia stack* of $X /\!\!/ G$ is

$$I_{X/\!\!/G} = \bigcup_{r>0} \mathrm{Hom}^{\mathrm{rep}}(\mathbb{P}(r), X/\!\!/G) = \bigcup_{[g]} X^{g,\mathrm{ss}}/Z_g.$$

In the first union, $\mathrm{Hom}^{\mathrm{rep}}(\mathbb{P}(r), \cdot)$ denotes representable morphisms from $\mathbb{P}(r) = B\mathbb{Z}_r$ and the second union is over conjugacy classes $[g]$ of elements $g \in G$, with $Z_g \subset G$ the centralizer of g and $X^{g,\mathrm{ss}}$ the intersection of the semistable locus X^{ss} with the fixed point set

$$X^g := \{x \in X \mid gx = x\}.$$

The *rigidified inertia stack* is

$$\overline{I}_{X/\!\!/G} = \bigcup_{r>0} \mathrm{Hom}^{\mathrm{rep}}(\mathbb{P}(r), X/G)/\mathbb{P}(r) = \bigcup_{[g]} X^{g,\mathrm{ss}}/(Z_g/\langle g \rangle)$$

where $\langle g \rangle$ denotes the subgroup generated by g, as in Abramovich-Graber-Vistoli [1], Chen-Ruan [10].

(d) (Orbifold quantum cohomology of a git quotient) Let

$$QH(X/\!\!/G) := H(I_{X/\!\!/G},) \otimes \Lambda$$

denote the orbifold quantum cohomology of $X/\!\!/G$, or $QH(X/\!\!/G, \mathbb{Q})$ the version with rational coefficients. Enumeration of twisted stable maps to $X/\!\!/G$ (representable maps from orbifold curves to $X/\!\!/G$) defines a Frobenius manifold structure on $QH(X/\!\!/G)$ [1], [10] given by a family of products

$$\star_\alpha : T_\alpha QH(X/\!\!/G, \mathbb{Q})^2 \to T_\alpha QH(X/\!\!/G, \mathbb{Q}).$$

These products are defined in a formal neighborhood of an equivariant symplectic class $\omega \in H^2(X/\!\!/G, \mathbb{Q})$ by

$$\langle \beta \star_{\omega+\alpha} \gamma, \delta \rangle := \sum_{\substack{d \in H_2(X/\!\!/G,\mathbb{Q}) \\ n \geq 0}} \frac{q^{\langle d,\omega\rangle}}{n!} \int_{[\overline{\mathcal{M}}_{0,n+3}(X/\!\!/G,d)]} \mathrm{ev}^*(\alpha, \ldots, \alpha, \beta, \gamma, \delta^\vee) \tag{2}$$

for $\alpha, \beta, \gamma \in H(I_{X/\!\!/G})$, extended by linearity over Λ. The pairing on the left-hand-side is a certain re-scaled Poincaré pairing on the inertia stack $I_{X/\!\!/G}$, see [1].

Example 1.2. To connect with the notation in [1], [10] (where one works with different Novikov fields) consider the following examples.

(a) (Stacky half-point) Let $G =^\times$ act on $X =$ with weight two so that $X/\!\!/G = \mathbb{P}(2)$. The inertia stack $I_{X/\!\!/G}$ is the union of two copies of $\mathbb{P}(2)$ corresponding to the elements ± 1 of $\mathbb{Z}_2$. Thus

$$QH(X/\!\!/G) = \Lambda \oplus \Lambda\theta_-$$

the sum of two copies of Λ, where θ_- is the additive generator of the twisted sector. Representable morphisms from a stacky curve C to $X/\!\!/G = \mathbb{P}(2)$ correspond to double covers of the coarse moduli space C, with ramification at the stacky points. Since there is a unique double cover of the projective line with two ramification points (up to isomorphism) multiplication is given by $\theta_- \star_\omega \theta_- = 1$.

(b) (Teardrop orbifold) Suppose that $G =^\times$ acts on $X =^2$ with weights $1, 2$. Then $X/\!\!/G = \mathbb{P}(1,2)$ is a weighted projective line, $QH_G(X) \cong \Lambda[\xi]$ is a polynomial ring in a single generator, while

$$QH(X/\!\!/G) = \Lambda \oplus \Lambda\theta_+ \oplus \Lambda\theta_-$$

where θ_+ is the point class in $H(X/\!\!/G) \subset H(I_{X/\!\!/G})$ and θ_- is the class of the fixed point set $X^{-1}/\langle -1 \rangle = \mathbb{P}(2)$ in the twisted sector. Identify $H_2^G(X, \mathbb{Q}) \cong \mathbb{Q}$ corresponding to the dual of the Euler class of the representation with weight one. The fundamental class in $H_2(X/\!\!/G, \mathbb{Q}) \cong H_2^G(X, \mathbb{Q})$ then maps to $1/2$. The moduli space of twisted stable maps $u : C \to \mathbb{P}(1,2)$ of genus and class zero is either isomorphic to $\mathbb{P}(1,2)$ for no stacky points in the domain C, or isomorphic to $\mathbb{P}(2)$, for two stacky points in the domain C. Furthermore there is a unique (up to isomorphism) homology class $1/2$ twisted map with two smooth marked points and one stacky marked point with $\mathbb{Z}_2$ automorphism group. It follows that if the symplectic class ω has area $1/2$ on the fundamental class of $\mathbb{P}(1,2)$ then the quantum product is defined by

$$\theta_+ \star_\omega \theta_+ = q^{1/2}\theta_-/2, \quad \theta_- \star_\omega \theta_+ = q^{1/2}/2, \quad \theta_- \star_\omega \theta_- = \theta_+.$$

Thus after inverting $q^{1/2}$, the orbifold quantum cohomology is generated by θ_+ with the relation $\theta_+^3 = q/4$.

Remark 1.3. (Alternative power series rings) Some confusion may be caused by the multitude of formal power series rings that one can work over; unfortunately almost every set of authors has a different convention.

(a) The equivariant quantum cohomology $QH_G(X)$ can be defined over the larger *equivariant Novikov field* $\Lambda_X^G \subset \operatorname{Map}(H_2^G(X,\mathbb{Z}),\mathbb{Q})$ consisting of infinite sums $\sum_{i=1}^{\infty} c_i q^{d_i}$ with $\langle d_i, \omega\rangle \to \infty$, where q^{d_i} is the delta function at $d_i \in H_2^G(X,\mathbb{Z})$. Similarly, the quantum cohomology of the quotient $QH(X/\!\!/G)$ can be defined over the Novikov field $\Lambda_{X/\!\!/G} \subset \operatorname{Map}(H(X/\!\!/G,\mathbb{Q}),\mathbb{Q})$ consisting of infinite sums $\sum_{i=1}^{\infty} c_i q^{d_i}$ with $\langle d_i, \omega\rangle \to \infty$, where q^{d_i} is the delta function at $d_i \in H_2(X/\!\!/G,\mathbb{Q})$. The advantage of these rings is that the equivariant quantum cohomology $QH_G(X)$ becomes $\mathbb{Z}$-graded.

(b) $QH_G(X)$ is also defined over the *universal Novikov ring* Λ_0. If ω is integral, then $QH_G(X)$ is defined over $\mathbb{Q}[[q]]$. Similarly, $QH(X/\!\!/G)$ is defined over the Novikov ring Λ_0, and if ω is integral, over $\mathbb{Q}[[q^{1/n}]]$ for n equal to the least common multiple of the orders of the automorphism groups in $X/\!\!/G$. However, it is convenient to work over the field Λ. Invariance under Hamiltonian perturbation only holds for Floer/quantum cohomology over the Novikov field Λ, and so working over Λ is more natural for the purposes of symplectic geometry.

(c) Unfortunately, Λ and Λ_0 are not finitely generated over and so some care is required when talking about intersection multiplicities. In practice, when we wish to talk about intersection multiplicities we assume that the symplectic form is integral in which case our algebras are defined over $[q, q^{-1}]$.

(d) In algebraic geometry, one often uses the monoid-algebra of effective curve classes, but we prefer Novikov fields because of the better invariance properties. In fact, the cone of effective curve classes is not any more explicit than working over the Novikov field since it is the classes of *connected curves* that appear in the Gromov-Witten potentials, and these are rather hard to determine.

In [50, 51, 52] the second author studied the relationship between $QH_G(X)$ and $QH(X/\!\!/G)$ given by virtual enumeration of affine gauged maps, called the *quantum Kirwan map*. An n-marked *affine gauged map* is a representable morphism from a weighted projective line $\mathbb{P}(1,r)$ for some $r > 0$ to the quotient stack X/G mapping $\mathbb{P}(r) \subset \mathbb{P}(1,r)$ to the semistable locus $X/\!\!/G$. Some of the results of [50, 51, 52] are:

Theorem 1.4. (Definition and properties of the quantum Kirwan map)

(a) *The stack $\mathcal{M}_{n,1}^G(\mathbb{A},X,d)$ of n-marked affine gauged maps of class $d \in H_2^G(X,\mathbb{Q})$ has a natural compactification $\overline{\mathcal{M}}_{n,1}^G(\mathbb{A},X,d)$. Denote by* $\operatorname{ev}, \operatorname{ev}_\infty$ *the evaluation maps*

$$\begin{array}{ccc} & \overline{\mathcal{M}}_{n,1}^G(\mathbb{A},X) & \\ \swarrow^{\operatorname{ev}} & & \searrow^{\operatorname{ev}_\infty} \\ (X/G)^n & & \overline{I}_{X/\!\!/G} \end{array}$$

and ev_d, $\mathrm{ev}_{d,\infty}$ *their restrictions to maps of class* d. *The moduli stack* $\overline{\mathcal{M}}_{n,1}^G(\mathbb{A}, X, d)$ *has a perfect relative obstruction theory over* $\overline{\mathcal{M}}_{n,1}(\mathbb{A})$ *(the case of* X *and* G *trivial) where* $\overline{\mathcal{M}}_{n,1}(\mathbb{A})$ *is the complexification of Stasheff's multiplihedron.*

(b) *For any* $n \geq 0$, *the map defined by virtual enumeration of stable* n*-marked affine gauged maps*

$$\kappa_X^{G,n} : QH_G(X, \mathbb{Q}) \to QH(X/\!\!/G, \mathbb{Q}) \qquad (3)$$

$$\alpha \mapsto \sum_{d \in H_2^G(X,\mathbb{Q})} q^{\langle d, \omega \rangle} \mathrm{ev}_{d,\infty,*} \mathrm{ev}_d^*(\alpha, \ldots, \alpha)$$

is well-defined.

(c) *The sum*

$$\kappa_X^G : \ QH_G(X, \mathbb{Q}) \to QH(X/\!\!/G, \mathbb{Q}), \quad \alpha \mapsto \sum_{n \geq 0} \frac{\kappa_X^{G,n}(\alpha)}{n!}$$

defines a formal map from $QH_G(X, \mathbb{Q})$ *to* $QH(X/\!\!/G, \mathbb{Q})$ *in a neighborhood of the symplectic class* $\omega \in H_G^2(X, \mathbb{Q})$ *with the property that each linearization*

$$D_\alpha \kappa_X^G : T_\alpha QH_G(X, \mathbb{Q}) \to T_{\kappa_X^G(\alpha)} QH(X/\!\!/G, \mathbb{Q})$$

is a $\star$*-homomorphism with respect to the quantum products.*

By analogy with the classical case one hopes to obtain a presentation of the quantum cohomology algebra $T_{\kappa_X^G(\alpha)} QH(X/\!\!/G, \mathbb{Q})$ by showing that $D_\alpha \kappa_X^G$ is surjective and computing its kernel. This hope leads to the following strong and weak quantum version of Kirwan surjectivity. In the strong form, one might hope that κ_X^G has infinite radius of convergence, κ_X^G is surjective, and $D_\alpha \kappa_X^G$ is surjective for any $\alpha \in QH_G(X, \mathbb{Q})$. More modestly, one might hope that $D_\alpha \kappa_X^G$ is surjective for α in a formal neighborhood of a rational symplectic class $\omega \in H_2^G(X, \mathbb{Q})$.

We now specialize to the toric case. Suppose that G is a complex torus with Lie algebra $\mathfrak{g}$ acting on a finite-dimensional complex vector space X.

Notation 1.5. (a) (Weights) Let $X_1, \ldots, X_k \subset X$ be the weight spaces of X where $\dim(X_j) = 1$ and G acts on X_j with weight $\mu_j \in \mathfrak{g}^\vee$ in the sense that for $x \in X_j$ and $\xi \in \mathfrak{g}$ we have $\exp(\xi)x = \exp(i\langle \xi, \mu_j \rangle), j = 1, \ldots, k$. We assume that the weights $\mu_j \in \mathfrak{g}^\vee$ are contained in an open half-space, that is, for some $\nu \in \mathfrak{g}$ we have $\langle \nu, \mu_i \rangle \in \mathbb{R}_{>0}$, $i = 1, \ldots, k$. We also assume that the weights μ_i span $\mathfrak{g}^\vee$, so that G acts generically locally free on X.

(b) (Polarization and semistable locus) We assume that X is equipped with a polarization, that is, an ample G-line bundle $L \to X$, which we may allow to be rational, that is, an integer root of an honest G-line bundle. Let $\omega \in \mathfrak{g}_\mathbb{Q}^\vee$ be the vector representing the first Chern class of the polarization $c_1^G(L) \in H_2^G(X, \mathbb{Q})$ under the isomorphism $\mathfrak{g}_\mathbb{Q}^\vee \cong H_G^2(X, \mathbb{Q})$. The point ω determines a rational polarization on

X with semistable locus given as follows. Let

$$\mathcal{I}(\omega) = \left\{ I \subset \{1, \ldots, k\} \mid \omega \notin \sum_{i \in I} \mathbb{R}_{\geq 0} \mu_i \right\} \tag{4}$$

be the set of subsets so that ω is not in the span of the corresponding weights. Let X^I be the intersection of coordinate hyperplanes

$$X^I = \{(x_1, \ldots, x_k) | x_i = 0, \ \forall i \notin I\}.$$

Then

$$X^{\mathrm{ss}} = X \setminus \bigcup_{I \in \mathcal{I}(\omega)} X^I.$$

The stable=semistable condition assumption translates to the condition for each $I \notin \mathcal{I}(\omega)$ the weights $\mu_i, i \in I$ span $\mathfrak{g}^\vee$. In this case the quotient $X /\!\!/ G = X^{\mathrm{ss}}/G$ is then a smooth (possibly empty) proper Deligne-Mumford stack. We suppose that $X /\!\!/ G$ is non-empty.

(c) (Quantum Stanley-Reisner ideal) The *quantum Stanley-Reisner ideal* is

$$QSR_{X,G}(\alpha) := \langle QSR_{X,G}(d, \alpha), d \in H_2^G(X, \mathbb{Z}) \rangle \subset QH_G(X, \mathbb{Q})$$

where

$$QSR_{X,G}(d, \alpha) := \prod_{\langle \mu_j, d \rangle \geq 0} \mu_j^{\langle \mu_j, d \rangle} - q^{\langle d, \alpha \rangle} \prod_{\langle \mu_j, d \rangle \leq 0} \mu_j^{-\langle \mu_j, d \rangle}.$$

If α is the given symplectic class ω, we write $QSR_{X,G} := QSR_{X,G}(\omega)$. The quotient $T_\omega QH_G(X, \mathbb{Q})/QSR_{X,G}$ is the *quantum Stanley-Reisner a.k.a Batyrev ring.*

Example 1.6. (a) (Batyrev ring for projective space) Let $G =^\times$ act on $X =^k$ by scalar multiplication. All weights $\mu_1, \ldots, \mu_k$ are equal to $1 \in \mathfrak{g}_{\mathbb{Z}}^\vee \cong \mathbb{Z}$ and the polarization vector $\omega = 1 \in \mathfrak{g}_{\mathbb{Q}}^\vee \cong H_G^2(X, \mathbb{Q})$. There is a unique subset $I = \emptyset$ in $\mathcal{I}(\omega)$ and $X^I = \{0\} \subset X$. Thus the semistable locus is $X^{\mathrm{ss}} = X - X^\emptyset = X - \{0\}$ and the git quotient is $X /\!\!/ G = X^{\mathrm{ss}}/G = \mathbb{P}^{k-1}$. The quantum Stanley-Reisner ideal is generated by the single element $QSR_{X,G}(1) = \xi^k - q$. The Batyrev ring is $\Lambda[\xi]/\langle \xi^k - q \rangle$.

(b) (Batyrev ring for the teardrop orbifold) Continuing Example 1.2 (b), suppose that $G =^\times$ acts on $X =^2$ with weights $1, 2$ so that $X /\!\!/ G = \mathbb{P}(1, 2)$ is a weighted projective line. The Batyrev ring is $\Lambda[\xi]/\langle (\xi)(2\xi)^2 - q \rangle$.

(c) (Batyrev ring for the $B\mathbb{Z}_2$) Continuing Example 1.2 (b), suppose that $G =^\times$ acts on $X =$ with weights 2 so that $X /\!\!/ G = \mathbb{P}(2) \cong B\mathbb{Z}_2$. The Batyrev ring is $\Lambda[\xi]/\langle (2\xi)^2 - q \rangle$. After specializing q, the Batyrev ring is isomorphic to the group ring of $\mathbb{Z}_2$.

Our main result says that Batyrev's original suggestion [4] for the quantum cohomology is true, after passing to a suitable formal version of the equivariant cohomology and "quantizing" the divisor classes:

Theorem 1.7. *For a suitable formal version $\widehat{QH}_G(X)$ of the equivariant quantum cohomology $QH_G(X)$ (see Section 2) the linearized quantum Kirwan map $D_\omega\kappa_X^G$ induces an isomorphism*

$$T_\alpha\widehat{QH}_G(X,\mathbb{Q})/\widehat{QSR}_{X,G}(\omega) \to T_{\kappa_X^G(\omega)}QH(X/\!\!/G,\mathbb{Q}).$$

at the tangent space to the rational symplectic class $\omega \in H^2_G(X,\mathbb{Q})$.

Remark 1.8. (a) Many earlier cases of this theorem were known. Batyrev [4] proved a similar presentation in the case of convex toric manifolds, that is, in the case that the deformations of any stable map are un-obstructed. In the semi-Fano case (that is, $c_1(X/\!\!/G)$ is non-negative on any curve class) a presentation was given by Givental [24]. For non-weak-Fano toric manifolds, Iritani [34, 5.11] gave an isomorphism with the Batyrev ring, see also Brown [8]. From the symplectic point of view a presentation for the quantum cohomology of toric manifolds was given in Fukaya et al [21], using results of McDuff-Tolman [43] on the Seidel representation. The latter approach uses open-closed Gromov-Witten invariants to define a potential counting holomorphic disks whose leading order terms are the potential above. The quantum Stanley-Reisner relations were proved by Coates, Corti, Iritani, and Tseng [13, Theorem 5.13], see also Woodward [50, 51, 52], in papers that appeared after the first version of this manuscript. That these relations generate the ideal was expected for some time, see Iritani [33]. Thus the main content of this paper is that these relations suffice. A quantization of the Borisov-Chen-Smith presentation of the orbifold cohomology [7] was given in Tseng-Wang [48]. The latter is *not* a presentation in terms of divisor classes; for example, for weighted projective spaces the typical number of generators is much larger than one, while the Batyrev ring has a single generator.

(b) For the result above to hold the quantum cohomology must be defined over the Novikov *field*, or at least, that a suitable rational power of the formal parameter q has been inverted: over a polynomial ring such as $[q]$, one does not obtain an surjection because certain elements in twisted sectors are not contained in the image for $q = 0$. Thus one sees a Batyrev presentation of the quantum cohomology only for non-zero q. The necessity of corrections to Batyrev's original conjecture, which involved the divisor classes as generators, was noted in Cox-Katz [14, Example 11.2.5.2] for the second Hirzebruch surface and Spielberg [46] for a toric threefold. The fact that the change of coordinates restores the original presentation was noted in Guest [30] for semi-Fano toric varieties, and Iritani [34, Section 5], for not-necessarily-Fano toric varieties in general, after passing to a formal completion. See Iritani [34, Example 5.5] and González-Iritani [26, Example 3.5] for examples in the toric manifold case.

(c) Note that Danilov's results [16] do not require projectivity of the coarse moduli space. It seems possible that quantum cohomology might also be defined for non-projective toric varieties. Namely certain convergence conditions would remove the necessity of working over a Novikov ring, and one might have a theorem similar to 1.7, but we lack any results in this direction.