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Quantum supergroups VI: roots of 1

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Abstract

A quantum covering group is an algebra with parameters q and π subject to $\pi^2 = 1$, and it admits an integral form; it specializes to the usual quantum group at $\pi = 1$ and to a quantum supergroup of anisotropic type at $\pi = -1$. In this paper we establish the Frobenius–Lusztig homomorphism and Lusztig–Steinberg tensor product theorem in the setting of quantum covering groups at roots of 1. The specialization of these constructions at $\pi = 1$ recovers Lusztig's constructions for quantum groups at roots of 1.

Keywords Quantum groups · Quantum covering groups · Roots of 1 · Frobenius–Lusztig homomorphism

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1 Introduction

1.1. A Drinfeld–Jimbo quantum group with the quantum parameter q admits an integral $\mathbb{Z}[q, q^{-1}]$-form; its specialization at q being a root of 1 was studied by Lusztig in [15,16], [17, Part V] and also by many other authors. In these works Lusztig developed the quantum group version of Frobenius homomorphism and Frobenius kernel (known as small quantum groups), as a quantum analogue of several classical concepts arising from algebraic groups in a prime characteristic. The quantum groups at roots of 1 and their representation theory form a substantial part of Lusztig's program on modular representation theory, and they have further impacted other areas including geometric representation theory and categorification.

A quantum covering group $\mathbf{U}$, which was introduced in [4] (cf. [12]), is an algebra defined via super Cartan datum, which depends on parameters q and π subject to $\pi^2 = 1$. A quantum covering group specializes at $\pi = 1$ to a quantum group and at $\pi = -1$ to a quantum supergroup of anisotropic type (see [3]). Half the quantum covering group with parameter π with $\pi^2 = 1$ appeared first in [12] in an attempt to clarify the puzzle why quantum groups are categorified once more by the (spin) quiver Hecke superalgebras introduced in [14]. There has been much further progress on odd/spin/super categorification of quantum covering groups; see [2,10,13].

For quantum covering groups, the (q, π)-integer

$$[n]_{q,\pi} = \frac{(\pi q)^n - q^{-n}}{\pi q - q^{-1}} \in \mathbb{N}[q, q^{-1}, \pi]$$

and the corresponding (q, π)-binomial coefficients are used, and they help to restore the positivity which is lost in the quantum supergroup with $\pi = -1$. The algebra $\mathbf{U}$ (and its modified form $\dot{\mathbf{U}}$, respectively) admits an integral $\mathbb{Z}[q, q^{-1}, \pi]$-form ${}_\mathcal{A}\mathbf{U}$ (and ${}_\mathcal{A}\dot{\mathbf{U}}$, respectively). In [5] and then in [7] the canonical bases arising from quantum covering groups à la Lusztig and Kashiwara were constructed, and this provided for the first time a systematic construction of canonical bases for quantum supergroups. The braid group action has been constructed in [8] for quantum covering groups, and the first step toward a geometric realization of quantum covering groups was taken in [11].

1.2. To date the main parts of the book of Lusztig [17] have been generalized to the quantum covering group setting, except part V on roots of 1 and part II on geometric realization in full generality. The goal of this paper is to fill a gap in this direction by presenting a systematic study of the quantum covering groups at roots of 1; we follow closely the blueprint in [17, Chapters 33–36].

1.3. We impose a mild *bar-consistent* assumption on the super Cartan datum in this paper, following [5,12]. This assumption ensures that the new super Cartan datum and root datum arising from considerations of roots of 1 work as smoothly as one hopes. The assumption turns out to be also most appropriate again for the existence of Frobenius–Lusztig homomorphisms for quantum covering groups.

We expect that the quantum covering groups of finite type at roots of 1 have very interesting representation theory, which has yet to be developed (compare [1]). The categorification of the quantum covering group *of rank one* at roots of 1 is already highly nontrivial as shown in the recent work of Egilmez and Lauda [9]. We hope our work on higher rank quantum covering groups could provide a solid algebraic foundation for further super categorification and connection to quantum topology.

Specializing at $\pi = -1$, we obtain the corresponding results for (half, modified) quantum supergroups of anisotropic type at roots of 1; this class of quantum supergroups includes the quantum supergroup of type $\mathfrak{osp}(1|2n)$ as the only finite type example. It will be very interesting to develop systematically the quantum supergroups at roots of 1 associated with the *basic* Lie superalgebras (i.e., the simple Lie superalgebras with non-degenerate supersymmetric bilinear forms).

1.4. Below we provide some more detailed descriptions of the results and the organization of the paper. In Sect. 2, we establish several basic properties of the (q, π)-binomial coefficients at roots of 1, generalizing Lusztig [17, Chapter 34].

In Sect. 3, we recall half the quantum covering group ${}_R\mathbf{f}$ and the whole (respectively, the modified) quantum covering group $\mathbf{U}$ (respectively, ${}_R\dot{\mathbf{U}}$) over some ring R^π, associated with a super Cartan datum. We give a presentation of ${}_R\dot{\mathbf{U}}$ and a presentation of the quasi-classical counterpart ${}_R\mathbf{f}^\diamond$ of ${}_R\mathbf{f}$, generalizing [17, 33.2].

Our Sect. 4 is a generalization of [17, Chapter 35]. We establish in Theorem 4.1 a R^π-superalgebra homomorphism $\mathrm{Fr}' : {}_R\mathbf{f}^\diamond \longrightarrow {}_R\mathbf{f}$, which sends the generators $\theta_i^{(n)}$ to $\theta_i^{(n\ell_i)}$ for all $i \in I, n$. This is followed by the Lusztig–Steinberg tensor product theorem for ${}_R\mathbf{f}$ which we prove in Theorem 4.5. Next we establish in Theorem 4.7 the Frobenius–Lusztig homomorphism $\mathrm{Fr} : {}_R\mathbf{f} \longrightarrow {}_R\mathbf{f}^\diamond$ which sends the generators $\theta_i^{(n)}$ to $\theta_i^{(n/\ell_i)}$ if ℓ_i divides n, and to 0 otherwise, for all $i \in I, n$. We further extend the homomorphism Fr to the modified quantum covering group in Theorem 4.8.

Finally in Sect. 5, we formulate the small quantum covering groups and show it is a Hopf algebra. In case of finite type (i.e., corresponding to $\mathfrak{osp}(1|2n)$ or $\mathfrak{so}(1+2n)$), we show that the small quantum covering group is finite dimensional.

2 The (q, π)-binomials at roots of 1

In this section, we establish several basic formulas of the (q, π)-binomial coefficients at roots of 1. They specialize to the formulas in [17, Chapter 34] at $\pi = 1$.

2.1. Let π and q be formal indeterminants such that $\pi^2 = 1$. Fix $\sqrt{\pi}$ such that $\sqrt{\pi}^2 = \pi$. In contrast to earlier papers on the quantum covering groups [4–7], it is often helpful and sometimes crucial for the ground rings considered in this paper to contain $\sqrt{\pi}$, and for the sake of simplicity we choose to do so uniformly from the outset. For any ring S with 1, define the new ring

$$S^\pi = S \otimes_{\mathbb{Z}} \mathbb{Z}[\sqrt{\pi}].$$

We shall use often the following two rings:

$$\mathcal{A} = \mathbb{Z}[q, q^{-1}], \qquad \mathcal{A}^\pi = \mathbb{Z}[q, q^{-1}, \sqrt{\pi}].$$

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. For $a \in \mathbb{Z}$ and $n \in \mathbb{N}$, we define the (q, π)-*integer*

$$[a]_{q,\pi} = \frac{(\pi q)^a - q^{-a}}{\pi q - q^{-1}} \in \mathcal{A}^{\pi},$$

and then define the corresponding (q, π)-factorials and (q, π)-binomial coefficients by

$$[n]^!_{q,\pi} = \prod_{i=1}^{n} [i]_{q,\pi}, \qquad \begin{bmatrix} a \\ n \end{bmatrix}_{q,\pi} = \frac{\prod_{i=1}^{n}[a+1-i]_{q,\pi}}{[n]^!_{q,\pi}}.$$

For an indeterminant v, we denote the v-integers

$$[a]_v = \frac{v^a - v^{-a}}{v - v^{-1}}$$

and we similarly define the v-factorials $[n]^!_v$ and v-binomial coefficients $\begin{bmatrix} a \\ n \end{bmatrix}_v$. We denote by $\binom{a}{n}$ the classical binomial coefficients.

2.2. In this paper, the notation v is auxiliary, and we will identify

$$v := \sqrt{\pi} q,$$

and hence, for $n, t \in \mathbb{N}$,

$$\begin{aligned} {}_{q,\pi} &= \sqrt{\pi}^{n-1} [n]_v, \qquad [n]^!_{q,\pi} = \sqrt{\pi}^{n(n-1)/2} [n]^!_v, \\ \begin{bmatrix} n \\ t \end{bmatrix}_{q,\pi} &= \sqrt{\pi}^{(n-t)t} \begin{bmatrix} n \\ t \end{bmatrix}_v. \end{aligned} \tag{2.1}$$

2.3. Fix $\ell \in \mathbb{Z}_{>0}$ and let $\ell' = \ell$ or 2ℓ if ℓ is odd and let $\ell' = 2\ell$ if ℓ is even. Let

$$\mathcal{A}' = \mathcal{A}/\langle f(q) \rangle,$$

where $\mathcal{A}/\langle f(q) \rangle$ denotes the ideal generated by the ℓ'-th cyclotomic polynomial $f(q)$; we denote by $\varepsilon \in \mathcal{A}'$ the image of $q \in \mathcal{A}$. Take R to be an $\mathcal{A}'$-algebra with 1 (and so also an $\mathcal{A}$-algebra). Introduce the following root of 1 in R^{π}:

$$\mathbf{q} = \sqrt{\pi} \varepsilon \in R^{\pi}. \tag{2.2}$$

Then, the element

$$\mathbf{v} := \sqrt{\pi} \mathbf{q} \in R^{\pi}$$

satisfies that

$$\mathbf{v}^{2\ell} = 1, \qquad \mathbf{v}^{2t} \neq 1 \quad (\forall t \in \mathbb{Z}, \ell > t > 0). \tag{2.3}$$

Consider the specialization homomorphism $\phi : \mathcal{A}^\pi \to R^\pi$ which sends q to $\mathbf{q}$ and $\sqrt{\pi}$ to $\sqrt{\pi}$. We shall denote by $[n]_{\mathbf{q},\pi}$ and $\begin{bmatrix} n \\ t \end{bmatrix}_{\mathbf{q},\pi}$ the images of $[n]_{q,\pi}$ and $\begin{bmatrix} n \\ t \end{bmatrix}_{q,\pi}$ under ϕ, respectively, and so on.

The following lemma is an analogue of [17, Lemma 34.1.2], which can be in turn recovered by setting $\pi = 1$ below.

Lemma 2.1 (a) *If $t \in \mathbb{Z}_{>0}$ is not divisible by ℓ and $n \in \mathbb{Z}$ is divisible by ℓ, then* $\begin{bmatrix} n \\ t \end{bmatrix}_{\mathbf{q},\pi} = 0.$

(b) *If $n_1 \in \mathbb{Z}$ and $t_1 \in \mathbb{N}$, then we have*

$$\begin{bmatrix} \ell n_1 \\ \ell t_1 \end{bmatrix}_{\mathbf{q},\pi} = \pi^{\ell^2 t_1 (n_1 - (t_1 - 1)/2)} \mathbf{q}^{\ell^2 t_1 (n_1 + 1)} \binom{n_1}{t_1}.$$

(c) *Let $n \in \mathbb{Z}$ and $t \in \mathbb{N}$. Write $n = n_0 + \ell n_1$ with $n_0, n_1 \in \mathbb{Z}$ such that $0 \le n_0 \le \ell - 1$ and write $t = t_0 + \ell t_1$ with $t_0, t_1 \in \mathbb{N}$ such that $0 \le t_0 \le \ell - 1$. Then, we have*

$$\begin{bmatrix} n \\ t \end{bmatrix}_{\mathbf{q},\pi} = \pi^{\ell(n_0 - t_0)t_1 + \ell^2 (n_1 - (t_1 - 1)/2) t_1} \mathbf{q}^{\ell(n_0 t_1 - n_1 t_0) + \ell^2 (n_1 + 1) t_1} \begin{bmatrix} n_0 \\ t_0 \end{bmatrix}_{\mathbf{q},\pi} \binom{n_1}{t_1}.$$

Proof One proof would be by imitating the arguments for [17, Lemma 34.1.2]. Below we shall use an alternative and quicker approach, which is to convert [17, Lemma 34.1.2] into our current statements using (2.1) via the substitution $\mathbf{v} = \sqrt{\pi}\mathbf{q}$. Part (a) immediately follows from [17, Lemma 34.1.2(a)].

(b) By applying [17, Lemma 34.1.2(b)] to $\begin{bmatrix} \ell n_1 \\ \ell t_1 \end{bmatrix}_{\mathbf{v}}$ and using (2.1), we have

$$\begin{bmatrix} \ell n_1 \\ \ell t_1 \end{bmatrix}_{\mathbf{q},\pi} = \sqrt{\pi}^{\ell t_1 (\ell n_1 - \ell t_1)} \begin{bmatrix} \ell n_1 \\ \ell t_1 \end{bmatrix}_{\mathbf{v}} = \sqrt{\pi}^{\ell^2 t_1 (n_1 - t_1)} \mathbf{v}^{\ell^2 t_1 (n_1 + 1)} \binom{n_1}{t_1},$$

which can be easily shown to be equal to the formula as stated in the lemma.

(c) Note that

$$\sqrt{\pi}^{(n-t)t} = \sqrt{\pi}^{\ell((n_0 - t_0)t_1 + (n_1 - t_1)t_0)} \sqrt{\pi}^{\ell^2 (n_1 - t_1) t_1} \sqrt{\pi}^{(n_0 - t_0) t_0}. \tag{2.4}$$

By applying [17, Lemma 34.1.2(c)] to $\begin{bmatrix} n \\ t \end{bmatrix}_{\mathbf{v}}$ and using (2.1)–(2.4), we have

$$
\begin{aligned}
\begin{bmatrix} n \\ t \end{bmatrix}_{\mathbf{q},\pi} &= \sqrt{\pi}^{(n-t)t} \begin{bmatrix} n \\ t \end{bmatrix}_{\mathbf{v}} \\
&= \sqrt{\pi}^{(n-t)t} \mathbf{v}^{\ell(n_0 t_1 - n_1 t_0) + \ell^2 (n_1+1) t_1} \begin{bmatrix} n_0 \\ t_0 \end{bmatrix}_{\mathbf{v}} \binom{n_1}{t_1} \\
&= \sqrt{\pi}^{\ell((n_0-t_0)t_1 + (n_1-t_1)t_0)} \sqrt{\pi}^{\ell^2(n_1-t_1)t_1} \sqrt{\pi}^{\ell(n_0 t_1 - n_1 t_0) + \ell^2(n_1+1)t_1} \\
&\quad \times \mathbf{q}^{\ell(n_0 t_1 - n_1 t_0) + \ell^2 (n_1+1) t_1} \left(\sqrt{\pi}^{(n_0-t_0)t_0} \begin{bmatrix} n_0 \\ t_0 \end{bmatrix}_{\mathbf{v}} \right) \binom{n_1}{t_1} \\
&= \pi^{\ell(n_0-t_0)t_1 + \ell^2(n_1 - (t_1-1)/2)t_1} \mathbf{q}^{\ell(n_0 t_1 - n_1 t_0) + \ell^2(n_1+1)t_1} \begin{bmatrix} n_0 \\ t_0 \end{bmatrix}_{\mathbf{q},\pi} \binom{n_1}{t_1}.
\end{aligned}
$$

The lemma is proved. □

Note that, due to our choice of $\mathbf{q} = \sqrt{\pi}\varepsilon$, we also have an analogue of equation (e) in the proof of [17, Lemma 34.1.2]:

$$
\mathbf{v}^{\ell^2+\ell} = \pi^{(\ell+1)\ell/2} \mathbf{q}^{\ell^2+\ell} = (-1)^{\ell+1}. \tag{2.5}
$$

2.4. The following is an analogue of [17, § 34.1.3(a)].

Lemma 2.2 *Let $b \geq 0$. Then,*

$$
\frac{[\ell b]^!_{\mathbf{q},\pi}}{([\ell]^!_{\mathbf{q},\pi})^b} = b!(\pi \mathbf{q})^{\ell^2 b(b-1)/2}.
$$

Proof Recall $\mathbf{v} = \sqrt{\pi}\mathbf{q}$. Using (2.1) and [17, § 34.1.3(a)], we have

$$
\begin{aligned}
[\ell b]^!_{\mathbf{q},\pi} / ([\ell]^!_{\mathbf{q},\pi})^b &= \sqrt{\pi}^{\ell b(\ell b-1)/2 - b\ell(\ell-1)/2} [\ell b]^!_{\mathbf{v}} / ([\ell]^!_{\mathbf{v}})^b \\
&= \sqrt{\pi}^{\ell^2 b(b-1)/2} b! \mathbf{v}^{\ell^2 b(b-1)/2} = b!(\pi \mathbf{q})^{\ell^2 b(b-1)/2}.
\end{aligned}
$$

The lemma is proved. □

Below is a π-enhanced version of [17, Lemma 34.1.4].

Lemma 2.3 *Suppose that $0 \leq r \leq a < \ell$. Then,*

$$
\begin{aligned}
&\sum_{s=0}^{\ell-a-1} (-1)^{\ell-r+1+s} \pi^{\binom{s+1}{2} + s(r-\ell)} \mathbf{q}^{-(\ell-r)(a-\ell+1+s)+s} \begin{bmatrix} \ell - r \\ s \end{bmatrix}_{\mathbf{q},\pi} \\
&= \pi^{\binom{r}{2} - \binom{l}{2} - a(r-l)} \mathbf{q}^{\ell(a-r)} \begin{bmatrix} a \\ r \end{bmatrix}_{\mathbf{q},\pi}.
\end{aligned}
$$

Proof Plugging $\mathbf{v} = \sqrt{\pi}\mathbf{q}$ into [17, Lemma 34.1.4] and using (2.1), we obtain

$$\sum_{s=0}^{\ell-a-1} (-1)^{\ell-r+1+s} \sqrt{\pi}^{-(\ell-r)(a-\ell+1+s)+s+s(s-\ell+r)} \mathbf{q}^{-(\ell-r)(a-\ell+1+s)+s} \begin{bmatrix} \ell-r \\ s \end{bmatrix}_{\mathbf{q},\pi}$$
$$= \sqrt{\pi}^{\ell(a-r)+r(r-a)} \mathbf{q}^{\ell(a-r)} \begin{bmatrix} a \\ r \end{bmatrix}_{\mathbf{q},\pi}.$$

Rearranging the $\sqrt{\pi}$ terms, we have

$$\sum_{s=0}^{\ell-a-1} (-1)^{\ell-r+1+s} \sqrt{\pi}^{s(s+1)+2s(r-\ell)} \mathbf{q}^{-(\ell-r)(a-\ell+1+s)+s} \begin{bmatrix} \ell-r \\ s \end{bmatrix}_{\mathbf{q},\pi}$$
$$= \sqrt{\pi}^{r(r-1)-\ell(\ell-1)-2a(r-l)} \mathbf{q}^{\ell(a-r)} \begin{bmatrix} a \\ r \end{bmatrix}_{\mathbf{q},\pi}.$$

from which the desired formula is immediate. □

3 Quantum covering groups at roots of 1

In this section we recall the notion of super Cartan/root datum and the quantum covering groups. Then, we obtain presentations of the modified quantum covering groups and their quasi-classical counterpart.

3.1. The following is an analogue of [17, §2.2.4–5].

A *Cartan datum* is a pair $(I, \cdot)$ consisting of a finite set I and a symmetric bilinear form $\nu, \nu' \mapsto \nu \cdot \nu'$ on the free abelian group $\mathbb{Z}[I]$ with values in $\mathbb{Z}$ satisfying

(a) $d_i = \frac{i \cdot i}{2} \in \mathbb{Z}_{>0}$;
(b) $2\frac{i \cdot j}{i \cdot i} \in -\mathbb{N}$ for $i \neq j$ in I.

If the datum can be decomposed as $I = I_0 \coprod I_1$ such that

(c) $I_1 \neq \emptyset$,
(d) $2\frac{i \cdot j}{i \cdot i} \in 2\mathbb{Z}$ if $i \in I_1$,

then it is called a *super Cartan datum*; cf. [4]. We denote the parity $p(i) = 0$ for $i \in I_0$ and $p(i) = 1$ for $i \in I_1$.

Following [4], we will always assume a super Cartan datum satisfies the additional *bar-consistent* condition:

(e) $\frac{i \cdot i}{2} \equiv p(i) \mod 2, \quad \forall i \in I$.

A root datum of type $(I, \cdot)$ consists of 2 finite rank lattices X, Y with a perfect bilinear pairing $\langle \cdot, \cdot \rangle : Y \times X \to \mathbb{Z}$, 2 embeddings $I \hookrightarrow X$ ($i \mapsto i'$) and $I \hookrightarrow Y$ ($i \mapsto i$) such that $\langle i, j' \rangle = 2\frac{i \cdot j}{i \cdot i}$, $\forall i, j \in I$. Moreover, we will assume throughout the paper that the root datum is X-*regular*, i.e., that the simple roots are linearly independent in X.

Define

$$\ell_i = \min\{r \in \mathbb{Z}_{>0} \mid r(i \cdot i)/2 \in \ell\mathbb{Z}\}.$$

The next lemma follows by the definition of ℓ_i and the bar-consistency condition of I.

Lemma 3.1 *For each $i \in I_1$, ℓ_i has the same parity as ℓ.*

Then, $(I, \diamond)$ is a new root datum by [17, 2.2.4], where we let

$$i \diamond j = (i \cdot j)\ell_i \ell_j, \quad \forall i, j \in I.$$

Note that if ℓ is odd, then $(I, \diamond)$ is a super Cartan datum with the same parity decomposition $I = I_0 \cup I_1$ as for $(I, \cdot)$ by Lemma 3.1; if ℓ is even, then $(I, \diamond)$ is a (non-super) Cartan datum with $I_1 = \emptyset$.

We shall write $Y^\diamond$, $X^\diamond$ in this paper what Lusztig [17, 2.2.5] denoted by Y^*, X^*, respectively, and we will use superscript $^\diamond$ in related notation associated with $(Y^\diamond, X^\diamond, I, \diamond)$ below. More explicitly, we set $X^\diamond = \{\zeta \in X | \langle i, \zeta\rangle \in \ell_i\mathbb{Z}, \forall i \in I\}$ and $Y^\diamond = \mathrm{Hom}_{\mathbb{Z}}(X^\diamond, \mathbb{Z})$ with the obvious pairing. The embedding $I \hookrightarrow X^\diamond$ is given by $i \mapsto i'^\diamond = \ell_i i' \in X$, while embedding $I \hookrightarrow Y^\diamond$ is given by $i \mapsto i^\diamond \in Y^\diamond$ whose value at any $\zeta \in X^\diamond$ is $\langle i, \zeta\rangle/\ell_i$. It follows that $\langle i^\diamond, j'^\diamond\rangle = 2i \diamond j/i \diamond i$.

If ℓ is odd, then $(Y^\diamond, X^\diamond, \ldots)$ is a new super root datum satisfying (a)–(d) above and in addition the bar-consistency condition (e). Indeed, we have $2\frac{i\diamond j}{i\diamond i} = 2\frac{i\cdot j}{i\cdot i}\frac{\ell_j}{\ell_i} \in 2\mathbb{Z}$ by Lemma 3.1, whence (d), and $\frac{i\diamond i}{2} = \frac{i\cdot i}{2}\ell_i^2 \equiv p(i) \mod 2$ by Lemma 3.1, whence (e). If ℓ is even, then $(Y^\diamond, X^\diamond, \cdots)$ is a new (non-super) root datum just as in [17, 2.2.5].

3.2. By [4, Propositions 1.4.1, 3.4.1], the unital $\mathbb{Q}(q)^\pi$-superalgebra $\mathbf{f}$ is generated by θ_i $(i \in I)$ subject to the super Serre relations

$$\sum_{n+n'=1-\langle i, j'\rangle} (-1)^{n'} \pi_i^{n'p(j)+\binom{n'}{2}} \theta_i^{(n)} \theta_j \theta_i^{(n')} = 0$$

for any $i \neq j$ in I; here a generator θ_i is even if and only if $i \in I_0$. There is an $\mathcal{A}^\pi$-form for $\mathbf{f}$, which we call ${}_{\mathcal{A}}\mathbf{f}$. It is generated by the divided powers $\theta_i^{(n)} = \theta_i^n/[n]^!_{q_i,\pi_i}$ for all $i \in I, n \geq 1$. As R^π is an $\mathcal{A}^\pi$-algebra (cf. Sect. 2.3), by a base change we define ${}_R\mathbf{f} = R^\pi \otimes_{\mathcal{A}^\pi} {}_{\mathcal{A}}\mathbf{f}$. The algebras ${}'\mathbf{f}^\diamond$, $\mathbf{f}^\diamond$ and ${}_R\mathbf{f}^\diamond$ are defined in the same way using the Cartan datum $(I, \diamond)$.

Let $\mathbf{U}$ denote the quantum covering group associated with the root datum $(Y, X, \ldots)$ introduced in [4]. By [4, Proposition 3.4.2], $\mathbf{U}$ is a unital $\mathbb{Q}(q)^\pi$-superalgebra with generators

$$E_i \quad (i \in I), \quad F_i \quad (i \in I), \quad J_\mu \quad (\mu \in Y), \quad K_\mu \quad (\mu \in Y),$$

subject to the relations (a)–(f) below for all $i, j \in I$, $\mu, \mu' \in Y$:

$$K_0 = 1, \quad K_\mu K_{\mu'} = K_{\mu+\mu'}, \tag{a}$$

$$J_{2\mu} = 1, \quad J_\mu J_{\mu'} = J_{\mu+\mu'}, \quad J_\mu K_{\mu'} = K_{\mu'} J_\mu, \tag{b}$$
$$K_\mu E_i = q^{\langle \mu, i' \rangle} E_i K_\mu, \quad J_\mu E_i = \pi^{\langle \mu, i' \rangle} E_i J_\mu,$$

$$K_\mu F_i = q^{-\langle \mu, i' \rangle} F_i K_\mu, \quad J_\mu F_i = \pi^{-\langle \mu, i' \rangle} F_i J_\mu, \tag{c}$$

$$E_i F_j - \pi^{p(i)p(j)} F_j E_i = \delta_{i,j} \frac{\widetilde{J}_i \widetilde{K}_i - \widetilde{K}_{-i}}{\pi_i q_i - q_i^{-1}}, \tag{d}$$

$$\sum_{n+n'=1-\langle i, j' \rangle} (-1)^{n'} \pi_i^{n' p(j) + \binom{n'}{2}} E_i^{(n)} E_j E_i^{(n')} = 0 \tag{e}$$

$$\sum_{n+n'=1-\langle i, j' \rangle} (-1)^{n'} \pi_i^{n' p(j) + \binom{n'}{2}} F_i^{(n)} F_j F_i^{(n')} = 0 \tag{f}$$

where for any element $\nu = \sum_i \nu_i i \in \mathbb{Z}[I]$ we have set $\widetilde{K}_\nu = \prod_i K_{d_i \nu_i i}$, $\widetilde{J}_\nu = \prod_i J_{d_i \nu_i i}$. In particular, $\widetilde{K}_i = K_{d_i i}$, $\widetilde{J}_i = J_{d_i i}$. (Under the bar-consistent condition (e), $\widetilde{J}_i = 1$ for $i \in I_{\bar{0}}$ while $\widetilde{J}_i = J_i$ for $i \in I_{\bar{1}}$.) We endow $\mathbf{U}$ with a $\mathbb{Z}[I]$-grading $|\cdot|$ by setting $|E_i| = i, \quad |F_i| = -i, \quad |J_\mu| = |K_\mu| = 0$. The parity on $\mathbf{U}$ is given by $p(E_i) = p(F_i) = p(i)$ and $p(K_\mu) = p(J_\mu) = 0$,

The algebra $\mathbf{U}$ has an $\mathcal{A}^\pi$-form ${}_\mathcal{A}\mathbf{U}$. By a base change, we obtain ${}_R\mathbf{U} = R^\pi \otimes_{\mathcal{A}^\pi} {}_\mathcal{A}\mathbf{U}$. Let ${}_R\mathbf{U}^+$ (resp. ${}_R\mathbf{U}^-$) denote the subalgebra of ${}_R\mathbf{U}$ generated by the $E_i^{(n)} = E_i^n/[n]^!_{\mathbf{q}_i,\pi_i}$ (resp. $F_i = F_i^n/[n]^!_{\mathbf{q}_i,\pi_i}$). As a R^π-algebra, ${}_R\mathbf{f}$ is isomorphic to ${}_R\mathbf{U}^+$ (resp. ${}_R\mathbf{U}^-$) via the map $x \mapsto x^+$ (resp. $x \mapsto x^-$), where $(\theta_i^{(n)})^+ = E_i^{(n)}$ (resp. $(\theta_i^{(n)})^- = F_i^{(n)}$.

Denote by $X^+ = \{\lambda \in X \mid \langle i, \lambda \rangle \in \mathbb{N}, \forall i \in I\}$, the set of dominant integral weights.

For $\lambda \in X$, let $M(\lambda)$ be the Verma module of $\mathbf{U}$, and we can naturally identify $M(\lambda) = \mathbf{f}$ as $\mathbb{Q}(q)^\pi$-modules. The ${}_\mathcal{A}\mathbf{U}$-submodule ${}_\mathcal{A}M(\lambda)$ can be identified with ${}_\mathcal{A}\mathbf{f}$ as $\mathcal{A}^\pi$-free modules. For $\lambda \in X^+$, we define the integrable $\mathbf{U}$-module $V(\lambda) = M(\lambda)/J_\lambda$, where J_λ is the left $\mathbf{f}$-module generated by $\theta_i^{\langle i,\lambda \rangle + 1}$ for all $i \in I$. Let ${}_R M(\lambda) = R^\pi \otimes_{\mathcal{A}^\pi} {}_\mathcal{A}M(\lambda)$ for $\lambda \in X$, and ${}_R V(\lambda) = R^\pi \otimes_{\mathcal{A}^\pi} {}_\mathcal{A}V(\lambda)$ for $\lambda \in X^+$.

The algebra $\mathbf{U}^\diamond$ is defined in the same way as $\mathbf{U}$ based on the root datum $(Y^\diamond, X^\diamond, \ldots)$.

Recall from [6, Definition 4.2] that the modified quantum covering group $\dot{\mathbf{U}}$ is a $\mathbb{Q}(q)^\pi$-algebra without unit which is generated by the symbols 1_λ, $E_i 1_\lambda$ and $F_i 1_\lambda$, for $\lambda \in X$ and $i \in I$, subject to the relations:

$$\begin{aligned}
&1_\lambda 1_{\lambda'} = \delta_{\lambda,\lambda'} 1_\lambda, \\
&(E_i 1_\lambda) 1_{\lambda'} = \delta_{\lambda,\lambda'} E_i 1_\lambda, \quad 1_{\lambda'}(E_i 1_\lambda) = \delta_{\lambda',\lambda+i'} E_i 1_\lambda, \\
&(F_i 1_\lambda) 1_{\lambda'} = \delta_{\lambda,\lambda'} F_i 1_\lambda, \quad 1_{\lambda'}(F_i 1_\lambda) = \delta_{\lambda',\lambda-i'} F_i 1_\lambda, \\
&(E_i F_j - \pi^{p(i)p(j)} F_j E_i) 1_\lambda = \delta_{ij} \, [\langle i, \lambda \rangle]_{v_i,\pi_i} \, 1_\lambda, \\
&\textstyle\sum_{n+n'=1-\langle i,j' \rangle} (-1)^{n'} \pi_i^{n' p(j) + \binom{n'}{2}} E_i^{(n)} E_j E_i^{(n')} 1_\lambda = 0 \quad (i \neq j), \\
&\textstyle\sum_{n+n'=1-\langle i,j' \rangle} (-1)^{n'} \pi_i^{n' p(j) + \binom{n'}{2}} F_i^{(n)} F_j F_i^{(n')} 1_\lambda = 0 \quad (i \neq j),
\end{aligned}$$

where $i, j \in I$, $\lambda, \lambda' \in X$, and we use the notation $xy1_\lambda = (x1_{\lambda+|y|})(y1_\lambda)$ for $x, y \in \mathbf{U}$.

The modified quantum covering group $\dot{\mathbf{U}}$ admits an $\mathcal{A}^\pi$-form, ${}_\mathcal{A}\dot{\mathbf{U}}$ and so we can define ${}_R\dot{\mathbf{U}} = R^\pi \otimes_{\mathcal{A}^\pi} {}_\mathcal{A}\dot{\mathbf{U}}$. Let us give a presentation for ${}_R\dot{\mathbf{U}}$.

Lemma 3.2 *The modified quantum covering group ${}_R\dot{\mathbf{U}}$ is generated as an R^π-algebra by $x^+\mathbf{1}_\lambda x'^-$ or equivalently by $x^-\mathbf{1}_\lambda x'^+$, where $x \in {}_R\mathbf{f}_\mu$, $x' \in {}_R\mathbf{f}_\nu$ and $\lambda \in X$, subject to the following relations:*

$$\left(\theta_i^{(N)}\right)^+ \mathbf{1}_\lambda \left(\theta_i^{(M)}\right)^-$$
$$= \sum_{t\geq 0} \pi_i^{MN-\binom{t+1}{2}} \left(\theta_i^{(M-t)}\right)^- \begin{bmatrix} M+N+\langle i,\lambda\rangle \\ t \end{bmatrix}_{\mathbf{q}_i,\pi_i} \mathbf{1}_{\lambda+(M+N-t)i'} \left(\theta_i^{(N-t)}\right)^+,$$
$$\left(\theta_i^{(N)}\right)^- \mathbf{1}_\lambda \left(\theta_i^{(M)}\right)^+$$
$$= \sum_{t\geq 0} \pi_i^{MN+t\langle i,\lambda\rangle-\binom{t}{2}} \left(\theta_i^{(M-t)}\right)^+ \begin{bmatrix} M+N-\langle i,\lambda\rangle \\ t \end{bmatrix}_{\mathbf{q}_i,\pi_i} \mathbf{1}_{\lambda-(M+N-t)i'} \left(\theta_i^{(N-t)}\right)^-,$$
$$\left(\theta_i^{(N)}\right)^+ \left(\theta_j^{(M)}\right)^- \mathbf{1}_\lambda = \pi^{MNp(i)p(j)} (\theta_j^{(M)})^- \left(\theta_i^{(N)}\right)^+ \mathbf{1}_\lambda, \text{ for } i \neq j,$$
$$x^+\mathbf{1}_\lambda = \mathbf{1}_{\lambda+\mu}x^+, \quad x^-\mathbf{1}_\lambda = \mathbf{1}_{\lambda-\mu}x^-,$$
$$(x^+\mathbf{1}_\lambda)(\mathbf{1}_{\lambda'}x'^-) = \delta_{\lambda,\lambda'}x^+\mathbf{1}_\lambda x'^-, \quad (x^-\mathbf{1}_\lambda)(\mathbf{1}_{\lambda'}x'^+) = \delta_{\lambda,\lambda'}x^-\mathbf{1}_\lambda x'^+,$$
$$(x^+\mathbf{1}_\lambda)(\mathbf{1}_{\lambda'}x'^-) = \delta_{\lambda,\lambda'}\mathbf{1}_{\lambda+\mu}x^+x'^-, \quad (x^-\mathbf{1}_\lambda)(\mathbf{1}_{\lambda'}x'^+) = \delta_{\lambda,\lambda'}\mathbf{1}_{\lambda-\mu}x^-x'^+,$$
$$(rx+r'x')^\pm\mathbf{1}_\lambda = rx^\pm\mathbf{1}_\lambda + r'x'^\pm\mathbf{1}_\lambda, \text{ where } r, r' \in R^\pi.$$

Proof This is proved in the same way as [17, § 31.1.3]. Let A be the R^π-algebra with the above generators and relations. All of these relations are known to hold in ${}_R\dot{\mathbf{U}}$. The first three are shown to hold in ${}_R\dot{\mathbf{U}}$ by a direct application of [4, Lemma 2.2.3] as in [7, Lemma 4], while the remaining ones are clear. However, there was an error in the second relation of [7, Lemma 4], so we derive that relation from [4, Lemma 2.2.3] here. We have

$$\left(\theta_i^{(N)}\right)^- \mathbf{1}_\lambda \left(\theta_i^{(M)}\right)^+$$
$$= \left(\theta_i^{(N)}\right)^- (\theta_i^{(M)})^+ \mathbf{1}_{\lambda-Mi'}$$
$$= \sum_{t\geq 0} (-1)^t \pi_i^{(M-t)(N-t)-t^2} \left(\theta_i^{(M-t)}\right)^+ \begin{bmatrix} \tilde{K}_i; M+N-(t+1) \\ t \end{bmatrix}_{\mathbf{q}_i,\pi_i} \left(\theta_i^{(N-t)}\right)^- \mathbf{1}_{\lambda-Mi'}$$
$$= \sum_{t\geq 0} (-1)^t \pi_i^{(M-t)(N-t)-t^2} \left(\theta_i^{(M-t)}\right)^+ \begin{bmatrix} \langle i,\lambda\rangle - M - N + t - 1 \\ t \end{bmatrix}_{\mathbf{q}_i,\pi_i} \mathbf{1}_{\lambda-(M+N-t)i'} \left(\theta_i^{(N-t)}\right)^-$$
$$= \sum_{t\geq 0} \pi_i^{MN+t\langle i,\lambda\rangle-\binom{t}{2}} \left(\theta_i^{(M-t)}\right)^+ \begin{bmatrix} M+N-\langle i,\lambda\rangle \\ t \end{bmatrix}_{\mathbf{q}_i,\pi_i} \mathbf{1}_{\lambda-(M+N-t)i'} \left(\theta_i^{(N-t)}\right)^-$$

where in the last step, we used [4, (1.10)] with $a = M+N-\langle i,\lambda\rangle$. Hence, the natural homomorphism $A \longrightarrow {}_R\dot{\mathbf{U}}$ is surjective. Let $\mathbf{S}$ be an R^π-basis of ${}_R\mathbf{f}$ consisting of weight vectors. Then, $\{x^+\mathbf{1}_\lambda x'^- | x, x' \in \mathbf{S}, \lambda \in X\}$ can be seen to be an R^π-basis for A, and it is known to be one for ${}_R\dot{\mathbf{U}}$ (cf. [7, Lemma 5]). Thus, the natural homomorphism is, in fact, an isomorphism. □

3.3. The algebra $\dot{\mathbf{U}}^\diamond$ is defined in the same way using $\mathbf{U}^\diamond$ and $(Y^\diamond, X^\diamond, \ldots)$, and so it also has an $\mathcal{A}^\pi$-form ${}_\mathcal{A}\dot{\mathbf{U}}^\diamond$ and we can define ${}_R\dot{\mathbf{U}}^\diamond = R^\pi \otimes_{\mathcal{A}^\pi} {}_\mathcal{A}\dot{\mathbf{U}}^\diamond$.

Remark 3.3 If ℓ is even, then ${}_R\mathbf{f}^\diamond$ is a (non-super) algebra; if ℓ is odd, then the θ_i in ${}_R\mathbf{f}^\diamond$ and ${}_R\mathbf{f}$ for any given i have the same parity.

For $i \in I$, we denote

$$q_i^\diamond = q^{i\diamond i/2} = q_i^{\ell_i^2}, \qquad \mathbf{q}_i^\diamond = \mathbf{q}^{i\diamond i/2} = \mathbf{q}_i^{\ell_i^2}, \qquad \pi_i^\diamond = \pi^{i\diamond i/2} = \pi_i^{\ell_i^2}. \tag{3.1}$$

Lemma 3.4 *Let $i \in I_1$.*

(a) *If ℓ is odd, then $\pi_i^\diamond = \pi_i$.*
(b) *If ℓ is even, then $\pi_i^\diamond = 1$.*

Proof Recall from Lemma 3.1 that ℓ_i must have the same parity as ℓ. The claim on $\pi_i^\diamond$ follows now from (3.1). □

For each $i \in I$, we have

$$\pi_i^\diamond \mathbf{q}_i^{\diamond 2} = (\pi_i \mathbf{q}_i^2)^{\ell_i^2} = 1. \tag{3.2}$$

Following Lusztig [17], we will refer to the quantum supergroup ${}_R\mathbf{f}^\diamond$ associated with $(Y^\diamond, X^\diamond, \cdots)$ as *quasi-classical*; cf. (3.2).

Proposition 3.5 *Let R be the fraction field of $\mathcal{A}'$. The quasi-classical algebra ${}_R\mathbf{f}^\diamond$ is isomorphic to ${}_R\tilde{\mathbf{f}}^\diamond$, the R^π-algebra generated by θ_i, $i \in I$, subject to the super Serre relations:*

$$\sum_{n+n'=1-\langle i,j'\rangle^\diamond} (-1)^{n'} (\pi_i^\diamond)^{np(j)+\binom{n}{2}} \theta_i^{(n)} \theta_j \theta_i^{(n')} = 0 \qquad (i \neq j \in I).$$

Proof When $\pi_i = 1$ or ℓ is even, $\pi_i^\diamond = 1$ and $\mathbf{q}_i^\diamond = \pm 1$ for each $i \in I$. Hence, in this case the lemma reduces to [17, § 33.2].

Now let ℓ be odd and $\pi = -1$. We make use of the *weight-preserving* automorphism $\dot{\Psi}$ of ${}_R\dot{\mathbf{U}}^\diamond$ (called a twistor) given in [6, Theorem 4.3] when the base ring contains $\sqrt{-1}$. We will only recall the basic property of $\dot{\Psi}$ which we need, and refer to [6] for details. Note that for all $i \in I$, $\mathbf{q}_i^\diamond$ is a power of $\sqrt{-1}$ with at least one of the $\mathbf{q}_i^\diamond = \pm\sqrt{-1}$. Thus, $\pm\sqrt{-1}$ will play the role played by the v in [6, Theorem 4.3], which we will denote by $\tilde{v}$ in this proof so as not to confuse it with the v defined in this paper. Recall $\dot{\Psi}$ takes π to $-\pi$ and $\tilde{v}$ to $\sqrt{-1}\tilde{v}$. When we specialize $\pi = -1$

and $\tilde{v} = \pm\sqrt{-1}$, we obtain an R-linear isomorphism of that specialization of ${}_R\dot{\mathbf{U}}^\diamond$, denoted by ${}_R\dot{\mathbf{U}}^\diamond|_{-1}$, with the (quasi-classical) modified quantum group corresponding to the specialization $\pi = 1$ and $\mathbf{q}_j^\diamond = \pm 1$, denoted by ${}_R\dot{\mathbf{U}}^\diamond|_1$.

Write

▷ ${}_{R_{-1}}\mathbf{f}$ for the half quantum (super)group over R corresponding to the former (i.e., $\pi = -1$);

▷ ${}_{R_1}\mathbf{f}^\diamond$ for the half (quasi-classical) quantum group over R corresponding to the latter (i.e., $\pi = 1$); cf. [17, 33.2].

Recall that ${}_R\mathbf{f}^\diamond$ is a direct sum of finite-dimensional weight spaces ${}_R\mathbf{f}_\nu^\diamond$, where $\nu \in \mathbb{Z}_{\geq 0}[I]$. The weight-preserving isomorphism $\dot{\Psi}$ above implies that

$$\dim_{R^\pi}({}_R\mathbf{f}_\nu^\diamond) = \dim_R({}_{R_{-1}}\mathbf{f}_\nu^\diamond) = \dim_R {}_{R_1}\mathbf{f}_\nu^\diamond, \quad \forall \nu.$$

As ${}_{R_1}\mathbf{f}^\diamond$ is quasi-classical in the sense of [17, 33.2], we have $\dim_R {}_{R_1}\mathbf{f}_\nu^\diamond = \dim_R {}_{R_1}\mathbf{f}_\nu$ for all ν, by [17, 33.2.2], where ${}_{R_1}\mathbf{f}$ is the enveloping algebra of the half KM algebra over R. Hence, we have

$$\dim_{R^\pi}({}_R\mathbf{f}_\nu^\diamond) = \dim_R({}_{R_1}\mathbf{f}_\nu), \quad \forall \nu. \tag{3.3}$$

Since the super Serre relations hold in ${}_R\mathbf{f}^\diamond$ (cf. [4, Proposition 1.7.3]), we have a surjective algebra homomorphism $\varphi : {}_R\tilde{\mathbf{f}}^\diamond \longrightarrow {}_R\mathbf{f}^\diamond$ mapping $\theta_i \mapsto \theta_i$ for all i. Then, φ maps each weight space ${}_R\tilde{\mathbf{f}}_\nu^\diamond$ onto the corresponding weight space ${}_R\mathbf{f}_\nu^\diamond$. As ${}_R\tilde{\mathbf{f}}^\diamond$ has a Serre-type presentation by definition, it follows by [5,13] that $\dim_{R^\pi}({}_R\tilde{\mathbf{f}}_\nu) = \dim_R({}_{R_1}\mathbf{f}_\nu)$ for each ν. This together with (3.3) implies that $\dim_{R^\pi}({}_R\tilde{\mathbf{f}}_\nu) = \dim_{R^\pi}({}_R\mathbf{f}_\nu^\diamond)$. Therefore, φ is a linear isomorphism on each weight space and thus an isomorphism. □

3.4. Below we provide an analogue of [17, 35.1.5].

Lemma 3.6 *Assume that both $n \in \mathbb{Z}$ and $t \in \mathbb{N}$ are divisible by ℓ_i. Then,*

$$\begin{bmatrix} n \\ t \end{bmatrix}_{\mathbf{q}_i, \pi_i} = \begin{bmatrix} n/\ell_i \\ t/\ell_i \end{bmatrix}_{\mathbf{q}_i^\diamond, \pi_i^\diamond}.$$

(Setting $\pi = 1$ in the above formula recovers [17, 35.1.5].)

Proof By Lemma 2.1(b), we have

$$\begin{bmatrix} n \\ t \end{bmatrix}_{\mathbf{q}_i, \pi_i} = \pi_i^{t(n-(t-\ell_i)/2)} \mathbf{q}_i^{t(n+\ell_i)} \binom{n/\ell_i}{t/\ell_i}.$$

Note that $\pi_i^\diamond \mathbf{q}_i^{\diamond 2} = (\pi \mathbf{q}^2)^{\frac{i \cdot i}{2} \ell_i^2}$. Since $(\pi \mathbf{q}^2)^{2\ell} = 1$ and ℓ divides $\frac{i \cdot i}{2} \ell_i^2$ by the definition of ℓ_i, we have $(\pi_i^\diamond \mathbf{q}_i^{\diamond 2})^2 = 1$. Hence, by (3.1) and Lemma 2.1(b) with $\ell = 1$ we have

$$\begin{bmatrix} n/\ell_i \\ t/\ell_i \end{bmatrix}_{\mathbf{q}_i^\diamond, \pi_i^\diamond} = \pi_i^{t(n-(t-\ell_i)/2)} \mathbf{q}_i^{t(n+\ell_i)} \binom{n/\ell_i}{t/\ell_i}.$$

The lemma follows. ◻

4 The Frobenius–Lusztig homomorphism

In this section we establish the Frobenius–Lusztig homomorphism between the quasi-classical covering group and the quantum covering group at roots of 1. We also formulate Lusztig–Steinberg tensor product theorem in this setting.

4.1. Following [17, 35.1.2], in this and following sections we shall assume

(a) for any $i \neq j \in I$ with $\ell_j \geq 2$, we have $\ell_i \geq -\langle i, j' \rangle + 1$.
(b) $(I, \cdot)$ has no odd cycles.

4.2. Below is a generalization of [17, Theorem 35.1.8].

Theorem 4.1 *There is a unique R^π-superalgebra homomorphism*

$$\mathrm{Fr}' : {}_R\mathbf{f}^\diamond \longrightarrow {}_R\mathbf{f}, \qquad \mathrm{Fr}'(\theta_i^{(n)}) = \theta_i^{(n\ell_i)} \quad (\forall i \in I, n \in \mathbb{Z}_{>0}).$$

(Be aware that the two θ_i's above belong to different algebras and hence are different. Theorem 4.1 is consistent with Remark 3.3.)

The rest of the section is devoted to a proof of Theorem 4.1. The same remark as in [17, 35.1.11] allows us to reduce the proof to the case when R is the quotient field of $\mathcal{A}'$, which we will assume in the remainder of this and the next section.

4.3. Recall from (2.3) that $\pi^\ell \mathbf{q}^{2\ell} = 1$ and $\pi^t \mathbf{q}^{2t} \neq 1$ for $0 < t < \ell$. By the definition of ℓ_i, we have $\pi_i^\ell \mathbf{q}_i^{2\ell} = 1$ and $\pi_i^t \mathbf{q}_i^{2t} \neq 1$ for $0 < t < \ell_i$. Then, $[t]_{\mathbf{q}_i}^\pi!$ is invertible in R^π, for $0 < t < \ell_i$.

The following is an analogue of [17, Lemma 35.2.2] and the proof uses now Lemmas 2.1 and 2.2.

Lemma 4.2 *The R^π-superalgebra ${}_R\mathbf{f}$ is generated by the elements $\theta_i^{(\ell_i)}$ for all $i \in I$ and the elements θ_i for $i \in I$ with $\ell_i \geq 2$.*

Proof By definition the algebra ${}_R\mathbf{f}$ is generated by $\theta_i^{(n)}$ for all $i \in I$ and $n \geq 0$. We can write $n = a + \ell_i b$, for $0 \leq a < \ell_i$ and $b \in \mathbb{N}$. We note the following three identities in ${}_R\mathbf{f}$:

$$\theta_i^{(a+\ell_i b)} = \mathbf{q}_i^{\ell_i ab} \theta_i^{(a)} \theta_i^{(\ell_i b)}, \tag{4.1}$$

$$\theta_i^{(a)} = [a]_{\mathbf{q}_i,\pi_i}^{-1} \theta_i^a, \tag{4.2}$$

$$\theta_i^{(\ell_i b)} = (b!)^{-1} (\pi_i \mathbf{q}_i)^{-\ell_i^2 \binom{b}{2}} (\theta_i^{(\ell_i)})^b, \tag{4.3}$$

where (4.1) follows by Lemma 2.1 and (4.3) follows by Lemma 2.2, respectively. (Note that a sign in the power of $\mathbf{v}_i$ in the identity (b) in [17, proof of Lemma 35.2.2] is optional, but the sign cannot be dropped from the power of $\mathbf{q}_i$ in (4.3).) The lemma follows. ◻

We shall prove Theorem 4.1 in this subsection. The uniqueness is clear.
By Lemma 2.2 (with $\ell = 1$), we have

$$[n]^!_{\mathbf{q}_i^\diamond,\pi_i^\diamond} = (\pi_i \mathbf{q}_i)^{\ell_i^2 \binom{n}{2}} n!. \tag{4.4}$$

We first observe that the existence of a homomorphism Fr' such that $\mathrm{Fr}'(\theta_i) = \theta_i^{(\ell_i)}$ implies that $\mathrm{Fr}'(\theta_i^{(n)}) = \theta_i^{(n\ell_i)}$ for all $n \geq 0$. Indeed, using (4.3)–(4.4) we have

$$\mathrm{Fr}'(\theta_i^{(n)}) = ([n]_{\mathbf{q}_i^\diamond,\pi_i^\diamond}!)^{-1}\mathrm{Fr}'(\theta_i)^n = \big((\pi_i \mathbf{q}_i)^{\ell_i^2 n(n-1)/2} n!\big)^{-1}\mathrm{Fr}'(\theta_i)^n = \theta_i^{(n\ell_i)}.$$

Hence, it remains to show that there exists an algebra homomorphism $\mathrm{Fr}' : {}_R\mathbf{f}^\diamond \to {}_R\mathbf{f}$ such that $\theta_i \to \theta_i^{(\ell_i)}, \forall i \in I$. By Proposition 3.5 (also cf. [4]), the algebra ${}_R\mathbf{f}^\diamond$ has the following defining relations:

$$\sum_{n+n'=1-\langle i,j'\rangle^\diamond} (-1)^{n'} (\pi_i^\diamond)^{np(j)+\binom{n}{2}} \theta_i^{(n)} \theta_j \theta_i^{(n')} = 0 \qquad (i \neq j \in I).$$

By (4.4) it suffices to check the following identity in ${}_R\mathbf{f}$: for $i \neq j \in I$,

$$\sum_{n+n'=1-\langle i,j'\rangle \ell_j/\ell_i} (-1)^{n'} \pi_i^{\ell_i^2(np(j)+n(n-1)/2)} (\pi_i \mathbf{q}_i)^{-\ell_i^2\binom{n}{2}} (\pi_i \mathbf{q}_i)^{-\ell_i^2\binom{n'}{2}}$$
$$\frac{(\theta_i^{(\ell_i)})^n}{n!} \theta_j^{(\ell_j)} \frac{(\theta_i^{(\ell_i)})^{n'}}{n'!} = 0,$$

which, by the identity (4.3), is equivalent to checking the following identity in ${}_R\mathbf{f}$:

$$\sum_{n+n'=1-\langle i,j'\rangle \ell_j/\ell_i} (-1)^{n'} \pi_i^{\ell_i^2(np(j)+n(n-1)/2)} \theta_i^{(\ell_i n)} \theta_j^{(\ell_j)} \theta_i^{(\ell_i n')} = 0. \tag{4.5}$$

It remains to prove (4.5). Set $\alpha = -\langle i, j' \rangle$. For any $0 \leq t \leq \ell_i - 1$, we set

$$g_t = \sum_{\substack{r,s \\ r+s=\ell_j\alpha+\ell_i-t}} (-1)^r \pi_i^{\ell_j r p(j)+r(r-1)/2} q_i^{r(\ell_i-1-t)} \theta_i^{(r)} \theta_j^{(\ell_j)} \theta_i^{(s)} \in \mathcal{A}\mathbf{f}.$$

This is basically $f'_{i,j;\ell_j,\ell_j\alpha+\ell_i-t}$ in [4, 4.1.1(d)] in the notation of θ's. By the higher super Serre relations (see [4, Proposition 4.2.4] and [4, 4.1.1(e)]), we have $g_t = 0$ for all $0 \leq t \leq \ell_i - 1$. Set

$$g = \sum_{t=0}^{\ell_i-1} (-1)^t \pi_i^{t(t-1)/2} \mathbf{q}_i^{\ell_j\alpha t+\ell_i t-t} g_t \theta_i^{(t)},$$

which must be 0. On the other hand, setting $s' = s + t$, we have

$$(0 =)\ g = \sum_{\substack{r,s' \\ r+s'=\ell_j\alpha+\ell_i}} c_{r,s'}\theta_i^{(r)}\theta_j^{(\ell_j)}\theta_i^{(s')}, \tag{4.6}$$

where

$$c_{r,s'} = \sum_{t=0}^{\ell_i-1}(-1)^{r+t}\pi_i^{\ell_j rp(j)+r(r-1)/2+t(t-1)/2} q_i^{r(\ell_i-1-t)+\ell_j\alpha t+\ell_i t-t} \begin{bmatrix} s' \\ t \end{bmatrix}_{q_i,\pi_i}.$$

Taking the image of the identity (4.6) under the map ${}_{\mathcal{A}}\mathbf{f} \to {}_R\mathbf{f}$, we have

$$\sum_{\substack{r,s' \\ r+s'=\ell_j\alpha+\ell_i}} \phi(c_{r,s'})\theta_i^{(r)}\theta_j^{(\ell_j)}\theta_i^{(s')} = 0 \ \in {}_R\mathbf{f}.$$

For a fixed s', we write $s' = a + \ell_i n$, where $a, n \in \mathbb{Z}$ and $0 \le a \le \ell_i - 1$. Note by Lemma 2.1(c) that $\begin{bmatrix} s' \\ t \end{bmatrix}_{\mathbf{q}_i,\pi_i} = \mathbf{q}_i^{-\ell_i nt}\begin{bmatrix} a \\ t \end{bmatrix}_{\mathbf{q}_i,\pi_i}$. Now using $r + s' = \ell_j\alpha + \ell_i$ we compute

$$\begin{aligned}
\phi(c_{r,s'}) &= (-1)^r\mathbf{q}_i^{r(\ell_i-1)}\sum_{t=0}^{\ell_i-1}(-1)^t\pi_i^{\ell_j rp(j)+r(r-1)/2+t(t-1)/2}\mathbf{q}_i^{t(s'-1)-\ell_i nt}\begin{bmatrix} a \\ t \end{bmatrix}_{\mathbf{q}_i,\pi_i} \\
&= (-1)^r\mathbf{q}_i^{r(\ell_i-1)}\sum_{t=0}^{a}(-1)^t\pi_i^{\ell_j rp(j)+r(r-1)/2+t(t-1)/2}\mathbf{q}_i^{t(a-1)}\begin{bmatrix} a \\ t \end{bmatrix}_{\mathbf{q}_i,\pi_i} \\
&\overset{(a)}{=} \delta_{a,0}(-1)^{\ell_j\alpha+\ell_i-\ell_i n}\pi_i^{\ell_j rp(j)+r(r-1)/2}\mathbf{q}_i^{(\ell_i-1)(\ell_j\alpha+\ell_i-\ell_i n)} \\
&\overset{(b)}{=} \delta_{a,0}(-1)^{\alpha\ell_j/\ell_i+1-n}\pi_i^{\ell_j rp(j)+r(r-1)/2-r(\ell_i-1)/2}.
\end{aligned} \tag{4.7}$$

The identity (a) above follows by the identity $\sum_{t=0}^{a}(-1)^t\pi_i^{t(t-1)/2}\mathbf{q}_i^{t(a-1)}\begin{bmatrix} a \\ t \end{bmatrix}_{\mathbf{q}_i,\pi_i} = \delta_{a,0}$ (see [4, 1.4.4]), and (b) follows by the identity $\pi_i^{(\ell_i-1)\ell_i/2}\mathbf{q}_i^{\ell_i^2-\ell_i} = (-1)^{\ell_i+1}$ (which is an i-version of (2.5) with the help of $\pi_i^{\ell_i}\mathbf{q}_i^{2\ell_i} = 1$).

Inserting (4.7) into (4.6) and comparing with (4.5), we reduce the proof of (4.5) to verify that $\pi_i^{\ell_i^2(np(j)+n(n-1)/2)} = \pi_i^{\ell_j\ell_i np(j)+\ell_i n(\ell_i n-1)/2-\ell_i n(\ell_i-1)/2}$, which is equivalent to verifying $\pi_i^{\ell_i^2 np(j)} = \pi_i^{\ell_j\ell_i np(j)}$. The latter identity is trivial unless both i and j are in I_1; when both i and j are in I_1, the identity follows from Lemma 3.1. Therefore, we have proved (4.5) and hence Theorem 4.1.

4.5. We develop in this subsection the analogue of [17, 35.3]; recall we are still working under the assumption that R is the quotient field of $\mathcal{A}'$.

Proposition 4.3 *Let $\lambda \in X^\diamond$, i.e., $\langle i, \lambda\rangle \in \ell_i\mathbb{Z}$ for all $i \in I$. Let M denote the simple highest weight module with highest weight λ in the category of R^π-free weight $\mathbf{U}$-modules, and let η be a highest weight vector of M^λ.*

(a) *If $\zeta \in X$ satisfies $M^\zeta \neq 0$, then $\zeta = \lambda - \sum_i \ell_i n_i i'$, where $n_i \in \mathbb{N}$. In particular, $\langle i, \zeta\rangle \in \ell_i\mathbb{Z}$ for all $i \in I$.*
(b) *If $i \in I$ is such that $\ell_i \geq 2$, then E_i, F_i act as zero on M.*
(c) *For any $r \geq 0$, let M'_r be the subspace of M spanned by $F_{i_1}^{(\ell_{i_1})} F_{i_2}^{(\ell_{i_2})} \dots F_{i_r}^{(\ell_{i_r})}\eta$ for various sequences $i_1, i_2, \dots, i_r$ in I. Let $M' = \sum_r M'_r$. Then, $M' = M$.*

Proof The proof is completely analogous to [17]. All computations are similar except that we are now working over R^π instead of R; and the results follow from Lemma 2.1, [4, (4.1) and Proposition 4.2.4], and Lemma 4.2.

First, we show that
(d) $E_i M'_r = 0$, $F_i M'_r = 0$ for any $i \in I$ such that $\ell_i \geq 2$,
which is similarly proved by induction on $r \geq 0$. The base case $r = 0$ follows from the fact that $\begin{bmatrix}\langle i, \lambda\rangle \\ t\end{bmatrix}_{\mathbf{q}_i,\pi_i} = 0$ since $\lambda \in X^\diamond$ (using Lemma 2.1) and the fact that $E_j^{(n)} F_i \eta$ is an R^π-linear combination of $F_i E_j^{(n)}$ and $E_j^{(n-1)}$. For the inductive step, we want to show that $E_i F_j^{(\ell_j)} m = 0$ and $F_i F_j^{(\ell_j)} m = 0$ for any $i, j \in I$ such that $\ell_i \geq 2$ and any $m \in M'_{r-1}\zeta$. For the first one we use the fact that $E_i F_j^{(\ell_j)} m$ is an R^π-linear combination of $F_j^{(\ell_j)} E_i m$ and $F_j^{\ell_j - 1}$ in the case $\ell_j \geq 2$, and for $\ell_j = 1$ we again use $\begin{bmatrix}\langle i, \lambda\rangle \\ t\end{bmatrix}_{\mathbf{q}_i,\pi_i} = 0$ from Lemma 2.1. For the second one, we may use [4, (4.1) and Proposition 4.2.4] to write $F_i F_j^{(\ell_j)} m$ as a R^π-linear combination of $F_j^{(\ell_j - r)} F_i F_j^{(r)} m$ for various r with $0 \leq r < \ell_j$, and for such r we have $F_i F_j^{(r)} m = 0$ by the induction hypothesis.

Next, we may show by induction on $r \geq 0$ that
(e) $E_i^{(l_i)} M'_r \subset M'_{r-1}$ for any $i \in I$,
(by convention $M'_{-1} = 0$); again for $m' \in M'_{r-1}$ we can use the fact that $E_i^{(l_i)} F_j^{(\ell_j)} m'$ is an R^π-linear combination of $F_j^{(\ell_j)} E_i^{(\ell_i)} m'$ (which is in M'_{r-1} by the induction hypothesis), and elements of the form $F_j^{(\ell_j - t)} E_i^{(\ell_i - t)} m'$ with $t > 0$ and $t \leq \ell_i, t \leq \ell_j$ (which as before are zero if $t < \ell_i$ or if $t = \ell_i$ and $t < \ell_j$, by (d), and are in M'_{r-1} if $t = \ell_i = \ell_j$).

The statements (d), (e) together with Lemma 4.2 show that $\sum_r M'_r$ is an ${}_R\dot{\mathbf{U}}$-submodules of M, and by simplicity of M it follows that $M = \sum_r M'_r$, from which (a) and (b) also follow. □

Corollary 4.4 *There is a unique weight ${}_R\dot{\mathbf{U}}^\diamond$-module structure on M (as in Proposition 4.3) in which the ζ-weight space is the same as that in the ${}_R\dot{\mathbf{U}}^\diamond$-modules M, for any $\zeta \in X^\diamond \subset X$, and such that $E_i, F_i \in {}_R\mathbf{f}^\diamond$ act as $E_i^{(\ell_i)}, F_i^{(\ell_i)} \in {}_R\mathbf{f}$. Moreover, this is a simple (R^π-free) highest weight module for ${}_R\dot{\mathbf{U}}^\diamond$ with highest weight $\lambda \in X^\diamond$.*

Proof We define operators $e_i, f_i : M \to M$ for $i \in I$ by $e_i = E_i^{(\ell_i)}$, $f_i = F_i^{(\ell_i)}$. Using Theorem 4.1 we see that e_i and f_i satisfy the Serre-type relations of ${}_R\mathbf{f}^\diamond$.

If $\zeta \in X\backslash X^\diamond$, we have $M^\zeta = 0$ by Proposition 4.3(a) above. If $\zeta \in X^\diamond$ and $m \in M^\zeta$, then we have that $(e_i f_j - f_j e_i)(m)$ is equal to $\delta_{i,j}\begin{bmatrix}\langle i,\lambda\rangle\\ \ell_i\end{bmatrix}_{\mathbf{q}_i,\pi_i}\cdot m$ plus an R^π-linear combination of elements of the form $F_i^{\ell_i-t}E_i^{\ell_i-t}(m)$ with $0<t<\ell_i$ (this follows by [7, Lemma 4]) which are zero by Proposition 4.3(b). Since $\langle i,\zeta\rangle \in \ell_i\mathbb{Z}$, we see from Lemma 3.6 that

$$\begin{bmatrix}\langle i,\lambda\rangle\\ \ell_i\end{bmatrix}_{\mathbf{q}_i,\pi_i} = \begin{bmatrix}\langle i,\lambda\rangle/\ell_i\\ 1\end{bmatrix}_{\mathbf{q}_i^\diamond,\pi_i^\diamond}$$

and so $(e_i f_j - f_j e_i)m = \delta_{i,j}[\langle i,\lambda\rangle/\ell_i]_{\mathbf{q}_i^\diamond,\pi_i^\diamond}\cdot m$. We also have that $e_i(M^\zeta)\subset M^{\zeta+\ell_i i'}$ and $f_i(M^\zeta)\subset M^{\zeta-\ell_i i'}$. Thus, we have a unital ${}_R\dot{\mathbf{U}}^\diamond$-module structure on M, and by Proposition 4.3(c) this is a highest weight module of ${}_R\dot{\mathbf{U}}^\diamond$ with highest weight λ and simplicity also follows using Lemma 4.2 in the same argument as in [17]. □

4.6. Now we are ready to state our analogue of the main result of [17, 35.4] on a tensor product decomposition. Let $\mathfrak{f}$ be the R-subalgebra of ${}_R\mathbf{f}$ generated by the elements θ_i for various i such that $\ell_i \ge 2$. We have $\mathfrak{f} = \oplus_\nu \mathfrak{f}_\nu$ where $\mathfrak{f} = {}_R\mathbf{f}_\nu \cap \mathfrak{f}$.

Theorem 4.5 (Lusztig–Steinberg tensor product theorem) *The R^π-linear map*

$$\chi : {}_R\mathbf{f}^\diamond \otimes_R \mathfrak{f} \to {}_R\mathbf{f}, \qquad x\otimes y \mapsto \mathrm{Fr}'(x)y$$

is an isomorphism of R^π-modules.

Proof First, we make the following statement which is similar to (but slightly less precise than) [17, 35.4.2(a)].

Claim. For any $i \in I$ and $y \in \mathfrak{f}_\nu$, there exists some $a(y), b(y) \in \mathbb{Z}$ such that the difference $\theta_i^{(\ell_i)}y - \pi_i^{a(y)}\mathbf{q}_i^{b(y)}y\theta_i^{(\ell_i)}$ belongs to $\mathfrak{f}$.

For $y = y'y''$ one easily reduces the claim to the same type of claim for y' and y''. Hence, it suffices to show this claim when y is a generator of $\mathfrak{f}$, i.e., $y = \theta_j$ where $\ell_j \ge 2$. Recall our assumption (a) in Sect. 4.1 that $\ell_i \ge -\langle i, j'\rangle + 1$. Hence, we may use the higher Serre relation in [4, (4.1) and Proposition 4.2.4] (but with θ_i's instead of F_i's) to show that for some $a(j), b(j)$, the difference $\theta_i^{(\ell_i)}\theta_j - \pi_i^{a(j)}\mathbf{q}_i^{b(j)}\theta_j\theta_i^{(\ell_i)}$ is an R^π-linear combination of products of the form $\theta_i^{(r)}\theta_j\theta_i^{(\ell_i-r)}$ with $0<r<\ell_i$, which are contained in $\mathfrak{f}$ by definition. The claim is proved.

By Lemma 4.2, ${}_R\mathbf{f}$ is generated by $\theta_i^{(\ell_i)}$ and θ_j with $\ell_j \ge 2$. The surjectivity of χ follows as the claim allows us to move factors θ_j to the right which produces lower terms in $\mathfrak{f}$.

The injectivity is proved by exactly the same argument as in [17, 35.4.2] using now Proposition 4.3 and Corollary 4.4; the details will be skipped. □

The following is an analogue of [17, Proposition 35.4.4], which follows by the same argument now using the anti-involution σ of ${}_R\mathbf{f}$ which fixes each θ_i (cf. [4, § 1.4]). We omit the detail to avoid much repetition.

Proposition 4.6 *Assume that the root datum is simply connected. Then, there is a unique $\lambda \in X^+$ such that $\langle i, \lambda \rangle = \ell_i - 1$ for all i. Let η be the canonical generator of ${}_R V(\lambda)$. The map $x \mapsto x^- \eta$ is an R^π-linear isomorphism $\mathfrak{f} \longrightarrow {}_R V(\lambda)$.*

4.7. The following is a generalization of [17, Theorem 35.1.7]. As with Theorem 4.1, we may reduce the proof to the case when R is the quotient field of $\mathcal{A}'$ (cf. [17, 35.1.11]).

Theorem 4.7 *There is a unique R^π-superalgebra homomorphism* $\mathrm{Fr} : {}_R\mathbf{f} \longrightarrow {}_R\mathbf{f}^\diamond$ *such that, for all $i \in I$, $n \in \mathbb{N}$,*

$$\mathrm{Fr}(\theta_i^{(n)}) = \begin{cases} \theta_i^{(n/\ell_i)}, & \textit{if } \ell_i \textit{ divides } n, \\ 0, & \textit{otherwise}. \end{cases}$$

(We call Fr the Frobenius–Lusztig homomorphism.)

Proof The proof proceeds essentially like that of [17, Theorem 35.1.7]. Uniqueness is clear; we need only prove the existence. By Theorem 4.5, there is an R^π-linear map $P : {}_R\mathbf{f} \longrightarrow {}_R\mathbf{f}^\diamond$, such that for all $i_k \in I$ and for $j_p \in I$ where $\ell_{j_p} \geq 2$

$$P(\theta_{i_1}^{(\ell_{i_1})} \cdots \theta_{i_n}^{(\ell_{i_n})} \theta_{j_1} \cdots \theta_{j_r}) = \begin{cases} \theta_{i_1} \cdots \theta_{i_n}, & \text{if } r = 0, \\ 0, & \text{otherwise}. \end{cases}$$

We now check that P is a homomorphism of R^π-algebras. Because ${}_R\mathbf{f}$ is generated as an R^π-module by elements of the form $x = \theta_{i_1}^{(\ell_{i_1})} ... \theta_{i_n}^{(\ell_{i_n})} \theta_{j_1} \cdots \theta_{j_r}$, we need to check that for any such x,

$$P(x\theta_j) = P(x)P(\theta_j) \tag{4.8}$$

for $j \in I$ such that $\ell_j \geq 2$ and

$$P(x\theta_i^{(\ell_i)}) = P(x)P\left(\theta_i^{(\ell_i)}\right) \tag{4.9}$$

for all $i \in I$. As (4.8) is obvious, we will concern ourselves with (4.9). Note that (4.9) is clear when $r = 0$. Assume now $r > 0$. Let us write $x' = \theta_{i_1}^{(\ell_{i_1})} ... \theta_{i_n}^{(\ell_{i_n})} \theta_{j_1} \cdots \theta_{j_{r-1}}$ and $\theta_j = \theta_{j_r}$ so that $x = x'\theta_j$. For $i = j$, we have $P(x)P\left(\theta_i^{(\ell_i)}\right) = 0$ and

$$P(x\theta_i^{(\ell_i)}) = P(x'\theta_i\theta_i^{(\ell_i)}) = P(x'\theta_i^{(\ell_i)}\theta_i) = P(x'\theta_i^{(\ell_i)})P(\theta_i) = 0,$$

where the third equality is due to (4.8). Now suppose that $i \neq j$. As $\ell_i > -\langle i, j' \rangle$, we may use the higher-order Serre relations for quantum covering groups (cf. [4, (4.1) and Proposition 4.2.4]) to write $\theta_j \theta_i^{(\ell_i)}$ as a linear combination of terms of the form $\theta_i^{(m)} \theta_j \theta_i^{(n)}$ where $m + n = \ell_i$ and $m \geq 1$. Because of (4.2) and (4.8), $P(x'\theta_i^{(m)}\theta_j\theta_i^{(n)}) = 0$ for $1 \leq m < \ell_i$, and $P(x'\theta_i^{(\ell_i)}\theta_j) = 0$.

Now that we know that P is an R^π-algebra homomorphism, and it remains to compute $P(\theta_i^{(n)})$ for all $n \in \mathbb{Z}_{\geq 0}$. Write $n = b\ell_i + a$, where $0 \leq a < \ell_i$ and $b \in \mathbb{Z}_{\geq 0}$. Using (4.1), (4.2) and (4.3), for $a > 0$ we have

$$P(\theta^{(b\ell_i+a)}) = \mathbf{q}_i^{\ell_i ab} P(\theta_i^{(a)}) P(\theta_i^{(b\ell_i)}) = \mathbf{q}_i^{\ell_i ab}([a]^!_{\mathbf{q}_i,\pi_i})^{-1} P\left(\theta_i^a\right) P(\theta_i^{(b\ell_i)}) = 0.$$

Similarly, for $a = 0$ we have

$$\begin{aligned} P\left(\theta_i^{(b\ell_i)}\right) &= (b!)^{-1}(\pi_i\mathbf{q}_i)^{-\ell_i^2\binom{b}{2}} P\left(\theta_i^{(\ell_i)}\right)^b \\ &= (b!)^{-1}(\pi_i^\diamond\mathbf{q}_i^\diamond)^{-\binom{b}{2}}\theta_i^b = ([b]^!_{\mathbf{q}_i^\diamond,\pi_i^\diamond})^{-1}\theta_i^b = \theta_i^{(b)}, \end{aligned}$$

where, in the third equality we used Lemma 2.2, with $\ell = 1$. Hence, P is the desired homomorphism Fr. □

4.8. We extend the Frobenius–Lusztig homomorphism $\mathrm{Fr} : {}_R\mathbf{f} \longrightarrow {}_R\mathbf{f}^\diamond$ in Theorem 4.7 to ${}_R\dot{\mathbf{U}}$. In contrast to the quantum group setting, we have to twist Fr slightly on one half of the quantum covering group.

Theorem 4.8 *There is a unique R^π-superalgebra homomorphism* $\mathrm{Fr} : {}_R\dot{\mathbf{U}} \longrightarrow {}_R\dot{\mathbf{U}}^\diamond$ *such that for all $i \in I$, $n \in \mathbb{Z}$, $\lambda \in X$,*

$$\mathrm{Fr}(E_i^{(n)}\mathbf{1}_\lambda) = \begin{cases} \pi_i^{\binom{\ell_i}{2}n/\ell_i} E_i^{(n/\ell_i)}\mathbf{1}_\lambda, & \textit{if } \ell_i \textit{ divides } n \textit{ and } \lambda \in X^\diamond, \\ 0, & \textit{otherwise} \end{cases} \tag{4.10}$$

and

$$\mathrm{Fr}(F_i^{(n)}\mathbf{1}_\lambda) = \begin{cases} F_i^{(n/\ell_i)}\mathbf{1}_\lambda, & \textit{if } \ell_i \textit{ divides } n \textit{ and } \lambda \in X^\diamond, \\ 0, & \textit{otherwise.} \end{cases}$$

(We also call Fr in this theorem the Frobenius–Lusztig homomorphism.)

Proof Let $\mathrm{Fr} : {}_R\mathbf{f} \longrightarrow {}_R\mathbf{f}^\diamond$ be the homomorphism from Theorem 4.7. Consider the homomorphism $\tilde{\mathrm{Fr}} = \psi \circ \mathrm{Fr}$, where $\psi : {}_R\mathbf{f}^\diamond \longrightarrow {}_R\mathbf{f}^\diamond$ is the algebra automorphism such that $\theta_i^{(n)} \mapsto \pi_i^n\theta_i^{(n)}$. The proof, much like that of [17, Theorem 35.1.9], amounts to checking that for $x, x' \in {}_R\mathbf{f}$ the assignment

$$x^+\mathbf{1}_\lambda x'^- \mapsto \tilde{\mathrm{Fr}}(x^+)\mathbf{1}_\lambda Fr(x'^-), \quad x^-\mathbf{1}_\lambda x'^+ \mapsto \mathrm{Fr}(x^-)\mathbf{1}_\lambda\tilde{\mathrm{Fr}}(x'^+),$$

for $\lambda \in X^\diamond$, and

$$x^+\mathbf{1}_\lambda x'^- \mapsto 0, \quad x^-\mathbf{1}_\lambda x'^+ \mapsto 0,$$

for $\lambda \in X \backslash X^\diamond$ satisfies the appropriate relations. These are the relations of Lemma 3.2 for ${}_R\dot{\mathbf{U}}$ and for ${}_R\dot{\mathbf{U}}^\diamond$, using Lemma 3.6 to deal with the $(\mathbf{q}, \pi)$-binomial coefficients.

The use of the homomorphism $\tilde{\mathrm{Fr}}$ (in place of Fr) on $\mathbf{U}^+$ is necessitated by the first and second relations in Lemma 3.2. Both sides of the first relation are mapped to zero by Fr unless $N, M \in \ell_i\mathbb{Z}$ and $\lambda \in X^\diamond$, so we focus on this case. Recalling $\mathbf{q}_i^\diamond, \pi_i^\diamond$ from (3.1), we have

$$
\begin{aligned}
&\mathrm{Fr}\left(\sum_{t\geq 0}\pi_i^{MN-\binom{t+1}{2}}F_i^{(M-t)}\begin{bmatrix}M+N+\langle i,\lambda\rangle\\ t\end{bmatrix}_{\mathbf{q}_i,\pi_i}\mathbf{1}_{\lambda+(M+N-t)i'}E_i^{(N-t)}\right)\\
&=\sum_{t\geq 0}\pi_i^{MN-\binom{t+1}{2}}\mathrm{Fr}\left(F_i^{(M-t)}\right)\begin{bmatrix}M+N+\langle i,\lambda\rangle\\ t\end{bmatrix}_{\mathbf{q}_i,\pi_i}\mathbf{1}_{\lambda+(M+N-t)i'}Fr\left(E_i^{(N-t)}\right)\\
&=\sum_{t\geq 0, t\in\ell_i\mathbb{Z}}(\pi_i^\diamond)^{(M/\ell_i)(N/\ell_i)-\binom{t/\ell_i+1}{2}}\pi_i^{t/\ell_i\binom{\ell_i}{2}}F_i^{((M-t)/\ell_i)}\begin{bmatrix}(M+N+\langle i,\lambda\rangle)/\ell_i\\ t/\ell_i\end{bmatrix}_{\mathbf{q}_i^\diamond,\pi_i^\diamond}\\
&\quad\cdot\mathbf{1}_{\lambda+(M+N-t)i'}\pi_i^{(N-t)/\ell_i\binom{\ell_i}{2}}E_i^{((N-t)/\ell_i)}\\
&=\pi_i^{N/\ell_i\binom{\ell_i}{2}}\sum_{t\geq 0, t\in\ell_i\mathbb{Z}}(\pi_i^\diamond)^{(M/\ell_i)(N/\ell_i)-\binom{t/\ell_i+1}{2}}F_i^{((M-t)/\ell_i)}\begin{bmatrix}(M+N+\langle i,\lambda\rangle)/\ell_i\\ t/\ell_i\end{bmatrix}_{\mathbf{q}_i^\diamond,\pi_i^\diamond}\\
&\quad\cdot\mathbf{1}_{\lambda+(M+N-t)i'}E_i^{((N-t)/\ell_i)}\\
&=\pi_i^{N/\ell_i\binom{\ell_i}{2}}E_i^{(N/\ell_i)}\mathbf{1}_\lambda F_i^{(M/\ell_i)}\\
&=\mathrm{Fr}\big(E_i^{(N)}\mathbf{1}_\lambda F_i^{(M)}\big),
\end{aligned}
$$

where we have used $\pi_i^{-\binom{t+1}{2}} = (\pi_i^\diamond)^{-\binom{t/\ell_i+1}{2}}\pi_i^{t/\ell_i\binom{\ell_i}{2}}$ and Lemma 3.6 in the second equality above.

The verification of the second relation of Lemma 3.2 is entirely similar, and the other relations therein are straightforward. □

5 Small quantum covering groups

In this section, we construct and study the small quantum covering groups. We take $R^\pi = \mathbb{Q}(\mathbf{q})^\pi$, where $\mathbf{q}$ is as in (2.2).

5.1. Let ${}_R\dot{\mathfrak{u}}$ be the subalgebra of ${}_R\dot{\mathbf{U}}$ generated by $E_i\mathbf{1}_\lambda$ and $F_i\mathbf{1}_\lambda$ for all $i \in I$ with $\ell_i \geq 2$ and $\lambda \in X$. It is clear then that ${}_R\dot{\mathfrak{u}}$ is spanned by terms of the form $x^+\mathbf{1}_\lambda x'^-$ where $x, x' \in \mathfrak{f}$. We follow the construction of [17, § 36.2.3] in extending ${}_R\dot{\mathbf{U}}$ to a new algebra ${}_R\hat{\mathbf{U}}$. Any element of ${}_R\dot{\mathbf{U}}$ can be written as a sum of the form $\sum_{\lambda,\mu\in X} x_{\lambda,\mu}$ where $x_{\lambda,\mu} \in \mathbf{1}_\lambda {}_R\dot{\mathbf{U}}\mathbf{1}_\mu$ is zero for all but finitely many pairs λ, μ. We relax this condition in ${}_R\hat{\mathbf{U}}$ by allowing such sums to have infinitely many nonzero terms provided that the corresponding $\lambda-\mu$ are contained in a finite subset of X. The algebra structure extends in the obvious way. We define ${}_R\hat{\mathfrak{u}}$ to be the subalgebra of ${}_R\hat{\mathbf{U}}$ with $x_{\lambda,\mu} \in \mathbf{1}_\lambda {}_R\dot{\mathfrak{u}}\mathbf{1}_\mu$.

Let $2\tilde{\ell}$ be the smallest positive integer such that $\mathbf{q}^{2\tilde{\ell}} = 1$. Hence, $\tilde{\ell} = 2\ell$ for ℓ odd and $\tilde{\ell} = \ell$ for ℓ even. We define the cosets

$$\mathbf{c_a} = \{\lambda \in X \mid \langle i, \lambda\rangle \equiv a_i \pmod{2\tilde{\ell}}, \quad \forall i \in I\}, \tag{5.1}$$

for $\mathbf{a} = (a_i | i \in I)$ with $0 \le a_i \le 2\tilde{\ell} - 1$. Note that there are at most $(2\tilde{\ell})^{|I|}$ such cosets and they partition X. Moreover, for each coset $\mathbf{c}$, $\mathbf{1_c} := \sum_{\lambda\in\mathbf{c}} \mathbf{1}_\lambda$ is an element of ${}_R\hat{\mathfrak{u}}$.

Let ${}_R\mathfrak{u}$ (resp. ${}_R\mathfrak{u}'$) be the R^π-submodule of ${}_R\hat{\mathfrak{u}}$ generated by the elements $x^+\mathbf{1_c}x'^-$ (resp. $x^-\mathbf{1_c}x'^+$) where $x, x' \in \mathfrak{f}$. The following is an analogue of [17, Lemma 36.2.4].

Lemma 5.1 (1) *For any $u \in {}_R\mathfrak{u}$ and $0 \le M \le \ell_i - 1$, $F_i^{(M)}u$ lies in ${}_R\mathfrak{u}$.*
(2) *We have ${}_R\mathfrak{u} = {}_R\mathfrak{u}'$, and ${}_R\mathfrak{u}$ is a subalgebra of ${}_R\hat{\mathfrak{u}}$.*

The algebra ${}_R\mathfrak{u}$ is called the *small quantum covering group*.

Proof We follow the proof in [17]. We prove the first statement by induction on p, where our $u = E_{i_1}^{(n_1)} \ldots E_{i_p}^{(n_p)} x'^-$. The result is obvious for $p = 0$, so we now consider $p \ge 1$ and rewrite u as

$$u = \mathbf{1}_{\mathbf{c}'} E_{i_1}^{(n_1)} x_1^+ x'^-$$

where $x_1 = \theta_{i_2}^{(n_2)} \ldots \theta_{i_p}^{(n_p)}$. When $i \neq i_1$, the result is immediate, so we consider $i = i_1$. In that case, using the relations of Lemma 3.2, we have

$$\begin{aligned} F_i^{(M)} u = \sum_{\lambda\in\mathbf{c}'} \sum_{t\le n_1, t\le M} \pi_i^{MN + t\langle i,\lambda\rangle - \binom{t}{2}} \begin{bmatrix} n_1 + M - \langle i, \lambda\rangle \\ t \end{bmatrix}_{\mathbf{q}_i,\pi_i} \\ \cdot E_i^{(a_1 - t)} \mathbf{1}_{\lambda-(n_1+M-t)i'} F_i^{(M-t)} x_1^+ x'^-. \end{aligned}$$

Fix $\mu \in \mathbf{c}'$. Then for any $\lambda \in \mathbf{c}'$, $n_1 + M - \langle i, \lambda\rangle \equiv n_1 + M - \langle i, \mu\rangle \bmod(\ell_i)$. Using Lemma 2.1 and noting that $t < \ell_i$, we have that

$$\begin{aligned} \begin{bmatrix} n_1 + M - \langle i, \lambda\rangle \\ t \end{bmatrix}_{\mathbf{q}_i,\pi_i} &= \mathbf{q}_i^{-\ell_i t(\langle i,\lambda\rangle - \langle i,\mu\rangle)} \begin{bmatrix} n_1 + M - \langle i, \mu\rangle \\ t \end{bmatrix}_{\mathbf{q}_i,\pi_i} \\ &= \begin{bmatrix} n_1 + M - \langle i, \mu\rangle \\ t \end{bmatrix}_{\mathbf{q}_i,\pi_i}, \end{aligned}$$

where we used in the second equality the condition that $\langle i, \lambda\rangle - \langle i, \mu\rangle \equiv 0 \bmod(2\tilde{\ell})$. Hence, $F_i^{(M)}u$ is equal to

$$\begin{aligned} &\sum_{t\le n_1, t\le M} \pi_i^{MN + t\langle i,\mu\rangle - \binom{t}{2}} \begin{bmatrix} n_1 + M - \langle i, \mu\rangle \\ t \end{bmatrix}_{\mathbf{q}_i,\pi_i} E_i^{(a_1 - t)} \left(\sum_{\lambda\in\mathbf{c}'} \mathbf{1}_{\lambda-(n_1+M-t)i'} \right) F_i^{(M-t)} x_1^+ x'^- \\ &= \sum_{t\le n_1, t\le M} \pi_i^{MN + t\langle i,\mu\rangle - \binom{t}{2}} \begin{bmatrix} n_1 + M - \langle i, \mu\rangle \\ t \end{bmatrix}_{\mathbf{q}_i,\pi_i} E_i^{(a_1 - t)} \mathbf{1}_{\mathbf{c}''} F_i^{(M-t)} x_1^+ x'^-, \end{aligned}$$

for some other $\mathbf{c}''$. Hence, $F_i^{(M)} u \in {}_R\mathfrak{u}$ by induction. Finally, the second statement is shown by repeated application of this result as in [17, Lemma 36.2.4]. $\square$

5.2. Recall there are a comultiplication Δ and an antipode S on $\mathbf{U}$ as defined in [4, Lemmas 2.2.1, 2.4.1]. Write ${}_\lambda\mathbf{U}_\mu$ for the subspace of ${}_R\dot{\mathbf{U}}$ spanned by elements of the form $\mathbf{1}_\lambda x \mathbf{1}_\mu$, where $x \in {}_R\mathbf{U}$ and write $p_{\lambda,\mu}$ for the canonical projection ${}_R\mathbf{U} \to {}_\lambda\mathbf{U}_\mu$. As in [17, 23.1.5, 23.1.6], Δ and S induce R^π-linear maps

$$\Delta_{\lambda,\mu,\lambda',\mu'} : {}_{\lambda+\lambda'}\mathbf{U}_{\mu+\mu'} \longrightarrow {}_\lambda\mathbf{U}_\mu \otimes {}_{\lambda'}\mathbf{U}_{\mu'}$$

given by $\Delta_{\lambda,\mu,\lambda',\mu'}(p_{\lambda+\lambda',\mu+\mu'}(x)) = (p_{\lambda,\mu} \otimes p_{\lambda',\mu'})(\Delta(x))$, for $\lambda, \mu, \lambda', \mu' \in X$, and

$$\dot{S} : {}_R\dot{\mathbf{U}} \longrightarrow {}_R\dot{\mathbf{U}}$$

defined by $\dot{S}(\mathbf{1}_\lambda x \mathbf{1}_\mu) = \mathbf{1}_{-\mu} S(x) \mathbf{1}_{-\lambda}$ for $x \in {}_R\mathbf{U}$. For example, $\Delta(E_i) = E_i \otimes 1 + \tilde{J}_i \tilde{K}_i \otimes E_i$ in ${}_R\mathbf{U}$, and hence, we obtain

$$\Delta_{\lambda-\nu+i',\lambda-\nu,\nu,\nu}(E_i\mathbf{1}_\lambda) = p_{\lambda-\nu+i',\lambda-\nu} \otimes p_{\nu,\nu}(E_i \otimes 1 + \tilde{J}_i\tilde{K}_i \otimes E_i) = E_i\mathbf{1}_{\lambda-\nu} \otimes \mathbf{1}_\nu.$$

This collection of maps is called the comultiplication on ${}_R\dot{\mathbf{U}}$, and it can be formally regarded as a single linear map

$$\dot{\Delta} = \prod_{\lambda,\mu,\lambda',\mu' \in X} \hat{\Delta}_{\lambda,\mu,\lambda',\mu'} : {}_R\dot{\mathbf{U}} \longrightarrow \prod_{\lambda,\mu,\lambda',\mu' \in X} {}_\lambda\mathbf{U}_\mu \otimes {}_{\lambda'}\mathbf{U}_{\mu'}.$$

A comultiplication $\dot{\Delta}^\diamond$ on ${}_R\dot{\mathbf{U}}^\diamond$ can be defined in the same way.

Proposition 5.2 *The Frobenius–Lusztig homomorphism* Fr *is compatible with the comultiplications on ${}_R\dot{\mathbf{U}}$ and ${}_R\dot{\mathbf{U}}^\diamond$, i.e.,* $\dot{\Delta}^\diamond \circ \mathrm{Fr} = (\mathrm{Fr} \otimes \mathrm{Fr}) \circ \dot{\Delta}$.

(In the usual quantum group setting this was noted by [17, 35.1.10].)

Proof It suffices to check on the generators $E_i^{(n)}\mathbf{1}_\lambda$ and $F_i^{(n)}\mathbf{1}_\lambda$. Let $n = m\ell_i \in \ell_i\mathbb{Z}$, and recall that $\mathrm{Fr}(E_i^{(m\ell_i)}\mathbf{1}_\lambda) = \pi_i^{\binom{\ell_i}{2}m} E_i^{(m)}\mathbf{1}_\lambda$ in ${}_R\dot{\mathbf{U}}^\diamond$. Using the formula (above [4, Proposition 2.2.2])

$$\Delta(E_i^{(m)}) = \sum_{p+r=m} (\pi_i q_i)^{pr} E_i^{(p)} (\tilde{J}_i\tilde{K}_i)^r \otimes E_i^{(r)}$$

we see that the nonzero parts in $\dot{\Delta}^\diamond(\mathrm{Fr}(E_i^{(m\ell_i)}\mathbf{1}_\lambda))$ computed via (4.10) are of the form

$$\pi_i^{\binom{\ell_i}{2}m} (\pi_i^\diamond q_i^\diamond)^{(p+\langle i,\nu\rangle^\diamond)r} E_i^{(p)}\mathbf{1}_\nu \otimes E_i^{(r)}\mathbf{1}_{\lambda-\nu}, \qquad p+r=m$$

for various $\nu \in X^\diamond$, which coincides with $\mathrm{Fr} \otimes \mathrm{Fr}$ applied to terms in $\dot{\Delta}(E_i^{(m\ell_i)}\mathbf{1}_\lambda))$ of the form

$$(\pi_i q_i)^{(p\ell_i+\langle i,\nu\rangle)(r\ell_i)} E_i^{(p\ell_i)}\mathbf{1}_\nu \otimes E_i^{(r\ell_i)}\mathbf{1}_{\lambda-\nu}, \qquad p+r=m,$$

where we note there is a factor contributing from (4.10) which matches up with the previous part thanks to $\pi_i^{\binom{\ell_i}{2}p+\binom{\ell_i}{2}r} = \pi_i^{\binom{\ell_i}{2}m}$; the remaining terms are zero under $\mathrm{Fr}\otimes\mathrm{Fr}$ since at least one of the divided powers of E_i appearing in either tensor factor must be not divisible by ℓ_i.

On the other hand, if n is not divisible by ℓ_i, then the right-hand side will also be zero, since all the nonzero parts of $\dot{\Delta}(E_i^{(n)}\mathbf{1}_\lambda))$ will have a tensor factor containing some divided power of E_i not divisible by ℓ_i.

A similar verification takes care of $F_i^{(n)}1_\lambda$. □

5.3. The maps $\dot{\Delta}$ and $\dot{S}$ restrict to maps on ${}_R\dot{\mathfrak{u}}$, which extend to R^π-linear maps $\hat{\Delta}$ and $\hat{S}$ on ${}_R\hat{\mathfrak{u}}$ in the obvious way. Henceforth, when we refer to $\hat{\Delta}$ and $\hat{S}$, we mean the restrictions to ${}_R\mathfrak{u}$.

Additionally, for any basis $\mathbf{B}$ of $\mathfrak{f}$ consisting of weight vectors, with unique zero weight element equal to 1, we define an R^π-linear map $\hat{e} : {}_R\mathfrak{u} \to R^\pi$ by:

$$\hat{e}(rb^+b'^-\mathbf{1}_{\mathbf{c_a}}) = \begin{cases} r, & \text{if } b, b' = 1 \text{ and } \mathbf{a} = \mathbf{0}, \\ 0, & \text{otherwise.} \end{cases}$$

where $b, b' \in \mathbf{B}$, $r \in R^\pi$, and $\mathbf{c_a}$ in (5.1).

Define the following elements:

$$K_i = \sum_{\lambda\in X} \mathbf{q}^{\langle i,\lambda\rangle}\mathbf{1}_\lambda, \quad J_i = \sum_{\lambda\in X} \pi^{\langle i,\lambda\rangle}\mathbf{1}_\lambda, \quad 1 = \sum_{\lambda\in X}\mathbf{1}_\lambda. \tag{5.2}$$

Proposition 5.3 (1) *The R^π-algebra ${}_R\mathfrak{u}$ has a generating set $\{E_i, F_i$ ($\forall i$ with $\ell_i \geq 2$), K_i, J_i ($\forall i \in I$)$\}$.*
(2) $({}_R\mathfrak{u}, \hat{\Delta}, \hat{e}, \hat{S})$ *forms a Hopf superalgebra.*

Proof The elements in (5.2) can be written as

$$K_i = \sum_{\mathbf{c}} \mathbf{q}_{\mathbf{c},i}\mathbf{1}_{\mathbf{c}}, \quad J_i = \sum_{\mathbf{c}} \pi_{\mathbf{c},i}\mathbf{1}_{\mathbf{c}}, \quad 1 = \sum_{\mathbf{c}}\mathbf{1}_{\mathbf{c}},$$

where we have defined $\mathbf{q}_{\mathbf{c},i} = \mathbf{q}^{\langle i,\lambda\rangle}$ and $\pi_{\mathbf{c},i} = \pi^{\langle i,\lambda\rangle}$ for any $\lambda \in \mathbf{c}$. This implies that these elements are also in ${}_R\mathfrak{u}$. Moreover, we have

$$\mathbf{1}_{\mathbf{c}} = \prod_{i\in I}(2\tilde{\ell})^{-1}(1+\pi_{\mathbf{c},i}J_i)(1+\mathbf{q}_{\mathbf{c},i}^{-1}K_i+\mathbf{q}_{\mathbf{c},i}^{-2}K_i^2+\cdots+\mathbf{q}_{\mathbf{c},i}^{1-\tilde{\ell}}K_i^{\tilde{\ell}-1}).$$

This proves (1).

A direct computation using these generators shows that $\hat{\Delta}$, $\hat{e}$ and $\hat{S}$ are given by the same formulas as Δ, e and S; the former maps inherit the following properties of the latter: $\hat{\Delta}$ is a homomorphism which satisfies the coassociativity (cf. [4, Lemmas 2.2.1 and 2.2.3]), $\hat{e}$ is a homomorphism (cf. [4, Lemma 2.2.3]), and

$\hat{S}(xy) = \pi^{p(x)p(y)}\hat{S}(y)\hat{S}(x)$ (cf. [4, Lemma 2.4.1]). Moreover, the image of $\hat{\Delta}$ (respectively, $\hat{S}$) lies in ${}_R\mathfrak{u} \otimes {}_R\mathfrak{u}$ (respectively, ${}_R\mathfrak{u}$). Hence, (2) holds. □

5.4. We consider the Cartan datum associated with the Lie superalgebra $\mathfrak{osp}(1|2n)$, where $n = |I|$, with the following Dynkin diagram:

The black node denotes the (only) odd simple root. We set

$$i \cdot i = \begin{cases} 2, & \text{if } i \text{ is odd,} \\ 4, & \text{if } i \text{ is even.} \end{cases}$$

The above Cartan datum on I is a super Cartan datum satisfying the bar-consistent condition in the sense of Sect. 3.1.

Proposition 5.4 *The small quantum covering group ${}_R\mathfrak{u}$ of type $\mathfrak{osp}(1|2n)$ is a finite-dimensional R^π-module. In particular,*

$$\dim_{R^\pi}({}_R\mathfrak{u}) = \frac{\ell^{2n^2}}{\gcd(2,\ell)^{2n^2-2n}}(2\tilde{\ell})^n = \begin{cases} \ell^{2n^2}(4\ell)^n, & \textit{for } \ell \textit{ odd,} \\ \dfrac{\ell^{2n^2}}{2^{2n^2-2n}}(2\ell)^n, & \textit{for } \ell \textit{ even,} \end{cases}$$

when X is the weight lattice, and similarly,

$$\dim_{R^\pi}({}_R\mathfrak{u}) = \frac{\ell^{2n^2}}{\gcd(2,\ell)^{2n^2-2n}}2^{n-1}\tilde{\ell}^n = \begin{cases} \ell^{2n^2}2^{2n-1}\ell^n, & \textit{for } \ell \textit{ odd,} \\ \dfrac{\ell^{2n^2}}{2^{2n^2-2n}}2^{n-1}\ell^n, & \textit{for } \ell \textit{ even,} \end{cases}$$

when X is the root lattice.

Proof Note that ${}_R\mathfrak{u}$ is a $\mathfrak{f} \otimes \mathfrak{f}^{\mathrm{opp}}$ module with basis given by the $\mathbf{1_c}$ defined above. This basis has at most $(2\tilde{\ell})^n$ elements for any X. In particular, it has $(2\tilde{\ell})^n$ elements when X is the weight lattice, and $2^{n-1}\tilde{\ell}^n$ elements when X is the root lattice, as the root lattice is index 2 in the weight lattice. Moreover, by Proposition 4.6, we have that $\dim_{R^\pi}(\mathfrak{f}^\pm) = \dim_{R^\pi}({}_RV(\lambda))$, where λ is the unique weight such that $\langle i, \lambda\rangle = \ell_i - 1$ for each $i \in I$. Let $V(\lambda)_1$ (respectively, $V(\lambda)_{-1}$) be the quotient of the Verma module of highest weight λ by its maximal ideal for the quantum group (resp. quantum supergroup) to which the quantum covering group specializes at $\pi = 1$ (respectively, $\pi = -1$) with base field $R = \mathbb{Q}(\varepsilon)$ (recall from Sect. 2.3 that ε is an ℓ'-th root of unity). Because

$${}_RV(\lambda) = (\pi + 1){}_RV(\lambda) \oplus (\pi - 1){}_RV(\lambda) \cong V(\lambda)_1 \oplus V(\lambda)_{-1}$$

and the characters of $V(\lambda)_1$ and $V(\lambda)_{-1}$ coincide for dominant weights (cf. [13], [5, Remark 2.5]), we have

$$\dim_{R^\pi} \mathfrak{f}^{\pm} = \dim_{R^\pi}\ {}_R V(\lambda) = \dim_R V(\lambda)_1 = \dim_R \mathfrak{f}_1^{\pm} = \frac{\ell^{n^2}}{\gcd(2,\ell)^{n^2-n}}$$

where $\mathfrak{f}_1$ is the (non-super) half small quantum group, i.e., $\mathfrak{f}$ specialized at $\pi = 1$. The last equality is due to [16, Theorem 8.3(iv)]. □

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