ON THE OBERLIN AFFINE CURVATURE CONDITION

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Abstract

We generalize the well-known notions of affine arclength and affine hypersurface measure to submanifolds of any dimension d in $\mathbb{R}^n$, $1 \leq d \leq n-1$. We show that a canonical equiaffine-invariant measure exists and that, modulo sufficient regularity assumptions on the submanifold, the measure satisfies the affine curvature condition of Oberlin with an exponent which is best possible. The proof combines aspects of geometric invariant theory, convex geometry, and frame theory. A significant new element of the proof is a generalization to higher dimensions of an earlier result concerning inequalities of reverse Sobolev type for polynomials on arbitrary measurable subsets of the real line.

Contents

1. Introduction

Many of the deep questions in harmonic analysis, such as Fourier restriction, decoupling theory, or L^p-improving estimates for geometric averages, deal with certain operators associated to submanifolds of Euclidean space. In most cases, the "nicest possible" submanifolds are, informally, as far as possible from lying in any affine hyperplane. Many of these problems also exhibit natural equiaffine invariance, meaning that when the underlying Euclidean space is transformed by a measure-preserving affine linear mapping, the relevant quantities (i.e., norms and so forth) are unchanged.

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This simple observation leads naturally to the question of how in general to properly quantify this sort of well-curvedness in a way that respects equiaffine invariance. Of the many approaches to this question, one particularly successful strategy has been the use of the so-called *affine arclength measure* for curves and the analogous notion of *affine hypersurface measure* (sometimes called *equiaffine measure*). Affine arclength measure is defined for the curve parameterized by $\gamma : I \to \mathbb{R}^n$ by

$$\int_{\mathbb{R}^n} f \, d\mu_{\mathcal{A}} := \int_I f\big(\gamma(t)\big)\big|\det\big(\gamma'(t), \dots, \gamma^{(n)}(t)\big)\big|^{\frac{2}{n(n+1)}} \, dt,$$

and equiaffine measure for the graph $\{(x, \varphi(x)) \in \mathbb{R}^n | x \in U \subset \mathbb{R}^{n-1}\}$ is

$$\int_{\mathbb{R}^n} f \, d\mu_{\mathcal{A}} := \int_U f\big(x, \varphi(x)\big)\big|\det \nabla^2 \varphi(x)\big|^{\frac{1}{n+1}} \, dx,$$

where $\nabla^2 \varphi$ is the Hessian matrix of second derivatives of φ. Although these measures were well known outside harmonic analysis for quite some time (see, e.g., [16], [23]), their first appearances within the field are somewhat more recent, in work of Sjölin [29] (in two dimensions, generalized later by Drury and Marshall [10]) and Carbery and Ziesler [4], respectively. Both measures have the property that they are independent of the parameterization and that they are suitably invariant when the curve or surface is transformed by an equiaffine mapping. These measures and certain "variable coefficient" generalizations to families of curves and hypersurfaces have played a central role in the Fourier restriction problem as well as the problem of characterizing the L^p–L^q mapping properties of geometrically constructed convolution operators, two problems which have been of sustained interest for many years (see, e.g., [1], [2], [5]–[9], [15], [24], [27], [30], [31]).

The deep connections between analysis and geometry enjoyed by affine arclength and hypersurface measures naturally lead to the problem of generalizing these objects to manifolds of arbitrary dimension or even to abstract measure-theoretic settings. One particularly interesting approach is due to Oberlin [25] (which generalizes an earlier observation of Graham, Hare, and Ritter [12] in one dimension), who introduced the following condition on nonnegative measures μ associated to submanifolds: a Borel measure μ on $\mathbb{R}^n$ which is supported on a d-dimensional immersed submanifold of $\mathbb{R}^n$ will be said to satisfy the Oberlin condition with exponent $\alpha > 0$ when there exists a finite positive constant C such that, for every K in the set $\mathcal{K}_n$ of compact convex subsets of $\mathbb{R}^n$,

$$\mu(K) \leq C|K|^{\alpha}, \tag{1}$$

where $|K|$ represents the usual Lebesgue measure of K in $\mathbb{R}^n$. When restricted to the class of balls with respect to the standard metric on $\mathbb{R}^n$, the condition (1) becomes a familiar inequality from geometric measure theory. Unlike in that setting, here the exponent α measures not just the dimension of the measure, but also a certain kind of curvature, for the simple reason that (1) cannot hold for any $\alpha > 0$ when μ is supported on a hyperplane, which can be seen by taking K to be increasingly thin in the direction transverse to such a hyperplane. Oberlin observed that this condition is necessary for Fourier restriction or L^p-improving estimates to hold; in particular,

$$\left(\int |\hat{f}|^q \, d\mu\right)^{\frac{1}{q}} \lesssim \|f\|_{L^p(\mathbb{R}^n)} \quad \forall f \in L^p(\mathbb{R}^n) \Rightarrow \mu(K) \lesssim |K|^{\frac{q}{p'}} \quad \forall K \in \mathcal{K}_n$$

and

$$\|f * \mu\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)} \quad \forall f \in L^p(\mathbb{R}^n) \Rightarrow \mu(K) \lesssim |K|^{\frac{1}{p}-\frac{1}{q}} \quad \forall K \in \mathcal{K}_n,$$

where $\hat{\cdot}$ is the Fourier transform and $*$ is convolution. (Here and throughout the present article, the notation $A \lesssim B$ means that there is a finite positive constant C such that $A \leq CB$, and this constant C is independent of the relevant variables, functions, sets, and so forth, appearing in the expressions or quantities A and B.) By virtue of known results for these two problems, the affine arclength and hypersurface measures must satisfy (1) for appropriate exponents α when suitable regularity hypotheses on the submanifolds are imposed.

The significance of the Oberlin condition (1) for curves and hypersurfaces in $\mathbb{R}^n$ is that, up to a constant factor, the affine arclength and affine hypersurface measures on immersed submanifolds are the unique largest measures on such manifolds satisfying (1) when $\alpha = 2/(n^2+n)$ and $\alpha = (n-1)/(n+1)$, respectively. More precisely, for hypersurfaces satisfying certain algebraic constraints, Oberlin [25] showed that any sufficiently regular measure μ which is supported on an immersed hypersurface $\mathcal{M} \subset \mathbb{R}^n$ and which satisfies (1) with $\alpha = (n+1)/(n-1)$ must also satisfy $\mu \lesssim \mu_{\mathcal{A}}$ for affine hypersurface measure $\mu_{\mathcal{A}}$ (where $\lesssim$ here means $\mu(E) \lesssim \mu_{\mathcal{A}}(E)$ uniformly for all Borel sets E). Moreover, under the same algebraic limits on the complexity of the immersion, $\mu_{\mathcal{A}}$ itself satisfies (1) for this same exponent. The condition (1) also turns out to be equivalent to the boundedness of certain geometrically constructed multilinear determinant functionals (see [14]). For curves in the plane, the Oberlin condition (1) has also been shown to be connected to an affine generalization of the classical Hausdorff measure, which as Oberlin showed in [26], reduces to affine arclength on sufficiently regular convex curves.

In this article we examine the Oberlin condition for arbitrary d-dimensional submanifolds of $\mathbb{R}^n$ (where $1 \leq d < n$) and we characterize it in the case of maximal nondegeneracy. Specifically, results analogous to those just mentioned above are established in all dimensions and codimensions: an equiaffine-invariant measure[1] is

[1]The precise definition of this measure is delayed until Section 2.2 because there are a number of items of notation and auxiliary geometric objects which must first be defined and understood.

constructed which is essentially the largest possible measure satisfying the Oberlin condition for the largest nontrivial choice of α. To say that α is nontrivial means simply that there is a nonzero measure satisfying (1) for this α on some immersed submanifold of the given dimension and codimension. As in the case of curves and hypersurfaces, the largest nontrivial α can be understood as a ratio of the intrinsic dimension of the submanifold and its *h*omogeneous dimension, which captures information about scaling and curvature-like properties to be measured. The correct value of homogeneous dimension is defined as follows: when d and n are fixed, let the homogeneous dimension $Q = Q(d,n)$ be defined to be the smallest positive integer which equals the sum of the degrees of some collection of n distinct, nonconstant monomials in d variables (see Figure 1). The main result of this paper is Theorem 1.

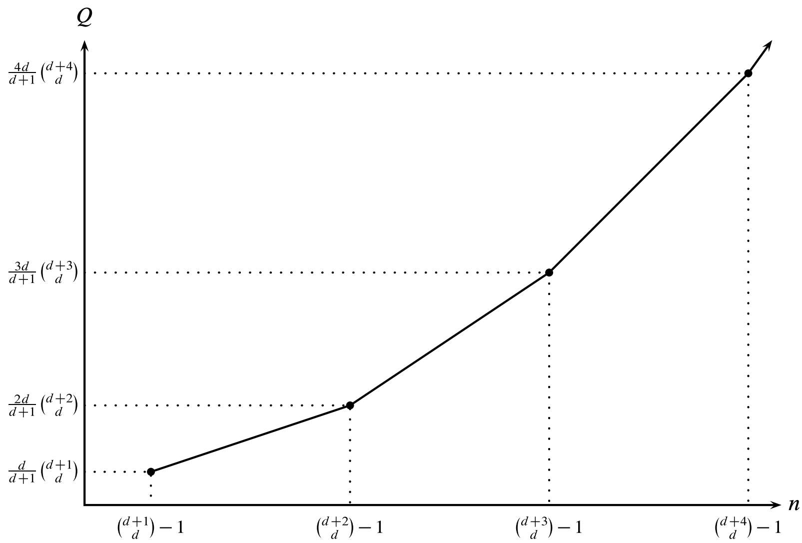

Figure 1. This plot shows the homogeneous dimension Q as a function of n for d fixed. The graph is piecewise linear with slope $k+1$ from the point $\left(\binom{d+k}{d}-1, \frac{kd}{d+1}\binom{d+k}{d}\right)$ to the point $\left(\binom{d+k+1}{d}-1, \frac{(k+1)d}{d+1}\binom{d+k+1}{d}\right)$ for each $k \geq 1$.

THEOREM 1

Suppose that $\mathcal{M}$ is an immersed d-dimensional submanifold of $\mathbb{R}^n$. To any such $\mathcal{M}$, one may associate a nonnegative Borel measure $\mu_{\mathcal{A}}$ on $\mathbb{R}^n$, defined by the formula (15) in Section 2.2, which is supported on $\mathcal{M}$. Then the following are true:

(1) *If μ is any nonnegative Borel measure supported on $\mathcal{M}$ which satisfies (1) with exponent $\alpha > d/Q$, then μ is the zero measure.*

(2) *If μ is any nonnegative Borel measure supported on $\mathcal{M}$ which satisfies (1) with exponent $\alpha = d/Q$, then $\mu \lesssim \mu_{\mathcal{A}}$.*

(3) *If $\mathcal{M}$ is the image of an immersion $f : \Omega \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^d$ is open with compact closure $\overline{\Omega}$ and f extends to be real analytic on a neighborhood of $\overline{\Omega}$, then the measure $\mu_{\mathcal{A}}$ satisfies (1) with exponent $\alpha = d/Q$ and is consequently the largest such measure, up to a multiplicative constant.*

Furthermore, it is also the case that:

(4) *The measure $\mu_{\mathcal{A}}$ agrees up to normalization with affine arclength and affine hypersurface measure when $d = 1$ and $n - 1$, respectively.*

(5) *The correspondence sending $\mathcal{M}$ to $\mu_{\mathcal{A}}$ is equiaffine invariant, meaning that if $\mathcal{M}$ and $\mathcal{M}'$ are immersed submanifolds such that $\mathcal{M}'$ is the image of $\mathcal{M}$ under some equiaffine transformation T of $\mathbb{R}^n$, then the measure $\mu'_{\mathcal{A}}$ corresponding to $\mathcal{M}'$ and the measure $\mu_{\mathcal{A}}$ corresponding to $\mathcal{M}$ satisfy $\mu'_{\mathcal{A}}(T(E)) = \mu_{\mathcal{A}}(E)$ for all Borel sets E.*

Theorem 1 extends Oberlin's result for equiaffine hypersurface measure in [25] to submanifolds of any dimension. To certify the nontriviality of Theorem 1—that is, to demonstrate that $\mu_{\mathcal{A}}$ is not simply the zero measure for all possible choices of $\mathcal{M}$—it is necessary to carry out some additional careful study of $\mu_{\mathcal{A}}$ for submanifolds $\mathcal{M}$ parameterized by $f : \Omega \to \mathbb{R}^n$ of the form

$$f(t) := \big((t^\alpha)_{1 \leq |\alpha| < \kappa}, p_1(t), \ldots, p_m(t)\big), \tag{2}$$

where $p_1, \ldots, p_m$ are linearly independent, real homogeneous polynomials of degree κ in d variables, and m and κ are chosen (as functions of n and d only) so that the right-hand side of (2) is an element of $\mathbb{R}^n$. For such polynomials, let $P_{j\ell}(t) := \partial_j p_\ell(t)$, $j = 1, \ldots, d$, $\ell = 1, \ldots, m$. Such an embedding f will be called a *model form* when there exist real numbers c and c' such that

$$\sum_{j=1}^{d} P_{j\ell}(\partial) P_{j\ell'}(t)|_{t=0} = c\delta_{\ell,\ell'} \qquad \text{and} \qquad \sum_{\ell=1}^{m} P_{j\ell}(\partial) P_{j'\ell}(t)|_{t=0} = c'\delta_{j,j'}, \tag{3}$$

where δ is the Kronecker delta. The main result for model forms is Theorem 2.

THEOREM 2

The following are true for embeddings f of the form (2).

(1) *The closure of the orbit $\{Nf(M^T t)\}_{N \in \mathrm{SL}(n,\mathbb{R}), M \in \mathrm{SL}(d,\mathbb{R})}$ in the space of n-tuples of polynomials of degree at most κ always contains an embedding of the form (2) which is a model form (3). If any embedding in the closure of the orbit is degenerate, then all are degenerate (i.e., $\mu_{\mathcal{A}} = 0$ for each embedding in the orbit closure or $\mu_{\mathcal{A}} \neq 0$ for each embedding).*

(2) *For any $p := (p_1, \dots, p_m)$ satisfying (3), the measure $\mu_{\mathcal{A}}$ associated to the submanifold of $\mathbb{R}^n$ parameterized by (2) is a nonzero constant times the push-forward of Lebesgue measure via f if and only if c and c' are nonzero.*

(3) *For any pair (d,n) with $1 \leq d < n$, there is some $p := (p_1, \dots, p_m)$ satisfying (3) for nonzero c and c'. Consequently, in any dimension and codimension, there is a submanifold $\mathcal{M}$ whose affine measure $\mu_{\mathcal{A}}$ is everywhere nonzero on $\mathcal{M}$.*

Outline

The structure of this article is as follows. In Section 2, the measure $\mu_{\mathcal{A}}$ is constructed by combining ideas of Kempf and Ness [22] from geometric invariant theory together with a simple but far-reaching observation that any covariant tensor field on a manifold can be used to construct an associated measure on that manifold in a way that generalizes the relationship between the Riemannian metric tensor and the Riemannian volume. In particular, the measure $\mu_{\mathcal{A}}$ will be the measure associated to an "affine curvature tensor" on the manifold $\mathcal{M}$ immersed in $\mathbb{R}^n$. The general process of passing from a tensor to an associated measure is detailed in Section 2.1, and the construction of the affine curvature tensor is given in Section 2.2. Section 2.2 then explicitly gives a definition of the measures (15) (via the constructions from Section 2.1) which are the subject of Theorem 1 and gives proofs of parts (4) and (5) of that theorem (i.e., that $\mu_{\mathcal{A}}$ is intrinsic, equiaffine-invariant, and agrees up to constants with affine arclength and equiaffine measure in dimension and codimension 1, respectively).

Section 3 is devoted to an in-depth analysis of the measure $\mu_{\mathcal{A}}$ constructed in Section 2 with a particular emphasis on developing a host of computational tools to use when establishing the triviality or nontriviality of $\mu_{\mathcal{A}}$ in both general and concrete cases. The number and variety of results in Section 3 highlight the wealth of possibilities for understanding affine curvature which results from the meeting of several seemingly disjoint areas of mathematics. Readers primarily interested in the proofs of Theorems 1 and 2 rather than applications can proceed to Section 3.3, as it contains proofs of parts (1) and (2) of Theorem 2 and lays additional groundwork for the later proof of part (3).

Section 4 returns to our main thread, with proofs of parts (1) and (2) of Theorem 1, which are based on what are essentially elementary observations concerning scaling, Taylor approximation, and convexity.

Section 5 is devoted to the proofs of part (3) of Theorem 1 and part (3) of Theorem 2. Part (3) of Theorem 1 is proved by first generalizing Theorem 1 of [13] to higher dimensions. This generalization, accomplished by Theorem 3 and Lemma 7, is interesting in its own right and will likely have important implications for the theory of L^p-improving estimates for averages over submanifolds in much the same way that [13, Theorem 1] formed the basis for a new proof of a restricted version of Tao and Wright's result in [33] for averages over curves. To formulate these results, it is convenient to make the following definition. Let $\mathcal{M}$ be any real analytic manifold of dimension d, and let $\mathcal{F}$ be a finite-dimensional vector space of real analytic functions on $\mathcal{M}$ whose differentials span the cotangent space at every point of $\mathcal{M}$. Any such pair $(\mathcal{M}, \mathcal{F})$ will be called a *geometric function system*. Such a system will be called *compact* when either $\mathcal{M}$ is compact or has a compact closure in some larger real analytic manifold $\mathcal{M}^+$ such that the functions of $\mathcal{F}$ extend to functions $\mathcal{F}^+$ on $\mathcal{M}^+$ in such a way that $(\mathcal{M}^+, \mathcal{F}^+)$ is also a geometric function system. The "zeroth order" generalization of the results of [13] are the following.

THEOREM 3
Suppose that $(\mathcal{M}, \mathcal{F})$ is a compact geometric function system. Then for any finite positive measure μ on $\mathcal{M}$ absolutely continuous with respect to Lebesgue measure and any measurable set $E \subset \mathcal{M}$ of positive measure there is a measurable subset $E' \subset E$ such that $\mu(E') \gtrsim \mu(E)$ and

$$\sup_{p \in E'} \left| f(p) \right| \lesssim \frac{1}{\mu(E)} \int_E |f| \, d\mu \quad \text{for all } f \in \mathcal{F}.$$

The implicit constants in both inequalities depend only on the pair $(\mathcal{M}, \mathcal{F})$.

The proof of this theorem and its generalization appear in Section 5.1. The use of these results to prove part (3) of Theorem 1 appears in Section 5.2. Section 5.3 shows that the measure $\mu_{\mathcal{A}}$ from Theorems 1 and 2 is not trivial by proving part (3) of Theorem 2—that is, by constructing submanifolds in every possible dimension and codimension such that $\mu_{\mathcal{A}}$ is comparable to the pushforward of Lebesgue measure.

Finally, Section 5.3 establishes uniform estimates for the number of nondegenerate solutions—that is, solutions where the Jacobian determinant is nonvanishing—of certain systems of equations encountered in Section 5.1. These estimates are important for part (3) of Theorem 1; furthermore, they establish not only that the Oberlin condition is satisfied for submanifolds with algebraic or real analytic parameterizations, but also give an indication as to how one can extend the same result to more general situations like global polynomial embeddings or o-minimal structures.

Notation
As already noted, this paper will make frequent use of the notation $A \lesssim B$ to indicate that there is a finite positive constant C such that $A \leq CB$ with C independent of the relevant functions, sets, and so on, appearing in the expressions or quantities A and B. Also, $A \approx B$ means $A \lesssim B$ and $B \lesssim A$.

The symbol ∂ will generally represent the d-tuple $(\partial_1, \ldots, \partial_d)$ of standard coordinate partial derivatives on $\mathbb{R}^d$. If $M \in \mathbb{R}^{d \times d}$, then $M\partial$ will represent the d-tuple of coordinate derivatives given by

$$(M\partial)_i = \sum_{j=1}^{d} M_{ij} \partial_j .$$

In a few places, the notation ∇f will denote the standard coordinate gradient of f; that is, $\nabla f := (\partial_1 f, \ldots, \partial_d f)$. This paper also makes extensive use of multi-index notation: for any d-tuple $\alpha := (\alpha_1, \ldots, \alpha_d)$ of nonnegative integers, let

$$t^\alpha := t_1^{\alpha_1} \cdots t_d^{\alpha_d} \quad \text{for all } t \in \mathbb{R}^d, \qquad \partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d},$$
$$\alpha! := \alpha_1! \cdots \alpha_d!, \qquad \text{and} \qquad |\alpha| := \alpha_1 + \cdots + \alpha_d .$$

The quantity $|\alpha|$ will be referred to as either the *degree* or the *order* of $|\alpha|$ depending on context.

The space of real polynomials of degree at most κ in d variables will be denoted P_d^κ, and the subspace of homogeneous polynomials of degree κ will be denoted $\dot{P}_d^\kappa$. This space comes equipped with the inner product $\langle \cdot, \cdot \rangle_\kappa$ given by

$$\begin{aligned} \langle q, r \rangle_\kappa &:= \sum_{i_1, \ldots, i_\kappa = 1}^{d} \partial_{i_1} \cdots \partial_{i_\kappa} q|_0 \partial_{i_1} \cdots \partial_{i_\kappa} r|_0 \\ &= \sum_{|\alpha| = \kappa} \frac{k!}{\alpha!} \partial^\alpha q|_0 \partial^\alpha r|_0 \\ &= k! q(\partial) r(t)|_{t=0}, \end{aligned} \tag{4}$$

where the middle identity follows because there are $k!/\alpha!$ distinct ways to expand the mixed derivative ∂^α as a product of first coordinate derivatives, and the final identity follows easily by direct computation together with the observation that monomials form an orthogonal basis.

2. Affine geometry and necessity

2.1. Geometric invariant theory

The main ideas and results from geometric invariant theory that we will use come from the seminal work of Kempf and Ness [22] and its subsequent extension to real reductive algebraic groups by Richardson and Slodowy [28]. The idea of interest is that, for suitable representations of such groups, one can study group orbits by understanding the infimum over the orbit of a certain vector space norm. For our purposes here, it suffices to consider only representations of $\mathrm{SL}(d,\mathbb{R})$ or $\mathrm{SL}(m,\mathbb{R}) \times \mathrm{SL}(d,\mathbb{R})$ on vector spaces of tensors. In this context, the associated minimum vectors can be understood as model forms of tensors and the actual numerical value of the infimum carries meaningful and important quantitative information about these tensors (in contrast with the usual situation in geometric invariant theory, in which one cares only about whether the infimum is zero or nonzero and whether or not it is attained).

To begin the construction, suppose that $\mathcal{A}$ is any k-linear functional on a real vector space V of dimension d. Appropriating the Kempf–Ness minimum vector calculations of geometric invariant theory, it becomes possible to canonically associate a density functional $\delta : V^d \to \mathbb{R}_{\geq 0}$ to any such $\mathcal{A}$. Specifically, for any such $\mathcal{A}$ and any vectors $v_1, \ldots, v_d$, let $\delta(v_1, \ldots, v_d)$ be the quantity given by

$$\delta(v_1, \ldots, v_d) := \Big[\inf_{M \in \mathrm{SL}(d,\mathbb{R})} \sum_{j_1,\ldots,j_k=1}^{d} \Big| \sum_{i_1,\ldots,i_k=1}^{d} \mathcal{A}(M_{j_1 i_1} v_{i_1}, \ldots, M_{j_k i_k} v_{i_k}) \Big|^2 \Big]^{\frac{d}{2k}}. \quad (5)$$

Before showing that the quantity (5) is a density functional, it is worth acknowledging the algebraic structure that lies behind it. When the vectors in the d-tuple $v := (v_1, \ldots, v_d) \in V^d$ are linearly independent, one may define a representation $\rho^v_{\cdot} : \mathrm{SL}(d,\mathbb{R}) \times V \to V$ by setting

$$\rho^v_M(v_j) := \sum_{i=1}^{d} M_{ij} v_i \quad (6)$$

for each $j = 1, \ldots, d$ and then extending to all of V by linearity. This representation extends to act on k-linear functionals by duality; that is,

$$(\rho^v_M \mathcal{A})(v_{j_1}, \ldots, v_{j_k}) := \mathcal{A}(\rho^v_{M^T} v_{j_1}, \ldots, \rho^v_{M^T} v_{j_k}),$$

where M^T is the transpose of M. If one further defines a norm on the space of k-linear functionals by means of the formula

$$\|\mathcal{A}\|_v^2 := \sum_{j_1=1}^{d} \cdots \sum_{j_k=1}^{d} \big| \mathcal{A}(v_{j_1}, \ldots, v_{j_k}) \big|^2,$$

then the definition (5) may be restated as

$$\delta(v_1, \ldots, v_d) = \big(\inf_{M \in \mathrm{SL}(d,\mathbb{R})} \|\rho^v_M \mathcal{A}\|_v \big)^{\frac{d}{k}}.$$

PROPOSITION 1

The quantity (5) is a density functional; that is, if T is any linear transformation of V and $v_1, \ldots, v_d \in V$, then

$$\delta(Tv_1, \ldots, Tv_d) = |\det T| \delta(v_1, \ldots, v_d). \tag{7}$$

Furthermore, δ is intrinsic in the sense that it depends only on $\mathcal{A}$ and V and not on any other objects, such as choices of bases.

Proof

The fact that δ is intrinsic is immediately visible from its definition (5). To see that δ is a density functional as promised, the first step is to demonstrate that $\delta(v_1, \ldots, v_d) = 0$ when $v_1, \ldots, v_d$ are linearly dependent. In this case, there must exist an invertible matrix M such that $\sum_{i=1}^d M_{1i} v_i = 0$, and without loss of generality, one may assume that this matrix M has been normalized so as to belong to $\mathrm{SL}(d, \mathbb{R})$. Now for each $t > 0$, let $M^{(t)}$ be the matrix obtained by scalar multiplying the first row of M by t^{d-1} and all remaining rows by t^{-1}. These matrices $M^{(t)}$ belong to $\mathrm{SL}(d, \mathbb{R})$ for all $t > 0$, and

$$\|\rho^v_{M^{(t)}} \mathcal{A}\|_v = t^{-k} \|\rho^v_M \mathcal{A}\|_v \tag{8}$$

by multilinearity of $\mathcal{A}$ because

$$\mathcal{A}(M^{(t)}_{j_1 i_1} v_{i_1}, \ldots, M^{(t)}_{j_k i_k} v_{i_k}) = t^{-k} \mathcal{A}(M_{j_1 i_1} v_{i_1}, \ldots, M_{j_k i_k} v_{i_k})$$

by homogeneity provided that each $j_1, \ldots, j_k$ is not equal to 1; if any index j_ℓ does equal 1, then both sides vanish when summed over i, making the equality (8) true in all cases. Taking $t \to \infty$ shows that the infimum in (5) over all $\mathrm{SL}(d, \mathbb{R})$ must vanish when $v_1, \ldots, v_d$ are linearly dependent.

Now let T be any linear transformation of V. When $v_1, \ldots, v_d$ are linearly dependent or when T is not invertible, $Tv_1, \ldots, Tv_d$ will be linearly dependent, so it must hold that

$$\delta(Tv_1, \ldots, Tv_d) = |\det T| \delta(v_1, \ldots, v_d) = 0.$$

Otherwise, when $v_1, \ldots, v_d$ are linearly independent and T is invertible, there is a matrix $P \in \mathrm{GL}(d, \mathbb{R})$ with $\det P = \det T$ such that $T v_j = \sum_{i=1}^d P_{ji} v_i$ for each $j = 1, \ldots, d$. Factor P as $\pm |\det P|^{\frac{1}{d}} P'$ for some P' with $|\det P'| = 1$ in general and $\det P' = 1$ when d is odd. Once again, by multilinearity of $\mathcal{A}$,

$$\sum_{j_1, \ldots, j_k = 1}^d \Big| \sum_{i_1, \ldots, i_k = 1}^d \mathcal{A}(M_{j_1 i_1} T v_{i_1}, \ldots, M_{j_k i_k} T v_{i_k}) \Big|^2$$

$$= |\det T|^{\frac{2k}{d}}$$

$$\times \sum_{j_1,\dots,j_k=1}^{d} \left| \sum_{i_1,\dots,i_k,\ell_1,\dots,\ell_k=1}^{d} \mathcal{A}(M_{j_1 i_1} P'_{i_1 \ell_1} v_{\ell_1}, \dots, M_{j_k i_k} P'_{i_k \ell_k} v_{\ell_k}) \right|^2$$

$$= |\det T|^{\frac{2k}{d}} \sum_{j_1,\dots,j_k=1}^{d} \left| \sum_{i_1,\dots,i_k=1}^{d} \mathcal{A}\big((MP')_{j_1 i_1} v_{i_1}, \dots, (MP')_{j_k i_k} v_{i_k}\big) \right|^2. \quad (9)$$

Since $\mathrm{SL}(d,\mathbb{R})$ is a group, the set of matrices of the form MP' when $M \in \mathrm{SL}(d,\mathbb{R})$ is itself exactly $\mathrm{SL}(d,\mathbb{R})$, assuming that $\det P' = 1$. If $\det P' = -1$, then the matrices MP' for $M \in \mathrm{SL}(d,\mathbb{R})$ are exactly those matrices N which belong to $\mathrm{SL}(d,\mathbb{R})$ after the first two rows of N are interchanged. Since (9) is invariant under permutations of the rows of MP', it follows in both cases ($\det P' = \pm 1$) that

$$\inf_{M \in \mathrm{SL}(d,R)} \sum_{j_1,\dots,j_k=1}^{d} \left| \sum_{i_1,\dots,i_k=1}^{d} \mathcal{A}\big((MP')_{j_1 i_1} v_{i_1}, \dots, (MP')_{j_k i_k} v_{i_k}\big) \right|^2$$

$$= \big[\delta(v_1,\dots,v_d)\big]^{\frac{2k}{d}},$$

which gives the desired identity

$$\delta(Tv_1,\dots,Tv_d) = |\det T| \delta(v_1,\dots,v_d)$$

for any $v_1,\dots,v_d$ and any linear transformation T. □

Example

It is illuminating to compute δ in the special case when $\mathcal{A}$ is a symmetric bilinear form. Fix linearly independent vectors $v_1,\dots,v_d$ and define the matrix A by $A_{ij} := \mathcal{A}(v_i, v_j)$. It follows that

$$(\rho_M^v \mathcal{A})(v_{j_1}, v_{j_2}) = (MAM^T)_{j_1 j_2} \qquad \text{and} \qquad \|\rho_M^v \mathcal{A}\|_v^2 = \mathrm{tr}(MAM^T MA^T M^T).$$

Now $MAM^T MA^T M^T$ is symmetric and positive-semidefinite, so its eigenvalues are all nonnegative. Thus the *a*rithmetic mean-geometric mean (AM-GM) inequality implies that

$$d\big(\det(MAM^T MA^T M^T)\big)^{\frac{1}{d}} \le \mathrm{tr}(MAM^T MA^T M^T)$$

with equality when all eigenvalues are equal (which, when A is invertible, must hold for some $M \in \mathrm{SL}(d,\mathbb{R})$ by building M from a basis of unit-length eigenvectors of A with respect to some inner product and then rescaling the eigenvectors appropriately). Because $\det M = 1$, $\det(MAM^T MA^T M^T) = (\det A)^2$, and so

$$\delta(v_1, \ldots, v_d) = d^{\frac{d}{4}} |\det A|^{\frac{1}{2}}. \tag{10}$$

In particular, on a Riemannian manifold, setting $\mathcal{A}$ equal to the metric tensor g yields a density δ which is exactly equal to a dimensional constant times the corresponding Riemannian volume density functional.

2.2. *Construction of the affine curvature tensor and associated measure*

We move now to the construction of a covariant tensor which captures the affine curvature of interest here. This tensor gives rise to an associated density using (5) which can be integrated to give a canonical measure on immersed submanifolds $\mathcal{M} \subset \mathbb{R}^n$.

To be precise, suppose that $\mathcal{M}$ is a manifold of dimension d which is equipped with a smooth immersion $f : \mathcal{M} \to \mathbb{R}^n$. For convenience, let the values of f be regarded as column vectors. For any positive integer j, let κ_j be the smallest integer such that the dimension of the space $P_d^{\kappa_j}$ of real polynomials of degree κ_j in d variables has dimension at least $j + 1$, and let $\Lambda_{d,n}$ be the index set

$$\Lambda_{d,n} := \left\{(j,k) \in \mathbb{Z} \times \mathbb{Z} \,\middle|\, 1 \leq j \leq n \text{ and } 1 \leq k \leq \kappa_j\right\}.$$

The index set $\Lambda_{d,n}$ is represented pictorially in Figure 2 as the first n columns of boxes. The cardinality of $\Lambda_{d,n}$ is exactly the homogeneous dimension Q defined in the Introduction. We will define a Q-linear covariant tensor $\mathcal{A}_p$ at each point $p \in \mathcal{M}$ which captures the affine geometry of the immersion f. We will denote the action of $\mathcal{A}_p$ on Q-tuples of tangent vectors at the point p by either

$$\mathcal{A}_p(X_1, \ldots, X_Q) \qquad \text{or} \qquad \mathcal{A}_p\big((X_\lambda)_{\lambda \in \Lambda_{d,n}}\big)$$

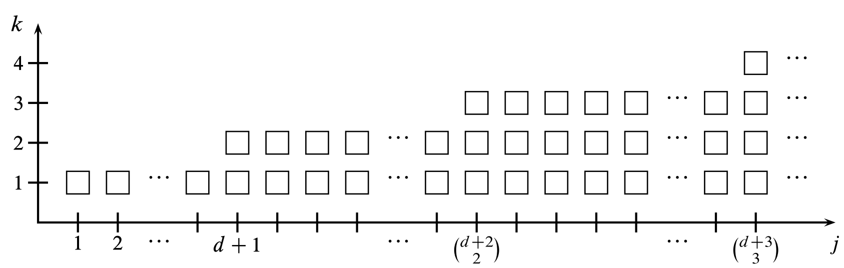

Figure 2. In the diagram above, squares represent points in $\mathbb{Z} \times \mathbb{Z}$. The index set $\Lambda_{d,n}$ is simply the union of the first n columns, and the homogeneous dimension Q is simply the cardinality of $\Lambda_{d,n}$. In terms of the tensor $\mathcal{A}_p$, the number of boxes in each column indicates how many derivatives are applied to f in the corresponding factor of the wedge product (or equivalently, the corresponding column of the matrix).

depending on which approach is most convenient at the moment (where we lexicographically order the elements of $\Lambda_{d,n}$ when such an order is not specified).

Now for any finite sequence of vector fields X_λ indexed by $\lambda \in \Lambda_{d,n}$, let

$$\mathcal{A}_p\big((X_\lambda)_{\lambda\in\Lambda_{d,n}}\big) := \det\big(X_{(1,1)} f(p) \wedge \cdots \wedge X_{(j,1)} \cdots X_{(j,\kappa_j)} f(p) \wedge \cdots \wedge X_{(n,1)} \cdots X_{(n,\kappa_n)} f(p)\big). \tag{11}$$

In other words, (11) equals the determinant of an $n \times n$ matrix whose jth column is the column vector $X_{(j,1)} \cdots X_{(j,\kappa_j)} f(p)$. (Note also that the lexicographic order on $\Lambda_{d,n}$ corresponds exactly to the order that each $\lambda \in \Lambda_{d,n}$ appears in the above formula when moving from left to right; with respect to Figure 2, the order is left to right followed by bottom to top.) This object $\mathcal{A}_p$ will be called the *affine curvature tensor at p*.

PROPOSITION 2

The affine curvature tensor is a tensor in the usual sense, that is, a multilinear function of the vector fields X_λ which depends only on the pointwise values of these vector fields at the point p. The tensor $\mathcal{A}_p$ is also equiaffine-invariant, by which we mean that $\mathcal{A}_p$ is invariant up to a factor of ± 1 under the action of equiaffine (i.e., measure-preserving) transformations of $\mathbb{R}^n$ applied to f.

Proof

Equiaffine invariance of $\mathcal{A}_p$ follows immediately from (11). Linearity in X_λ for each $\lambda \in \Lambda_{d,n}$ is a trivial consequence of the multilinearity of wedge products and the determinant. To see that $\mathcal{A}_p$ depends only on the pointwise values of the X_λ's at p and not on any derivatives of these vector fields, it suffices to show that any single one of the vector fields X_λ may be replaced by any other vector field X'_λ agreeing with X_λ at p without changing the value of $\mathcal{A}_p$. For any indices $\lambda = (j,k)$ such that $\kappa_j = 1$, this invariance under replacement follows immediately from the fact that these vector fields appear alone in their own column (i.e., the formula (11) contains no derivatives of X_λ to begin with). For any $\lambda = (j,k)$ with $\kappa_j > 1$, the identity

$$\begin{aligned} &X_{(j,1)} \cdots X_{(j,k)} \cdots X_{(j,\kappa_j)} f(p) - X_{(j,1)} \cdots X'_{(j,k)} \cdots X_{(j,\kappa_j)} f(p) \\ &\quad = X_{(j,1)} \cdots X_{(j,k-1)} [X_{(j,k-1)}, X_{(j,k)} - X'_{(j,k)}] X_{(j,k+1)} \cdots X_{(j,\kappa_j)} f(p) \\ &\qquad + \cdots + [X_{(j,1)}, X_{(j,k)} - X'_{(j,k)}] X_{(j,2)} \cdots \widehat{X_{(j,k)}} \cdots X_{(j,\kappa_j)} f(p) \end{aligned}$$

(where $\widehat{\cdot}$ indicates omission of $X_{(j,k)}$ in its usual place) shows that $\mathcal{A}_p$ vanishes when X_λ is replaced by $X_\lambda - X'_\lambda$; the difference between $\mathcal{A}_p((X_\lambda)_{\lambda\in\Lambda_{d,n}})$ and the corresponding quantity where one $X_{(j,k)}$ is replaced by a corresponding $X'_{(j,k)}$ can, by virtue of the identity, be written as a linear combination of determinants of matrices such that, for each matrix, the number of columns in which f is differentiated to order $\kappa_j - 1$ is strictly greater than the dimension of the vector space of homogeneous order $\kappa_j - 1$ differential operators. Thus each such matrix must have linearly dependent columns, which forces the difference in values of $\mathcal{A}_p$ to vanish. $\square$

The distinguished measure $\mu_{\mathcal{A}}$ of Theorem 1 can now be defined as the pushforward via f of the measure on $\mathcal{M}$ associated to the density (5) generated by the affine curvature tensor (11). In other words, for $\mathcal{A}_p$ as in (11), let

$$\delta_p(X_1, \ldots, X_d) := \inf_{M \in \mathrm{SL}(d,\mathbb{R})} \Big[\sum_{j_1,\ldots,j_Q=1}^{d} \Big| \sum_{i_1,\ldots,i_Q=1}^{d} \mathcal{A}_p(M_{j_1 i_1} X_{i_1}, \ldots, M_{j_Q i_Q} X_{i_Q}) \Big|^2 \Big]^{\frac{d}{2Q}}. \quad (12)$$

Since δ is a density on $\mathcal{M}$, it uniquely defines a measure $\nu_{\mathcal{A}}$ on $\mathcal{M}$ such that, for any coordinate chart $\varphi : B \to \mathcal{M}$ (where $B \subset \mathbb{R}^d$ is any open ball) and any nonnegative Borel function g supported on $\varphi(B)$,

$$\int g \, d\nu_{\mathcal{A}} = \int_B g\big(\varphi(t)\big) \delta_{\varphi(t)} \Big(d\varphi\Big(\frac{\partial}{\partial t_1}\Big), \ldots, d\varphi\Big(\frac{\partial}{\partial t_d}\Big)\Big) \, dt. \quad (13)$$

The transformation law (7) and the change of variables formula guarantee consistency of the definition on the overlap of coordinate charts. In particular, this means that in any local coordinates $(t_1, \ldots, t_d)$ near a point $p \in \mathcal{M}$, the Radon–Nikodym derivative of $\nu_{\mathcal{A}}$ with respect to the Lebesgue measure dt is given by evaluating δ_p on the standard d-tuple of coordinate vectors. By (11) (recalling the definition of $M\partial$ from the Introduction), this gives the identity

$$\Big[\frac{d\nu_{\mathcal{A}}}{dt}\Big|_p \Big]^{\frac{2Q}{d}} = \inf_{M \in \mathrm{SL}(d,\mathbb{R})} \sum_{\substack{|\alpha_1| = \kappa_1, \\ \ldots, |\alpha_n| = \kappa_n}} \frac{\kappa_1! \cdots \kappa_n!}{\alpha_1! \cdots \alpha_n!} \Big| \det\big((M\partial)^{\alpha_1} f(p) \wedge \cdots \wedge (M\partial)^{\alpha_n} f(p) \big) \Big|^2 \quad (14)$$

for almost every p, where the factorial factors merely count the number of ways that the monomial ∂^α may be written as a product of first-order operators. To avoid the minor irritation of constantly excluding exceptional sets of measure zero, (14) will be taken to *define* a unique representative of the Radon–Nikodym derivative within the usual equivalence class. In other words, $d\nu_{\mathcal{A}}/dt$ will always be taken so as to satisfy (14) for all p.

Finally, the measure $\mu_{\mathcal{A}}$ from Theorems 1 and 2 is given by the pushforward of $\nu_{\mathcal{A}}$ via f; that is, for any nonnegative Borel function F on $\mathbb{R}^n$,

$$\int_{\mathbb{R}^n} F \, d\mu_{\mathcal{A}} := \int_{\mathcal{M}} (F \circ f) \, d\nu_{\mathcal{A}}. \tag{15}$$

It is now a trivial task to prove parts (4) and (5) of Theorem 1.

Proof of part (5) of Theorem 1
The affine curvature tensor (11) is clearly invariant (up to a factor of ± 1) under the action of equiaffine transformations of $\mathbb{R}^n$ because it only depends on derivatives of f (so is trivially insensitive to translation in $\mathbb{R}^n$) and because of the elementary transformation law

$$\begin{aligned}\det\big(&X_{(1,1)} N f(p) \wedge \cdots \wedge X_{(n,1)} \cdots X_{(n,\kappa_n)} N f(p)\big) \\ &= (\det N) \det\big(X_{(1,1)} f(p) \wedge \cdots \wedge X_{(n,1)} \cdots X_{(n,\kappa_n)} f(p)\big)\end{aligned}$$

which holds for any $N \in \mathbb{R}^{n \times n}$. By (12), the density δ_p is therefore unchanged when f is acted on by an equiaffine transformation of $\mathbb{R}^n$. □

To prove part (4) of Theorem 1, the following auxiliary lemma is needed.

LEMMA 1
Suppose that $\alpha_1, \ldots, \alpha_N$ is an enumeration of all multi-indices of order k in d variables. Then

$$(M\partial)^{\alpha_1} f(p) \wedge \cdots \wedge (M\partial)^{\alpha_N} f(p) = \partial^{\alpha_1} f(p) \wedge \cdots \wedge \partial^{\alpha_N} f(p)$$

for any $M \in \mathrm{SL}(d, \mathbb{R})$.

Proof
Let T_M be the operator on the vector space of homogeneous differential operators of order k which is given by $T_M p(\partial) := p(M\partial)$ for all homogeneous polynomials p of degree k in d variables. By multilinearity of the wedge product, the lemma will follow once it is shown that $\det T_M = 1$ for all $M \in \mathrm{SL}(d, \mathbb{R})$. Because $\mathrm{SL}(d, \mathbb{R})$ is connected and $\det T_M = 1$ when M is the identity, it suffices to prove that the determinant equals ± 1 for any $M \in \mathrm{SL}(d, \mathbb{R})$.

For convenience, we regard T_M as simply acting on $\dot{P}^k_d$. By the chain rule,

$$\partial_{i_1} \cdots \partial_{i_k} (T_M p)|_0 = \sum_{j_1=1}^{d} \cdots \sum_{j_k=1}^{d} M_{j_1 i_1} \cdots M_{j_k i_k} (\partial_{j_1} \cdots \partial_{j_k} p|_0) \tag{16}$$

for each $M \in \mathbb{R}^{n \times n}$, each $p \in \dot{P}_d^k$, and any indices $i_1, \ldots, i_k$. If one supposes that O is an orthogonal matrix, then by (4) and (16), the inner product $\langle T_O p, T_O q \rangle_k$ is equal to the sum over all indices $i_1, \ldots, i_k, j_1, \ldots, j_k, j'_1, \ldots, j'_k$ of the quantities

$$O_{j_1 i_1} \cdots O_{j_k i_k} O_{j'_1 i_1} \cdots O_{j'_k i_1} (\partial_{j_1} \cdots \partial_{j_k} p|_0)(\partial_{j'_1} \cdots \partial_{j'_k} q|_0).$$

Summing over $i_1, \ldots, i_k$ first and using orthogonality of O proves that

$$\langle T_O p, T_O q \rangle_k = \langle p, q \rangle_k. \tag{17}$$

Since T_O is orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle_k$, $\det T_O = \pm 1$.

On the other hand, every $p \in \dot{P}_d^k$ may also be uniquely expressed as a sum

$$p(t) = \sum_{|\alpha|=k} p_\alpha t^\alpha$$

with constant coefficients p_α. If D is diagonal with entries $(s_1, \ldots, s_d)$, then

$$(T_D p)(t) = \sum_{|\alpha=k} (s^\alpha p_\alpha) t^\alpha,$$

which means that T_D is also diagonal in the monomial basis. Thus

$$\det T_D = \prod_{|\alpha|=k} s^\alpha = (\det D)^{\binom{k+d-1}{k-1}},$$

where the second identity holds because symmetry dictates that the product must have the form $(s_1 \cdots s_d)^r$ for some r; subsequently, r can be easily computed from the degree of the polynomial.

The lemma follows by the singular-value decomposition, since every M of determinant 1 may be factored as $O_1 D O_2$ for orthogonal matrices O_1, O_2 and a diagonal matrix D of determinant 1. □

Proof of part (4) of Theorem 1

When $d = 1$, the expression (14) reduces immediately to the usual torsion determinant for affine arclength because the action of M is trivial and because $\kappa_j = j$ for each j.

To compute (14) in the case of hypersurfaces (i.e., $d = n - 1$), by virtue of Lemma 1, if A is the $(n-1) \times (n-1)$ matrix given by

$$A_{ii'} := \det\bigl(\partial_1 f(p) \wedge \cdots \wedge \partial_{n-1} f(p) \wedge \partial^2_{ii'} f(p)\bigr),$$

then (14) simplifies to

$$\left.\frac{d\nu_{\mathcal{A}}}{dt}\right|_p = \inf_{M \in \mathrm{SL}(n-1,\mathbb{R})} \Big[(n-1)! \sum_{i,i'=1}^{n-1} \big|(MAM^T)_{ii'}\big|^2 \Big]^{\frac{n-1}{2(n+1)}}$$

because when $|\alpha_1| = \cdots = |\alpha_{n-1}| = 1$, the wedge product $(M\partial)^{\alpha_1} f(p) \wedge \cdots \wedge (M\partial)^{\alpha_{n-1}} f(p)$ vanishes unless $\alpha_1, \ldots, \alpha_{n-1}$ are distinct, in which case the wedge product is independent of M. The factor of $(n-1)!$ counts the total number of ways to assign distinct values to the multi-indices $\alpha_1, \ldots, \alpha_{n-1}$, which are all of order 1. By the example calculation (10) from the previous section, this infimum is explicitly computable:

$$\left.\frac{d\nu_{\mathcal{A}}}{dt}\right|_p = \big((n-1)!\big)^{\frac{n-1}{2(n+1)}} \big[(n-1)^{\frac{n-1}{4}} |\det A|^{\frac{1}{2}} \big]^{\frac{2}{n+1}} = C_n |\det A|^{\frac{1}{n+1}}. \tag{18}$$

Aside from the constant factor C_n, this corresponds exactly to equiaffine measure on the hypersurface parameterized by f. □

3. Working with affine curvature

In contrast to the essentially trivial cases of affine arclength and equiaffine hypersurface measure, the presence of the infimum in the definition (12) of the density δ_p presents an added layer of difficulty when explicitly computing the associated measure $\mu_{\mathcal{A}}$. However, the richness of existing tools from algebra, geometry, and analysis provides a variety of ways to overcome this difficulty. In this section, a number of different approaches to the computational problem are identified along with illustrative examples of how these approaches may be applied to determine well-curvedness (as defined by the nonvanishing of $\mu_{\mathcal{A}}$).

3.1. Algebraic approaches

The first and simplest observation to make is that Lemma 1 facilitates the further simplification of the expression used to define $d\nu_{\mathcal{A}}/dt$ via the right-hand side of (14). Given n and d, let m be the number of entries in the ordered list $\kappa_1, \ldots, \kappa_n$ which equal κ_n. This number m will be called the *r*elative codimension of a manifold of dimension d in $\mathbb{R}^n$. Analytically, m counts the number of highest-order derivatives in (14). One can explicitly see that $m = n + 1 - \dim P_d^{\kappa_n - 1} = n + 1 - \binom{\kappa_n + d - 1}{d}$, but this identity will not be particularly useful. By Lemma 1, the dependence on M of the determinants inside the sum (14) is much less than it would seem. For each k, let

$$L^k f(p) := \bigwedge_{1 \leq |\alpha| \leq k} \frac{1}{\alpha!} \partial^\alpha f(p) \tag{19}$$

(where the monomials are ordered lexicographically). By Lemma 1 and (14),

$$\frac{d\nu_{\mathcal{A}}}{dt}\bigg|_p = C\Big[\inf_{M\in \mathrm{SL}(d,\mathbb{R})} \sum_{|\beta_1|=\cdots=|\beta_m|=\kappa_n} \frac{(\kappa_n!)^m}{\beta_1!\cdots\beta_m!}\big|\det\big(L^{\kappa_n-1}f(p) \wedge (M\partial)^{\beta_1}f(p)\wedge\cdots\wedge(M\partial)^{\beta_m}f(p)\big)\big|^2\Big]^{\frac{d}{2Q}} \tag{20}$$

for some constant C that depends only on d and n. In certain special cases, the simplification proceeds even further.

Example

When $n+1 = \dim P_d^{\kappa_n}$ (including curves for all n), the measure $\nu_{\mathcal{A}}$ has Radon–Nikodym derivative with respect to Lebesgue measure equal to

$$\frac{d\nu_{\mathcal{A}}}{dt}\bigg|_p = C\big|\det L^{\kappa_n} f(p)\big|$$

for some constant C depending only on d and n. In such cases, $d\nu_{\mathcal{A}}/dt$ is nonzero at a point if and only if the vectors $\{\partial^\alpha f(p)\}_{1\leq|\alpha|\leq\kappa_n}$ span $\mathbb{R}^n$.

The reader may wonder if it is always possible to evaluate the infimum (20) explicitly in terms of a determinant as in the example. The answer is in general no, because for generic d and n, one must appeal to a richer and deeper family of algebraic operations than merely the determinant. As observed in Section 2.1, the infimum (20) can be understood as the infimum of a norm over an orbit of an $\mathrm{SL}(d,\mathbb{R})$ representation. In the specific instance at hand, for $s_1,\ldots,s_m \in \mathbb{R}^d$, at every point p, one may define the polynomial

$$\mathcal{P}_p(s_1,\ldots,s_m) := (\kappa_n!)^{-m}\det\big(L^{\kappa_n-1}f(p)\wedge(s_1\cdot\nabla)^{\kappa_n}f(p)\wedge\cdots\wedge(s_m\cdot\nabla)^{\kappa_n}f(p)\big) \tag{21}$$

which is homogeneous of degree κ_n in each set of variables $s_1,\ldots,s_m$. The group $\mathrm{SL}(d,\mathbb{R})$ acts on the vector space of such multihomogeneous polynomials via the group representation

$$(\rho_M\mathcal{P}_p)(s_1,\ldots,s_m) := \mathcal{P}_p(M^T s_1,\ldots,M^T s_m). \tag{22}$$

If one defines the inner product $\langle\cdot,\cdot\rangle_{\kappa_n,m}$ on such polynomials by the formula

$$\langle q,r\rangle_{\kappa_n,m} := \sum_{|\beta_1|=\cdots=|\beta_m|=\kappa_n}\frac{(\kappa_n!)^m}{\beta_1!\cdots\beta_m!}\partial_{s_1}^{\beta_1}\cdots\partial_{s_m}^{\beta_m}q|_0\,\partial_{s_1}^{\beta_1}\cdots\partial_{s_m}^{\beta_m}r|_0, \tag{23}$$

then by (20),

$$\left.\frac{d\nu_{\mathcal{A}}}{dt}\right|_p = C' \inf_{M \in \mathrm{SL}(d,\mathbb{R})} \|\rho_M \mathcal{P}_p\|_{\kappa_n,m}^{\frac{d}{Q}} \tag{24}$$

for a suitable constant C' that depends only on d and n.

Quantities like the right-hand side of (24) have been thoroughly studied in the context of geometric invariant theory. In this vast literature, it is well understood that the algebra of $\mathrm{SL}(d,\mathbb{R})$-invariant polynomials in the coefficients of $\mathcal{P}_p$ is fundamentally connected to the right-hand side of (24). Hilbert [17] showed that, when the group $\mathrm{SL}(d,\mathbb{R})$ is replaced by $\mathrm{SL}(d,\mathbb{C})$, this algebra is finitely generated. From this fact it is easy to see that the same result must be true for $\mathrm{SL}(d,\mathbb{R})$ itself. Subsequent work has shown that this algebra must also be finitely generated for representations of any group G which is a real reductive algebraic group (which, as far as the present work is concerned, is a class which includes $\mathrm{SL}(d,\mathbb{R})$ and is closed under Cartesian products). The relevance of Hilbert's theorem to the quantity (24) is as follows.

LEMMA 2

Suppose that G is a real reductive algebraic group and that ρ is a G-representation on some finite-dimensional real vector space V equipped with a norm $\|\cdot\|$. Let $p_1,\ldots,p_N$ be any collection of homogeneous polynomial functions on V, with positive degrees $d_1,\ldots,d_N$, which generates the algebra of all G-invariant polynomials. Then there exist constants $0 < C_1 \le C_2 < \infty$ such that

$$C_1 \max_{j=1,\ldots,N} \left|p_j(v)\right|^{\frac{1}{d_j}} \le \inf_{M \in G} \|\rho_M v\| \\ \le C_2 \max_{j=1,\ldots,N} \left|p_j(v)\right|^{\frac{1}{d_j}} \quad \textit{for all } v \in V. \tag{25}$$

Proof

To prove the first inequality, observe by scaling that

$$\|p_j\|_\infty^{-\frac{1}{d_j}} \left|p_j(v)\right|^{\frac{1}{d_j}} \le \|v\|$$

for all j and all $v \in V$, where $\|p_j\|_\infty$ is the supremum of p_j on the unit sphere of $\|\cdot\|$. Moreover, because each p_j is invariant under ρ,

$$\|p_j\|_\infty^{-\frac{1}{d_j}} \left|p_j(v)\right|^{\frac{1}{d_j}} = \|p_j\|_\infty^{-\frac{1}{d_j}} \left|p_j(\rho_M v)\right|^{\frac{1}{d_j}} \le \|\rho_M v\|,$$

so taking an infimum in M and a supremum in j gives

$$\Big[\min_{j=1,\ldots,N} \|p_j\|_\infty^{-\frac{1}{d_j}}\Big] \max_{j=1,\ldots,N} \left|p_j(v)\right|^{\frac{1}{d_j}} \le \inf_{M \in G} \|\rho_M v\|.$$

To prove the reverse inequality, suppose for the sake of contradiction that the second inequality of (25) does not hold for any finite C_2. Because the inequality is homogeneous in the norm $\|\cdot\|$, its failure would imply that one could find a sequence v_k with $\inf_M \|\rho_M v_k\| = 1$ for all k such that

$$\max_j \left|p_j(v_k)\right|^{\frac{1}{d_j}} \leq k^{-1} \inf_{M \in G} \|\rho_M v_k\| = k^{-1}.$$

Moreover, by replacing v_k by $\rho_{M_k} v_k$ for suitable M_k and taking a subsequence by compactness, it may be assumed that v_k converges to some v in the unit sphere as $k \to \infty$. By continuity of the polynomials p_j, $p_j(v) = 0$ for all j. Therefore, v belongs to the so-called *nullcone* of the representation, and by the real Hilbert–Mumford criterion, first proved by Birkes [3], there must exist a 1-parameter subgroup $\rho_{\exp(tX)}$ of G such that $\rho_{\exp(tX)} v \to 0$ as $t \to \infty$. This, of course, implies that $\inf_M \|\rho_M v\| = 0$. However, $\inf_M \|\rho_M v_k\| = 1$ for all k implies that $\|\rho_M v_k\| \geq 1$ for all $M \in G$ and all k, which means by continuity that $\|\rho_M v\| \geq 1$ for all M, so $\inf_M \|\rho_M v\| = 0$ must be contradicted. □

There are certain cases in which minimal generating families of the algebra are explicitly known, which means that the measure $\mu_{\mathcal{A}}$ may consequently be computed up to factors depending on d and n. We have the following examples.

Example
When $d = 3$ and $n = 3 + 6 + 1 = 10$, $\mathcal{P}_p$ is a ternary cubic form (i.e., cubic polynomial of degree 3). The algebra of $\mathrm{SL}(3, \mathbb{R})$-invariant polynomials of ternary cubic forms is known to be generated by Aronhold's invariants S and T (see [32]) of degrees 4 and 6, respectively. Consequently, for any 3-surface in $\mathbb{R}^{10}$,

$$\left.\frac{d\nu_{\mathcal{A}}}{dt}\right|_p \approx \left|S(\mathcal{P}_p)\right|^{\frac{1}{4}} + \left|T(\mathcal{P}_p)\right|^{\frac{1}{6}}$$

with universal implicit constants that do not depend on $\mathcal{P}_p$. For example, the 3-surface parameterized by

$$(t_1, t_2, t_3, t_1^2, t_2^2, t_3^2, t_1 t_2, t_2 t_3, t_1 t_3, t_1^3 + t_2^3 + t_3^3)$$

has $\mathcal{P}_p(t) = t_1^3 + t_2^3 + t_3^3$, where $t = (t_1, t_2, t_3) \in \mathbb{R}^3$ and $S(\mathcal{P}_p) = 1$, $T(\mathcal{P}_p) = 0$, so $\nu_{\mathcal{A}}$ is a nonzero constant times Lebesgue measure dt.

Example
When $d = 2$ and $m = 1$, $\mathcal{P}_p$ is a homogeneous polynomial of degree κ_n on $\mathbb{R}^2$. By Hilbert [17], it is known that the nullcone of the representation consists exactly of all

polynomials which have a (projective) zero of order greater than $\kappa_n/2$. Thus for any 2-surface parameterized by

$$\left((t^\alpha)_{1\leq|\alpha|<\kappa}, q(t)\right)$$

for some homogeneous polynomial q of degree κ in two variables, it is an elementary exercise to determine whether or not $\mu_{\mathcal{A}} = 0$.

Example

When $m = 1$ and d and n are otherwise arbitrary, it is known that the discriminant is an $\mathrm{SL}(d,\mathbb{R})$-invariant polynomial on the space of homogeneous polynomials of degree κ for any κ. While the discriminant alone does not characterize $d\nu_{\mathcal{A}}/dt$, when the discriminant of $\mathcal{P}_p$ is nonzero, $(d\nu_{\mathcal{A}}/dt)|_p$ will also be nonzero. The discriminant vanishes if and only if the gradient of the polynomial vanishes at some nonzero complex point. Thus one can immediately see that the d-surface given by

$$(t_1,\ldots,t_d) \mapsto \left((t^\alpha)_{1\leq|\alpha|<\kappa}, t_1^\kappa + \cdots + t_d^\kappa\right)$$

will also have $\nu_{\mathcal{A}}$ equal to a nonzero constant times Lebesgue measure for any d and κ. The same will be true for any small-coefficient perturbations $t_1^\kappa + \cdots + t_d^\kappa$ as well.

It is always possible in principle to compute a complete, finite collection of generators of the algebra of invariant polynomials explicitly in finite time (see Sturmfels [32]), which means by (25) that for any individual values of d and n of particular interest, it is always possible to compute the magnitude of $d\nu_{\mathcal{A}}/dt$ directly up to unimportant multiplicative factors. Carrying out this computation in parallel for many different values of d and n is, with the exception of special combinations, somewhat unwieldy and akin to the attempted computation of the determinant via the permutation expansion rather than by more efficient, symmetry-exploiting techniques. As implied by the above examples, it is also worth observing that when many invariant polynomials exist (which, unlike for curves and hypersurfaces, is typically the case), the nullcone of tensors $\mathcal{A}$ such that the density (12) vanishes has codimension greater than 1, which means that for general submanifolds of dimension d in $\mathbb{R}^n$, it is typically "easier" to have nonvanishing affine curvature than it is for hypersurfaces because the space of "flat" Taylor polynomial jets to be avoided is of codimension greater than 1.

It is substantially easier to directly compute the entire algebra of $\mathrm{SL}(d,\mathbb{R})$-invariant polynomial functions of $\mathcal{P}_p$ than it is to determine a finite set of generators because the Reynolds operator, which projects polynomials in $\mathcal{P}_p$ onto the space of $\mathrm{SL}(d,\mathbb{R})$-invariant polynomials, can be explicitly expressed in terms of Cayley's Ω

operator. For any polynomial function of the entries of $M \in \mathbb{R}^{d \times d}$, let

$$\Omega_M := \sum_{\sigma \in \mathcal{S}_d} (-1)^\sigma \frac{\partial^d}{\partial M_{1\sigma_1} \cdots \partial M_{d\sigma_d}}. \tag{26}$$

THEOREM 4

Let $\dot{V}^{\kappa_n}_{d,m}$ be the real vector space of real polynomials in variables $s_1, \ldots, s_m \in \mathbb{R}^d$ which are homogeneous of degree κ_n as a function of each variable s_i, $i = 1, \ldots, m$. A homogeneous polynomial F of degree k on $\dot{V}^{\kappa_n}_{d,m}$ is called an $\mathrm{SL}(d,\mathbb{R})$ invariant *if $F(\rho_M \mathcal{P}) = F(\mathcal{P})$ for all $M \in \mathrm{SL}(d,\mathbb{R})$ and all $\mathcal{P} \in \dot{V}^{\kappa_n}_{d,m}$, where ρ_M is as defined in (22). If d divides $\kappa_n m k$, then let*

$$\mathcal{R}[F](\mathcal{P}) := (\Omega_M)^{\frac{\kappa_n k m}{d}} \big[F(\rho_M \mathcal{P}) \big] \big|_{M=0}. \tag{27}$$

Then the following are true.

(1) *If d does not divide $\kappa_n m k$, then there are no nonzero homogeneous* $\mathrm{SL}(d,\mathbb{R})$ *invariants F of degree k on $\dot{V}^{\kappa_n}_{d,m}$.*

(2) *If d divides $\kappa_n m k$ and F is any homogeneous polynomial of degree k on $\dot{V}^{\kappa_n}_{d,m}$, then $\mathcal{R}[F]$ is an* $\mathrm{SL}(d,\mathbb{R})$ *invariant; that is,*

$$\mathcal{R}[F](\rho_M \mathcal{P}) = \mathcal{R}[F](\mathcal{P}) \quad \text{for all } M \in \mathrm{SL}(d,\mathbb{R}) \text{ and all } \mathcal{P} \in \dot{V}^{\kappa_n}_{d,m}.$$

(3) *If d divides $\kappa_n m k$, then there is a positive constant c depending on (d, k, m, κ_n) such that every homogeneous* $\mathrm{SL}(d,\mathbb{R})$ *invariant F of degree k on $\dot{V}^{\kappa_n}_{d,m}$ satisfies $\mathcal{R}[F] = c\mathcal{R}[F]$.*

Proof

To establish the first conclusion (for which it may be assumed without loss of generality that $d > 1$), let M be a diagonal matrix with entries $t_1, \ldots, t_d$ such that $t_1 \cdots t_d = 1$. The quantity $F(\rho_M \mathcal{P})$ is necessarily a polynomial of degree $\kappa_n m k$ in $t_1, \ldots, t_d$. If any term $c_\alpha t^\alpha$ of this polynomial has a nonzero coefficient for a multi-index $\alpha := (\alpha_1, \ldots, \alpha_d)$ with entries that are not all equal (i.e., $\alpha_i \neq \alpha_{i'}$ for some i, i'), then substituting $t_j = (t_1 \cdots t_{j-1} t_{j+1} \cdots t_d)^{-1}$ for an appropriate choice of index j will necessarily yield a nonconstant rational function of the remaining variables, which contradicts $\mathrm{SL}(d,\mathbb{R})$ invariance. However, the entries of α can only be equal to one another if d divides $|\alpha| = \kappa_n m k$.

The remaining conclusions of this theorem are simply applications of more general results of Sturmfels [32] for relative $\mathrm{GL}(d,\mathbb{C})$ invariants. A polynomial F as above is called a *r*elative $\mathrm{GL}(d,\mathbb{C})$ invariant of index i when

$$F(\rho_M \mathcal{P}) = (\det M)^i F(\mathcal{P}) \quad \text{for all } M \in \mathrm{GL}(d,\mathbb{C})$$

and all $\mathcal{P}$ in the complexification of $\dot{V}^{\kappa_n}_{d,m}$. By Theorem 4.3.7 of [32], $\mathcal{R}[F]$ is always a relative $\mathrm{GL}(d,\mathbb{C})$ invariant of index $\kappa_n mk/d$, which in particular forces it to be an $\mathrm{SL}(d,\mathbb{R})$ invariant. Conversely, if F is an $\mathrm{SL}(d,\mathbb{R})$ invariant, then any $M \in \mathrm{GL}(d,\mathbb{R})$ with positive determinant may be factored as

$$M = (\det M)^{\frac{1}{d}} M'$$

for $M' \in \mathrm{SL}(d,\mathbb{R})$. By homogeneity,

$$\begin{aligned} F(\rho_M \mathcal{P}) &= F\big((\det M)^{\frac{\kappa_n m}{d}} \rho_{M'} \mathcal{P}\big) \\ &= (\det M)^{\frac{\kappa_n mk}{d}} F(\rho_{M'} \mathcal{P}) \\ &= (\det M)^{\frac{\kappa_n mk}{d}} F(\mathcal{P}) \end{aligned}$$

for all $\mathcal{P} \in \dot{V}^{\kappa_n}_{d,m}$. Now both $F(\rho_M \mathcal{P})$ and $(\det M)^{\frac{\kappa_n mk}{d}} F(\mathcal{P})$ are polynomial functions of M and $\mathcal{P}$ which are equal for any real $\mathcal{P}$ and any M in an open subset of real matrices. By analytic continuation, they must also agree for all $M \in \mathrm{GL}(d,\mathbb{C})$ for all complex $\mathcal{P}$, so F must be a relative $\mathrm{GL}(d,\mathbb{C})$ invariant of index $\kappa_n km/d$; by [32, Corollary 4.3.6],

$$\begin{aligned} \mathcal{R}[F](\mathcal{P}) &= (\Omega_M)^{\frac{\kappa_n km}{d}} \big[F(\rho_M \mathcal{P})\big]\big|_{M=0} \\ &= F(\mathcal{P})(\Omega_M)^{\frac{\kappa_n km}{d}} \big[(\det M)^{\frac{\kappa_n km}{d}}\big]\big|_{M=0} \\ &= cF(\mathcal{P}) \end{aligned}$$

for some $c > 0$. This finishes the proof. □

By Lemma 2, one has the following immediate corollary of Theorem 4.

COROLLARY 1
For any point p, $(dv_{\mathcal{A}}/dt)|_p = 0$ if and only if, for every k such that d divides $\kappa_n mk$, every homogeneous polynomial F of degree k on $\dot{V}^{\kappa_n}_{d,m}$ satisfies

$$(\Omega_M)^{\frac{\kappa_n km}{d}} \big[F(\rho_M \mathcal{P}_p)\big]\big|_{M=0} = 0. \tag{28}$$

Proof
This condition is immediately equivalent to the condition that every $\mathrm{SL}(d,\mathbb{R})$ invariant F vanishes at $\mathcal{P}_p$, which is equivalent to every generator of the algebra vanishing at $\mathcal{P}_p$. □

Example
The 3-surface in $\mathbb{R}^{12}$ given by

$$(t_1, t_2, t_3, t_1^2, t_2^2, t_3^2, t_1 t_2, t_1 t_3, t_2 t_3, t_1^3, t_2^2 t_3, t_2 t_3^2)$$

is well curved. If $s_1, s_2, s_3 \in \mathbb{R}^3$, then

$$\mathcal{P}_p(s_1, s_2, s_3) = \det \begin{bmatrix} s_{11}^3 & s_{21}^3 & s_{31}^3 \\ s_{12}^2 s_{13} & s_{22}^2 s_{23} & s_{32}^2 s_{33} \\ s_{12} s_{13}^2 & s_{22} s_{23}^2 & s_{32} s_{33}^2 \end{bmatrix}.$$

Let $F(\mathcal{P})$ be the coefficient of $s_{11}^3 s_{22}^3 s_{23}^3$ when $\mathcal{P}$ is expressed in the standard basis of the variables s_{ij} (i.e., $s_i = (s_{i1}, s_{i2}, i3)$ for each i). Then

$$F\big(\mathcal{P}_p(M^T s_1, M^T s_2, M^T s_3)\big) = \det \begin{bmatrix} M_{11}^3 & M_{21}^3 & M_{31}^3 \\ M_{12}^2 M_{13} & M_{22}^2 M_{23} & M_{32}^2 M_{33} \\ M_{12} M_{13}^2 & M_{22} M_{23}^2 & M_{32} M_{33}^2 \end{bmatrix}.$$

Because Ω_M is antisymmetric under permutation of the rows of M,

$$\Omega_M^3 F(\rho_M \mathcal{P}_p) = 6 \Omega_M^3 (M_{11}^3 M_{22}^2 M_{23} M_{32} M_{33}^2),$$

which readers may recognize as a transvectant. It follows that

$$\begin{aligned} \Omega_M^3 F(\rho_M \mathcal{P}_p) &= 6 \Omega_M^2 \big(3 M_{11}^2 [4 M_{22} M_{23} M_{32} M_{33} - M_{22}^2 M_{33}^2]\big) \\ &= 18 \Omega_M \big(2 M_{11} [4 M_{23} M_{32} - 4 M_{22} M_{33} - 4 M_{22} M_{33}]\big) \\ &= 144 \Omega_M \big(M_{11} [M_{23} M_{32} - 2 M_{22} M_{33}]\big) \\ &= 144[-2-1] = -432 \neq 0. \end{aligned}$$

Because the algebra of $\mathrm{SL}(d, \mathbb{R})$ invariants is finitely generated, Corollary 1 can be further refined to state that there is some k_0 depending only on d and n such that $(d\nu_{\mathcal{A}}/dt)|_p = 0$ if and only if (28) holds for all homogeneous polynomials F of degree $k \leq k_0$ such that d divides $\kappa_n m k$.

3.2. Sublevel set approaches
Because all norms on a given finite-dimensional vector space are comparable, it can be useful to replace the norm appearing in (24) by other more meaningful ones. One such example is the observation that

$$\frac{d\nu_{\mathcal{A}}}{dt}\bigg|_p \approx \inf_{M \in \mathrm{SL}(d,\mathbb{R})} \sup_{\|s_1\|, \dots, \|s_m\| \leq 1} \big|\mathcal{P}_p(M s_1, \dots, M s_m)\big|^{\frac{d}{Q}},$$

where $\|\cdot\|$ is any choice of norm on $\mathbb{R}^d$. Let K denote the unit ball of this norm, and let $Z_p := \{(s_1,\ldots,s_m) \mid |\mathcal{P}_p(s_1,\ldots,s_m)| \leq 1\}$. A trivial application of homogeneity gives

$$\left[\frac{d\nu_{\mathcal{A}}}{dt}\Big|_p\right]^{-1} \approx \sup_{M\in \mathrm{GL}(d,\mathbb{R})} \left\{|MK|^{\frac{m}{Q}} \;\Big|\; (MK)^m \subset Z_p\right\}, \tag{29}$$

where $|MK|$ denotes Lebesgue measure. This allows one to employ sublevel set estimates to establish nondegeneracy of $\mu_{\mathcal{A}}$. This idea is made precise by Theorem 5 which is stated and proved below after the following lemma.

LEMMA 3
Suppose that $K \subset \mathbb{R}^d$ is a centered ellipsoid. There exists a nonzero radius r satisfying $r \gtrsim |K|^{1/d}$ for an implicit constant depending only on d such that the $(d-1)$-dimensional Lebesgue measure of $K \cap \{x \in \mathbb{R}^d \mid |x| = r\}$ satisfies

$$\left|K \cap \left\{x \in \mathbb{R}^d \;\Big|\; |x| = r\right\}\right| \gtrsim |K| r^{-1}$$

for some second implicit constant depending only on d.

Proof
Applying an orthogonal transformation to K if necessary, it may be assumed that the axes of K are in the standard coordinate directions; that is,

$$K = \left\{(x_1,\ldots,x_d) \in \mathbb{R}^d \;\Big|\; \sum_{j=1}^d \frac{x_j^2}{R_j^2} \leq 1\right\}$$

for strictly positive $R_1,\ldots,R_d$. Moreover, it may be further assumed that $R_j \leq R_d$ for all $j \in \{1,\ldots,d-1\}$. Let $r := R_d/\sqrt{2}$ and

$$E_0 := \left[-\frac{R_1}{2\sqrt{d-1}}, \frac{R_1}{2\sqrt{d-1}}\right] \times \cdots \times \left[-\frac{R_{d-1}}{2\sqrt{d-1}}, \frac{R_{d-1}}{2\sqrt{d-1}}\right].$$

Every $y \in E_0$ satisfies

$$\sum_{j=1}^{d-1} |y_j|^2 \leq \sum_{j=1}^{d-1} \frac{R_j^2}{4(d-1)} \leq \frac{R_d^2}{4} = \frac{r^2}{2}.$$

Set $\pi(y) := (y, \sqrt{r^2 - |y|^2}) \in \mathbb{R}^d$. The map π is well defined on all of E_0 and maps it into the sphere $\{x \in \mathbb{R}^d \mid |x| = r\}$. The Lebesgue measure of $\pi(E_0)$ in this sphere satisfies the inequality

$$\left|\pi(E_0)\right| = \int_{E_0} \left(1 - \frac{|y|^2}{r^2}\right)^{-\frac{1}{2}} dy \geq |E_0| = (d-1)^{-\frac{d-1}{2}} R_1 \cdots R_{d-1} \gtrsim |K| r^{-1}.$$

Lastly, every point of $\pi(E_0)$ belongs to the ellipse K because

$$\Big(\sum_{j=1}^{d-1} \frac{|y_{j-1}|^2}{R_j^2}\Big) + \frac{r^2 - |y|^2}{R_d^2} \leq \Big(\sum_{j=1}^{d-1} \frac{1}{4(d-1)}\Big) + \frac{1}{2} \leq 1.$$

This completes the proof. □

THEOREM 5
For every $\epsilon > 0$, let $I(\epsilon)$ be the measure in $(\mathbb{S}^{d-1})^m$ of the set

$$\big\{(\omega_1, \ldots, \omega_m) \in (\mathbb{S}^{d-1})^m \bigm| \big|\mathcal{P}_p(\omega_1, \ldots, \omega_m)\big| \leq \epsilon\big\},$$

where $\mathcal{P}_p$ is as in (21). Suppose that there exist positive constants C_I and σ such that $I(\epsilon) \leq C_I \epsilon^{1/\sigma}$ for all $\epsilon > 0$. If $\sigma \leq \frac{\kappa_n}{d}$, then

$$\frac{d\nu_{\mathcal{A}}}{dt}\Big|_p \gtrsim C_I^{-\frac{\sigma d}{\kappa_n Q}} \tag{30}$$

with an implicit constant depending only on d and n.

Proof
The class of centered ellipsoids in $\mathbb{R}^d$ is closed under invertible linear transformations. Let K be any centered ellipsoid such that

$$K^m \subset \big\{(t_1, \ldots, t_m) \in (\mathbb{R}^d)^m \bigm| \big|\mathcal{P}_p(t_1, \ldots, t_m)\big| \leq 1\big\}.$$

Let r be the radius identified by Lemma 3. By Lemma 3 and the fact that K^m is contained in the sublevel set where $|\mathcal{P}_p| \leq 1$, it follows that the Lebesgue measure in $(r\mathbb{S}^{d-1})^m$ of the sublevel set $|P_p| \leq 1$ is greater than or comparable to $(|K| r^{-1})^m$ with an implicit constant depending only on d. By homogeneity,

$$r^{m(d-1)} I(r^{-\kappa_n m}) \gtrsim \big(|K| r^{-1}\big)^m.$$

By the sublevel set bound for $I(\epsilon)$, it must be the case that

$$|K| \lesssim C_I^{\frac{1}{m}} r^{-\frac{\kappa_n}{\sigma} + d} \lesssim C_I^{\frac{1}{m}} |K|^{-\frac{\kappa_n}{\sigma d} + 1},$$

where the last inequality follows because $-\frac{\kappa_n}{\sigma} + d \leq 0$ and $r \gtrsim |K|^{1/d}$. Therefore, $|K| \lesssim C_I^{\sigma d/(\kappa_n m)}$ with an implicit constant depending only on d and n. By (29), this establishes (30). □

Example
The 3-surface in $\mathbb{R}^5$ parameterized by

$$(t_1, t_2, t_3, t_1^2, t_2^2 + t_3^2)$$

is *not* well curved (i.e., $\mu_{\mathcal{A}} = 0$). It has

$$\mathcal{P}_p(t_1, t_2, t_3, s_1, s_2, s_3) = t_1^2(s_2^2 + s_3^2) - (t_2^2 + t_3^2)s_1^2.$$

The boxes $K_\epsilon = [-\epsilon/2, \epsilon/2] \times [-\epsilon^{-1}, \epsilon^{-1}]^2$ have volume tending to infinity as $\epsilon \to 0^+$, but $K_\epsilon \times K_\epsilon$ is contained in the set where $|\mathcal{P}_p| \leq 1$ for all $\epsilon > 0$. Thus (29) implies that $\mu_{\mathcal{A}} = 0$. In contrast, the 3-surface in $\mathbb{R}^{21}$ parameterized by

$$\big((t^\alpha)_{1 \leq |\alpha| \leq 3}, t_1^2|t|^2, (t_2^2 + t_3^2)|t|^2\big)$$

is well curved. It has

$$\begin{aligned}\mathcal{P}_p(t_1, t_2, t_3, s_1, s_2, s_3) &= \big(t_1^2(s_2^2 + s_3^2) - (t_2^2 + t_3^2)s_1^2\big)|t|^2|s|^2 \\ &= \big(t_1^2|s|^2 - s_1^2|t|^2\big)|t|^2|s|^2.\end{aligned}$$

By a simple change of variables, every set in $\mathbb{S}^2$ which is rotationally symmetric about the x_1-axis has measure equal to 2π times the measure of the projection onto that axis. Therefore, the sublevel integral $I(\epsilon)$ defined in Theorem 5 satisfies

$$I(\epsilon) = 4\pi^2 \Big|\big\{(u, v) \in [-1, 1]^2 \;\big|\; |u^2 - v^2| \leq \epsilon\big\}\Big|$$

which is easily shown to satisfy a nontrivial sublevel set inequality for any $\sigma > 1$.

3.3. *Model forms for well-curvedness and Theorem 2*

Recall from the Introduction that a polynomial map f of the form

$$f(t) := \big((t^\alpha)_{1 \leq |\alpha| < \kappa_n}, p_1(t), \ldots, p_m(t)\big) \tag{31}$$

for $p_1, \ldots, p_m \in \dot{P}_d^{\kappa_n}$ is a model form when there exist $\lambda_1, \lambda_2 \geq 0$ such that

$$\begin{aligned}\sum_{j=1}^{d} \langle \partial_j p_\ell, \partial_j p_{\ell'} \rangle_{\kappa_n - 1} &= \lambda_1 \delta_{\ell, \ell'}, \\ \sum_{\ell=1}^{m} \langle \partial_j p_\ell, \partial_{j'} p_\ell \rangle_{\kappa_n - 1} &= \lambda_2 \delta_{j, j'},\end{aligned} \tag{32}$$

where in both equations δ is the Kronecker delta. In this section, we will prove parts (1) and (2) of Theorem 2 concerning model forms. This section will also lay some additional groundwork for proof of part (3) of Theorem 2, which will be completed in Section 5.3.

The key observation from geometric invariant theory which plays a prominent role in this case is that critical points (as a function of M) in the infimum (14) must be points at which the infimum is attained. This will be the main observation to be exploited; a secondary observation, encapsulated in the following lemma, allows one to simplify the structure of $\mathscr{P}$ even further at the expense of taking an infimum over a larger group.

LEMMA 4
Let f have the form (2). Let $p := (p_1, \ldots, p_m) \in (\dot{P}_d^{\kappa_n})^m$ be the m-tuple of highest-order parts of f. For any $(N, M) \in \mathrm{SL}(m,\mathbb{R}) \times \mathrm{SL}(d,\mathbb{R})$, let

$$(R_{N,M}\, p)(t) := Np(M^T t). \tag{33}$$

Then the measure $\nu_{\mathcal{A}}$ defined by (14) is a constant times Lebesgue measure, and

$$\frac{d\nu_{\mathcal{A}}}{dt} = C'' \Big[\inf_{\substack{N \in \mathrm{SL}(m,\mathbb{R}) \\ M \in \mathrm{SL}(d,\mathbb{R})}} \sum_{j=1}^{m} \| (R_{N,M}\, p)_j \|_{\kappa_n}^2 \Big]^{\frac{md}{2Q}} \tag{34}$$

for some constant C'' depending only on d and n.

Before proving Lemma 4, a more fundamental lemma is necessary.

LEMMA 5
Let A be a real $m \times m'$ matrix where $m' \geq m$, and let $[A]_{i_1 \cdots i_m}$ be the $m \times m$ matrix formed by combining columns $i_1, \ldots, i_m$ of A into a square matrix; that is, the (j,k) entry of this matrix is $A_{j i_k}$. Then

$$\sum_{i_1,\ldots,i_m=1}^{m'} \left| \det [A]_{i_1 \cdots i_m} \right|^2 = \frac{m!}{m^m} \Big[\inf_{N \in \mathrm{SL}(m,\mathbb{R})} \sum_{j=1}^{m} \sum_{i=1}^{m'} \Big| \sum_{k=1}^{m} N_{jk} A_{ki} \Big|^2 \Big]^m. \tag{35}$$

Proof
First observe that both

$$A \mapsto \sum_{i_1,\ldots,i_m=1}^{m'} \left| \det [A]_{i_1 \cdots i_m} \right|^2 \qquad \text{and} \qquad A \mapsto \sum_{j=1}^{m} \sum_{i=1}^{m'} |A_{ji}|^2$$

are invariant under the left action of $O(m,\mathbb{R})$ on columns of A as well as the right action of $O(m',\mathbb{R})$ on rows of A (in both cases, the identity is established by expanding multilinear sums and directly exploiting the orthogonality identity as was done in

Lemma 1). In particular, this means that we may, by the singular-value decomposition, assume without loss of generality that

$$A_{ji} = \sigma_j \delta_{j,i},$$

where σ_j is the jth singular value of A and δ is the Kronecker delta. Thus

$$\sum_{i_1,\dots,i_m=1}^{m'} \left|\det[A]_{i_1\cdots i_m}\right|^2 = m!\sigma_1^2\cdots\sigma_m^2 \qquad \text{and} \qquad \sum_{j=1}^{m}\sum_{i=1}^{m'} |A_{ji}|^2 = \sigma_1^2+\cdots+\sigma_m^2.$$

By the AM-GM inequality,

$$\frac{1}{m!}\sum_{i_1,\dots,i_m=1}^{m'} \left|\det[A]_{i_1\cdots i_m}\right|^2 \leq \left[\frac{1}{m}\sum_{j=1}^{m}\sum_{i=1}^{m'} |A_{ji}|^2\right]^m \tag{36}$$

with equality if and only if the singular values of A are all equal. Now multiplication of A on the left by a matrix $N \in \mathrm{SL}(m,\mathbb{R})$ preserves the left-hand side of (36) but not necessarily the right-hand side; taking an infimum of the right-hand side over all N gives

$$\sum_{i_1=1,\dots,i_m=1}^{m'} \left|\det[A]_{i_1\cdots i_m}\right|^2 \leq \frac{m!}{m^m}\left[\inf_{N\in\mathrm{SL}(m,\mathbb{R})}\sum_{j=1}^{m}\sum_{i=1}^{m'}\left|\sum_{k=1}^{m} N_{jk}A_{ki}\right|^2\right]^m$$

for any $A \in \mathbb{R}^{m\times m'}$. To show equality, assume once again without loss of generality that A is diagonal in the standard basis of $\mathbb{R}^{m\times m'}$, and let N be the diagonal matrix such that $N_{ii} := \sigma_i^{-1}(\sigma_1\cdots\sigma_m)^{1/m}$ assuming none of the singular values are zero. In this case, NA has all diagonal entries equal, and consequently (36) holds with equality when A is replaced by NA, giving equality in (35) as well. If, on the other hand, some singular value $\sigma_{i'}$ of A is zero, let $N^{(t)}$ be another diagonal matrix such that $N_{ii}^{(t)} = t$ for all entries $i \neq i'$, and let $N_{i'i'}^{(t)} = t^{-m+1}$. Then for $t > 0$, $N^{(t)} \in \mathrm{SL}(m,\mathbb{R})$ and

$$\lim_{t\to 0^+}\left[\sum_{j=1}^{m}\sum_{i=1}^{m'}\left|\sum_{k=1}^{m} N_{jk}^{(t)}A_{ki}\right|^2\right]^m = \lim_{t\to 0^+}\left[\sum_{i\neq i'} t^2\sigma_i^2\right]^m = 0,$$

so (35) holds with equality again in this case as well. □

Proof of Lemma 4

The polynomial given by (21) is independent of the point at which it is based and takes the particularly simple form

$$\mathcal{P}(s_1,\dots,s_m) = \det\begin{bmatrix} p_1(s_1) & \cdots & p_1(s_m) \\ \vdots & \ddots & \vdots \\ p_m(s_1) & \cdots & p_m(s_m)\end{bmatrix}. \tag{37}$$

Independence of the basepoint means that the density δ_p given by (12) is either always zero or never zero. For any $M \in \mathrm{SL}(d,\mathbb{R})$, let A^M be the $m \times d^{\kappa_n}$ matrix (where columns are indexed by $\imath := (i_1, \ldots, i_{\kappa_n}) \in \{1, \ldots, d\}^{\kappa_n}$) given by

$$A^M_{j\imath} := (M\partial)_{i_1} \cdots (M\partial)_{i_{\kappa_n}} p_j|_0.$$

By virtue of (22), (23), and (37),

$$\|\rho_M \mathcal{P}\|^2_{\kappa_n,m} = \sum_{\imath_1,\ldots,\imath_m \in \{1,\ldots,d\}^{\kappa_n}} \left|\det[A^M]_{\imath_1,\ldots,\imath_m}\right|^2.$$

By (35), this implies that

$$\begin{aligned}
&\|\rho_M \mathcal{P}\|^2_{\kappa_n,m} \\
&\quad = \frac{m!}{m^m}\Big[\inf_{N\in\mathrm{SL}(m,\mathbb{R})} \sum_{j=1}^{m} \sum_{i_1,\ldots,i_{\kappa_n}=1}^{d} \Big| \sum_{k=1}^{m} N_{jk}(M\partial)_{i_1}\cdots(M\partial)_{i_{\kappa_n}} p_k|_0\Big|^2\Big]^m \\
&\quad = \frac{m!}{m^m}\Big[\inf_{N\in\mathrm{SL}(m,\mathbb{R})} \sum_{j=1}^{m} \big\|(R_{N,M}p)_j\big\|^2_{\kappa_n}\Big]^m,
\end{aligned}$$

where $\|\cdot\|_{\kappa_n}$ is the norm corresponding to the inner product (4) defined in the Introduction, and $R_{N,M}$ is the representation of $\mathrm{SL}(m,\mathbb{R}) \times \mathrm{SL}(d,\mathbb{R})$ on the space $(\dot{P}^{\kappa_n}_d)^m$ given by (33). Raising both sides to the power $\frac{d}{2Q}$ and taking an infimum over M gives the formula (34). □

Proof of part (2) of Theorem 2
Now given any $p, q \in (\dot{P}^{\kappa_n}_d)^m$, the quantity

$$\sum_{j=1}^{m} \langle p_j, q_j\rangle_{\kappa_j}$$

is an inner product on $(\dot{P}^{\kappa_n}_d)^m$ which is invariant under the action of R_{O_1,O_2} for orthogonal O_1 and O_2 by virtue of the invariance of the usual inner product on $\mathbb{R}^m$ and by the identity (17) from Lemma 1. By Theorem 4.3 of Richardson and Slodowy [28] (which is the real analogue of ideas introduced by Kempf and Ness [22]), it suffices to show that the map

$$(N,M) \mapsto \sum_{j=1}^{m} \big\|(R_{N,M}p)_j\big\|^2_{\kappa_n} \tag{38}$$

has a critical point at the identity since their theorem establishes that all critical points are points where the infimum over all $(N,M) \in \mathrm{SL}(m,\mathbb{R}) \times \mathrm{SL}(d,\mathbb{R})$ is attained. (In particular, there are no nontrivial local maxima. The proof of their theorem ultimately reduces to establishing that the function must be convex along any 1-dimensional exponential family starting at a critical point.) Differentiating N at the identity in the direction of $E \in \mathfrak{sl}(m,\mathbb{R})$ gives

$$2 \sum_{\ell,\ell'=1}^{m} E_{\ell\ell'} \langle p_j, p_{\ell'} \rangle_{\kappa_n} = 0$$

for all traceless $m \times m$ matrices E. A similar calculation differentiating M at the identity gives

$$2\kappa_n \sum_{j=1}^{m} \sum_{i,i'=1}^{d} E_{ii'} \langle \partial_i p_j, \partial_{i'} p_j \rangle_{\kappa_n - 1} = 0$$

for all traceless $E \in \mathbb{R}^{d \times d}$. From these two calculations, it must be the case that (38) has a critical point at the identity if and only if p satisfies the system (32). By summing each identity over the diagonal, it follows that

$$m\lambda_1 = d\lambda_2 = \sum_{j=1}^{m} \|p_j\|_{\kappa_n}^2 = \inf_{\substack{N \in \mathrm{SL}(m,\mathbb{R}) \\ M \in \mathrm{SL}(d,\mathbb{R})}} \sum_{j=1}^{m} \|R_{N,M} p_j\|_{\kappa_n}^2 .$$

By (34),

$$\frac{d\nu_{\mathcal{A}}}{dt} = C''(\lambda_2 d)^{\frac{md}{2Q}} .$$

Thus for any f of the form (2) satisfying the critical point equations (32), the measure $\mu_{\mathcal{A}}$ is a nonzero constant times the pushforward of Lebesgue measure if and only if $\lambda_1, \lambda_2 \neq 0$. □

Proof of part (1) of Theorem 2

Exactly as was computed in the proof of Lemma 1, the space $P_d^{\kappa_n - 1}$ is mapped into itself by the representation

$$\rho_M q(t) := q(M^T t)$$

and $\det \rho_M = 1$ when $M \in \mathrm{SL}(d,\mathbb{R})$. Therefore, it suffices to study only the highest-order part of f (namely, $(p_1, \ldots, p_m) \in (\dot{P}_d^{\kappa_n})^m$) and to show that the closure of the orbit $\{R_{N,M}\, p\}_{N \in \mathrm{SL}(m,\mathbb{R}), M \in \mathrm{SL}(d,\mathbb{R})}$ in $(\dot{P}_d^{\kappa_n})^m$ always contains an m-tuple satisfying (32) and that degeneracy or nondegeneracy must hold for every single $p' \in (\dot{P}_d^{\kappa_n})^m$ in the closure of the orbit. This latter fact is an immediate consequence of the identity (34). As for the fact that the closure of the orbit always contains a model form, this is a consequence of Lemma 3.3 and Theorem 4.4 of Richardson and Slodowy [28] since

- the orbit $\{R_{N,M}\, p\}$ meets the solution set of (32) when the orbit is Zariski-closed;
- when the orbit $\{R_{N,M}\, p\}$ is not Zariski-closed, there exist traceless self-adjoint N_0, M_0 such that

$$\lim_{t \to -\infty} R_{e^{tN_0}, e^{tM_0}} f$$

exists and does have Zariski-closed orbit.

In this latter case, the orbit of the limit meets the solution set of (32). In particular, there must exist $N_1 \in \mathrm{SL}(m, \mathbb{R})$ and $M_1 \in \mathrm{SL}(d, \mathbb{R})$ such that

$$\lim_{t \to -\infty} R_{N_1 e^{tN_0}, M_1 e^{tM_0}} f$$

exists and satisfies (32), which implies that the closure of the orbit $R_{N,M} f$ (in either the standard or Zariski topologies) always contains a solution of (32). □

Example

Suppose that $p(t) := (t^{\alpha_1}/\sqrt{\alpha_1!}, \ldots, t^{\alpha_m}/\sqrt{\alpha_m!})$ for multi-indices α_i, $i = 1, \ldots, m$, such that $|\alpha_i| = \kappa_n$. Suppose also that e_j is the multi-index of order 1 with entry 1 in position j and that the multi-indices $\{\alpha_i + e_j\}$ are distinct for every pair (i, j). Then it must be the case that $\langle \partial_j t^{\alpha_i}, \partial_{j'} t^{\alpha_{i'}} \rangle_{\kappa_n - 1} = 0$ when $(i, j) \neq (i', j')$ and that

$$\begin{aligned} \left\| \frac{1}{\sqrt{\alpha_i!}} \partial_j t^{\alpha_i} \right\|^2_{\kappa_n - 1} &= \frac{1}{\alpha_i!} (\alpha_i)_j^2 \langle t^{\alpha_i - e_j}, t^{\alpha_i - e_j} \rangle_{\kappa_n - 1} \\ &= \frac{1}{\alpha_i!} (\kappa_n - 1)! (\alpha_i)_j^2 (\alpha_i - e_j)! = (\kappa_n - 1)! (\alpha_i)_j . \end{aligned}$$

For this particular p, the first system of equations in (32) will always be satisfied for some nonzero λ_1 because each multi-index has the same order. The second system will be true with some nonzero λ_2 if $\sum_{i=1}^m \alpha_i$ is a multiple of the multi-index $(1, \ldots, 1)$. This immediately gives well-curvedness for examples such as the 4-surface in $\mathbb{R}^{129}$ parameterized by

$$\left((t^\alpha)_{1 \le |\alpha| \le 5}, t_1^2 t_2^2 t_3^2, t_1^4 t_4^2, t_2^4 t_4^2, t_3^4 t_4^2 \right)$$

(since rescaling in individual directions by constant nonzero factors preserves well-curvedness). A somewhat more sophisticated set of examples in a similar spirit is provided by Lemma 8 in Section 5.3.

3.4. Newton-type polyhedra and height

The final observation we make regarding well-curvedness of simple polynomial submanifolds of the form (2) is an analytic reinterpretation of the Hilbert–Mumford criterion, which in this case asserts that $d\nu_{\mathcal{A}}/dt \equiv 0$ if and only if there exist traceless self-adjoint matrices $N_0 \in \mathbb{R}^{m\times m}$, $M_0 \in \mathbb{R}^{d\times d}$ such that

$$R_{e^{-tN_0},e^{-tM_0}} p \to 0 \quad \text{as } t \to \infty.$$

This condition has a natural geometric interpretation similar to (but simpler than) the notion of the height of real analytic functions, which is ubiquitous in the application of resolution of singularities methods to oscillatory integral operators in harmonic analysis (see, e.g., [19]).

THEOREM 6
Let $\{e_1,\ldots,e_m\}$ be the standard basis of $\mathbb{R}^m$. For any $p \in (\dot{P}_d^{\kappa_n})^m$, let $\mathcal{N}(p) \subset [0,\infty)^m \times [0,\infty)^d$ be the convex hull of all points (e_j,α) such that

$$\partial^\alpha p_j(t)|_{t=0} \neq 0.$$

Then the associated submanifold defined by (2) is well curved if and only if

$$\Big(\frac{1}{m}\mathbf{1}_m, \frac{\kappa_n}{d}\mathbf{1}_d\Big) \in \mathcal{N}(R_{O_1,O_2} p) \tag{39}$$

for all real orthogonal matrices $O_1 \in O(m)$ and $O_2 \in O(d)$, where $\mathbf{1}_m := (1,\ldots,1) \in [0,\infty)^m$ and likewise for $\mathbf{1}_d \in [0,\infty)^d$.

Proof
Because the norm

$$\|p\|^2 := \sum_{j=1}^m \|p_j\|_{\kappa_n}^2$$

is invariant under the action of both of the orthogonal groups $O(m)$ and $O(d)$, it suffices in the calculation (34) of $d\nu_{\mathcal{A}}/dt$ to take the infimum only over $N \in \mathrm{SL}(m,\mathbb{R})$ and $M \in \mathrm{SL}(d,\mathbb{R})$ which have the form DO, where D is diagonal with nonnegative entries and O is orthogonal, by virtue of the singular-value decomposition. If $N = D_1O_1$ where the ordered diagonal entries of D_1 are $s_1,\ldots,s_m$, and if $M = D_2O_2$ where the ordered diagonal entries of D_2 are $t_1,\ldots,t_d$, then it must be the case that

$$\|R_{N,M}p\|^2 = \sum_{j=1}^m \sum_{|\alpha|=\kappa_n} s_j^2 t^{2\alpha} \frac{\kappa_n!}{\alpha!} \big|\big(O_1(O_2\partial)^\alpha p\big)_j\big|^2. \tag{40}$$

It follows that $d\nu_{\mathcal{A}}/dt$ is nonzero if and only if there is a nonzero lower bound for the right-hand side of (40) which holds for all O_1, O_2, s, and t assuming that O_1 and O_2 are orthogonal and $s_1\cdots s_m = t_1\cdots t_d = 1$.

If the condition (39) holds, then for every O_1 and O_2 there must be pairs $(e_{j_1}, \alpha_1), \ldots, (e_{j_k}, \alpha_k)$ and nonnegative $\theta_1, \ldots, \theta_k$ summing to 1 such that, for each i, we get $[O_1(O_2\partial)^{\alpha_i} p]_{j_i} \neq 0$ and

$$\sum_{i=1}^{k} \theta_i (e_{j_i}, \alpha_i) = \Big(\frac{1}{m}\mathbf{1}_m, \frac{\kappa_n}{d}\mathbf{1}_d\Big).$$

By the AM-GM inequality,

$$\begin{aligned}\sum_{i=1}^{k} s_{j_i}^2 t^{2\alpha_i} \frac{\kappa_n!}{\alpha_i!} \big|\big(O_1(O_2\partial)^{\alpha_i} p\big)_j\big|^2 &\geq \sum_{i=1}^{k} \theta_i s_{j_i}^2 t^{2\alpha_i} \frac{\kappa_n!}{\alpha_i!} \big|\big(O_1(O_2\partial)^{\alpha_i} p\big)_j\big|^2 \\ &\geq \prod_{i=1}^{k} \Big| s_{j_i}^2 t^{2\alpha_i} \frac{\kappa_n!}{\alpha_i!} \big|\big(O_1(O_2\partial)^{\alpha_i} p\big)_j\big|^2 \Big|^{\theta_i} \\ &= \prod_{i=1}^{k} \Big| \frac{\kappa_n!}{\alpha_i!} \big|\big(O_1(O_2\partial)^{\alpha_i} p\big)_j\big|^2 \Big|^{\theta_i} > 0 \qquad (41)\end{aligned}$$

uniformly for all admissible s and t. Because the quantities $O_1(O_2\partial)^{\alpha} p$ are continuous as a function of O_1 and O_2, the quantity on the right-hand side of (41) must be uniformly bounded below by a positive constant on some neighborhood of (O_1, O_2) in $O(m) \times O(d)$. Because this space is compact, there must be a universal bound from below, meaning by (34) that $\mu_{\mathcal{A}}$ is a nonzero constant times the pushforward of Lebesgue measure.

If on the other hand, there are O_1 and O_2 such that (39) does not hold, then in fact no point of the form $(c\mathbf{1}_m, c'\mathbf{1}_d)$ can belong to $\mathcal{N}(R_{O_1,O_2} p)$ since every (e_j, α) such that $[O_1(O_2\partial)^{\alpha} p]_j \neq 0$ has

$$e_j \cdot \mathbf{1}_m = 1 = \frac{1}{\kappa_n} \alpha \cdot \mathbf{1}_d,$$

and $c = 1/m, c' = \kappa_n/d$ is the only pair for which the point in question lies in both affine subspaces. Thus by the separating hyperplane theorem, there must exist vectors $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^d$,

$$u_j + v \cdot \alpha > 0 \quad \text{whenever } \big[O_1(O_2\partial)^{\alpha} p\big]_j \neq 0$$

and

$$u \cdot \mathbf{1}_m = \mathbf{1}_d \cdot v = 0.$$

(That is, there must exist a vector $(u, v) \in \mathbb{R}^{m+d}$ which has zero dot product with every vector of the form $(c\mathbf{1}_m, c'\mathbf{1}_d)$ and strictly positive dot product with every vector in $\mathcal{N}(R_{O_1,O_2} p)$.) Fixing $s_j := e^{-\tau u_j}$, $j = 1,\ldots,m$, and $t_j := e^{-\tau v_j}$, $j = 1,\ldots,d$, gives $s_1 \cdots s_m = 1 = t_1 \cdots t_d$ for all τ but also gives $s_j^2 t^{2\alpha} \to 0$ for each nonzero term in the sum (40) as $\tau \to \infty$, forcing $d\nu_{\mathcal{A}}/dt = 0$. □

Example
A homogeneous polynomial q of degree κ in two variables satisfies $\partial_1^\kappa q = \cdots = \partial_1^{\kappa-i} \partial_2^i q = 0$ if and only if $q, \partial_\theta q, \ldots, \partial_\theta^i q$ all vanish at the point $(1,0)$, where $\partial_\theta = -t_2 \partial_1 + t_1 \partial_2$ is the angular derivative. Using this fact, it is an elementary exercise to check that when $d = m = 2$, for example, the condition (39) for nondegeneracy of $p := (p_1, p_2)$ is equivalent to the condition that, for every orthogonal matrix $O \in \mathbb{R}^{2\times 2}$, there are no points on the unit circle at which functions $(Op)_1$ and $(Op)_2$ vanish in the angular direction to orders o_1 and o_2, respectively, with $o_1 + o_2 > \kappa_n$. The 2-surface in $\mathbb{R}^7$ given by

$$(t_1, t_2, t_1^2, t_2^2, t_1 t_2, t_1^3 - 3t_1 t_2^2, 3t_1^2 t_2 - t_2^3)$$

is well curved by this reasoning since no linear combination of the cubic polynomials vanishes to order more than 1 in the angular direction on the unit circle in $\mathbb{R}^2$.

4. Necessity and proofs of parts (1) and (2) of Theorem 1

It is now time to return attention to the proof of Theorem 1. We begin with the following elementary lemma, which gives an estimate for the volume of the convex hull of certain sets $S \subset \mathbb{R}^n$.

LEMMA 6
Suppose that $S \subset \mathbb{R}^n$ is a compact set containing the origin, and let K be its convex hull. There exist $v_1, \ldots, v_n \in S$ such that the sets

$$K_1 := \left\{ v \in \mathbb{R}^n \;\middle|\; v = \sum_{i=1}^n c_i v_i \text{ for coefficients } c_i \geq 0 \text{ such that } \sum_{i=1}^n c_i \leq 1 \right\}$$

and

$$K_\infty := \left\{ v \in \mathbb{R}^n \;\middle|\; v = \sum_{i=1}^n c_i v_i \text{ for coefficients } c_i \in [-1,1], i = 1, \ldots, n \right\}$$

satisfy

$$K_1 \subset K \subset K_\infty. \tag{42}$$

In particular,

$$\frac{1}{n!} \left| \det(v_1 \wedge \cdots \wedge v_n) \right| \leq |K| \leq 2^n \left| \det(v_1 \wedge \cdots \wedge v_n) \right|. \tag{43}$$

Proof

This lemma is essentially a minor variation of John's ellipsoid theorem in [20]. Let V be the unique vector subspace of $\mathbb{R}^n$ of smallest dimension which contains S (where uniqueness holds because the intersection of two subspaces containing S would be a subspace of smaller dimension also containing S). Let m denote the dimension of V, and let $\det_V$ be any nontrivial alternating m-linear form on V. Let $(v_1, \ldots, v_m) \in S^m$ be any m-tuple at which the maximum of the function

$$(v_1, \ldots, v_m) \to \left| \det_V (v_1 \wedge \cdots \wedge v_m) \right|$$

is attained. Since S is not contained in any subspace of smaller dimension, $|\det_V(v_1 \wedge \cdots \wedge v_m)| > 0$ unless $m = 0$ (in which case $S = \{0\}$ and the lemma is trivial). Now by Cramer's rule, for any $v \in V$,

$$v = \sum_{i=1}^{m} (-1)^{i-1} \frac{\det_V(v \wedge v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_m)}{\det_V(v_1 \wedge \cdots \wedge v_m)} v_i,$$

where, in this case, the circumflex $\widehat{\cdot}$ indicates that a vector is to be omitted from the determinant. In the particular case when $v \in S$, the m-tuple $(v, v_1, \ldots, \widehat{v_i}, \ldots, v_m)$ belongs to the set S^m over which the supremum of $|\det_V|$ was taken; therefore each numerator has magnitude less than or equal to the magnitude of the denominator. Thus S belongs to the parallelepiped

$$P := \left\{ v \in \mathbb{R}^n \;\middle|\; v = \sum_{i=1}^{m} c_i v_i \text{ for some } c_1, \ldots, c_m \in [-1, 1] \right\}.$$

Since P is convex and contains S, it must contain K as well. To establish the lemma, we extend the sequence $v_1, \ldots, v_m$ to a sequence of length n by fixing $v_j = 0$ for $j > m$. Trivially $P = K_\infty$ for this choice, so the containment $K \subset K_\infty$ must hold. For the remaining containment, observe that $0, v_1, \ldots, v_n$ must belong to K since they belong to S. Therefore, by convexity of K, the set K_1 must be contained in K. The volume inequality (43) follows from the elementary calculation of the volumes of K_1 and K_∞. □

With Lemma 6 in place, we turn now to the proofs of parts (1) and (2) of Theorem 1.

Proof of part (1) of Theorem 1

Pick any point $p \in \mathcal{M}$, and fix any smooth coordinate system $(t_1, \ldots, t_d)$ near p so that the immersion $f : \mathcal{M} \to \mathbb{R}^d$ may be regarded in these coordinates as a function from a 3δ neighborhood of the origin (chosen so that $t = 0$ are the coordinates of p)

into $\mathbb{R}^n$. It may also be assumed without loss of generality that f is an embedding on this neighborhood. By Taylor's formula, for all $t_0, t \in \mathbb{R}^d$ with $|t_0| \leq \delta, |t| \leq 2\delta$,

$$f(t) - f(t_0) = \sum_{0<|\alpha|\leq \ell} \frac{(t-t_0)^\alpha}{\alpha!} \partial_t^\alpha f(t_0) + \sum_{|\beta|=\ell+1} \frac{(t-t_0)^\beta}{\beta!} R_{t_0}^\beta(t), \tag{44}$$

for any finite ℓ, where each remainder term $R_{t_0}^\beta(t)$ is continuous on $|t| \leq 2\delta$ and equals $\partial_t^\beta f(t_0)$ when $t = t_0$. (For most of what follows, t_0 will be regarded as a fixed but otherwise arbitrary point with $|t_0| \leq \delta$.) For definiteness, let $\ell := \kappa_n$; that is, ℓ equals the highest order of differentiation that appears in a column of the matrix whose determinant forms $\mathcal{A}$ (or equivalently, ℓ is the number of boxes in column n of the diagram given in Figure 2). This choice of ℓ implies that the dimension of the space of polynomials of degree ℓ with no constant term is at least equal to n. For any $r \in (0, \delta]$, let $S_{t_0,r}$ be the compact subset of $\mathbb{R}^n$ given by

$$S_{t_0,r} := \{0\} \cup \bigcup_{|\alpha|\leq \ell} \left\{ \frac{r^{|\alpha|}}{\alpha!} \partial_t^\alpha f(t_0) \right\} \cup \bigcup_{|\beta|=\ell+1, |t|\leq 2\delta} \left\{ \frac{r^{\ell+1}}{\beta!} R_{t_0}^\beta(t) \right\},$$

and let $K_{t_0,r}$ be the convex hull of $S_{t_0,r} \cup (-S_{t_0,r})$. Now each term in either sum on the right-hand side of (44) belongs to $K_{t_0,r}$ whenever $|t| \leq 2\delta$ and $|t - t_0| \leq r$. Because the total number of summands on the right-hand side is at most some constant C depending only on d and n, the difference vector $f(t) - f(t_0)$ must belong to the dilated set $CK_{t_0,r}$ whenever $|t| \leq 2\delta$ and $|t - t_0| \leq r$. In particular, this implies that the translated set $CK_{t_0,r} + f(t_0)$ must contain the vector $f(t)$ whenever $|t| \leq 2\delta$ and $|t - t_0| \leq r \leq \delta$.

By virtue of (43), the Lebesgue measure of the set $CK_{t_0,r} + f(t_0)$ is $O(r^Q)$ as $r \to 0^+$ since it is dominated by a constant depending on d and n times a determinant $|\det(v_1 \wedge \cdots \wedge v_n)|$ for some $v_1, \ldots, v_n \in S_{t_0,r} \cup (-S_{t_0,r})$ and since Q is by definition the smallest integer which is possible to express as a sum of degrees of distinct, nonconstant monomials in d variables (thus Q corresponds to the smallest possible power of r which will appear via scaling in such determinants). In fact, a slightly stronger result is also true—namely, that it is possible to quantify the implied constant in this $O(r^Q)$ estimate in terms of the affine curvature tensor $\mathcal{A}$ at t_0. For any collection $\alpha_1, \ldots, \alpha_n$ of multi-indices such that $|\alpha_1| + \cdots + |\alpha_n| = Q$, it is possible to find indices i_λ for each $\lambda \in \Lambda_{d,n}$ (these indices being obtained by "expanding" each α_i as a composition of first-order coordinate derivatives) so that

$$\left|\det\left(\partial_t^{\alpha_1} f(t_0) \wedge \cdots \wedge \partial_t^{\alpha_n} f(t_0)\right)\right| = \left|\mathcal{A}_{t_0}\left((\partial_{t_{i_\lambda}})_{\lambda \in \Lambda_{d,n}}\right)\right|$$

whenever the left-hand side is nonzero. Therefore, it follows from (43) that when $r \leq \delta$, the image $f(B_r(t_0))$ is contained in $CK_{t_0,r} + f(t_0)$, which is a compact convex set with volume no greater than

$$C' r^Q \Big[\sum_{j_1,\ldots,j_Q=1}^{d} \big| \mathcal{A}_{t_0}(\partial_{t_{j_1}}, \ldots, \partial_{t_{j_Q}}) \big|^2 \Big]^{\frac{1}{2}} + O(r^{Q+1})$$

as $r \to 0^+$, where C' is some new constant depending only on d and n. Consequently, if μ is any measure on $\mathbb{R}^n$ supported on $\mathcal{M}$ satisfying the Oberlin condition (1) with exponent α and constant C_μ, then

$$\limsup_{r \to 0^+} r^{-\alpha Q} \mu\big(f\left(B_r(t_0)\right)\big) \lesssim C_\mu \Big| \Big[\sum_{j_1,\ldots,j_Q=1}^{d} \big| \mathcal{A}_{t_0}(\partial_{t_{j_1}}, \ldots, \partial_{t_{j_Q}}) \big|^2 \Big]^{\frac{1}{2}} \Big|^{\alpha} \tag{45}$$

for any t_0 with $|t_0| \leq \delta$ with an implied constant depending only on d and n. If $\alpha \geq d/Q$, then this implies that μ must be absolutely continuous with respect to the pushforward of Lebesgue measure on $\mathcal{M}$ on a δ-neighborhood of the chosen origin point p, and if $\alpha > d/Q$, then it further implies that μ must be the zero measure on the image of that neighborhood in $\mathbb{R}^n$ (since the Radon–Nikodym derivative of μ with respect to the pushforward of Lebesgue measure must vanish at every Lebesgue point, which is almost every point in the neighborhood), thus establishing part (1) of Theorem 1. □

Proof of part (2) of Theorem 1

Returning to (45) when $\alpha = d/Q$, the previous calculations show that on a δ-neighborhood of the point p in the given coordinates, μ restricted to the image of that neighborhood (with respect to the immersion f, which may be assumed to be an embedding on this neighborhood) must be absolutely continuous with respect to the pushforward of Lebesgue measure. It follows that the Radon–Nikodym derivative $d\mu/dt$, which for simplicity is taken as a function on $\mathcal{M}$ rather than $\mathbb{R}^n$, is well defined and satisfies

$$\limsup_{r \to 0^+} r^{-d} \mu\big(f\left(B_r(t_0)\right)\big) = c_d \frac{d\mu}{dt}(t_0)$$

for almost every t_0 with $|t_0| \leq \delta$. By (45), then,

$$\frac{d\mu}{dt}\Big|_q \lesssim C_\mu \Big[\sum_{j_1,\ldots,j_Q=1}^{d} \big| \mathcal{A}_q(\partial_{t_{j_1}}, \ldots, \partial_{t_{j_Q}}) \big|^2 \Big]^{\frac{d}{2Q}}$$

for almost every point q in some neighborhood of the original point p (where, once again, the implied constant depends only on d and n). Now, by transforming the coordinates $(t_1, \ldots, t_d)$ by matrices $M \in \mathrm{SL}(d, \mathbb{R})$ to produce new coordinate systems, it follows by the same reasoning as above that

$$\frac{d\mu}{dt}\Big|_q \lesssim C_\mu \Big[\sum_{j_1,\ldots,j_Q=1}^{d} \Big| \sum_{i_1,\ldots,i_Q=1}^{d} \mathcal{A}_q(M_{j_1 i_1}\partial_{t_{i_1}},\ldots,M_{j_Q i_Q}\partial_{t_{i_Q}}) \Big|^2 \Big]^{\frac{d}{2Q}}$$

for every $M \in \mathrm{SL}(d,\mathbb{R})$ and almost every q in a neighborhood of p (by continuity of the right-hand side as a function of M, it suffices to consider only some countable dense subset of $\mathrm{SL}(d,\mathbb{R})$ so that the set on which the inequality fails is clearly null). Taking an infimum over M gives that

$$\frac{d\mu}{dt} \lesssim C_\mu \frac{d\nu_{\mathcal{A}}}{dt}$$

almost everywhere on the coordinate patch. Because the coordinates and patch were arbitrary, it follows that part (2) of Theorem 1 must hold with an implicit constant which equals a dimensional quantity (depending only on d and n) times the Oberlin constant C_μ from (1) for the measure μ itself. □

5. Sufficiency and nontriviality

5.1. On the geometry of functions on measurable sets

This section begins with a construction generalizing the results of Theorem 1 of [13]. Roughly stated, that theorem indicated that for single-variable real polynomials of a given degree, every measurable subset of the real line has a "core" which contains a nontrivial fraction of the set such that the supremum of any such polynomial (or appropriately weighted derivatives) on the core is bounded above by the average of the polynomial on the entire set. The proof involved careful analysis of Vandermonde determinants and has no immediate generalization to other dimensions or families of functions. In the arguments below, an entirely different approach will be used which is based on convex geometry and admits extensions to a variety of new contexts. In particular, the setting of polynomials is no simpler to study than any other finite-dimensional family of real analytic functions, which will be the preferred formulation of the result.

Recall from Section 1 that a pair $(\mathcal{M},\mathcal{F})$ of a real analytic manifold $\mathcal{M}$ of dimension d and a finite-dimensional vector space $\mathcal{F}$ of real analytic functions on $\mathcal{M}$ whose differentials span the cotangent space at every point of $\mathcal{M}$ is called a *geometric function system*. Such a system is called *compact* when $\mathcal{M}$ is either compact or has a compact closure in some larger real analytic manifold $\mathcal{M}^+$ such that the functions of $\mathcal{F}$ extend to functions $\mathcal{F}^+$ on $\mathcal{M}^+$ in such a way that $(\mathcal{M}^+,\mathcal{F}^+)$ is also a geometric function system. Recall also Theorem 3.

THEOREM 3

Suppose that $(\mathcal{M},\mathcal{F})$ is a compact geometric function system. Then for any finite *positive measure μ on $\mathcal{M}$ absolutely continuous with respect to Lebesgue measure and any measurable set $E \subset \mathcal{M}$ of positive measure, there is a measurable subset $E' \subset E$ such that $\mu(E') \gtrsim \mu(E)$ and*

$$\sup_{p \in E'} \left| f(p) \right| \lesssim \frac{1}{\mu(E)} \int_E |f| \, d\mu \quad \textit{for all } f \in \mathcal{F}.$$

The implicit constants in both inequalities depend only on the pair $(\mathcal{M}, \mathcal{F})$.

Proof
For each $f \in \mathcal{F}$, consider the norm

$$\|f\| := \frac{1}{\mu(E)} \int_E |f| \, d\mu.$$

Compactness of the geometric function system implies that $\|f\|$ is finite for every $f \in \mathcal{F}$. Because each function $f \in \mathcal{F}$ is real-analytic and the measure of E is strictly positive, no $f \in \mathcal{F}$ aside from the zero function can have $\|f\| = 0$, which is what guarantees that $\| \cdot \|$ is a norm rather than merely a seminorm. Assuming that the dimension of $\mathcal{F}$ is k, applying Lemma 6 to the set S which is the unit ball of $\| \cdot \|$ and using homogeneity of the norm, there must be functions $f_1, \ldots, f_k$ with $\|f_i\| = 1$ for all i (none of the functions f_i will be identically zero because the unit sphere does not lie in any nontrivial subspace of $\mathcal{F}$) such that every $f \in \mathcal{F}$ has the property that

$$f = \sum_{i=1}^{k} c_i f_i$$

with $|c_i| \leq \|f\|$ for each i. In particular, this implies that

$$\left| f(p) \right| = \left| \sum_{i=1}^{k} c_i f_i(p) \right| \leq \|f\| \sum_{i=1}^{k} \left| f_i(p) \right|$$

for each $f \in \mathcal{F}$. Let E' be the subset of E on which $\sum_{i=1}^{k} |f_i(p)| \leq 2k$; by Chebyshev's inequality,

$$\mu(E') \geq \mu(E) - \frac{1}{2k} \int_E \sum_{i=1}^{k} \left| f_i(p) \right| d\mu \geq \frac{1}{2} \mu(E)$$

and

$$\sup_{p \in E'} \left| f(p) \right| \leq \sup_{p \in E'} \left[\|f\| \sum_{i=1}^{k} \left| f_i(p) \right| \right] \leq \frac{2k}{\mu(E)} \int_E |f| \, d\mu$$

for all $f \in \mathcal{F}$. □

The extension of the results of [13] to derivative estimates in higher dimensions is necessarily much more subtle than the 1-dimensional case because of inherent issues of anisotropy of differentiation in various directions. Any proper formulation will necessarily be phrased in terms of vector fields which capture (either implicitly or explicitly) this anisotropy. For convenience, given any vectors $X_1, \ldots, X_d$ at a point $p \in \mathcal{M}$, let $\mu(X_1 \wedge \cdots \wedge X_d)$ denote the associated density of μ, which is equal to the volume of the parallelepiped generated by $X_1, \ldots, X_d$ as measured by μ. In other words,

$$\mu(X_1 \wedge \cdots \wedge X_d)|_p := \frac{d\mu}{dt}\Big|_p |\det \mathbf{X}|$$

where the Radon–Nikodym derivative $d\mu/dt$ is with respect to Lebesgue measure in some coordinate system $(t_1, \ldots, t_d)$, and $\mathbf{X}$ is the matrix with entries $\mathbf{X}_{ij} = dt_i(X_j)$. With this definition in place, the formulation of the differential version of Theorem 3 to be proved here is as follows.

LEMMA 7

Suppose that $(\mathcal{M}, \mathcal{F})$ is a compact geometric function system, and let N be any positive integer. Then for any finite positive measure μ on $\mathcal{M}$ which is absolutely continuous with respect to Lebesgue measure and any measurable set $E \subset \mathcal{M}$ of positive measure, there is an open set $U \subset \mathcal{M}$, a family of smooth vector fields $\{X_{j,i}\}_{j,i}$ with $j \in \{1, \ldots, N\}$ and $i \in \{1, \ldots, d\}$, and a measurable set $E' \subset E \cap U$ such that the following are true with implicit constants depending only on the pair $(\mathcal{M}, \mathcal{F})$ and the integer N.

- *The subset $E' \subset E \cap U$ satisfies $\mu(E') \gtrsim \mu(E)$.*
- *The vector fields $X_{j,i}$ satisfy $\inf_{p \in E'} \mu(X_{j,1} \wedge \cdots \wedge X_{j,d})|_p \gtrsim \mu(E)$ and*

$$X_{j,i} = \sum_{i'=1}^{d} c_{j,i,i'} X_{j-1,i'} \tag{46}$$

with $|c_{j,i,i'}| \lesssim 1$ for each $j \in \{2, \ldots, N\}$ and each $i, i' \in \{1, \ldots, d\}$.
- *For any $k \in \{1, \ldots, N\}$, any indices $1 \leq j_1 < j_2 < \cdots < j_k \leq N$ and $i_1, \ldots, i_k \in \{1, \ldots, d\}$,*

$$\sup_{p \in E'} \left| X_{j_k, i_k} \cdots X_{j_1, i_1} f(p) \right| \lesssim \frac{1}{\mu(E)} \int_E |f| \, d\mu \tag{47}$$

uniformly for all $f \in \mathcal{F}$. Furthermore, the case corresponding to $k = 0$, that is,

$$\sup_{p \in E'} \left| f(p) \right| \lesssim \frac{1}{\mu(E)} \int_E |f| \, d\mu,$$

also holds uniformly for all $f \in \mathcal{F}$.

Proof

By induction (the base case of which is taken to be Theorem 3), for a given measurable set $E \subset \mathcal{M}$ of positive measure, we may assume that there exist a nested family of open sets $\mathcal{M} =: U_0 \supset U_1 \supset U_2 \supset \cdots \supset U_{N-1}$ and vector fields $\{X_{j,i}\}$, $i = 1, \ldots, d$, defined on U_j for each $j \in \{1, \ldots, N-1\}$ satisfying all the stated properties. Next, let $\mathcal{F}_0 := \mathcal{F}$, and then take $\mathcal{F}_j$ to be the vector space of real analytic functions on U_j spanned by $\mathcal{F}_{j-1}$ and all functions of the form $X_{j,i} f$ for $i = 1, \ldots, d$ and $f \in \mathcal{F}_{j-1}$. By construction, $\mu(E \cap U_{N-1}) \gtrsim \mu(E) > 0$, so in particular,

$$\|f\|_{N-1} := \frac{1}{\mu(E \cap U_{N-1})} \int_{E \cap U_{N-1}} |f| \, d\mu$$

will be a norm on $\mathcal{F}_{N-1}$. Let $f_1, \ldots, f_k$ be a basis of $\mathcal{F}_{N-1}$ given by applying Lemma 6 to the unit ball of $\|\cdot\|_{N-1}$, let $\mathcal{I}$ be the set of d-tuples $\beta := (\beta_1, \ldots, \beta_d)$ of indices satisfying $1 \leq \beta_1 < \beta_2 < \cdots < \beta_d \leq k$, and let V_β be the open set

$$\left\{ p \in U_{N-1} \,\middle|\, |df_{\beta_1} \wedge \cdots \wedge df_{\beta_d}|_p| > \frac{1}{2} |df_{\beta_1'} \wedge \cdots \wedge df_{\beta_d'}|_p| \forall \beta' \in \mathcal{I} \setminus \{\beta\} \right\},$$

where the bars $|\cdot|$ indicate any nontrivial norm on d-forms at p (the precise choice does not affect the set since any such norms must be multiples of each other because the space is 1-dimensional). Because the cardinality of $\mathcal{I}$ is bounded by a constant depending only on N and the dimensions of $\mathcal{F}$ and $\mathcal{M}$, there is at least one β such that $\mu(E \cap V_\beta) \gtrsim \mu(E \cap U_{N-1})$, where the implicit constant may simply be taken to be $(\#\mathcal{I})^{-1}$. If we now define the vector field $X_{N,i}$ on the set $U_N := V_\beta$ to equal

$$X_{N,i} f(p) := \frac{df_{\beta_1} \wedge \cdots \wedge df_{\beta_{i-1}} \wedge df \wedge df_{\beta_{i+1}} \wedge \cdots \wedge df_{\beta_d}|_p}{df_{\beta_1} \wedge \cdots \wedge df_{\beta_d}|_p}$$

(again, well defined because numerator and denominator belong to the same 1-dimensional vector space), then it must be the case that

$$X_{N,i} f(p) = \sum_{i=1}^{k} c_i \frac{df_{\beta_1} \wedge \cdots \wedge df_{\beta_{i-1}} \wedge df_i \wedge df_{\beta_{i+1}} \wedge \cdots \wedge df_{\beta_d}|_p}{df_{\beta_1} \wedge \cdots \wedge df_{\beta_d}|_p}$$

for constants c_i satisfying $|c_i| \leq \|f\|_{N-1}$. Since the ratio is bounded above by 2 on U_N, it follows that

$$\sup_{p \in U_N} \left| X_{N,i} f(p) \right| \leq \frac{2k}{\mu(E \cap U_{N-1})} \int_{E \cap U_{N-1}} |f| \, d\mu \tag{48}$$

for all $f \in \mathcal{F}_{N-1}$. Since $\mu(E \cap U_{N-1}) \gtrsim \mu(E)$, it follows by induction that

$$\sup_{p \in U_N} \left| X_{N,i_N} \cdots X_{1,i_1} f(p) \right| \lesssim \frac{1}{\mu(E)} \int_E |f| \, d\mu$$

for all $f \in \mathcal{F}$. This establishes (47) for any set $E' \subset E \cap U_N$ assuming that the top index j_k equals N. If not, then (47) follows simply by induction, provided that E' in the induction step N is a subset of the corresponding E' from induction step $N-1$.

Next observe that each vector field $X_{N,i}$ is locally a coordinate vector field relative to the coordinate functions $(f_{\beta_1}, \ldots, f_{\beta_d}) \in \mathcal{F}^d_{N-1}$ (i.e., the vector fields $X_{N,i]}$ equal the coordinate partial derivatives when $(f_{\beta_1}, \ldots, f_{\beta_d})$ are used as coordinate functions) and that the average value of $|f_{\beta_i}|$ on $E \cap U_{N-1}$ is 1. Without loss of generality (since the hypothesis is trivial in the base case $N = 0$), we may assume the additional induction hypothesis that for each $j \in \{1, \ldots, N-1\}$, the vector fields $X_{j,i}$ are local coordinate vector fields with respect to coordinate functions $(g_{\alpha_1}, \ldots, g_{\alpha_n}) \in \mathcal{F}^d_{j-1}$. In particular, if $(g_{\alpha_1}, \ldots, g_{\alpha_d})$ are the coordinate functions for the vector fields $X_{N-1,i}$, then it follows that

$$X_{N,i} = \sum_{i'=1}^{d} (X_{N,i} g_{\alpha_{i'}}) X_{N-1,i'}.$$

By the derivative estimate (48), assuming $N \geq 2$,

$$\begin{aligned} \sup_{p \in U_N} \left| X_{N,i} g_{\alpha_{i'}}(p) \right| &\leq \frac{2 \dim \mathcal{F}_{N-1}}{\mu(E \cap U_{N-1})} \int_{U_{N-1} \cap E} |g_{\alpha_{i'}}| \, d\mu \\ &\lesssim \frac{1}{\mu(E \cap U_{N-2})} \int_{E \cap U_{N-2}} |g_{\alpha_{i'}}| \, d\mu = 1, \end{aligned}$$

which is exactly the bound on the coefficients $c_{j,i,i'}$ claimed for (46) when $j = N$. For $j < N$, (46) follows analogously by virtue of the added induction hypothesis.

Lastly, the quantity $\mu(X_{N,1} \wedge \cdots \wedge X_{N,d})$ must be estimated. Observe that $\mu(X_{N,1} \wedge \cdots \wedge X_{N,d})$ is exactly the Radon–Nikodym derivative of μ with respect to Lebesgue measure in coordinates given by $f_{\beta_1}, \ldots, f_{\beta_d}$. This implies that

$$\int_{E'} \left[\mu(X_{N,1} \wedge \cdots \wedge X_{N,d}) \right]^{-1} d\mu = \int_{E'} |df_{\beta_1} \wedge \cdots \wedge df_{\beta_d}|.$$

Because the functions are real-analytic, we know that there is a finite number M independent of the choice of the functions f_{β_i} such that the system $f_{\beta_i}(p) = c_i$ has at most M nondegenerate solutions (at which the Jacobian is nonzero). For polynomial functions, this is a simple consequence of Bézout's theorem. In our case, however, even if the original function system $\mathcal{F}_0$ consists only of polynomial functions, the definition of the $X_{j,i}$'s leads naturally to the inclusion of certain rational functions in $\mathcal{F}_i$, at which point there is little additional difficulty in going to the more general context of real analytic functions. The algebraic argument is given in Section 5.3 and for now may be safely postponed.

Assuming the existence of such an M depending only on the geometric function system, by the change of variables formula and Fubini's theorem,

$$\int_{E'} |df_{\beta_1} \wedge \cdots \wedge df_{\beta_d}| \leq M \prod_{j=1}^{d} \left| f_{\beta_i}(E') \right|,$$

where $|f_{\beta_i}(E')|$ refers to the 1-dimensional Lebesgue measure of the image of E' via f_{β_i}. Because the average value of f_{β_i} on $E \cap U_{N-1}$ is 1, by Theorem 3 there is a subset $E' \subset E \cap U_N$ with $\mu(E') \gtrsim \mu(E \cap U_N) \gtrsim \mu(E)$ such that

$$\begin{aligned} \sup_{p \in E'} \left| f_{\beta_i}(p) \right| &\lesssim \frac{1}{\mu(E \cap U_N)} \int_{E \cap U_N} |f_{\beta_i}| \, d\mu \\ &\lesssim \frac{1}{\mu(E \cap U_{N-1})} \int_{E \cap U_{N-1}} |f_{\beta_i}| \, d\mu \\ &= 1 \end{aligned}$$

for each i, which implies that $|f_{\beta_i}(E')| \lesssim 1$ for each i as well. For this set E', it follows that

$$\int_{E'} \left[\mu(X_{N,1} \wedge \cdots \wedge X_{N,d}) \right]^{-1} d\mu \lesssim 1.$$

Further restricting E' using Chebyshev's inequality, we may assume that

$$\inf_{p \in E'} \mu(X_{N,1} \wedge \cdots \wedge X_{N,n})|_p \gtrsim \mu(E).$$

This completes the proof. □

5.2. *Proof of part (3) of Theorem 1*

We now return to the proof of part (3) of Theorem 1. The proof combines Lemma 7 with the geometric framework introduced in Section 2.2.

Suppose that $\mathcal{M}$ is a real analytic manifold of dimension d and that f is a real analytic immersion of $\mathcal{M}$ into $\mathbb{R}^n$ in such a way that the component functions $f_1, \ldots, f_n$ of the immersion together with the constant function belong to some compact geometric function system $(\mathcal{M}, \mathcal{F})$. Fix any compact convex set $K \in \mathcal{K}_n$, let $E := f^{-1}(K)$, and let $p_0 \in E$. Now the integral

$$\begin{aligned} I(E) := \frac{1}{\nu_{\mathcal{A}}(E)^n} \int_{E^n} \big| \det\big(& f(p_1) - f(p_0), \ldots, f(p_n) \\ & - f(p_0) \big) \big| \, d\nu_{\mathcal{A}}(p_1) \cdots d\nu_{\mathcal{A}}(p_n) \end{aligned}$$

(where $\nu_{\mathcal{A}}$ is the measure (13) whose pushforward (15) is the affine measure $\mu_{\mathcal{A}}$ on $\mathbb{R}^n$) must be bounded above by $n!|K|$ since the integrand equals $n!$ times the volume of the simplex generated by $f(p_0), \ldots, f(p_n)$, which has volume bounded by $|K|$ since each point $f(p_i)$ belongs to K and K is convex. In this case, Lemma 7 can be applied to each integral iteratively to prove a lower bound for the functional. Specifically, the lemma is applied to the innermost integral, which is then replaced by a supremum over some set E' of some derivative in the parameter p_1. As a result, the lemma establishes that

$$\begin{aligned} I(E) \gtrsim \sup_{(p_1,\ldots,p_n) \in (E')^n} \Big| \det\big(& X_{1,i_{(1,1)}} f(p_1) \wedge \cdots \\ & \wedge X_{\kappa_j, i_{(j,1)}} \cdots X_{1, i_{(j,\kappa_j)}} f(p_j) \wedge \cdots \\ & \wedge X_{\kappa_n, i_{(n,1)}} \cdots X_{1, i_{(n,\kappa_n)}} f(p_n) \big) \Big| \end{aligned}$$

for any choice of indices i_λ for $\lambda \in \Lambda_{d,n}$. Next replace the supremum over $(p_1, \ldots, p_n) \in E'^n$ by a supremum over $p \in E'$ assuming that $p_1 = \cdots = p_n = p$. It is also advantageous to use only vector fields $X_{\kappa_n, i'}$ rather than using any $X_{j,i}$ for $j < \kappa_n$. Thanks to (46) it must be the case that

$$\begin{aligned} & \det\big(X_{\kappa_n, i'_{(1,1)}} f(p) \wedge \cdots \wedge X_{\kappa_n, i_{(n,1)}} \cdots X_{\kappa_n, i'_{(n,\kappa_n)}} f(p) \big) \\ & \quad = \sum_i c_{i i'} \det\big(X_{1,i_{(1,1)}} f(p) \wedge \cdots \wedge X_{\kappa_n, i_{(n,1)}} \cdots X_{1, i_{(n,\kappa_n)}} f(p) \big) \end{aligned}$$

with coefficients $|c_{i i'}| \lesssim 1$, where the sum is over all possible choices of the indices i_λ. This identity holds because the change of basis formula may be simply substituted term-by-term in the left-hand side of the equation; any terms in which the coefficients of the change of basis happened to be differentiated by some subsequent vector field would ultimately have determinant zero since (assuming the column in which the derivative appears is column j) the number of derivatives acting directly on f would be strictly less than κ_j, which means that column j and all preceding columns would be linearly dependent. Therefore, by the triangle inequality, it must be the case that

$$I(E) \gtrsim \sup_{p \in E'} \Big| \det\big(X_{\kappa_n, i'_{(1,1)}} f(p) \wedge \cdots \wedge X_{\kappa_n, i'_{(n,1)}} \cdots X_{\kappa_n, i'_{(n,\kappa_n)}} f(p) \big) \Big|$$

uniformly for any choice of i'_λ. Taking an ℓ^2 norm over all such choices and invoking the definition (5) of the density (12) gives

$$I(E) \gtrsim \sup_{p \in E'} \big[\delta_p (X_{\kappa_n,1}, \ldots, X_{\kappa_n,d}) \big|_p \big]^{\frac{Q}{d}}.$$

To conclude, observe that for the measure $\nu_{\mathcal{A}}$, the quantity $\delta_p(X_{\kappa_n,1},\ldots,X_{\kappa_n,d})$ exactly equals the quantity which was shown to be bounded below in Lemma 7. Therefore,

$$|K| \gtrsim I(E) \gtrsim \big[\big|\delta_p(X_{\kappa_n,1},\ldots,X_{\kappa_n,n})\big|_p\big]^{\frac{Q}{d}} \gtrsim \big(\nu_{\mathcal{A}}(E)\big)^{\frac{Q}{d}} = \big(\mu_{\mathcal{A}}(K)\big)^{\frac{Q}{d}}$$

uniformly in K. This is exactly part (3) of Theorem 1. □

5.3. Proof of part (3) of Theorem 2

The remaining result is to show that $\alpha = d/Q$ is a nontrivial exponent in the sense that there is always some submanifold $\mathcal{M}$ of dimension d in $\mathbb{R}^n$ for which the Oberlin condition (1) is satisfied with exponent α for some nonzero measure supported on $\mathcal{M}$. The model case to be considered here is exactly the one laid out in Theorem 2: $\mathcal{M}$ is topologically a bounded open subset of $\mathbb{R}^d$ and the embedding f is a polynomial embedding into $\mathbb{R}^n$ of the form (2). In light of the inequalities (25), the measure $\mu_{\mathcal{A}}$ will vanish if and only if $\mathcal{P}$ belongs to the nullcone of the representation (22). Although the nullcone can be computed explicitly on a case-by-case basis, it can be challenging to compute in a very general or abstract way as we would seek to do here. Instead, we use the model form result, part (2) of Theorem 2, to construct a nontrivial example for every n and d. The specific examples to be constructed are laid out in the following lemma.

LEMMA 8

Suppose that $\mathcal{C}$ is a set of multi-indices of order κ_n in d variables such that $\mathcal{C}$ is closed under cyclic permutations of the entries of the multi-indices and such that each multi-index in $\mathcal{C}$ has at least two nonzero entries. Suppose further that $\{\varphi_j\}_{j=1,\ldots,d}$ is a uniform normalized tight frame (UNTF) on $\mathbb{R}^{d_0}$ for some $d_0 \leq d$, meaning that

$$\sum_{j=1}^{d} \big|\langle v,\varphi_j\rangle\big|^2 = \|v\|^2 \quad \text{for all } v \in \mathbb{R}^{d_0} \qquad \text{and} \qquad \|\varphi_j\|^2 = \frac{d_0}{d}, \quad j = 1,\ldots,d.$$

Then assuming that $\#\mathcal{C} + d_0 = m$, the collection of polynomials given by

$$\bigg(\Big(\frac{t^\alpha}{\sqrt{\alpha!}}\Big)_{\alpha\in\mathcal{C}}, \Big(\sum_{j=1}^{d} \frac{t_j^{\kappa_n}}{\sqrt{\kappa_n!}}\varphi_{j,k}\Big)_{k=1,\ldots,d_0}\bigg), \tag{49}$$

where $\varphi_{j,k}$ is the kth coordinate of φ_j in the standard basis, satisfies the critical point equations (32) for nonzero λ_1, λ_2.

Proof

For simplicity, fix $\kappa := \kappa_n$. Before beginning in earnest, note that

$$\langle \partial_i t^\alpha, \partial_{i'} t^\beta \rangle_{\kappa-1} = \alpha_i \beta_{i'} (\kappa-1)! (\alpha - e_i)! \delta_{\alpha-e_i,\beta-e_{i'}} \tag{50}$$

where e_i is the multi-index which is zero except in position i, where it equals 1 (and note that the right-hand side of (50) is to be interpreted as zero if $\alpha_i = 0$ or $\beta_{i'} = 0$).

To verify the first condition of (32), notice that when $\ell \neq \ell'$ and one of ℓ or ℓ' correspond to indices of a monomial-type polynomial, every inner product in the sum must be zero because $\partial_j p_\ell$ and $\partial_j p_{\ell'}$ have no monomials in common and are consequently orthogonal. If $\ell = \ell'$ and the polynomial p_ℓ is of monomial type, then

$$\sum_{j=1}^{d} \Big\langle \partial_j \frac{t^\alpha}{\sqrt{\alpha!}}, \partial_j \frac{t^\alpha}{\sqrt{\alpha!}} \Big\rangle_{\kappa-1} = \sum_{j=1}^{d} \frac{(\kappa-1)! \alpha_j^2 (\alpha - e_j)!}{\alpha!} \delta_{\alpha_j > 0} = \kappa!.$$

If, in the final case, both ℓ and ℓ' arise from UNTF terms, the left-hand side of the first equality of (32) must equal

$$\sum_{j=1}^{d} \frac{1}{\kappa!} \varphi_{j,\ell} \varphi_{j,\ell'} \langle \partial_j t_j^\kappa, \partial_j t_j^\kappa \rangle_{\kappa-1} = \sum_{j=1}^{d} \frac{1}{\kappa!} \varphi_{j,\ell} \varphi_{j,\ell'} (\kappa!)^2 = \kappa! \delta_{\ell,\ell'}$$

since the φ_j's are a normalized tight frame (NTF).

As for the second condition of (32), by (50), the polynomials p_ℓ of monomial type have norms that equal

$$\Big\langle \partial_j \frac{t^\alpha}{\sqrt{\alpha!}}, \partial_{j'} \frac{t^\alpha}{\sqrt{\alpha!}} \Big\rangle_{\kappa-1} = (\kappa-1)! \alpha! \delta_{j,j'} \delta_{\alpha_j > 0}.$$

Summing over all monomial-type polynomials gives a matrix (as a function of i and i') which is a multiple of the identity: simply by symmetry, any monomial appearing in the sum also appears with all its cyclic permutations, so all diagonal entries must be equal. As for the terms of the sum which arise from UNTF polynomials,

$$\Big\langle \partial_j \sum_{i=1}^{d} \frac{t_i^\kappa}{\sqrt{\kappa!}} \varphi_{i,k}, \partial_{j'} \sum_{i=1}^{d} \frac{t_i^\kappa}{\sqrt{\kappa!}} \varphi_{i,k} \Big\rangle_{\kappa-1} = \frac{|\varphi_{j,k}|^2}{\kappa!} \delta_{j,j'} \|t_j^\kappa\|_\kappa^2 = \kappa! \delta_{j,j'} |\varphi_{j,k}|^2,$$

which again sums to a multiple of the identity since, after the sum, the jth diagonal entry equals $\|\varphi_j\|^2$. □

Proof of part (3) of Theorem 2

By Lemma 8, to establish the existence of a well-curved d-dimensional submanifold of $\mathbb{R}^n$, it suffices to establish that there is a collection of the form (49) with $\#\mathcal{C} + d_0 = m$, where m is the relative codimension. Real UNTFs are guaranteed to exist for any $d_0 \leq d$ (see [11] for existence; a general algorithm based on Theorem 7 of [21] which can convert an NTF to a UNTF is also known; see [18]). Thus it suffices to establish the existence of a suitable collection $\mathscr{C}$ of monomials such that $m - \#\mathscr{C} \leq d$ (note that the case $d_0 = 0$ is explicitly allowed; omitting any UNTF terms in Lemma 8 still solves the critical point equations (32)).

Let S be the set of cardinalities $\#\mathscr{C}$ for any set of multi-indices of order κ which is closed under cyclic permutations and containing only monomials with at least two nonzero entries. The smallest element of S is of course zero, and the largest is $\dim \dot{P}^{\kappa}_{d} - d$, which corresponds to $\mathscr{C}$ being all monomials with at least two nonzero entries. The size of any gap (i.e., the difference between consecutive values in S) must be strictly less than d for the simple reason that no equivalence class of monomials modulo cyclic permutation has cardinality greater than d. In other words, if any non-pure power polynomials happen not already to belong to the collection $\mathscr{C}$, including any such monomial together with its cyclic permutations (which is a total of d or fewer new monomials) will again make a larger admissible set. Since the gaps in S are of size strictly less than d and since d_0 can be chosen as desired in $\{1, \dots, d\}$, for any d and n there must be a collection $\mathscr{C}$ and UNTF $\{\varphi_j\}_{j=1,\dots,d_0}$ to which Lemma 8 applies. Consequently, the associated f given by (2) must be well curved and have affine measure $\mu_{\mathcal{A}}$ which equals a nonzero constant times the pushforward of Lebesgue measure via the embedding f. $\square$

Appendix: Uniform bounds on the number of solutions of real analytic systems of equations

We finish with a brief discussion of the problem of uniformly bounding the number of nondegenerate solutions to any system of equations that arises in a geometric function system. The precise statement that is needed is the following.

LEMMA 9

For arbitrary positive integers d and n (no longer retaining their previous definitions), when $f_1, \dots, f_d$ are real analytic functions on a neighborhood of the unit cube $[0,1]^n$, then any system of equations

$$\big(\Phi_1(x), \dots, \Phi_n(x)\big) = (y_1, \dots, y_n)$$

must have bounded nondegenerate multiplicity when the functions Φ_i are rational functions of the f_i's and finitely many derivatives of each f_i. Here bounded nondegenerate multiplicity *is defined to mean that the number of solutions in $[0,1]^n$ at which the Jacobian determinant of the system is nonzero is bounded above by a constant that depends only on the functions f_i and the complexity of the system, that is, the degrees of the numerators and denominators and the order of the highest derivative of an f_i.*

Proof

To see why this lemma must be true, let S be the Cartesian product of $\{1, \dots, d\}$ with the set of multi-indices $\alpha := (\alpha_1, \dots, \alpha_n)$ such that $|\alpha| := \alpha_1 + \cdots + \alpha_n \leq N$. For any β which is a multi-index on S (i.e., a map from S into nonnegative integers), we define $s^\beta := \prod_{(j,\alpha)\in S} (s_{j,\alpha})^{\beta_{j,\alpha}}$ for every $s \in \mathbb{R}^S$ in analogy with the usual notation. Lastly, define P to be the Cartesian product of $\{1, \dots, n\}$ and multi-indices β of size at most N on the set S. We can then define a mapping F from $[0,1]^n \times \mathbb{R}^n \times \mathbb{R}^P \times \mathbb{R}^P \times \mathbb{R}^S$ into $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^P \times \mathbb{R}^P \times \mathbb{R}^S$ by means of the formula

$$\begin{aligned} F&(x,y,p,q,s) \\ &:= \Big(\Big(\sum_{(1,\beta)\in P} (p_{1,\beta} - y_1 q_{1,\beta}) s^\beta, \dots, \sum_{(n,\beta)\in P} (p_{n,\beta} - y_n q_{n,\beta}) s^\beta\Big), y, p, q, \\ &\qquad \big\{s_{j,\alpha} - \partial^\alpha f_j(x)\big\}_{(j,\alpha)\in S}\Big). \end{aligned}$$

For a given triple $(y_0, p_0, q_0) \in \mathbb{R}^n \times \mathbb{R}^P \times \mathbb{R}^P$ and any positive scalar C, nondegenerate solutions of the system

$$\frac{\sum_{|\beta|\leq N} (p_0)_{j,\beta} \prod_{(j',\alpha)\in S} (\partial^\alpha f_{j'}(x))^{\beta_{j',\alpha}}}{\sum_{|\beta|\leq N} (q_0)_{j,\beta} \prod_{(j',\alpha)\in S} (\partial^\alpha f_{j'}(x))^{\beta_{j',\alpha}}} = (y_0)_j, \quad j = 1, \dots, n, \tag{51}$$

will also be nondegenerate solutions of the system

$$F(x,y,p,q,s) = \Big(0, \frac{1}{C} y_0, \frac{1}{C^2} p_0, \frac{1}{C} q_0, 0\Big).$$

Choosing C so that the right-hand side always belongs to a fixed neighborhood of the origin with compact closure, we may use the fact that F is itself real-analytic in all parameters and so the number of connected components of the fiber $F^{-1}(0, y_0/C, p_0/C^2, q_0/C, 0)$ is bounded uniformly in y_0, p_0, and q_0 (which holds, in fact, for any analytic-geometric category in the sense of van den Dries and Miller [34]), which gives exactly the desired property that there is also a uniform bound on the number of isolated solutions of (51). If the functions f_j are all polynomial, Bézout's theorem gives a similar global bound on the number of nondegenerate solutions, that is, for all nondegenerate solutions $x \in \mathbb{R}^n$ rather than simply $[0,1]^n$. □

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References

[1] J.-G. BAK and S. HAM, *Restriction of the Fourier transform to some complex curves*, J. Math. Anal. Appl. **409** (2014), no. 2, 1107–1127. MR 3103222. DOI 10.1016/j.jmaa.2013.07.073. *(2076)*

[2] J.-G. BAK, D. M. OBERLIN, and A. SEEGER, *Restriction of Fourier transforms to curves: An endpoint estimate with affine arclength measure*, J. Reine Angew. Math. **682** (2013), 167–205. MR 3181503. *(2076)*

[3] D. BIRKES, *Orbits of linear algebraic groups*, Ann. of Math. (2) **93** (1971), 459–475. MR 0296077. DOI 10.2307/1970884. *(2094)*

[4] A. CARBERY and S. ZIESLER, *Restriction and decay for flat hypersurfaces*, Publ. Mat. **46** (2002), no. 2, 405–434. MR 1934361. DOI 10.5565/PUBLMAT_46202_04. *(2076)*

[5] Y. CHOI, *Convolution operators with the affine arclength measure on plane curves*, J. Korean Math. Soc. **36** (1999), no. 1, 193–207. MR 1669101. *(2076)*

[6] S. DENDRINOS, N. LAGHI, and J. WRIGHT, *Universal L^p improving for averages along polynomial curves in low dimensions*, J. Funct. Anal. **257** (2009), no. 5, 1355–1378. MR 2541272. DOI 10.1016/j.jfa.2009.05.011. *(2076)*

[7] S. DENDRINOS and B. STOVALL, *Uniform estimates for the X-ray transform restricted to polynomial curves*, J. Funct. Anal. **262** (2012), no. 12, 4986–5020. MR 2916059. DOI 10.1016/j.jfa.2012.03.020. *(2076)*

[8] S. DENDRINOS and J. WRIGHT, *Fourier restriction to polynomial curves, I: A geometric inequality*, Amer. J. Math. **132** (2010), no. 4, 1031–1076. MR 2663647. DOI 10.1353/ajm.0.0127. *(2076)*

[9] S. W. DRURY, *Degenerate curves and harmonic analysis*, Math. Proc. Cambridge Philos. Soc. **108** (1990), no. 1, 89–96. MR 1049762. DOI 10.1017/S0305004100068973. *(2076)*

[10] S. W. DRURY and B. P. MARSHALL, *Fourier restriction theorems for curves with affine and Euclidean arclengths*, Math. Proc. Cambridge Philos. Soc. **97** (1985), no. 1, 111–125. MR 0764500. DOI 10.1017/S0305004100062654. *(2076)*

[11] V. K. GOYAL, J. KOVAČEVIĆ, and J. A. KELNER, *Quantized frame expansions with erasures*, Appl. Comput. Harmon. Anal. **10** (2001), no. 3, 203–233. MR 1829801. DOI 10.1006/acha.2000.0340. *(2121)*

[12] C. C. GRAHAM, K. E. HARE, and D. L. RITTER, *The size of L^p-improving measures*, J. Funct. Anal. **84** (1989), no. 2, 472–495. MR 1001469. DOI 10.1016/0022-1236(89)90106-7. *(2076)*

[13] P. T. GRESSMAN, *L^p-improving properties of averages on polynomial curves and related integral estimates*, Math. Res. Lett. **16** (2009), no. 6, 971–989. MR 2576685. DOI 10.4310/MRL.2009.v16.n6.a5. *(2080, 2081, 2113, 2115)*

[14] ———, *On multilinear determinant functionals*, Proc. Amer. Math. Soc. **139** (2011), no. 7, 2473–2484. MR 2784813. DOI 10.1090/S0002-9939-2010-10656-1. *(2077)*

[15] ———, *Uniform sublevel Radon-like inequalities*, J. Geom. Anal. **23** (2013), no. 2, 611–652. MR 3023852. DOI 10.1007/s12220-011-9256-2. *(2076)*

[16] H. W. GUGGENHEIMER, *Differential Geometry*, corrected reprint of the 1963 edition Dover, New York, 1977. MR 0493768. *(2076)*

[17] D. HILBERT, *Ueber die vollen Invariantensysteme*, Math. Ann. **42** (1893), no. 3, 313–373. MR 1510781. DOI 10.1007/BF01444162. *(2093, 2094)*

[18] R. B. HOLMES and V. I. PAULSEN, *Optimal frames for erasures*, Linear Algebra Appl. **377** (2004), 31–51. MR 2021601. DOI 10.1016/j.laa.2003.07.012. *(2121)*

[19] I. A. IKROMOV and D. MÜLLER, *On adapted coordinate systems*, Trans. Amer. Math. Soc. **363** (2011), no. 6, 2821–2848. MR 2775788. DOI 10.1090/S0002-9947-2011-04951-2. *(2107)*

[20] F. JOHN, “Extremum problems with inequalities as subsidiary conditions” in *Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948*, Interscience, New York, 1948, 187–204. MR 0030135. *(2110)*

[21] R. V. KADISON, *The Pythagorean theorem, I: The finite case*, Proc. Natl. Acad. Sci. USA **99** (2002), no. 7, 4178–4184. MR 1895747. DOI 10.1073/pnas.032677199. *(2121)*

[22] G. KEMPF and L. NESS, “The length of vectors in representation spaces” in *Algebraic Geometry (Copenhagen, 1978)*, Lecture Notes in Math. **732**, Springer, Berlin, 1979, 233–243. MR 0555701. *(2080, 2082, 2104)*

[23] E. LUTWAK, *Extended affine surface area*, Adv. Math. **85** (1991), no. 1, 39–68. MR 1087796. DOI 10.1016/0001-8708(91)90049-D. *(2076)*

[24] D. M. OBERLIN, *Convolution with affine arclength measures in the plane*, Proc. Amer. Math. Soc. **127** (1999), no. 12, 3591–3592. MR 1690999. DOI 10.1090/S0002-9939-99-05462-3. *(2076)*

[25] ———, *Convolution with measures on hypersurfaces*, Math. Proc. Cambridge Philos. Soc. **129** (2000), no. 3, 517–526. MR 1780502. DOI 10.1017/S0305004100004552. *(2076, 2077, 2079)*

[26] ———, *Affine dimension: Measuring the vestiges of curvature*, Michigan Math. J. **51** (2003), no. 1, 13–26. MR 1960918. DOI 10.1307/mmj/1049832890. *(2077)*

[27] ———, *A uniform Fourier restriction theorem for surfaces in $\mathbb{R}^d$*, Proc. Amer. Math. Soc. **140** (2012), no. 1, 263–265. MR 2833538. DOI 10.1090/S0002-9939-2011-11218-8. *(2076)*

[28] R. W. RICHARDSON and P. J. SLODOWY, *Minimum vectors for real reductive algebraic groups*, J. Lond. Math. Soc. (2) **42** (1990), no. 3, 409–429. MR 1087217. DOI 10.1112/jlms/s2-42.3.409. *(2082, 2104, 2105)*

[29] P. SJÖLIN, *Fourier multipliers and estimates of the Fourier transform of measures carried by smooth curves in R^2*, Studia Math. **51** (1974), 169–182. MR 0385437. DOI 10.4064/sm-51-2-169-182. *(2076)*

[30] B. STOVALL, *Uniform L^p-improving for weighted averages on curves*, Anal. PDE **7** (2014), no. 5, 1109–1136. MR 3265961. DOI 10.2140/apde.2014.7.1109. *(2076)*

[31] ———, *Uniform estimates for Fourier restriction to polynomial curves in $\mathbb{R}^d$*, Amer. J. Math. **138** (2016), no. 2, 449–471. MR 3483472. DOI 10.1353/ajm.2016.0021. *(2076)*

[32] B. STURMFELS, *Algorithms in Invariant Theory*, 2nd ed., Texts Monogr. Symbol. Comput., Springer, Vienna, 2008. MR 2667486. *(2094, 2095, 2096, 2097)*

[33] T. TAO and J. WRIGHT, *L^p-improving bounds for averages along curves*, J. Amer. Math. Soc. **16** (2003), no. 3, 605–638. MR 1969206. DOI 10.1090/S0894-0347-03-00420-X. *(2081)*

[34] L. VAN DEN DRIES and C. MILLER, *Geometric categories and o-minimal structures*, Duke Math. J. **84** (1996), no. 2, 497–540. MR 1404337. DOI 10.1215/S0012-7094-96-08416-1. *(2123)*

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