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Asymptotically free mimetic gravity

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Abstract The idea of "asymptotically free" gravity is implemented using a constrained mimetic scalar field. The effective gravitational constant is assumed to vanish at some limiting curvature. As a result singularities in spatially flat Friedmann and Kasner universes are avoided. Instead, the solutions in both cases approach a de Sitter metric with limiting curvature. We show that quantum metric fluctuations vanish when this limiting curvature is approached.

1 Introduction

In [1] mimetic matter was introduced utilizing reparametrization of the physical metric $g_{\mu\nu}$ in terms of an auxiliary metric $h_{\mu\nu}$ and a scalar field ϕ in the form

$$g_{\mu\nu} = h_{\mu\nu} h^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} \tag{1}$$

This definition implies that ϕ identically satisfies

$$g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} = 1. \tag{2}$$

Because the physical metric is invariant under Weyl transformations of $h_{\mu\nu}$, the trace of the equations obtained by variation of the Einstein action with respect to the metric vanishes identically. In the absence of matter these equations become

$$G^{\mu}_{\nu} - G\phi^{,\mu}\phi_{,\nu} = 0, \tag{3}$$

where $G^{\mu}_{\nu} = R^{\mu}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu} R$ is the Einstein tensor, and they do not imply that $R = 0$ even in vacuum. Therefore, Eq. (3) taken together with (2) has additional solutions imitating dust-like cold dark matter. The scalar field ϕ satisfies a first order differential Eq. (2) and hence has half a degree of freedom which, when combined with the non-dynamical longitudinal mode of gravity, provides an extra degree of freedom in the form of mimetic "dust". Equivalently, the same theory is obtained by implementing Eq. (2) as a constraint added to the Einstein action:

$$S = \frac{1}{2} \int \mathrm{d}^4 x \sqrt{-g} \left(-\frac{1}{8\pi G} R + \lambda \left(g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - 1 \right) \right), \tag{4}$$

where λ is a Lagrange multiplier [2]. Unexpectedly, the concept of a mimetic field got a support in noncommutative geometry as a consequence of the volume quantization of compact three dimensional foliations of space time [3–5]. The mimetic field ϕ proved to be very robust. It could be used to modify Einstein Gravity in different possible ways. In particular, in [6] it was shown that adding appropriate potentials $V(\phi)$ to the action leads to many interesting cosmological solutions. Using instead gravity modification of the Born-Infeld type, where $\Box\phi$ is bounded by a limiting value, allowed to obtain bouncing solutions avoiding cosmological singularities [7] and to resolve black hole singularities [8]. Moreover, one can use the mimetic field to easily construct ghost free massive gravity with *non* Fierz-Pauli mass term [9,10].

In this paper we will explore the possibility of a running gravitational constant assuming that it depends on $\Box\phi$, that is, $G = G(\Box\phi)$. As we shall see, this quantity is the only measure of curvature G can depend on without introducing higher time derivatives in the modified Einstein equation. Assuming that G vanishes at some limiting curvature characterized by $(\Box\phi)^2_L$ we will implement in this way the idea of "asymptotic freedom" for gravity and investigate its possible consequences.

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2 Action and equations of motion

Let us consider the theory with action

$$S = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\,(-f\,(\Box\phi)\,R - 2\Lambda\,(\Box\phi) + \lambda\left(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - 1\right) + 2\mathcal{L}_m\right), \tag{5}$$

where

$$f\,(\Box\phi) = \frac{1}{8\pi G\,(\Box\phi)} \tag{6}$$

is the inverse running gravitational constant, $\mathcal{L}_m$ is the matter Lagrangian and for generality we also included a "cosmological-like term" $\Lambda\,(\Box\phi)$ [1]. Below we will use Planck units setting $8\pi G\,(\Box\phi = 0) = 8\pi G_0 = 1$. In these units $f\,(\Box\phi = 0) = 1$. Variation of the action with respect to the metric $g_{\mu\nu}$ gives

$$fG_{\mu\nu} + \left(\Box f - \Lambda + \tfrac{1}{2}\left(Z\,\phi^{;\alpha}\right)_{;\alpha}\right)g_{\mu\nu} - f_{;\mu\nu} - Z_{(,\mu}\phi_{,\nu)} = \lambda\phi_{,\mu}\phi_{,\nu} + T^{(\mathrm{m})}_{\mu\nu}, \tag{7}$$

where

$$Z := Rf' + 2\Lambda', \tag{8}$$

$T^{(\mathrm{m})}_{\mu\nu}$ is the energy momentum tensor for matter and the prime denotes derivative with respect to $\Box\phi$. The equation

$$\left(Z^{;\nu} + 2\lambda\phi^{;\nu}\right)_{;\nu} = 0 \tag{9}$$

follows from the variation of action (5) with respect to ϕ. Alternatively (9) can be obtained as a consequence of the Bianchi identities by taking the divergence of (7) and assuming that the energy momentum tensor $T^{(\mathrm{m})}_{\mu\nu}$ for ordinary matter is covariantly conserved. Taken together with the constraint

$$g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} = 1,$$

Equation (9) allows to determine the Lagrange multiplier λ.

3 The synchronous coordinate system

The assumption of global solvability of (2) is of course a restriction on admissible space-times. As shown in [18], the existence of a function whose gradient is everywhere time-like implies stable causality, i.e. there are no closed time-like curves also for small perturbations of the metric. Since the norm of the gradient of ϕ is not just positive but everywhere equal to unity, $t := \phi$ even qualifies to be used as the time coordinate of a synchronous coordinate system (see [17])

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \gamma_{ik}\mathrm{d}x^i\mathrm{d}x^k \tag{10}$$

where the above equations greatly simplify. In this coordinate system, the mimetic field ϕ defines the space-like hypersurfaces of constant time. The extrinsic curvature of these hypersurfaces,

$$\kappa_{ik} = \frac{1}{2}\frac{\partial}{\partial t}\gamma_{ik} \tag{11}$$

can be expressed as $\kappa_{ik} = -\phi_{;ik}$, while $\phi_{;0\alpha} = 0$. Thus,

$$\Box\phi = g^{\alpha\beta}\phi_{;\alpha\beta} = \gamma^{ik}\kappa_{ik} = \kappa = \frac{\partial}{\partial t}\ln\sqrt{\gamma}, \tag{12}$$

that is, in this coordinate system $\Box\phi$ is simply equal to the trace of the extrinsic curvature of the hypersurfaces of constant ϕ. In this paper, for the sake of simplicity, we will only consider a homogeneous metric with vanishing spatial curvature. In this case γ_{ik} depends only on time t and Eq. (9) simplifies to

$$\frac{1}{\sqrt{\gamma}}\partial_t\left[\sqrt{\gamma}\,(\partial_t Z + 2\lambda)\right] = 0, \tag{13}$$

and can be easily integrated to give

$$\lambda = -\frac{1}{2}\dot{Z} + \frac{C}{\sqrt{\gamma}} \tag{14}$$

where the dot denotes derivative with respect to time t and the constant of integration C describes the contribution of mimetic matter.

Substituting the expression (14) for λ in (7) and calculating the covariant derivatives of f and Z we find that the $0-0$ component of the equation becomes

$$fG_{00} + \left(\dot{\kappa} + \frac{1}{2}R\right)\kappa f' - \Lambda + \kappa\Lambda' = \varepsilon, \tag{15}$$

where

$$\varepsilon \equiv T_{00} + \frac{C}{\sqrt{\gamma}}, \tag{16}$$

is the total energy density of mimetic and ordinary matter. Assuming that the spatial components of the energy-momentum tensor satisfy $T^i_k \propto \delta^i_k$, subtracting from the spatial components of Eq. (7) one third of their trace gives

$$f\left(G^i_k - \frac{1}{3}G^m_m\delta^i_k\right) - \left(f^{;i}_{;k} - \frac{1}{3}f^{;m}_{;m}\delta^i_k\right) = 0. \tag{17}$$

[1] Please note that we have changed the notations used in [7,8] to more convenient ones.

For the spatially flat metric γ_{ik}

$$R_0^0 = -\dot{\kappa} - \kappa_k^i \kappa_i^k, \; R_k^i = -\frac{1}{\sqrt{\gamma}} \partial_0 \left(\sqrt{\gamma} \kappa_k^i \right), \tag{18}$$

where $\kappa_k^i = \gamma^{im} \kappa_{mk}$ (see, for example, [17]). Using these expression, Eqs. (15) and (17) become

$$\frac{1}{3} \left(f - 2\kappa f' \right) \kappa^2 - \Lambda + \kappa \Lambda' - \frac{1}{2} \left(f + \kappa f' \right) \tilde{\kappa}_k^i \tilde{\kappa}_i^k = \varepsilon \tag{19}$$

and

$$\partial_0 \left(f \sqrt{\gamma} \tilde{\kappa}_k^i \right) = 0, \tag{20}$$

correspondingly, where

$$\tilde{\kappa}_k^i = \kappa_k^i - \frac{1}{3} \kappa \delta_k^i, \tag{21}$$

is the traceless part of the extrinsic curvature.

The absence of higher time derivative terms in the modified Einstein equations can be understood by realizing that in the synchronous coordinate system

$$\begin{aligned} f R &= f \left(-2\dot{\kappa} - \kappa^2 - \kappa_k^i \kappa_i^k - {}^3R \right) \\ &= -2\dot{F} - f \left(\kappa^2 + \kappa_k^i \kappa_i^k + {}^3R \right) \end{aligned} \tag{22}$$

where f is assumed to be integrable with $f(\kappa) = F'(\kappa)$ and ${}^3R$ is the spatial curvature scalar. Hence the action contains, up to a total derivative only first order time derivatives of the metric.[2] This is a distinguishing feature of the $f(\Box\phi)$-theory which would not be present if $\Box\phi$ is replaced by any other non-constant, covariant expression containing first time derivatives of the metric like e.g. $\phi^{;\mu\nu}\phi_{;\mu\nu} = \kappa_k^i \kappa_i^k$.

Note that if we choose f and Λ to be symmetric functions, then the time reversal invariance of the Einstein equation is maintained. Hence the expanding counterparts for all the contracting solutions presented in the following can be found simply by reversing the arrow of time.

4 Asymptotic freedom and the fate of a collapsing universe

Equation (19) can be further simplified by making the choice

$$\Lambda = \frac{2}{3} \kappa^2 (f - 1) \tag{23}$$

such that it becomes

$$\left(f - \frac{2}{3} \right) \kappa^2 - \frac{1}{2} \left(f + \kappa f' \right) \tilde{\kappa}_k^i \tilde{\kappa}_i^k = \varepsilon. \tag{24}$$

In our units the inverse gravitational constant f is normalized to unity for $\kappa^2 = 0$. To guarantee that at low curvatures the corrections to General Relativity will be in the next order in curvature we have to assume that for $\kappa^2 \ll 1$, $f = 1 + \mathcal{O}\left(\kappa^2\right)$; in this case $\Lambda = \mathcal{O}\left(\kappa^4\right)$. In addition we assume that the gravitational constant $G(\kappa^2) \propto 1/f$ vanishes at some limiting curvature κ_0^2 (cf. [11–16]) and thus take the simplest possible function for f, namely

$$f = \frac{1}{1 - \left(\kappa^2 / \kappa_0^2 \right)}, \tag{25}$$

where κ_0^2 is a free parameter of the theory and can be taken well below the Planckian value.

Friedmann Universe First let us consider a flat contracting Friedmann universe with the metric

$$ds^2 = dt^2 - a^2(t) \delta_{ik} dx^i dx^k. \tag{26}$$

In this case

$$\kappa = 3 \frac{\dot{a}}{a} \tag{27}$$

and $\tilde{\kappa}_k^i$ vanishes. Therefore, Eq. (21) is satisfied identically and Eq. (24) can be rewritten as

$$\frac{1}{3} \kappa^2 \left(\frac{1 + 2\left(\kappa^2 / \kappa_0^2 \right)}{1 - \left(\kappa^2 / \kappa_0^2 \right)} \right) = \varepsilon. \tag{28}$$

Before writing the exact solution for Eq. (28), we first consider some of its asymptotic limits. For $\kappa^2/\kappa_0^2 \ll 1$ it reduces in the leading order to the usual Friedmann equation

$$\left(\frac{\dot{a}}{a} \right)^2 = \frac{1}{3} \varepsilon. \tag{29}$$

For a contracting universe dominated by matter with equation of state $p = w\varepsilon$ it has the solution

$$a \propto t^{\frac{2}{3(1+w)}}, \tag{30}$$

for large negative t. At the moment when the curvature approaches its limiting value, the gravitational constant begins to decrease and for $1 - \left(\kappa^2 / \kappa_0^2 \right) \ll 1$, Eq. (28) can be approximated by

$$\kappa^2 = \kappa_0^2 \left(1 - \frac{\kappa_0^2}{\varepsilon} + \cdots \right). \tag{31}$$

[2] This argument can, however, only serve as a heuristic explanation. Strictly speaking, it is not allowed to use $\Box\phi = \kappa$ and impose gauge conditions in the action before variation.

In a contracting universe the scale factor a decreases, while the energy density grows as $\varepsilon \propto a^{-3(1+w)}$. Hence, the solution of Eq. (31) approaches the contracting flat de Sitter universe with constant curvature where the scale factor decreases as

$$a \propto \exp\left(-\frac{\kappa_0 t}{3}\right) \tag{32}$$

for $\kappa_0 t \gg 1$. The gravitational constant $G \propto f^{-1}$ vanishes as $1/\varepsilon$ when $\varepsilon \to \infty$. The singularity is thus avoided as a result of the asymptotic freedom of gravity irrespective of the matter content of the universe.

For $\varepsilon \propto \gamma^{-\frac{1+w}{2}}$ the differential Eq. (28) can be integrated to obtain the exact implicit solution for $\kappa(t)$. In fact, differentiating the logarithm of Eq. (28) with respect to time and taking into account that $\partial \ln \gamma / \partial t = 2\kappa$, we obtain a first order differential equation which can be easily integrated to give

$$\frac{1+w}{2}\kappa_0 t = \frac{\kappa_0}{\kappa} - \operatorname{atanh}\frac{\kappa}{\kappa_0} - \sqrt{2}\arctan\left(\sqrt{2}\frac{\kappa}{\kappa_0}\right). \tag{33}$$

One can easily verify that the asymptotics (30) and (32) are smoothly connected in this solution. In particular, for large negative t the universe contracts according to (30). However, as it follows from (33), $\kappa(t=0) \simeq -0.6\kappa_0$ instead of blowing up as it would for solution (30) and for large positive t our solution approaches the de Sitter asymptotic (32).

In conclusion, the singularity is replaced by a smooth transition to a de Sitter metric. This qualitative behavior follows most naturally from our theory, independent of a specific choice of f and Λ. Note that the modified Friedmann equation is in general just a relation of the form $\kappa^2(\varepsilon)$. Demanding that this relation is smooth, one-to-one, bounded and has bounded slope, as it is necessary to ensure limiting curvature, the only remaining possibility is for κ to approach its constant limiting value as ε tends to infinity.

Kasner Universe We now consider a contracting anisotropic Kasner universe to find out what happens when the curvature approaches its limiting value for which the gravitational constant vanishes. To simplify the formulae we will set the energy density of matter to zero although all our conclusions survive also in the presence of the matter. For an anisotropic universe

$$\gamma_{ik} = \gamma_{(i)}(t)\,\delta_{ik} \tag{34}$$

and $\gamma = \gamma_{(1)}\gamma_{(2)}\gamma_{(3)}$. The traceless part of the extrinsic curvature in this case is nonvanishing and is determined by intergrating Eq. (21):

$$\tilde{\kappa}^i_k = \frac{\lambda^i_k}{f\sqrt{\gamma}}, \tag{35}$$

where λ^i_k are constants of integration satisfying $\lambda^i_i = 0$. Substituting this expression in Eq. (24) and using (25) we obtain

$$\frac{1}{3}\kappa^2\left(\frac{1+2\left(\kappa^2/\kappa_0^2\right)}{1-\left(\kappa^2/\kappa_0^2\right)}\right) = \frac{1}{2}\frac{\left(1+\left(\kappa^2/\kappa_0^2\right)\right)\bar{\lambda}^2}{\gamma}, \tag{36}$$

where $\bar{\lambda}^2 = \lambda^i_k\lambda^k_i$. Because $\kappa = \dot{\gamma}/2\gamma$, this equation allows us to determine how the determinant of the metric depends on time. Knowing $\gamma(t)$, the components of the metric can be found in the following way: without loss of generality we can diagonalize λ^i_k, so that, $\lambda^i_k = \lambda_{(i)}\delta^i_k$. Taking into account the definitions (11) and (21), Eq. (35) reduce to

$$\frac{\dot{\gamma}_{(i)}}{\gamma_{(i)}} - \frac{1}{3}\frac{\dot{\gamma}}{\gamma} = \frac{2\lambda_{(i)}}{f\sqrt{\gamma}}, \tag{37}$$

from which it follows that

$$\gamma_{(i)} = \gamma^{1/3}\exp\left(\int \frac{2\lambda_{(i)}}{f\sqrt{\gamma}}dt\right). \tag{38}$$

Before giving the exact solution of Eq. (36) it is more enlightening to study the asymptotic solutions. At low curvatures, that is, for $\kappa^2 \ll \kappa_0^2$, Eq. (36) simplifies to

$$\left(\frac{\dot{\gamma}}{\gamma}\right)^2 \simeq \frac{6\bar{\lambda}^2}{\gamma}, \tag{39}$$

and has the solution

$$\gamma = \frac{3}{2}\bar{\lambda}^2 t^2. \tag{40}$$

Taking into account that in this limit $f = 1$ and substituting this solution in (38) we find

$$\gamma_{(i)} = \left(\frac{3}{2}\bar{\lambda}^2\right)^{1/3} t^{2p_i}, \tag{41}$$

where

$$p_i = \frac{1}{3} \pm \sqrt{\frac{2}{3}}\frac{\lambda_{(i)}}{\bar{\lambda}}. \tag{42}$$

Since $\lambda_1 + \lambda_2 + \lambda_3 = 0$, the p_i satisfy the conditions

$$p_1 + p_2 + p_3 = 1, \quad p_1^2 + p_2^2 + p_3^2 = 1,$$

and at low curvatures we have either an expanding or a contracting Kasner universe [17].

In a contracting universe, at $|t| \simeq 1/\kappa_0$ the curvature becomes of the order of limiting curvature and for $1 - \left(\kappa^2/\kappa_0^2\right) \ll 1$, Eq. (36) is well approximated by

$$\kappa_0^2\left(\frac{1}{1-\left(\kappa^2/\kappa_0^2\right)}\right)=\frac{\bar{\lambda}^2}{\gamma}, \tag{43}$$

from which it follows that

$$\frac{\dot{\gamma}}{\gamma}=-2\kappa_0\left(1-\frac{\kappa_0^2\gamma}{\bar{\lambda}^2}\right)^{1/2} \tag{44}$$

in a contracting universe and for $\gamma \ll \bar{\lambda}^2/\kappa_0^2$ we have

$$\gamma \propto \exp\left(-2\kappa_0 t\right). \tag{45}$$

As follows from (44), in this limit

$$f=\frac{\bar{\lambda}^2}{\kappa_0^2\gamma} \tag{46}$$

and the integrals in (38) fast converge to some constants for $t \gg 1/\kappa_0$. These constants can be absorbed by redefinition of the spatial coordinates to give the asymptotic solution

$$\gamma_{(1)}=\gamma_{(2)}=\gamma_{(3)}=\gamma^{1/3}\propto \exp\left(-\frac{2}{3}\kappa_0 t\right), \tag{47}$$

that describes a contracting flat de Sitter universe with constant curvature.

The exact implicit solution of Eq. (36) for $\kappa\left(t\right)$ is given by

$$\kappa_0 t=\frac{\kappa_0}{\kappa}-\operatorname{atanh}\frac{\kappa}{\kappa_0}-\sqrt{2}\arctan\left(\sqrt{2}\frac{\kappa}{\kappa_0}\right)+\arctan\frac{\kappa}{\kappa_0}. \tag{48}$$

Note that in the anisotropic case we are forced to use asymptotic freedom if we want to obtain a non-singular modification where κ tends to its constant limiting value. Only in this way the anisotropy can disappear during contraction.

5 Quantum fluctuations

Now we look at what happens with quantum fluctuations of the gravitational field as we approach the limiting curvature where the gravitational constant vanishes. As it is well known (see, for example, [19]), in General Relativity the typical amplitude of quantum fluctuations of gravitational waves in Minkowski space and on curved background at scales l much smaller than the curvature scale is about

$$\delta h_l \simeq \frac{\sqrt{G}}{l}, \tag{49}$$

where G is the gravitational constant. Therefore, in our theory where this gravitational constant vanishes on the background with limiting curvature, one could expect that the quantum metric fluctuations must also vanish. We will now show that this is what really happens. Consider a slightly perturbed flat Friedmann Universe with metric

$$ds^2=a^2\left(\eta\right)\left(d\eta^2-\left(\delta_{ik}+h_{ik}\right)dx^i dx^k\right), \tag{50}$$

where we have introduced conformal time $\eta=\int\frac{dt}{a(t)}$ and h_{ik} is the traceless ($h_i^i=0$) and transverse ($h_{k,i}^i=0$) part of the metric perturbations. Substituting this metric in action (5) and expanding it to second order in h we obtain the following action for the gravitational waves:

$$S=\frac{1}{8}\int fa^2\left(h_k^{i\prime}h_i^{k\prime}-h_{k,m}^i h_i^{k,m}\right)d\eta d^3x, \tag{51}$$

where prime denotes the derivative with respect to conformal time η and the spatial indices are raised and lowered with δ_{ik}. This precisely coincides with the action for gravitational waves in a Friedmann universe with the "scale factor"

$$\tilde{a}:=a\sqrt{f}.$$

In this case the quantization procedure is well known and there is no need to repeat all the steps here. Referring to section 8.4 in [19] we find that the typical amplitude squared for the quantum fluctuations is

$$\delta h^2\left(k,\eta\right)\simeq\frac{|v_k|^2k^3}{\tilde{a}^2}=\frac{|v_k|^2k^3}{fa^2}, \tag{52}$$

where k is the co-moving wave number and the mode function v_k satisfies the equation

$$v_k''+\omega_k^2 v_k=0,\quad \omega_k^2\equiv k^2-\frac{\tilde{a}''}{\tilde{a}} \tag{53}$$

with initial conditions $v_k\left(\eta_{in}\right)=1/\sqrt{\omega_k}$, $v_k'\left(\eta_{in}\right)=i\sqrt{\omega_k}$ for quantum fluctuations. When the solution approaches the limiting curvature we have $f\propto\varepsilon\propto a^{-3(1+w)}$ and $\tilde{a}\propto a^{-\frac{1}{2}(1+3w)}$. Taking into account that in contracting de Sitter $a\left(\eta\right)=3/\kappa_0\eta$, where η grows, Eq. (53) becomes

$$v_k''+\left(k^2-\frac{\left(9w^2-1\right)}{4\eta^2}\right)v_k=0. \tag{54}$$

We can define quantum fluctuations only for short wave gravitational waves satisfying $k\eta\gg 1$, that is, for physical scales $l=a/k\ll\kappa_0^{-1}$. In this case $v_k\simeq\exp\left(ik\eta\right)/\sqrt{k}$ and, as follows from (52)

$$\delta h\left(l\right)\simeq\frac{1}{\sqrt{f}l}\simeq\frac{\sqrt{G\left(\kappa\right)}}{l}. \tag{55}$$

Hence, quantum fluctuations in a given physical scale $l\ll\kappa_0^{-1}$ vanish as $\kappa\to\kappa_0$ and correspondingly $G\left(\kappa\right)\to 0$.

This is in complete agreement with our expectations. The perturbations with $k\eta \ll 1$, which were outside the horizon κ_0^{-1} finally come inside because η grows in a contracting de Sitter space-time. The amplitude of metric perturbations h is constant before horizon crossing, but after entering the horizon it decays as $\tilde{a}^{-1} \propto a^{\frac{1}{2}(1+3w)}$. Thus we have shown that the de Sitter space-time with limiting curvature is completely classical, with no quantum metric fluctuations present.

6 Conclusions

The simple observation that the conformal part of the metric in General Relativity can be extracted covariantly via a constrained scalar field ϕ has proven to be very fruitful. The resulting modified gravity theory does not induce any additional degrees of freedom for the graviton, but at the same time makes the longitudinal mode dynamical even in the absence of matter. This mode can serve as a viable candidate for dark matter in our Universe. Moreover the constrained scalar field allows us to build invariants which in synchronous coordinates can be expressed exclusively in terms of first order time derivatives of the metric. This opens the possibility to modify General Relativity in a simple way avoiding problematic higher order time derivative terms which generically lead to ghost degrees of freedom. Such a generalization of Einstein theory happens to be very interesting and allows us for example to implement the idea of limiting curvature and resolve spacelike singularities in Friedmann and Kasner universes as well as in black holes. The limiting curvature, which is a parameter of the theory, can be taken well below the Planckian curvature. Potentially, this would make the difficult unresolved problem of non-perturbative quantum gravity obsolete for all practical purposes.

In this paper we have investigated the possibility of implementing the idea of classical asymptotic freedom just assuming that the gravitational constant vanishes at the limiting curvature. As it was shown, in this case the singularities in flat contracting Friedmann and Kasner universes are resolved and close to the limiting curvature the de Sitter solution is approached. Moreover, quantum metric fluctuations asymptotically vanish and the spacetime becomes fully classical at this limiting curvature. This opens an interesting possibility to resolve the longstanding singularity problem in General Relativity via a simple modification of Einstein theory at large curvatures without referring this problem to a yet unknown non-perturbative theory of quantum gravity.

For the sake of simplicity and to highlight the most important aspects first, in this paper we focused mainly on the homogeneous, spatially flat sector of the theory proposed above. In another soon to appear paper we will extend our analysis and consider applications to spatially non-flat space-times, including Black Holes.

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