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Stabilization of Switched Systems on Non-Uniform Time Domain with Dwell Time

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Abstract— **In this paper, we will present new stability conditions for a special class of linear switched systems, that evolves on non-uniform time domain. The considered systems switch between continuous-time subsystems on intervals with variable lengths, and discrete-time subsystems with variable step sizes. Time scales theory is introduced to derive conditions for exponential stability of this special class of switched systems by using the dwell time approach. The conditions are based on the existence of a multiple Lyapunov function. This shows that this class of switched systems can be stabilized if the dwell time of each continuous-time subsystem is greater than some bound, and if the gap of the discrete-time subsystem is bounded by some specific values. Numerical examples are presented to show the effectiveness of the proposed scheme.**

I. INTRODUCTION

Switched systems are systems composed of a finite number of continuous or discrete-time subsystems and the corresponding switching signal orchestrating the switching between them. In the past decade, switched systems have drawn considerable attention and they have been widely studied, because they describe a wide range of engineering systems and control fields [1], [9]. The stability under arbitrary switching and the stability under constrained switching have been addressed [1], [6], [7]. The existence of a common Lyapunov function (CLF) for all subsystems guarantees the stability of the switched system under arbitrary switching. In this case, it is necessary to require that all subsystems are asymptotically stable [1]. Noting that, finding a CLF is not an easy task, except for certain special cases [8], [12]. To seek less conservative results, the multiple Lyapunov functions (MLF) approach was introduced to analyze the stability of switched systems under constrained switching [1], [3], [11]. It was shown that switched systems, without exception, are prone to instability problems and an arbitrary fast switching may cause large state transients at the switching points. Much effort has centralized on time-controlled switching [1], [11], where it is shown that a dwell time may then be required for these transients. The dwell time approach is demonstrated to be a successful and an effective technique to analyze switched system stability and controller design. The idea behind it is that the time interval between any two consecutive switching is not smaller (or in average) than $\tau_d \in \mathbb{R}^+$ [5], [10].

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Most of the existing dwell time methods to analyze the stability of switched systems can only be applied to systems operating in the continuous-time [1], [10], [14], or uniform discrete-time domains [13], [2]. In contrast, in engineering, there are many switched systems that evolve on a non-uniform time domain, such subsystems can be discrete with non-uniform sampling time, or a combination of discrete and continuous time domains. A set of discrete-time controllers and switching among the controllers is one example. Impulsive systems in which non-instantaneous jumps occur at some time instances is a second example. In these cases, the time domain is neither continuous ($\mathbb{R}$) nor uniformly discrete ($h\mathbb{Z}$). To overcome this difficulty and extend existing results for switched systems evolving on a non-uniform time domain, time scales theory has been introduced and shown promising results. This theory unifies and demonstrates the interplay between continuous-time and discrete-time dynamics [15]. The concept of exponential stability of dynamical systems on time scale $\mathbb{T}$, has been derived in [16], [17]. Such results have been recently generalized including switched systems on arbitrary time scales [18], [19].

The stability of a special class of systems that switches between continuous-time dynamics that evolve on intervals with variable lengths, and discrete-time dynamics with variable discrete step-size has been studied in [20], [21], [22], [23]. The stability conditions are derived using the generalized exponential function on time scale. The motivation to study such systems is that, there are many applications involving such switched systems, such impulsive systems with non-instantaneous jumps. In [22], the problem of consensus for multi-agent systems with intermittent information transmission has been converted to an asymptotic stabilization problem of this class of switched systems, and stability conditions have been derived using the general solution of the switched systems. In [21], the asymptotic stability of this class of switched system has been studied by designing a CLF. However, in these works, the time scale $\mathbb{T}$ is supposed to be given in advance, and pairwise commutative matrices are assumed. In this paper, we will consider this special class of continuous/discrete switched systems. The aim of this work is to derive a new dwell time conditions (i.e design the time scale $\mathbb{T}$) to establish a stabilizing switching law for the considered switched system. The results are derived by introducing MLF and by considering that unstable subsystems may exist. A numerical example is presented to show the effectiveness of the proposed method.

II. Preliminaries on time scale theory

We present here, for convenience, a few preliminaries regarding time scales calculus.

The time scale is a closed nonempty subset of real numbers denoted by $\mathbb{T}$. For the calculus of time scales we refer the readers to [15]. The *forward jump operator* is given by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$. The *backward jump* operator is defined by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$. The mapping $\mu : \mathbb{T} \to \mathbb{R}^+$, called the *graininess function*, is defined by $\mu(t) = \sigma(t) - t$. For $\mathbb{T} = \mathbb{R}$, we have $\rho(t) = t = \sigma(t)$, and $\mu(t) = 0$, for all t, while for $\mathbb{T} = h\mathbb{Z}$, we have $\rho(t) = t - h$, $\sigma(t) = t + h$, and $\mu(t) = h$. A point $t \in \mathbb{T}$ is called *right-scattered* (resp. *left-scattered*) if $\sigma(t) > t$ (resp. $\rho(t) < t$) and *right-dense* (resp. *left-dense*), if $\sigma(t) = t$ (resp. $\rho(t) = t$).

If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$; otherwise $\mathbb{T}^\kappa = \mathbb{T}$. For $f : \mathbb{T} \to \mathbb{R}$, $t \in \mathbb{T}^\kappa$, the Δ-derivative of $f(t)$ is defined by

$$f^\Delta(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}.$$

The Δ-derivative unifies the derivative in the continuous sense, and the difference operator in the discrete sense. A function $f : \mathbb{T} \to \mathbb{R}$ is said right-dense continuous (*rd-continuous*), if it is continuous at every right-dense points in $\mathbb{T}$ and its left-hand limit exists at every left-dense points in $\mathbb{T}$.

A function $p : \mathbb{T} \to \mathbb{R}$ is *regressive* (resp. *positively regressive*), if $1+\mu(t)p(t) \neq 0, \forall t \in \mathbb{T}^\kappa$ (resp. $1+\mu(t)p(t) > 0, \ \forall t \in \mathbb{T}^\kappa$). We denote the set of regressive (resp. positively regressive) and rd-continuous functions by $\mathcal{R}$ (resp, $\mathcal{R}^+$). Similarly, a matrix function $A : \mathbb{T} \to \mathbb{R}^{n\times n}$ is called *regressive*, if and only if all its eigenvalues, $\lambda_j(t)$, are regressive (i.e; $1 + \mu(t)\lambda_i(t) \neq 0, \ \forall 1 \leq j \leq n, \ \forall t \in \mathbb{T}^\kappa$).

Let the transformation $\xi_\mu(z)$, with $\mu > 0$, be defined on the set $\{z \in \mathbb{C} : z \neq \frac{-1}{\mu}\}$ by $\xi_\mu(z) := \frac{\log(1+\mu z)}{\mu}$, and $\xi_0(z) := z$.
The *generalized exponential function* of $p \in \mathcal{R}$, on a time scale $\mathbb{T}$, is expressed by

$$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right), \quad s,t \in \mathbb{T}.$$

For $\mathbb{T} = \mathbb{R}$ and p constant, $e_p(t,t_0) = e^{p(t-t_0)}$, and for $\mathbb{T} = h\mathbb{Z}$, $e_p(t,s) = \prod_{\tau=s}^{t-h}(1 + hp(\tau))$.

Let a regressive constant matrix $A \in \mathbb{R}^{n\times n}$. The unique solution of the first order dynamical system

$$x^\Delta(t) = Ax(t), \quad x(t_0) = x_0, \quad t, t_0 \in \mathbb{T}, \tag{1}$$

is expressed by the generalized exponential function, such that $x(t) = e_A(t,t_0)x_0$ (see [15]). Note that, the regressivity of A is needed in order to $e_A(t,t_0)$ to be well defined. For the continuous case $\mathbb{T} = \mathbb{R}$, the matrix A is always regressive.

System (1) is exponentially stable on $\mathbb{T}$, if there exists a constant $\beta \geq 1$ and a negative constant $\lambda \in \mathcal{R}^+$, such that the corresponding solution satisfies

$$\|x(t)\| \leq \beta\|x_0\|e_\lambda(t,t_0), \quad \forall t, t_0 \in \mathbb{T}.$$

This characterization is a generalization of the definition of exponential stability for dynamical systems defined in $\mathbb{R}$ or $h\mathbb{Z}$. More specifically, the condition that $\lambda < 0$ and $\lambda \in \mathcal{R}^+$ is reduced to $\lambda < 0$ for $\mathbb{T} = \mathbb{R}$, to $0 < 1 + h\lambda < 1$ for $\mathbb{T} = h\mathbb{Z}$, and to $0 < 1 + \mu(t)\lambda < 1, \ \forall t \in \mathbb{T}$ for any discrete time scale $\mathbb{T}$. To study the stability of linear dynamical systems on a time scale $\mathbb{T}$, a particular open set of the complex plane called the *Hilger circle* is defined for all $t \in \mathbb{T}$ as

$$\mathcal{H}_{\mu(t)} := \left\{z \in \mathbb{C} : \ |1 + z\mu(t)| < 1, \ z \neq -\frac{1}{\mu(t)}\right\},$$

with $\mathcal{H}_0 = \mathbb{C}^-$. For $\mathbb{T} = h\mathbb{Z}$, the Hilger circle is determined by the disc of center $(\frac{-1}{h}, 0)$ and radius $\frac{1}{h}$. The smallest Hilger circle (denoted $\mathcal{H}_{\min}$) is the Hilger circle associated with $\mu_{\max} = \sup_{t\in\mathbb{T}} \mu(t)$. A regressive time-invariant matrix A is called Hilger stable if spec$(A) \subset \mathcal{H}_{\min}$ (i.e all the eigenvalues of A are in $\mathcal{H}_{\min}$) [24].

The Lyapunov stability on time scales was studied in several works [19], [21]. Let $\mathbb{T}$ be an arbitrary time scale and $P = P^T > 0$ a positive definite matrix. The Δ-derivative of the Lyapunov function $V = x^TPx$ along the trajectories of system (1) on $\mathbb{T}$ is given by $A^TP + PA + \mu(t)A^TPA = -Q$, where $Q = Q^T > 0$. In [19], the authors shows that, if A has all its eigenvalues in the corresponding Hilger circle for every $t \geq t_0$, then for each $t \in \mathbb{T}$, the matrix P is determined by

$$P = \int_{t_0}^t e_{A^T}(s,t_0)Qe_A(s,t_0)\Delta s,$$

and the existence of such Lyapunov function implies the exponential stability of the dynamical system (1), which is a generalization of the Lyapunov criteria for exponential stability of discrete and continuous linear systems.

III. Problem statement

It is known that the trajectories of a continuous-time or discrete-time switched system can be unstable under arbitrary switching, even if all the subsystems are stable. One method to stabilize such switched system is by using the dwell time switching approach, where some constraints are imposed on the duration of each subsystem (slow switching). Note that, in the literature, the dwell time conditions are derived for continuous-time and discrete-time switched systems separately. In this paper, we will derive stabilizing dwell time conditions, using MLF, for a special class of switched systems evolving on a particular time scale,

$$\mathbb{T} = \mathbb{P}_{\{\sigma(t_k),t_{k+1}\}} = \cup_{k=0}^\infty[\sigma(t_k), t_{k+1}],$$

which is the union of disjoint closed intervals with variable lengths and variable gaps, where $\sigma(.)$ is the forward jump operator, such that, $\sigma(t_0) = t_0$, $t_k < \sigma(t_k) < t_{k+1}, \ \forall k \in \mathbb{N}^*$ and the graininess function $\mu(t_k) = \sigma(t_k) - t_k, \forall k \in \mathbb{N}^*$ (Fig. 1). This time scale arises for example in the consensus of multi-agent systems with intermittent information transmission [22], or impulsive systems with non-instantaneous jump. It is assumed throughout the paper that $\mathbb{T}$ is unbounded above and $\mu(t)$ is bounded. Let $\{A_c, A_d\}$ be a set of two constant

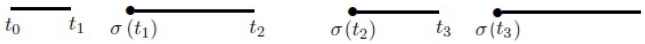

Fig. 1. Time scale $\mathbb{T} = \mathbb{P}_{\{\sigma(t_k), t_{k+1}\}}$.

regressive matrices in $\mathbb{R}^{n\times n}$. The eigenvalues of A_c (resp. A_d) are denoted by $\lambda_c^j \in spec(A_c)$ (resp. $\lambda_d^j \in spec(A_d)$), $1 \leq j \leq n$. The considered switched linear system on $\mathbb{T} = \mathbb{P}_{\{\sigma(t_k), t_{k+1}\}}$, is given by

$$x^{\Delta}(t) = \begin{cases} A_c x(t), & \text{for} \quad t \in \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}[\\ \\ A_d x(t), & \text{for} \quad t \in \cup_{k=0}^{\infty}\{t_{k+1}\}. \end{cases} \tag{2}$$

The first equation in (2) describes the continuous-time dynamics of the system. At times t_{k+1}, $k \in \mathbb{N}$, the state $x(t_{k+1})$ will jump non instantaneously to the state $x(\sigma(t_{k+1}))$. The duration of this jump is described by $\mu(t_k)$ which is considered to be variable in time. At the instants $t = t_{k+1}$, $k \in \mathbb{N}$, we determine a discrete dynamic of the state by using the Δ-derivative as follows,

$$x^{\Delta}(t_{k+1}) = \frac{x(\sigma(t_{k+1})) - x(t_{k+1})}{\mu(t_{k+1})} = A_d x(t_{k+1}), \quad \forall k \in \mathbb{N}.$$

So, the second equation of (2) can be seen as the non-instantaneous state jump dynamic.

Hilger stability is related to the Hilger circle. If all the eigenvalues of A_c lies in $\mathcal{H}_0 = \mathbb{C}^-$, the continuous-time subsystem is Hilger stable (exponentially stable). The discrete-time subsystem is Hilger stable, if all the eigenvalues of A_d lies strictly within the Hilger circle $\mathcal{H}_{\min}$, which means that,

$$|1 + \mu(t)\lambda_d^j| < 1, \ \forall 1 \leq j \leq n, \ \forall t \in \cup_{k=0}^{\infty}\{t_{k+1}\}. \tag{3}$$

Condition (3) implies that,

$$0 < \mu(t) < \gamma_d, \quad \text{where} \quad \gamma_d = \min_{1\leq j \leq n}\left\{\frac{-2\Re(\lambda_d^j)}{|\lambda_d^j|^2}\right\}. \tag{4}$$

On the other hand, A_d is unstable if there exists at least one eigenvalue λ_d^j of A_d such that $|1 + \mu(t)\lambda_d^j| > 1, \ \forall t \in \cup_{k=0}^{\infty}\{t_{k+1}\}$. In all the following, A_d is assumed to be regressive (i.e $\mu(t) \neq \frac{-1}{\lambda_d^j}, \ \forall 1 \leq j \leq n, \ \forall t \in \cup_{k=0}^{\infty}\{t_{k+1}\}$). Note that, the stability and instability of A_d do not depend only on λ_d^j, but also on $\mu(t_k)$, which makes the novelty in studying such switched systems and leads us to choose carefully the dwell time of the discrete-time subsystems.

IV. Dwell time conditions using MLF

We will study in this section the exponential stability of the switched system (2), by considering the cases where A_c and A_d can be Hilger stable or unstable. The objective is to determine dwell time conditions in order to guarantee the exponential stability of (2). We will introduce the framework of MLF, which may correspond to each single subsystem, and is non-increasing at the switching instants. This non-traditional Lyapunov function may not be monotonically decreasing along the state trajectories.

A. Formulation of the problem

Consider the switched system (2). Let us first present some necessary background. Note that if the matrix A_c is exponentially stable, so there exists a constant $\lambda_c > 0$, such that $(A_c + \lambda_c I)$ remains exponentially stable. On the other hand, if A_c is unstable, there exists a positive constant $\lambda_c > 0$, such that $(A_c - \lambda_c I)$ is exponentially stable. If A_d is Hilger stable with respect to time scale $\mathbb{T}$ (i.e; condition (3) is satisfied), there exists a constant λ_d with $|\lambda_d| < 1$, such that

$$A_s(t) = \left[\frac{1}{\lambda_d}A_d + \left(\frac{1-\lambda_d}{\mu(t)\lambda_d}\right)I\right], \tag{5}$$

is Hilger stable, $\forall t \in \cup_{k=0}^{\infty}\{t_{k+1}\}$. This is shown as:

$$\begin{aligned} I + \mu(t)A_s &= I + \mu(t)\left[\frac{1}{\lambda_d}A_d + \left(\frac{1-\lambda_d}{\mu(t)\lambda_d}\right)I\right] \\ &= \frac{1}{\lambda_d}(I + \mu(t)A_d). \end{aligned}$$

Let λ_s^j be the eigenvalues of A_s, $1 \leq j \leq n$, so they satisfies,

$$|1 + \mu(t)\lambda_s^j| = \frac{1}{|\lambda_d|}|1 + \mu(t)\lambda_d^j|, \ \forall t \in \cup_{k=0}^{\infty}\{t_{k+1}\}. \tag{6}$$

Since, A_d is supposed to be Hilger stable, and from (3), we can always choose $|\lambda_d| < 1$, such that

$$|1 + \mu(t)\lambda_s^j| < 1, \quad \forall t \in \cup_{k=0}^{\infty}\{t_{k+1}\}, \quad \forall 1 \leq j \leq n,$$

which means that A_s is Hilger stable with respect to $\mathbb{T}$. In the same way, if A_d is unstable (i.e, $\exists \lambda_d^j \in spec(A_d)$), such that $|1 + \mu(t)\lambda_d^j| > 1, \forall t \in \cup_{k=0}^{\infty}\{t_{k+1}\}$), so there exists a constant λ_d with $|\lambda_d| > 1$, such that $A_s = \left[\frac{1}{\lambda_d}A_d + \left(\frac{1-\lambda_d}{\mu(t)\lambda_d}\right)I\right]$ is Hilger stable. This means that, and similar to (6),

$$|1 + \mu(t)\lambda_s^j| = \frac{1}{|\lambda_d|}|1 + \mu(t)\lambda_d^j| < 1, \quad \forall t \in \cup_{k=0}^{\infty}\{t_{k+1}\}. \tag{7}$$

Let A_c and A_d be Hilger stable. By considering the Hilger stable matrices $(A_c + \lambda_c I)$ (with $\lambda_c > 0$) and A_s, so there exists a Lyapunov quadratic functions

$$V_i(x(t)) = x^T(t)P_i x(t) \quad \text{with} \quad P_i = P_i^T > 0, \ i \in \{c, d\},$$

such that, $\forall t \in \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}[$,

$$V_c^{\Delta}(x(t)) = x^T[(A_c + \lambda_c I)^T P_c + P_c(A_c + \lambda_c I)]x < 0, \tag{8}$$

and $\forall t \in \cup_{k=0}^{\infty}\{t_{k+1}\}$ (see [21]),

$$V_d^{\Delta}(x(t)) = x^T(A_s^T P_d + P_d A_s + \mu(t)A_s^T P_d A_s)x < 0. \tag{9}$$

Substituting A_s by its value in (9), we get the following inequality, $\forall t \in \cup_{k=0}^{\infty}\{t_{k+1}\}$

$$(A_d^T P_d + P_d A_d + \mu(t)A_d^T P_d A_d) < \left(\frac{\lambda_d^2 - 1}{\mu(t)}\right)P_d, \tag{10}$$

with $|\lambda_d| < 1$. The Δ-derivative of $V_c(x(t))$ along the trajectories of the continuous-time subsystem of (2) is given by $V_c^{\Delta}(x(t)) = x^T(t)(A_c^T P_c + P_c A_c)x(t)$. From (8), we get

$$V_c^{\Delta}(x(t)) < -2\lambda_c x^T(t)P_c x(t) = -2\lambda_c V_c(x(t)). \tag{11}$$

The Δ-derivative of $V_d(x(t))$ along the trajectories of the discrete-time subsystem of (2) is given by

$$V_d^{\Delta}(x(t)) = x^T(t)(A_d^T P + PA_d + \mu(t)A_d^T PA_d)x(t).$$

From (10), we get

$$V_d^{\Delta}(x(t)) < \left(\frac{\lambda_d^2 - 1}{\mu(t)}\right) V_d(x(t)), \quad \forall t \in \cup_{k=0}^{\infty}\{t_{k+1}\}. \tag{12}$$

Let the following MLF of (2),

$$V_i(x(t)) = x^T(t)P_i x(t), \quad i \in \{c, d\}. \tag{13}$$

From (11) and (12), we get

$$V_i^{\Delta}(x(t)) \leq \begin{cases} -2\lambda_c V_c(x(t)), & t \in \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}[, \\ \left(\frac{\lambda_d^2 - 1}{\mu(t)}\right) V_d(x(t)), & t \in \cup_{k=0}^{\infty}\{t_{k+1}\}. \end{cases}$$

Using differential inequalities on time scale [15], we obtain, for $t \in [\sigma(t_k), t_{k+1}[, \forall k \in \mathbb{N}$,

$$V_c(x(t)) \leq e^{-2\lambda_c(t-\sigma(t_k))} V_c(x(\sigma(t_k))), \tag{14}$$

and for $t = t_{k+1}, \forall k \in \mathbb{N}$

$$V_d(x(\sigma(t))) \leq \left[1 + \mu(t)\left(\frac{\lambda_d^2 - 1}{\mu(t)}\right)\right] V_d(x(t)) = \lambda_d^2\, V_d(x(t)). \tag{15}$$

Notice that, the following inequalities are always satisfied [4],

$$\lambda_{\min}(P_i)\|x\|^2 \leq V_i(x) \leq \lambda_{\max}(P_i)\|x\|^2, \ i \in \{c, d\} \tag{16}$$

which implies that, for all $k \in \mathbb{N}$,

$$V_d(x(t_k)) \leq \frac{\lambda_{\max}(P_d)}{\lambda_{\min}(P_c)} V_c(x(t_k)) = \delta V_c(x(t_k)),$$

and

$$V_c(x(\sigma(t_k))) \leq \frac{\lambda_{\max}(P_c)}{\lambda_{\min}(P_d)} V_d(x(\sigma(t_k))) = \delta' V_d(x(\sigma(t_k))).$$

Let $\beta = \max\{\delta, \delta'\} \geq 1$, we get, $\forall t \in \mathbb{T}$

$$V_i(x(t)) \leq \beta V_j(x(t)), \ i \neq j, \ i, j \in \{c, d\}. \tag{17}$$

B. Main results

Let us now, derive the stabilizing dwell time conditions for system (2). From (14), (15) and (17), we can derive an upper bound of the solution of the switched system (2) as follows:

For $t_0 \leq t \leq t_1$,

$$V_c(x(t) \leq e^{-2\lambda_c(t-t_0)} V_c(x(\sigma(t_0))) = e^{-2\lambda_c(t-t_0)} V_c(x(t_0)).$$

For $t = t_1$, $\quad V_c(x(t_1)) \leq e^{-2\lambda_c(t_1-t_0)} V_c(x(t_0))$.

For $t = \sigma(t_1)$,

$$\begin{aligned} V_d(x(\sigma(t_1))) &\leq (\lambda_d^2) V_d(x(t_1)) \\ &\leq \beta\ (\lambda_d^2) V_c(x(t_1)) \\ &\leq \beta\ (\lambda_d^2)\ e^{-2\lambda_c(t_1-t_0)} V_c(x(t_0)). \end{aligned}$$

For $\sigma(t_1) \leq t \leq t_2$,

$$\begin{aligned} V_c(x(t)) &\leq e^{-2\lambda_c(t-\sigma(t_1))} V_c(x(\sigma(t_1))) \\ &\leq \beta\ e^{-2\lambda_c(t-\sigma(t_1))} V_d(x(\sigma(t_1))) \\ &\leq \beta\ e^{-2\lambda_c(t-\sigma(t_1))}\ \beta\ (\lambda_d^2)\ e^{-2\lambda_c(t_1-t_0)} V_c(x(t_0)). \end{aligned}$$

By induction, for $\sigma(t_k) \leq t \leq t_{k+1}$ and $\tau_k := t_{k+1} - \sigma(t_k), \forall k \in \mathbb{N}$, we have

$$V_c(x(t)) \leq e^{-2\lambda_c(t-\sigma(t_k))} \prod_{i=0}^{k-1} e^{(\log(\beta) - 2\lambda_c\tau_i)}\ (\beta\lambda_d^2)^k\ V_c(x(t_0)). \tag{18}$$

From (16), we derive

$$\begin{aligned} \|x(t)\| &\leq \xi\ e^{-\lambda_c(t-\sigma(t_k))} \textstyle\prod_{i=0}^{k-1} e^{(\log(\sqrt{\beta}) - \lambda_c\tau_i)}\ (\sqrt{\beta}|\lambda_d|)^k \\ &\quad \times \|x_0\|, \end{aligned} \tag{19}$$

with $\xi = \sqrt{\dfrac{\lambda_{\max}(P_c)}{\lambda_{\min}(P_c)}}$. One can get the following upper bound

$$\|x(t_{k+1})\| \leq \xi \prod_{i=0}^{k} e^{(\log(\sqrt{\beta}) - \lambda_c\tau_i)}\ (\sqrt{\beta}|\lambda_d|)^k\ \|x_0\|. \tag{20}$$

Consider the switched linear system (2) such that A_c and A_d are Hilger stable. Let $\lambda_c > 0$, $|\lambda_d| < 1$, and $\beta \geq 1$ be defined as in (8), (5) and (17) respectively. Suppose that the dwell times of each continuous-time subsystem τ_k and of each discrete-time subsystem $\mu(t_k)$, satisfy one of the following conditions:

(i) For all $k \in \mathbb{N}$, and $\forall 1 \leq j \leq n$,

$$|1 + \mu(t_k)\lambda_d^j| < |\lambda_d| \ \text{ and } \ |\lambda_d| < \frac{1}{\sqrt{\beta}} < 1, \tag{21}$$

$$\tau_k > \frac{\log(\sqrt{\beta})}{\lambda_c}, \ \forall k \in \mathbb{N}. \tag{22}$$

(ii) For all $k \in \mathbb{N}$, and $\forall 1 \leq j \leq n$,

$$\tau_k > \frac{\log(\beta)}{\lambda_c} \text{ and } \ |1 + \mu(t_k)\lambda_d^j| < |\lambda_d| < 1. \tag{23}$$

Then, the switched system (2) is exponentially stable.

Proof:

(i) Let the upper bound of the solution of system (2) given by (20). Since $\lambda_c > 0$, we have

$$\begin{aligned} &\|x(t_{k+1})\| \\ &\leq \xi\ e^{\sum_{i=0}^{k}(\log(\sqrt{\beta}) - \lambda_c\tau_i)}\ e^{k\log(\sqrt{\beta}|\lambda_d|)} \|x_0\| \\ &\leq \xi\ e^{\sum_{i=0}^{k}(\log(\sqrt{\beta}) - \lambda_c\tau_{\min})}\ e^{k\log(\sqrt{\beta}|\lambda_d|)} \|x_0\| \\ &= \xi e^{k[\log(\sqrt{\beta}) - \lambda_c\tau_{\min} + \log(\sqrt{\beta}|\lambda_d|)]}\ \|x_0\|. \end{aligned} \tag{24}$$

From (22) and (21), we get

$$\log(\sqrt{\beta}) - \lambda_c\tau_{\min} < 0 \text{ and } \log(\sqrt{\beta}|\lambda_d|) < 0.$$

We deduce that the terms at the exponential in (24) are negative, which implies that the solution converges exponentially to zero when $k \to \infty$ (i.e; $t \to \infty$).

(ii) If $\sqrt{\beta}|\lambda_d| > 1$, inequality (24) can be written as,

$$\|x(t_{k+1})\| \le \xi e^{k[\log(\beta) - \lambda_c \tau_{\min} + \log(|\lambda_d|)]} \|x_0\|. \quad (25)$$

So, from (23), we have $\log(\beta) - \lambda_c \tau_{\min} < 0$ and since $|\lambda_d| < 1$, all the terms at the exponential function in (25) are negative and the solution converges exponentially to zero when $k \to \infty$ (i.e; $t \to \infty$).

Suppose now that A_c is exponentially stable and A_d may be unstable. Consider the switched linear system (2) such that A_c is stable and A_d may be Hilger stable or unstable. Let $\lambda_c > 0$, $|\lambda_d| > 1$ and $\beta \le 1$ defined as in (8), (7) and (17) respectively. Suppose that the following assumptions are fulfilled:

(i) $\mu(t_k)$ satisfies, $\forall k \in \mathbb{N}^*$,

$$|1+\mu(t_k)\lambda_d^j| < |\lambda_d|, \text{ and } |\lambda_d| > 1, \ \forall 1 \le j \le n. \quad (26)$$

(ii) The duration of each continuous-time subsystem satisfies

$$\tau_k > \frac{\log(\beta|\lambda_d|)}{\lambda_c}, \ \forall k \in \mathbb{N}. \quad (27)$$

Then, system (2) is exponentially stable under (26), (27). *Proof:* Since A_d can be unstable, so there exists a constant $|\lambda_d| > 1$, such that A_s defined as in (5) is Hilger stable, which leads to the condition (26). Since A_c is exponentially stable, so there exists $\lambda_c > 0$ such that $(A_c + \lambda_c I)$ is stable, and an upper bound of the solution of (2) is given by (20). Similar to (24), we get

$$\|x(t_{k+1})\| \le \xi \ e^{k[-\lambda_c \tau_{\min} + \log(\beta|\lambda_d|)]} \ \|x_0\|, \quad (28)$$

with $|\lambda_d| > 1$. From conditions (27), we conclude that the term at the exponential in (28) is always negative, and the solution converges exponentially to zero when $k \to \infty$ (i.e. $t \to \infty$).

Suppose now that A_c is unstable and A_d is Hilger stable. Consider the switched linear system (2) such that A_c is unstable and A_d is Hilger stable. Let $\lambda_c > 0$, and let $|\lambda_d| < 1$ and $\beta \ge 1$ defined as in (5) and (17), respectively. Suppose that the following assumptions are fulfilled:

(i) $\mu(t_k)$ satisfies, $\forall k \in \mathbb{N}^*$

$$|1 + \mu_k \lambda_d^j| < |\lambda_d|, \text{ with } |\lambda_d| < \frac{1}{\beta}, \ \forall 1 \le j \le n. \quad (29)$$

(ii) The duration of each continuous-time subsystem satisfies

$$0 < \tau_k < \frac{-\log(\beta|\lambda_d|)}{\lambda_c}, \ \forall k \in \mathbb{N}. \quad (30)$$

Then, the switched system (2) is exponentially stable. *Proof:* Since A_c is unstable, so there exists a constant $\lambda_c > 0$ such that $(A_c - \lambda_c I)$ is stable. Suppose that A_d is Hilger stable such that $\mu(t)$ satisfies condition (3), so there exists a constant $|\lambda_d| < 1$, such that A_s defined as in (5) is Hilger stable. Similar to the above analysis, the upper bound of the solution of (2) is given by

$$\|x(t_{k+1})\| \le \xi \prod_{i=0}^{k} e^{(\log(\sqrt{\beta}) + \lambda_c \tau_i)} \ (\sqrt{\beta}|\lambda_d|)^k \ \|x_0\|, \quad (31)$$

with $|\lambda_d| < 1$ and $\lambda_c > 0$. We can derive the following

$$\begin{aligned} \|x(t_{k+1})\| &\le \xi \ e^{\sum_{i=0}^{k}(\log(\sqrt{\beta}) + \lambda_c \tau_i)} \ e^{k \log(\sqrt{\beta}|\lambda_d|)} \ \|x_0\| \\ &\le \xi e^{k[\lambda_c \tau_{\max} + \log(\beta|\lambda_d|)]} \ \|x_0\|. \end{aligned} \quad (32)$$

From conditions (29), (30), the term at the exponential in (32) is negative, which implies the exponential stability of (2).

The matrix P_d in (10) is computed by fixing $\mu_{\max}$, with $0 < \mu(t) < \mu_{\max}$, such that condition (3) is satisfied (i.e, A_d is Hilger stable with respect to $\mu(t)$). By fixing $|\lambda_d| < 1$, the matrix P_d satisfies

$$A_d^T P_d + P_d A_d + \mu_{\max} A_d^T P_d A_d < \left(\frac{\lambda_d^2 - 1}{\mu_{\max}} \right) P_d, \quad (33)$$

which is equivalent to solving the LMI

$$-\lambda_d^2 P_d + (I + \mu_{\max} A_d)^T P_d (I + \mu_{\max} A_d) < 0.$$

Note that, if (33) is satisfied, so inequality (10) is satisfied for any $\mu(t) \le \mu_{\max}$, since, we have

$$\begin{aligned} &A_d^T P_d + P_d A_d + \mu(t) A_d^T P_d A_d + (\mu_{\max} - \mu(t)) A_d^T P_d A_d < \\ &\left(\frac{\lambda_d^2 - 1}{\mu(t)} \right) P_d + \left[\left(\frac{\lambda_d^2 - 1}{\mu_{\max}} \right) - \left(\frac{\lambda_d^2 - 1}{\mu(t)} \right) P_d \right], \end{aligned} \quad (34)$$

which implies that,

$$\begin{aligned} &A_d^T P_d + P_d A_d + \mu(t) A_d^T P_d A_d < \left(\frac{\lambda_d^2 - 1}{\mu(t)} \right) P_d \\ &+ (\mu_{\max} - \mu(t)) \left[-A_d^T P_d A_d + \left(\frac{\lambda_d^2 - 1}{\mu(t)\mu_{\max}} \right) \right] P_d. \end{aligned} \quad (35)$$

Inequality (33) implies that

$$A_d^T P_d + P_d A_d < -\mu_{\max} A_d^T P_d A_d + \left(\frac{\lambda_d^2 - 1}{\mu_{\max}} \right) P_d < 0.$$

A_d is Hilger stable and $A_d^T P_d A_d > 0$, which leads to

$$-A_d^T P_d A_d + \left(\frac{\lambda_d^2 - 1}{\mu_{\max}^2} \right) P_d < 0. \quad (36)$$

One can conclude from (35) and (36), that (10) is always satisfied.

V. A Numerical example

Consider the switched system

$$x^{\Delta} = \begin{cases} \begin{pmatrix} -3 & \frac{-3}{2}, \\ 2 & \frac{1}{2} \end{pmatrix} x, & t \in \cup_{k=0}^{\infty} [\sigma(t_k), t_{k+1}[, \\ \begin{pmatrix} \frac{-1}{2} & \frac{1}{10} \\ 0 & -1 \end{pmatrix} x, & t \in \cup_{k=0}^{\infty} \{t_{k+1}\}, \end{cases} \quad (37)$$

The matrix A_c is exponentially stable with eigenvalues, $\lambda_c^1 = -1.5$ and $\lambda_c^2 = -1$. The eigenvalues of A_d are $\lambda_d^1 = -0.5$ and $\lambda_d^2 = -1$, so A_d is Hilger stable if $0 < \mu < 2$ and it is regressive if $\mu \neq 1$. Let $\lambda_c = 0.5$, so $(A_c + \lambda_c I) = \begin{pmatrix} -2.5 & -1.5 \\ 2 & 1 \end{pmatrix}$ and $P_c = \begin{pmatrix} 1.0534 & 0.8466 \\ 0.8466 & 0.9801 \end{pmatrix}$. Let $\mu_{\max} = 1.8$ and $\lambda_d = 0.4$, so $P_d = \begin{pmatrix} 1.2735 & 0.0937 \\ 0.0937 & 1.695 \end{pmatrix}$. We have $\beta = 10.1294$, so for $1.2 \leq \mu(t) \leq 1.4$ and $\tau_k > 4.6309$, condition (23) is satisfied and the switched system (37) is exponentially stable. Let the time scale $\mathbb{T} = \cup_{k=0}^{\infty}[4.5k + \frac{7k}{5k+0.8}\ ,\ 4.7(k+1)]$, which verifies the dwell times conditions. Fig. 2 show that the switched system (37) is stable.

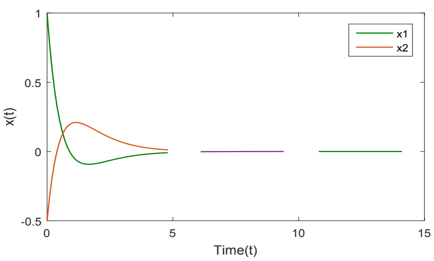

Fig. 2. Stable trajectories of system (37) with $x_0 = [-0.5, 1]^T$.

VI. CONCLUSION

Time scale theory was introduced to derive new dwell time conditions for stability of continuous/discrete switched systems which evolve on a non-uniform time domain in the presence of unstable subsystems. Note that, the stability of discrete-time subsystems with variable discrete steps is not only related to their dynamics, but also to the step sizes. Using Lyapunov-like functions, we have shown that this special class of switched systems can be stabilized if the dwell time of each continuous-time and the gaps of the discrete-time subsystems are bounded by specific values.

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REFERENCES

[1] D. Liberzon, *Switching in System and Control*. Systems and Control: Foundations and Applications, Boston: Birkhauser, (2003).

[2] G. Zhai, B. Hu, K. Yasuda, and A. N. Michel, *Qualitative Analysis of Discrete-time Switched Systems*, In American Control Conference, (2002).

[3] M. S. Branicky, *Multiple Lyapunov Functions and Other Analysis Tools for Switched and Hydrid Systems*, IEEE Transactions on Automatic Control, 43(4), pp. 475–482, (1998).

[4] H. K. Khalil, *Nonlinear Systems*, (Second Edition), Prentice-Hall, (1996).

[5] X. Zhao, L. Zhang, and P. Shi, M. Liu, *Stability and Stabilization of Switched Linear Systems with Mode-Dependent Average Dwell Time*, IEEE Trans. Autom. Control, 57(7), pp. 1809–1815, (2012).

[6] J. Geromel and P. Colaneri, *Stability and Stabilization of Continuous-Time Switched Linear Systems*, SIAM J. Control Optim, 45, pp. 1915–1930, (2006).

[7] J. Geromel and P. Colaneri, *Stability and Stabilization of Discrete-Time Switched Systems*, Int. J. Control, 79, pp. 719–728, (2006).

[8] G. Zhai, H. Xu, H. Lin and A. N. Michel, *Analysis and Design of Switched Normal Systems*, Nonlinear Anal, Theory, Methods Appl, 65(12), pp. 2218–2259, (2006).

[9] J. Daafouz, R. Riedinger, and C. Iung, *Stability Analysis and Control Synthesis for Switched Systems: A Switched Lyapunov Function Approach*, IEEE Trans. Automat. Control, 47(11), pp. 1883–1887, (2002).

[10] J. P Hespanha and A. S. Morse, *Stability of Switched Systems with Average Dwell Time*, 38th IEEE Conf. Decision. Control, pp. 2655–2660, (1999).

[11] H. Lin, and P. J. Antsaklis, *Stability and Stabilizability of Switched Linear Systems: A Survey of Recent Results*, IEEE Trans. Autom. Control, pp. 308–322, (2009).

[12] K. S. Narendra and J. Balakrishnana, *A Common Lyapunov Function For Stable LTI Systems with Commuting A-matrices*, IEEE trans. Automat. Control, 39(12), pp. 2469–2471, (1994).

[13] H. Zhang, D. Xie, Hongyu Zhang, and G. Wang, *Stability Analysis for Discrete-time Switched Systems with Unstable Subsystems by a Mode-dependent Average Dwell Time Approach*, ISA Transaction, 53, pp. 1081–1086, (2014).

[14] G. Zhai, B. Hu, K. Yasuda, and A. N. Michel, *Stability Analysis of Switched Systems with Stable and Unstable Subsystems: An Average Dwell Time Approach*, Journal of the Franklin Institute, 338(7), pp. 765–779, (2001).

[15] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales, An Introduction with Applications*, Birkhauser, Boston, (2001).

[16] J. DaCuhna, *Stability for Time Varying Linear Dynamic Systems on Time Scales*, J. Comput. Appl. Math., 176(2), pp. 381–410, (2005).

[17] A. C. Peterson and Y. N. Raffoul, *Exponential Stability of Dynamic Equations on Time Scales*, Advances in Difference Equations, 2, pp. 133–144, (2005).

[18] J. M. Davis, I. A. Gravagne, and A. A. Ramos, *Stability of Switched Linear Systems on Non-uniform Time Domains*, IEEE Southeastern Symposium on Systems Theory, Texas, (2010).

[19] G. Eisenbarth, J. Davis, and I. Gravagne, *Stability of Simultaneously Triangularizable Switched Systems on Hybrid Domains*, Electron. J. Differ. Eq., 63, pp. 1–19, (2014).

[20] F. Z. Taousser, M. Defoort, and M. Djemai, *Stability Analysis of a Class of Switched Linear Systems on Non-Uniform Time Domains*, Systems and Control Letter, 74, pp. 24–31, (2014).

[21] F. Z. Taousser, M. Defoort, and M. Djemai, *Stability Analysis of a Class of Uncertain Switched Systems on Time Scale Using Lyapunov Functions*, Nonlinear Analysis: Hybrid Systems, 16, pp. 13-23, (2015).

[22] F. Z. Taousser, M. Defoort, and M. Djemai, *Consensus for Linear Multi-Agent System with Intermittent Information Transmissions using the Time Scale Theory*, International Journal of Control, 89, pp. 210–220, (2016).

[23] F. Z. Taousser, M. Defoort, and M. Djemai, Seddik. M. Djouadi and Kevin Tomsovic, *Stability Analysis of a Class of Switched Nonlinear Systems using the Time Scale Theory*, Nonlinear Analysis: Hybrid Systems, early access, (2019).

[24] T. Gard and J. Hoffacker, *Asymptotic Behavior of Natural Growth on Time Scales*, Dynamic Systems and Applications, 12(1), pp. 131–147, (2003).