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Asymptotic Sign Coherence Conjecture

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ABSTRACT

The sign coherence phenomenon is an important feature of *c*-vectors in cluster algebras with principal coefficients. In this note, we consider a more general version of *c*-vectors defined for arbitrary cluster algebras of geometric type and formulate a conjecture describing their asymptotic behavior. This conjecture, which is called *the asymptotic sign coherence conjecture*, states that for any infinite sequence of matrix mutations that satisfies certain natural conditions, the corresponding *c*-vectors eventually become sign coherent. We prove this conjecture for rank 2 cluster algebras of infinite type and for a particular sequence of mutations in a cluster algebra associated with the Markov quiver.

KEYWORDS

cluster mutations; *c*-vectors; sign coherence

1. Introduction

A study of *c*-vectors for cluster algebras with principal coefficients was initiated in [Fomin and Zelevinsky 07], the fourth in the series of foundational papers that gave rise to the theory of cluster algebras. There, *c*-vectors appeared, together with related concepts such as *g*-vectors and *F*-polynomials, as tools for understanding a deeper structure of cluster variables. Also in [Fomin and Zelevinsky 07], a conjecture that later became known as *the sign coherence conjecture* was first formulated. It states that in any cluster mutation equivalent to the initial one, each *c*-vector is nonzero and has either all nonnegative or all nonpositive coefficients. It was soon realized that this conjecture has important implications in various aspects of the theory of cluster algebras, notably, in establishing duality properties as was done in [Nakanishi and Zelevinsky 12]. The conjecture was proved in [Derksen et al. 10] for the skew-symmetric cluster algebras and in [Gross et al. 17] for the skew-symmetrizable ones.

It is natural to wonder if there is an analog of sign coherence that remains valid if the condition on coefficients being principal is relaxed, in particular, if a similar phenomenon occurs in cluster algebras of *geometric type* in the sense of [Fomin and Zelevinsky 07, Definition 2.12], where their coefficients take value in tropical semifields. Clearly, this property cannot be satisfied as is-one can simply start with the initial coefficients for which it is not valid. This note proposes how sign coherence can be treated in arbitrary cluster algebras of geometric type: after a sufficiently generic sequence of mutation, *c*-vectors, defined in a more general context, become sign coherent.

In the next section, after providing a background on matrix mutations and providing an example illustrating the phenomenon described above, we formulate a conjecture that describes this behavior. We call it *the asymptotic sign coherence conjecture*. In Section 3, we verify our conjecture in the rank 2 case. The final section deals with a particular sequences of mutations in a cluster algebra associated with the Markov quiver—we show that the conjecture holds true in this case as well.

2. Preliminaries and the main conjecture

2.1. Matrix mutations

We say that an $N \times N$ integer matrix $B = (b_{ij})_{i,j=1}^{N}$ is *skew-symmetrizable* if there is a diagonal matrix $D = \mathrm{diag}(d_1, \ldots, d_N)$ with positive integer diagonal

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entries $d_1, \dots, d_N$ such that DB is skew-symmetric, i.e., $d_i b_{ij} = -d_j b_{ji}$. We call such D a *skew-symmetrizer* of B. B is called *irreducible* if there does not exist a pair I, J of nonempty subsets of $[1, N] := \{1, \dots, N\}$ such that $b_{ij} = 0$ for any $i \in I$ and $j \in J$. If B is skew-symmetric, it can be realized as an adjacency matrix of a quiver with N vertices. The irreducibility property in this case is equivalent to connectedness of the quiver.

The notion of a *matrix mutation* is one of the key ingredients in the definition of a cluster algebra. It is also the only ingredient of that definition that we will need in this note. We will be interested in matrix mutations of integer matrices with skew-symmetrizable principal parts. Namely, let $\hat{B}$ be an integer $(N+M) \times N$ matrix such that its principal submatrix $B := \hat{B}_{[N]}$ formed by the first N rows is skew-symmetrizable. Columns of $\hat{B}$ determine the rules of transformations of cluster variables $x_1, \dots, x_N$ with the variables $x_{N+1}, \dots, x_{N+M}$ being frozen. Alternatively, we may view the columns of the bottom $M \times N$ submatrix of $\hat{B}$ as encoding expressions for coefficients in a cluster algebra of geometric type. See [Fomin and Zelevinsky 07, Section 2] for the explanation.

For each $k = 1, \dots, N$, the *mutation of $\hat{B}$ at k* is another integer $(N+M) \times N$ matrix $\hat{B}' = \mu_k(\hat{B})$, which is obtained from $\hat{B}$ by the following formulas:

$$b'_{ij} = \begin{cases} -b_{ij} & i = k \text{ or } j = k \\ b_{ij} + [-\varepsilon b_{ik}]_+ b_{kj} + b_{ik} [\varepsilon b_{kj}]_+ & i, j \neq k, \end{cases} \tag{2–1}$$

where for $a \in \mathbb{R}$, we denote $[a]_+ = \max(a, 0)$, and ε is a sign, $+$ or $-$, which is naturally identified with 1 or -1, respectively. Then we have the following properties:

1. The right hand side of (2–1) is independent of the choice of sign ε.
2. If D is a skew-symmetrizer of B, then it is also a skew-symmetrizer of $B' = \hat{B}'_{[N]}$.
3. The mutation μ_k is involutive, namely,

$$\mu_k \circ \mu_k = \mathrm{id}.$$

More generally, if $\hat{B}'$ is obtained from $\hat{B}$ by a sequence of mutations (2–1), then $\hat{B}'$ is said to be *mutation equivalent* to $\hat{B}$.

If $M = N$ and the block formed by the last N rows of $\hat{B}$ is equal to the identity matrix 1_N then the cluster algebra associated with $\hat{B}$ is said to have *principal coefficients*. In this case, the bottom $N \times N$ submatrix of any matrix $\hat{B}'$ mutation equivalent to $\hat{B}$ is called a C-matrix and its columns are called c-vectors.

2.2. Main conjecture

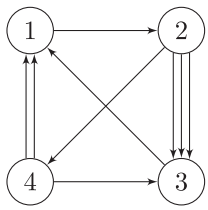
The following key property of c-vectors that proved to be of fundamental importance in the theory and applications of cluster algebras was conjectured in [Fomin and Zelevinsky 07, Conjecture 5.5 and Proposition 5.6] and later proved in [Derksen et al. 10, Theorem 1.7] in the skew-symmetric case and [Gross et al. 17, Corollary 5.5] in the skew-symmetrizable case with both proofs using [Fomin and Zelevinsky 07, Proposition 5.6].

Theorem 2.1 (**Sign-coherence of c-vectors**). *Each c-vector is a nonzero vector, and its components are either all nonnegative or all nonpositive.*

In this note, we are interested in a behavior of a more general version of c-vectors: namely, we will denote by C the bottom $M \times N$ submatrix of $\hat{B}$ and refer to its columns as c-vectors. Whenever we will need to invoke the original definition of c-vectors, we will call them *principal* c-vectors.

We begin with the following example.

Example 2.2. Let B be the adjacency matrix for the quiver below. This quiver is a rather randomly chosen one with four vertices.



Consider a sequence, also rather randomly chosen, of C-matrix mutations, together with the initial C-matrix $C[0]$, specified below:

$$C[0]=\begin{pmatrix}1&-1&2&1\\-2&1&3&-2\\-1&1&-1&0\\0&-1&-1&-2\end{pmatrix}\overset{1}{\to}C[1]=\begin{pmatrix}-1&0&2&1\\2&1&1&-6\\-1&2&-1&0\\0&-1&-1&-2\end{pmatrix}\overset{3}{\to}$$

$$C[2]=\begin{pmatrix}-1&0&-2&1\\2&1&-1&-6\\-2&0&1&-1\\-1&-3&1&-3\end{pmatrix}\overset{4}{\to}C[3]=\begin{pmatrix}-1&3&-2&-1\\-10&1&-7&6\\-4&0&0&1\\-7&-3&-2&3\end{pmatrix}\overset{2}{\to}$$

$$C[4]=\begin{pmatrix}-1&-3&-2&8\\-10&-1&-7&9\\-4&0&0&1\\-22&3&-17&3\end{pmatrix}\overset{3}{\to}C[5]=\begin{pmatrix}-1&-13&2&8\\-10&-36&7&9\\-4&0&0&1\\-22&-82&17&3\end{pmatrix}\overset{1}{\to}$$

$$C[6]=\begin{pmatrix}1&-23&2&8\\10&-136&7&9\\4&-40&0&1\\22&-302&17&3\end{pmatrix}\overset{4}{\to}C[7]=\begin{pmatrix}105&-23&114&-8\\127&-136&133&-9\\17&-40&14&-1\\61&-302&59&-3\end{pmatrix}\overset{1}{\to}$$

$$C[8]=\begin{pmatrix}-105&-23&114&1357\\-127&-136&133&1642\\-17&-40&14&220\\-61&-302&59&790\end{pmatrix}\overset{\cdots}{\to}$$

Example above is just one of many we have considered and all of these examples exhibited the same phenomenon: *after sufficiently many random mutations, C-vectors become sign-coherent*.

To frame these observations as an explicit conjecture, we will only need to consider cluster algebras with a single frozen variable since mutations of rows of a C-matrix are independent of each other by (2–1). We fix a skew-symmetrizable $N\times N$ matrix B and an integer vector $\mathbf{a}=(a_1,\ldots,a_N)$ and define an $(N+1)\times N$ matrix $\hat{B}$ by appending $\mathbf{a}$ to B as the $(N+1)$th row.

Let $\mu=(\mu_{k_j})_{j=1}^{\infty}$ $(k_j\in[1,N])$ be a sequence of matrix mutations applied to $\hat{B}$. We denote $\mu_{k_n}\circ\cdots\circ\mu_{k_2}\circ\mu_{k_1}$ by $\mu^{(n)}$, $\mu^{(n)}(\hat{B})$ by $\hat{B}^{(n)}$ and the last row of $\hat{B}^{(n)}$ by $\mathbf{a}^{(n)}=(a_1^{(n)},\ldots,a_N^{(n)})$. We use a convention $\hat{B}^{(0)}=\hat{B},\mathbf{a}^{(0)}=\mathbf{a}$. Note that if $\mathbf{a}=0$, then $\mathbf{a}^{(n)}=0$ for any n by (2–1). Therefore, from now on we assume that $\mathbf{a}$ is a *nonzero* vector.

Let $\hat{B}_{pr}$ denote an $N\times 2N$ matrix $\begin{bmatrix}B\\\mathbf{1}_N\end{bmatrix}$ (in other words, $\hat{B}_{pr}$ is the initial exchange matrix for the cluster algebra with principal coefficients defined by B). We define *the distance* $\mathrm{dist}(\hat{B}_{pr},\mu^{(n)}(\hat{B}_{pr}))\le n$ as the *minimal* number of matrix mutations needed to obtain $\mu^{(n)}(\hat{B}_{pr})$ from $\hat{B}_{pr}$. This notion of distance agrees with *the distance in the exchange graph of labeled seeds* in the cluster algebra associated with B (with any coefficients), see, e.g., [Nakanishi 19].

We say that the sequence of mutatons μ is *monotone* (with respect to B) if

$$\begin{aligned}&\mathrm{dist}\left(\hat{B}_{pr},\mu^{(n+1)}\left(\hat{B}_{pr}\right)\right)\\&>\mathrm{dist}\left(\hat{B}_{pr},\mu^{(n)}\left(\hat{B}_{pr}\right)\right)(n=1,2,\ldots).\end{aligned}\tag{2–2}$$

We say that μ is *balanced* if for every $k\in[1,N]$,

$$\liminf_{n\to\infty}\frac{1}{n}\#\{j\in[1,n]:k_j=k\}>0.$$

For an integer a, we define

$$\mathrm{sign}(a)=\begin{cases}+&a>0\\0&a=0\\-&a<0\end{cases}.$$

We say that a sign vector $\omega=(\omega_1,\ldots,\omega_N)\in\{+,0,-\}^N$ is *strict* if $\omega_i\neq 0$ for any i. Define *the sign vector corresponding to* $\mu^{(n)}$ as

$$\sigma_\mu^{(n)}(\mathbf{a}) = \left(\operatorname{sign}\left(a_1^{(n)}\right), \ldots, \operatorname{sign}\left(a_N^{(n)}\right)\right). \tag{2–3}$$

The *sign pattern corresponding to* μ is defined by

$$\sigma_\mu(\mathbf{a}) = \left(\sigma_\mu^{(n)}(\mathbf{a})\right)_{n=0}^{\infty}. \tag{2–4}$$

Conjecture 2.3 (**Asymptotic sign coherence**). *Let* $B = \hat{B}_{[N]}$ *be irreducible and let* μ *be a monotone and balanced sequence of mutations applied to* $\hat{B}$. *Then there exists a sequence of strict sign vectors*

$$\sigma_{\mathrm{reg}} = \left(\sigma_{\mathrm{reg}}^{(n)}\right)_{n=0}^{\infty}, \sigma_{\mathrm{reg}}^{(n)} \in \{\pm\}^N$$

such that for any nonzero $\mathbf{a} \in \mathbb{Z}^N$ *there is* $T \in \mathbb{N}$ *such that* $\sigma_\mu^{(n)}(\mathbf{a}) = \sigma_{\mathrm{reg}}^{(n)}$ *for* $n > T$.

Remarks 2.4. 1. If the C-matrix part of $\hat{B}$ has multiple rows (i.e., $M \geq 2$), Conjecture 2.3 means that for a monotone and balanced sequence of mutations, for all n greater than certain T, the sign patterns of any two nonzero rows of the C-matrix $C^{(n)}$ obtained on the nth step coincide. But that means that the columns of $C^{(n)}$ are sign-coherent. This justifies the term *asymptotic sign coherence*.

2. The monotonicity assumption precludes the initial exchange matrix B from being an exchange matrix of a cluster algebra of finite type.

3. Both conditions in Conjecture 2.3 are used to ensure that the mutation sequence μ is sufficiently random. Both of these conditions could be relaxed or modified in various ways. In particular, one could require that a magnitude of every entry of the C-matrix tends to infinity as n goes to infinity. One could also weaken the condition of μ being balanced by replacing it with the requirement that every mutation direction appears in μ infinitely many times. In this case, we call μ *weakly balanced* (see Remark 4.2 below).

3. Rank 2 case

In this section, we will verify Conjecture 2.3 in the infinite type rank 2 case. The matrix $\hat{B}$ in this case has a form

$$\hat{B} = \hat{B}^{(0)} = \begin{bmatrix} 0 & -p \\ q & 0 \\ a_1^{(0)} & a_2^{(0)} \end{bmatrix}, \tag{3–1}$$

where positive integer parameters p, q satisfy $pq \geq 4$ [Fomin and Zelevinsky 02, Section 6]. We assume that at least one of $a_1^{(0)}, a_2^{(0)}$ is nonzero.

In the rank 2 case, the monotonicity assumption for an infinite sequence of mutations amounts to a requirement that mutations μ_1 and μ_2 alternate. Instead of considering two possible monotone sequences of mutations, we will unify them into a single double-infinite sequence. Namely, we consider a sequence of mutations $\mu_1, \mu_2, \mu_1, \mu_2, \ldots$ and denote the exchange matrix obtained at the nth step of this sequence by $\hat{B}^{(n)}$. Similarly, the exchange matrix obtained at the nth step of a sequence of mutations $\mu_2, \mu_1, \mu_2, \mu_1, \ldots$ will be denoted by $\hat{B}^{(-n)}$. Then, for any $n \in \mathbb{Z}$,

$$\hat{B}^{(n)} = \begin{bmatrix} 0 & (-)^{n-1}p \\ (-)^n q & 0 \\ a_1^{(n)} & a_2^{(n)} \end{bmatrix}. \tag{3–2}$$

Choosing $\varepsilon = 1$ in (2–1), we obtain the following recursion for $a_1^{(n)}, a_2^{(n)} (n \geq 0)$:

$$\begin{aligned} a_1^{(2n+1)} &= -a_1^{(2n)}, \\ a_2^{(2n+1)} &= a_2^{(2n)} - \left[-a_1^{(2n)}\right]_+ p, \end{aligned} \tag{3–3}$$

$$\begin{aligned} a_1^{(2n+2)} &= a_1^{(2n+1)} - \left[-a_2^{(2n+1)}\right]_+ q, \\ a_2^{(2n+2)} &= -a_2^{(2n+1)}. \end{aligned} \tag{3–4}$$

For $n \leq 0$, we choose $\varepsilon = -1$, and obtain

$$\begin{aligned} a_1^{(-2n-1)} &= a_1^{(-2n)} + \left[a_2^{(-2n)}\right]_+ q, \\ a_2^{(-2n-1)} &= -a_2^{(-2n)}, \end{aligned} \tag{3–5}$$

$$\begin{aligned} a_1^{(-2n-2)} &= -a_1^{(-2n-1)}, \\ a_2^{(-2n-2)} &= a_2^{(-2n-1)} + \left[a_1^{(-2n-1)}\right]_+ p. \end{aligned} \tag{3–6}$$

We want to investigate a dependence of signs of $a_1^{(n)}, a_2^{(n)}$ on initial values $a_1^{(0)}, a_2^{(0)}$. As in (2–3), we define a *sign vector*

$$\sigma^{(n)} = \left(\operatorname{sign}\left(a_1^{(n)}\right), \operatorname{sign}\left(a_2^{(n)}\right)\right).$$

If an expression is only known to be nonnegative (resp. nonpositive), we will denote its sign by $+/0$ (resp. $-/0$). Following the definition (2.4), we call a sequence

$$\sigma\left(a_1^{(0)}, a_2^{(0)}\right) = \left(\sigma^{(n)}\right)_{n\in\mathbb{Z}} \tag{3–7}$$

the *sign pattern* with initial values $a_1^{(0)}, a_2^{(0)}$.

We define the sequence σ_{reg} of strict sign vectors as

$$\sigma_{\mathrm{reg}} = \left(\left((-)^{n-1}, (-)^n\right)\right)_{n\in\mathbb{Z}}. \tag{3–8}$$

We are going to show that for any $a_1^{(0)}, a_2^{(0)}$, the sequence $\sigma(a_1^{(0)}, a_2^{(0)})$ differs from σ_{reg} for only finitely many components.

To proceed, we need to recall some of the properties of *the Chebyshev polynomials* $U_n(t)(n \in \mathbb{Z}_{\geq -1})$ *of the second kind* (see, e.g. [Doman 16, Chapter 6]). They satisfy a three-term recursion of the form

$$2tU_n(t) = U_{n-1}(t) + U_{n+1}(t)(n \geq 0); U_{-1}(t) = 0, U_0(t) = 1, \tag{3–9}$$

and are orthogonal with respect to the positive weight $\sqrt{1-t^2}$ on the interval $[-1, 1]$. As a result,

$$U_n(t) > 0 (n \geq 0, t > 1). \tag{3–10}$$

Furthermore,

$$U_n^2(t) - U_{n-1}(t)U_{n+1}(t) = 1 (n = 0, 1, \ldots), \tag{3–11}$$

and therefore

$$\frac{U_n(t)}{U_{n-1}(t)} > \frac{U_{n+1}(t)}{U_n(t)} (n > 0). \tag{3–12}$$

There is an explicit formula for $U_n(t)$,

$$U_n(t) = \frac{\left(t + \sqrt{t^2-1}\right)^{n+1} - \left(t - \sqrt{t^2-1}\right)^{n+1}}{2\sqrt{t^2-1}}, \tag{3–13}$$

which implies, in particular, that

$$\lim_{n\to\infty} \frac{U_{n+1}(t)}{U_n(t)} = t + \sqrt{t^2-1} (|t| \geq 1). \tag{3–14}$$

The role played by Chebyshev polynomials in the study of rank 2 cluster algebras was previously observed by several authors, notably in [Lee and Schiffler 13, Lee and Schiffler 15] where it was utilized in initial steps of the proof of the Laurent positivity conjecture, and, more recently, in [Reading 18] where closely related polynomials were used in a description of rank 2 infinite type cluster scattering diagrams. Below, we use Chebyshev polynomials to investigate the sign pattern $\sigma(a_1^{(0)}, a_2^{(0)})$.

Denote

$$\kappa = \sqrt{pq}, \nu = \sqrt{\frac{p}{q}}.$$

Proposition 3.1. *If* $a_1^{(0)} = a_1, a_2^{(0)} = a_2 \geq 0$, *then*

$$a_1^{(1)} = -a_1, a_2^{(1)} = a_2, a_1^{(2)} = -a_1, a_2^{(2)} = -a_2,$$

for $n \geq 2$

$$\begin{aligned} a_1^{(2n)} &= -a_1 U_{2n-2}\left(\frac{\kappa}{2}\right) - a_2 \nu^{-1} U_{2n-3}\left(\frac{\kappa}{2}\right) < 0, \\ a_2^{(2n)} &= a_1 \nu U_{2n-3}\left(\frac{\kappa}{2}\right) + a_2 U_{2n-4}\left(\frac{\kappa}{2}\right) > 0, \end{aligned} \tag{3–15}$$

for $n \geq 1$

$$\begin{aligned} a_1^{(2n+1)} &= a_1 U_{2n-2}\left(\frac{\kappa}{2}\right) + a_2 \nu^{-1} U_{2n-3}\left(\frac{\kappa}{2}\right) \geq 0, \\ a_2^{(2n+1)} &= -a_1 \nu U_{2n-1}\left(\frac{\kappa}{2}\right) - a_2 U_{2n-2}\left(\frac{\kappa}{2}\right) < 0, \end{aligned} \tag{3–16}$$

with the first inequality in (3–16) *strict for* $n > 1$, *and, for* $n \geq 0$

$$\begin{aligned} a_1^{(-2n-2)} &= -a_1 U_{2n}\left(\frac{\kappa}{2}\right) - a_2 \nu^{-1} U_{2n+1}\left(\frac{\kappa}{2}\right) < 0, \\ a_2^{(-2n-2)} &= a_1 \nu U_{2n+1}\left(\frac{\kappa}{2}\right) + a_2 U_{2n+2}\left(\frac{\kappa}{2}\right) > 0, \end{aligned} \tag{3–17}$$

$$\begin{aligned} a_1^{(-2n-1)} &= a_1 U_{2n}\left(\frac{\kappa}{2}\right) + a_2 \nu^{-1} U_{2n+1}\left(\frac{\kappa}{2}\right) > 0, \\ a_2^{(-2n-1)} &= -a_1 \nu U_{2n-1}\left(\frac{\kappa}{2}\right) - a_2 U_{2n}\left(\frac{\kappa}{2}\right) \leq 0, \end{aligned} \tag{3–18}$$

with the second inequality in (3–18) *strict for* $n > 0$.

Therefore, $\sigma^{(-1)} = (+,-/0), \sigma^{(0)} = (+/0,+/0)$, $\sigma^{(1)} = (-/0,+/0), \sigma^{(2)} = (-/0,-/0), \sigma^{(3)} = (+/0,-)$ *and*

$$\sigma^{(n)} = \left((-)^{n-1}, (-)^n\right) (n \neq -1, 0, 1, 2, 3). \tag{3–19}$$

Proof. By $\frac{\kappa}{2} \geq 1$ and (3–10), the expressions in (3–15)–(3–18) imply the inequalities therein, therefore we obtain (3–19).

We will only verify equations in (3–16). The rest of the formulas in Proposition 3.1 can be treated the same way. Recursions (3–3), (3–4) combine into

$$\begin{aligned} a_1^{(2n+3)} &= -a_1^{(2n+1)} + \left[-a_2^{(2n+1)}\right]_+ q, \\ a_2^{(2n+3)} &= -a_2^{(2n+1)} - \left[a_1^{(2n+3)}\right]_+ p. \end{aligned} \tag{3–20}$$

For $n = 0$, this gives

$$\begin{aligned} a_1^{(3)} &= a_1 = a_1 U_0\left(\frac{\kappa}{2}\right) + a_2 \nu^{-1} U_{-1}\left(\frac{\kappa}{2}\right), \\ a_2^{(3)} &= -a_1 p - a_2 = -a_1 \nu \kappa - a_2 \\ &= -a_1 \nu U_1\left(\frac{\kappa}{2}\right) - a_2 U_0\left(\frac{\kappa}{2}\right), \end{aligned}$$

which is consistent with (3–16). We can now proceed by induction while taking into an account that the induction hypothesis stating that (3–16) is valid for $n \leq m$ also ensures that $a_1^{(2n+1)} \geq 0 \geq a_2^{(2n+1)}$ for $n \leq m$. (Here we use the inequality $\frac{\kappa}{2} \geq 1$ and (3–10).) Then equations in (3–20) become

$$
\begin{aligned}
a_1^{(2n+3)} &= -a_1 U_{2n-2}\left(\frac{\kappa}{2}\right) - a_2\nu^{-1} U_{2n-3}\left(\frac{\kappa}{2}\right) \\
&\quad + \left(a_1\nu U_{2n-1}\left(\frac{\kappa}{2}\right) + a_2 U_{2n-2}\left(\frac{\kappa}{2}\right)\right) q \\
&= a_1\left(\kappa U_{2n-1}\left(\frac{\kappa}{2}\right) - U_{2n-2}\left(\frac{\kappa}{2}\right)\right) \\
&\quad + a_2\nu^{-1}\left(-U_{2n-3}\left(\frac{\kappa}{2}\right) + \kappa U_{2n-2}\left(\frac{\kappa}{2}\right)\right) \\
&\overset{(3.9)}{=} a_1 U_{2n}\left(\frac{\kappa}{2}\right) + a_2\nu^{-1} U_{2n-1}\left(\frac{\kappa}{2}\right), \\
a_2^{(2n+3)} &= a_1\nu U_{2n-1}\left(\frac{\kappa}{2}\right) + a_2 U_{2n-2}\left(\frac{\kappa}{2}\right) \\
&\quad - \left(a_1 U_{2n}\left(\frac{\kappa}{2}\right) + a_2\nu^{-1} U_{2n-1}\left(\frac{\kappa}{2}\right)\right) p \\
&= a_1\left(U_{2n-1}\left(\frac{\kappa}{2}\right) - \kappa U_{2n}\left(\frac{\kappa}{2}\right)\right)\nu \\
&\quad + a_2\left(U_{2n-2}\left(\frac{\kappa}{2}\right) - \kappa U_{2n-1}\left(\frac{\kappa}{2}\right)\right) \\
&\overset{(3.9)}{=} -a_1\nu U_{2n+1}\left(\frac{\kappa}{2}\right) - a_2 U_{2n}\left(\frac{\kappa}{2}\right),
\end{aligned}
$$

which proves (3–16). □

Proposition 3.1 shows that if $a_1^{(0)}, a_2^{(0)}$ are both nonnegative then the sign pattern $\sigma(a_1^{(0)}, a_2^{(0)})$ does not depend on precise values of $a_1^{(0)}, a_2^{(0)}$ and we are justified in using a notation σ_{++} for such sign pattern.

Furthermore, if $a_1^{(0)}, a_2^{(0)}$ are both nonpositive, then Proposition 3.1 also implies that $\sigma(a_1^{(0)}, a_2^{(0)}) = (\sigma_{++}^{(n+2)})_{n\in\mathbb{Z}}$ also does not depend on precise values of $a_1^{(0)}, a_2^{(0)}$ and can be denoted by σ_{--}.

The case $a_1^{(0)} = a_1 > 0, a_2^{(0)} = -a_2 < 0$ is still covered by Proposition 3.1: if one switches the roles of $a_1^{(n)}, a_2^{(n)}$ and also the roles of p and q, then the corresponding matrix

$$
B^{(0)} = \begin{bmatrix} 0 & q \\ -p & 0 \\ -a_2 & a_1 \end{bmatrix}
$$

can be seen as $\tilde{B}^{(1)}$ for

$$
\tilde{B}^{(0)} = \begin{bmatrix} 0 & -q \\ p & 0 \\ a_2 & a_1 \end{bmatrix}
$$

and so

$$
\sigma_{+-} := \sigma\left(a_1^{(0)}, a_2^{(0)}\right) = \left(\tau\left(\sigma_{++}^{(n+1)}\right)\right)_{n\in\mathbb{Z}}, \tag{3–21}
$$

where τ permutes the entries of a two-vector.

The situation is different and more complicated if $a_1^{(0)} = -a_1 < 0, a_2^{(0)} = a_2 > 0$. In view of (3–12) and (3–13), we separate it into three cases:

Case 1. $\frac{2\nu}{\kappa+\sqrt{\kappa^2-4}} \le \frac{a_2}{a_1} \le \frac{\nu}{2}(\kappa+\sqrt{\kappa^2-4})$.
Case 2. $\frac{\nu}{2}(\kappa+\sqrt{\kappa^2-4}) < \frac{a_2}{a_1}$.
Case 3. $\frac{a_2}{a_1} < \frac{2\nu}{\kappa+\sqrt{\kappa^2-4}}$.
Let us start with Case 1.

Proposition 3.2. *Let $a_1^{(0)} = -a_1 < 0, a_2^{(0)} = a_2 > 0$.*

If $\frac{a_2}{a_1} \in \left[\frac{2\nu}{\kappa+\sqrt{\kappa^2-4}}, \frac{\nu}{2}(\kappa+\sqrt{\kappa^2-4})\right]$, then for $n \ge 1$,

$$
\begin{aligned}
a_1^{(2n)} &= -a_1 U_{2n}\left(\frac{\kappa}{2}\right) + a_2\nu^{-1} U_{2n-1}\left(\frac{\kappa}{2}\right) < 0, \\
a_2^{(2n)} &= a_1\nu U_{2n-1}\left(\frac{\kappa}{2}\right) - a_2 U_{2n-2}\left(\frac{\kappa}{2}\right) > 0,
\end{aligned} \tag{3–22}
$$

and for $n \ge 0$,

$$
\begin{aligned}
a_1^{(2n+1)} &= a_1 U_{2n}\left(\frac{\kappa}{2}\right) - a_2\nu^{-1} U_{2n-1}\left(\frac{\kappa}{2}\right) > 0, \\
a_2^{(2n+1)} &= -a_1\nu U_{2n+1}\left(\frac{\kappa}{2}\right) + a_2 U_{2n}\left(\frac{\kappa}{2}\right) < 0,
\end{aligned} \tag{3–23}
$$

$$
\begin{aligned}
a_1^{(-2n-2)} &= a_1 U_{2n}\left(\frac{\kappa}{2}\right) - a_2\nu^{-1} U_{2n+1}\left(\frac{\kappa}{2}\right) < 0, \\
a_2^{(-2n-2)} &= -a_1\nu U_{2n+1}\left(\frac{\kappa}{2}\right) + a_2 U_{2n+2}\left(\frac{\kappa}{2}\right) > 0,
\end{aligned} \tag{3–24}
$$

$$
\begin{aligned}
a_1^{(-2n-1)} &= -a_1 U_{2n}\left(\frac{\kappa}{2}\right) + a_2\nu^{-1} U_{2n+1}\left(\frac{\kappa}{2}\right) > 0, \\
a_2^{(-2n-1)} &= a_1\nu U_{2n-1}\left(\frac{\kappa}{2}\right) - a_2 U_{2n}\left(\frac{\kappa}{2}\right) < 0.
\end{aligned} \tag{3–25}
$$

Therefore,

$$
\sigma\left(a_1^{(0)}, a_2^{(0)}\right) = \sigma_{\mathrm{reg}}, \tag{3–26}
$$

where σ_{reg} is the one in (3–8).

Proof. Our assumption for $\frac{a_2}{a_1}$ together with properties (3–12) and (3–14) of the Chebyshev polynomials ensure that

$$
\nu\frac{U_n\left(\frac{k}{2}\right)}{U_{n+1}\left(\frac{k}{2}\right)} < \frac{a_2}{a_1} < \nu\frac{U_{n+1}\left(\frac{k}{2}\right)}{U_n\left(\frac{k}{2}\right)} \tag{3–27}
$$

for any $n > 0$. Then the expressions (3–22)–(3–25) imply the inequalities therein, therefore we obtain (3–26). As in the proof of Proposition 3.1, we verify only one of the expressions (3–22)–(3–25), for example, (3–23). The rest can be treated similarly. First, note that $a_1^{(1)} = a_1 = a_1 U_0(\frac{\kappa}{2}) - a_2\nu^{-1} U_{-1}(\frac{\kappa}{2})$, $a_2^{(1)} = -a_1 p + a_2 = -a_1\nu U_1(\frac{\kappa}{2}) + a_2 U_0(\frac{\kappa}{2})$ satisfy (3–23)

for $n=0$ and that $a_1^{(1)}>0>a_2^{(1)}$. Arguing by induction and using (3–20), we obtain

$$\begin{aligned} a_1^{(2n+3)} &= -a_1 U_{2n}\left(\frac{\kappa}{2}\right) + a_2\nu^{-1}U_{2n-1}\left(\frac{\kappa}{2}\right) \\ &\quad + \left(a_1\nu U_{2n+1}\left(\frac{\kappa}{2}\right) - a_2 U_{2n}\left(\frac{\kappa}{2}\right)\right)q \\ &\overset{(3.9)}{=} a_1 U_{2n+2}\left(\frac{\kappa}{2}\right) - a_2\nu^{-1}U_{2n+1}\left(\frac{\kappa}{2}\right) > 0, \\ a_2^{(2n+3)} &= a_1\nu U_{2n+1}\left(\frac{\kappa}{2}\right) - a_2 U_{2n}\left(\frac{\kappa}{2}\right) \\ &\quad - \left(a_1 U_{2n+2}\left(\frac{\kappa}{2}\right) - a_2\nu^{-1}U_{2n+1}\left(\frac{\kappa}{2}\right)\right)p \\ &\overset{(3.9)}{=} -a_1\nu U_{2n+3}\left(\frac{\kappa}{2}\right) + a_2 U_{2n+2}\left(\frac{\kappa}{2}\right) < 0, \end{aligned}$$

and the claim follows. Note that here we used the right inequality in (3–27). It plays the same role in the proof of (3–22), while for (3–24), (3–25) the left inequality in (3–27) is needed. □

Next we consider Case 2.

Proposition 3.3. *If there exists $N>0$ such that*

$$\nu\frac{U_{N+1}\left(\frac{\kappa}{2}\right)}{U_N\left(\frac{\kappa}{2}\right)} \le \frac{a_2}{a_1} < \nu\frac{U_N\left(\frac{\kappa}{2}\right)}{U_{N-1}\left(\frac{\kappa}{2}\right)}, \tag{3–28}$$

then equations (3–24), (3–25) remain valid for $n\ge 0$, and equations (3–22), (3–23) remain valid for $a_1^{(k)}, a_2^{(k)} (k=0,1\ldots,N)$. Furthermore, $a_1^{(N+1)}$ and $a_2^{(N+1)}$ are nonnegative and not both zero and

$$\sigma\left(a_1^{(0)}, a_2^{(0)}\right) = \begin{cases} \left(\sigma_{++}^{(n-N-1)}\right)_{n\in\mathbb{Z}} & \text{for odd } N \\ \left(\tau\left(\sigma_{++}^{(n-N-1)}\right)\right)_{n\in\mathbb{Z}} & \text{for even } N, \end{cases} \tag{3–29}$$

where τ is defined in (3–21).

Proof. In the proof of Proposition 3.2, our argument relied on the inequalities (3–27). Under the current assumptions, the left inequality remains valid for all n, while the right inequality is valid for $n<N$. This explains the claim about the validity of (3–22)–(3–25) in this situation.

Next, if N is odd, $N=2m+1$, then (3–22), (3–23) are valid for $n=m$ with $a_1^{(2m)}<0<a_2^{(2m)}$ and $a_1^{(2m+1)}>0>a_2^{(2m+1)}$. This means that the expressions in (3–22) are also valid for $n=m+1$, however, due to (3–28), $a_1^{(2m+2)}\ge 0$ and $a_2^{(2m+2)}>0$. This puts us in the situation covered by Proposition 3.1 and the first line in (3–29) follows. The case of N even is treated in the same way. □

Finally, we consider Case 3.

Proposition 3.4. *If there exists $N>0$ such that*

$$\nu\frac{U_{N-1}\left(\frac{\kappa}{2}\right)}{U_N\left(\frac{\kappa}{2}\right)} \le \frac{a_2}{a_1} < \nu\frac{U_N\left(\frac{\kappa}{2}\right)}{U_{N+1}\left(\frac{\kappa}{2}\right)},$$

then equations (3–22), (3–23) remain valid for $n\ge 0$, and equations (3–24), (3–25) remain valid for $a_1^{(k)}, a_2^{(k)} (k=0,-1\ldots,-N)$. Furthermore, $a_1^{(-N-1)}$ and $a_2^{(-N-1)}$ are nonpositive and not both zero, and

$$\sigma\left(a_1^{(0)}, a_2^{(0)}\right) = \begin{cases} \left(\sigma_{++}^{(n+N+3)}\right)_{n\in\mathbb{Z}} & \text{for odd } N \\ \left(\tau\left(\sigma_{++}^{(n+N+3)}\right)\right)_{n\in\mathbb{Z}} & \text{for even } N, \end{cases}$$

where τ is defined in (3–21).

Proof. The proof is completely analogous to that of a previous proposition. □

Combining Propositions 3.1–3.4, we arrive at the following conclusion.

Theorem 3.5. *Conjecture 2.3 is valid in the rank 2 case.*

Proof. One can see that each of the sign patterns $\sigma_{++}, \sigma_{--}, \sigma_{+-}$, as well as every sign pattern that appears in Propositions 3.3, 3.4, differs from σ_{reg} in (3–8) for exactly three consecutive components. □

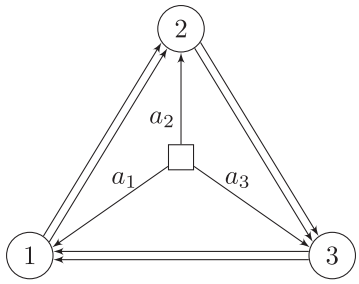
4. Rank 3 example: The Markov quiver

To present additional evidence in support of Conjecture 2.3, we consider the rank 3 case with B being the adjacency matrix of the celebrated *Markov quiver*. The corresponding cluster algebra served as a test case for several important phenomena in the theory of cluster algebras. In particular, the principal c-vectors and g-vectors associated with the Markov quiver were described in [Najera Chavez 12].

We add an additional frozen vertex to the Markov quiver and investigate possible sign patterns, for example, for a monotone and balanced sequence

$$\mu = (1,2,3,1,2,3,\ldots).$$

The figure below illustrates the case when the initial vector $\mathbf{a}^{(0)}=(a_1,a_2,a_3)$ is component wise nonnegative. In this figure, we assume that there are a_i arrows pointing from the frozen vertex to the vertex i.



Since the sequence μ consists in a consecutive mutations applied to the quiver in a clockwise order starting with the vertex 1 and since, as is well-known, the Markov quiver transforms into its opposite after a mutation at any vertex, we can replace μ with iterations of *the same* transformation ρ that consists of a quiver mutation at the vertex 1, followed by rotation of the quiver counter clockwise by 120 degrees (equivalently, applying a cyclic permutation $\tau = (132)$ to the mutable vertices) and then reversing all arrows. Using (2–1), we compute the result of ρ acting on the initial nonzero vector $\mathbf{a}^{(0)} = (a_1^{(0)}, a_2^{(0)}, a_3^{(0)}) \in \mathbb{Z}^3$:

$$\begin{aligned}\rho(\mathbf{a}^{(0)}) &= \left(\rho(\mathbf{a}^{(0)})_1, \rho(\mathbf{a}^{(0)})_2, \rho(\mathbf{a}^{(0)})_3\right)\\ &= \left(-2\left[a_1^{(0)}\right]_+ - a_2^{(0)}, 2\left[-a_1^{(0)}\right]_+ - a_3^{(0)}, a_1^{(0)}\right).\end{aligned} \tag{4–1}$$

For a (not necessarily strict) sign vector $\tilde{\omega} = (\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3)$ and a strict sign vector $\omega = (\omega_1, \omega_1, \omega_3)$, we write $\tilde{\omega} \approx \omega$ if $(\tilde{\omega}_i, \omega_i) \neq (+,-), (-,+)$ for any i.

Observe that if $\omega^{(0)} = (-,+,-)$ then $\omega^{(n)} = \omega^{(0)}$ and so $\omega^{(n)}$ is stable. More generally, if $\omega^{(0)} \approx (-,+,-)$, i.e., $\mathbf{a}^{(0)} = (-a_1, a_2, -a_3)$, where a_1, a_2, a_3 are nonnegative integers and not all zero, then $\omega^{(n)} \approx (-,+,-)$ as well and, moreover, $\omega^{(n)} = (-,+,-)$ for $n \geq 5$. This follows from a straightforward computation that gives

$$\begin{aligned}\rho^5(\mathbf{a}^{(0)}) = (&-4a_1 - 4a_2 - a_3, 9a_1 + 4a_2 + 4a_3,\\ &-4a_1 - a_2 - 2a_3).\end{aligned}$$

Thus, to establish Conjecture 2.3 for $\mu = (1,2,3,1,2,3,\ldots)$, it suffices to show that $\omega^{(n)} = \omega^{(n)}(\mathbf{a}^{(0)})$ eventually stabilizes at $(-,+,-)$ for any $\mathbf{a}^{(0)}$.

Denote by $\omega^{(n)} = \omega^{(n)}(\mathbf{a}^{(0)})$ the sign vector of $\rho^n(\mathbf{a}^{(0)})$ defined similarly to (2–3). For the rest of this section, we fix a_1, a_2, a_3 to be nonnegative integers and not all zero. In what follows, the choice of $\omega^{(0)} \approx (\epsilon_1, \epsilon_2, \epsilon_3)$ signifies that $\mathbf{a}^{(0)} = (\epsilon_1 a_1, \epsilon_2 a_2, \epsilon_3 a_3)$.

Let us first consider the case when $\omega^{(0)} \approx (+,+,-)$, which leads to

$$\begin{aligned}\rho(\mathbf{a}^{(0)}) &= (-2a_1 - a_2, a_3, a_1), &\omega^{(1)} &\approx (-,+,+),\\ \rho^2(\mathbf{a}^{(0)}) &= (-a_3, 3a_1 + 2a_2, -2a_1 - a_2), &\omega^{(2)} &\approx (-,+,-),\end{aligned}$$

after which the desired sign stabilization occurs. Similarly, starting with $\omega^{(0)} \approx (-,-,-)$, we obtain

$$\rho(\mathbf{a}^{(0)}) = (a_2, 2a_1 + a_3, -a_1), \quad \omega^{(1)} \approx (+,+,-),$$

which reduces to the case we just considered, as does the situation depicted in the figure above that corresponds to $\omega^{(0)} \approx (+,+,+)$, which results in

$$\begin{aligned}\rho(\mathbf{a}^{(0)}) &= (-2a_1 - a_2, -a_3, a_1), &\omega^{(1)} &\approx (-,-,+),\\ \rho^2(\mathbf{a}^{(0)}) &= (a_3, 3a_1 + 2a_2, -2a_1 - a_2), &\omega^{(2)} &\approx (+,+,-).\end{aligned}$$

For $\omega^{(0)} \approx (-,+,+)$, we get $\rho(\mathbf{a}^{(0)}) = (-a_2, 2a_1 - a_3, -a_1)$ and so $\omega^{(1)} \approx (-,+,-)$ or $(-,-,-)$, depending on the sign of $2a_1 - a_3$. Both of these cases were already covered above.

We are left with three remaining choices for $\omega^{(0)}$: $(+,-,-/0), (-/0,-,+)$ and $(+,-,+)$. The first of these, depending on the sign of $-2a_1 + a_2$, results in $\omega^{(1)} = (+,+,+)$ or in $\omega^{(1)} = (-/0,+,+)$—both situations have been already treated above.

The case $\omega^{(0)} = (-/0,-,+)$, depending on the sign of $2a_1 - a_3$, leads to $\omega^{(1)} = (+,+,-)$ or in $\omega^{(1)} = (+,-/0,-)$, also covered by now.

The last case requires a more delicate analysis. Indeed, if $\mathbf{a}^{(0)} = (a_1, -a_2, a_3)$ and if $a_2 > 2a_1$, then the sign pattern for $\rho(\mathbf{a}^{(0)}) = (-2a_1 + a_2, -a_3, a_1)$ is also $(+,-,+)$. Persistence of such situation would contradict our claim that $\omega^{(n)}$ stabilizes at $(-,+,-)$.

Lemma 4.1. *Suppose a_1, a_2, a_3 are positive numbers such that for $\mathbf{a}^{(0)} = (a_1, -a_2, a_3), \omega^{(n)} = (+,-,+)$ for all n. Then the first component of $\rho^n(\mathbf{a}^{(0)})$ is*

$$\begin{aligned}\rho^n(\mathbf{a}^{(0)})_1 = (-1)^n((&f_{n+3} - 1)a_1 - (f_{n+2} - 1)a_2\\ &+ (f_{n+1} - 1)a_3),\end{aligned} \tag{4–2}$$

where f_n denotes the nth Fibonacci's number.

Proof. The claim (4–2) is checked directly for $n = 0$, 1, 2, where

$$\begin{aligned}(\mathbf{a}^{(0)})_1 = a_1, \rho(\mathbf{a}^{(0)})_1 &= -2a_1 + a_2, \rho^2(\mathbf{a}^{(0)})_1\\ &= 4a_1 - 2a_2 + a_3\end{aligned}$$

and $f_1 = f_2 = 1, f_3 = 2, f_4 = 3, f_5 = 5$.

Under the assumptions of the lemma, (4–1) implies

$$\rho^{n+1}(\mathbf{a}^{(0)})_2 = -\rho^n(\mathbf{a}^{(0)})_3 = -\rho^{n-1}(\mathbf{a}^{(0)})_1$$

and then

$$\rho^{n+2}(\mathbf{a}^{(0)})_1 = -2\rho^{n+1}(\mathbf{a}^{(0)})_1 + \rho^{n-1}(\mathbf{a}^{(0)})_1.$$

The claim follows by induction from a relation $2f_{n+2} - f_n = f_{n+3}$ which is an easy consequence of the Fibonacci recursion. □

As a corollary of Lemma 4.1, we conclude that in order for a sign pattern $(+,-,+)$ to persists, the constants a_1, a_2, a_3 must satisfy inequalities

$$(f_{2n+1}-1)a_1 - (f_{2n}-1)a_2 + (f_{2n-1}-1)a_3 > 0 > (f_{2n+2}-1)a_1 - (f_{2n+1}-1)a_2 + (f_{2n}-1)a_3 \tag{4–3}$$

for all $n > 0$. Since $\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \varphi$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the *golden ratio*, (4–3) implies that $a_1\varphi^2 - a_2\varphi + a_3 = \frac{1}{2}((3a_1 - a_2 + 2a_3) + (a_1 - a_2)\sqrt{5}) = 0$. This is where the integrality of a_1, a_2, a_3 comes into play, since for the last equation to hold, we must have $a_1 = a_2 = -a_3$, which contradicts our positivity assumption for a_1, a_2, a_3. The conclusion is that if $\omega^{(0)} = (+,-,+)$ then there exists such n that $\omega^{(n)} = (-/0,-,+)$. This concludes the proof of Conjecture 2.3 for the Markov quiver and $\mu = (1,2,3,1,2,3,\ldots)$.

Remark 4.2. The condition of a sequence of mutations μ being at least weakly balanced is necessary for Conjecture 2.3 to hold true in the case of the Markov quiver. Indeed, let $\mu = (1,2,1,2,\ldots.)$ and $\mathbf{a}^{(0)} = (1,-1,a)$, where a is any integer. Then it is easy to check that $\mathbf{a}^{(n)} = ((-1)^n, (-1)^{n+1}, a)$, $\sigma^{(n)} = ((-1)^n, (-1)^{n+1}, \mathrm{sign}(a))$ and the claim in Conjecture 2.3 fails.

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