Poisson percolation on the oriented square lattice

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Abstract

In Poisson percolation each edge becomes open after an independent exponentially distributed time with rate that decreases in the distance from the origin. As a sequel to our work on the square lattice, we describe the limiting shape of the component containing the origin in the oriented case. We show that the density of occupied sites at height y in the cluster is close to the percolation probability in the corresponding homogeneous percolation process, and we study the fluctuations of the boundary.

1 Introduction

Percolation was introduced by Broadbent and Hammersley a little over 60 years ago to model a porous medium [3]. The model goes by including each edge of the integer lattice $\mathbb{Z}^d$ independently with probability p. One of the most fundamental questions is whether the subgraph contains an infinite component. There is known to be a critical value $p_c(d)$ such that for $p > p_c(d)$ such a component exists almost surely. A vast amount of literature is devoted to understanding the geometry of this component for different values of p. See Grimmett's book [11] for a thorough introduction or the article by Beffara and Sidoravicius [1] for a briefer overview.

The subgraph obtained via homogeneous percolation is static. In [4] we introduced Poisson percolation, which has a stochastically growing set of open edges. This could potentially model a medium that becomes more porous over time. Each edge in the unoriented square lattice $\mathbb{Z}^2$ with midpoint $x \in \mathbb{Z}^2$ becomes open at rate $\|x\|_\infty^{-\beta}$. Thus, the probability an edge is open at time t is equal to $\rho(x,t) = 1 - \exp(-t\|x\|^{-\beta})$. We studied three aspects of the structure of $\mathbb{C}_0$, the connected component containing 0. See Figure 1 for a simulation.

Size and shape of the cluster. For fixed t, the probability an edge beyond distance $N = t^{1/\beta}(\log 2)^{-1/\beta}$ is open is smaller than $p_c(2) = 1/2$. Accordingly, we show [4, Theorem 1] that with high probability $\mathbb{C}_0 \subseteq (1+\epsilon)[-N,N]^2$ for all $\epsilon > 0$.

Cluster density. Fix $1/2 < a < 1$ and tile $(1-\epsilon)[-N,N]^2$ with boxes $R_{i,j}$ with side-length N^a. In [4, Theorem 2] we proved that with high probability the density in each box $|\mathbb{C}_0 \cap R_{i,j}|/N^{2a}$ is close to the density of the giant component in homogeneous percolation with parameter $\rho(x_{i,j}, t)$.

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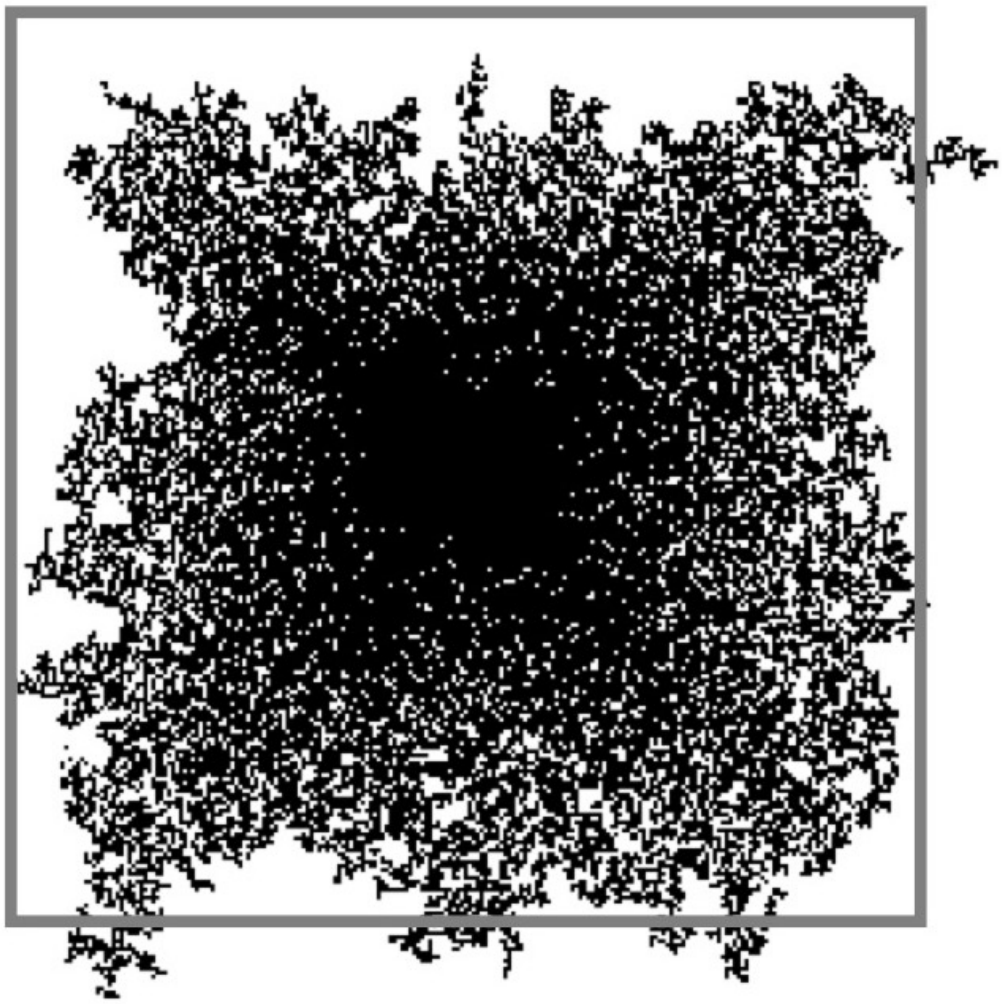

Figure 1: $\mathbb{C}_0$ on the unoriented square lattice for $\beta = 1$ and $t = 104$. The gray box has radius $N = 150$.

Fluctuations of the boundary. The fluctuations of $\mathbb{C}_0$ in the e_1 direction are defined as $|N - \max\{x\colon (x,0) \in \mathbb{C}_0(t)\}|$. Our understanding of this quantity comes from work of Nolin [15] on gradient percolation, in which the probability p that a bond (or site) is open decreases linearly from 1 to 0 as the height is increased from 0 to N. Rather than edge percolation on a square lattice, he considered site percolation on the triangular lattice in order to take advantage of the rigorous computation of critical exponents by Smirnov and Werner [18]. In this setting, he studied the shape of the boundary of the cluster containing the base of a trapezoidal region of length ℓ_N and height N. He found that the edge stays in a strip of width $N^{4/7+\delta}$ centered at $N/2$, and the length of the front is $\ell_N N^{(3/7)\pm\delta}$. These results are expected to hold on the square lattice. In our system, the change in the density is nonlinear but differentiable. Since the position of the front is dictated by bonds that are open with probabilities that have non-zero derivative close to p_c, the boundary behavior should be the same. See also the work of Nolin on the geometry of diffusion fronts [16].

In this article we study Poisson percolation on oriented lattice $\mathcal{L} = \{(m,n) \in \mathbb{Z}^2 \colon m + n \text{ is even}\}$ with oriented edges from $(m,n) \to (m+1, n+1)$ and $(m,n) \to (m-1, n+1)$. This is $\mathbb{Z}^2$ rotated 45°. We will again study the size and shape of the cluster, its density, and edge fluctuations. We think of m as space and of n as time. To avoid conflict with the parameter t, we think of it as giving the age of the cluter.

Fix $\beta > 0$. An edge with midpoint (x,y) and $y > 0$ is open with probability

$$\rho(y,t) = 1 - \exp(-ty^{-\beta}). \tag{1}$$

Let $n(p,t) = \max\{y \colon \rho(y,t) \geq p\}$ be the largest height at which edges are open with probability $\geq p$. A little algebra shows that

$$n(p,t) \sim c_{p,\beta} t^{1/\beta} \quad \text{where} \quad c_{p,\beta} = (-\log(1-p))^{-1/\beta}. \tag{2}$$

We write $(x,m) \to (y,n)$ if there is a path of open edges from (x,m) to (y,n). Let

$$\mathbb{C}_0(t) = \{(x,n) \colon (0,0) \to (x,n)\},$$

and let

$$f(y) = \rho(yt^{1/\beta}, t) = 1 - \exp(-y^{-\beta}). \tag{3}$$

Define y_c by $f(y_c) = p_c$, where $p_c \approx .64470019$ (see page 5242 of [13]) is the critical value $\inf\{p \colon P(|\mathcal{C}_0| = \infty\}$ where $\mathcal{C}_0$ is the cluster containing the origin. Note that $\mathcal{C}_0$ is the cluster containing 0 in homogeneous oriented percolation, whereas $\mathbb{C}_0$ is for Poisson percolation. The oriented case has fewer symmetries so the shape is more interesting than a square (see Figure 2).

1.1 Size and shape of the cluster

To define the limiting shape of $\mathbb{C}_0$ we need to introduce the *right-edge speed* in homogeneous percolation. Consider homogeneous percolation with parameter p on oriented $\mathcal{L}$. Following [5], we let

$$r_k = \sup\{x \colon \exists y \leq 0 \text{ with } (y,0) \to (x,k)\}$$

be the right most site at height k that can be reached from a site in $(-\infty, 0] \times \{0\}$. The subadditive ergodic theorem guarantees the existence of a limiting speed $r_k/k \to \alpha(p)$ for $p \geq p_c$. Obviously $\alpha(1) = 1$. It is known that $\alpha(p_c) = 0$. When $p < p_c$ the system dies out exponentially fast so $\alpha(p) = -\infty$.

Letting $g(0) = 0$ and $g'(y) = \alpha(f(y))$ for $0 \leq y \leq y_c$ we define our limiting shape

$$\Gamma = \{(x,y) \colon |x| < g(y), 0 \leq y \leq y_c\} \subseteq \mathbb{R}^2.$$

Intuitively the shape result is

$$\mathbb{C}_0(t)/t^{1/\beta} \to \Gamma.$$

To prove the result it is convenient to work on the unscaled lattice. Let $r_t^0(k) = \max\{x \colon (x,k) \in \mathbb{C}_0(t)\}$ be the right edge of $\mathbb{C}_0(t)$ at height k, and let $\ell_t^0(k) = \min\{x \colon (x,k) \in \mathbb{C}_0(t)\}$ be the left edge at height k. It is convenient to have g defined for all $y > 0$ so we let $g(y) = g(y_c)$ for $y > y_c$. Let

$$\Gamma_t(y) = t^{1/\beta} g(y/t^{1/\beta}).$$

Throughout the paper we will let $N = n(p_c, t)$.

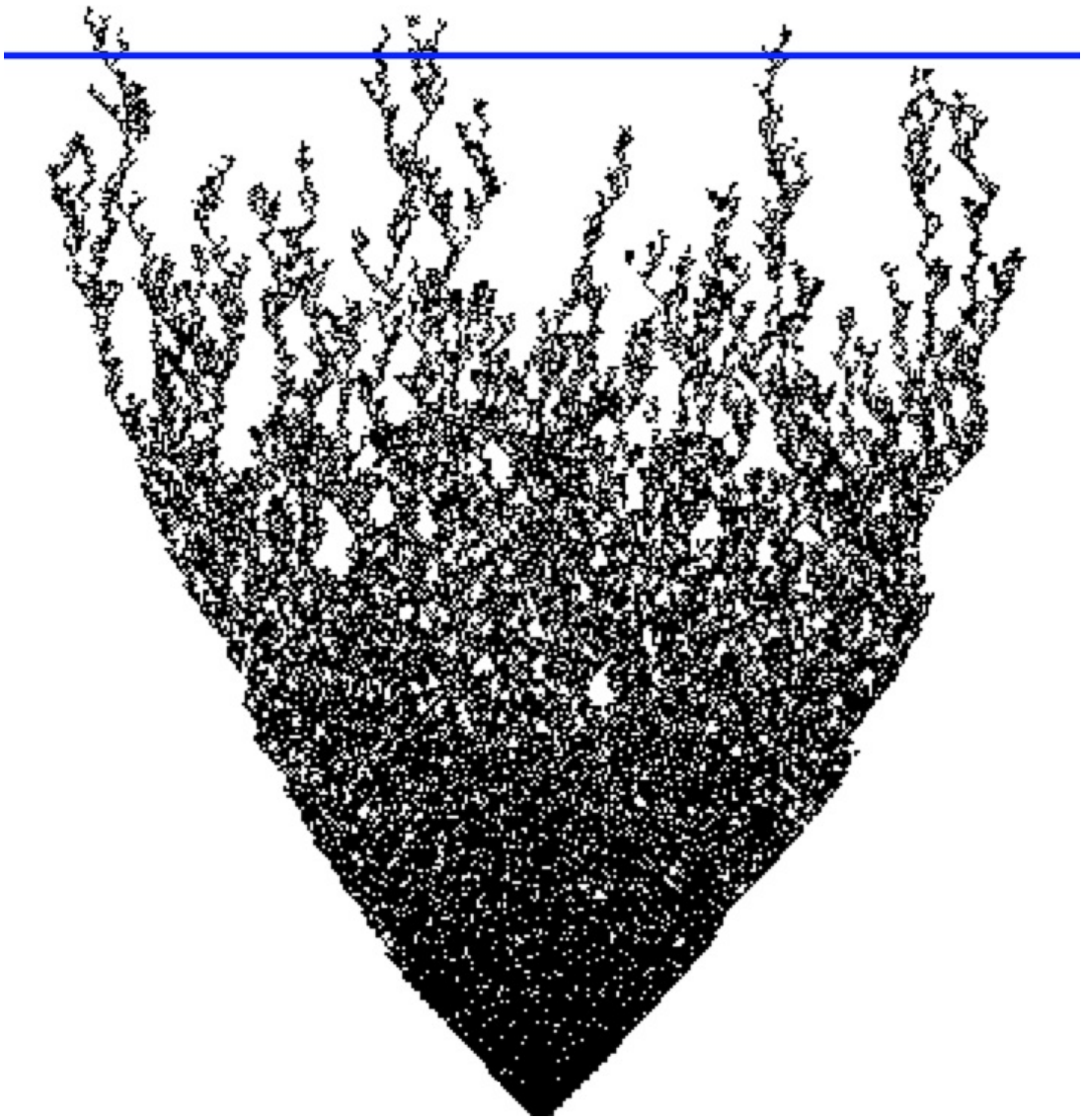

Figure 2: $\mathbb{C}_0$ on the oriented lattice with $\beta = 1/2$ and $t = 30$. The blue line is at height $N \approx 839$.

Theorem 1. *For any $\eta > 0$, as $t \to \infty$,*

(i) $P(\mathbb{C}_0(t) \subset \mathbb{R} \times [0, (1+\eta)N]) \to 1$, *and*

(ii) $P\Big(-(1+\eta)\Gamma_t(k) \le \ell_t^0(k),\ r_t^0(k) \le (1+\eta)\Gamma_t(k)$ *for all* $k \le (1+\eta)N\ \Big) \to 1.$

Proving Theorem 1 (i) is easy because our percolation process is subcritical when $y > N$. To prove Theorem 1 (ii) we fix m (it does not grow with t) and decompose $\mathbb{Z} \times [0, (1+\eta)N]$ into m strips, $\mathbb{Z} \times [z_i, z_{i+1})$, so that $\alpha_i = \alpha(\rho(z_i, t)) = 1 - i/m$ for $i < m$. We dominate the process in each strip by using homogeneous percolation with probability $p_i = \rho(z_i, t)$. Large deviation estimates on the distance of the right edge from α from [6] allow us to prove that $\mathbb{C}_0$ lies to the left of a piecewise linear function whose slope is α_i in each strip.

The next result gives a corresponding lower bound.

Theorem 2. *For any $\eta > 0$, as $t \to \infty$*

$$P(\ell_t^0(k) \le -(1-\eta)\Gamma_t(k) \text{ and } (1-\eta)\Gamma_t(k) \le r_t^0(k) \text{ for all } k \le (1-\eta)N\ ) \to 1.$$

Again we divide space into strips $\mathbb{Z} \times [z_i, z_{i+1})$, but now we lower bound the process by using homogeneous percolation with probability $p_{i+1} = \rho(z_{i+1}, t)$. In each strip we use a block construction to relate our process to a 1-dependent oriented bond percolation on $\mathbb{Z}^2$ with parameter $p = 1 - \epsilon$. On the renormalized lattice the right edge has speed close to 1. When we map the path of the right edge back to the Poisson percolation process we get a piecewise linear function that serves as a lower bound on the location of $r_t^0(k)$.

1.2 Cluster density

Let P_p be the probability measure for oriented bond percolation on $\mathcal{L}$, when edges are open with probability p. Let $\mathcal{C}_0$ be the open cluster containing the origin, and let $\theta(p) = P_p(|\mathcal{C}_0| = \infty)$. Let

$$G(t, \eta) = \cup_{y=0}^{(1-\eta)N} [-(1-\eta)\Gamma_t(y), (1-\eta)\Gamma_t(y)] \times \{y\}.$$

Intuitively, our next result says that near $(x, y) \in G(t, \eta)$ the density of points in $\mathbb{C}_0(t)$ will be close to $\theta(\rho(y, t))$. To state this precisely, fix $1/2 < a < 1$ and tile the plane with boxes of side length N^a:

$$R_{i,j} = [iN^a, (i+1)N^a] \times [jN^a, (j+1)N^a],$$

and let $(x_{i,j}, y_{i,j})$ be the center of $R_{i,j}$. Let $D_{i,j} = |\mathbb{C}_0(t) \cap R_{i,j}|/N^{2a}$ be the fraction of points in $R_{i,j}$ that belong to $\mathbb{C}_0(t)$ and let $\Lambda(t, \eta) = \{(i, j) \colon R_{i,j} \subset G(t, \eta)\}$ be the indices of boxes that fit inside $G(t, \eta)$.

Theorem 3. *For any $\eta, \delta > 0$, as $t \to \infty$,*

$$P\left(\sup_{(i,j) \in \Lambda(t,\eta)} |D_{i,j}(t) - \theta(\rho(y_{i,j}, t))| > \delta\right) \to 0.$$

1.3 Boundary fluctuations

The first three results were laws of large numbers. We will now consider the fluctuations of the right edge $r_t(k)$. When

$$\mathbb{C}_0^k(t) \equiv \mathbb{C}_0(t) \cap (\mathbb{Z} \times \{k\}) \neq \emptyset$$

we have $r_t^0(k) = r_t(k)$. $r_t(k)$ has the advantage that it is well defined even if $\mathbb{C}_0^k(t) = \emptyset$. In the homogeneous case, Galves and Presutti [10] were the first to prove such a central limit theorem for the supercritical contact process. Letting r_t be the rightmost occupied site at time t in the contact process with birth rate λ when initially all points $y \leq 0$ are occupied, they showed that

$$\frac{r_{ns} - \alpha(\lambda)ns}{\sigma(\lambda)\sqrt{n}} \Rightarrow B_s$$

Here B_s standard Brownian motion and $\Rightarrow$ is weak convergence of stochastic processes. Their proof also applies to oriented percolation. It implies that, if we start with the nonpositive integers occupied, then there is a constant $\sigma(p)$ so that for all $k > 0$ as $n \to \infty$

$$\frac{r_{[ns]} - \alpha(p)ns}{\sigma(p)\sqrt{n}} \Rightarrow B_s.$$

Two years later Kuczek [14] simplified the proof by introducing what he called break points: times T_i at which the right-most particle starts an oriented percolation that does not die out. In this case for $i \geq 1$, the increments $(r_{T_{i+1}} - r_{T_i}, T_{i+1} - T_i)$ are i.i.d. Using his method we prove the analogue for Poisson percolation.

Theorem 4. *As $t \to \infty$,*

$$\frac{r_t(\lceil Nu \rceil) - \int_0^{Nu} \alpha(p(y,t))dy}{N^{1/2}} \Rightarrow W_u,$$

where $W_u, 0 \leq u < 1$ is a Gaussian process with independent increments. It holds that $EW_u = 0$ and

$$EW_u^2 = \frac{1}{N} \int_0^{Nu} \sigma^2(p(y,t))\, dy.$$

Given the result for the homogeneous case this conclusion is what one would expect to hold; if we divide the space into a large number of thin strips we have a sequence of homogeneous oriented percolation processes

Very little is known rigorously about critical exponents for oriented percolation, so we are not able to prove mathemtically an analogue of Nolin's result. However, we can give a physicist style derivation of the following:

Conjecture 5. *Fluctuations in the height of $\mathbb{C}_0(t)$ are of order $N^{0.634}$.*

First, recall that oriented percolation has two correlation lengths. The correlation length in time, $L_\parallel$ can be defined simultaneously for the subcritical and supercritical cases by

$$\gamma_\parallel(p) = -\lim_{t\to\infty} \frac{1}{n} \log P_p(n \leq \tau^0 < \infty) \qquad L_\parallel(p) = 1/\gamma_\parallel(p),$$

where τ^0 is the extinction time of the process starting from only 0 occupied. The correlation length in space $L_\perp$ has two different definitions for $p < p_c$ and $p > p_c$. Let ξ_n^0 be the the set of occupied sites at time n, and define $R^0 = \sup\{x : (x, n) \in \xi_n^0 \text{ for some } n\}$. Also define

$$\gamma_\perp(p) = -\lim_{t\to\infty} \frac{1}{n} \log P_p(R^0 \geq n) \qquad L_\perp(p) = 1/\gamma_\perp(p)$$
$$\gamma_\perp(p) = -\lim_{t\to\infty} \frac{1}{n} \log P_p(\tau^{\{-n,\dots n\}} < \infty) \qquad L_\perp(p) = 1/\gamma_\perp(p).$$

where $\tau^{\{-n,\dots n\}}$ is the extinction time of the process starting from $\{-n, \dots n\}$ occupied. The last two limits and the one that defines $\gamma_\parallel$ when $p < p_c$ exist due to supermultiplicativity (i.e. $P(R^0 \geq m + n) \geq P(R^0 \geq m)P(R^0 > n)$). See [8] for more details, and some other definitions.

The corresponding critical exponents are defined by

$$L_\parallel(p) \approx |p - p_c|^{-\gamma_\parallel} \qquad L_\perp(p) \approx |p - p_c|^{-\gamma_\perp}.$$

Here $\approx$ could be something as precise as $\sim C|p - p_c|^{-\gamma}$ or $\log L(p)/\log|p - p_c| \to -\gamma$. Numerically, see [13, equation (15)]

$$\gamma_\parallel = 1.733847 \qquad 2\gamma_\perp = 2.193708.$$

Nolin gives the following "hand-waving" argument for his result [15, page 1756]. If we are at distance N^b behind the front then $p - p_c = O(N^{b-1})$ and the correlation length is

$$|p - p_c|^{-\nu_\parallel} = O(N^{(1-b)\nu_\parallel})$$

if $b = (1-b)\nu_\parallel$, i.e., $b = \nu_\parallel/(1+\nu_\parallel)$, then the correlation length matches the distance behind the front. In this case the physical interpretation of the correlation length implies that the percolation process will look like the critical case. Nolin's proof of the localization of the front, see [15, Theorem 6], is not long, but it is based on properties of sponge crossing, which will be difficult to generalize to the oriented case. However, there has been some recent work in that direction by Duminil-Copin et. al. [9], as well as Sakai [17].

2 Oriented percolation toolbox

Here we state additional definitions and facts that we will need in the proofs of our theorems. The first is a simple observation that percolation is monotonic in the parameter.

Fact 1. *Let $G_p \subseteq \updownarrow$ be the random subgraph obtained in homogeneous oriented percolation with parameter p. There exists a coupling such that if $p < p'$, then $G_p \subseteq G_{p'}$.*

This follows by coupling the Bernoulli random variables on each each edge in the canonical way. A similar statement holds in Poisson percolation.

Fact 2. *Let $G(t)$ be the set of all open edges at time t in Poisson percolation. Fix a subset of edges H in $\mathcal{L}$ and let*

$$p^- = \min\{\rho(\mathbf{x}, t) \colon \mathbf{x} \in H)\}, \quad p^+ = \max\{\rho(\mathbf{x}, t) \colon \mathbf{x} \in H\}.$$

There exists a coupling such that $G_{p^-} \cap H \subseteq G(t) \cap H \subseteq G_{p^+} \cap H$.

The estimate in [6, (1) Section 7] bounds the probability that there is a path from 0 to $\mathbb{Z} \times \{k\}$.

Fact 3. *For any $\delta > 0$. there is a constant $\gamma = \gamma(\delta) > 0$ so that*

$$P_{p_c-\delta}(\xi_n^0 \neq \emptyset) \leq e^{-\gamma n}.$$

We are also interested in the speed of the rightmost particle in supercritical homogeneous percolation where we assume all edges in $(-\infty, 0] \times \{0\}$ are open. Recall that $\alpha(p) = \lim_{k\to\infty} r_k/k$ is the limiting speed. [6, (3) Section 11 and Section 14] gives that

Fact 4. *$\alpha(\cdot)$ is a continuous, strictly increasing function of p for p in $[p_c, 1)$. Moreover, for any $\delta > 0$, the function $\theta(p) = P_p(|\mathcal{C}_0| = \infty)$ is Lipschitz continuous on $[p_c + \delta, 1]$.*

[6, (2) Section 7] also gives the following estimate.

Fact 5. *If $p > p_c$ and $\beta > \alpha(p)$, then there are constants $0 < \gamma, C < \infty$ that depend on p, and β so that*

$$P_p(r_n > \beta n) \leq Ce^{-\gamma n}.$$

Results in [6, Section 12] imply that

Fact 6. *If $p \neq p_c$ there exists $\gamma > 0, C < \infty$ such that $P(n \leq \tau^0 < \infty) \leq Ce^{-\gamma n}$.*

We will make use of the *dual process* to oriented homogeneous percolation when proving Theorem 2. This is the process obtained by keeping the same edges open, but reversing the orientation so that edges point southwest and southeast. Note that the dual process has the same law as usual oriented percolation. Thus, Fact 6 also holds for the survival time of a component started at w in the dual percolation.

Supercritical percolation almost surely contains an infinite component. Translation invariance of the lattice ensures that an individual edge has probability $\theta(p)$ of belonging to this component. Let τ^H denote the length of the longest surviving path started from a site in H. This is proven in [6, Section 10].

Fact 7. *There exists $0 < \gamma, C < \infty$ that depend on p such that for any $A \subseteq \mathbb{Z}^2$ it holds that*

$$P(\tau^A < \infty) \leq Ce^{-\gamma|A|}.$$

Some of our proofs involve comparison with *one-dependent oriented percolation.* One-dependence means that the values on edges that share a common vertex are correlated, but edges without a common vertex are independent. This type of percolation is analyzed in [6]. Consider one-dependent oriented percolation in which the marginal distribution for each edge is such that it is open with probability at least $1 - \epsilon$. Let $\mathcal{C} = \{w\colon \text{ for some } x \leq 0, (x, 0) \to w\}$, and let $s_k = \sup\{x\colon (x, k) \in \mathcal{C}\}$. According to [6, Theorem 2; Section 11],

Fact 8. *If $0 < q < 1$ and $\epsilon < 3^{-36/(1-q)}$, then there are constants $0 < \gamma, C < \infty$ so that*

$$P(s_k \leq qk) \leq Ce^{-\gamma k}.$$

3 Proof of the Theorem 1

We start by proving (i). Let $\delta > 0$ be small. For $i = 1, 2$, let $y_i = n(p_c - i\delta, t)$. On (y_1, ∞) we use Fact 1 to dominate Poisson percolation by homogeneous percolation in which bonds are open with probability $p_c - \delta$. We have $y_i \sim c_i t^{1/\alpha}$ for constants $c_1 < c_2$. Let $k = y_2 - y_1$. Note that at height y_1, all the x-coordinates of points of $\mathbb{C}_0(t)$ are in $[-y_1, y_1]$. It follows from Fact 3 that for large t

$$P(\mathbb{C}_0(t) \cap (\mathbb{Z} \times \{y_2\}) \neq \emptyset) \leq 2c_1 t^{1/\alpha} \exp(-\gamma(c_2 - c_1)t^{1/\alpha}) \to 0. \tag{4}$$

If δ is small, then $y_2 < (1 + \eta)N$ and we have the desired upper bound on the height.

Theorem 1 (ii) follows from the following two lemmas. We subdivide time by introducing probabilities p_i, $1 \leq i \leq m - 1$ so that $\alpha(p_i) = 1 - i/m$, and let $p_0 = 1$, $p_m = p_c - 2\delta$. Note that these values are well defined by Fact 4. We will choose the value of m appropriately for η in just moment. Let $z_0 = 0$ and $z_i = n(p_i, t)$ for $i \geq 1$. The last interval $(z_{m-1}, z_m]$ is longer so that $z_m = y_2$. When $z_i < y \leq z_{i+1}$, we use Fact 2 to bound our system from above by oriented percolation with probability p_i, which has edge speed $= 1 - i/m$.

We define sequences u_i, v_i for $0 \leq i \leq m - 1$ inductively by $u_0 = \delta$

$$v_i = u_i + (1 - i/m)(z_{i+1} - z_i), \qquad u_{i+1} = v_i + \delta.$$

Now define a function $h_t(x)$ to be linear on $[z_i, z_{i+1})$, with $h_t(z_i) = u_i$ and

$$\lim_{y \uparrow z_{i+1}} h_t(y) := h_t(z_{i+1}-) = v_i.$$

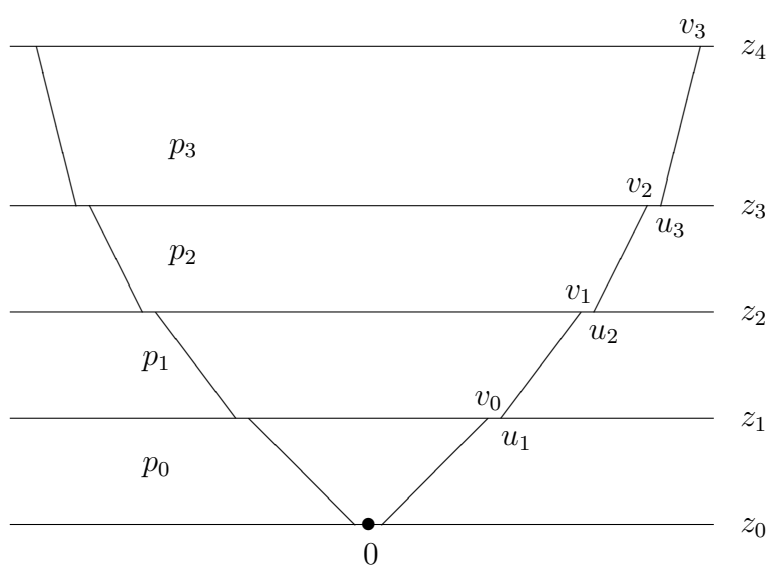

Figure 3: Region defined by $h_t(k)$ when $m = 4$. Notice that the slopes of the segments (u_i, v_i) are 1, 4/3, 2, and 4, i.e., 1 over the maximum edge speed in the interval.

Lemma 6. *As $t \to \infty$, $P(r_k^0(t) \le h_t(k)$ for all $k \le z_m) \to 1$.*

Proof. Let $1 \le i < m$. Suppose that $r_{z_i}^0(t) \le v_{i-1}$. To prove the result it is enough to show that as $t \to \infty$

$$P(r_k(t) \le h_t(k) \text{ for } z_i \le k < z_{i+1}\ ) \to 1. \tag{5}$$

When $i = 0$, the dominating process has $p_0 = 1$ so

$$P(r_k^0 \le h_t(k) \text{ for } z_0 \le k < z_1\ ) = 1.$$

Now suppose $i > 0$. When $k \in [z_i, z_i + u_i - v_{i-1})$, it is impossible for the process to reach $h_t(k)$ since the x-coordinate of the right-most particle can increase by at most 1 on each step. In order to get from v_{i-1} to v_i in time $z_{i+1} - z_i$, the right edge would have to travel at an average speed of more than $1 - (i-1)/m$. Using Fact 5, and summing over $k \in [z_i + u_i - v_{i-1}, z_{i+1}]$ proves (5). □

Lemma 7. *Let $\eta > 0$. If we take m large enough and δ small then $h_t(y) \le (1+\eta)\Gamma_t(y)$ for all $y \in [0, (1+\eta)N]$.*

Proof. We begin by noting that Fact 4 implies that $\alpha(f(z))$ is decreasing while $f(z) > p_c$. If m is large enough then $\alpha(p_i) - \alpha(p_{i-1}) < \eta/2$ for $1 \le i < m$ and $\alpha(p_{m-1}) = 1/m < \eta/2$. To prove the result now note that if $i < m$ then

$$\Gamma_t(z_i) - \Gamma_t(z_{i-1}) = \int_{z_{i-1}}^{z_i} \alpha(f(y))\, dy$$
$$h(z_i-) - h(z_{i-1}) = \alpha(f(z_{i-1}))(z_i - z_{i-1}).$$

So, by the choices we have made above,

$$h(z_i-) - h(z_{i-1}) < (1+\eta/2)(\Gamma_t(z_i) - \Gamma_t(z_{i-1})).$$

Now, if δ is small enough $h(y) < (1+\eta/2)\Gamma_t(y)$ for $y < z_{m-1}$. On $[z_{m-1}, z_m]$,

$$h(y) - h(z_{m-1}) < (\eta/2)(y - z_{m-1}).$$

If δ is small enough we have $h(y) < (1+\eta)\Gamma_t(y)$ for $y < (1+\eta)N$. □

4 Proof of Theorem 2

To get the cluster at 0 started, we observe that it with high probability contains all possible sites within distance $t^{b/\beta}$ with $0 < b < 1$.

Lemma 8. *Let $\mathcal{K}(n) = \{(x,y)\colon 0 \le y \le n, \text{ and } |x| \le y\}$. For any $0 < b < 1$, as $t \to \infty$*

$$P(\mathcal{K}(t^{b/\beta}) \subseteq \mathbb{C}_0) \to 1.$$

Proof. By (1), each edge in $\mathcal{K}(t^{b/\beta})$ is closed with probability $\le \exp(-t^{1-b})$. Since there are fewer than $t^{2b/\beta}$ edges, the result follows from a union bound. □

4.1 Constructing the renormalized lattice

The next step is renormalizing the lattice to compare Poisson percolation with 1-dependent oriented percolation with parameter $1-\epsilon$. As in the previous section, we introduce probabilities p_i, $1 \le i \le m-1$ so that $\alpha(p_i) = 1 - i/m$. We let $z_0 = t^{b/\beta}$ and for $1 \le i \le m-1$ let $z_i = n(p_i, t)$. The key ingredient for describing the density is to show that the rightmost edge of $\mathbb{C}_0$ stays to the right of $(1-\eta)\Gamma$. When $1 \le i \le m-1$ and $z_{i-1} < y \le z_i$, we bound our system from below by oriented percolation in which edges are open with probability p_i, and the edge speed is $\alpha_i = 1 - i/m$.

To lower bound the process in which each edge is open with probability p_i we will use a block construction. So that the lattices associated with different strips will fit together nicely, the x coordinates of the sites in the renormalized lattice will always be at integer multiples of some fixed constant L, and we will vary the heights of the blocks. In the ith strip $z_{i-1} < y \le z_i$, we let $A^i_{0,0}$ be the parallelogram with vertices

$$\begin{aligned} u_0 &= (-1.5\delta L, 0) \qquad & u_1 &= ((1+1.5\delta)L, (1+3\delta)L/\alpha_i) \\ v_0 &= (-0.5\delta L, 0) \qquad & v_1 &= ((1+2.5\delta)L, (1+3\delta L)/\alpha_i) \end{aligned}$$

and let $B^i_{0,0} = -A^i_{0,0}$,

To begin to define the renormalized lattice, we let $T_1 = z_0$. In the ith strip, the points in the renormalized lattice are

$$(c^i_m, d^i_n) = (mL, T_i + n(1+\delta)L/\alpha_i)$$

where m and n are integers so that $m+n$ is even, $n \ge 0$ and $T_i + n(1+3\delta)L/\alpha_i \le z_i$. The last condition is to guarantee that all the edges we consider in the ith part of the construction are open with probability at least p_i. Note that in each strip the vertical index n begins at 0.

To continue the construction when $i < m-1$ we let

$$T_{i+1} = \max\{T_i + n(1+\delta)L/\alpha_i : T_i + n(1+3\delta)L/\alpha_i \le z_i\}.$$

Let $A^i_{m,n} = (c^i_m, d^i_n) + A^i_{0,0}$, let $B^i_{m,n} = (c^i_m, d^i_n) + B^i_{0,0}$ and let $I^i_m = c^i_m + (-0.5\delta L, 0.5\delta L)$. The parallelograms are designed so that (see Figure 4)

(i) at height $d^i_{n+1} = d^i_n + (1+\delta L)/\alpha$, $A^i_{m,n}$ fits in I^i_{m+1}.

(ii) at height $d^i_n + (1+3\delta L)/\alpha$ the x component of the left edge of $A^i_{m,n}$ is the same as that of the right edge of $B^i_{m+1,n+1}$.

We say that the good event $G^i_{0,0}$ occurs if

(I) in $A^i_{0,0}$ there is a path from the bottom edge to the top edge.

(II) in $B^i_{0,0}$ there is a path from the bottom edge to the top edge.

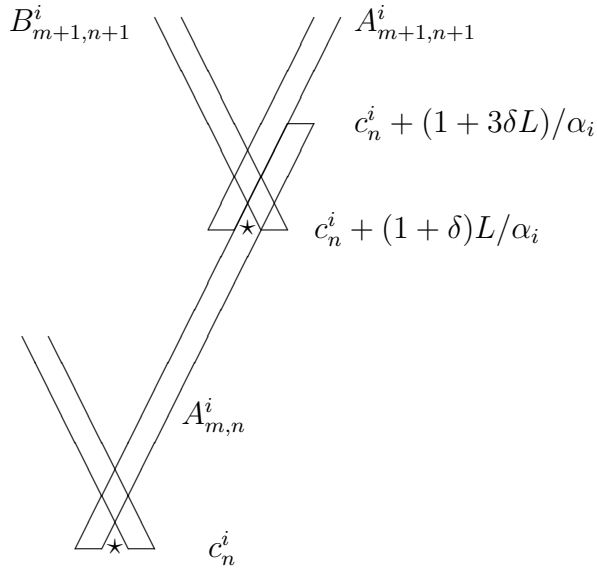

Figure 4: Picture of the block construction. Stars mark points of the renormalized lattice.

Note that the existence of the paths in (I) and (II) are determined by the edges in $A^i_{0,0}$ and $B^i_{0,0}$ respectively. The parallelograms are constructed to overlap in such a way (see Figure 4) that, if there is a path in $A^i_{m,n}$, and there are paths in $B^i_{m+1,n+1}$ and $A^i_{m+1,n+1}$, then there is a path from the bottom edge of $A^i_{m,n}$ to the top edges of $A^i_{m+1,n+1}$ and $B_{m+1,n+1}$.

We define $G^i_{m,n}$ by translation. In [6, Section 9] it was shown that, given $\epsilon > 0$, for $L \geq L_i$ it holds that $P(G^i_{0,0}) \geq 1 - \epsilon$. Let $\bar{L} = \max_{1\leq i\leq m-1} L_i$. Suppose $\delta < 0.01$, let $R^i_{0,0} = [-1.5L, 1.5L] \times [0, (1+3\delta)L/\alpha_i]$, and let

$$R^i_{m,n} = (c^i_m, c^i_n) + R^i_{0,0}.$$

The existence of paths in parallelograms that do not overlap is independent. The box $R^i_{0,0}$ intersects $R^i_{2,1}$, $R^i_{-2,1}$, $R^i_{-1,0}$, $R^i_{1,0}$, $R^i_{2,-1}$, and $R^i_{-2,-1}$, so the construction is one dependent (as described after Fact 7).

4.2 Lower bound for the right-most particle

To facilitate comparison with oriented percolation, we will renumber the rows of renormalized sites with $z_0 \leq y \leq z_{m-1}$ by the nonnegative integers $0, 1, 2, \ldots M$ and let $\tau_0, \tau_1, \ldots, \tau_M = \inf\{k\colon z_k \geq (1-\eta)N\}$ be the corresponding heights in Poisson percolation on the usual lattice. In our construction, we will pick L large and then let $t \to \infty$, so there are constants C_1 and C_2 so that $C_1 t^{1/\beta} \leq M \leq C_2 t^{1/\beta}$. Also, fix $0 < b < 1$ and let $K = K(t,b) = \max\{j\colon \tau_j \leq t^{b/\beta}\}$ be the last parallelogram below height $t^{b/\beta}$. Note also that $K \to \infty$ since L is fixed.

Consider 1-dependent oriented percolation in which edges are open with probability $1-\epsilon$. Fix two numbers $0 < q < q' < 1$. Define the set of edges $\mathcal{E}_K = [q'K, K] \times \{0\}$, and

$$s'_k = \max\{x\colon \text{ there exists } w \in \mathcal{E}_K \text{ with } w \to (x,k)\}$$

to be the rightmost edge at height k accessible from $\mathcal{E}_K$. By Lemma 8 we know that $\mathcal{E}_K$ will have all edges open with probability going to 1. Moreover, we claim that as $t \to \infty$,

$$P(s'_k \geq qk \text{ for all } k \geq 0) \to 1. \tag{6}$$

Fact 7 guarantees that the probability $\mathcal{E}_K$ contains a path to infinity goes to 1 as $t \to \infty$. Since a path can displace at most one unit to the left at each height, the first time we could have $s'_k < qk$ is at height $(q'-q)K/2$. Applying the bound from Fact 8 to the rightmost edge started from $\mathcal{E}_K$, we then have

$$P(s'_k \leq qk \text{ for some } k \geq 0) \leq \sum_{k=(q'-q)K/2}^{M} Ce^{-\gamma k} \to 0,$$

since $K \to \infty$.

To get a lower bound on the right-edge in the Poisson percolation process, we consider the mapping $(s'_k, k) \to (Ls'_k, \tau_k)$ from the renormalized lattice back to the original lattice. Because of (6), we consider the image of the line $y = qk$ under this map. It is given by a piecewise linear function with

- $h(0) = 0$, and $h(t) = qk$ for $k \in [0, z_0]$, and
- $h(k) = h(z_{i-1}) + q\alpha_i(k - z_{i-1})$ for $k \in [z_{i-1}, z_i]$ with $1 \leq i \leq m-1$.

The renormalized sites that make up the right edge will map to the right of this curve. The paths that connect them will lie in the associated parallelogram from Section 4.1, so they cannot go further than $(1+3\delta)L/\alpha_i$ to the left of h. It follows that

$$P(r_t(k) \geq h(k) - (1+3\delta)L/\alpha_i \text{ for all } z_{i-1} \leq t \leq z_i) \to 1.$$

On $[z_{i-1}, z_i]$, h has slope $q\alpha_i$ while Γ_t increases at rate $\leq \alpha_{i-1} = \alpha_i + 1/m$. If m is large enough then $\alpha_i \geq (1-\eta/2)\alpha_{i-1}$ for $1 \leq i \leq M$. It follows that if q is chosen close enough to 1 then $h(k) - (1+3\delta)L/\alpha_i \geq (1-\eta)\Gamma_t(k)$ for all $z_{i-1} \leq k \leq z_i$ and $1 \leq i \leq m-1$. The proof for the left edge is similar.

5 Proof of Theorem 3

Consider the site $w = (x, y)$ with $z_i \leq y < z_{i+1}$, so that it is in the ith strip of the unscaled lattice. Fact 6 implies

($\star$) if $n_i = (1/\gamma_i)\log(C_i N^4)$ and the dual process started from w survives for n_i units of time then the probability $w \notin \mathbb{C}_0$ is $\leq 1/N^4$.

This says that $\mathbb{C}_0$ is closely approximated by the points whose dual survives for time n_i. Let

$$R_{j,k} = [jN^a, (j+1)N^a] \times [kN^a, (k+1)N^a]$$

and suppose that all the points in $R_{j,k}$ are in the ith strip.

Let $A_w = \{\tau^w \geq k_i\}$, and count the number of points in $R_{j,k}$ with a long-surviving dual with

$$S_{j,k} = \sum_{w \in R_{j,k}} \mathbf{1}\{A_w\}.$$

Since $|R_{j,k}| = N^{2s}$, $(\star)$ ensures that

$$P(S_{j,k} \neq |\mathbb{C}_0 \cap R_{j,k}|) \leq N^{2a-4}.$$

Since there are no more than N^{2-2a} boxes with high probability this holds for all of them.

So, it suffices to study $S_{j,k}$. We start by centering it. Let $\theta_w = P(A_w)$, and define

$$\bar{S}_{j,k} = S_{j,k} - ES_{j,k} = \sum_{w \in R_{j,k}} \mathbf{1}\{A_w\} - \theta_w.$$

The advantage of considering A_w is that if $w = (x, y)$ the event A_w is determined by edges in $[x - n_i, x + n_i] \times [y - n_i, y]$ so if $\|w - w'\| > 2n_i$ the indicator random variables are independent. Using the bound

$$|\mathbf{1}\{A_w \cap A_{w'}\} - P(A_w)P(A_{w'})| \leq 1$$

when $\|w - w'\| \leq 2n_i$, we obtain

$$E\bar{S}_{j,k}^2 \leq N^{2a} \cdot 4n_i^2 \leq \frac{4}{\gamma_i} N^{2a} \log^2 C_i N^4. \tag{7}$$

Using (7) with Chebyshev's inequality gives for $\delta > 0$ and some $C_i' > 0$

$$P(|\bar{S}_{j,k}| > \delta N^{2a}) \leq \frac{C_i' N^{2a} \log^2 N}{\delta^2 N^{4a}} = O(N^{-2a} \log^2 N). \tag{8}$$

Since there are $O(N^{2-2a})$ many different boxes $R_{j,k}$, it follows from (8) that

$$P\left(\sup_{(j,k) \in \Lambda(\eta,\eta)} |\bar{S}_{j,k}| > \delta N^{2a}\right) = O(N^{2-4a} \log^2 N).$$

The right term is $o(1)$ since $a > 1/2$. To relate this back to $\mathbb{C}_0$ we note that $f(y)$ defined in (3) is Lipschitz continuous, and by Fact 4 so is $\theta(p)$ on $[p_c + \delta, 1]$. Thus,

$$\sup\{|\theta(\rho(w,t)) - \theta(\rho(w',t))| \colon w, w' \in R_{j,k}\} \leq CN^{a-1} \to 0.$$

Using this and Fact 6, we can replace the $P(A_w)$ terms in $S_{j,k}$ with a representative $\theta_{j,k} = \theta(\rho((x_j, y_k), t))$, and Theorem 3 follows.

6 Proof of Theorem 4

Recall $N = n(p_c, t)$. In our process, the right edge particle cannot be part of an infinite cluster, so we define renewals to be times at which the rightmost particle lives for time at least $\log^2 N$. This is motivated by the bound from Fact 6. To get started, if $b < 1$ then the state at time $t^{b/\alpha}$ is an interval and the rightmost particle survives for $\log^2 N$ with probability $\to 1$ by Lemma 8. Suppose t_i is the time of the ith renewal and let p_i be the probability bonds are open at that time. On $[t_i, t_i + 2\log^2 N]$ bonds are open with probability $\geq p_i - c(\log^2 N)/N$. The 2 is to allow us to find the renewal point and then verify it works. The bonds of interest are in a triangle with point at (r_i, T_i), sides with slope 1, and height $2\log^2 N$ so we can by Fact 2 couple the inhomogenenous system with a system with probabilities p_i so that with high probability there are no errors.

Unfortunately the increments in the right-edge defined in this way are not independent. If $r_i - r_{i-1}$ is large then the p for the next increment will be smaller. To fix this we will again divide $[0, N]$ into strips by choosing $\alpha(p_i) = 1 - i/m$ and $z_i = n(p_i, t)$ but now we will use $m = N^{0.6}$ strips. For renewals that begin in the strip $z_i \leq y < z_{i+1}$ we will upper bound the movement of the right edge by using $p = p_i$ and lower bound by using $p = p_{i+1}$. The large number of strips guarantees that the difference between the upper and lower bounds on $E(r_k - r_{k-1})$ will be $N^{-0.6}$ so when we sum N of these terms the result is $O(N^{0.4}) = o(N^{0.5})$

Kuczek [14] has shown that when p is fixed $r_i - r_{i-1}$ has an exponential tail, so using the Lindberg-Feller theorem, see e.g., Theorem 3.4.5 in [7], on the upper bound and on the lower bound

$$\frac{\sum_{k=1}^n (r_k - r_{k-1}) - E(r_k - r_{k-1})}{\sqrt{\sum_{k=1}^n \operatorname{var}(r_k - r_{k-1})}} \Rightarrow \chi \tag{9}$$

where χ is standard normal. To convert this to continuous time note that for homogeneous percolation

$$\begin{aligned} E(r_i - r_{i-1}) &= \alpha(p) E_p(t_i - t_{i-1}) \quad \text{because } Er(t)/t \to \alpha(p), \\ \operatorname{var}(r_i - r_{i-1}) &= \sigma^2(p) E_p(t_i - t_{i-1}) \quad \text{because } \operatorname{var} r(t)/t \to \sigma^2(p). \end{aligned}$$

Let $M(s)$ be the number of renewals needed to get to height s. Replacing n by $M(s)$ in (9) the result is

$$\frac{r_t(s) - \int_0^s \alpha(p(y,t))\, dy}{\int_0^s \sigma^2(p(y,t))\, dy} \Rightarrow \chi.$$

Taking $s = Nu$ and replacing the denominator by $\sqrt{N}$ we have convergence of the one dimensional distributions to the desired limit. Since the increments of the limit process are independent, convergence of finite dimensional distributions follows easily. Since

$$\sum_{k=1}^n (r_k - r_{k-1}) - E(r_k - r_{k-1})$$

is a square integrable martingale it is not hard to use the L^2 maximal inequality to check that the tightness criteria that can be found for example in Section 8 of Billingsley [2]. Alternatively one can invoke Theorem 4.13 on page 322 of Jacod and Shiryaev [12].

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