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Asymptotic behavior of orbits of holomorphic semigroups

Filippo Bracci [a,1], Manuel D. Contreras [b,2], Santiago Díaz-Madrigal [b,2], Hervé Gaussier [c,*,3], Andrew Zimmer [d,4]

[a] *Dipartimento di Matematica, Università di Roma "Tor Vergata", Via della Ricerca Scientifica 1, 00133, Roma, Italy*
[b] *Camino de los Descubrimientos, s/n, Departamento de Matemática Aplicada II and IMUS, Universidad de Sevilla, Sevilla, 41092, Spain*
[c] *Univ. Grenoble Alpes, CNRS, IF, F-38000 Grenoble, France*
[d] *Department of Mathematics, Louisiana State University, Baton Rouge, LA, USA*

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A B S T R A C T

Let (ϕ_t) be a holomorphic semigroup of the unit disc (*i.e.*, the flow of a semicomplete holomorphic vector field) without fixed points in the unit disc and let Ω be the starlike at infinity domain image of the Koenigs function of (ϕ_t). In this paper we characterize the type of convergence of the orbits of (ϕ_t) to the Denjoy-Wolff point in terms of the shape of Ω. In particular we prove that the convergence is non-tangential if and only if the domain Ω is "quasi-symmetric with respect to vertical axis". We also prove that such conditions are equivalent to the curve $[0,\infty) \ni t \mapsto \phi_t(z)$ being a quasi-geodesic in the sense of Gromov. Also, we characterize the tangential convergence in terms of the shape of Ω.

R É S U M É

Soit (ϕ_t) un semi-groupe holomorphe du disque unité (i.e. le flot d'un champ de vecteur holomorphe semi-complet), sans point fixe dans le disque unité, et soit Ω le domaine étoilé à l'infini, image du disque unité par la fonction de Koenigs de (ϕ_t). Nous caractérisons le type de convergence des orbites de (ϕ_t) au point de Denjoy-Wolff en termes de forme de Ω. Nous démontrons notamment que la convergence est non tangentielle si et seulement si le domaine Ω est "quasi-symétrique par rapport à l'axe vertical". Nous démontrons aussi que de telles conditions sont équivalentes au

* Corresponding author.
E-mail addresses: fbracci@mat.uniroma2.it (F. Bracci), contreras@us.es (M.D. Contreras), madrigal@us.es (S. Díaz-Madrigal), herve.gaussier@univ-grenoble-alpes.fr (H. Gaussier), amzimmer@lsu.edu (A. Zimmer).
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fait que la courbe $[0, \infty) \ni t \mapsto \phi_t(z)$ est une quasi-géodésique au sens de Gromov. Enfin, nous caractérisons la convergence tangentielle en termes de forme de Ω.

1. Introduction and statements of the main results

A holomorphic vector field G on the unit disc $\mathbb{D}$ is (real) semicomplete if the Cauchy problem $\dot{x}(t) = G(x(t)), x(0) = z$ has a solution defined for all $t \geq 0$ and for all $z \in \mathbb{D}$. The flow of a semicomplete vector field, (ϕ_t), is a continuous semigroup of holomorphic self-maps of $\mathbb{D}$—or simply a semigroup in $\mathbb{D}$. Namely, (ϕ_t) is a continuous homomorphism of the real semigroup $[0, +\infty)$ endowed with the Euclidean topology to the semigroup under composition of holomorphic self-maps of $\mathbb{D}$ endowed with the topology of uniform convergence on compacta.

It appears that semigroups in $\mathbb{D}$ were first considered in the 1930's by J. Wolff [20], although it was only with a paper of E. Berkson and H. Porta [3] in the 1970's that the modern study of semigroups in $\mathbb{D}$ initiated. Since their work, interest in semigroups in $\mathbb{D}$ has expanded due to their connections with branching stochastic processes (see, *e.g.*, [14,15]), biology [16] and their connections to composition operators and Loewner's theory (we refer the reader to the books [1,19,18,12] and [5] for more details).

In this paper we study the asymptotic behavior of semigroups in $\mathbb{D}$ via the Euclidean geometry of the image of an associated Koenigs function. Aside being motived by the study of the dynamics of semigroups, our main results also give a complete answer to the following question from geometric function theory.

Let $f : \mathbb{D} \to \mathbb{C}$ be a Riemann map such that $\Omega := f(\mathbb{D})$ is starlike at infinity, that is $\Omega + it \subset \Omega$ for every $t \geq 0$. Let $p \in \Omega$ and let $\{t_n\}$ be a sequence of positive real numbers converging to $+\infty$. Looking only at the shape of Ω, how can one decide whether the sequence $\{f^{-1}(p + it_n)\}$ converges to a point $\tau \in \partial\mathbb{D}$ non-tangentially or tangentially?

Starlike at infinity domains are also sometimes called "vertically invariant" (e.g. [1]) or "convex in the positive direction of the real axis" (e.g. [12]).

If (ϕ_t) is a semigroup in $\mathbb{D}$, which is not a group of hyperbolic rotations, then there exists a unique $\tau \in \overline{\mathbb{D}}$, the *Denjoy-Wolff point* of (ϕ_t), such that $\lim_{t \to +\infty} \phi_t(z) = \tau$, and the convergence is uniform on compacta. In case $\tau \in \mathbb{D}$, the semigroup is called *elliptic*.

In case the semigroup (ϕ_t) is non-elliptic, the action is conjugate to linear translation on an unbounded simply connected domain. More precisely, there exists an (essentially unique) univalent function h, called the *Koenigs function* of (ϕ_t), such that $h(\mathbb{D})$ is starlike at infinity, $h(\phi_t(z)) = h(z) + it$ for all $t \geq 0$ and $z \in \mathbb{D}$ (see, *e.g.*, [1,2,11]).

The *slope* of a non-elliptic semigroup (ϕ_t) at $z \in \mathbb{D}$ is the cluster set of $\mathsf{Arg}(1 - \overline{\tau}\phi_t(z))$ as $t \to +\infty$. The slope is a compact connected subset of $[-\pi/2, \pi/2]$.

Given $z \in \mathbb{D}$, we say that the orbit $[0, +\infty) \ni t \mapsto \phi_t(z)$ converges *non-tangentially* to the Denjoy-Wolff point if the slope of (ϕ_t) at z is contained in $(-\pi/2, \pi/2)$. In case the slope is $\{-\pi/2\}$ or $\{\pi/2\}$, the convergence is *tangential*.

For one-parameter groups of automorphisms there are two possible behaviors. Either $h(\mathbb{D})$ is a vertical strip (and the group is called *hyperbolic*) or $h(\mathbb{D})$ is a vertical half-plane (and the group is called *parabolic*). In the hyperbolic group case, $h(\mathbb{D})$ is symmetric with respect to the line of symmetry of the vertical strip, and "quasi-symmetric" with respect to any vertical line contained in the strip, and, in fact, the orbits of the group converge non-tangentially to the Denjoy-Wolff point. While, in the parabolic case, $h(\mathbb{D})$ is highly non-symmetric with respect to any line contained in the half-plane and the orbits of the group converge tangentially to the Denjoy-Wolff point.

For the general case, we will show that non-tangential convergence is equivalent to the image of the Koenigs function being "quasi-symmetric" about a vertical line. Suppose $\Omega \subsetneq \mathbb{C}$ is a domain starlike at infinity and $p \in \mathbb{C}$. Then for $t \geq 0$ define

$$\delta^+_{\Omega,p}(t) := \min\{t,\ \inf\{|z-(p+it)| : z \in \partial\Omega, \mathsf{Re}\, z \geq \mathsf{Re}\, p\}\},$$

and

$$\delta^-_{\Omega,p}(t) := \min\{t,\ \inf\{|z-(p+it)| : z \in \partial\Omega, \mathsf{Re}\, z \leq \mathsf{Re}\, p\}\}.$$

Then, the first main result we prove is the following:

Theorem 1.1. *Let (ϕ_t) be a non-elliptic semigroup in $\mathbb{D}$ with Denjoy-Wolff point $\tau \in \partial\mathbb{D}$ and Koenigs function h and let $\Omega := h(\mathbb{D})$. Suppose that $\{t_n\}$ is a sequence converging to $+\infty$. Then*

(1) the sequence $\{\phi_{t_n}(z)\}$ converges non-tangentially to τ as $n \to \infty$ for some—and hence any—$z \in \mathbb{D}$ if and only if for some—and hence any—$p \in \Omega$ there exist $0 < c < C$ such that for all $n \in \mathbb{N}$

$$c\delta^+_{\Omega,p}(t_n) \leq \delta^-_{\Omega,p}(t_n) \leq C\delta^+_{\Omega,p}(t_n).$$

(2) $\lim_{n\to\infty} \mathsf{Arg}(1-\overline{\tau}\phi_{t_n}(z)) = \frac{\pi}{2}$ (in particular, $\{\phi_{t_n}(z)\}$ converges tangentially to τ as $n \to \infty$) for some—and hence any—$z \in \mathbb{D}$ if and only if for some—and hence any—$p \in \Omega$,

$$\lim_{n\to+\infty} \frac{\delta^+_{\Omega,p}(t_n)}{\delta^-_{\Omega,p}(t_n)} = 0,$$

while, $\lim_{n\to\infty} \mathsf{Arg}(1-\overline{\tau}\phi_{t_n}(z)) = -\frac{\pi}{2}$ (in particular, $\{\phi_{t_n}(z)\}$ converges tangentially to τ as $n \to \infty$) for some—and hence any—$z \in \mathbb{D}$ if and only if for some—and hence any—$p \in \Omega$,

$$\lim_{n\to+\infty} \frac{\delta^+_{\Omega,p}(t_n)}{\delta^-_{\Omega,p}(t_n)} = +\infty.$$

The proof of this result is very involved, and it is based almost entirely on Gromov's theory of negatively curved metric spaces. In particular, let k_Ω denote the hyperbolic distance on Ω. When $0 \notin \Omega$ and $it \in \Omega$ for all $t > 0$, we show that the 2-Lipschitz curve

$$\sigma : [1, +\infty) \ni t \mapsto \frac{\delta^+_{\Omega,0}(t) - \delta^-_{\Omega,0}(t)}{2} + it \tag{1.1}$$

can be reparametrized to be a quasi-geodesic in (Ω, k_Ω) (see Section 3 for details on quasi-geodesics). Thus, by Gromov's shadowing lemma, σ always stays within a finite hyperbolic distance from a geodesic "converging to ∞." Theorem 1.1 then follows by noticing that non-tangential convergence is equivalent to staying at finite hyperbolic distance from σ (see Section 5 for details).

This argument also shows that an orbit of a semigroup converges non-tangentially if and only if it can be reparameterized to be a quasi-geodesic in the unit disc.

Theorem 1.2. *Let (ϕ_t) be a non-elliptic semigroup in $\mathbb{D}$ with Denjoy-Wolff point $\tau \in \partial\mathbb{D}$ and Koenigs function h and let $\Omega := h(\mathbb{D})$. Then the following are equivalent:*

(1) for some—and hence any—$z \in \mathbb{D}$, the orbit $[0, +\infty) \ni t \mapsto \phi_t(z)$ converges non-tangentially to τ as $t \to +\infty$,
(2) for some—and hence any—$z \in \mathbb{D}$, the curve $[0, +\infty) \ni t \mapsto \phi_t(z)$ can be reparametrized to be a quasi-geodesic,
(3) for some—and hence any—$p \in \Omega$ there exist $0 < c < C$ such that for all $t \geq 0$,

$$c\delta^+_{\Omega,p}(t) \leq \delta^-_{\Omega,p}(t) \leq C\delta^+_{\Omega,p}(t).$$

The proof actually implies more: the orbit $(\phi_t(z))$ converges non-tangentially if and only if for every $0 \leq t_1 \leq t_2$, the hyperbolic length of the orbit of $(\phi_t(z))$ between t_1 and t_2 is, up to uniform multiplicative and additive error, the hyperbolic distance between $\phi_{t_1}(z)$ and $\phi_{t_2}(z)$. The fact that these orbits are close to length minimizing is somewhat surprising given the examples constructed in [4,8,6]. In particular, there exist examples of parabolic semigroups whose slope is an interval $[a, b]$ with $-\pi/2 < a < b < \pi/2$. Despite this oscillation, which can only increase the hyperbolic length, Theorem 1.2 implies that the orbits in these examples are almost length minimizing.

We also give a geometric characterization of when an orbit converges tangentially.

Theorem 1.3. *Let (ϕ_t) be a non-elliptic semigroup in $\mathbb{D}$ with Denjoy-Wolff point $\tau \in \partial\mathbb{D}$ and Koenigs function h and let $\Omega := h(\mathbb{D})$. Then the following are equivalent:*

(1) $\lim_{t\to+\infty} \mathsf{Arg}(1 - \phi_t(z)) = \pi/2$ (respectively $= -\pi/2$) for some—and hence any—$z \in \mathbb{D}$, and, in particular, $[0, +\infty) \ni t \mapsto \phi_t(z)$ converges tangentially to τ as $t \to +\infty$,
(2) $\lim_{t\to+\infty} \frac{\delta^+_{\Omega,p}(t)}{\delta^-_{\Omega,p}(t)} = 0$ (respect. $\lim_{t\to+\infty} \frac{\delta^+_{\Omega,p}(t)}{\delta^-_{\Omega,p}(t)} = +\infty$).

As we will show, Theorem 1.2 and Theorem 1.3 are consequences of Theorem 1.1 and of its proof.

Recall that a non-elliptic semigroup (ϕ_t) is *hyperbolic* if $h(\mathbb{D})$ is contained in a vertical strip, it is *parabolic of positive hyperbolic step* if $h(\mathbb{D})$ is contained in a vertical half-plane but not in a vertical strip and *parabolic of zero hyperbolic step* otherwise. We mention that, although our proofs do not rely on previous results about dynamics of semigroups, it was already known (see [7,9]) that if (ϕ_t) is a hyperbolic semigroup then the trajectory $t \mapsto \phi_t(z)$ always converges non-tangentially to its Denjoy-Wolff point as $t \to +\infty$ for every $z \in \mathbb{D}$ and the slope is a single point which depends harmonically on z, while, if it is parabolic of positive hyperbolic step then $\phi_t(z)$ always converges tangentially to its Denjoy-Wolff point as $t \to +\infty$ for every $z \in \mathbb{D}$ and the slope is independent of z (and it is either $\{\pi/2\}$ or $\{-\pi/2\}$).

Therefore, Theorem 1.3 gives the new information that every orbit of a hyperbolic semigroup is a quasi-geodesic, while, in the case of parabolic semigroups of positive hyperbolic step, the orbits are never quasi-geodesics.

In the case of parabolic semigroups of zero hyperbolic step, all cases can happen. In Section 2 we give some examples illustrating the possible behaviors.

The paper is organized as follows. In Section 2 we provide some examples of possible behavior of orbits. In Section 3 we state some preliminaries we need in this paper. In Section 4 we show that the curve σ defined in Equation (1.1) can indeed be reparametrized to be a quasi-geodesic and also estimate its hyperbolic distance to the vertical axis at p. Finally, in Section 5 we prove the theorems.

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2. Examples

In this section we construct some examples of parabolic semigroups of zero hyperbolic step illustrating possible cases. We define domains Ω starlike at infinity, and, if $h : \mathbb{D} \to \Omega$ is a Riemann map, the semigroup is given by $\phi_t(z) := h^{-1}(h(z) + it)$.

Example 2.1. The model domain Ω_1 is defined by $\Omega_1 := \{\zeta \in \mathbb{C} : \mathsf{Im}\,(\zeta) > (\mathsf{Re}\,(\zeta))^2\}$ (see Fig. 1).

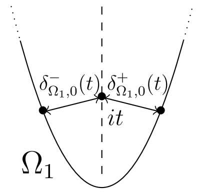

Fig. 1. Semigroup with orthogonal convergence.

Then Ω_1 is symmetric with respect to the imaginary axis, $\delta^+_{\Omega_1,0}(t) = \delta^-_{\Omega_1,0}(t)$ for $t > 0$ and $\gamma : [1, +\infty) \ni t \mapsto it$ can be reparametrized as a geodesic in Ω_1. Hence, for every $z \in \mathbb{D}$, the semigroup $\phi_t(z)$ converges orthogonally to the Denjoy-Wolff point $\tau \in \partial\mathbb{D}$.

Example 2.2. The model domain Ω_2 (see Fig. 2) is defined by

$$\Omega_2 := \{\zeta \in \mathbb{C} : \mathsf{Re}\,(\zeta) > 0\} \cup \{\zeta \in \mathbb{C} : \mathsf{Im}\,(\zeta) > (\mathsf{Re}\,(\zeta))^2\}.$$

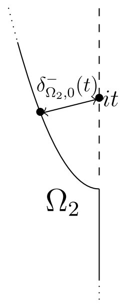

Fig. 2. Semigroup with tangential convergence.

Then for every $t > 4$, $\delta^+_{\Omega_2,0}(t) = t$ and $\delta^-_{\Omega_2,0}(t) = \sqrt{t - \frac{1}{4}}$. Hence

$$\lim_{t\to\infty} \frac{\delta^+_{\Omega_2,0}(t)}{\delta^-_{\Omega_2,0}(t)} = +\infty.$$

It follows from Theorem 1.3 that for every $z \in \mathbb{D}$, the semigroup $\phi_t(z)$ converges tangentially to the Denjoy-Wolff point $\tau \in \partial\mathbb{D}$.

Example 2.3. Fix two sequence of negative numbers (a_n) and (b_n). Then consider the model domain Ω_3 (see Fig. 3) defined by

$$\Omega_3 := \Omega_2 \cup_{n\geq 1} S_n,$$

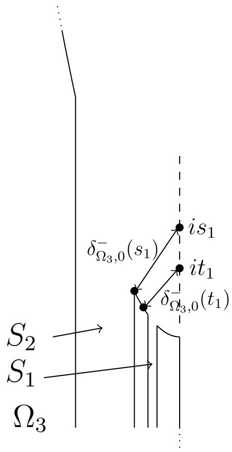

Fig. 3. Semigroup with slope $[-\pi/2, \alpha]$ for some $-\pi/2 < \alpha < \pi/2$.

where for every $n \geq 1$, S_n is a vertical strip $S_n := \{\zeta \in \mathbb{C} : a_n < \mathsf{Re}\,(\zeta) < b_n < 0\}$. We claim that we can select the sequences (a_n) and (b_n) such that the slope of the associated semigroup (ϕ_t) is $[-\pi/2, \alpha]$ for some $-\pi/2 < \alpha < \pi/2$.

First notice that for any choice of (a_n) and (b_n) we can find $t_n \to +\infty$ such that

$$\delta^-_{\Omega_3,0}(t_n) = \delta^+_{\Omega_2,0}(t_n)$$

and hence

$$\frac{\delta^+_{\Omega_3,0}(t_n)}{\delta^-_{\Omega_3,0}(t_n)} = \frac{t_n}{\sqrt{t_n - \frac{1}{4}}} \to +\infty \quad \text{as } n \to \infty,$$

which, by Theorem 1.1, implies that $\phi_{t_n}(z) \to \tau$ with slope $-\pi/2$.

On the other hand, by making the gap between a_n and b_n sufficiently large we can find $s_n \in (a_n, b_n)$ such that

$$\delta^-_{\Omega_3,0}(s_n) \geq s_n/2.$$

Then

$$\frac{\delta^-_{\Omega_3,0}(s_n)}{\delta^+_{\Omega_3,0}(s_n)} \geq \frac{s_n/2}{s_n} = \frac{1}{2}$$

and hence for every $z \in \mathbb{D}$ the sequence $\{\phi_{s_n}(z)\}$ converges non-tangentially to τ. In particular, the slope of (ϕ_t) is $[-\pi/2, \alpha]$ for some $-\pi/2 < \alpha < \pi/2$.

3. Preliminaries on hyperbolic and Euclidean geometry

3.1. Hyperbolic geometry of simply connected domains

Let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain. Recall that the hyperbolic metric κ_Ω is defined for $z \in \Omega$ and $v \in \mathbb{C}$ by

$$\kappa_\Omega(z; v) := \frac{|v|}{f'(0)},$$

where $f : \mathbb{D} \to \Omega$ is the Riemann map such that $f(0) = z$ and $f'(0) > 0$. The hyperbolic distance between $z, w \in \Omega$ is defined as

$$k_\Omega(z, w) := \inf \int_0^1 \kappa_\Omega(\gamma(\tau); \gamma'(\tau)) d\tau,$$

where the infimum is taken over all piecewise C^1-smooth curves $\gamma : [0, 1] \to \Omega$ such that $\gamma(0) = z$ and $\gamma(1) = w$.

A curve $\gamma : [a, b] \to \Omega$ is rectifiable if

$$\ell_\Omega(\gamma; [a, b]) := \sup_{\mathcal{P}} \sum_{j=0}^{N} k_\Omega(\gamma(t_j), \gamma(t_{j+1})) < +\infty,$$

where the supremum is taken over all partitions $\mathcal{P}$ of $[a, b]$ of type $a = t_0 < t_1 < \ldots < t_{N+1} = b$, $N \in \mathbb{N}$.

The number $\ell_\Omega(\gamma; [a, b])$ is the hyperbolic length of γ and, by definition,

$$\ell_\Omega(\gamma; [a, b]) \geq k_\Omega(\gamma(a), \gamma(b)).$$

Every rectifiable curve can be reparametrized by hyperbolic arc length. If γ is a Lipschitz curve then

$$\ell_\Omega(\gamma; [s, t]) = \int_s^t \kappa_\Omega(\gamma(\tau); \gamma'(\tau)) d\tau.$$

3.2. Geodesics and non-tangential convergence

Let $-\infty \leq a < b \leq +\infty$. A smooth curve $\eta : (a, b) \to \Omega$ is a (unit speed) *geodesic* if

$$t - s = k_\Omega(\eta(s), \eta(t))$$

for all $a < s < t < b$.

Given $R > 0$ and a geodesic $\eta : [0, +\infty) \to \Omega$, the *hyperbolic sector around* η *of amplitude* R is given by

$$S_\Omega(\eta, R) := \{z \in \Omega : k_\Omega(z, \eta([0, +\infty))) < R\}.$$

We can use hyperbolic sectors to detect non-tangential convergence (see for instance [6, Proposition 4.5]):

Proposition 3.1. *Let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain and let $f : \mathbb{D} \to \Omega$ be a Riemann map.*

(1) Suppose $\gamma : [0, +\infty) \to \Omega$ be a continuous curve such that $\lim_{t\to+\infty} k_\Omega(\gamma(0), \gamma(t)) = +\infty$, then $f^{-1}(\gamma(t))$ converges non-tangentially to a point $\sigma \in \partial\mathbb{D}$ if and only if there exist $R > 0$ and a geodesic $\eta : [0, +\infty) \to \Omega$ such that $\gamma(t) \in S_\Omega(\eta, R)$ for all t sufficiently large.

(2) Suppose $\{w_n\} \subset \Omega$ be a sequence such that $\lim_{n\to\infty} k_\Omega(w_0, w_n) = \infty$, then w_n converges non-tangentially to a point $\sigma \in \partial\mathbb{D}$ if and only if there exist $R > 0$ and a geodesic $\eta : [0, +\infty) \to \Omega$ such that $w_n \in S_\Omega(\eta, R)$ for all n sufficiently large.

3.3. Quasi-geodesics

Given a general simply connected domain $\Omega \subsetneq \mathbb{C}$, it is essentially impossible to determine the geodesics in the hyperbolic metric. However, it is sometimes possible to find so-called quasi-geodesics which, by Gromov's shadowing lemma (also called Morse lemma, or the geodesic stability lemma), turn out to approximate geodesics.

Definition 3.2. Let $-\infty < a < b \leq +\infty$. Let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain and $\gamma : [a, b) \to \Omega$. Let $A \geq 1$, $B \geq 0$. We say that γ is a (A, B)-*quasi-geodesic* if for all $a \leq s \leq t < b$,

$$\frac{1}{A}(t-s) - B \leq \ell_\Omega(\gamma; [s,t]) \leq A(t-s) + B.$$

For short, we say that γ is a *quasi-geodesic* if there exist $A \geq 1, B \geq 0$ such that γ is a (A, B)-quasi-geodesic.

By Gromov's shadowing lemma (see, *e.g.*, [10, Théorème 3.1, pag. 41]) there exists $M > 0$ (which depends only on A, B) such that if $\gamma : [0, +\infty) \to \Omega$ is a (A, B)-quasi-geodesic then there exists a geodesic $\eta : [0, +\infty) \to \Omega$ such that $\eta(0) = \gamma(0)$ and for every $t \in [0, +\infty)$

$$k_\Omega(\gamma(t), \eta([0, +\infty))) < M, \quad k_\Omega(\eta(t), \gamma([0, +\infty))) < M. \tag{3.1}$$

Remark 3.3. Let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain and let $f : \mathbb{D} \to \Omega$ be a Riemann map. By the previous argument and Proposition 3.1 it follows that if $\gamma : [0, +\infty) \to \Omega$ is a quasi-geodesic then $f^{-1}(\gamma(t))$ converges non-tangentially to a point $\sigma \in \partial\mathbb{D}$ as $t \to +\infty$.

From the previous discussion, we have the following result which allows to detect quasi-geodesics:

Proposition 3.4. *Suppose that $\Omega \subsetneq \mathbb{C}$ is a simply connected domain and $\gamma : [0, +\infty) \to \Omega$ is a Lipschitz curve. If there exists $A \geq 1$ and $B \geq 0$ such that*

$$\ell_\Omega(\gamma; [s,t]) \leq A k_\Omega(\gamma(s), \gamma(t)) + B$$

for all $0 \leq s \leq t$, then γ can be reparametrized to be a (A, B)-quasi-geodesic.

3.4. Estimates on the hyperbolic distance

As customary, for $p \in \Omega$ we let

$$\delta_\Omega(p) = \inf\{|z-p| : z \in \mathbb{C} \setminus \Omega\}.$$

In this paper we will use the following estimates for the hyperbolic metric and distance (see [6, Section 3] for details):

Theorem 3.5 *(Distance Lemma). Let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain. Then for every $z \in \Omega$ and $v \in \mathbb{C}$,*

$$\frac{|v|}{4\delta_\Omega(z)} \leq \kappa_\Omega(z; v) \leq \frac{|v|}{\delta_\Omega(z)}.$$

Moreover, for every $w_1, w_2 \in \Omega$,

$$\frac{1}{4}\log\left(1+\frac{|w_1-w_2|}{\min\{\delta_\Omega(w_1),\delta_\Omega(w_2)\}}\right)\leq k_\Omega(w_1,w_2)\leq\int_\Gamma\frac{|dw|}{\delta_\Omega(w)},$$

where Γ *is any absolutely continuous curve in* Ω *joining* w_1 *to* w_2*.*

Note that Theorem 3.5 implies immediately that for all $z, w \in \Omega$,

$$k_\Omega(z,w)\geq\sup_{\zeta\in\mathbb{C}\setminus\Omega}\frac{1}{4}\left|\log\frac{|z-\zeta|}{|w-\zeta|}\right|. \tag{3.2}$$

3.5. Euclidean geometry of domains starlike at infinity

Let Ω be a simply connected domain which is starlike at infinity and $p \in \mathbb{C}$. For $t > 0$, let

$$\tilde{\delta}^+_{\Omega,p}(t) := \inf\{|z-(p+it)| : \mathsf{Re}\, z \geq \mathsf{Re}\, p, z \in \mathbb{C}\setminus\Omega\},$$
$$\tilde{\delta}^-_{\Omega,p}(t) := \inf\{|z-(p+it)| : \mathsf{Re}\, z \leq \mathsf{Re}\, p, z \in \mathbb{C}\setminus\Omega\}.$$

Note that, if $p + it \in \mathbb{C}\setminus\Omega$ then $\tilde{\delta}^+_{\Omega,p}(t) = \tilde{\delta}^-_{\Omega,p}(t) = 0$. While, for $p \in \Omega$ and $t > 0$, $\delta_\Omega(p + it) = \min\{\tilde{\delta}^+_{\Omega,p}(t), \tilde{\delta}^-_{\Omega,p}(t)\}$.

Moreover, for $t > 0$ we let

$$\delta^+_{\Omega,p}(t) := \min\{\tilde{\delta}^+_{\Omega,p}(t), t\}, \quad \delta^-_{\Omega,p}(t) := \min\{\tilde{\delta}^-_{\Omega,p}(t), t\}.$$

Note that, since Ω is starlike at infinity, then $(0, +\infty) \ni t \mapsto \delta^\pm_{\Omega,p}(t)$ is non-decreasing.

Simple geometric considerations allow to prove the following lemma:

Lemma 3.6. *Let* Ω *be a simply connected domain starlike at infinity. For all* $p, q \in \Omega$ *there exist* $0 < c < C$ *such that for all* $t > 0$

$$c\delta^\pm_{\Omega,p}(t) \leq \delta^\pm_{\Omega,q}(t) \leq C\delta^\pm_{\Omega,p}(t).$$

4. Quasi-geodesics in starlike at infinity domains

The aim of this section is to construct a quasi-geodesic in a domain $\Omega \subsetneq \mathbb{C}$ starlike at infinity which converges in the Carathéodory topology to "$+\infty$" and to get useful estimates on the hyperbolic distance from this curve to a vertical axis.

In all this section, we assume that $\Omega \subset \mathbb{C}$ is a domain starlike at infinity such that $0 \notin \Omega$ and $it \in \Omega$ for all $t > 0$.

We define $\sigma : [1, +\infty) \to \Omega$ by

$$\sigma(t) := \frac{\delta^+_{\Omega,0}(t) - \delta^-_{\Omega,0}(t)}{2} + it. \tag{4.1}$$

Lemma 4.1. *The curve* σ *is* 2*-Lipschitz. In particular,* $|\sigma'(t)| \leq 2$ *for almost every* $t \geq 1$*.*

Proof. For all $s, t \geq 1$, using the triangle inequality we have $\delta^\pm_{\Omega,0}(t) \leq |t-s| + \delta^\pm_{\Omega,0}(s)$ and $\delta^\pm_{\Omega,0}(t) \geq -|t-s| + \delta^\pm_{\Omega,0}(s)$. Therefore,

$$|\delta^\pm_{\Omega,0}(t) - \delta^\pm_{\Omega,0}(s)| \leq |t-s|.$$

From this it follows immediately that σ is 2-Lipschitz. $\square$

4.1. The curve σ is up to reparametrization a quasi-geodesic

The aim of this subsection is to prove the following result:

Theorem 4.2. *The curve $[1,+\infty) \ni t \mapsto \sigma(t)$ can be reparametrized to be a quasi-geodesic in Ω.*

The proof is rather long and technical and requires many lemmas.

Let

$$\omega(t) := \delta^+_{\Omega,0}(t) + \delta^-_{\Omega,0}(t).$$

Lemma 4.3. *For $t \geq 1$*

$$\delta_\Omega(\sigma(t)) \geq \frac{1}{2\sqrt{2}}\omega(t).$$

Proof. Fix $t \geq 1$. First consider the case $\delta^+_{\Omega,0}(t) \geq \delta^-_{\Omega,0}(t)$, which implies that $\mathsf{Re}\,\sigma(t) \geq 0$.

If $z \in \partial\Omega$ and $\mathsf{Re}\,(z) > 0$, then

$$|z-\sigma(t)| \geq |z-it| - |it-\sigma(t)| \geq \delta^+_{\Omega,0}(t) - \frac{\delta^+_{\Omega,0}(t) - \delta^-_{\Omega,0}(t)}{2} = \frac{\omega(t)}{2}.$$

Now, for $z \in \mathbb{C}$ define

$$\|z\|_1 = |\mathsf{Re}\,z| + |\mathsf{Im}\,z|\,.$$

Then

$$|z| \leq \|z\|_1 \leq \sqrt{2}\,|z|$$

If $z \in \partial\Omega$ and $\mathsf{Re}\,z \leq 0$, then

$$|z-\sigma(t)| \geq \frac{1}{\sqrt{2}}\|z-\sigma(t)\|_1.$$

Further, since $\mathsf{Re}\,z \leq 0 \leq \mathsf{Re}\,\sigma(t)$ we have

$$\begin{aligned}\|z-\sigma(t)\|_1 &= |\mathsf{Re}\,z - \mathsf{Re}\,\sigma(t)| + |\mathsf{Im}\,z - \mathsf{Im}\,\sigma(t)| = \mathsf{Re}\,\sigma(t) - \mathsf{Re}\,z + |\mathsf{Im}\,z - t| \\ &= \mathsf{Re}\,\sigma(t) + \|z-it\|_1 \geq \mathsf{Re}\,\sigma(t) + |z-it| \geq \mathsf{Re}\,\sigma(t) + \delta^-_{\Omega,0}(t) \\ &= \frac{\delta^+_{\Omega,0}(t) - \delta^-_{\Omega,0}(t)}{2} + \delta^-_{\Omega,0}(t) = \frac{1}{2}\omega(t).\end{aligned}$$

Hence

$$|z-\sigma(t)| \geq \frac{1}{2\sqrt{2}}\omega(t).$$

The case when $\delta^+_{\Omega,0}(t) \leq \delta^-_{\Omega,0}(t)$ is similar. $\square$

As a direct consequence of the previous lemma, Lemma 4.1 and Theorem 3.5, we have:

Lemma 4.4. *If $1 \le a < b < \infty$, then*

$$\ell_\Omega\left(\sigma;[a,b]\right) \le 4\sqrt{2}\int_a^b \frac{1}{\omega(t)}dt.$$

We can now prove Theorem 4.2 in a simple case.

Proposition 4.5. *Suppose that there exist $\alpha, T_0 > 0$ such that*

$$\omega(t) \ge \alpha t$$

for all $t \ge T_0$. Then σ can be reparametrized to be a quasi-geodesic in Ω.

Proof. We have $\delta^{\pm}_{\Omega,0}(t) \le t$ for all $t \ge 1$, hence,

$$\frac{|\delta^{+}_{\Omega,0}(t) - \delta^{-}_{\Omega,0}(t)|}{t} \le \frac{|\delta^{+}_{\Omega,0}(t)|}{t} + \frac{|\delta^{-}_{\Omega,0}(t)|}{t} \le 2.$$

Therefore, for all $t \ge 1$,

$$t \le |\sigma(t)| \le 2t.$$

So, by (3.2), for all $1 \le a \le b$,

$$k_\Omega(\sigma(a),\sigma(b)) \ge \frac{1}{4}\left|\log\frac{|\sigma(b)|}{|\sigma(a)|}\right| \ge \frac{1}{4}\log\frac{b}{a} - \frac{1}{4}\log 2. \tag{4.2}$$

On the other hand, if $T_0 \le a \le b$, then by Lemma 4.4,

$$\ell_\Omega(\sigma;[a,b]) \le 4\sqrt{2}\int_a^b \frac{dt}{\omega(t)} \le \frac{4\sqrt{2}}{\alpha}\int_a^b \frac{dt}{t} = \frac{4\sqrt{2}}{\alpha}\log\frac{b}{a}. \tag{4.3}$$

From this last inequality, (4.2), and Proposition 3.4 it follows at once that σ can be reparametrized to be a quasi-geodesic in Ω. □

Remark 4.6. For future reference, we make the following observations. If there exist $\alpha, T_0 > 0$ such that

$$\omega(t) \ge \alpha t$$

for all $t \ge T_0$, then

(1) by the same token we obtained (4.2), we have

$$\max\{k_\Omega(ia,\sigma(b)), k_\Omega(\sigma(a), ib)\} \ge \frac{1}{4}\log\frac{b}{a} - \frac{1}{4}\log 2.$$

Hence, by (4.3), there exist constants $A, B > 0$ such that for every $T_0 \le a \le b$ we have

$$k_\Omega(\sigma(a),\sigma(b)) \le \ell_\Omega(\sigma;[a,b]) \le A\min\{k_\Omega(ia,\sigma(b)), k_\Omega(\sigma(a), ib)\} + B. \tag{4.4}$$

(2) Also, again arguing as in (4.2), we have

$$\int_a^b \frac{dt}{\omega(t)} \leq \frac{4}{\alpha} k_\Omega(ia, ib). \tag{4.5}$$

Now we make the following assumption:

Assumption. *There does not exist $\alpha, T_0 > 0$ such that $\omega(t) \geq \alpha t$ for all $t \geq T_0$.*

Assuming this condition, there exists $T_0 > 0$ such that $\omega(T_0) < T_0$. In particular, $\max\{\delta^+_{\Omega,0}(T_0), \delta^-_{\Omega,0}(T_0)\} < T_0$. Hence, for every $t \geq T_0$ we have

$$\delta^\pm_{\Omega,0}(t) \leq t - T_0 + \delta^\pm_{\Omega,0}(T_0) < t - T_0 + T_0 = t.$$

Therefore, for every $t \geq T_0$,

$$\delta^\pm_{\Omega,0}(t) < t. \tag{4.6}$$

4.1.1. Step 1: constructing sequences

Fix $a, b \in [T_0, \infty)$ with $a < b$. We define a sequence of positive numbers $\{t_n\}$

$$a = t_0 < t_1 < t_2 < \dots$$

and complex numbers $\{z_n^\pm\} \subset \mathbb{C} \setminus \Omega$ such that for all $n \geq 0$

(0) $\mathsf{Im}\, z_n^+ = \mathsf{Im}\, z_n^-$,
(1) $\mathsf{Re}\, z_n^- < 0 < \mathsf{Re}\, z_n^+$,
(2) $|\mathsf{Re}\, z_n^\pm| \leq \delta^\pm_{\Omega,0}(t_n)$,
(3) $y_n \leq t_n$, where $y_n := \Im z_n^+ = \Im z_n^-$,
(4) $\max\{|\sigma(t_n) - z_n^+|, |\sigma(t_n) - z_n^-|\} \leq 2\omega(t_n)$,

and for all $n \geq 1$,

(5) $\min\{|\sigma(t_n) - z_{n-1}^+|, |\sigma(t_n) - z_{n-1}^-|\} = 6\omega(t_n)$.

We first explain the construction of these sequences and then verify that they have the desired properties. We define t_n, z_n^+, and z_n^- sequentially as follows. If $n = 0$, then define $t_0 := a$. Otherwise, define

$$t_n := \max\left\{t \geq t_{n-1} : \omega(s) \geq \frac{1}{6} \min\{\left|\sigma(s) - z_{n-1}^+\right|, \left|\sigma(s) - z_{n-1}^-\right|\} \text{ for all } s \in [t_{n-1}, t]\right\}. \tag{4.7}$$

Next pick $a_n, b_n \in \mathbb{C} \setminus \Omega$ such that

$$\begin{aligned} &\mathrm{Re}(a_n) \leq 0 \leq \mathrm{Re}(b_n), \\ &|a_n - it_n| = \delta^-_{\Omega,0}(t_n), \text{ and} \\ &|b_n - it_n| = \delta^+_{\Omega,0}(t_n). \end{aligned}$$

Since $t_n \geq T_0$, by (4.6) we have $\mathsf{Re}\,(a_n) < 0 < \mathsf{Re}\,(b_n)$. Then let

$$y_n := \min\{\mathrm{Im}(a_n), \mathrm{Im}(b_n)\}.$$

Since Ω is starlike at infinity, $\max\{\mathrm{Im}(a_n), \mathrm{Im}(b_n)\} \leq t_n$, hence $y_n \leq t_n$. Then define

$$z_n^+ := \mathsf{Re}\,(b_n) + iy_n, \quad z_n^- := \mathsf{Re}\,(a_n) + iy_n.$$

We now verify that the resulting sequences have the desired properties.

Claim. $a = t_0 < t_1 < t_2 < \dots$

Proof. First, note that Property (4) implies that the set in (4.7) is non-empty. Hence each t_n exists. We next show that $t_n < +\infty$. If $n = 0$, then $t_n = a < +\infty$. If $n > 1$, then the definition of σ implies that

$$\min\{\left|\sigma(t) - z_{n-1}^+\right|, \left|\sigma(t) - z_{n-1}^-\right|\} \geq t - t_{n-1}$$

for all $t \geq t_{n-1}$. Then, since we assume that there does not exist $\alpha, T_0 > 0$ such that $\omega(t) \geq \alpha t$ for all $t \geq T_0$, we see that $t_n < +\infty$. Finally, we show that if $n > 0$, then $t_{n-1} < t_n$. By Property (4)

$$\min\left\{\left|\sigma(t_{n-1}) - z_{n-1}^+\right|, \left|\sigma(t_{n-1}) - z_{n-1}^-\right|\right\} \leq 2\omega(t_{n-1}).$$

So by the continuity of ω we see that $t_n > t_{n-1}$. $\square$

Now Properties (0)-(3) and (5) hold by the construction. So we only have to verify Property (4).

Claim. $\max\{|\sigma(t_n) - z_n^+|, |\sigma(t_n) - z_n^-|\} \leq 2\omega(t_n)$ *for all* $n \geq 0$.

Proof. We first argue that

$$\max\{|it_n - z_n^+|, |it_n - z_n^-|\} \leq \omega(t_n). \tag{4.8}$$

Indeed, assume that $y_n = \mathsf{Im}\, b_n$ (a similar argument works in case $y_n = \mathsf{Im}\, a_n$). Then, $|it_n - z_n^+| = \delta^+_{\Omega,0}(t_n)$, while

$$\begin{aligned} |it_n - z_n^-| &\leq |it_n - a_n| + |a_n - (\mathsf{Re}\, a_n + iy_n)| = \delta^-_{\Omega,0}(t_n) + (\mathsf{Im}\, a_n - y_n) \\ &\leq \delta^-_{\Omega,0}(t_n) + (t_n - y_n) \leq \delta^-_{\Omega,0}(t_n) + |it_n - b_n| = \delta^-_{\Omega,0}(t_n) + \delta^+_{\Omega,0}(t_n) = \omega(t_n). \end{aligned}$$

Also, clearly $|\sigma(t_n) - it_n| \leq \omega(t_n)$. This last inequality, together with (4.8), implies

$$|\sigma(t_n) - z_n^\pm| \leq |\sigma(t_n) - it_n| + |it_n - z_n^\pm| \leq 2\omega(t_n). \quad \square$$

This completes the construction of the sequences.

4.1.2. Step 2: key estimates

We now establish key estimates on the sequences constructed in the previous step.

Lemma 4.7. *For* $n \geq 1$ *we have*

$$3\omega(t_n) \leq y_n - t_{n-1} \leq \min\{t_n - t_{n-1}, y_n - y_{n-1}\}.$$

In particular,

$$t_0 < y_1 \le t_1 < y_2 \le t_2 < \dots$$

and $\lim_{n\to\infty} y_n = \infty$.

Proof. Fix $n \ge 1$. By property (5),

$$\min\{|\sigma(t_n) - z_{n-1}^+|, |\sigma(t_n) - z_{n-1}^-|\} = 6\omega(t_n).$$

First assume that $|\sigma(t_n) - z_{n-1}^+| = 6\omega(t_n)$. Then, by (4.8) and taking into account that $\omega(t_n) \ge \omega(t_{n-1})$, we have

$$\begin{aligned} y_n - t_{n-1} &= |iy_n - it_{n-1}| \\ &\ge |\sigma(t_n) - z_{n-1}^+| - |\sigma(t_n) - iy_n| - |it_{n-1} - z_{n-1}^+| \\ &\ge 6\omega(t_n) - 2\omega(t_n) - \omega(t_{n-1}) \ge (6-3)\,\omega(t_n) = 3\omega(t_n). \end{aligned}$$

By property (3), $y_n - t_{n-1} \le \min\{t_n - t_{n-1}, y_n - y_{n-1}\}$. The case when $|\sigma(t_n) - z_{n-1}^-| = 6\omega(t_n)$ is essentially the same.

Finally, the previous estimates show that $\{y_n\}$ is an increasing sequence and

$$0 < 3\omega(t_0) \le 3 \lim_{n\to\infty} \omega(t_n) \le \lim_{n\to\infty} (y_n - y_{n-1}).$$

Hence $\lim_{n\to\infty} y_n = \infty$. □

As straightforward consequence of the previous lemma and taking into account that $\omega(t_n) \ge \omega(t_{n-1})$, we see that

$$\log\left(\frac{y_n - y_{n-1}}{\omega(t_{n-1})}\right) \ge \log 3 > 1 \tag{4.9}$$

for every $n \ge 1$.

Lemma 4.8. *If* $n \ge 1$ *and* $t \in [y_n, t_n]$, *then*

$$\omega(t) \le \omega(t_n) \le 2\omega(t).$$

Proof. The first inequality follows from the fact that Ω is starlike at infinity.

Since $t_{n-1} < y_n \le t_n$ it follows from (4.7) and the fact that σ is 2-Lipschitz (see Lemma 4.1) that

$$\begin{aligned} \omega(t) &\ge \frac{1}{6} \min\{\left|\sigma(t) - z_{n-1}^+\right|, \left|\sigma(t) - z_{n-1}^-\right|\} \\ &\ge \frac{1}{6} \min\{\left|\sigma(t_n) - z_{n-1}^+\right|, \left|\sigma(t_n) - z_{n-1}^-\right|\} - \frac{1}{6}\,|\sigma(t_n) - \sigma(t)| \\ &\ge \omega(t_n) - 2\frac{1}{6}(t_n - t) \ge \omega(t_n) - \frac{1}{3}|it_n - iy_n| \ge \left(1 - \frac{1}{3}\right)\omega(t_n) \ge \frac{1}{2}\omega(t_n), \end{aligned}$$

and the proof is completed. □

4.1.3. Step 3: A lower bound on distance

Define

$$\delta_n := \mathrm{Re}(z_n^+) - \mathrm{Re}(z_n^-).$$

By property (2) in the definition of the sequence $\{z_n^\pm\}$,

$$\delta_n \le \delta^+_{\Omega,0}(t_n) + \delta^-_{\Omega,0}(t_n) = \omega(t_n).$$

Recall that we fixed $a, b \in [T_0, \infty)$ with $a < b$. Lemma 4.7 implies that $\lim_{n\to\infty} y_n = \infty$ and $y_0 \le t_0 = a < b$, so there exists a unique $N \ge 0$ such that

$$y_N \le b < y_{N+1}.$$

Lemma 4.9. *Suppose $u \in \{ia, \sigma(a)\}$ and $v \in \{ib, \sigma(b)\}$. If $N = 0$, then*

$$k_\Omega(u, v) \ge -\frac{1}{4}\log(2) + \frac{1}{4}\log\left(\max\left\{1, \frac{b - y_0}{\omega(a)}\right\}\right).$$

If $N \ge 1$, then

$$k_\Omega(u, v) \ge \frac{1}{4}\left(-\log 2 + \log\left(\frac{y_1 - y_0}{\omega(a)}\right) + \sum_{k=1}^{N-1}\log\left(\frac{y_{k+1} - y_k}{\delta_k}\right) + \log\left(\max\left\{1, \frac{b - y_N}{\delta_N}\right\}\right)\right).$$

Proof. First suppose that $N = 0$. If $b - y_0 \le \omega(a)$ there is nothing to prove. So suppose that

$$\frac{b - y_0}{\omega(a)} \ge 1.$$

By (4.8) and property (4) in Step 1,

$$\max\left\{|u - z_0^+|, |u - z_0^-|\right\} \le 2\omega(a). \tag{4.10}$$

Next, since $|v - z_0^\pm| \ge |\mathsf{Im}\, v - \mathsf{Im}\, z_0^\pm| = b - y_0$, we have

$$\min\left\{|v - z_0^+|, |v - z_0^-|\right\} \ge b - y_0. \tag{4.11}$$

Putting together (3.2) with (4.10) and (4.11), we have

$$k_\Omega(u, v) \ge \frac{1}{4}\log\left|\frac{v - z_0^+}{u - z_0^+}\right| \ge -\frac{1}{4}\log(2) + \frac{1}{4}\log\left(\frac{b - y_0}{\omega(a)}\right).$$

Next suppose that $N > 0$. Let $\gamma : [0, T] \to \Omega$ be a unit speed geodesic with $\gamma(0) = u$ and $\gamma(T) = v$. For $k = 1, \dots, N$ define

$$\tau_k := \min\{t \ge 0 : \mathrm{Im}(\gamma(t)) = y_k\}.$$

Note that $a < \tau_1 < \tau_2 < \cdots < \tau_N < b$.

Then, since Ω is starlike at infinity,

$$\mathsf{Re}\,(z_k^-) < \mathsf{Re}\,(\gamma(\tau_k)) < \mathsf{Re}\,(z_k^+). \tag{4.12}$$

Also, since $|\gamma(\tau_{k+1}) - z_k^{\pm}| \geq |\mathsf{Im}\,\gamma(\tau_{k+1}) - \mathsf{Im}\, z_k^{\pm}| = y_{k+1} - y_k$, we have

$$\min\left\{|\gamma(\tau_{k+1}) - z_k^+|, |\gamma(\tau_{k+1}) - z_k^-|\right\} \geq y_{k+1} - y_k. \tag{4.13}$$

Moreover, by (4.12) we have $|\gamma(\tau_k) - z_k^{\pm}| = |\mathsf{Re}\,\gamma(\tau_k) - \mathsf{Re}\, z_k^{\pm}| \leq \delta_k$, hence

$$\max\left\{|\gamma(\tau_k) - z_k^+|, |\gamma(\tau_k) - z_k^-|\right\} \leq \delta_k. \tag{4.14}$$

Now, by (3.2), (4.13) and (4.10) we have

$$k_\Omega(u, \gamma(\tau_1)) \geq \frac{1}{4}\log\left|\frac{\gamma(\tau_1) - z_0^+}{u - z_0^+}\right| \geq -\frac{1}{4}\log(2) + \frac{1}{4}\log\left(\frac{y_1 - y_0}{\omega(a)}\right). \tag{4.15}$$

For $k \geq 1$, (3.2), (4.13) and (4.14) imply that

$$k_\Omega(\gamma(\tau_{k+1}), \gamma(\tau_k)) \geq \frac{1}{4}\log\left|\frac{\gamma(\tau_{k+1}) - z_k^+}{\gamma(\tau_k) - z_k^+}\right| \geq \frac{1}{4}\log\left(\frac{y_{k+1} - y_k}{\delta_k}\right). \tag{4.16}$$

Finally, (3.2), (4.14) implies that

$$k_\Omega(\gamma(\tau_N), v) \geq \frac{1}{4}\log\left|\frac{v - z_N^+}{\gamma(\tau_N) - z_N^+}\right| \geq \frac{1}{4}\log\left(\frac{b - y_N}{\delta_N}\right),$$

and hence

$$k_\Omega(\gamma(\tau_N), v) \geq \frac{1}{4}\log\left(\max\left\{1, \frac{b - y_N}{\delta_N}\right\}\right). \tag{4.17}$$

Since γ is a geodesic, we have

$$k_\Omega(u, v) = k_\Omega(u, \gamma(\tau_1)) + \sum_{k=1}^{N-1} k_\Omega(\gamma(\tau_k), \gamma(\tau_{k+1})) + k_\Omega(\gamma(\tau_N), v).$$

The statement then follows from (4.15), (4.16), (4.17). □

4.1.4. Proof of Theorem 4.2

By Lemma 4.4 we have

$$\ell_\Omega(\sigma; [a, b]) \leq 4\sqrt{2}\int_a^b \frac{1}{\omega(t)}dt.$$

Hence to prove Theorem 4.2 it is enough to show that $\int_a^b \omega(t)^{-1}dt$ is comparable to the lower bounds in Lemma 4.9.

Lemma 4.10. *If $T \in [a, y_1]$, then*

$$\int_a^T \frac{dt}{\omega(t)} \leq 1 + 6\log\left(\max\left\{1, \frac{T - y_0}{\omega(a)}\right\}\right)$$

Proof. Notice that by (4.8), $(a - y_0) = (t_0 - y_0) \le |it_0 - z_0^{\pm}| \le \omega(t_0)$, hence,

$$y_0 \le a \le y_0 + \omega(a)$$

and if $a \le t$, then $\omega(a) \le \omega(t)$. So

$$\int_a^{y_0+\omega(a)} \frac{dt}{\omega(t)} \le \int_a^{y_0+\omega(a)} \frac{dt}{\omega(a)} \le 1.$$

Now if $t \in [a, y_1]$, then by (4.7),

$$\omega(t) \ge \frac{1}{6}\min\{\left|\sigma(t) - z_0^+\right|, \left|\sigma(t) - z_0^-\right|\} \ge \frac{1}{6}(t - y_0).$$

So if $T \ge y_0 + \omega(a)$, then

$$\int_{y_0+\omega(a)}^{T} \frac{dt}{\omega(t)} \le 6 \int_{y_0+\omega(a)}^{T} \frac{dt}{t - y_0} = 6\log\left(\frac{T - y_0}{\omega(a)}\right). \quad \square$$

Lemma 4.11. *For $k \ge 1$,*

$$\int_{y_k}^{y_{k+1}} \frac{dt}{\omega(t)} \le 8\log\left(\frac{y_{k+1} - y_k}{\omega(t_k)}\right).$$

Proof. By Lemma 4.7 and the fact that ω is an increasing function,

$$y_k + \omega(t_k) \le y_k + 3\omega(t_k) \le y_k + 3\omega(t_{k+1}) \le t_k + 3\omega(t_{k+1}) \le y_{k+1}.$$

Further, by Lemma 4.8, if $t \in [y_k, t_k]$, then

$$\omega(t) \ge \omega(t_k)/2$$

and, since Ω is starlike at infinity, if $t \ge t_k$, then $\omega(t) \ge \omega(t_k)$. Therefore, $\omega(t) \ge \omega(t_k)/2$ when $t \ge y_k$. Thus

$$\int_{y_k}^{y_k+\omega(t_k)} \frac{dt}{\omega(t)} \le \int_{y_k}^{y_k+\omega(t_k)} \frac{2dt}{\omega(t_k)} = 2. \tag{4.18}$$

Next consider $t \in [y_k + \omega(t_k), y_{k+1}]$. By (4.8), we have

$$t_k - y_k = |it_k - iy_k| \le |it_k - z_k^{\pm}| \le \omega(t_k).$$

Then $y_k + \omega(t_k) \ge t_k$. So $t \in [t_k, y_{k+1}]$ and $y_{k+1} \le t_{k+1}$. Hence, by (4.7),

$$\omega(t) \ge \frac{1}{6}\min\{|\sigma(t) - z_k^+|, |\sigma(t) - z_k^-|\} \ge \frac{1}{6}(t - y_k).$$

Therefore,

$$\int_{y_k+\omega(t_k)}^{y_{k+1}} \frac{dt}{\omega(t)} \le 6 \int_{y_k+\omega(t_k)}^{y_{k+1}} \frac{dt}{t-y_k} = 6\log\left(\frac{y_{k+1}-y_k}{\omega(t_k)}\right). \tag{4.19}$$

Thus by (4.18) and (4.19) and (4.9),

$$\int_{y_k}^{y_{k+1}} \frac{dt}{\omega(t)} \le 2 + 6\log\left(\frac{y_{k+1}-y_k}{\omega(t_k)}\right) \le 8\log\left(\frac{y_{k+1}-y_k}{\omega(t_k)}\right),$$

and we are done. □

Repeating the proof of the previous lemma one can prove:

Lemma 4.12. *If $N \ge 1$, then*

$$\int_{y_N}^{b} \frac{dt}{\omega(t)} \le 2 + 6\log\left(\max\left\{1, \frac{b-y_N}{\omega(t_N)}\right\}\right).$$

Combining the estimates in the previous three lemmas we can estimate $\int_a^b \omega(t)^{-1}dt$.

Recall that $a, b \in [T_0, \infty)$ with $a < b$ and $N \ge 0$ is a natural number such that $y_N \le b < y_{N+1}$.

If $N = 0$, then Lemma 4.10 implies

$$\ell_\Omega(\sigma;[a,b]) \le 4\sqrt{2} + 24\sqrt{2}\log\left(\max\left\{1, \frac{b-y_0}{\omega(a)}\right\}\right), \tag{4.20}$$

while if $N > 0$, then Lemma 4.11 and Lemma 4.12 imply

$$\begin{aligned}\ell_\Omega(\sigma;[a,b]) &\le 12\sqrt{2} + 24\sqrt{2}\log\left(\frac{y_1-y_0}{\omega(a)}\right) \\ &\quad + 32\sqrt{2}\sum_{k=1}^{N-1}\log\left(\frac{y_{k+1}-y_k}{\omega(t_k)}\right) + 24\sqrt{2}\log\left(\max\left\{1, \frac{b-y_N}{\omega(t_N)}\right\}\right).\end{aligned} \tag{4.21}$$

Then Lemma 4.9 and the fact that $\delta_k \le \omega(t_k)$ imply that there exist $A > 1$ and $B > 0$ such that for every $T_0 \le a \le b$,

$$\ell_\Omega(\sigma;[a,b]) \le A k_\Omega(\sigma(a), \sigma(b)) + B.$$

Now, since $\sigma([1, T_0])$ is compact, possibly taking a larger B, the previous estimate holds for every $1 \le a \le b$, and Theorem 4.2 is finally proved.

Remark 4.13. We also notice that by (4.20), (4.21) and Lemma 4.9, there exist constants $A, B > 0$ such that for every $1 \le a \le b$,

$$k_\Omega(\sigma(a), \sigma(b)) \le \ell_\Omega(\sigma;[a,b]) \le A\min\{k_\Omega(\sigma(a), ib), k_\Omega(\sigma(b), ia)\} + B. \tag{4.22}$$

As a consequence of the previous results, we have the following:

Proposition 4.14. *Assume there exist $c, C > 0$ such that $c\delta^-_{\Omega,0}(t) \le \delta^+_{\Omega,0}(t) \le C\delta^-_{\Omega,0}(t)$ for all $t \ge 1$. Then $\beta_i : [0, +\infty) \ni t \mapsto i + it$ can be reparametrized to be a quasi-geodesic.*

Proof. Since $\delta_\Omega(it) = \min\{\delta^-_{\Omega,0}(t), \delta^+_{\Omega,0}(t)\}$ and $\delta^-_{\Omega,0}(t)$ is comparable to $\delta^+_{\Omega,0}(t)$, there exists $C' > 1$ such that for every $t \geq 1$,

$$\omega(t) \leq C'\delta_\Omega(it).$$

In particular, by Theorem 3.5, we have for every $0 \leq a \leq b$,

$$\ell_\Omega(\beta_i; [a, b]) \leq \int_a^b \frac{d\tau}{\delta_\Omega(i\tau)} \leq \frac{1}{C'} \int_a^b \frac{d\tau}{\omega(\tau)}.$$

Therefore, in case there exist $\alpha, T_0 > 0$ such that $\omega(t) \geq \alpha t$ for all $t \geq T_0$, equation (4.5) implies that β_i can be reparametrized to be a quasi-geodesic.

On the other hand, if there exist no $\alpha, T_0 > 0$ such that $\omega(t) \geq \alpha t$ for all $t \geq T_0$, Lemmas 4.9, 4.10, 4.11, 4.12 imply again that β_i can be reparametrized to be a quasi-geodesic. □

4.2. Estimates on the distance between σ and the vertical axis

For $t \geq 1$, let $s_t \in [1, +\infty)$ be such that

$$k_\Omega(\sigma(s_t), it) = \min_{r\in[1,+\infty)} k_\Omega(\sigma(r), it). \tag{4.23}$$

Proposition 4.15. *There exist $\alpha > 1, \beta > 0$ such that for every $t \geq 1$,*

$$k_\Omega(\sigma(t), it) \leq \alpha k_\Omega(\sigma(s_t), it) + \beta.$$

Proof. Either by (4.4), or by (4.22), for $t \geq 1$, we have

$$k_\Omega(\sigma(t), \sigma(s_t)) \leq Ak_\Omega(it, \sigma(s_t)) + B.$$

Therefore

$$k_\Omega(it, \sigma(t)) \leq k_\Omega(it, \sigma(s_t)) + k_\Omega(\sigma(s_t), \sigma(t)) \leq (A + 1)k_\Omega(it, \sigma(s_t)) + B,$$

and we are done. □

5. Proof of Theorem 1.1, Theorem 1.2 and Theorem 1.3

5.1. Proof of Theorem 1.1

Let $(\phi_t), \tau, h, \Omega, \{t_n\}$ be as in Theorem 1.1.

We can suppose that (ϕ_t) is not a group of automorphisms of $\mathbb{D}$, for otherwise the result is clear.

In this case, there exists $p \in \mathbb{C}$ such that $p \notin \Omega$ and $p + it \in \Omega$ for all $t > 0$. Up to a translation, we can assume $p = 0$. In particular, this implies that $\tilde{\delta}^\pm_{\Omega,0}(t) = \delta^\pm_{\Omega,0}(t)$ for every $t > 0$.

Lemma 5.1. *The sequence $\{\phi_{t_n}(h^{-1}(i))\}$ converges to τ as $n \to +\infty$ non-tangentially (respectively, tangentially) if and only if for every $z \in \mathbb{D}$ the sequence $\{\phi_{t_n}(z)\}$ converges to τ as $n \to +\infty$ non-tangentially (respect., tangentially).*

Proof. Since $k_{\mathbb{D}}(\phi_{t_n}(h^{-1}(i)), \phi_{t_n}(z)) \leq k_{\mathbb{D}}(h^{-1}(i), z) < +\infty$ for every $n \in \mathbb{N}$, it follows that $\phi_{t_n}(z)$ is contained in a fixed hyperbolic neighborhood of $\{\phi_{t_m}(h^{-1}(i)) : m \in \mathbb{N}\}$ for all $n \in \mathbb{N}$. Therefore the result follows at once from the triangle inequality and from Proposition 3.1. □

Let σ be the curve defined in (4.1).

Lemma 5.2. $\lim_{t\to+\infty} h^{-1}(\sigma(t)) = \tau$.

Proof. By Remark 3.3, the limit $x := \lim_{t\to+\infty} h^{-1}(\sigma(t))$ exists. Suppose for a contradiction that $x \neq \tau$.

For $n \in \mathbb{N}$ consider the segments $\tilde{C}_n(s) := in + s\frac{\delta^+_{\Omega,0}(n)-\delta^-_{\Omega,0}(n)}{2}$, $0 \leq s \leq 1$. Note that $\tilde{C}_n \subset \Omega$ for all $n \in \mathbb{N}$.

Let $C_n := h^{-1}(\tilde{C}_n)$, $n \in \mathbb{N}$. Since $x \neq \tau$, the Euclidean diameter of (C_n) is bounded from below by a constant $K > 0$.

Moreover, for every sequence $\{z_n\}$ such that $z_n \in C_n$, it holds $\lim_{n\to+\infty} |h(z_n)| = \infty$.

Therefore, (C_n) is a sequence of Koebe's arcs for h, contradicting the no Koebe arcs theorem (see, *e.g.*, [17, Corollary 9.1]). □

Corollary 5.3. *The sequence $\{\phi_{t_n}(z_0)\}$ converges non-tangentially to τ as $n \to +\infty$ for all $z_0 \in \mathbb{D}$ if and only if there exists $C > 0$ such that for every $n \in \mathbb{N}$*

$$k_\Omega(it_n, \sigma([1, +\infty))) \leq C.$$

Conversely, the sequence $\{\phi_{t_n}(z_0)\}$ converges tangentially to τ as $n \to +\infty$ for all $z_0 \in \mathbb{D}$ if and only if for every $M > 0$ there exists $n_M \geq 1$ such that for all $n \geq n_M$,

$$k_\Omega(it_n, \sigma([1, +\infty))) > M.$$

Proof. By Theorem 4.2 the curve σ can be reparametrized to be a quasi-geodesic in Ω, hence by (3.1), it is "shadowed" by a geodesic γ in Ω. The curve $h^{-1}(\gamma)$ is then a geodesic in $\mathbb{D}$ and by Lemma 5.2 it converges to τ. Hence, by the triangle inequality and Proposition 3.1, the sequence $\{\phi_{t_n}(h^{-1}(i))\}$ converges non-tangentially to τ as $n \to +\infty$ if and only if it is contained in a hyperbolic neighborhood of $h^{-1}(\sigma[1, \infty))$. Since h is an isometry for the hyperbolic distance, it follows that $\{\phi_{t_n}(h^{-1}(i))\}$ converges non-tangentially to τ as $n \to +\infty$ if and only if there exists $C > 0$ such that for every $n \in \mathbb{N}$

$$k_\Omega(it_n, \sigma([1, +\infty))) \leq C.$$

Conversely, since (again by Proposition 3.1) $\{\phi_{t_n}(h^{-1}(i))\}$ converges tangentially to τ as $n \to +\infty$ if and only if it is eventually outside any hyperbolic sector around $h^{-1}(\gamma)$, by the same token as before, we get that $\{\phi_{t_n}(h^{-1}(i))\}$ converges tangentially to τ as $n \to +\infty$ if and only if for every $M > 0$ there exists $n_M \in \mathbb{N}$ such that for all $n \geq n_M$,

$$k_\Omega(it_n, \sigma([1, +\infty))) > M,$$

and we are done. □

Now, for $t \geq 1$ let s_t be defined as in (4.23). Notice that

$$k_\Omega(it, \sigma([1, +\infty))) = k_\Omega(it, \sigma(s_t))).$$

Then, by Proposition 4.15 and the Distance Lemma (see Theorem 3.5), we have for all $t \geq 1$,

$$k_\Omega(it,\sigma([1,+\infty)))\geq \frac{1}{\alpha}k_\Omega(it,\sigma(t))-\frac{\beta}{\alpha}\geq \frac{1}{4\alpha}\log\left(\frac{|\delta^+_{\Omega,0}(t)-\delta^-_{\Omega,0}(t)|}{2\min\{\delta_\Omega(it),\delta_\Omega(\sigma(t))\}}\right)-\frac{\beta}{\alpha}.$$

In other words, there exist $A, B > 0$ such that for every $t \geq 1$,

$$k_\Omega(it,\sigma([1,+\infty)))\geq A\log\left(\frac{|\delta^+_{\Omega,0}(t)-\delta^-_{\Omega,0}(t)|}{2\delta_\Omega(it)}\right)-B. \tag{5.1}$$

Now, for $t \geq 1$ let $\eta_t : [0, 1] \to \Omega$ be defined as

$$\eta_t(r) := it + r\frac{\delta^+_{\Omega,0}(t)-\delta^-_{\Omega,0}(t)}{2}.$$

For all $t \geq 1$ we have

$$k_\Omega(\sigma(s_t), it) \leq k_\Omega(it, \sigma(t)) \leq \ell_\Omega(\eta_t; [0, 1]). \tag{5.2}$$

We compute $\ell_\Omega(\eta_t; [0, 1])$. In order to do so, we need a lemma:

Lemma 5.4. *For every $t \geq 1$ and for every $r \in [0, 1]$ we have*

$$\delta_\Omega(\eta_t(r)) \geq \delta_\Omega(it).$$

Proof. Fix $t \geq 1$ and assume that $\delta^+_{\Omega,0}(t) \geq \delta^-_{\Omega,0}(t)$ (the case $\delta^+_{\Omega,0}(t) \leq \delta^-_{\Omega,0}(t)$ is similar and we omit it).

Fix $r \in [0, 1]$. Notice that $\mathsf{Re}\,\eta_t(r) \geq 0$. Therefore, if $z \in \mathbb{C} \setminus \Omega$ and $\mathsf{Re}\, z \leq 0$, then

$$|\eta_t(r) - z| \geq |it - z| \geq \delta^-_{\Omega,0}(t) = \delta_\Omega(it).$$

On the other hand, if $z \in \mathbb{C} \setminus \Omega$ and $\mathsf{Re}\, z > 0$, then $|it - z| \geq \delta^+_{\Omega,0}(t)$. Therefore,

$$\begin{aligned}|\eta_t(r)-z| &\geq \inf_{|w-it|=\delta^+_{\Omega,0}(t),\mathsf{Re}\,w>0}|\eta_t(r)-w| = \delta^+_{\Omega,0}(t)-\mathsf{Re}\,\eta_t(r)\\ &\geq \delta^+_{\Omega,0}(t)-\mathsf{Re}\,\sigma(t)=\frac{1}{2}\left(\delta^+_{\Omega,0}(t)+\delta^-_{\Omega,0}(t)\right)\geq\delta_\Omega(it),\end{aligned}$$

and we are done. □

By Lemma 5.4 and the Distance Lemma (Theorem 3.5), we have for every $t \geq 1$,

$$\begin{aligned}\ell_\Omega(\eta_t;[0,1]) &= \int_0^1 \kappa_\Omega(\eta_t(r);\eta_t'(r))dr \leq \frac{|\delta^+_{\Omega,0}(t)-\delta^-_{\Omega,0}(t)|}{2}\int_0^1\frac{dr}{\delta_\Omega(\eta_t(r))}\\ &\leq \frac{|\delta^+_{\Omega,0}(t)-\delta^-_{\Omega,0}(t)|}{2\delta_\Omega(it)}.\end{aligned}$$

This latter equation together with (5.2) and (5.1) implies that for every $t \geq 1$,

$$A\log\left(\frac{|\delta^+_{\Omega,0}(t)-\delta^-_{\Omega,0}(t)|}{2\delta_\Omega(it)}\right)-B\leq k_\Omega(it,\sigma([1,+\infty)))\leq\frac{|\delta^+_{\Omega,0}(t)-\delta^-_{\Omega,0}(t)|}{2\delta_\Omega(it)}. \tag{5.3}$$

The first part (the "non-tangential part") of Theorem 1.1 follows now directly from Corollary 5.3 and (5.3).

Also, by the same token, we see that $\phi_{t_n}(z) \to \tau$ tangentially if and only if $\frac{\delta^+_{\Omega,0}(t_n)}{\delta^-_{\Omega,0}(t_n)}$ converges either to 0 or $+\infty$ as $n \to \infty$.

We are left to show that

$$\lim_{n\to\infty} \frac{\delta^+_{\Omega,0}(t_n)}{\delta^-_{\Omega,0}(t_n)} = +\infty \tag{5.4}$$

if and only if

$$\lim_{n\to\infty} \mathsf{Arg}(1 - \overline{\tau}\phi_{t_n}(z)) = -\frac{\pi}{2}. \tag{5.5}$$

To this aim, we extend σ to all of $(0,\infty)$ in the obvious way:

$$\sigma(t) = \frac{\delta^+_{\Omega,0}(t) - \delta^-_{\Omega,0}(t)}{2} + it.$$

Since $0 \notin \Omega$ and $it \in \Omega$ for all $t > 0$, $\lim_{t\to 0^+} \sigma(t) = 0$. Then $\sigma((0,\infty))$ divides Ω into the connected domains

$$U^+ = \{x + iy \in \Omega : x > \mathsf{Re}\,\sigma(y)\}$$

and

$$U^- = \{x + iy \in \Omega : x < \mathsf{Re}\,\sigma(y)\}.$$

Hence, $\Gamma := \overline{h^{-1}(\sigma(0,+\infty))}$ divides $\mathbb{D}$ into two connected components $D^+ := h^{-1}(U^+)$ and $D^- := h^{-1}(U^-)$.

Also, there exists $\tilde{\tau} \in \partial\mathbb{D}$, $\tilde{\tau} \neq \tau$ such that $\lim_{t\to 0^-} h^{-1}(\sigma(t)) = \tilde{\tau}$ (see, *e.g.*, [13, Theorem 1, p. 37]).

By Remark 3.3 and Lemma 5.2, $h^{-1}(\sigma(t))$ converges to τ non-tangentially as $t \to +\infty$. This implies that Γ is contained in the set

$$\{z \in \mathbb{D} : |\mathsf{Arg}(1 - \overline{\tau}z)| \leq \theta\} \cup \{\tau, \tilde{\tau}\}$$

for some $\theta \in (0, \pi/2)$. Notice that the last set is an angular sector of amplitude 2θ with vertex τ symmetric with respect to segment joining $-\tau$ with τ.

Since h preserves orientation, it follows that D^+ contains all the sequences converging tangentially to τ with slope $\pi/2$ while D^- contains all the sequences converging tangentially to τ with slope $-\pi/2$.

Therefore, if (5.4) holds, then $it_n \in U^-$ for n sufficiently big, hence, $\phi_{t_n}(z) \in D^-$ eventually and (5.5) holds. Conversely, if (5.5) holds then $\phi_{t_n}(z) \in D^-$ eventually, hence, $it_n \in U^-$ eventually and (5.4) holds.

This concludes the proof of the theorem.

5.2. Proof of Theorem 1.2

The part "(1) if and only if (3)" follows immediately from Theorem 1.1. By Remark 3.3 it is clear that (2) implies (1) in Theorem 1.2. In order to end the proof, we show that (3) implies (2).

We need to prove that the orbit $[0, +\infty) \ni t \mapsto \phi_t(z)$ can be reparametrized to be a quasi-geodesic for every $z \in \mathbb{D}$. Since h is an isometry between $k_{\mathbb{D}}$ and k_Ω, the latter statement is equivalent to proving that, setting $p = h(z)$, the curve $\beta_p : [0, +\infty) \ni t \mapsto p + it$ can be reparametrized to be a quasi-geodesic in Ω.

As before, we can assume $0 \notin \Omega$ and $it \in \Omega$ for all $t > 0$. Hence, by Proposition 4.14, the curve β_i can be reparametrized to be a quasi-geodesic in Ω.

In order to complete the proof, we will prove the following:

Lemma 5.5. *For every $p \in \Omega$, there exists $A_p > 1$ and $B_p > 0$ such that*

$$\ell_\Omega(\beta_p; [s,t]) \leq A_p k_\Omega(\beta_p(s), \beta_p(t)) + B_p,$$

for all $0 \leq s \leq t$. Hence, by Proposition 3.4, β_p can be reparametrized to be a quasi-geodesic in Ω.

Proof. Fix $p \in \Omega$. By Proposition 4.14, there exists $A \geq 1$ and $B \geq 0$ such that

$$\ell_\Omega(\beta_i; [s,t]) \leq A k_\Omega(\beta_i(s), \beta_i(t)) + B, \tag{5.6}$$

for all $0 \leq s \leq t$. Now, for $0 \leq s \leq t$,

$$\begin{aligned} k_\Omega(i + is, i + it) &\leq k_\Omega(p + is, i + is) + k_\Omega(p + is, p + it) + k_\Omega(i + it, p + it) \\ &\leq k_\Omega(p + is, p + it) + 2k_\Omega(p, i), \end{aligned}$$

where the last inequality follows from the fact that $\Omega \ni z \mapsto z + it$ is a holomorphic self-map of Ω. Therefore, there exists $B_1 > 0$ such that for all $s, t \geq 0$,

$$k_\Omega(i + is, i + it) \leq k_\Omega(p + is, p + it) + B_1. \tag{5.7}$$

By Lemma 3.6 there exists $c > 0$ such that $\delta_\Omega(i + it) \leq c\delta_\Omega(p + it)$ for all $t \geq 0$. Hence, by the Distance Lemma (Theorem 3.5), for $0 \leq s \leq t$,

$$\begin{aligned} \ell_\Omega(\beta_p; [s,t]) &= \int_s^t \kappa_\Omega(\beta_p(r); \beta_p'(r))dr \leq \int_s^t \frac{dr}{\delta_\Omega(p + ir)} \\ &\leq c \int_s^t \frac{dr}{\delta_\Omega(i + ir)} \leq 4c \int_s^t \kappa_\Omega(\beta_i(r); \beta_i'(r))dr = 4c\ell_\Omega(\beta_i; [s,t]). \end{aligned}$$

Therefore, by (5.6) and (5.7)

$$\begin{aligned} \ell_\Omega(\beta_p; [s,t]) &\leq 4c\ell_\Omega(\beta_i; [s,t]) \leq 4cAk_\Omega(i + is, i + it) + 4cB \\ &\leq 4cAk_\Omega(p + is, p + it) + 4cAB_1 + 4cB, \end{aligned}$$

for all $0 \leq s \leq t$. □

5.3. Proof of Theorem 1.3

It follows directly from Theorem 1.1.

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