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The space of convex domains in complex Euclidean space

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Abstract

In this mostly expository article, we describe some properties of the space of convex domains in complex Euclidean space (endowed with the local Hausdorff topology). In particular, we give careful proofs that the Kobayashi metric, the Bergman kernel/metric, and the Kähler–Einstein metric are all continuous on the space of convex domains. The group of affine automorphisms acts on this space and we also describe the orbit closures for some special classes of domains.

Keywords Convex domains · Invariant metrics and pseudodistances

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1 Introduction

A convex domain $\Omega \subset \mathbb{C}^d$ is called $\mathbb{C}$-*properly convex* if every complex affine map $\mathbb{C} \to \Omega$ is constant. Let $\mathbb{X}_d$ denote the space of all $\mathbb{C}$-properly convex domains in $\mathbb{C}^d$ endowed with the local Hausdorff topology (see Sect. 3). Then let $\mathbb{X}_{d,0}$ denote the space of pointed $\mathbb{C}$-properly convex domains, that is

$$\mathbb{X}_{d,0} := \{(\Omega, z) : \Omega \in \mathbb{X}_d, z \in \Omega\} \subset \mathbb{X}_d \times \mathbb{C}^d .$$

Next let $\mathrm{Aff}(\mathbb{C}^d)$ denote the group of affine automorphisms of $\mathbb{C}^d$, that is maps $A : \mathbb{C}^d \to \mathbb{C}^d$ of the form

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$$A(z) = b + Mz$$

where $b \in \mathbb{C}^d$ and $M \in \mathrm{GL}_d(\mathbb{C})$. Then $\mathrm{Aff}(\mathbb{C}^d)$ has a natural continuous action on $\mathbb{X}_d$ and $\mathbb{X}_{d,0}$ given by

$$A \cdot \Omega = A\Omega \quad \text{and} \quad A \cdot (\Omega, z) = (A\Omega, Az).$$

In this mostly expository article, we will describe some properties of these spaces and actions.

The space $\mathbb{X}_{d,0}$ is quite large—by any reasonable definition it has infinitely many dimensions, while the group $\mathrm{Aff}(\mathbb{C}^d)$ is a finite dimensional Lie group. Despite this difference in size, Frankel proved that the action of $\mathrm{Aff}(\mathbb{C}^d)$ on $\mathbb{X}_{d,0}$ is nearly transitive, more precisely:

Theorem 1.1 (Frankel [15]) *The action of* $\mathrm{Aff}(\mathbb{C}^d)$ *on* $\mathbb{X}_{d,0}$ *is co-compact, that is there exists a compact set* $K \subset \mathbb{X}_{d,0}$ *such that* $\mathrm{Aff}(\mathbb{C}^d) \cdot K = \mathbb{X}_{d,0}$.

We provide a proof of this theorem in Sect. 4 and construct an explicit compact subset $\mathbb{K}_{d,0} \subset \mathbb{X}_{d,0}$ such that $\mathrm{Aff}(\mathbb{C}^d) \cdot \mathbb{K}_{d,0} = \mathbb{X}_{d,0}$.

In Sects. 5 and 6 we give two applications of Frankel's result. In Sect. 5, we use Frankel's co-compactness theorem to construct holomorphic embeddings of convex domains with certain uniform properties. In Sect. 6, we use these embeddings to show that convex domains are holomorphic homogeneous regular domains (see Definition 6.1). This result was established by Frankel [15], but was recently rediscovered independently by Kim–Zhang [25] and Nikolov–Andreev [32].

One important property of holomorphic homogeneous regular domains is that the standard invariant metrics are all complete and uniformly bi-Lipschitz to each other. In particular, given a $\mathbb{C}$-properly convex domain $\Omega \subset \mathbb{C}^d$ let b_Ω, c_Ω, g^Ω_{KE}, and k_Ω denote the Bergman metric, Carathéodory metric, Kähler–Einstein metric, and Kobayashi metric, respectively. Then we have the following.

Theorem 1.2 *For any* $d \in \mathbb{N}$ *there exists* $A = A(d) > 1$ *such that: if* $\Omega \subset \mathbb{C}^d$ *is a* $\mathbb{C}$*-properly convex domain, then the metrics* b_Ω, c_Ω, g^Ω_{KE}, *and* k_Ω *induce proper geodesic metric spaces and are all* A*-bi-Lipschitz to each other.*

Remark 1.3 It appears that Theorem 1.2 was first observed by Frankel [15].

In Sect. 7, we study the $\mathrm{Aff}(\mathbb{C}^d)$-orbit closure of certain types of bounded convex domains $\Omega \subset \mathbb{C}^d$. In particular, given a domain $\Omega \in \mathbb{X}_d$, let

$$\overline{\mathrm{Aff}(\mathbb{C}^d) \cdot \Omega}^{\mathbb{X}_d} \subset \mathbb{X}_d$$

denote the closure of the $\mathrm{Aff}(\mathbb{C}^d)$-orbit of Ω in $\mathbb{X}_d$. We will show that geometric properties of $\partial\Omega$ can be magnified by considering limits of the form $D = \lim_{n\to\infty} A_n\Omega$ where $A_n \in \mathrm{Aff}(\mathbb{C}^d)$. For instance, we will show that if Ω is a bounded convex domain with C^∞ boundary and $\partial\Omega$ contains a point of infinite type in the sense of D'Angelo, then there exists some

$$D \in \overline{\mathrm{Aff}(\mathbb{C}^d) \cdot \Omega}^{\mathbb{X}_d}$$

such that ∂D contains a non-trivial holomorphic disk (that is, there exists a non-constant map $\mathbb{D} \to \partial D$).

Sections 8–11 are devoted to proving that the four standard invariant metrics (Bergman, Carathéodory, Kähler–Einstein, and Kobayashi) are continuous on the space of convex domains. The continuity of Kobayashi metric uses standard techniques and is already known, see for instance the Appendix in [5]. The continuity of the Kähler–Einstein metric follows from a compactness result for families of Kähler metrics of quasi-bounded geometry established in [5] using tools from [37]. The continuity of the Bergman metric in the general setting of $\mathbb{C}$-properly convex domains appears to be new.

In particular, we will prove the following.

Theorem 1.4 *Suppose that Ω_n converges to Ω in $\mathbb{X}_d$. Then*

(1) $b_\Omega = \lim_{n\to\infty} b_{\Omega_n}$ *locally uniform on Ω in the C^∞ topology,*
(2) $c_\Omega = \lim_{n\to\infty} c_{\Omega_n}$ *locally uniform on Ω in the C^0 topology,*
(3) $g_{KE}^{\Omega} = \lim_{n\to\infty} g_{KE}^{\Omega_n}$ *locally uniform on Ω in the C^∞ topology, and*
(4) $k_\Omega = \lim_{n\to\infty} k_{\Omega_n}$ *locally uniform on Ω in the C^0 topology.*

Remark 1.5 Here "$f = \lim_{n\to\infty} f_n$ locally uniformly on Ω in the C^k topology" means that for every compact subset K of Ω, all the derivatives of f_n of order less than or equal to k converge to the corresponding derivative of f, uniformly on K.

Part (1) is established in Sect. 10. Part (2) is a consequence of Part (4) and deep results of Lempert. Part (3) is established in Sect. 11. Finally, Part (4) is established in Sects. 8 and 9.

We end the introduction by explaining two situations where studying the space of convex domains can lead to insight into the complex geometry of a particular convex domain.

1.1 Domains with Non-compact Automorphism Groups

There has been considerable interest in the following question (see the survey [21]):

Problem *Characterize the bounded pseudoconvex domains with smooth boundary and non-compact automorphism group.*

The first major result in this direction is the Wong–Rosay Theorem [35,36] which characterizes the ball, up to biholomorphism, as the unique strongly pseudoconvex domain with non-compact automorphism group.

Later, Bedford–Pinchuk [6] established a similar result for finite type convex domains. In particular, they proved: if $\Omega \subset \mathbb{C}^d$ is a smoothly bounded convex domain with finite type with non-compact automorphism group, then Ω is biholomorphic to a domain of the form

$$\{(z, w) \in \mathbb{C}^{d-1} \times \mathbb{C} : \mathrm{Re}(w) + H(z, \overline{z}) < 0\}, \quad (1)$$

where H is "balanced, weighted homogeneous convex polynomial." Further, in case $d = 2$, $H(z, \bar{z}) = |z|^{2m}$ for some integer $m \geq 0$.

In the same vein, Frankel [14] characterized the bounded symmetric domains as the only $\mathbb{C}$-properly convex domains in $\mathbb{C}^d$ admitting a co-compact, free, discrete action of a subgroup of its automorphism group.

The results of Bedford–Pinchuk and Frankel have the same (implicit) starting point: one considers a sequence φ_n of automorphisms of Ω and a point $p_0 \in \Omega$ where $\varphi_n(p_0)$ converges to a boundary point. Using Frankel's co-compactness theorem one can then select affine maps $A_n \in \mathrm{Aff}(\mathbb{C}^d)$ such that the set

$$\{A_n(\Omega, \varphi_n(p_0)) : n \geq 0\} \subset \mathbb{X}_{d,0}$$

is relatively compact. Then by passing to a subsequence one can assume that $A_n(\Omega, \varphi_n(p_0))$ converges to some $(\Omega_\infty, z_\infty)$ in $\mathbb{X}_{d,0}$. A normal family argument can then be used to show that a subsequence of the maps

$$f_n = A_n \varphi_n : \Omega \to A_n \Omega$$

converges to a biholomorphism $\Omega \to \Omega_\infty$. In both cases, the domain Ω_∞ can be chosen to have special structure which is then analyzed (in highly non-trivial ways!).

1.2 Geometric Properties of Domains

Frankel's co-compactness theorem can also be used as a starting point to study the interior complex geometry of a particular domain. The general philosophy is as follows: suppose that you want to show that a convex domain Ω does not have some particular property-($\star$). One can first try to show that if property-($\star$) holds for Ω, then property-($\star$) holds for every domain in

$$\overline{\mathrm{Aff}(\mathbb{C}^d) \cdot \Omega}^{\mathbb{X}_d} \subset \mathbb{X}_d \,.$$

One then can try and construct a domain in the orbit closure where it is easier to show that property-($\star$) does not hold.

This general scheme was the starting point of the following result of the authors joint work with Bracci.

Theorem 1.6 [5] *Suppose $\Omega \subset \mathbb{C}^d$ is a convex domain and Ω has a complete Kähler metric, with pinched negative holomorphic bisectional curvature outside a (possibly empty) compact subset of Ω. Then:*

(1) *Ω is a $\mathbb{C}$-properly convex domain,*
(2) *$\partial\Omega$ does not contain an analytic disk (that is, there does not exist a non-constant holomorphic map $\mathbb{D} \to \partial\Omega$), and*
(3) *if $\partial\Omega$ is a C^∞ smooth hypersurface, then $\partial\Omega$ is of finite type in the sense of D'Angelo.*

This theorem is a generalization of a classical result of Yang [38] who proved that the bidisk does not admit any complete Kähler metric with negative pinched holomorphic bisectional curvature.

This general scheme was also used by the second author to show that (2) implies (1) in the following theorem.

Theorem 1.7 [40] *Suppose that $\Omega \subset \mathbb{C}^d$ is a bounded convex domain with C^∞ boundary. Then the following are equivalent:*

(1) *$\partial\Omega$ has finite type in the sense of D'Angelo,*
(2) *the Kobayashi metric on Ω is Gromov hyperbolic.*

In some cases, one can apply the opposite argument: given a $\mathbb{C}$-properly convex domain Ω, if every domain in

$$\overline{\mathrm{Aff}(\mathbb{C}^d) \cdot \Omega}^{\mathbb{X}_d} \setminus \mathrm{Aff}(\mathbb{C}^d) \cdot \Omega$$

has some particular property-($\star$), then it is sometimes possible to show that Ω also has property-($\star$). For instance, this strategy was used to show that (1) implies (2) in Theorem 1.7.

1.3 The Purpose of this Article

In order to use the strategies described in the previous two subsection, one needs to understand the space of convex domains and its basic properties. Of particular importance is understanding the types of domains which can be found in the orbit closures

$$\overline{\mathrm{Aff}(\mathbb{C}^d) \cdot \Omega}^{\mathbb{X}_d} \subset \mathbb{X}_d$$

and verifying that the "interior complex geometry" of a domain is continuous on the space of domains $\mathbb{X}_d$. In this article, we explain some aspects of this theory.

More exposition along these lines can be found in [4,13,15,19,23,33].

1.4 Some Notations

Let us first fix some notations.

(1) For $z \in \mathbb{C}^d$ let $\|z\|$ be the standard Euclidean norm.
(2) For $z_0 \in \mathbb{C}^d$ and $r > 0$, let

$$\mathbb{B}_d(z_0; r) = \left\{ z \in \mathbb{C}^d : \|z - z_0\| < r \right\}.$$

Then let $\mathbb{B}_d = \mathbb{B}_d(0; 1)$ and $\mathbb{D} = \mathbb{B}_1$.

(3) Given a domain $\Omega \subset \mathbb{C}^d$ and a point $z \in \Omega$, let $\delta_\Omega(z)$ denote the distance from z to the boundary of Ω, that is

$$\delta_\Omega(z) = \inf\{\|z - \xi\| : \xi \in \partial\Omega\}.$$

Given a non-zero vector $v \in \mathbb{C}^d$ let $\delta_\Omega(z; v)$ denote the distance from z to the boundary of Ω in the complex direction of v, that is

$$\delta_\Omega(z; v) = \inf\{\|z - \xi\| : \xi \in \partial\Omega \cap (z + \mathbb{C} \cdot v)\}.$$

2 Invariant Metrics

In this section we define the Bergman, Carathéodory, Kähler–Einstein, and Kobayashi metrics. These metrics are all known to exist and be non-degenerate on bounded pseudoconvex domains. To obtain that these metrics also exist on $\mathbb{C}$-properly convex domains we will use the following observation.

Observation 2.1 Suppose $\Omega \subset \mathbb{C}^d$ is a convex domain. Then Ω is $\mathbb{C}$-properly convex if and only if Ω is biholomorphic to a bounded domain.

Proof If Ω is biholomorphic to a bounded domain, then every holomorphic map $\mathbb{C} \to \Omega$ is constant. In particular, every complex affine map $\mathbb{C} \to \Omega$ is constant and so Ω is $\mathbb{C}$-properly convex.

If Ω is $\mathbb{C}$-properly convex, then Ω is biholomorphic to a bounded domain by Proposition 6.2. □

2.1 The Kobayashi Metric

Definition 2.2 Let z, z' be two points in a complex manifold M and $v \in T_z M$, where $T_z M$ is the space of tangent vectors at z.

- The *infinitesimal Kobayashi (pseudo-)metric* $k_M(z; v)$ is given by

$$k_M(z; v) = \inf\left\{|\alpha| : \exists f \in \mathrm{Hol}(\mathbb{D}, M),\ f(0) = z,\ df_0\alpha = v\right\}.$$

- The *Kobayashi (pseudo-)distance* is the length function defined by

$$K_M(z, z') = \inf\left\{\int_0^1 k_M(\gamma(t); \gamma'(t))\mathrm{d}t\right\},$$

  where the infimum is taken over all piecewise $\mathcal{C}^1$ curves $\gamma : [0, 1] \to M$, joining z to z'.

The Kobayashi (pseudo-)metric is a complex Finsler (pseudo-)metric that has only weak regularity in general. It is the largest complex Finsler metric on a complex manifold that coincides with the Poincaré metric on the unit disk $\mathbb{D} \subset \mathbb{C}$ and that is decreasing under the composition by holomorphic maps, meaning that if $f : M \to N$ is holomorphic, then for every $z \in M$ and every $v \in T_z M$,

$$k_N(f(z), df_z(v)) \leq k_M(z, v).$$

When $\Omega \subset \mathbb{C}^d$ is a bounded domain, it is fairly easy to show that K_Ω is a non-degenerate metric, but in general it is very difficult to determine if K_Ω is a Cauchy complete metric. In the convex case, things are easier and we have the following result of Barth.

Theorem 2.3 (Barth [3]) *Suppose $\Omega \subset \mathbb{C}^d$ is a convex domain. Then the following are equivalent:*

(1) *Ω is $\mathbb{C}$-properly convex,*
(2) *K_Ω is a non-degenerate metric on Ω,*
(3) *(Ω, K_Ω) is a proper geodesic metric space.*

Remark 2.4 Recall that a metric space is called *proper* if closed bounded sets are compact. A proper metric space is always Cauchy complete. A metric space (X, d) is called *geodesic* if every two points can be joined by a geodesic segment, i.e., for all $x, y \in X$ there exists a map $\sigma : [0, T] \to X$ with $\sigma(0) = x$, $\sigma(T) = y$, and

$$d(\sigma(s), \sigma(t)) = |s - t|$$

for all $s, t \in [0, T]$.

2.2 The Carathéodory Metric

Definition 2.5 Let z, z' be two points in a complex manifold M and $v \in T_z M$.

- The *infinitesimal Carathéodory (pseudo-)metric* $k_M(z; v)$ is given by

$$c_M(z; v) = \sup\left\{|df_z(v)| : \ f \in \mathrm{Hol}(M, \mathbb{D}), \ f(z) = 0\right\}.$$

- The *Carathéodory (pseudo-)distance* is the length function defined by

$$C_M(z, z') = \inf\left\{\int_0^1 c_M(\gamma(t); \gamma'(t))\mathrm{d}t\right\},$$

  where the infimum is taken over all piecewise $\mathcal{C}^1$ curves $\gamma : [0, 1] \to M$, joining z to z'.

Notice that by definition $c_M \leq k_M$ and hence $C_M \leq K_M$. A deep result of Lempert [26] shows that on $\mathbb{C}$-properly convex domains the Carathéodory and Kobayashi metrics coincide.

Theorem 2.6 (Lempert [26]) *If $\Omega \subset \mathbb{C}^d$ is a $\mathbb{C}$-properly convex domain, then $c_\Omega = k_\Omega$.*

Remark 2.7 To be precise, in [26] the above theorem is only established for bounded convex domains, but a simple argument extends the result to all $\mathbb{C}$-properly convex domains (see for instance [8, Lemma 3.1]).

2.3 The Bergman Metric

Let Ω be a domain in $\mathbb{C}^d$ and let

$$\mathcal{H}^2(\Omega) := \left\{ f \in \mathrm{Hol}(\Omega, \mathbb{C}) : \int_\Omega |f|^2 \mathrm{d}\mu < +\infty \right\},$$

where $d\mu$ denotes the Lebesgue measure on $\mathbb{C}^d$. If $\mathcal{H}^2(\Omega) \neq \{0\}$, then it is a non-trivial Hilbert space, equipped with the L^2-inner product. The *Bergman kernel of* Ω, denoted by κ_Ω, is the function defined on $\Omega \times \Omega$ by

$$\kappa_\Omega(z, w) = \sum_{j \geq 0} \phi_j(z) \overline{\phi_j(w)},$$

where $\{\phi_j, \ j = 1, 2, \ldots\}$ is an orthonormal basis of the Hilbert space $\mathcal{H}^2(\Omega)$. It is uniquely defined and does not depend on the choice of an orthonormal basis of $\mathcal{H}^2(\Omega)$, see [22, Chapter 12].

The Bergman (pseudo-)metric is then defined as follows.

Definition 2.8 Suppose $\Omega \subset \mathbb{C}^d$ is a domain and $\kappa_\Omega(z, z) > 0$ for all $z \in \Omega$. Then the *Bergman (pseudo-)metric on* Ω is the smooth (1,1)-Hermitian form

$$b_\Omega(z; \cdot, \cdot) := \sum_{j,k=1}^{n} b^\Omega_{j\bar{k}}(z) \mathrm{d}z_j \otimes \mathrm{d}\overline{z_k}$$

where

$$b^\Omega_{j\bar{k}}(z) := \frac{\partial^2(\zeta \mapsto \log \kappa_\Omega(\zeta, \zeta))}{\partial \zeta_j \partial \overline{\zeta_k}}(z).$$

According to [22, Corollary 12.7.6, p. 486], $b_\Omega(z; \cdot, \cdot)$ defines a metric (i.e., is positive definite), if for every $v \in \mathbb{C}^d \setminus \{0\}$, there exists $f \in \mathcal{H}^2(\Omega)$ such that $df_z(v) \neq 0$. This is the case, for instance, if Ω is biholomorphic to a bounded domain in $\mathbb{C}^d$.

In the rest of this subsection, we will recall some basic properties of the Bergman metric and kernel.

We first explain why this definition is invariant under biholomorphisms. When $\Phi : \Omega_1 \to \Omega_2$ is a biholomorphism and $\{\phi_j, \ j = 1, 2, \ldots\}$ is an orthonormal basis of the Hilbert space $\mathcal{H}^2(\Omega_2)$, then $\{J(\Phi)\phi_j \circ \Phi, \ j = 1, 2, \ldots\}$ is an orthonormal basis of the Hilbert space $\mathcal{H}^2(\Omega_1)$, where $J(\Phi)(z) := \det(\partial \Phi_i / \partial z_j(z))_{ij}$. Hence

$$\kappa_{\Omega_2}(\Phi(z), \Phi(w)) J(\Phi)(z) \overline{J(\Phi)(w)} = \kappa_{\Omega_1}(z, w) \tag{2}$$

for all $z, w \in \Omega_1$. Using Eq. (2) it is straightforward to show that

$$b_{\Omega_2}(\Phi(z); d(\Phi)_z v, d(\Phi)_z w) = b_{\Omega_1}(z; v, w)$$

for all $z \in \Omega_1$ and $v, w \in \mathbb{C}^d$. Thus the Bergman metric is invariant under biholomorphisms.

We next state some basic estimates. The Bergman kernel is a reproducing kernel (see [22, Chapter 12]) and hence

$$f(z) = \int_\Omega f(w)\kappa_\Omega(z, w)\mathrm{d}\mu(w). \tag{3}$$

for every $f \in \mathcal{H}^2(\Omega)$ and every $z \in \Omega$. In particular,

$$\kappa_\Omega(z, z) = \int_\Omega \kappa_\Omega(w, z)\kappa_\Omega(z, w)\,\mathrm{d}\mu(w) = \int_\Omega |\kappa_\Omega(\cdot, z)|^2 \mathrm{d}\mu = \|\kappa_\Omega(\cdot, z)\|_2^2 \tag{4}$$

for every $z \in \Omega$. Further, if $z_1, z_2 \in \Omega$, then the Cauchy–Schwarz inequality implies that

$$|\kappa_\Omega(z_1, z_2)| = \left| \int_\Omega \kappa(w, z_1)\kappa(z_2, w)\mathrm{d}\mu(w) \right| \leq \|\kappa_\Omega(\cdot, z_1)\|_2 \, \|\kappa_\Omega(\cdot, z_2)\|_2$$

and hence

$$|\kappa_\Omega(z_1, z_2)|^2 \leq \kappa_\Omega(z_1, z_1)\kappa_\Omega(z_2, z_2). \tag{5}$$

Next we describe how $\kappa_\Omega(\cdot, z)$ is the solution to a certain optimization problem. For every $z \in \Omega$, let

$$I_0^\Omega(z) := \inf \left\{ \int_\Omega |f|^2 \mathrm{d}\mu : \; f \in \mathcal{H}^2(\Omega), \; f(z) = 1 \right\}$$

with the convention $I_0^\Omega(z) = \infty$ if $\left\{ f \in \mathcal{H}^2(\Omega) : \; f(z) \neq 0 \right\} = \emptyset$.

The following lemma gives the existence and the uniqueness, for every $z \in \Omega$, of a function $f \in \mathcal{H}^2(\Omega)$ such that $f(z) = 1$ and $I_0^\Omega(z) = \int_\Omega |f|^2 \mathrm{d}\mu$.

Lemma 2.9 *Suppose $\Omega \subset \mathbb{C}^d$ is domain, $z \in \Omega$, $\kappa_\Omega(z, z) > 0$, and $f \in \mathcal{H}^2(\Omega)$. Then $f(z) = 1$ and $\int_\Omega |f|^2 \,\mathrm{d}\mu = I_0^\Omega(z)$ if and only if*

$$f = \frac{1}{\kappa_\Omega(z, z)} \kappa_\Omega(\cdot, z).$$

In particular,

$$\kappa_\Omega(z, z) = \frac{1}{I_0^\Omega(z)}.$$

Proof Notice that if $f \in \mathcal{H}^2(\Omega)$ and $f(z) = 1$, then

$$1 = f(z) = \int_\Omega f(w)\kappa_\Omega(w, z)\mathrm{d}\mu(w) \leq \|f\|_2 \, \|\kappa_\Omega(\cdot, z)\|_2 = \|f\|_2 \, \kappa_\Omega(z, z).$$

Hence

$$I_0^\Omega(z) \geq \frac{1}{\kappa_\Omega(z, z)}. \tag{6}$$

($\Leftarrow$): Define

$$f_z := \frac{1}{\kappa_\Omega(z, z)}\kappa_\Omega(\cdot, z).$$

Then $f_z(z) = 1$. Further Eqs. (6) and (4) imply

$$I_0^\Omega(z) \leq \int_\Omega |f_z|^2 \, \mathrm{d}\mu = \frac{1}{\kappa_\Omega(z, z)^2} \int_\Omega |\kappa_\Omega(\cdot, z)|^2 \, \mathrm{d}\mu = \frac{1}{\kappa_\Omega(z, z)} \leq I_0^\Omega(z).$$

So $\int_\Omega |f_z|^2 \, \mathrm{d}\mu = I_0^\Omega(z)$.
($\Rightarrow$): Suppose that $f \in \mathcal{H}^2(\Omega)$, $f(z) = 1$, and $\int_\Omega |f|^2 \, \mathrm{d}\mu = I_0^\Omega(z)$. Consider $h = \frac{1}{2}(f + f_z)$. Then $h(z) = 1$ and

$$\begin{aligned} I_0^\Omega(z) \leq \int_\Omega |h|^2 \, \mathrm{d}\mu &= \frac{1}{4}\left(\|f\|_2^2 + \|f_z\|_2^2 + 2\,\mathrm{Re}\,\langle f, f_z\rangle\right) \\ &\leq \frac{1}{4}\left(\|f\|_2^2 + \|f_z\|_2^2 + 2\,\|f\|_2\,\|f_z\|_2\right) = I_0^\Omega(z). \end{aligned}$$

So we are in the equality case in the Cauchy–Schwarz inequality. Hence $f = \lambda f_z$ for some $\lambda \in \mathbb{C}$. Since $f(z) = 1 = f_z(z)$ we must have $\lambda = 1$ and so $f = f_z$. □

As a consequence of Lemma 2.9 we have the following estimate.

Lemma 2.10 *If $\Omega_1 \subset \Omega_2 \subset \mathbb{C}^d$ are domains, then*

$$\kappa_{\Omega_2}(z, z) \leq \kappa_{\Omega_1}(z, z) \tag{7}$$

for every $z \in \Omega_1$.

We end our discussion of Bergman kernel by recalling a classical result of Ramadanov (see [22, Theorem 12.1.23, p. 428] for the case where Ω is unbounded).

Theorem 2.11 (Ramadanov [34]) *Suppose that*

$$\Omega_1 \subset \Omega_2 \subset \cdots$$

is an increasing sequence of domains and $\Omega := \cup_{n\geq 1}\Omega_n$. *Then*

$$\kappa_\Omega = \lim_{n\to\infty} \kappa_{\Omega_n},$$

locally uniformly in the C^∞ *topology on* $\Omega \times \Omega$.

2.4 The Kähler–Einstein Metric

Let $g := (g_{i\bar{j}})_{i\bar{j}}$ be a Hermitian metric on a complex manifold M, of class C^∞, and let ω_g be the associated symplectic form given in local holomorphic coordinates by

$$\omega_g := \sqrt{-1}\sum_{i,j=1}^{n} g_{i\bar{j}} \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_j.$$

The Ricci form of ω_g, denoted $\mathrm{Ric}(\omega_g)$, is the $(1, 1)$-form defined by

$$\mathrm{Ric}(\omega_g) := -\sqrt{-1}\partial\bar{\partial}\log(\det(g_{i\bar{j}})).$$

Definition 2.12 Let g be a Hermitian metric on a complex manifold M. We say that

- g is a *Kähler metric* (or equivalently that (M, ω_g) is a *Kähler manifold*) if $d\omega_g = 0$ on M,
- g is an *Einstein metric* if there exists $\lambda \in \mathbb{R}$ such that $\mathrm{Ric}(\omega_g) = \lambda\omega_g$ and we call λ the *Ricci curvature* (or *Ricci constant*) of the Einstein metric g,
- g is a *Kähler–Einstein metric* if it is both a Kähler and an Einstein metric.

It is a deep result of Cheng and Yau [9] and Mok and Yau [31] that any bounded pseudoconvex domain $\Omega \subset \mathbb{C}^d$ admits a unique Kähler–Einstein metric with Ricci constant equal to $-(d + 1)$ (and hence a unique Kähler–Einstein metric with Ricci constant λ for every $\lambda < 0$).

Since every $\mathbb{C}$-properly convex domain is biholomorphic to a bounded pseudoconvex domain, the following definition makes sense.

Definition 2.13 If $\Omega \subset \mathbb{C}^d$ is a $\mathbb{C}$-properly convex domain, then let g^{Ω}_{KE} denote the unique Kähler–Einstein metric with Ricci constant equal to $-(d + 1)$.

3 Topology on $\mathbb{X}_d$

In this section we describe the local Hausdorff topology on the set of $\mathbb{C}$-properly convex domains.

Let A and B be two sets in $\mathbb{C}^d$. The *Hausdorff distance* between A and B is defined by

$$d_H(A, B) = \max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\}.$$

Despite the name, this does not yield a distance on the set of all sets. For instance,

$$d_H\left(\mathbb{D}, \overline{\mathbb{D}}\right) = d_H\left(\mathbb{D}\setminus\{0\}, \mathbb{D}\right) = d_H\left(\mathbb{D}\setminus\{0\}, \overline{\mathbb{D}}\right) = 0.$$

However, when restricted to compact subsets we do obtain a distance.

Theorem 3.1 *Let $\mathcal{K}$ be the set of all compact subsets of $\mathbb{C}^d$. Then $(\mathcal{K}, d_H)$ is a complete metric space.*

The Hausdorff distance also does not behave very well on the set of all closed sets. For instance, if

$$C_n := \{x + iy \in \mathbb{C} : |x| \leq n \text{ and } y \geq 0\},$$

then one would hope that the sequence C_n converges to

$$C := \{x + iy \in \mathbb{C} : y \geq 0\}.$$

However,

$$d_H\left(C, C_n\right) = \infty$$

for every $n \geq 0$.

These types of examples can be handled by considering the local Hausdorff semi-norm. For $R > 0$ and A a closed set in $\mathbb{C}^d$, define $A^{(R)} := A \cap \overline{\mathbb{B}_d(0; R)}$. Then, we define the *local Hausdorff semi-norms* by

$$d_H^{(R)}(A, B) := d_H\left(A^{(R)}, B^{(R)}\right).$$

Then we say that a sequence C_n of closed sets *converges in the local Hausdorff topology* to a closed set C if there exists $R_0 \geq 0$ such that

$$\lim_{n\to\infty} d_H^{(R)}(C_n, C) = 0$$

for all $R \geq R_0$.

Since an open convex domain is completely determined by its closure, we can use the local Hausdorff topology to obtain a topology on the set of all convex domains.

Definition 3.2 (1) A sequence Ω_n in $\mathbb{X}_d$ converges to Ω in $\mathbb{X}_d$ if $\overline{\Omega}_n$ converges to $\overline{\Omega}$ in the local Hausdorff topology.

(2) A sequence (Ω_n, z_n) in $\mathbb{X}_{d,0}$ converges to (Ω, z) in $\mathbb{X}_{d,0}$ if $\overline{\Omega}_n$ converges to $\overline{\Omega}$ in the local Hausdorff topology and z_n converges to z.

The following facts about this notion of convergence will be useful.

Proposition 3.3 *Suppose that Ω_n converges to Ω in $\mathbb{X}_d$. Then:*

(1) *If* $K \subset \Omega$ *is compact, then there exists* $N \geq 0$ *such that* $K \subset \Omega_n$ *for all* $n \geq N$.
(2) *If* $z_n \in \Omega_n$ *and* $\lim_{n\to\infty} z_n = z$, *then* $z \in \overline{\Omega}$.
(3) *If* $z_n \in \mathbb{C}^d \setminus \Omega_n$ *and* $\lim_{n\to\infty} z_n = z$, *then* $z \in \mathbb{C}^d \setminus \Omega$.

Proof Part (2) and (3) are immediate from the definition. To prove Part (1) fix a compact set $K \subset \Omega$. Assume for a contradiction that there exist $n_k \to \infty$ and $x_{n_k} \in K$ such that $x_{n_k} \notin \Omega_{n_k}$ for every $k \geq 0$. Up to passing to a subsequence we may assume that $\lim_{k\to\infty} x_{n_k} = x_\infty \in K$. Since each Ω_{n_k} is convex, each Ω_{n_k} is contained in a real half space passing through x_{n_k}. In particular, if $\epsilon > 0$ is such that $\overline{\mathbb{B}_d(x_\infty; \epsilon)} \subset \Omega$, then for k sufficiently large there is a point $y_{n_k} \in \mathbb{B}_d(x_\infty; \epsilon)$ such that $d_{\mathrm{Euc}}(y_{n_k}, \Omega_{n_k}) \geq \epsilon/2$.

Now fix $R > \|x_\infty\| + \epsilon$ so that

$$\lim_{n\to\infty} d_H^{(R)}(\Omega_n, \Omega) = 0.$$

Since $y_{n_k} \in \mathbb{B}_d(x_\infty; \epsilon) \subset \Omega$ and $d_{\mathrm{Euc}}(y_{n_k}, \Omega_{n_k}) \geq \epsilon/2$, we see that

$$d_H^{(R)}(\Omega_n, \Omega) \geq \epsilon/2,$$

which is impossible. □

4 Frankel's Co-compactness Theorem

In this section we show that the action of $\mathrm{Aff}(\mathbb{C}^d)$ on $\mathbb{X}_{d,0}$ is proper and co-compact.

Proposition 4.1 *The group* $\mathrm{Aff}(\mathbb{C}^d)$ *acts properly on* $\mathbb{X}_{d,0}$, *that is if* $K \subset \mathbb{X}_{d,0}$ *is compact, then the set*

$$\left\{A \in \mathrm{Aff}(\mathbb{C}^d) : AK \cap K \neq \emptyset\right\}$$

is compact in $\mathrm{Aff}(\mathbb{C}^d)$.

Proof Suppose not, then there exists a compact set $K \subset \mathbb{X}_{d,0}$ such that the set

$$\left\{A \in \mathrm{Aff}(\mathbb{C}^d) : AK \cap K \neq \emptyset\right\}$$

is not compact in $\mathrm{Aff}(\mathbb{C}^d)$. Then we can find sequences $(\Omega_n, z_n) \in K$ and $A_n \in \mathrm{Aff}(\mathbb{C}^d)$ such that

(1) $A_n \to \infty$ in $\mathrm{Aff}(\mathbb{C}^d)$ (that is, the sequence A_n leaves every compact subset of $\mathrm{Aff}(\mathbb{C}^d)$),
(2) (Ω_n, z_n) converges to some (U_1, u_1) in K, and
(3) $A_n(\Omega_n, z_n)$ converges to some (U_2, u_2) in K.

For $(\Omega, z) \in \mathbb{X}_{d,0}$ let

$$I(\Omega, z) = \{v \in \mathbb{C}^d : v = 0 \text{ or } \|v\| \leq \delta_\Omega(z; v)\}.$$

Notice that this is always a compact set.

Since (Ω_n, z_n) and $A_n(\Omega_n, z_n)$ are both convergent sequences in $\mathbb{X}_{d,0}$ there exists $0 < c < C$ such that

$$c\,\mathbb{B}_d \subset I(\Omega_n, z_n) \subset C\,\mathbb{B}_d$$

and

$$c\,\mathbb{B}_d \subset I(A_n\Omega_n, A_n z_n) \subset C\,\mathbb{B}_d$$

for all $n \geq 0$.

Now suppose that $A_n(z) = b_n + g_n(z)$ where $b_n \in \mathbb{C}^d$ and $g_n \in \mathrm{GL}_d(\mathbb{C})$. Then

$$g_n I(\Omega_n, z_n) = I(A_n\Omega, A_n z_n)$$

and so

$$\frac{c}{C}\,\mathbb{B}_d \subset g_n\,\mathbb{B}_d \subset \frac{C}{c}\,\mathbb{B}_d$$

for $n \geq 0$. Hence $\{g_n : n \in \mathbb{N}\}$ is relatively compact in $\mathrm{GL}_d(\mathbb{C})$. Then since $z_n \to u_1$ and $A_n z_n \to u_2$ we see that $\{b_n : n \in \mathbb{N}\}$ must be relatively compact in $\mathbb{C}^d$. So $\{A_n : n \in \mathbb{N}\}$ is relatively compact in $\mathrm{Aff}(\mathbb{C}^d)$. Hence we have a contradiction. □

Theorem 4.2 (Frankel [15]) *There exists a compact set $K \subset \mathbb{X}_{d,0}$ such that* $\mathrm{Aff}(\mathbb{C}^d) \cdot K = \mathbb{X}_{d,0}$.

Then rest of the section is devoted to a proof of Frankel's theorem.

Definition 4.3 Let $(e_1, \ldots, e_d)$ denote the standard basis of $\mathbb{C}^d$. Then, let $\mathbb{K}_d \subset \mathbb{X}_d$ denote the set of convex domains $\Omega \subset \mathbb{C}^d$ where $\mathbb{D} \cdot e_j \subset \Omega$ and

$$\left(e_j + \mathrm{Span}_{\mathbb{C}}\{e_{j+1}, \ldots, e_d\}\right) \cap \Omega = \emptyset$$

for all $j = 1, \ldots, d$. Also, define

$$\mathbb{K}_{d,0} = \{(\Omega, 0) : \Omega \in \mathbb{K}_d\}.$$

Proposition 4.4 *The set $\mathbb{K}_d$ is a compact subset of $\mathbb{X}_d$.*

Proof Suppose Ω_n is a sequence in $\mathbb{K}_d$. For each m, the set

$$\left\{K \subset \overline{\mathbb{B}_d(0; m)} : K \text{ is compact}\right\}$$

is compact in the Hausdorff topology. So we can find nested subsequences

$$(n_{1,j})_{j=1}^{\infty} \supset (n_{2,j})_{j=1}^{\infty} \supset \cdots$$

such that

$$\lim_{j\to\infty} \overline{\Omega}_{n_{m,j}} \cap \overline{\mathbb{B}_d(0;m)} = C_m,$$

where C_m is a closed convex domain. Then $\overline{\Omega}_{n_{m,m}}$ converges in the local Hausdorff topology to $C := \cup_{m=1}^{\infty} C_m$.

Let Ω_∞ denote the interior of C. Since $\mathbb{D}\cdot e_j \subset \Omega_n$ for every n, we see that $\mathbb{D}\cdot e_j \subset C$. So C has non-empty interior. So Ω_∞ is non-empty and $\overline{\Omega}_\infty = C$. Then $\Omega_{n_{m,m}}$ converges to Ω_∞ in the local Hausdorff topology.

We claim that $\Omega_\infty \in \mathbb{K}_d$. Since each Ω_n is in $\mathbb{K}_d$, we see that $\mathbb{D}\cdot e_j \subset \Omega_\infty$ and

$$\left(e_j + \mathrm{Span}_{\mathbb{C}}\{e_{j+1},\dots,e_d\}\right) \cap \Omega_\infty = \emptyset$$

for all $j = 1,\dots,d$. So we just have to show that $\Omega_\infty \in \mathbb{X}_d$. To show this it is enough to show that every affine map $\mathbb{C} \to \Omega_\infty$ is constant. Let $\ell : \mathbb{C} \to \Omega_\infty$ be such a map. Then $\ell(z) = a + bz$ for some $a, b \in \mathbb{C}^d$. Since Ω_∞ is open, convex, and $0 \in \Omega$ we then see that $bz \in \Omega_\infty$ for every $z \in \mathbb{C}$.

Since

$$\left(e_1 + \mathrm{Span}_{\mathbb{C}}\{e_2,\dots,e_d\}\right) \cap \Omega_\infty = \emptyset$$

we must have $b_1 = 0$. Then since

$$\left(e_2 + \mathrm{Span}_{\mathbb{C}}\{e_3,\dots,e_d\}\right) \cap \Omega_\infty = \emptyset$$

we must have $b_2 = 0$. Repeating the same argument shows that $b_3 = b_4 = \cdots = b_d = 0$. Thus ℓ is constant and since ℓ was an arbitrary affine map, we see that $\Omega_\infty \in \mathbb{X}_d$. □

Proposition 4.5 *For every $(\Omega, z) \in \mathbb{X}_{d,0}$, there exists an affine map $A \in \mathrm{Aff}(\mathbb{C}^d)$ such that $A(\Omega, z) \in \mathbb{K}_{d,0}$.*

Proof Fix $(\Omega, z) \in \mathbb{X}_{d,0}$. By applying an initial affine automorphism we can assume that $z = 0$.

We begin by picking points $\xi_1,\dots,\xi_d \in \partial\Omega$ as follows: first let ξ_1 be a point in $\partial\Omega$ closest to 0. Then assuming $\xi_1,\dots,\xi_j$ have already been selected, let V_j be the maximal complex linear subspace through 0 orthogonal to the complex lines $\mathbb{C}\cdot\xi_1,\dots,\mathbb{C}\cdot\xi_j$. Then let ξ_{j+1} be a point in $V_j \cap \partial\Omega$ closest to 0.

Notice that by construction $\mathbb{D}\cdot\xi_j \subset \Omega$,

$$\left(\xi_j + V_{j+1}\right) \cap \Omega = \emptyset,$$

and

$$V_{j+1} = \mathrm{Span}_{\mathbb{C}}\left\{\xi_{j+1},\dots,\xi_d\right\}$$

for every $1 \leq j \leq d$.

Once $\xi_1, \ldots, \xi_d$ have been selected let $\tau_j = \left\| \xi_j \right\|$ for $1 \leq i \leq d$ and define $\Lambda \in \mathrm{GL}_d(\mathbb{C})$ to be the linear map

$$\begin{pmatrix} \tau_1^{-1} & & \\ & \ddots & \\ & & \tau_d^{-1} \end{pmatrix}.$$

Next let U be the unitary map such that

$$\Lambda U(\xi_i) = e_i.$$

Notice that if $\Omega' = (\Lambda U)\Omega$, then $\mathbb{D} \cdot e_j \subset \Omega'$ and

$$\left(e_j + \mathrm{Span}_{\mathbb{C}}\{e_{j+1}, \ldots, e_d\}\right) \cap \Omega = \emptyset$$

for all $1 \leq j \leq d$. Hence $(\Lambda U)(\Omega, 0) \in \mathbb{K}_{d,0}$.

□

5 Uniform Bounded Embeddings

In this section we show that every $\mathbb{C}$-properly convex domain is biholomorphic to a bounded domain (in a uniform way).

5.1 Supporting Vectors

Given a convex domain $\Omega \in \mathbb{K}_d$, we say that vectors $(v_1, \ldots, v_d)$ are Ω-*supporting* if

$$e_j + \mathrm{Span}_{\mathbb{C}}\{e_{j+1}, \ldots, e_d\} \subset \{z \in \mathbb{C}^d : \mathrm{Re}\left\langle z, v_j \right\rangle = 1\}$$

and

$$\Omega \subset \{z \in \mathbb{C}^d : \mathrm{Re}\left\langle z, v_j \right\rangle < 1\}$$

for every $j = 1, \ldots, d$.

Lemma 5.1 *If $\Omega \in \mathbb{K}_d$, then there exist Ω-supporting vectors.*

Proof Fix $1 \leq j \leq d$. Since

$$\left(e_j + \mathrm{Span}_{\mathbb{C}}\{e_{j+1}, \ldots, e_d\}\right) \cap \Omega = \emptyset$$

and Ω is convex, there exists a real hyperplane H_j such that

$$\left(e_j + \mathrm{Span}_{\mathbb{C}}\{e_{j+1}, \ldots, e_d\}\right) \subset H_j$$

and $H_j \cap \Omega = \emptyset$. Then there exists a vector $v_j = (v_{j,1}, \ldots, v_{j,d})$ such that

$$H_j = \{z \in \mathbb{C}^d : \mathrm{Re}\langle z, v_j\rangle = \mathrm{Re}\langle e_j, v_j\rangle\} = \{z \in \mathbb{C}^d : \mathrm{Re}\langle z, v_j\rangle = \mathrm{Re}\, v_{j,j}\}$$

and

$$\Omega \subset \{z \in \mathbb{C}^d : \mathrm{Re}\langle z, v_j\rangle < \mathrm{Re}\, v_{j,j}\}$$

Since $0 \in \Omega$, we see that $0 < \mathrm{Re}\, v_{j,j}$. So $v_{j,j} \neq 0$. Since $\mathbb{D} \cdot e_j \subset \Omega$,

$$\left|v_{j,j}\right| = \mathrm{Re}\left\langle \frac{v_{j,j}}{\left|v_{j,j}\right|} e_j, v_j \right\rangle \leq \mathrm{Re}\, v_{j,j}.$$

So we must have $v_{j,j} \in (0, \infty)$ and then we can scale v_j so that $v_{j,j} = 1$. Then

$$H_j = \{z \in \mathbb{C}^d : \mathrm{Re}\langle z, v_j\rangle = 1\}$$

and

$$\Omega \subset \{z \in \mathbb{C}^d : \mathrm{Re}\langle z, v_j\rangle < 1\}.$$

□

Proposition 5.2 *The set*

$$\{(\Omega, v_1, \ldots, v_d) : \Omega \in \mathbb{K}_d, (v_1, \ldots, v_d) \textit{ is } \Omega\textit{-supporting}\}$$

is compact in $\mathbb{X}_d \times \mathbb{C}^d$.

Since $\mathbb{K}_d$ is compact in $\mathbb{X}_d$, to prove Proposition 5.2 it is enough to establish uniform bounds on the supporting vectors. This is accomplished in the next lemma.

Lemma 5.3 *Suppose that* $\Omega \in \mathbb{K}_d$ *and* $(v_1, \ldots, v_d)$ *are* Ω*-supporting. Then*

(1) $v_{j,j} = 1$,
(2) $v_{j,k} = 0$ *if* $k > j$,
(3) $\left|v_{j,k}\right| \leq 1$ *when* $k < j$.

Proof Since

$$e_j \in \{z \in \mathbb{C}^d : \mathrm{Re}\langle z, v_j\rangle = 1\}$$

we see that $\mathrm{Re}(v_{j,j}) = 1$. Since $\mathbb{D} \cdot e_j \subset \Omega$ we see that

$$1 = \mathrm{Re}(v_{j,j}) \leq \left|v_{j,j}\right| = \mathrm{Re}\left\langle \frac{v_{j,j}}{\left|v_{j,j}\right|} e_j, v_j \right\rangle \leq 1.$$

Hence $v_{j,j} = 1$.

If $k > j$, then

$$e_j + \lambda e_k \in \left\{ z \in \mathbb{C}^d : \mathrm{Re} \left\langle z, v_j \right\rangle = 1 \right\}$$

for every $\lambda \in \mathbb{C}$. So

$$1 = \mathrm{Re} \left\langle e_j + \lambda e_k, v_j \right\rangle = 1 + \lambda \overline{v}_{j,k}$$

and hence $v_{j,k} = 0$.

If $k < j$, then $\mathbb{D} \cdot e_k \subset \Omega$ and so

$$1 > \mathrm{Re} \left\langle \lambda e_k, v_j \right\rangle = \lambda \overline{v}_{j,k}$$

for every $\lambda \in \mathbb{D}$. So $\left| v_{j,k} \right| \leq 1$. □

5.2 Uniform Bounded Embeddings

Now we use these supporting vectors to provide nice bounded embeddings.

Proposition 5.4 *Suppose that $\Omega \in \mathbb{K}_d$ and $V = (v_1, \ldots, v_d)$ are Ω-supporting. Define the function $F_{\Omega,V} : \Omega \to \mathbb{C}^d$ by*

$$F_{\Omega,V}(z) = \left(\frac{\langle z, v_1 \rangle}{2 - \langle z, v_1 \rangle}, \ldots, \frac{\langle z, v_d \rangle}{2 - \langle z, v_d \rangle} \right).$$

Then $F_{\Omega,V}$ is a holomorphic embedding and $F_{\Omega,V}(\Omega) \subset \mathbb{D}^d$. Moreover, there exists $\epsilon > 0$, independent of Ω and V, such that $\mathbb{B}_d(0; \epsilon) \subset F_{\Omega,V}(\Omega)$.

Proof Define $H := \{z \in \mathbb{C} : \mathrm{Re}(z) < 1\}$. Then the map $f : H \to \mathbb{D}$ given by

$$f(z) = \frac{z}{2 - z}$$

is a biholomorphism. By Lemma 5.3 the vectors $(v_1, \ldots, v_d)$ form a basis of $\mathbb{C}^d$. So the map $g : \Omega \to H^d$ given by

$$g(z) = (\langle z, v_1 \rangle, \ldots, \langle z, v_d \rangle)$$

is a holomorphic embedding. So $F_{\Omega,V} = (f \circ g_1, \ldots, f \circ g_d)$ is a holomorphic embedding of Ω into $\mathbb{D}^d$.

The "moreover" part follows from the fact that the set

$$\{(\Omega, v_1, \ldots, v_d) : \Omega \in \mathbb{K}_d, (v_1, \ldots, v_d) \text{ is } \Omega\text{ -supporting}\}$$

is compact in $\mathbb{X}_d \times \mathbb{C}^d$ and each domain in Ω contains the set

$$\text{ConvHull}\,(\mathbb{D}\, e_1 \cup \cdots \cup \mathbb{D}\, e_d)\,. \qquad \square$$

Notice that in the context of the last proposition, if $z_n \in \Omega$ is a sequence with $\lim_{n\to\infty} \|z_n\| = \infty$, then there exists some component of $F_{\Omega,V}(z_n)$ that converges to -1. The next result provides a uniform version of this behavior for convergent sequences in $\mathbb{X}_{d,0}$.

Let $\mathbf{1} = (1, \ldots, 1)$ and for $\delta \geq 0$ let

$$Z_\delta = \cup_{j=1}^d \left\{ (z_1, \ldots, z_d) \in \overline{\mathbb{D}}^d : \left| z_j - (-1) \right| \leq \delta \right\}.$$

Notice that each Z_δ is a star shaped set with center $-\mathbf{1}$ and $\{Z_\delta\}_{\delta>0}$ is a neighborhood basis of

$$Z_0 = \left\{ (z_1, \ldots, z_d) \in \overline{\mathbb{D}}^d : z_j = -1 \text{ for some } 1 \leq j \leq d \right\}$$

in $\overline{\mathbb{D}}^d$.

Corollary 5.5 *Suppose that* (Ω_n, z_n) *converges to* $(\Omega_\infty, z_\infty)$ *in* $\mathbb{X}_{d,0}$. *Then there exist holomorphic embeddings* $F_n : \Omega_n \to \mathbb{D}^d$ *and functions* $\tau_\pm : (0, 1] \to [0, \infty)$ *such that* $F_n(z_n) = 0$ *and*

$$\Omega_n \cap \mathbb{B}_d(0; \tau_-(\delta)) \subset F_n^{-1}\left(\mathbb{D}^d \setminus Z_\delta \right) \subset \Omega_n \cap \mathbb{B}_d(0; \tau_+(\delta))$$

for all $\delta \in (0, 1]$ *and* $n \in \mathbb{N} \cup \{\infty\}$.

Proof Using Proposition 4.5, we can find a sequence $A_n \in \text{Aff}(\mathbb{C}^d)$ such that $A_n(\Omega_n, z_n) \in \mathbb{K}_{d,0}$. Since $\text{Aff}(\mathbb{C}^d)$ acts properly on $\mathbb{X}_{d,0}$, the set $\{A_n : n \in \mathbb{N}\}$ must be relatively compact in $\text{Aff}(\mathbb{C}^d)$. Thus there exist $\alpha > 1$, $\beta > 0$ such that

$$\frac{1}{\alpha} \|z\| - \beta \leq \|A_n z\| \leq \alpha \|z\| + \beta \tag{8}$$

for all $n \geq 0$ and $z \in \mathbb{C}^d$.

For each n, let $V_n = \left(v_1^{(n)}, \ldots, v_d^{(n)} \right)$ be $(A_n\Omega_n)$-supporting. Then let $F_n = F_{A_n\Omega_n, V_n} \circ A_n$. Then by construction $F_n(z_n) = 0$. Since

$$\left| \frac{\lambda}{2-\lambda} - (-1) \right| = 2\frac{1}{|2-\lambda|}$$

when $\lambda \in \mathbb{C} \setminus \{2\}$, the existence of the desired functions $\tau_\pm : (0, 1] \to [0, \infty)$ follows from Eq. (8) and Lemma 5.3. $\square$

6 The HHR Condition and Applications

In this section we recall the definition of HHR domains and their basic properties. We then observe, from Frankel's co-compactness theorem, that convex domains are HHR domains.

Definition 6.1 (Liu et al. [27,28]) A domain $\Omega \subset \mathbb{C}^d$ is said to be *holomorphic homogeneous regular (HHR)* if there exists $s > 0$ with the following property: for every $z \in \Omega$ there exists a holomorphic embedding $\varphi : \Omega \to \mathbb{C}^d$ such that $\varphi(z) = 0$ and

$$s\,\mathbb{B}_d \subset \varphi(\Omega) \subset \mathbb{B}_d,$$

where $\mathbb{B}_d \subset \mathbb{C}^d$ is the unit ball.

A HHR domain is sometimes called a domain with the *uniform squeezing property*, see for instance [39].

Examples of HHR domains include:

(1) $\mathcal{T}_{g,n}$, the Teichmüller space of hyperbolic surfaces with genus g and n punctures [27],
(2) bounded convex domains or more generally bounded $\mathbb{C}$-convex domains [15,25,32],
(3) bounded domains where $\mathrm{Aut}(\Omega)$ acts co-compactly on Ω, and
(4) strongly pseudoconvex domains [10,11].

Yeung proved that every HHR domain is pseudoconvex [39, Theorem 1] but not every pseudoconvex domain is an HHR domain. For instance, Fornæss and Rong have constructed smoothly bounded pseudoconvex domains in $\mathbb{C}^3$ which are not HHR [12].

As a consequence of Frankel's co-compactness theorem, convex domains satisfy the HHR condition.

Proposition 6.2 (Frankel [15], Kim–Zhang [25]) *For every $d \in \mathbb{N}$, there exists $s_d > 0$ such that: if Ω is a $\mathbb{C}$-properly convex domain, then Ω is a HHR domain with parameter $s \geq s_d$.*

Proof Fix a holomorphic embedding $G : \mathbb{D}^d \to \mathbb{B}_d$ such that $G(0) = 0$. Let $\epsilon > 0$ be the constant from Proposition 5.4. Then fix $s > 0$ such that $s\,\mathbb{B}_d \subset G(\epsilon\,\mathbb{B}_d)$.

We claim that if $\Omega \subset \mathbb{C}^d$ is a $\mathbb{C}$-properly convex domain and $z \in \Omega$, then there exists a holomorphic embedding $\varphi : \Omega \to \mathbb{C}^d$ such that $\varphi(z) = 0$ and

$$s\,\mathbb{B}_d \subset \varphi(\Omega) \subset \mathbb{B}_d\,.$$

First by Proposition 4.5 there exists an affine transformation A such that $A(\Omega, z) \in \mathbb{K}_{d,0}$. Then by Proposition 5.4, there exists a holomorphic embedding $F : A\Omega \to \mathbb{D}^d$ such that $F(0) = 0$ and $\epsilon\,\mathbb{B}_d \subset F(A\Omega)$. Then simply set $\varphi = G \circ F$. □

The HHR condition is useful because it implies that the various important interior geometries on a HHR domain exist and are well behaved.

Theorem 6.3 (Yeung [39]) *If $\Omega \subset \mathbb{C}^d$ is a HHR domain, then*

(1) *Ω is pseudoconvex,*
(2) *the Bergman, Kobayashi, and Kähler–Einstein metric are proper (and hence complete) geodesic metrics on Ω.*

Theorem 6.4 (Liu et al. [27], Yeung [39]) *For any $s \in (0, 1]$ and $d \in \mathbb{N}$ there exists $A > 0$ such that: if $\Omega \subset \mathbb{C}^d$ is a HHR domain with parameter at least s, then the Kobayashi, Bergman, and Kähler–Einstein metrics are A-bi-Lipschitz on Ω.*

Theorem 6.5 (Yeung [39]) *For any $s \in (0, 1]$ and $d \in \mathbb{N}$ there exist $I > 0$ and $\{C_q\}_{q\in\mathbb{N}}$ such that: if $\Omega \subset \mathbb{C}^d$ is a HHR domain with parameter at least s and g is either the Bergman or Kähler–Einstein metric, then:*

(1) *The injectivity radius of g is at least I,*
(2) *For any $q \in \mathbb{N}$*

$$\sup_{\Omega} \left\| \nabla^q R \right\|_g \leq C_q$$

where R is the curvature tensor of g.

7 Examples of Orbit Closures

Given a domain $\Omega \in \mathbb{X}_d$, let

$$\overline{\mathrm{Aff}(\mathbb{C}^d) \cdot \Omega}^{\mathbb{X}_d} \subset \mathbb{X}_d$$

denote the closure of the $\mathrm{Aff}(\mathbb{C}^d)$-orbit of Ω in $\mathbb{X}_d$. In this section we compute this orbit closure for some special examples. We also consider a special subset of this orbit closure which we call the affine limits.

Definition 7.1 Suppose $\Omega \in \mathbb{X}_d$.

(1) A domain $D \in \mathbb{X}_d$ is an *affine limit* of Ω if there exists $A_j \in \mathrm{Aff}(\mathbb{C}^d)$ such that
   (a) $A_j \to \infty$ in $\mathrm{Aff}(\mathbb{C}^d)$ (that is, the sequence A_n leaves every compact subset of $\mathrm{Aff}(\mathbb{C}^d)$),
   (b) $A_j\Omega$ converges to D in $\mathbb{X}_d$.
(2) Let $\mathrm{AL}(\Omega) \subset \overline{\mathrm{Aff}(\mathbb{C}^d) \cdot \Omega}^{\mathbb{X}_d}$ denote the set of all affine limits of Ω.

Notice that when $\Omega \in \mathbb{X}_d$ we have

$$\overline{\mathrm{Aff}(\mathbb{C}^d) \cdot \Omega}^{\mathbb{X}_d} = \left(\mathrm{Aff}(\mathbb{C}^d) \cdot \Omega \right) \cup \mathrm{AL}(\Omega).$$

When Ω is bounded this is a disjoint union, but when Ω is unbounded it is possible for the sets to have non-empty intersection.

We first observe that the set of affine limits is non-empty.

Proposition 7.2 *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded convex domain. If $z_n \in \Omega$ is a sequence with $\lim_{n\to\infty} \delta_\Omega(z_n) = 0$, then there exists a subsequence $(z_{n_j})_{j\geq 0}$ and affine maps $A_j \in \mathrm{Aff}(\mathbb{C}^d)$ such that $A_j(\Omega, z_{n_j})$ converges to some (U, u) in $\mathbb{X}_{d,0}$. Further, $U \in \mathrm{AL}(\Omega)$ and hence $\mathrm{AL}(\Omega) \neq \emptyset$.*

Proof This is a simple consequence of Frankel's co-compactness theorem: by Theorem 4.2 we can find a sequence $A_n \in \mathrm{Aff}(\mathbb{C}^d)$ such that the set

$$\{A_n(\Omega, z_n) : n \geq 0\} \subset \mathbb{X}_{d,0}$$

is relatively compact in $\mathbb{X}_{d,0}$. Then by passing to a subsequence we can suppose that $A_n(\Omega, z_n)$ converges to some (U, u) in $\mathbb{X}_{d,0}$.

It remains to show that $U \in \mathrm{AL}(\Omega)$ or equivalently that $\{A_n : n \geq 0\}$ is unbounded in $\mathrm{Aff}(\mathbb{C}^d)$. Suppose $A_n(\cdot) = b_n + g_n(\cdot)$ where $b_n \in \mathbb{C}^d$ and $g_n \in \mathrm{GL}_d(\mathbb{C})$. Then

$$\delta_{A_n\Omega}(A_n z_n) \leq \|g_n\|_{\mathrm{op}}\, \delta_\Omega(z_n).$$

Then, since $\lim_{n\to\infty} \delta_\Omega(z_n) = 0$ and

$$\delta_U(u) = \lim_{n\to\infty} \delta_{A_n\Omega}(A_n z_n) > 0,$$

we must have $\lim_{n\to\infty} \|g_n\|_{\mathrm{op}} = +\infty$. Hence $\{A_n : n \geq 0\}$ is unbounded in $\mathrm{Aff}(\mathbb{C}^d)$. □

The next two results show that passing to the affine limit "magnifies" certain good and bad properties.

Theorem 7.3 *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded convex domain.*

(1) *If Ω is strongly pseudoconvex domain with $\mathcal{C}^2$ boundary and $D \in \mathrm{AL}(\Omega)$, then there exists $A \in \mathrm{Aff}(\mathbb{C}^d)$ such that*

$$AD = \left\{ (z_1, \dots, z_d) \in \mathbb{C}^d : \mathrm{Im}(z_1) > \sum_{j=2}^{d} \left|z_j\right|^2 \right\}.$$

(2) *If $\Omega \subset \mathbb{C}^d$ has $\mathcal{C}^\infty$ boundary of finite type in the sense of D'Angelo and $D \in \mathrm{AL}(\Omega)$, then there exist $A \in \mathrm{Aff}(\mathbb{C}^d)$ and a non-degenerate non-negative convex polynomial $P : \mathbb{C}^{d-1} \to \mathbb{R}$ such that $P(0) = 0$ and*

$$AD = \left\{ (z_1, \dots, z_d) \in \mathbb{C}^d : \mathrm{Im}(z_1) > P(z_2, \dots, z_d) \right\}.$$

(3) *If $\Omega \subset \mathbb{C}^d$ has $\mathcal{C}^1$ boundary and $D \in \mathrm{AL}(\Omega)$, then $\mathrm{Aut}(D)$ contains a one-parameter subgroup.*

Part (1) is due to Pinchuk [33], Part (2) is due to Bedford–Pinchuk [6], and Part (3) follows from work of Frankel [14]. In Appendix A we sketch the argument.

For the next result, let

$$\mathcal{H} := \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}.$$

Theorem 7.4 *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded convex domain. If either*

(1) *there exists a non-constant holomorphic map $\varphi : \mathbb{D} \to \partial\Omega$ or*
(2) *Ω has C^∞ boundary and $\partial\Omega$ contains a point of infinite type,*

then there exists a domain $D \in \mathrm{AL}(\Omega)$ such that

$$D \cap \{(z_1, z_2, 0, \dots, 0) : z_1, z_2 \in \mathbb{C}\} = \mathcal{H} \times \mathcal{H} \times \{(0, \dots, 0)\}.$$

This result follows from arguments in [40] and [5]. In Appendix A we give the complete argument.

8 Estimates on the Kobayashi Metric

In this section we state and prove some standard estimates for the Kobayashi metric on a convex domain.

We begin by considering the following example.

Example 8.1 If $\mathcal{H} := \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$, then

$$k_{\mathcal{H}}(z; v) = \frac{|v|}{2\operatorname{Im}(z)}$$

and

$$K_{\mathcal{H}}(z_1, z_2) = \frac{1}{2}\operatorname{arcosh}\left(1 + \frac{|z_1 - z_2|^2}{2\operatorname{Im}(z_1)\operatorname{Im}(z_2)}\right)$$

for $z_1, z_2 \in \mathcal{H}$ and $v \in \mathbb{C}$.

Using this simple example and linear projections we establish the next two lemmas.

Theorem 8.2 (Graham [18, Theorem 5]) *Suppose $\Omega \subset \mathbb{C}^d$ is a convex domain. Then*

$$\frac{\|v\|}{2\delta_\Omega(z; v)} \leq k_\Omega(z; v) \leq \frac{\|v\|}{\delta_\Omega(z; v)}$$

for any $z \in \Omega$ and non-zero $v \in \mathbb{C}^d$.

Proofs of this estimate can also be found in [6, Theorem 4.1] and [15, Theorem 2.2].

Proof The second inequality is valid without the convexity assumption and simply follows from considering the map $\varphi : \mathbb{D} \to \mathbb{C}$ defined by

$$\varphi(\lambda) = z + \frac{\lambda}{\delta_\Omega(z; v)} v.$$

The first inequality is a consequence of the supporting hyperplane property of convex domains. Let $L := z + \mathbb{C} \cdot v$ and pick $\xi \in \partial\Omega \cap L$ such that $\|\xi - x\| = \delta_\Omega(z; v)$. Let H be a real hyperplane through ξ which does not intersect Ω. By rotating and translating we may assume $\xi = 0$, $z = (z_1, 0, \ldots, 0)$, $H = \{(w_1, \ldots, w_d) \in \mathbb{C}^d : \operatorname{Im}(w_1) = 0\}$, and $\Omega \subset \{(w_1, \ldots, w_d) \in \mathbb{C}^d : \operatorname{Im}(w_1) > 0\}$. With this choice of normalization $v = (v_1, 0, \ldots, 0)$ for some $v_1 \in \mathbb{C}$.

Then if $P : \mathbb{C}^d \to \mathbb{C}$ is the projection onto the first component we have

$$k_\Omega(z; v) \geq k_{P(\Omega)}(z_1; v_1) \geq k_{\mathcal{H}}(z_1; v_1) = \frac{|v_1|}{2\operatorname{Im}(z_1)} \geq \frac{|v_1|}{2|z_1|}.$$

Since $|z_1| = \|\xi - z\| = \delta_\Omega(z; v)$ and $|v_1| = \|v\|$ this completes the proof. □

Essentially the same argument provides a lower bound on the Kobayashi distance.

Lemma 8.3 *Suppose $\Omega \subset \mathbb{C}^d$ is a convex domain and $x, y \in \Omega$ are distinct. If L is the complex line containing x, y and $\xi \in L \setminus L \cap \Omega$, then*

$$\frac{1}{2}\left|\log\left(\frac{\|x - \xi\|}{\|y - \xi\|}\right)\right| \leq K_\Omega(x, y).$$

Proof Notice that since x, y, ξ are all co-linear, both sides of the desired inequality are invariant under affine transformations. In particular, we can replace Ω by $A\Omega$ for some affine map A. Now let H be a real hyperplane through ξ which does not intersect Ω. Using an affine transformation we may assume $\xi = 0$, $x = (x_1, 0, \ldots, 0)$, $y = (y_1, 0, \ldots, 0)$, $H = \{(z_1, \ldots, z_d) \in \mathbb{C}^d : \operatorname{Im}(z_1) = 0\}$, and $\Omega \subset \{(z_1, \ldots, z_d) \in \mathbb{C}^d : \operatorname{Im}(z_1) > 0\}$.

Then if $P : \mathbb{C}^d \to \mathbb{C}$ is the projection onto the first coordinate, we have

$$\begin{aligned} K_\Omega(x, y) \geq K_{P(\Omega)}(x_1, y_1) \geq K_{\mathcal{H}}(x_1, y_1) &= \frac{1}{2}\operatorname{arcosh}\left(1 + \frac{|x_1 - y_1|^2}{2\operatorname{Im}(x_1)\operatorname{Im}(y_1)}\right) \\ &\geq \frac{1}{2}\operatorname{arcosh}\left(1 + \frac{(|x_1| - |y_1|)^2}{2|x_1||y_1|}\right) = \frac{1}{2}\operatorname{arcosh}\left(\frac{|x_1|}{2|y_1|} + \frac{|y_1|}{2|x_1|}\right) \\ &= \frac{1}{2}\left|\log\left(\frac{|x_1|}{|y_1|}\right)\right|. \end{aligned}$$

Since $\|x - \xi\| = |x_1|$ and $\|y - \xi\| = |y_1|$ the lemma follows. □

9 Continuity of the Kobayashi Metric

Theorem 9.1 *If Ω_n converges to Ω in $\mathbb{X}_d$, then*

$$\lim_{n\to\infty} k_{\Omega_n} = k_\Omega \text{ and } \lim_{n\to\infty} K_{\Omega_n} = K_\Omega,$$

locally uniformly on compact sets of $\Omega \times \mathbb{C}^d$ and $\Omega \times \Omega$, respectively.

The rest of the section is devoted to the proof of Theorem 9.1. So fix a sequence Ω_n which converges to some Ω in $\mathbb{X}_d$. Let $\mathbb{S} := \{v \in \mathbb{C}^d : \|v\| = 1\}$. To prove the Theorem, it is enough to prove the uniform convergence of the Kobayashi metrics k_{Ω_n} on compact subsets of $\Omega \times \mathbb{S}$.

Fix a compact subset $K \subset \Omega$. Then since $K \times \mathbb{S}$ is compact, it is enough to consider a sequence $(p_n, v_n) \in K \times \mathbb{S}$ with

$$\lim_{n\to\infty} (p_n, v_n) = (p, v)$$

and show that

$$\lim_{n\to\infty} k_{\Omega_n}(p_n; v_n) = k_\Omega(p; v).$$

Notice that Proposition 3.3 implies that $p_n \in \Omega_n$ for n sufficiently large and hence $k_{\Omega_n}(p_n; v_n)$ is well defined for n sufficiently large.

Lemma 9.2

$$\limsup_{n\to\infty} k_{\Omega_n}(p_n; v_n) \le k_\Omega(p; v).$$

Proof Fix some $r \in (0, 1)$ and let $D_r := \{\zeta \in \mathbb{C} : |\zeta| < r\}$. Then the set

$$\widehat{K} = \left\{ g(\zeta) : g : \mathbb{D} \to \Omega \text{ holomorphic, } g(0) \in K, \text{ and } \zeta \in \overline{D_r} \right\}$$

is compact in Ω since the Kobayashi distance is proper. By Proposition 3.3 there exists some $N_r \ge 0$ such that $\widehat{K} \subset \Omega_n$ for all $n \ge N_r$.

For every n, let $g_n : \mathbb{D} \to \Omega$ be a holomorphic map and $\alpha_n \in \mathbb{C}$ be such that $g_n(0) = p_n$, $g_n'(0)\alpha_n = v_n$, and

$$|\alpha_n| = k_\Omega(p_n, v_n).$$

Since $g_n(D_r) \subset \widehat{K}$, we have $g_n(D_r) \subset \Omega_n$ for all $n \ge N_r$. Then define $g_{n,r} : \mathbb{D} \to \Omega_n$ by $g_{n,r}(z) = g_n(rz)$. Then $g_{n,r}(0) = p_n$ and

$$g_{n,r}'(0)\frac{\alpha_n}{r} = v_n.$$

So

$$k_{\Omega_n}(p_n; v_n) \leq \frac{|\alpha_n|}{r} = \frac{1}{r} k_\Omega(p_n; v_n)$$

when $n \geq N_r$.

Since the Kobayashi distance on Ω is proper, Ω is a taut complex manifold. So k_Ω is continuous by [1, Proposition 2.3.34]. Hence

$$\lim_{n \to \infty} k_\Omega(p_n; v_n) = k_\Omega(p; v)$$

and so

$$\limsup_{n \to \infty} k_{\Omega_n}(p_n; v_n) \leq \frac{1}{r} \limsup_{n \to \infty} k_\Omega(p_n, v_n) = \frac{1}{r} k_\Omega(p; v).$$

Then since $r \in (0, 1)$ is arbitrary,

$$\limsup_{n \to \infty} k_{\Omega_n}(p_n; v_n) \leq k_\Omega(p; v).$$

□

Lemma 9.3

$$k_\Omega(p; v) \leq \liminf_{n \to \infty} k_{\Omega_n}(p_n; v_n).$$

Proof Let $f_n : \mathbb{D} \to \Omega_n$ be a holomorphic map and $\alpha_n \in \mathbb{C}$ be such that $f_n(0) = p_n$, $f_n'(0)\alpha_n = v_n$, and

$$|\alpha_n| = k_{\Omega_n}(p_n; v_n).$$

Next pick $n_j \to \infty$ such that

$$\alpha := \liminf_{n \to \infty} k_{\Omega_n}(p_n; v_n) = \lim_{j \to \infty} k_{\Omega_{n_j}}(p_{n_j}; v_{n_j}) = \lim_{j \to \infty} \left| \alpha_{n_j} \right|.$$

Notice that Lemma 9.2 implies

$$\alpha = \liminf_{n \to \infty} k_{\Omega_n}(p_n; v_n) \leq \limsup_{n \to \infty} k_{\Omega_n}(p_n; v_n) \leq k_\Omega(p; v) < +\infty.$$

Claim *After passing to a subsequence (if necessary), we may assume that f_{n_j} converges locally uniformly to a holomorphic map $f : \mathbb{D} \to \Omega$.*

Proof of the Claim By Montel's theorem, it is enough to fix a compact set $Y \subset \mathbb{D}$ and show that

$$\sup_{y \in Y} \sup_{j \geq 0} \left\| f_{n_j}(y) \right\| < +\infty.$$

By Corollary 5.5, there exists holomorphic embedding $G_n : \Omega_n \to \mathbb{D}^d$ with $G_n(p_n) = 0$ and functions $\tau_\pm : (0, 1] \to [0, \infty)$ such that

$$\Omega_n \cap \mathbb{B}_d(0; \tau_-(\delta)) \subset G_n^{-1}\left(\mathbb{D}^d \setminus Z_\delta\right) \subset \Omega_n \cap \mathbb{B}_d(0; \tau_+(\delta))$$

for all $\delta \in (0, 1]$ and $n \in \mathbb{N}$.

Now

$$\sup_{y \in Y} \sup_{j \geq 0} K_{\mathbb{D}^d}\left(G_{n_j}(f_{n_j}(y)), 0\right) \leq \sup_{y \in Y} K_{\mathbb{D}}(y, 0) < +\infty$$

since $G_{n_j}(f_{n_j}(0)) = G_{n_j}(p_{n_j}) = 0$. Since $K_{\mathbb{D}^d}$ is a proper distance on $\mathbb{D}^d$ this implies that there exists $\delta > 0$ such that

$$\left\{G_{n_j}(f_{n_j}(y)) : j \geq 0, y \in Y\right\} \subset \mathbb{D}^d \setminus Z_\delta.$$

Then

$$\sup_{y \in Y} \sup_{j \geq 0} \left\| f_{n_j}(y) \right\| \leq \tau_+(\delta) < +\infty.$$

Thus by Montel's theorem and passing to a subsequence if necessary, we may assume that f_{n_j} converges locally uniformly to a holomorphic map $f : \mathbb{D} \to \Omega$.

Then

$$f'(0)\alpha = \lim_{j \to \infty} f'_{n_j}(0)\alpha_{n_j} = \lim_{j \to \infty} v_{n_j} = v.$$

So

$$k_\Omega(p; v) \leq |\alpha| = \lim_{j \to \infty} \left|\alpha_{n_j}\right| = \liminf_{n \to \infty} k_{\Omega_n}(p_n; v_n).$$

□

Finally, it follows from Lemmas 9.2 and 9.3 that

$$\lim_{n \to \infty} k_{\Omega_n}(p_n; v_n) = k_\Omega(p; v).$$

10 Continuity of the Bergman Kernel and Metric

The aim of this section is to prove the following.

Theorem 10.1 *If Ω_n converges to Ω in $\mathbb{X}_d$, then*

$$\lim_{n \to \infty} \kappa_{\Omega_n} = \kappa_\Omega$$

locally uniformly on $\Omega \times \Omega$ in the C^∞ topology.

As an immediate corollary we obtain

Corollary 10.2 *If Ω_n converges to Ω in $\mathbb{X}_d$, then*

$$\lim_{n\to\infty} b_{\Omega_n} = b_\Omega$$

locally uniformly on Ω in the C^∞ topology.

The main idea in the proof of Theorem 10.1 is to establish uniform estimates on the Bergman kernel on subdomains of the polydisk and then use the embeddings constructed in Sect. 5 to establish uniform estimates on our sequence of convex domains. The main reason to consider bounded realizations is so that Hörmander's L^2-estimates for solutions to the $\overline{\partial}$-equation can be used.

10.1 Subdomains of the Polydisk

As before, let $\mathbf{1} = (1, \dots, 1)$ and for $\delta \geq 0$ let

$$Z_\delta = \cup_{j=1}^d \left\{ (z_1, \dots, z_d) \in \overline{\mathbb{D}}^d : \left| z_j - (-1) \right| \leq \delta \right\}.$$

Notice again that each Z_δ is a star shaped set with center $-\mathbf{1}$ and $\{Z_\delta\}_{\delta>0}$ is a neighborhood basis of

$$Z_0 = \left\{ (z_1, \dots, z_d) \in \overline{\mathbb{D}}^d : z_j = -1 \text{ for some } 1 \leq j \leq d \right\}$$

in $\overline{\mathbb{D}}^d$.

If $D \subset \mathbb{D}^d$ is a domain, then Ramadanov's theorem (Theorem 2.11) implies that

$$\lim_{\delta\to 0^+} \kappa_{D\setminus Z_\delta} = \kappa_D$$

uniformly on compact subsets of D. The next proposition gives a uniform version of this convergence over all subdomains of $\mathbb{D}^d$.

Proposition 10.3 *For any compact set $K \subset \overline{\mathbb{D}}^d \setminus Z_0$ and any $\epsilon > 0$ there exists $\delta > 0$ such that $K \subset \overline{\mathbb{D}}^d \setminus Z_\delta$ and*

$$\kappa_D(z, z) \leq \kappa_{D\setminus Z_\delta}(z, z) \leq (1+\epsilon)\kappa_D(z, z)$$

for all pseudoconvex domains $D \subset \mathbb{D}^d$ and $z \in K \cap D$.

Remark 10.4 The first inequality follows immediately from Eq. (7).

Proposition 10.3 requires the following lemma.

Lemma 10.5 *For any $\epsilon > 0$ there exists $r \in (0,1)$ such that: if $D \subset \mathbb{D}^d$ is a pseudoconvex domain and $z \in \mathbb{B}_d(\mathbf{1}; r) \cap D$, then*

$$\kappa_{D \setminus Z_1}(z,z) \leq (1+\epsilon)\kappa_D(z,z).$$

The following argument is a modification of the proof of [24, Theorem 4].

Proof Consider the holomorphic function $h : \mathbb{C}^d \to \mathbb{C}$ given by

$$h(z) = \prod_{j=1}^{d} \frac{1+z_j}{2}.$$

Then $h(\mathbf{1}) = 1$ and

$$|h(z)| \leq 2^{-d}$$

for every $z \in Z_1$.

Let V be an open neighborhood of $\mathbf{1}$ with $\overline{V} \cap Z_1 = \emptyset$. Fix $\chi \in C_o^\infty(\mathbb{C}^d)$ satisfying $\chi = 1$ on V, $0 \leq \chi \leq 1$ on $\mathbb{C}^d$, and $\chi = 0$ on Z_1. Then fix $r_0 \in (0,1)$ such that

$$\overline{\mathbb{B}_d(\mathbf{1}; r_0)} \subset V.$$

Next let

$$c := \sup_{z \in \mathbb{D}^d \setminus V} \ \sup_{\zeta \in \overline{\mathbb{B}_d(\mathbf{1}; r_0)}} \left|\bar\partial \chi(z)\right| \left(\frac{2}{\|z-\zeta\|} \right)^d$$

and

$$a := \max_{z \in \overline{\mathbb{D}}^d \setminus V} |h(z)|.$$

Notice that $a < 1$ since V is an open neighborhood of $\mathbf{1}$. Then let $k \geq 1$ be a positive integer such that

$$(1+2ca^k)^2 \leq (1+\epsilon)^{1/2}$$

and let $r \in (0, r_0)$ be such that

$$|h(z)|^{2k} \geq (1+\epsilon)^{-1/2}$$

for all $z \in \mathbb{B}_d(\mathbf{1}; r)$.

Now fix a pseudoconvex domain $D \subset \mathbb{D}^d$. We claim that

$$\kappa_{D \setminus Z_1}(z,z) \leq (1+\epsilon)\kappa_D(z,z).$$

for all $z \in D \cap \mathbb{B}_d(\mathbf{1}; r)$. If $D \cap \mathbb{B}_d(\mathbf{1}; r) = \emptyset$ there is nothing to prove, so we may assume that $D \cap \mathbb{B}_d(\mathbf{1}; r) \neq \emptyset$.

Fix $\zeta \in D \cap \mathbb{B}_d(\mathbf{1}; r)$. Let $f \in \mathcal{H}^2(D \setminus Z_1)$ be such that $f(\zeta) = 1$ and

$$I_0^{D \setminus Z_1}(\zeta) = \|f\|^2_{D \setminus Z_1}.$$

Then define $\alpha := \overline{\partial}(\chi f h^k)|_D$. Then α is a smooth closed $(1, 0)$-form on D with

$$\operatorname{supp}(\alpha) \subset D \setminus (Z_1 \cup V).$$

Since $D \subset \mathbb{D}^d$, Theorem 4.4.2 in [20] with the plurisubharmonic weight

$$\phi(z) = 2d \log\left(\frac{\|z - \zeta\|}{2}\right)$$

implies that there exists a solution u to the equation $\overline{\partial} u = \alpha$ on D such that

$$\int_D |u(z)|^2 e^{-\phi(z)} \mathrm{d}\mu \leq 4 \int_D |\alpha(z)|^2 e^{-\phi(z)} \mathrm{d}\mu.$$

Since $\phi < 0$ on $\mathbb{D}^d$ and $\alpha \equiv 0$ on $Z_1 \cup V$ we have

$$\begin{aligned} \|u\|_D^2 &\leq \int_D |u(z)|^2 e^{-\phi(z)} \mathrm{d}\mu \leq 4 \int_{D \setminus (Z_1 \cup V)} |\bar{\partial}\chi(z)|^2 \, |f(z)|^2 \, |h(z)|^{2k} e^{-\phi(z)} \mathrm{d}\mu \\ &\leq 4c^2 a^{2k} \|f\|^2_{D \setminus Z_1} < +\infty. \end{aligned} \tag{9}$$

Notice that, since

$$\int_D |u(z)|^2 \left(\frac{2}{\|z - \zeta\|}\right)^{2d} \mathrm{d}\mu = \int_D |u(z)|^2 e^{-\phi(z)} \mathrm{d}\mu < +\infty,$$

we must have $u(\zeta) = 0$.

Next consider the function $F_k = \chi f h^k - u$. By construction F_k is holomorphic on D. Further, using Eq. (9) we have

$$\|F_k\|_D \leq \left\|\chi f h^k\right\|_D + \|u\|_D \leq (1 + 2ca^k) \|f\|_{D \setminus Z_1}.$$

Finally, let $g = F_k / h^k(\zeta)$. Then $g \in \mathcal{H}^2(D)$ and $g(\zeta) = 1$. Further,

$$\begin{aligned} I_0^D(\zeta) &\leq \|g\|_D^2 \leq \frac{(1 + 2ca^k)^2}{|h(\zeta)|^{2k}} \|f\|_{D \setminus Z_1} = \frac{(1 + 2ca^k)^2}{|h(\zeta)|^{2k}} I_0^{D \setminus Z_1}(\zeta) \\ &\leq (1 + \epsilon) I_0^{D \setminus Z_1}(\zeta). \end{aligned}$$

Thus

$$\kappa_{D\setminus Z_1}(\zeta,\zeta) \leq (1+\epsilon)\kappa_D(\zeta,\zeta).$$

Since $\zeta \in D \cap \mathbb{B}_d(\mathbf{1}; r)$ was arbitrary, this completes the proof of the lemma. □

Proof of Proposition 10.3 Let $a_t : \mathbb{D} \to \mathbb{D}$ be the one-parameter group of biholomorphisms given by

$$a_t(z) = \left(\frac{\cosh(t)z + \sinh(t)}{\sinh(t)z + \cosh(t)}\right).$$

Then a_t extends smoothly to $\overline{\mathbb{D}}$ and $a_t(\pm 1) = \pm 1$. Moreover, if $z \in \overline{\mathbb{D}} \setminus \{-1\}$, then

$$\lim_{t\to\infty} a_t(z) = 1$$

and the convergence is uniform on compact subsets of $\overline{\mathbb{D}} \setminus \{-1\}$.

Next let $\psi_t : \mathbb{D}^d \to \mathbb{D}^d$ be the one-parameter group

$$\psi_t = (a_t, \ldots, a_t).$$

Then ψ_t extends smoothly to $\overline{\mathbb{D}}^d$ and $\psi_t(\pm\mathbf{1}) = \pm\mathbf{1}$. Further, if $z \in \overline{\mathbb{D}}^d \setminus Z_0$, then

$$\lim_{t\to\infty} \psi_t(z) = \mathbf{1}$$

and the convergence is uniform on compact subsets of $\overline{\mathbb{D}}^d \setminus Z_0$.

Suppose that $K \subset \overline{\mathbb{D}}^d \setminus Z_0$ is compact and $\epsilon > 0$. By Lemma 10.5 there exists $r \in (0,1)$ such that: if $D \subset \mathbb{D}^d$ is a pseudoconvex domain and $z \in \mathbb{B}_d(\mathbf{1}; r) \cap D$, then

$$\kappa_{D\setminus Z_1}(z,z) \leq (1+\epsilon)\kappa_D(z,z).$$

Since $K \subset \overline{\mathbb{D}}^d \setminus Z_0$ is compact there exists $T > 0$ such that

$$\psi_T(K) \subset \mathbb{B}_d(\mathbf{1}; r).$$

Further $\psi_{-T}(Z_1)$ is a neighborhood of Z_0 in $\overline{\mathbb{D}}^d$. So there exists $\delta > 0$ such that $Z_\delta \subset \psi_{-T}(Z_1)$.

Now suppose that $D \subset \mathbb{D}^d$ is a pseudoconvex domain and $z \in K \cap D$. Then by Eqs. (7) and (2)

$$\kappa_{D\setminus Z_\delta}(z,z) \leq \kappa_{D\setminus\psi_{-T}(Z_1)}(z,z) = |J(\psi_T)(z)|^2\, \kappa_{\psi_T(D)\setminus Z_1}(\psi_T(z), \psi_T(z)).$$

By construction $\psi_T(z) \in \mathbb{B}_d(\mathbf{1}; r)$ so by our choice of r and Eq. (2)

$$\begin{aligned}\kappa_{D\setminus Z_\delta}(z,z) &\leq |J(\psi_T)(z)|^2\,(1+\epsilon)\kappa_{\psi_T(D)}(\psi_T(z),\psi_T(z))\\ &= (1+\epsilon)\kappa_D(z,z).\end{aligned}$$

□

10.2 Proof of Theorem 10.1

The proof will require a series of lemmas. We start with establishing the result when Ω is bounded.

Lemma 10.6 *Suppose that Ω_n converges to Ω in $\mathbb{X}_d$. If Ω is bounded, then*

$$\lim_{n\to\infty}\kappa_{\Omega_n} = \kappa_\Omega$$

locally uniformly on $\Omega \times \Omega$ in the C^∞ topology.

Proof Without loss of generality we can assume that $0 \in \Omega$ and each Ω_n is bounded. Then we can pick a sequence $r_n \in (0,1)$ such that $\lim_{n\to\infty} r_n = 1$ and $\overline{r_n\Omega_n} \subset \Omega$. Then the domains $D_n := r_n\Omega_n$ also converge to Ω in $\mathbb{X}_d$. Further, the sequence $D_n := r_n\Omega_n$ is eventually increasing, that is for every $n \geq 0$ there exists $N \geq 0$ such that $D_n \subset D_m$ for all $m \geq N$. So we can apply Ramadanov's theorem (Theorem 2.11) and deduce that

$$\lim_{n\to\infty}\kappa_{D_n} = \kappa_\Omega$$

locally uniformly on $\Omega \times \Omega$ in the C^∞ topology. However, according to Eq. (2)

$$\kappa_{\Omega_n}(z,w) = \kappa_{D_n}(r_n z, r_n w) r_n^{2d}$$

and hence

$$\lim_{n\to\infty}\kappa_{\Omega_n} = \kappa_\Omega$$

locally uniformly on $\Omega \times \Omega$ in the C^∞ topology. □

Lemma 10.7 *Suppose that Ω_n converges to Ω in $\mathbb{X}_d$. If $K \subset \Omega$ is compact, then*

$$\limsup_{n\to\infty}\sup_{z,w\in K}\left|\kappa_{\Omega_n}(z,w)\right| < +\infty.$$

Proof Since $K \subset \Omega$ is compact, there exists $\epsilon > 0$ such that

$$\widehat{K} := \{z \in \mathbb{C}^d : \|z-k\| \leq \epsilon \text{ for some } k \in K\}$$

is contained in Ω. By Proposition 3.3 there exists $N \geq 0$ such that $\widehat{K} \subset \Omega_n$ for all $n \geq N$. Then Eqs. (5), (7), and (2) imply that

$$\begin{aligned}\left|\kappa_{\Omega_n}(z,w)\right|^2 &\leq \kappa_{\Omega_n}(z,z)\kappa_{\Omega_n}(w,w) \leq \kappa_{\mathbb{B}_d(z;\epsilon)}(z,z)\kappa_{\mathbb{B}_d(w;\epsilon)}(w,w)\\ &= \kappa_{\mathbb{B}_d(0;\epsilon)}(0,0)^2\end{aligned}$$

for all $n \geq N$ and $z, w \in K$.

So

$$\limsup_{n\to\infty} \sup_{z,w\in K} \left|\kappa_{\Omega_n}(z,w)\right| \leq \kappa_{\mathbb{B}_d(0;\epsilon)}(0,0) < +\infty.$$

□

Lemma 10.8 *Suppose that Ω_n converges to Ω in $\mathbb{X}_d$. If $K \subset \Omega$ is compact and $\epsilon > 0$, then there exist $N, R > 0$ such that*

$$\kappa_{\Omega_n}(z,z) \leq \kappa_{\Omega_n\cap\mathbb{B}_d(0;r)}(z,z) \leq (1+\epsilon)\kappa_{\Omega_n}(z,z).$$

for all $n \geq N$, $r \geq R$, and $z \in K$.

Proof Fix

$$R_0 > \max_{z\in K} \|z\|.$$

Proposition 3.3 implies that there exists $N_0 \geq 0$ such that $K \subset \Omega_n$ for all $n \geq N_0$. Then Eq. (7) implies that

$$\kappa_{\Omega_n}(z,z) \leq \kappa_{\Omega_n\cap\mathbb{B}_d(0;r)}(z,z)$$

for all $n \geq N_0$ and $r \geq R_0$. This gives the left inequality.

For the right inequality, fix holomorphic embeddings $F_n : \Omega_n \to \mathbb{D}^d$ with $F_n(z_n) = 0$ and a function $\tau_+ : (0,1] \to [0,\infty)$ satisfying Corollary 5.5. Since $F_n(z_n) = 0$ for all n and $K \subset \Omega_n$ for $n \geq N_0$, Montel's theorem implies that there exists a compact set $\widehat{K} \subset \mathbb{D}^d$ such that

$$\bigcup_{n\geq N_0} F_n(K) \subset \widehat{K}.$$

By Proposition 10.3 there exists $\delta > 0$ such that: $\widehat{K} \subset \overline{\mathbb{D}}^d \setminus Z_\delta$ and

$$\kappa_D(z,z) \leq \kappa_{D\setminus Z_\delta}(z,z) \leq (1+\epsilon)\kappa_D(z,z) \tag{10}$$

for all domains $D \subset \mathbb{D}^d$ and $z \in \widehat{K} \cap D$. Define

$$R := \max\{R_0, \tau_+(\delta)\}.$$

Then if $n \geq N$ and $r \geq R$, then

$$F_n^{-1}(\mathbb{D}^d \setminus Z_\delta) \subset \Omega_n \cap \mathbb{B}_d(0; r).$$

So if $z \in K$, then Eqs. (7), (2), and (10) imply

$$\begin{aligned}\kappa_{\Omega_n \cap \mathbb{B}_d(0;r)}(z, z) \leq \kappa_{F_n^{-1}(\mathbb{D}^d \setminus Z_\delta)}(z, z) &= |J(F_n)(z)|^2 \kappa_{F_n(\Omega_n)\setminus Z_\delta}(F_n(z), F_n(z)) \\ &\leq |J(F_n)(z)|^2 (1+\epsilon)\kappa_{F_n(\Omega_n)}(F_n(z), F_n(z)) \\ &= (1+\epsilon)\kappa_{\Omega_n}(z, z).\end{aligned}$$

This gives the right inequality. □

Lemma 10.9 *Suppose that Ω_n converges to Ω in $\mathbb{X}_d$. Then*

$$\lim_{n\to\infty} \kappa_{\Omega_n}(z, z) = \kappa_\Omega(z, z)$$

for every $z \in \Omega$.

Proof Fix $z \in \Omega$ and some $R_0 > \|z\|$. Using Proposition 3.3 and passing to a tail of the sequence $(\Omega_n)_{n\in\mathbb{N}}$, we can assume that $z \in \Omega_n$ for every n. By possibly increasing R_0 we can assume that $\Omega_n \cap \mathbb{B}_d(0; R)$ converges to $\Omega \cap \mathbb{B}_d(0; R)$ for every $R \geq R_0$.

Now fix $\epsilon > 0$. By Lemma 10.8, passing to a tail of the sequence $(\Omega_n)_{n\in\mathbb{N}}$, and increasing R_0 again we can assume that

$$\kappa_{\Omega_n}(z, z) \leq \kappa_{\Omega_n\cap\mathbb{B}_d(0;R)}(z, z) \leq (1+\epsilon)\kappa_{\Omega_n}(z, z)$$

for all $n \geq 0$ and $R \geq R_0$. By Ramadanov's theorem (Theorem 2.11), Eq. (7), and possibly increasing R_0 again, we can also assume that

$$\kappa_\Omega(z, z) \leq \kappa_{\Omega\cap\mathbb{B}_d(0;R)}(z, z) \leq (1+\epsilon)\kappa_\Omega(z, z)$$

for all $R \geq R_0$.

Now Lemma 10.6 implies that

$$\lim_{n\to\infty} \kappa_{\Omega_n\cap\mathbb{B}_d(0;R)}(z, z) = \kappa_{\Omega\cap\mathbb{B}_d(0;R)}(z, z)$$

and so

$$\frac{1}{1+\epsilon} \limsup_{n\to\infty} \kappa_{\Omega_n}(z, z) \leq \kappa_\Omega(z, z) \leq (1+\epsilon) \liminf_{n\to\infty} \kappa_{\Omega_n}(z, z).$$

Then since $\epsilon > 0$ was arbitrary we see that

$$\lim_{n\to\infty} \kappa_{\Omega_n}(z, z) = \kappa_\Omega(z, z).$$

□

Proof of Theorem 10.1 Notice that the real and imaginary parts of each κ_{Ω_n} are harmonic on $\Omega_n \times \Omega_n$, further Lemma 10.7 implies that the sequence κ_{Ω_n} is locally bounded in the following sense: for every compact set $K \subset \Omega$ there exists $N \geq 0$ such that

$$\sup_{n\geq N} \max_{z,w\in K} \left|\kappa_{\Omega_n}(z, w)\right| < +\infty.$$

Then by the Lebesgue dominated convergence theorem and the mean value property of harmonic functions, to show that κ_{Ω_n} converges to κ_Ω locally uniformly on $\Omega \times \Omega$ in the C^∞ topology, it is enough to verify that κ_{Ω_n} converges pointwise to κ_Ω.

Fix $(z_0, w_0) \in \Omega \times \Omega$. Then pick $n_j \to \infty$ such that

$$\limsup_{n\to\infty} \left|\kappa_{\Omega_n}(z_0, w_0) - \kappa_\Omega(z_0, w_0)\right| = \lim_{j\to\infty} \left|\kappa_{\Omega_{n_j}}(z_0, w_0) - \kappa_\Omega(z_0, w_0)\right|.$$

Since the real and imaginary parts of each κ_{Ω_n} are harmonic on $\Omega_n \times \Omega_n$ and the sequence κ_{Ω_n} is locally bounded (in the sense above), we can replace n_j by a subsequence and assume that $\kappa_{\Omega_{n_j}}$ converges to some κ locally uniformly on $\Omega \times \Omega$ in the C^∞ topology (see for instance [2, Theorem 2.6]).

Now consider the functions $f_j := \kappa_{\Omega_{n_j}}(\cdot, w_0)$ and

$$f := \kappa(\cdot, w_0) = \lim_{j\to\infty} \kappa_{\Omega_{n_j}}(\cdot, w_0).$$

Then Lemma 10.9 implies

$$f(w_0) = \lim_{j\to\infty} \kappa_{\Omega_{n_j}}(w_0, w_0) = \kappa_\Omega(w_0, w_0). \tag{11}$$

Further

$$\int_{\Omega_{n_j}} \left|f_j\right|^2 \mathrm{d}\mu = \kappa_{\Omega_{n_j}}(w_0, w_0) = f_j(w_0).$$

So Fatou's lemma and Eq. (11) imply

$$\int_\Omega |f|^2 \mathrm{d}\mu \leq \liminf_{j\to\infty} \int_{\Omega_{n_j}} \left|f_j\right|^2 \mathrm{d}\mu = \liminf_{j\to\infty} f_j(w_0) = f(w_0) = \kappa_\Omega(w_0, w_0).$$

In particular, $f \in \mathcal{H}^2(\Omega)$. Then if $g = \frac{1}{\kappa_\Omega(w_0,w_0)} f$ we have $g(w_0) = 1$ and

$$\int_\Omega |g|^2 \mathrm{d}\mu = \frac{1}{\kappa_\Omega(w_0, w_0)^2} \int_\Omega |f|^2 \mathrm{d}\mu \leq \frac{1}{\kappa_\Omega(w_0, w_0)}.$$

Then Lemma 2.9 implies that $\kappa(\cdot, w_0) = f = \kappa_\Omega(\cdot, w_0)$. So

$$\limsup_{n\to\infty} \left|\kappa_{\Omega_n}(z_0, w_0) - \kappa_\Omega(z_0, w_0)\right| = \lim_{j\to\infty} \left|\kappa_{\Omega_{n_j}}(z_0, w_0) - \kappa_\Omega(z_0, w_0)\right|$$
$$= |\kappa(z_0, w_0) - \kappa_\Omega(z_0, w_0)| = 0.$$

So

$$\lim_{n\to\infty} \kappa_{\Omega_n}(z_0, w_0) = \kappa_\Omega(z_0, w_0).$$

Since $(z_0, w_0) \in \Omega \times \Omega$ was arbitrary, we see that κ_{Ω_n} converges pointwise to κ_Ω. Then by the remarks at the start of the proof, κ_{Ω_n} converges to κ_Ω locally uniformly on $\Omega \times \Omega$ in the C^∞ topology. □

11 Continuity of the Kähler–Einstein Metric

Recall that if $\Omega \subset \mathbb{C}^d$ is a $\mathbb{C}$-properly convex domain, then g^Ω_{KE} denotes the unique Kähler–Einstein metric on Ω with Ricci constant $-(d+1)$. This section is devoted to the proof of the following result.

Theorem 11.1 *If Ω_n converges to Ω in $\mathbb{X}_d$, then*

$$\lim_{n\to\infty} g^{\Omega_n}_{KE} = g^\Omega_{KE}$$

locally uniformly on Ω in the C^∞ topology.

Theorem 11.1 is a direct application of the following result.

Proposition 11.2 [5, Proposition 6.1] *Let Ω_n converge to Ω in $\mathbb{X}_d$. Suppose that for every n, g_n is a Kähler metric on Ω_n such that:*

(1) *there exists $A > 1$, independent of n, such that*

$$\frac{1}{A} k_{\Omega_n}(z; v) \leq \sqrt{g_n(z)(v, v)} \leq A k_{\Omega_n}(z; v)$$

for all $z \in \Omega_n$ and $v \in \mathbb{C}^d$,

(2) *for every $q \geq 0$ there exists $C_q > 0$, independent of n, such that*

$$\sup_{\Omega_n} \left\|\nabla^q R(g_n)\right\|_{g_n} \leq C_q.$$

Then after passing to a subsequence the metrics g_n converge locally uniformly in the C^∞ topology to a metric g_∞ on Ω.

Proof of Theorem 11.1 It is enough to show that every subsequence of $\left(g_{KE}^{\Omega_n}\right)_{n\geq 0}$ admits a subsequence that converges to g_{KE}^{Ω} locally uniformly on Ω, in the C^∞ topology.

Fix a subsequence $\left(g_{KE}^{\Omega_{n_j}}\right)_{j\geq 0}$. It follows from Proposition 6.2, Theorems 6.4 and 6.5 that the metrics $g_{KE}^{\Omega_{n_j}}$ satisfy the assumptions of Proposition 11.2. Hence, there is a subsequence of $\left(g_{KE}^{\Omega_{n_j}}\right)_{j\geq 0}$ that converges locally uniformly in the C^∞ topology to a complete Kähler metric g_∞ on Ω. Further

$$\mathrm{Ric}(g_\infty) = \lim_{j\to\infty} \mathrm{Ric}(g_{KE}^{\Omega_{n_j}}) = \lim_{j\to\infty} -(d+1)g_{KE}^{\Omega_{n_j}} = -(d+1)g_\infty.$$

So g_∞ is the unique complete Kähler–Einstein metric g_{KE}^{Ω} on Ω with Ricci curvature $-(d+1)$. □

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Appendix A: Computing Orbit Closures

In this appendix we sketch the proof of Theorem 7.3 and prove Theorem 7.4. We begin by making the following observation.

Proposition A.1 *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded convex domain and $z_n \in \Omega$, $A_n \in \mathrm{Aff}(\mathbb{C}^d)$ are sequences where $A_n(\Omega, z_n)$ converges to some (D, z_∞) in $\mathbb{X}_{d,0}$. Then the following are equivalent:*

(1) $A_n \to \infty$ *in* $\mathrm{Aff}(\mathbb{C}^d)$ *(that is, the sequence A_n leaves every compact subset of* $\mathrm{Aff}(\mathbb{C}^d)$*),*
(2) $\delta_\Omega(z_n) \to 0$.

Proof With $\Omega \in \mathbb{X}_d$ fixed, for any $\epsilon > 0$ the set

$$\{(\Omega, z) : \delta_\Omega(z) \geq \epsilon\}$$

is compact in $\mathbb{X}_{d,0}$. So the proposition follows immediately from the fact that the action of $\mathrm{Aff}(\mathbb{C}^d)$ on $\mathbb{X}_{d,0}$ is continuous and proper. □

A.1 Strongly Pseudoconvex and Finite Type Domains

It will be convenient to introduce the notion of line type.

Given a function $f : \mathbb{C} \to \mathbb{R}$ with $f(0) = 0$ let $\nu(f)$ denote the order of vanishing of f at 0.

Definition A.2 Suppose that $\Omega \subset \mathbb{C}^d$ is a bounded convex domain with $\mathcal{C}^m$ boundary and $r : \mathbb{C}^d \to \mathbb{R}$ is a defining function for Ω, that is r is a $\mathcal{C}^m$ function, $\Omega = \{r < 0\}$, and $\nabla r \neq 0$ in a neighborhood of $\partial\Omega$. Then the *line type of* $x \in \partial\Omega$ is

$$\ell(\Omega, x) := \sup \left\{ \mu(r \circ \ell) : \ell : \mathbb{C} \to \mathbb{C}^d \text{ is a non-trivial affine map with } \ell(0) = x \right\}.$$

Then the *line type of* Ω is

$$\sup_{x \in \partial\Omega} \ell(\Omega, x).$$

McNeal [29] proved that if Ω is a bounded convex domain with $\mathcal{C}^\infty$ boundary, then $x \in \partial\Omega$ has line type m if and only if the DAngelo type at x is also m (see also [7]).

Proposition A.3 *Suppose m is a positive integer and $\Omega \subset \mathbb{C}^d$ is a bounded convex domain with line type m (in particular, Ω has $\mathcal{C}^m$ boundary). If $D \in \mathrm{AL}(\Omega)$, then there exists $A \in \mathrm{Aff}(\mathbb{C}^d)$ such that*

$$AD = \left\{ (z_1, \dots, z_d) \in \mathbb{C}^d : \mathrm{Im}(z_1) > P(z_2, \dots, z_d) \right\},$$

where $P : \mathbb{C}^{d-1} \to \mathbb{R}$ is non-degenerate non-negative convex polynomial with $P(0) = 0$ and $\deg(P) \leq m$.

For a careful proof of Proposition A.3 see for instance [40, Theorem 10.1] which is based on arguments in [6,17,30].

Using Proposition A.3, one can deduce the following.

Proposition A.4 *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded convex domain with strongly pseudo-convex boundary. If $D \in \mathrm{AL}(\Omega)$, then there exists $A \in \mathrm{Aff}(\mathbb{C}^d)$ such that*

$$AD = \left\{ (z_1, \dots, z_d) \in \mathbb{C}^d : \mathrm{Im}(z_1) > \sum_{j=2}^{d} \left| z_j \right|^2 \right\}.$$

Proof Notice that Ω has line type 2 so we can use Proposition A.3. Suppose that $D \in \mathrm{AL}(\Omega)$. By Proposition A.3 there exists $A_0 \in \mathrm{Aff}(\mathbb{C}^d)$ such that

$$A_0 D = \left\{ (z_1, \dots, z_d) \in \mathbb{C}^d : \mathrm{Im}(z_1) > P(z_2, \dots, z_d) \right\},$$

where $P : \mathbb{C}^{d-1} \to \mathbb{R}$ is non-degenerate non-negative convex polynomial with $P(0) = 0$ and $\deg(P) \leq 2$.

Since P is non-negative and $P(0) = 0$, we must have $\nabla P(0) = 0$. So P is a homogeneous polynomial of degree two. Since P is real valued, it must be Hermitian and since P is non-degenerate, it must be positive definite. So by changing A we may assume that $P(z) = \sum_{j=2}^{d} \left| z_j \right|^2$. □

A.2 Smoothly Bounded Domains

Proposition A.5 *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded convex domain. If $\Omega \subset \mathbb{C}^d$ has $\mathcal{C}^1$ boundary and $D \in \mathrm{AL}(\Omega)$, then $\mathrm{Aut}(D)$ contains a one-parameter subgroup.*

The following argument is essentially due to Frankel [14].

Proof We will show that there exists $z_0 \in D$ and a non-zero vector $v \in \mathbb{C}^d$ such that $z_0 + \mathbb{R} \cdot v \subset D$. Then, since D is open and convex,

$$z + \mathbb{R} \cdot v \subset D$$

for every $z \in D$. So $\mathrm{Aut}(D)$ contains the one-parameter group

$$u_t(z) = z + tv.$$

By definition there exists a sequence $A_n \in \mathrm{Aff}(\mathbb{C}^d)$ such that $A_n \to \infty$ in $\mathrm{Aff}(\mathbb{C}^d)$ and $A_n\Omega$ converges to D in $\mathbb{X}_d$. Fix some $z_0 \in D$. By passing to tail of $(A_n)_{n\in\mathbb{N}}$ we can assume that $z_0 \in A_n\Omega$ for every n. Then define $z_n := A_n^{-1}(z_0) \in \Omega$. Notice that Proposition A.1 implies that $\lim_{n\to\infty} \delta_\Omega(z_n) = 0$.

Next pick $\xi_n \in \partial\Omega$ such that $\|\xi_n - z_n\| = \delta_\Omega(z_n)$. Notice that the real affine line

$$\xi_n + \mathbb{R} \cdot i(\xi_n - z_n)$$

is tangent to $\partial\Omega$ at ξ_n. Then, since Ω_n has $\mathcal{C}^1$ boundary, there exists $r_n \to \infty$ such that

$$\{z_n + it(\xi_n - z_n) : t \in (-r_n, r_n)\} \subset \Omega.$$

Let

$$v_n := \frac{i}{\|A_n(\xi_n) - z_0\|}(A_n(\xi_n) - z_0).$$

By passing to a subsequence we can assume that $\lim_{n\to\infty} v_n = v$. We claim that $z_0 + \mathbb{R} \cdot v \subset D$. Since A_n is a complex affine transformation

$$\begin{aligned} A_n\Big(z_n + it(\xi_n - z_n)\Big) &= A_n(z_n) + itA_n(\xi_n) - itA_n(z_n) = z_0 + it(A_n(\xi_n) - z_0) \\ &= z_0 + t\,\|A_n(\xi_n) - z_0\|\, v_n. \end{aligned}$$

Since $A_n\xi_n \in \partial A_n\Omega$ we have

$$0 < \delta_D(z_0) = \lim_{n\to\infty} \delta_{A_n\Omega}(z_0) \le \liminf_{n\to\infty} \|A_n(\xi_n) - z_0\|.$$

So

$$\epsilon := \inf_{n\ge 1} \|A_n(\xi_n) - z_0\|$$

is positive and

$$\{z_0 + tv_n : t \in (-r_n\epsilon, r_n\epsilon)\} \subset A_n\Omega.$$

Thus $z_0 + \mathbb{R} \cdot v \subset D$ which completes the proof by the remarks above. □

A.3 Proof of Theorem 7.4

A result of Frankel will allow us to reduce to lower dimensional cases.

Theorem A.6 (Frankel [15, Theorem 9.3]) *Suppose $\Omega \in \mathbb{X}_d$ and V is a complex affine k-plane intersecting Ω. Let $D = \Omega \cap V$ and suppose there exists affine maps $A_n \in \mathrm{Aff}(V)$ such that $A_n(D)$ converges to a $\mathbb{C}$-properly convex domain D_∞ in V in the local Hausdorff topology. Then there exist affine maps $B_n \in \mathrm{Aff}(\mathbb{C}^d)$ such that $B_n(\Omega)$ converges to Ω_∞ in $\mathbb{X}_d$ with*

$$\Omega_\infty \cap V = D_\infty.$$

We will also use the following observation of Fu and Straube.

Lemma A.7 (Fu-Straube [16, Theorem 1.1]) *Suppose $\Omega \in \mathbb{X}_d$. If there exists a non-constant holomorphic map $\mathbb{D} \to \partial\Omega$, then there exists a non-constant affine map $\mathbb{D} \to \partial\Omega$.*

Lemma A.8 *If $\Omega \subset \mathbb{C}$ is convex and $\Omega \neq \mathbb{C}$, then $\mathcal{H} \subset \mathrm{AL}(\Omega)$.*

Proof Since the boundary is differentiable almost everywhere, by applying an initial affine transformation we can assume that $0 \in \partial\Omega$ is a $\mathcal{C}^1$ point of $\partial\Omega$ and the real axis is tangent to Ω at 0. Then let $A_n \in \mathrm{Aff}(\mathbb{C})$ be the affine map $z \to nz$. Then $A_n\Omega = n\Omega$ converges to $\mathcal{H}$ in $\mathbb{X}_1$. □

Proposition A.9 *Suppose $\Omega \in \mathbb{X}_d$ and there exists a non-constant holomorphic map $\varphi : \mathbb{D} \to \partial\Omega$, then there exists a domain $D \in \mathrm{AL}(\Omega)$ such that*

$$D \cap \{(z_1, z_2, 0, \ldots, 0) : z_1, z_2 \in \mathbb{C}\} = \mathcal{H} \times \mathcal{H} \times \{(0, \ldots, 0)\}.$$

The following argument comes from [5, Sect. 5].

Proof By Lemma A.7 there exists a non-constant affine map $\mathbb{D} \to \partial\Omega$. Then we can find a complex affine 2-plane V intersecting Ω such that there exist a non-constant affine map $\mathbb{D} \to \partial(\Omega \cap V)$. Then by Theorem A.6 we can assume that $d = 2$.

By applying an initial affine transformation to Ω, we can assume that

(1) $\Omega \subset \{(z_1, z_2) \in \mathbb{C}^2 : \mathrm{Im}(z_1) > 0\}$,
(2) $\{0\} \times \mathbb{D} \subset \partial\Omega$, and
(3) $(i, 0) \in \Omega$.

For every n, let $z_n = (i/n, 0) \in \Omega$. Then pick

$$\xi_n \in (\{i/n\} \times \mathbb{C}) \cap \partial\Omega$$

such that

$$\|\xi_n - z_n\| = \inf\{\|\xi - z_n\| : \xi \in (\{i/n\} \times \mathbb{C}) \cap \partial\Omega\}.$$

Since $\overline{\Omega}$ contains no complex affine line, we must have

$$\limsup_{n\to\infty} \|\xi_n - z_n\| < +\infty.$$

Suppose $\xi_n = (i/n, a_n)$. By passing to a subsequence we can suppose that $a_n \to a$. Then

$$\lim_{n\to\infty} \|\xi_n - z_n\| = \lim_{n\to\infty} |a_n| = |a|$$

and $(0, a) \in \partial\Omega$. Since $\{0\} \times \mathbb{D} \subset \partial\Omega$ and Ω is convex, we also have $|a| \geq 1$.

Then consider the matrix

$$A_n = \begin{pmatrix} n & 0 \\ 0 & a_n^{-1} \end{pmatrix}.$$

Let $T \in \mathrm{Aff}(\mathbb{C}^2)$ be the affine map

$$T(z_1, z_2) = (i(z_1 - i), z_2).$$

By construction, $TA_n(\Omega, z_n) \in \mathbb{K}_2$ where $\mathbb{K}_2 \subset \mathbb{X}_2$ is the subset from Proposition 4.4. So by passing to a subsequence we can assume that $A_n\Omega$ converges to some Ω_1 in $\mathbb{X}_2$.

Let $C_2 \subset \mathbb{C}$ be the open convex set such that

$$\{0\} \times \overline{C_2} = (\{0\} \times \mathbb{C}) \cap \partial\Omega.$$

Then define $D_2 = a^{-1} \cdot C_2$.

Claim $\{0\} \times D_2 \subset \partial\Omega_1$ *and* $\Omega_1 \subset \mathcal{H} \times D_2$. □

Proof of Claim If $(x, y) \in \Omega_1$, then there exists $(x_n, y_n) \in \Omega$ such that $A_n(x_n, y_n) \to (x, y)$. Thus $nx_n \to x$ and $y_n/a_n \to y$. So $x_n \to 0$ and $y_n \to ay$. Thus $y \in a^{-1} \cdot C_2$. So $\Omega_1 \subset \mathbb{C} \times D_2$. Since $\Omega \subset \mathcal{H} \times \mathbb{C}$ we also have $\Omega_1 \subset \mathcal{H} \times \mathbb{C}$. So

$$\Omega_1 \subset (\mathbb{C} \times D_2) \cap (\mathcal{H} \times \mathbb{C}) = \mathcal{H} \times D_2.$$

Since $a_n^{-1} \cdot C_2 \times \{0\} \subset A_n\partial\Omega$ and $a_n^{-1} \to a^{-1}$, the definition of the local Hausdorff topology implies that $\{0\} \times D_2 \subset \overline{\Omega}_1$. Since $\Omega_1 \subset \mathcal{H} \times D_2$, we must have $\{0\} \times D_2 \subset \partial\Omega_1$. □

Let $C_1 \subset \mathbb{C}$ be the open convex set such that

$$C_1 \times \{0\} = (\mathbb{C} \times \{0\}) \cap \Omega.$$

Next define $D_1 = \cup_{n=1}^{\infty} nC_1$. Then D_1 is a non-empty convex open cone since $0 \in \partial C_1$.
Claim: $D_1 \times D_2 \subset \Omega_1$.

Proof of Claim By construction

$$nC_1 \times \{0\} \subset A_n \Omega$$

so, by the definition of the local Hausdorff topology, $D_1 \times \{0\} \subset \overline{\Omega}_1$. Now suppose that $(x, y) \in D_1 \times D_2$. Since D_1 is a cone, $(nx, 0) \in \overline{\Omega}_1$ for all n. Further, the previous claim implies that $(0, y) \in \overline{\Omega}_1$. Thus by convexity

$$(x, y) = \lim_{n\to\infty} \frac{1}{n}(nx, 0) + \frac{n-1}{n}(0, y) \in \overline{\Omega}_1.$$

Thus $D_1 \times D_2 \subset \overline{\Omega}_1$. Since Ω_1 has complex dimension 2, $D_1 \times D_2 \subset \Omega_1$. □

Next consider the matrices

$$B_n = \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}.$$

Then since D_1 and $\mathcal{H}$ are cones we have

$$D_1 \times D_2 \subset B_n\Omega_1 \subset \mathcal{H} \times D_2.$$

So by passing to a subsequence we can assume that $B_n\Omega_1$ converges to some Ω_2 in $\mathbb{X}_2$.
Claim: $\Omega_2 = D_1 \times D_2$.

Proof of Claim Notice that $D_1 \times D_2 \subset \Omega_2$ since $D_1 \times D_2 \subset B_n\Omega_1$ for any n.

For every $z \in D_2$ let $S_z \subset \mathbb{C}$ be the convex open set such that

$$S_z \times \{z\} = (\mathbb{C} \times \{z\}) \cap \Omega_1.$$

Then define $\mathcal{C}_z = \bigcup_{n\in\mathbb{N}} n \cdot S_z$. Then $\mathcal{C}_z$ is a convex open cone since $0 \in \overline{S_z}$. Further

$$\mathcal{C}_z \times \{z\} = (\mathbb{C} \times \{z\}) \cap \Omega_2.$$

Since $D_1 \times D_2 \subset \Omega_2$ we see that $\overline{D_1} \subset \overline{\mathcal{C}}_z$. Suppose, for a contradiction, that $\overline{D_1} \neq \overline{\mathcal{C}}_z$ for some $z \in D_2$. Then there exists some $w \in \mathcal{C}_z \setminus D_1$. Then $(tw, z) \in \Omega_2$ for all $t > 0$. Then by convexity

$$(w, 0) = \lim_{n\to\infty} \frac{1}{n}(nw, z) + \frac{n-1}{n}(0, 0) \in \overline{\Omega}_2.$$

So $w \in \overline{D}_1$. So we have a contradiction. Thus $\mathcal{C}_z = D_1$ for all $z \in D_2$ and hence $\Omega_2 = D_1 \times D_2$. □

Next Lemma A.8 implies that $\mathcal{H} \in \mathrm{AL}(D_j)$ for $j = 1, 2$. So $\mathcal{H} \times \mathcal{H} \in \mathrm{AL}(\Omega_2)$. Then, since

$$\mathrm{AL}(\Omega_2) \subset \mathrm{AL}(\Omega_1) \subset \mathrm{AL}(\Omega),$$

we see that $\mathcal{H} \times \mathcal{H} \in \mathrm{AL}(\Omega)$. □

Proposition A.10 *If $\Omega \subset \mathbb{C}^d$ is a bounded convex domain, $\partial\Omega$ is $\mathcal{C}^\infty$, and there exists a point of infinite type in $\partial\Omega$, then there exists a domain $D \in \mathrm{AL}(\Omega)$ such that*

$$D \cap \{(z_1, z_2, 0, \ldots, 0) : z_1, z_2 \in \mathbb{C}\} = \mathcal{H} \times \mathcal{H} \times \{(0, \ldots, 0)\}.$$

The following argument comes from [40, Sect. 6].

Proof Using Theorem A.6, it is enough to consider the case when $d = 2$ and show that

$$\mathcal{H} \times \mathcal{H} \in \mathrm{AL}(\Omega).$$

We first show that there exists a domain

$$\Omega_1 \in \overline{\mathrm{Aff}(\mathbb{C}^d) \cdot \Omega}^{\mathbb{X}_d}$$

with a non-constant holomorphic map $\mathbb{D} \to \partial\Omega_1$.

Let

$$B := \{x + iy : |x| \leq 1, |y| \leq 1\} \subset \mathbb{C}\,.$$

By applying an initial affine transformation to Ω, we can assume that $0 \in \partial\Omega$ is a point of infinite type, $\mathbb{R} \times \mathbb{C}$ is tangent to Ω at 0, and there exists a function $f : [-1, 1] \times B \to [0, 1]$ such that $f(0) = 0$ and

$$\Omega \cap (B \times B) = \{(x + iy, z) : y > f(x, z)\}\,.$$

Since $\mathbb{R} \times \mathbb{C}$ is tangent to Ω at 0, we see that ie_1 is the inward pointing normal vector of $\partial\Omega$ at 0. Then, since $\partial\Omega$ is $\mathcal{C}^2$ smooth, there exists $r > 0$ such that

$$rie_1 + (r\,\mathbb{D}) \cdot e_1 \subset \Omega.$$

By scaling Ω we can assume that

$$ie_1 + \mathbb{D} \cdot e_1 \subset \Omega. \tag{12}$$

Since Ω is convex and $\mathbb{R} \times \mathbb{C}$ is tangent to Ω at 0 we have

$$(\mathbb{R} \times \mathbb{C}) \cap \Omega = \emptyset. \tag{13}$$

Finally since $0 \in \partial\Omega$ is a point of infinite type, for every $m \in \mathbb{N}$

$$\lim_{z \to 0} \frac{|f(0, z)|}{|z|^m} = 0.$$

Then we can pick $w_m \in B \setminus \{0\}$ and $\epsilon_m \to 0$ such that $|f(0, w_m)| = \epsilon_m |w_m|^m$ and

$$|f(0, z)| \leq \epsilon_m |z|^m \text{ for all } |z| \leq |w_m|.$$

If $\epsilon_m = 0$ for some m, then $f(0, z) = 0$ for $|z| \leq |w_m|$ and so $\partial\Omega$ contains the disk

$$\{(0, z) : |z| \leq |w_m|\}.$$

Then

$$z \in \mathbb{D} \to \left(0, \frac{1}{w_m} z\right) \in \partial\Omega$$

is a non-constant holomorphic map and we can simply define $\Omega_1 := \Omega$.

It remains to consider the case when $\epsilon_m > 0$ for all m. Let $T \in \text{Aff}(\mathbb{C}^2)$ be the affine map

$$T(z_1, z_2) = (i(z_1 - i), z_2).$$

Then consider the affine maps $A_m \in \text{Aff}(\mathbb{C}^2)$ given by

$$A_m(z_1, z_2) = \left(\frac{1}{f(0, w_m)} z_1, \frac{1}{w_m} z_2\right).$$

We claim that $T A_m \Omega \in \mathbb{K}_2$ for every m, that is

(1) $ie_1 + \mathbb{D} \cdot e_1 \subset A_m\Omega$ and $(\mathbb{C} \cdot e_2) \cap A_m\Omega = \emptyset$,
(2) $ie_1 + \mathbb{D} \cdot e_2 \subset A_m\Omega$ and $(i, 1) \in \partial A_m\Omega$.

First, by Eq. (12)

$$ie_1 + \mathbb{D} \cdot e_1 = A_m(f(0, w_m)ie_1 + f(0, w_m)\mathbb{D} \cdot e_1) \subset A_m(ie_1 + \mathbb{D} \cdot e_1) \subset A_m\Omega$$

and by Eq. (13)

$$(\mathbb{C} \cdot e_2) \cap A_m\Omega = A_m\Big((\mathbb{C} \cdot e_2) \cap \Omega\Big) = \emptyset.$$

Since

$$|f(0,z)| \le \epsilon_m |z|^m \le \epsilon_m |w_m|^m = f(0,w_m)$$

when $|z| \le |w_m|$ we see that

$$f(0,w_m)ie_1 + (|w_m|\,\mathbb{D}) \cdot e_2 \subset \overline{\Omega}.$$

Since $f(0,w_m)ie_1 \in \Omega$, convexity then implies that

$$f(0,w_m)ie_1 + (|w_m|\,\mathbb{D}) \cdot e_2 \subset \Omega.$$

Thus

$$ie_1 + \mathbb{D} \cdot e_2 = A_m\Big(f(0,w_m)ie_1 + (|w_m|\,\mathbb{D}) \cdot e_2\Big) \subset A_m\Omega.$$

Finally, by definition $(f(w_m)i, w_m) \in \partial\Omega$ so

$$(i,1) = A_m(f(w_m)i, w_m) \in \partial A_m\Omega.$$

Thus $TA_m\Omega \in \mathbb{K}_2$.

Since $\mathbb{K}_2$ is compact in $\mathbb{X}_2$, we can pass to a subsequence and suppose that $A_m\Omega$ converges to Ω_1 in $\mathbb{X}_2$. We claim that $\{0\} \times \mathbb{D} \subset \partial\Omega_1$. Notice that

$$(A_m\Omega) \cap (B \times B) = \{(x+iy, z) \in B \times B : y > f_m(x,z)\},$$

where

$$f_m(x,z) = \frac{1}{f(0,w_m)} f\Big(f(0,w_m)x, w_m z\Big).$$

In particular, if $|z| < 1$, then

$$\begin{aligned} f_n(0,z) = \quad & \frac{1}{f(0,w_m)} f\Big(0, w_m z\Big) = \frac{1}{\epsilon_m |w_m|^m} f\Big(0, w_m z\Big) \\ & \le \frac{1}{\epsilon_m |w_m|^m} \epsilon_m |w_m z|^m = |z|^m . \end{aligned}$$

So

$$\Big\{(iy, z) : |z| < 1,\ |z|^m < y < 1\Big\} \subset A_m\Omega$$

and thus

$$\{(iy,z) : |z| < 1,\ 0 < y < 1\} \subset \Omega_1.$$

In particular, $\{0\} \times \mathbb{D} \subset \overline{\Omega_1}$. Since $(\mathbb{C} \cdot e_2) \cap A_m \Omega = \emptyset$ for all m, we also have $(\mathbb{C} \cdot e_2) \cap \Omega_1 = \emptyset$. Hence $\{0\} \times \mathbb{D} \subset \partial \Omega_1$.

Now Proposition A.9 implies that $\mathcal{H} \times \mathcal{H} \in \mathrm{AL}(\Omega_1)$. Since

$$\mathrm{AL}(\Omega_1) \subset \mathrm{AL}(\Omega),$$

we see that $\mathcal{H} \times \mathcal{H} \in \mathrm{AL}(\Omega)$. □

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