

Analysis and Control of Stochastic Systems using Semidefinite Programming over Moments

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Abstract—This technical note develops a unified methodology for probabilistic analysis and optimal control design for jump diffusion processes defined by polynomials. The statistical moments of these systems can be described by a system of linear ordinary differential equations. Typically, however, the low-order moments depend on higher-order moments, thus requiring an infinite system of equations to compute any moment exactly. Here we develop a methodology for bounding statistical moments by using the higher-order moments as inputs to an auxiliary convex optimal control problem with semidefinite constraints. For steady-state problems, the auxiliary optimal control problem reduces to a static semidefinite program. The method applies to both controlled and uncontrolled stochastic processes. For stochastic optimal control problems, the method gives bounds on achievable performance and can be used to compute approximately optimal solutions. For uncontrolled problems, both upper and lower bounds on desired moments can be computed. While the accuracy of most moment approximations cannot be quantitatively characterized, our method guarantees that the moment of interest is between the computed bounds.

I. INTRODUCTION

Two ubiquitous problems for stochastic systems are: 1) the optimization of cost functions [1], [2] and 2) the estimation of statistical moments [3]–[5]. Aside from linear systems with quadratic costs, few stochastic optimal control problems can be solved analytically. Approximation methods for more general problems can get stuck in local minima [6]. Similarly, for estimation of moments, exact statistics can rarely be computed analytically. Typically, estimates of the statistics are either found through Monte Carlo simulation or approximation schemes. However, the accuracy of estimates can rarely be deduced quantitatively.

This technical note defines a unified methodology for approximating both analysis and control problems via convex optimization methods. The paper focuses on continuous-time jump diffusions defined by polynomials. We study polynomials because they are rich enough to capture interesting problems from chemical analysis [4] and resource management [7], [8], while remaining analytically tractable. Both finite-horizon and steady-state problems are considered. In the finite-horizon case, approximations are achieved by solving an auxiliary linear optimal control problem with semidefinite constraints. In the steady-state case, this problem reduces to a static semidefinite program (SDP).

The most closely related methods to convex relaxations of stochastic optimal control problem focus on occupation measures [9]–[11]. In particular, these methods optimize the joint distribution of the state and control processes over both space and time. Our work can be interpreted as an “instantaneous”

version of occupation measures. A more removed relaxation method uses convex duality to relax causality constraints [12], [13]. Also related is the work of [14], [15] which uses a combination of moment dynamics and deterministic optimal control to find approximate solutions to stochastic optimal control problems. That work, however, only considers systems with closed moments.

For uncontrolled systems, our method provides provable upper and lower bounds on statistical quantities of interest. The method can be viewed as an alternative to moment closure methods [3], [5], [16]–[19], which give point approximations to moments, but whose accuracy cannot be guaranteed. In contrast, the method of this technical note describes an interval in which the moment must lie. The special case of stationary moments was independently studied in [20].

This work is an extension of the conference paper [21]. That paper only considered the case of finite-horizon optimal control for stochastic differential equations. This current work provides a unified treatment of finite-horizon and steady state problems, jump processes, and systems with and without control inputs. Finally, changes in presentation highlight that the optimization problem for bounding moments is a type of linear optimal control problem.

Section II defines the general problems of interest. Background results on stochastic processes are given in Section III. The main results and numerical examples are given in Section IV. Finally, Section V concludes.

II. PROBLEMS

A. Notation

Random variables are denoted in bold. For a stochastic process, $\mathbf{x}(t)$, $d\mathbf{x}(t)$ denotes the increment $\mathbf{x}(t + dt) - \mathbf{x}(t)$. If $\mathbf{x}$ is a random variable, its expectation is denoted by $\langle \mathbf{x} \rangle$. The notation $M \succeq 0$ denotes that M is a symmetric, positive semidefinite matrix.

B. Finite-Horizon Stochastic Optimal Control

The general problem has the form:

$$\text{minimize } \left\langle \int_0^T c(\mathbf{x}(t), \mathbf{u}(t)) dt + h(\mathbf{x}(T), \mathbf{u}(T)) \right\rangle \quad (1a)$$

subject to

$$d\mathbf{x}(t) = f(\mathbf{x}(t), \mathbf{u}(t))dt + g(\mathbf{x}(t), \mathbf{u}(t))d\mathbf{w}(t) \quad (1b)$$

$$+ \sum_{i=1}^J (\phi_i(\mathbf{x}(t), \mathbf{u}(t)) - \mathbf{x}(t)) d\mathbf{N}_i(\mathbf{x}(t), \mathbf{u}(t)) \quad (1c)$$

$$d\mathbf{N}_i(\mathbf{x}(t), \mathbf{u}(t)) = \begin{cases} 1 & \text{with probability } \lambda_i(\mathbf{x}(t), \mathbf{u}(t))dt \\ 0 & \text{otherwise} \end{cases} \quad (1d)$$

$$b_i(\mathbf{x}(t), \mathbf{u}(t)) \geq 0 \quad \text{for } i \in \{1, \dots, n_b\} \quad (1e)$$

$$\mathbf{x}(0) \sim \mathcal{D} \quad \text{and } \mathbf{u}(t) \text{ is admissible.} \quad (1f)$$

Here $\mathbf{x}(0) \sim \mathcal{D}$ means that $\mathbf{x}(0)$ has some specified initial distribution, $\mathcal{D}$. In the problems studied, $\mathcal{D}$ will be a distribution with known moments. By admissible, we mean that $\mathbf{u}(t)$ is measurable with respect to the filtration $\mathcal{F}(t)$ generated by $\{\mathbf{x}(\tau) : 0 \leq \tau \leq t\}$. We will assume that the functions c , h , f , g , ϕ_i , λ_i , and b_i are polynomials.

Let V^* be the optimal cost. The method to be described in Subsection IV-A gives a semidefinite programming method for computing an increasing sequence of lower bounds, $\mathfrak{L}_i$ on the optimal cost: $\mathfrak{L}_0 \leq \dots \leq \mathfrak{L}_i \leq V^*$.

For uncontrolled systems, both upper and lower bounds can be found: $\mathfrak{L}_0 \leq \dots \leq \mathfrak{L}_i \leq V^* \leq \mathfrak{U}_i \leq \dots \leq \mathfrak{U}_0$.

Furthermore, In Subsection IV-C, we show how to use the result of the semidefinite programs to produce feasible controllers. The cost associated with any feasible controller will necessarily be at least as high as the optimal cost, and so the corresponding controller gives an upper bound $\mathfrak{U}_i$. So, we must have that $\mathfrak{L}_i \leq V^* \leq \mathfrak{U}_i$.

C. Steady State

The results for finite-horizon problems can be modified to bound steady-state moments:

$$\text{minimize } \lim_{T \rightarrow \infty} \langle h(\mathbf{x}(T), \mathbf{u}(T)) \rangle \quad (2a)$$

$$\text{subject to } (1b) - (1f) \quad (2b)$$

III. BACKGROUND RESULTS

A. Generators

In this technical note, we will examine the behavior of moments of a stochastic process. The dynamics of the moments can be derived using standard tools from stochastic processes. For more details, see [22], [23].

Consider the dynamics from (1b). Let $h(\mathbf{x})$ be a smooth scalar valued function of the state. The *generator* of the process is given by:

$$Lh(\mathbf{x}, \mathbf{u}) = \frac{\partial h(\mathbf{x})}{\partial \mathbf{x}} f(\mathbf{x}, \mathbf{u}) + \frac{1}{2} \text{Tr} \left(\frac{\partial^2 h(\mathbf{x})}{\partial \mathbf{x}^2} g(\mathbf{x}, \mathbf{u}) g(\mathbf{x}, \mathbf{u})^\top \right) + \sum_{i=1}^J (h(\phi_i(\mathbf{x}, \mathbf{u})) - h(\mathbf{x})) \lambda_i(\mathbf{x}, \mathbf{u}). \quad (3)$$

A standard result shows that $\langle h(\mathbf{x}(t)) \rangle$ can be expressed using the generator as

$$\frac{d}{dt} \langle h(\mathbf{x}(t)) \rangle = \langle Lh(\mathbf{x}(t), \mathbf{u}(t)) \rangle. \quad (4)$$

Say that $h(\mathbf{x}(t))$ is a polynomial. Then, since all of the functions, f , g , ϕ_i , and λ_i are polynomials, term inside the expectation on the right hand side of (4) must also be a polynomial.

B. The Auxiliary Linear System

This section shows how the dynamics, constraints, and costs can all be studied in terms of an auxiliary linear control system. The state, $\mathcal{X}(t)$, and input $\mathcal{U}(t)$, of the auxiliary control system will be collections of moments:

$$\mathcal{X}(t) = \begin{bmatrix} 1 \\ \langle \mathbf{x}(t)^{(m_1)} \rangle \\ \langle \mathbf{x}(t)^{(m_2)} \rangle \\ \vdots \\ \langle \mathbf{x}(t)^{(m_N)} \rangle \end{bmatrix}, \quad \mathcal{U}(t) = \begin{bmatrix} \langle \mathbf{x}(t)^{(q_1)} \mathbf{u}(t)^{(r_1)} \rangle \\ \langle \mathbf{x}(t)^{(q_2)} \mathbf{u}(t)^{(r_2)} \rangle \\ \vdots \\ \langle \mathbf{x}(t)^{(q_P)} \mathbf{u}(t)^{(r_P)} \rangle \end{bmatrix}. \quad (5)$$

Here, we use the notation $\mathbf{x}(t)^{(m)}$ such that if m is a collection of non-negative integers $m = (d_1, d_2, \dots, d_n)$, then $\mathbf{x}(t)^{(m)}$ denotes the product:

$$\mathbf{x}(t)^{(m)} = (\mathbf{x}_1(t))^{d_1} (\mathbf{x}_2(t))^{d_2} \dots (\mathbf{x}_n(t))^{d_n}. \quad (6)$$

Note that $1 = \langle \mathbf{x}(t)^{(0)} \rangle$, in $\mathcal{X}(t)$. For simplicity, $\mathcal{X}(t)$ will often include all moments of $\mathbf{x}(t)$ up to some degree.

In the case of an uncontrolled system, the moments from $\mathcal{U}(t)$ will all take the form $\langle \mathbf{x}(t)^{(q_i)} \rangle$ for some moment not appearing in $\mathcal{X}(t)$. The moments contained in $\mathcal{U}(t)$ will be chosen so that the Lemmas 1 – 3 below hold.

Lemma 1: Consider the dynamics from (1b) and let $\mathcal{X}(t)$ be the vector of moments defined in (5). There exist constant matrices, A , and B , such that

$$\dot{\mathcal{X}}(t) = A\mathcal{X}(t) + B\mathcal{U}(t). \quad (7)$$

Proof: This is an immediate consequence of (3), provided that all of the moments on the right hand side that do not appear in $\mathcal{X}(t)$ are contained in $\mathcal{U}(t)$. ■

Remark 1: The result from Lemma 1 is well known, and commonly arises in works on *moment closure* for stochastic dynamic systems [3], [5], [16]–[19]. We say that a stochastic process has non-closed moments when the dynamics of a given moment depend on a higher order moments. In this case, infinitely many differential equations are required to describe any one moment exactly. Moment closure methods approximate this infinite set of differential equations with a finite set of differential equations.

Lemma 2: Assume that the moments in $\mathcal{X}(t)$ and $\mathcal{U}(t)$ are bounded. There exist constant matrices, C , D , H , and K such that

$$\left\langle \int_0^T c(\mathbf{x}(t), \mathbf{u}(t)) dt + h(\mathbf{x}(T), \mathbf{u}(T)) \right\rangle = \int_0^T (C\mathcal{X}(t) + D\mathcal{U}(t)) dt + H\mathcal{X}(T) + K\mathcal{U}(T). \quad (8)$$

Proof: This follows by the assumption that c and h are polynomials, provided that all moments from c and h that do not appear in $\mathcal{X}(t)$ are contained in $\mathcal{U}(t)$. The exchange of integration and expectation is justified by boundedness of the moments and Fubini's theorem. ■

Lemma 3: Let $v_1(\mathbf{x}(t), \mathbf{u}(t)), \dots, v_m(\mathbf{x}(t), \mathbf{u}(t))$ be any collection of polynomials. There is an affine matrix-valued function M such that the following holds:

$$\left\langle \begin{bmatrix} v_1(\mathbf{x}(t), \mathbf{u}(t)) \\ \vdots \\ v_m(\mathbf{x}(t), \mathbf{u}(t)) \end{bmatrix} \begin{bmatrix} v_1(\mathbf{x}(t), \mathbf{u}(t)) \\ \vdots \\ v_m(\mathbf{x}(t), \mathbf{u}(t)) \end{bmatrix}^\top \right\rangle = M(\mathcal{X}(t), \mathcal{U}(t)) \succeq 0. \quad (9)$$

Furthermore, if $b_i(\mathbf{x}(t), \mathbf{u}(t)) \geq 0$ for all t , and $s_1(\mathbf{x}(t), \mathbf{u}(t)), \dots, s_{m_i}(\mathbf{x}(t), \mathbf{u}(t))$ is a collection of polynomials, then there is a different affine matrix-valued function M_{b_i} such that

$$\left\langle b_i(\mathbf{x}(t), \mathbf{u}(t)) \begin{bmatrix} s_1(\mathbf{x}(t), \mathbf{u}(t)) \\ \vdots \\ s_{m_i}(\mathbf{x}(t), \mathbf{u}(t)) \end{bmatrix} \begin{bmatrix} s_1(\mathbf{x}(t), \mathbf{u}(t)) \\ \vdots \\ s_{m_i}(\mathbf{x}(t), \mathbf{u}(t)) \end{bmatrix}^\top \right\rangle = M_{b_i}(\mathcal{X}(t), \mathcal{U}(t)) \succeq 0. \quad (10)$$

Proof: The existence of an affine M follows from the polynomial assumption, provided that all moments that do not appear in $\mathcal{X}(t)$ are contained in $\mathcal{U}(t)$. The matrix, $M(\mathcal{X}(t), \mathcal{U}(t))$ is positive definite, since it is a convex combination of outer products which are positive semidefinite. Similarly, if the positive semidefinite outer product is multiplied by a non-negative scalar, and its mean is positive semidefinite. ■

Remark 2: The LMI from (9) must hold for any stochastic process for which the required moments are finite. However, there could potentially be values $\mathcal{X}$ and $\mathcal{U}$ such that $M(\mathcal{X}, \mathcal{U}) \succeq 0$, but no random variables $\mathbf{x}$, $\mathbf{u}$ satisfy $\langle v(\mathbf{x}, \mathbf{u})v(\mathbf{x}, \mathbf{u})^\top \rangle = M(\mathcal{X}, \mathcal{U})$. Here v is the vector of polynomials from (9). See [24] and references therein.

IV. RESULTS

A. Bounds via Linear Optimal Control

The following theorem shows how to compute bounds on stochastic control problems using the auxiliary system.

Theorem 1: Let A, B, C, D, H, K, M , and M_{b_i} be the terms defined in Lemmas 1 – 3. Consider the corresponding continuous-time semidefinite program:

$$\begin{aligned} \underset{\mathcal{X}(t), \mathcal{U}(t)}{\text{minimize}} \quad & \int_0^T (C\mathcal{X}(t) + D\mathcal{U}(t)) dt + H\mathcal{X}_T + K\mathcal{U}_T \\ \text{subject to} \quad & \dot{\mathcal{X}}(t) = A\mathcal{X}(t) + B\mathcal{U}(t) \end{aligned} \quad (11a)$$

$$\mathcal{X}(0) = \mathcal{X}(0) \quad (11b)$$

$$M(\mathcal{X}(t), \mathcal{U}(t)) \succeq 0 \quad \text{for all } t \in [0, T] \quad (11c)$$

$$M_{b_i}(\mathcal{X}(t), \mathcal{U}(t)) \succeq 0 \quad (11d)$$

$$\text{for all } t \in [0, T], i \in \{1, \dots, n_b\}.$$

The optimal value for this problem is always a lower bound on the optimal value for the stochastic control problem, (1), when the corresponding moments exist. If the number of constraints in (11) is increased, either by adding more moments to $\mathcal{X}(t)$ or by adding more semidefinite constraints, the value of (11) cannot decrease.

Proof: From classical results in stochastic control, optimal strategies can be found of the form $\mathbf{u}(t) = \gamma^*(t, \mathbf{x}(t))$, [1]. Thus, the function γ^* induces a joint distribution over the trajectories $\mathbf{x}(t)$ and $\mathbf{u}(t)$. Denote the corresponding moments by $\mathcal{X}^*(t)$ and $\mathcal{U}^*(t)$, provided that they exist. Lemmas 1 and 3 imply that $\mathcal{X}^*(t)$ and $\mathcal{U}^*(t)$ satisfy all of the constraints of the semidefinite program. Furthermore, Lemma 2 implies that the optimal cost of the original stochastic control problem is given by (8) / (11a) applied to $\mathcal{X}^*(t)$ and $\mathcal{U}^*(t)$. Thus the optimal cost of the semidefinite program is a lower bound on the cost of the original stochastic control problem.

Say now that the problem is augmented by adding more semidefinite constraints. In this case, the set of feasible solutions can only get smaller, and so the optimal value cannot decrease. Similarly, say that more moments are added to $\mathcal{X}(t)$ giving rise to a larger state vector, $\mathcal{X}(t)$. In this case, the number of variables increases, but the original variables must still satisfy the constraints of the original problem. So, again the optimal value cannot decrease. ■

Corollary 1: For an uncontrolled system, maximizing and minimizing (11), are both convex SDPs. Thus (11) gives upper and lower bounds on the value: $\left\langle \int_0^T c(\mathbf{x}(t))dt + h(\mathbf{x}(T)) \right\rangle$.

The process of constructing the semidefinite programs can be automated, which we briefly sketch. First, we fix a moment matrix, $M(\mathcal{X}(t), \mathcal{U}(t))$, containing all moments up to a given order. The state vector $\mathcal{X}(t)$ is constructed by adding all of the moments $\langle \mathbf{x}(t)^{(m)} \rangle$ such that the moment itself and corresponding moments from the generator are all contained in $M(\mathcal{X}(t), \mathcal{U}(t))$. For a more detailed discussion, see [25].

B. Steady State Bounds

If the process has converged to a stationary distribution with finite moments, then the moments must be constant. This implies that the true state satisfies $\frac{d}{dt}\mathcal{X}(t) = 0$.

Theorem 2: Let A, B, H, K, M , and M_{b_i} be the terms defined in Lemmas 1 – 3. Consider the following semidefinite program:

$$\underset{\mathcal{X}, \mathcal{U}}{\text{minimize}} \quad H\mathcal{X} + K\mathcal{U} \quad (12a)$$

$$\text{subject to} \quad 0 = A\mathcal{X} + B\mathcal{U} \quad (12b)$$

$$M(\mathcal{X}, \mathcal{U}) \succeq 0 \quad (12c)$$

$$M_{b_i}(\mathcal{X}, \mathcal{U}) \succeq 0 \quad \text{for all } i \in \{1, \dots, n_b\}. \quad (12d)$$

The optimal value for this problem is always a lower bound on the optimal value for the steady-state stochastic control problem, (2), provided that the stationary moments exist and are finite. If the number of constraints in the linear problem, (12), is increased, either by adding more moments to $\mathcal{X}$ or by adding more semidefinite constraints, the value of (12) cannot decrease.

Remark 3: A special case of this theorem is studied in detail in [20]. Specifically, that work considers uncontrolled stochastic differential equations with no jumps. They prove that in some cases, the upper and lower bounds converge.

Remark 4: In some pathological cases, the stationary moments may not exist, [26]. Requiring finite moments avoids these pathologies, [20].

C. Constructing a Feasible Controller

The SDPs (11) and (12) approximate statistical moments of the control inputs. This subsection gives a heuristic method for computing controllers from the SDP results.

Assume that $\mathbf{u}(t)$ is a polynomial function of $\mathbf{x}(t)$:

$$\mathbf{u}(t) = \sum_{i=1}^p k_i(t) \mathbf{x}(t)^{(d_i)}, \quad (13)$$

where $k_i(t)$ is a vector of coefficients and $\mathbf{x}(t)^{(d_i)}$ is a monomial. In this case, the correlation between $\mathbf{u}(t)$ and any other monomial $\mathbf{x}(t)^{(m)}$ can be expressed as:

$$\langle \mathbf{u}(t) \mathbf{x}(t)^{(m)} \rangle = \sum_{i=1}^p k_i(t) \langle \mathbf{x}(t)^{(d_i+m)} \rangle. \quad (14)$$

For moments computed using the SDP, (14) will not hold exactly. To make (14) hold approximately, we 1) solve the SDP from (11) and 2) solve the least squares problem:

$$\begin{aligned} & \text{minimize } \left\| \begin{bmatrix} \langle \mathbf{u}(t) \mathbf{x}(t)^{(m_1)} \rangle & \dots & \langle \mathbf{u}(t) \mathbf{x}(t)^{(m_q)} \rangle \end{bmatrix} - \right. \\ & \left. \sum_{i=1}^p k_i(t) \begin{bmatrix} \langle \mathbf{x}(t)^{(d_i+m_1)} \rangle & \dots & \langle \mathbf{x}(t)^{(d_i+m_q)} \rangle \end{bmatrix} \right\|_F^2, \quad (15) \end{aligned}$$

where $\|\cdot\|_F$ denotes the Frobenius norm.

The controller computed from (13) and (15) suffices for problems with no constraints. However, in the general formulation, the inputs and states could have constraints, defined by the inequalities $b_i(\mathbf{x}(t), \mathbf{u}(t)) \geq 0$. In the special case of input constraints, the inequalities define a set U such that $\mathbf{u}(t) \in U$. In the case of a convex U , the control strategy can be modified by using the convex projection onto U :

$$\mathbf{u}(t) = \Pi_U \left(\sum_{i=1}^p k_i(t) \mathbf{x}(t)^{(d_i)} \right). \quad (16)$$

Proposition 1: Let $\mathbf{u}(t)$ be the controller computed according to (13) / (16) and (15). The cost (1a) induced by this controller is an upper bound on the optimal value for the original optimal control problem (1).

Proof: The controller from (13) and (15) is feasible for (1), so its value cannot be less than the optimum. ■

Theorem 1 and Proposition 1 give the following corollary.

Corollary 2: Let $\mathcal{L}$ be the optimal value of (11) and let $\mathcal{U}$ be the expected cost induced by the controller computed from (13) and (15). If V^* is the optimal value (1), then V^* satisfies: $\mathcal{L} \leq V^* \leq \mathcal{U}$.

D. Examples

In this subsection we will discuss how the results of the previous subsection can be applied to specific examples. Throughout this subsection, we will set $\mathbf{v}_i^j(t) = [\mathbf{x}(t)^i \ \mathbf{x}(t)^{i+1} \ \dots \ \mathbf{x}(t)^j]^\top$ for compact notation.

Example 1 (Stochastic Logistic Model): Consider the stochastic logistic model studied in [4]:

$$\mathbf{x}(t+dt) = \begin{cases} \mathbf{x}(t) + 1 & \text{with probability } (a_1 \mathbf{x} - b_1 \mathbf{x}^2)dt \\ \mathbf{x}(t) - 1 & \text{with probability } (a_2 \mathbf{x} + b_2 \mathbf{x}^2)dt \\ \mathbf{x}(t) & \text{otherwise,} \end{cases} \quad (17)$$

Here we assume that the coefficients satisfy:

$$\Omega := \frac{a_1}{b_1}, \quad a_1 > 0, \quad a_2 > 0, \quad b_1 > 0, \quad b_2 \geq 0, \quad (18)$$

where Ω is a positive integer. These assumptions guarantee that if $\mathbf{x}(0) \in \{0, 1, 2, \dots, \Omega\}$ then $\mathbf{x}(t) \in \{0, 1, 2, \dots, \Omega\}$ for all $t \geq 0$. Our goal is to find bounds on the second moment of the state at the final time, $\langle \mathbf{x}(T)^2 \rangle$.

The differential equations corresponding to the moments have the form:

$$\begin{aligned} \frac{d}{dt} \langle \mathbf{x}(t)^k \rangle = & \langle ((\mathbf{x}(t) + 1)^k - \mathbf{x}(t)^k)(a_1 \mathbf{x}(t) - b_1 \mathbf{x}(t)^2) + \\ & \langle ((\mathbf{x}(t) - 1)^k - \mathbf{x}(t)^k)(a_2 \mathbf{x}(t) + b_2 \mathbf{x}(t)^2) \rangle \end{aligned}$$

Because of a cancellation in the terms $(x+1)^k - x^k$ and $(x-1)^k - x^k$, the degree of the right hand side is $k+1$. Here, the auxiliary control variable $\mathcal{U}(t)$ will represent a higher order moment that is not part of $\mathcal{X}(t)$.

For $d \geq 1$, we take the state, input, and LMIs as:

$$\mathcal{X}(t) = \langle \mathbf{v}_0^{2d-1}(t) \rangle, \quad \mathcal{U}(t) = \langle \mathbf{x}(t)^{2d} \rangle. \quad (19)$$

$$\langle \mathbf{v}_0^d(t) \mathbf{v}_0^d(t)^\top \rangle \succeq 0, \quad \langle \mathbf{x}(t) \mathbf{v}_0^{d-1}(t) \mathbf{v}_0^{d-1}(t)^\top \rangle \succeq 0, \quad (20)$$

$$\langle (\Omega - \mathbf{x}(t)) \mathbf{v}_0^{d-1}(t) \mathbf{v}_0^{d-1}(t)^\top \rangle \succeq 0 \quad (21)$$

The first LMI encodes the basic outer product constraint, while the other two encode the constraint that $\mathbf{x}(t) \in [0, \Omega]$. See Fig. 1 for numerical results.

Example 2 (Fishery Management): A problem of fishery management, modified from [7], [8], is given by:

$$\text{maximize} \quad \left\langle \int_0^T \mathbf{u}(t) dt \right\rangle \quad (22a)$$

$$\text{subject to} \quad d\mathbf{x}(t) = (\mathbf{x}(t) - \gamma \mathbf{x}(t)^2 - \mathbf{u}(t)) dt + \sigma \mathbf{x}(t) d\mathbf{w}(t) \quad (22b)$$

$$\mathbf{x}(0) = x_0, \quad \mathbf{x}(t) \geq 0, \quad \mathbf{u}(t) \geq 0. \quad (22c)$$

Here $\mathbf{x}(t)$ models the population in a fishery and $\mathbf{u}(t)$ models the rate of harvesting. As in the earlier works, a non-negative population, $\mathbf{x}(t) \geq 0$, is required. The constraint that $\mathbf{u}(t) \geq 0$ encodes the idea that fish are only being taken out, not put into the fishery. The primary difference between this formulation and that of [8] and [7], is that the cost is not discounted, and operates over a fixed, finite horizon.

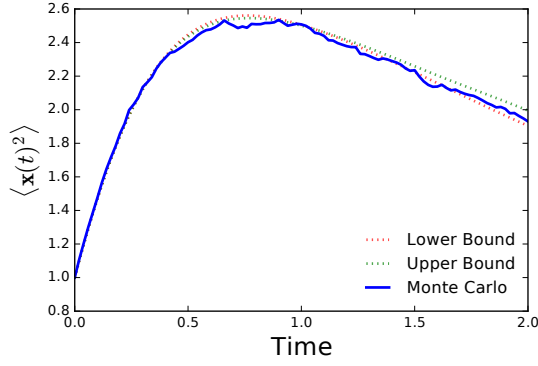


Fig. 1: For the logistic model, (17), we used parameters $a_1 = 3$, $b_1 = 1$, $a_2 = 1$, $b_2 = 0$ and initial condition $\mathbf{x}(0) = 1$. Using the SDP of Theorem 1, we get upper and lower bounds on $\langle x(T)^2 \rangle$. While we only penalized the final value, the bounds and averaged trajectory (5000 runs) are similar over the horizon.

The moment dynamics are given by:

$$\frac{d}{dt} \langle x(t)^k \rangle = k \langle x(t)^k - \gamma x(t)^{k+1} - x(t)^{k-1} u(t) \rangle + \frac{1}{2} k(k-1) \langle x(t)^k \rangle \quad (23)$$

For $d \geq 2$, the states, inputs, and LMIs are given by:

$$\mathcal{X}(t) = \langle v_0^{d+1}(t) \rangle, \quad \mathcal{U}(t) = \left\langle \begin{bmatrix} v_{d+2}^{2d}(t) \\ v_0^d(t) u(t) \\ u(t)^2 \end{bmatrix} \right\rangle \quad (24)$$

$$\left\langle \begin{bmatrix} v_0^d(t) \\ u(t) \end{bmatrix} \begin{bmatrix} v_0^d(t) \\ u(t) \end{bmatrix}^\top \right\rangle \succeq 0, \quad \langle x(t) v_0^{d-1}(t) v_0^{d-1}(t)^\top \rangle \succeq 0, \quad (25)$$

$$\langle u(t) v_0^{d-1}(t) v_0^{d-1}(t) \rangle \succeq 0 \quad (26)$$

An interesting control strategy appears to emerge, whereby the population is held constant for most of the interval and then fished to extinction at the end of the horizon. See Fig. 2.

Example 3 (Jump Rate Control): In the next problem, the size of a Brownian motion is controlled by instantaneously jumping the state to zero, with jump rate $u(t)$:

$$\text{minimize} \quad \lim_{T \rightarrow \infty} \langle Qx(T)^2 + Ru(T) \rangle \quad (27a)$$

$$\text{subject to} \quad d\mathbf{x}(t) = d\mathbf{w}(t) - \mathbf{x}(t)d\mathbf{N}(t) \quad (27b)$$

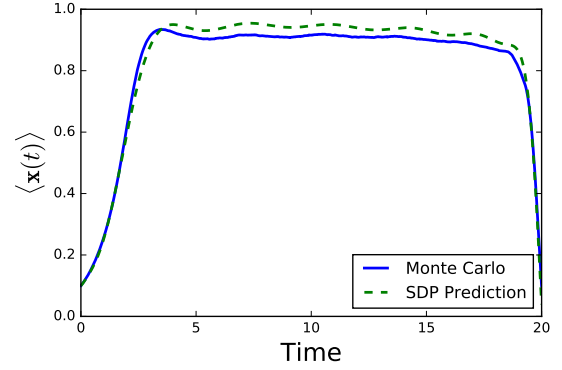
$$d\mathbf{N}(t) = \begin{cases} 1 & \text{with probability } u(t)dt \\ 0 & \text{with probability } 1 - u(t)dt \end{cases} \quad (27c)$$

$$0 \leq u(t) \leq \Omega. \quad (27d)$$

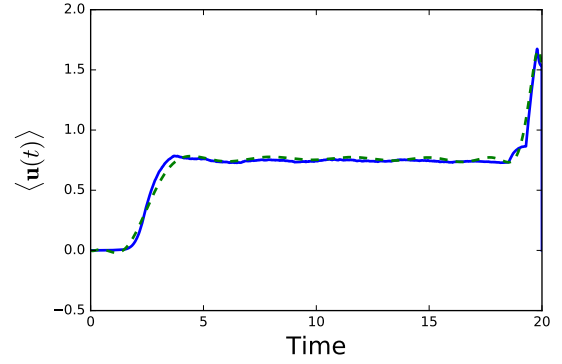
The constraints on $u(t)$ ensure that the jump rate is non-negative and finite. The moments have dynamics given by:

$$\frac{d}{dt} \langle x(t)^k \rangle = \frac{1}{2} k(k-1) \langle x(t)^{k-2} \rangle - \langle x(t)^k u(t) \rangle \quad (28)$$

for $k \geq 1$. Thus, the state moments do not depend on higher-order moments of the state, but they do depend on correlations



(a) Average Population



(b) Harvesting Strategy

Fig. 2: Fig. 2a shows the average population while Fig. 2b shows the average harvest. The SDP predicts a maximum yield of 13.77, while the corresponding controller achieves a yield of 13.55, when averaged over 5000 simulations.

with the input. Using the $v_i^j(t)$ notation defined above, we take our augmented state and input to be:

$$\mathcal{X}(t) = \langle v_0^{2d}(t) \rangle, \quad \mathcal{U}(t) = \langle v_0^{2d}(t) u(t) \rangle. \quad (29)$$

We enforce the constraint that $0 \leq u(t) \leq \Omega$ using LMIs:

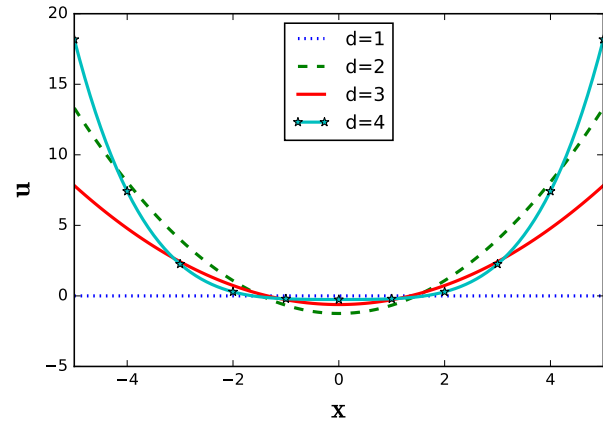
$$\langle u(t) v_0^d(t) v_0^d(t)^\top \rangle \succeq 0, \quad \langle (\Omega - u(t)) v_0^d(t) v_0^d(t)^\top \rangle \succeq 0$$

See Fig. 3 for numerical results.

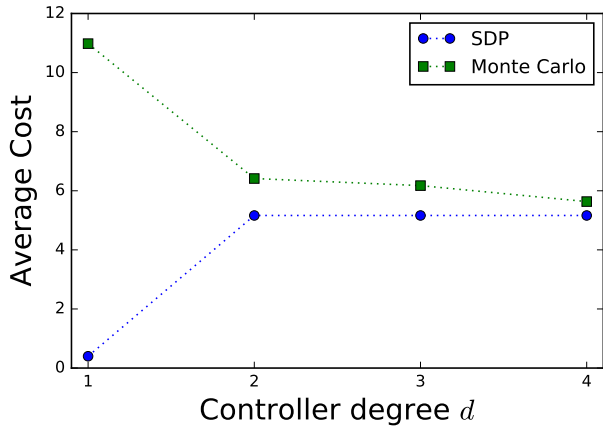
V. CONCLUSION

This technical note presented a method based on semidefinite programming for computing bounds on stochastic process moments and stochastic optimal control problems in a unified manner. The method is flexible, in that it can be applied to controlled and uncontrolled stochastic processes driven by Brownian motion and jumps. The key insight behind the method is the interpretation of the dynamics of the moments as a linear control problem. The auxiliary state consists of a collection of moments of the original state, while the auxiliary input consists of higher order state moments and terms involving inputs. Then all of the desired bounds can be computed in terms of optimal control problems on this auxiliary system.

Future work will focus on algorithmic improvements and theoretical extensions. A simple algorithmic extension would



(a) Jump Rate Strategies



(b) Average Cost

Fig. 3: Fig 3a shows the control policies computed as in Subsection IV-C for various orders. Here $Q = 1$, $R = 10$ and the upper bound on the jump rate $\Omega = 10$. As the degree of the controller increases, the strategy becomes flatter near $x(t) = 0$ and then increases rapidly for $|x(t)| > 3$. To enforce the constraint that $0 \leq u(t) \leq \Omega = 10$, the value is simply clipped to the correct range. Fig 3b compares the predicted bound with the achieved value (1000 runs each). With increasing controller degrees, the bounds become increasingly tight.

be to use alternative basis polynomials to improve numerical stability. Currently, all of the SDPs were solved using CVXPY [27] and off-the-shelf SDP solvers [28], [29]. However, the SDP from (11) has a structure which could be potentially be exploited by custom solvers. Furthermore, the general methodology could be extended to functions beyond polynomials.

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