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The $\mathbb{A}_{q,t}$ algebra and parabolic flag Hilbert schemes

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Abstract

The earlier work of the first and the third name authors introduced the algebra $\mathbb{A}_{q,t}$ and its polynomial representation. In this paper we construct an action of this algebra on the equivariant K-theory of certain smooth strata in the flag Hilbert scheme of points on the plane. In this presentation, the fixed points of the torus action correspond to generalized Macdonald polynomials, and the matrix elements of the operators have an explicit presentation.

1 Introduction

In the earlier article the first and the third name authors [4] introduced a new and interesting algebra called the algebra $\mathbb{A}_{q,t}$. It acts on the space $V = \bigoplus_{k=0}^{\infty} V_k$, where $V_k = \Lambda \otimes \mathbb{C}[y_1, \ldots, y_k]$ and Λ is the ring of symmetric functions in infinitely many variables. The algebra has generators y_i, z_i, T_i, d_+ and d_-. On each subspace V_k, y_i act as multiplication operators, T_i as Demazure–Lusztig operators, so together they form an affine Hecke algebra. The operators z_i and T_i also form an affine Hecke algebra (in particular, z_i commute). Finally, the most interesting operators $d_+ : V_k \to V_{k+1}$ and $d_- : V_k \to V_{k-1}$ intertwine different subspaces.

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The algebra $\mathbb{A}_{q,t}$ was used in [4] to prove a long-standing *Shuffle Conjecture* in algebraic combinatorics [14]. Later, it was also used in [19] to prove a "rational" version of Shuffle conjecture introduced in [10]. The latter yields a combinatorial expression for certain matrix elements of the generators $P_{m,n}$ of the elliptic Hall algebra [27] acting in its polynomial representation. In particular, the operator $P_{m,n} : V_0 \to V_0$ was realized in [19] inside the algebra $\mathbb{A}_{q,t}$.

It is known from the work of Schiffmann and Vasserot [27], Feigin and Tsymbaliuk [7] and Neguț [23] that the elliptic Hall algebra acts on the equivariant K-theory of the Hilbert schemes of points on the plane. In particular, [23] realized $P_{m,n}$ by an explicit geometric correspondence. This leads to a natural question: is there a geometric interpretation of the algebra $\mathbb{A}_{q,t}$ and its representation $V_\bullet$? We answer this question in the present paper.

The key geometric object is the *parabolic flag Hilbert scheme* $\mathrm{PFH}_{n,n-k}$ which is defined as the moduli space of flags $\{I_{n-k} \supset \cdots \supset I_n\}$, where I_s are ideals in $\mathbb{C}[x, y]$ of codimension s and $yI_{n-k} \subset I_n$. We prove that this is in fact a smooth quasiprojective variety. The following theorem is the main result of the paper.

Theorem 1.1 *Let* $U_k = \bigoplus_{n=k}^{\infty} K_{\mathbb{C}^*\times\mathbb{C}^*}(\mathrm{PFH}_{n,n-k})$ *and let* $U_\bullet = \bigoplus_{k=0}^{\infty} U_k$. *Then there is an action of the algebra* $\mathbb{A}_{q,t}$ *on* $U_\bullet$ *and isomorphisms* $U_k \simeq V_k$ *for all* k *compatible with the* $\mathbb{A}_{q,t}$*-algebra action.*

The construction of the action of the generators of $\mathbb{A}_{q,t}$ is quite natural. The action of z_i and T_i follows the classical work of Lusztig on the action of affine Hecke algebras on flag varieties [18]. In particular, z_i correspond to natural line bundles $\mathcal{L}_i = I_{n-i-1}/I_{n-i}$ on $\mathrm{PFH}_{n,n-k}$. The operators $d_\pm$ change the length of the flag and correspond to natural projections $\mathrm{PFH}_{n+1,n-k} \to \mathrm{PFH}_{n,n-k}$ and $\mathrm{PFH}_{n,n-k} \to \mathrm{PFH}_{n,n-k+1}$. Finally, the operators y_i can be obtained using the commutation relations between d_+, d_- and T_i.

We compare this geometric construction with [7,23,27]. The key operator in [7,27] is realized by a simple Nakajima correspondence $\mathrm{Hilb}^{n,n+1}$ with some power $\mathcal{L}^k$ of a line bundle on it, which naturally projects to Hilb^n and Hilb^{n+1}. This yields an operator $P_{1,k} : K(\mathrm{Hilb}^n) \to K(\mathrm{Hilb}^{n+1})$. We regard $\mathrm{Hilb}^{n,n+1}$ as a cousin of $\mathrm{PFH}_{n+1,n}$, and decompose $P_{1,k}$ as a composition of three operators $P_{1,k} = d_- z_1^k d_+$. Here $d_+ : U_0 \to U_1$ and $d_- : U_1 \to U_0$ correspond to the pullback and the pushforward under projections, and $z_1 : U_1 \to U_1$ corresponds to the line bundle $\mathcal{L}$. In Sect. 7.2 we make a similar comparison with the construction of [23] for more complicated operators $P_{m,n}$ in the elliptic Hall algebra.

A combinatorial consequence of this work is the construction of generalized Macdonald basis corresponding to the fixed points of the torus action in $\mathrm{PFH}_{n,n-k}$. For $k = 0$ we recover the modified Macdonald basis corresponding to the fixed points on the Hilbert scheme of points [12]. We explicitly compute the matrix elements for all the generators of $\mathbb{A}_{q,t}$ in this basis, see Eqs. (4.2), (4.3) and Lemma 4.2. In fact, we prove that these new elements have a triangularity property with respect to a version of the Bruhat order for affine permutations, generalizing the triangularity in the dominance order for usual Macdonald polynomials.

Finally, we would like to outline some future directions. First, the construction of the spaces $\mathrm{PFH}_{n,n-k}$ is very similar to the construction of so-called affine Laumon spaces

[6]. Tsymbaliuk [28] constructed an action of the quantum toroidal algebras $\ddot{U}(gl_k)$ on the K-theory of Laumon spaces. In particular, for $k = 1$ this action coincides with the action of the elliptic Hall algebra (which is known to be isomorphic to $\ddot{U}(gl_1)$) on the K-theory of the Hilbert scheme of points. However, it appears that for $k > 1$ his representation is larger than U_{k-1}. We plan to investigate the relations between $\mathbb{A}_{q,t}$ and quantum toroidal algebras in the future.

Second, the results of [10,11,20] suggest a deep relation between Hilbert schemes, the elliptic Hall algebra, and categorical link invariants such as Khovanov-Rozansky homology. In particular, a precise relation between the Khovanov-Rozansky homology of (m, n) torus knots and the operators $P_{m,n}$ was proved for $m = n+1$ by Hogancamp [16] and for general coprime (m, n) by the third author in [20]. It is expected [19] that $\mathbb{A}_{q,t}$ can be realized as the skein algebra of certain more general tangles in the thickened torus, so it would be interesting to extend the approach of [11] to this more general framework.

Finally, while our proofs of the relations in the algebra are fixed point formulas which make sense only in K-theory, the definitions of the operators are K-theoretic reductions of well-defined operators in the derived category. Identifying these definitions is therefore the first step towards categorification. One of our motivations in doing this is to find a geometric proof of the shuffle theorem, which we expect would have broader implications, to Khovanov-Rozansky homology, for instance. As supporting evidence, we conclude the paper by showing in Theorem 7.2 that the contribution to the shuffle formula from Dyck paths with exactly k touch points, has an explicit formula in the fixed point basis under the identification $U_k \cong V_k$. The form of these formulas is nearly identical to Haiman's formula for the resolution of the structure sheaf of the punctual Hilbert scheme [13], suggesting that we have found a compactly supported sheaf on $\mathrm{PFH}_{n,n-k}$. This observation would be hidden without the fixed point description.

In Sect. 2, we begin by recalling the construction of the $\mathbb{A}_{q,t}$ algebra. We then identify a subalgebra $\mathbb{B}_{q,t}$ which also admits a homomorphism $\mathbb{A}_{q,t} \to \mathbb{B}_{q,t}$. This is the algebra which is given a geometric construction. In Sect. 3, we define the parabolic flag Hilbert scheme $\mathrm{PFH}_{n,n-k}$, and prove properties such as smoothness. In Sect. 4, we define operators on $K_T(\mathrm{PFH}_{n,n-k})$ as pullback and proper pushforwards of natural projection maps. We show that these operators satisfy the relations of $\mathbb{B}_{q,t}$ using fixed point formulas in Sect. 5. They therefore define a representation of $\mathbb{A}_{q,t}$ via $\mathbb{A}_{q,t} \to \mathbb{B}_{q,t} \to \mathrm{End}(U_k)$, where U_k is the localization $K_T(\mathrm{PFH}_{n,n-k}) \otimes \mathbb{Q}(q, t)$. In Sect. 6, we prove that U_k is isomorphic to V_k, as representations of $\mathbb{A}_{q,t}$. We finally conclude with some example applications in Sect. 7. This includes the aforementioned promising fixed point formula for the contribution of Dyck paths with k touch points to the combinatorial side of the shuffle theorem, which has the appearance of the fundamental class of a compact subscheme, similar to class of the punctual Hilbert scheme $[\mathcal{O}_{Z_n}] \in K_T(\mathrm{Hilb}_n)$.

2 The algebra

2.1 $\mathbb{A}_q$

The algebras under consideration can be viewed as path algebras of quivers with vertex set $\mathbb{Z}_{\geq 0}$.[1] So we implicitly assume that all our algebras contain orthogonal idempotents Id_i ($i \in \mathbb{Z}_{\geq 0}$) and when we speak of an element $R : i \to j$ for $i, j \in \mathbb{Z}_{\geq 0}$ we impose the relation $R = R\,\mathrm{Id}_i = \mathrm{Id}_j\,R$. When we have a representation V of such an algebra we always assume that $V = \bigoplus_{i=0}^{\infty} V_i$ where $V_i = \mathrm{Id}_i\,V$. Then any element $R : i \to j$ as above induces a linear map $V_i \to V_j$. To stress the direct sum decomposition above we denote such a representation by $V_\bullet$.

First we define the "half algebra" $\mathbb{A}_q$ depending on one parameter $q \in \mathbb{Q}(q)$:

Definition 2.1 $\mathbb{A}_q$ is the $\mathbb{Q}(q)$-linear algebra generated by a collection of orthogonal idempotents labeled by $\mathbb{Z}_{\geq 0}$ and elements

$$d_+ : k \to k+1,\; d_- : k \to k-1,\; T_i : k \to k \quad (1 \leq i < k),\; y_i : k \to k \quad (1 \leq i \leq k)$$

subject to relations

$$(T_i - 1)(T_i + q) = 0, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad T_i T_j = T_j T_i \quad (|i-j| > 1), \tag{2.1}$$

$$T_i y_{i+1} T_i = q y_i \;(1 \leq i \leq k-1), \tag{2.2}$$

$$y_i T_j = T_j y_i \;(i \notin \{j, j+1\}),\; y_i y_j = y_j y_i \;(1 \leq i, j \leq k),$$

$$d_-^2 T_{k-1} = d_-^2,\; d_- T_i = T_i d_- \;(1 \leq i \leq k-2),\; d_- y_i = y_i d_- \;(1 \leq i \leq k-1), \tag{2.3}$$

$$T_1 d_+^2 = d_+^2,\; d_+ T_i = T_{i+1} d_+ \;(1 \leq i \leq k-1),$$

$$d_+ y_i = T_1 T_2 \cdots T_i y_i T_i^{-1} \cdots T_1^{-1} d_+,\; (1 \leq i \leq k) \tag{2.4}$$

$$d_+ d_- - d_- d_+ = (q-1) T_1 T_2 \cdots T_{k-1} y_k. \tag{2.5}$$

Remark 2.2 Note that relations (2.1) define the Hecke algebra, and relations (2.1) + (2.2) define the affine Hecke algebra AH_k.

In what follows we will need a slightly different description of the algebra $\mathbb{A}_q$. Let the AH_k be the affine Hecke algebra generated by $T_1, \ldots, T_{k-1}, y_1, \ldots, y_k$ modulo relations (2.1) and (2.2). The following lemma gives another presentation of the algebra AH_k similar to the Iwahori-Matsumoto presentation of the affine Hecke algebra, although in our definition y_i are not invertible. The proof is similar to [4, Lemma 5.4], but we present it here for completeness.

[1] A categorically inclined reader can view our algebras as categories with object set $\mathbb{Z}_{\geq 0}$. Then a representation of a category is a simply a functor to the category of vector spaces.

Lemma 2.3 *Consider the algebra AH'_k generated by $T_1, \dots, T_{k-1}$ and an element φ modulo relations (2.1) and*

$$\varphi T_i = T_{i+1}\varphi \ (i \le k-2),\ \varphi^2 T_{k-1} = T_1\varphi^2. \tag{2.6}$$

Then the algebras AH_k and AH'_k are isomorphic.

Proof Define $\varphi = T_1 \dots T_{k-1} y_k$. Let us prove that (2.2) imply (2.6). For $i \le k-2$ one has:

$$\varphi T_i = T_1 \dots T_{k-1} y_k T_i = T_1 \dots T_{k-1} T_i y_k = T_{i+1} T_1 \dots T_{k-1} y_k = T_{i+1}\varphi,$$

while

$$\begin{aligned}\varphi^2 T_{k-1} &= T_1 \dots T_{k-1} y_k T_1 \dots T_{k-1} y_k T_{k-1} = q(T_1 \dots T_{k-1})(T_1 \dots T_{k-2}) y_k y_{k-1},\\ T_1\varphi^2 &= T_1(T_1 \dots T_{k-1}) y_k (T_1 \dots T_{k-1}) y_k = T_1(T_1 \dots T_{k-1})(T_1 \dots T_{k-2}) y_k T_{k-1} y_k\\ &= T_1(T_2 \dots T_{k-1})(T_1 \dots T_{k-2}) T_{k-1} y_k T_{k-1} y_k = q(T_1 \dots T_{k-1})(T_1 \dots T_{k-2}) y_{k-1} y_k.\end{aligned}$$

Conversely, let us prove that (2.6) imply (2.2). Define

$$y_i = q^{i-k} T_{i-1}^{-1} \dots T_1^{-1} \varphi T_{k-1} \dots T_i. \tag{2.7}$$

Then, clearly, $T_i y_{i+1} T_i = q y_i$. If $j > i$ then

$$\begin{aligned} y_i T_j &= q^{i-k} T_{i-1}^{-1} \dots T_1^{-1} \varphi T_{k-1} \dots T_i T_j = q^{i-k} T_{i-1}^{-1} \dots T_1^{-1} \varphi T_{j-1} T_{k-1} \dots T_i\\ &= q^{i-k} T_{i-1}^{-1} \dots T_1^{-1} T_j \varphi T_{k-1} \dots T_i = T_j y_i.\end{aligned}$$

If $j < i-1$, the proof of $y_i T_j = T_j y_i$ is similar. Finally,

$$\begin{aligned} y_1 y_k &= \varphi T_{k-1} \dots T_1 T_{k-1}^{-1} \dots T_1^{-1} \varphi = \varphi T_{k-2}^{-1} \dots T_1^{-1} T_{k-1} \dots T_2 \varphi,\\ y_k y_1 &= T_{k-1}^{-1} \dots T_1^{-1} \varphi^2 T_{k-1} \dots T_1 = T_{k-1}^{-1} \dots T_1^{-1} T_1 \varphi^2 T_{k-2} \dots T_1\\ &= T_{k-1}^{-1} \dots T_2 \varphi^2 T_{k-2} \dots T_1 = \varphi T_{k-2}^{-1} \dots T_1^{-1} T_{k-1} \dots T_2 \varphi.\end{aligned}$$

The proof of other commutation relations $y_i y_j = y_j y_i$ is similar. □

Lemma 2.4 *The algebra $\mathbb{A}_q$ is generated by $T_1, \dots, T_{k-1}, d_+, d_-$ modulo relations (2.1), all relations in (2.3) and (2.4) not involving y_i, and two additional relations:*

$$q\varphi d_- = d_- \varphi T_{k-1}, \quad T_1 \varphi d_+ = q d_+ \varphi, \tag{2.8}$$

where $\varphi = \frac{1}{q-1}[d_+, d_-]$. All other relations follow from these.

Proof Let us check that φ satisfies (2.6) on V_k. Clearly, for $i \le k-2$ one has

$$(d_+d_- - d_-d_+)T_i = d_+T_id_- - d_-T_{i+1}d_+ = T_{i+1}(d_+d_- - d_-d_+).$$

Furthermore,

$$d_+d_-\varphi T_{k-1} = qd_+\varphi d_- = T_1\varphi d_+d_-,$$

and

$$d_-d_+\varphi T_{k-1} = q^{-1}d_-T_1\varphi d_+T_{k-1} = q^{-1}T_1d_-\varphi T_kd_+ = T_1\varphi d_-d_+,$$

so

$$\varphi^2T_{k-1} = \frac{1}{q-1}(d_+d_- - d_-d_+)\varphi T_{k-1} = \frac{1}{q-1}T_1\varphi(d_+d_- - d_-d_+) = T_1\varphi^2.$$

Therefore by Lemma 2.3 we can define y_i and check the commutation relations (2.2). Let us check the remaining relations:

$$\begin{aligned} d_-y_i &= d_-T_{i-1}^{-1}\dots T_1^{-1}\varphi T_{k-1}\dots T_i = T_{i-1}^{-1}\dots T_1^{-1}d_-\varphi T_{k-1}\dots T_i \\ &= T_{i-1}^{-1}\dots T_1^{-1}\varphi d_-T_{k-2}\dots T_i = T_{i-1}^{-1}\dots T_1^{-1}\varphi T_{k-2}\dots T_id_- = y_id_-. \end{aligned}$$

The last identity $d_+y_i = T_1\dots T_iy_iT_i^{-1}\dots T_1^{-1}d_+$ is also straightforward, see [4, Lemma 5.4]. □

2.2 $\mathbb{A}_{q,t}$

The "double algebra" $\mathbb{A}_{q,t}$ depends on two parameters $q, t \in \mathbb{Q}(q,t)$ and is obtained from two copies of $\mathbb{A}_q$ by imposing more relations:

Definition 2.5 $\mathbb{A}_{q,t}$ is the $\mathbb{Q}(q,t)$-linear algebra generated by a collection of orthogonal idempotents labelled by $\mathbb{Z}_{\ge 0}$ and elements:

$$\begin{aligned} &d_+, d_+^* : k \to k+1,\ d_- : k \to k-1,\ T_i : k \\ &\to k \quad (1 \le i < k),\ y_i, z_i : k \to k \quad (1 \le i \le k) \end{aligned}$$

subject to the

- relations of $\mathbb{A}_q$ for d_-, d_+, T_i, y_i,
- relations of $\mathbb{A}_{q^{-1}}$ for $d_-, d_+^*, T_i^{-1}, z_i$,

and

$$d_+z_i = z_{i+1}d_+, \quad d_+^*y_i = y_{i+1}d_+^* \quad (1 \le i \le k), \quad z_1d_+ = -tq^{k+1}y_1d_+^*. \tag{2.9}$$

Remark 2.6 One is tempted to say that the generators T_i, y_i and z_i form some sort of double affine Hecke algebra as in Remark 2.2, but this is not the case. The problem stems from the fact that double affine Hecke algebras of [3] do not embed into one another in the way that the affine Hecke algebras do. There is a way, however, to relate $\mathbb{A}_{q,t}$ to double affine Hecke algebras by making sense of limits of the form $\lim_{n\to\infty} e_n \mathrm{DAHA}_{n+k} e_n$, where $e_n \in \mathrm{DAHA}_{n+k}$ is the partial symmetrization operator on the indices $k+1, k+2, \ldots, k+n$. In particular, we expect that the degree zero part of $\mathbb{A}_{q,t}$ coincides with the positive part of the elliptic Hall algebra which is the stable limit of spherical DAHAs as shown in [27]. See also Remark 7.2 in [4]. In 7.2 we express the positive generators $P_{m,n}$ of the elliptic Hall algebra in terms of the generators of $\mathbb{A}_{q,t}$.

We also note that the generators of $\mathbb{A}_{q,t}$ are closely related to the braid group $\mathscr{B}_k(\mathbb{T}_0)$ of the punctured torus. Indeed, the latter has generators $T_i^{\pm}, y_i^{\pm}, z_i^{\pm}$ and one can define a related monoid $\mathscr{B}_k^+(\mathbb{T}_0)$ generated by $T_i^{\pm}, y_i, z_i$ (see [19] for details). In [19] the third author constructed homomorphisms from $\mathscr{B}_k^+(\mathbb{T}_0)$ to $\mathbb{A}_{q,t}$ for all k. This is similar to the homomorphism from $\mathscr{B}_k(\mathbb{T}_0)$ to the DAHA (e. g. [17]).

In what follows we will need a certain subalgebra of $\mathbb{A}_{q,t}$ which, nevertheless, contains an isomorphic copy of $\mathbb{A}_{q,t}$.

Definition 2.7 The algebra $\mathbb{B}_{q,t}$ is generated by a collection of orthogonal idempotents labelled by $\mathbb{Z}_{\geq 0}$, generators d_+, d_-, T_i and z_i modulo relations:

$$\begin{aligned}
&(T_i-1)(T_i+q)=0, \quad T_iT_{i+1}T_i=T_{i+1}T_iT_{i+1}, \quad T_iT_j=T_jT_i \quad (|i-j|>1),\\
&T_i^{-1}z_{i+1}T_i^{-1}=q^{-1}z_i \ (1\leq i\leq k-1),\\
&z_iT_j=T_jz_i \ (i\notin\{j,j+1\}), \ z_iz_j=z_jz_i \ (1\leq i,j\leq k),\\
&d_-^2T_{k-1}=d_-^2, \ d_-T_i=T_id_- \ (1\leq i\leq k-2),\\
&T_1d_+^2=d_+^2, \ d_+T_i=T_{i+1}d_+ \ (1\leq i\leq k-1),\\
&q\varphi d_-=d_-\varphi T_{k-1}, \quad T_1\varphi d_+=qd_+\varphi,\\
&z_id_-=d_-z_i, \quad d_+z_i=z_{i+1}d_+,\\
&z_1(qd_+d_--d_-d_+)=qt(d_+d_--d_-d_+)z_k.
\end{aligned}$$

Remark 2.8 By (2.5), one can define the elements $y_i \in \mathbb{B}_{q,t}$ and prove that y_i, T_i, d_+ and d_- generate a copy of $\mathbb{A}_q$.

Proposition 2.9 *There is a homomorphism $\alpha : \mathbb{B}_{q,t} \to \mathbb{A}_{q,t}$ which sends d_+, d_-, T_i and z_i to the corresponding generators of $\mathbb{A}_{q,t}$.*

Proof Let us check that the last defining relation for $\mathbb{B}_{q,t}$ holds in $\mathbb{A}_{q,t}$:

$$z_1(qd_+d_--d_-d_+)=qz_1d_+d_--z_1d_-d_+=q(z_1d_+)d_--d_-(z_1d_+).$$

We can replace z_1d_+ by a multiple of $y_1d_+^*$ and obtain:

$$q(-tq^k)y_1d_+^*d_--d_-(-tq^{k+1})y_1d_+^*=-tq^{k+1}y_1[d_+^*,d_-].$$

Since $d_-, d_+^*, T_i^{-1}, z_i$ satisfy the relations for $\mathbb{A}_{q^{-1}}$, by (2.5) we get:

$$[d_+^*, d_-] = (q^{-1} - 1)T_1^{-1} \cdots T_{k-1}^{-1} z_k,$$

so

$$\begin{aligned} -tq^{k+1} y_1 [d_+^*, d_-] &= -tq^{k+1}(q^{-1} - 1) y_1 T_1^{-1} \dots T_{k-1}^{-1} z_k \\ &= tq^k (q-1) y_1 T_1^{-1} \dots T_{k-1}^{-1} z_k = qt(q-1) T_1 \dots T_{k-1} y_k z_k \\ &= qt[d_+, d_-] z_k. \end{aligned}$$

It follows from the definition and Theorem 2.4 that all other defining relations of $\mathbb{B}_{q,t}$ are satisfied in $\mathbb{A}_{q,t}$. □

Theorem 2.10 *There is an algebra homomorphism $\beta : \mathbb{A}_{q,t} \to \mathbb{B}_{q,t}$ such that*

$$\beta(T_i) = T_i, \;\; \beta(d_-) = d_-, \;\; \beta(d_+) = d_+, \;\; \beta(d_+^*) = q^{-k} z_1 d_+$$

and $\beta(z_1) = -qty_1z_1$. There is a chain of homomorphisms:

$$\mathbb{A}_{q,t} \xrightarrow{\beta} \mathbb{B}_{q,t} \xrightarrow{\alpha} \mathbb{A}_{q,t}.$$

Proof It is clear that all defining relations of $\mathbb{A}_q$ are satisfied for T_i, d_-, d_+ and hence for y_i. We proceed to check the relations of $\mathbb{A}_{q^{-1}}$ for $T_i^{-1}, d_-, \beta(d_+^*), z_i$ in $\mathbb{B}_{q,t}$. In order to apply Lemma 2.3 we will need the following computation:

$$\begin{aligned} (q^{-1} - 1)\varphi^* &= [\beta(d_+^*), d_-] = q^{1-k} z_1 d_+ d_- - q^{-k} z_1 d_- d_+ = q^{-k} z_1 (q d_+ d_- - d_- d_+) \\ &= tq^{1-k}(d_+ d_- - d_- d_+) z_k = tq^{1-k}(q-1)\varphi z_k. \end{aligned}$$

Thus we have

$$\varphi^* = -tq^{2-k} \varphi z_k,$$

so that we can check (2.8):

$$\begin{aligned} q^{-1}\varphi^* d_- &= -tq^{2-k}\varphi z_{k-1} d_- = -tq^{1-k} d_- \varphi T_{k-1} z_{k-1} = d_- \varphi^* T_{k-1}^{-1}, \\ T_1^{-1}\varphi^* \beta(d_+^*) &= -tq^{1-k} T_1^{-1} \varphi z_{k+1} q^{-k} z_1 d_+ = -tq^{1-2k} T_1^{-1} \varphi z_1 d_+ z_k \\ &= -tq^{1-2k} T_1^{-1} z_2 \varphi d_+ z_k = -tq^{-2k} z_1 T_1 \varphi d_+ z_k = -tq^{1-2k} z_1 d_+ \varphi z_k \\ &= q^{-1} \beta(d_+^*) \varphi^*, \end{aligned}$$

where we have used the following identity between elements $k \to k$ for $k \geq 2$:

$$\varphi z_1 = \frac{1}{q-1}(d_+ d_- - d_- d_+) z_1 = \frac{1}{q-1} z_2 (d_+ d_- - d_- d_+) = z_2 \varphi. \qquad (2.10)$$

Among prerequisites for Lemma 2.3 it remains to check the identities between $\beta(d_+^*)$ and T_i. We have

$$\beta(d_+^*)T_i = q^{-k}z_1d_+T_i = q^{-k}z_1T_{i+1}d_+ = q^{-k}T_{i+1}d_+ = T_{i+1}\beta(d_+^*),$$
$$\beta(d_+^*)^2 = q^{-2k-1}z_1d_+z_1d_+ = q^{-2k-1}z_1z_2d_+^2,$$

hence

$$T_1\beta(d_+^*)^2 = q^{-2k-1}z_1z_2T_1d_+^2 = q^{-2k-1}z_1z_2d_+^2 = \beta(d_+^*)^2.$$

Thus we can apply Lemma 2.3 and deduce that the relations of $\mathbb{A}_{q^{-1}}$ for T_i^{-1}, d_-, $\beta(d_+^*)$, z_i are satisfied.

It remains to check relations (2.9) for d_+, y_i, $\beta(d_+^*)$, $\beta(z_i)$. We have

$$\beta(z_k) = T_{k-1}\dots T_1\varphi^* = -tq^{2-k}T_{k-1}\dots T_1\varphi z_k.$$

Therefore

$$\beta(z_i) = -tq^{2-k}T_{i-1}\dots T_1\varphi T_{k-1}\dots T_iz_i = -qtT_{i-1}\dots T_1y_1T_1^{-1}\dots T_{i-1}^{-1}z_i.$$

Thus we have

$$\begin{aligned}d_+\beta(z_i) &= -qtd_+T_{i-1}\dots T_1y_1T_1^{-1}\dots T_{i-1}^{-1}z_i = -qtT_i\dots T_1y_1T_1^{-1}\dots T_i^{-1}z_{i+1}\\ &= \beta(z_{i+1})d_+.\end{aligned}$$

Using Lemma 2.3 and (2.10) we obtain

$$\begin{aligned}\beta(d_+^*)\varphi &= q^{-k}z_1d_+\varphi = q^{-1-k}z_1T_1\varphi d_+ = q^{-k}T_1^{-1}z_2\varphi d_+ = q^{-k}T_1^{-1}\varphi z_1d_+\\ &= T_1^{-1}\varphi\beta(d_+^*),\end{aligned}$$

which implies

$$\begin{aligned}\beta(d_+^*)y_i &= \beta(d_+^*)T_{i-1}^{-1}\dots T_1^{-1}\varphi T_{k-1}\dots T_i = T_i^{-1}\dots T_1^{-1}\varphi T_k\dots T_{i+1}\beta(d_+^*)\\ &= y_{i+1}\beta(d_+^*).\end{aligned}$$

Finally, we have

$$\beta(z_1)d_+ = -qty_1z_1d_+ = -tq^{k+1}y_1\beta(d_+^*).$$

Thus we finished verifying (2.9). □

2.3 Gradings

The algebras $\mathbb{A}_{q,t}$ and $\mathbb{B}_{q,t}$ are triply graded. The grading of d_+ is $(1, 0, 0)$, the grading of d_- is $(0, 1, 0)$, and the grading of T_i is $(0, 0, 0)$. The commutation relations imply that y_i have grading $(1, 1, 0)$. Next, we require that d_+^* has grading $(0, 0, 1)$ and z_i have grading $(0, 1, 1)$. It is easy to check that all relations are tri-homogeneous with respect to these gradings. In particular, the degrees of $z_1 d_+$ and $y_1 d_+^*$ are both equal to $(1, 1, 1)$.

In what follows we will use two specializations of this triple grading. The first projection $(a, b, c) \mapsto a - b + c$ assigns to d_+ and d_+^* degree 1, d_- has degree (-1) and y_i, z_i, T_i all have degree 0. This is just the standard grading which equals k in the idempotent e_k.

The more interesting projection $(a, b, c) \to a+b+c$ assigns to d_+, d_-, d_+^* degree 1, and to y_i, z_i degree 2.

2.4 Polynomial representation

Denote by Λ the ring of symmetric functions in infinitely many variables $x_1, x_2, \ldots$. We will use the following standard notations for plethystic substitutions: if A is an element in some λ-ring R, we consider the homomorphism $\Lambda \to R,\ F \mapsto F[A]$ which sends power sums p_n to $p_n(A)$. For example,

$$p_n[X + (q-1)y_{k+1}] = p_n + (q^n - 1)y_{k+1}^n.$$

Also, we use notations $\mathrm{Exp}[A] = \sum_{n=0}^{\infty} h_n[A]$ and $\mathrm{Res}_y \sum_m c_m y^m dy = c_{-1}$.

Following [4] we introduce spaces

$$V_k = \Lambda \otimes \mathbb{C}(q,t)[y_1, \ldots, y_k], \quad V_\bullet = \bigoplus_{k \geq 0} V_k.$$

One of the results of [4] is the following:

Proposition 2.11 *There is an action of $\mathbb{A}_{q,t}$ on $V_\bullet$ in which*

$$\begin{aligned} T_i F &= \frac{(q-1)y_{i+1}F + (y_{i+1} - q y_i)s_i F}{y_{i+1} - y_i}, \quad y_i F = y_i \cdot F, \\ d_- F &= -\mathrm{Res}_{y_k} F[X - (q-1)y_k]\,\mathrm{Exp}[-y_k^{-1} X] dy_k \quad (F \in V_k), \\ d_+ F &= T_1 T_2 \ldots T_k (F[X + (q-1)y_{k+1}]). \\ d_{+,CM}^* F &= \gamma F[X + (q-1)y_{k+1}], \end{aligned}$$

where $\gamma(y_i) = y_{i+1}$ and $\gamma(y_{k+1}) = t y_1$. Furthermore, we have a unique isomorphism

$$V_\bullet = \mathbb{A}_q \,\mathrm{Id}_0,$$

of left $\mathbb{A}_q$-modules in which $1 \in V_0$ maps to Id_0.

Consider the space

$$W_\bullet := \bigoplus W_k, \quad W_k = (y_1 \dots y_k)^{-1} V_k \subset \Lambda \otimes \mathbb{C}(q,t)[y_1^{\pm 1}, \dots, y_k^{\pm 1}].$$

Clearly, $V_k \subset W_k$.

Theorem 2.12 *The following statements hold:*

(1) *The operators T_i, d_+, d_- and $d_+^* = -(qty_1)^{-1} d_{+,CM}^*$ can be naturally extended to the space $W_\bullet$ and define a representation of $\mathbb{A}_{q,t}$.*
(2) *In this representation, $\alpha(\mathbb{B}_{q,t})$ preserves the subspace $V_\bullet \subset W_\bullet$, and hence defines a representation of $\mathbb{B}_{q,t}$ in $V_\bullet$.*
(3) *The composition $\alpha\beta(\mathbb{A}_{q,t})$ also preserves $V_\bullet$, and hence defines a representation of $\mathbb{A}_{q,t}$ in $V_\bullet$. This representation agrees with the one in Proposition 2.11.*

We illustrate all these representations in the following commutative diagram:

$$\begin{array}{ccccc}
\mathbb{A}_{q,t} & \xrightarrow{\beta} & \mathbb{B}_{q,t} & \xrightarrow{\alpha} & \mathbb{A}_{q,t} \\
\downarrow{\scriptstyle d_{+,CM}^*} & & \downarrow & & \downarrow{\scriptstyle d_+^*} \\
\mathrm{End}(V_\bullet) & \longleftarrow & \mathrm{End}_{V_\bullet}(W_\bullet) & \hookrightarrow & \mathrm{End}(W_\bullet)
\end{array}$$

Here $\mathrm{End}_{V_\bullet}(W_\bullet)$ denotes the set of endomorphisms of $W_\bullet$ preserving $V_\bullet$.

Proof Let us prove that T_i, d_+, d_- and $d_+^* = -(qty_1)^{-1} d_{+,CM}^*$ are well-defined on $W_\bullet$. If $F \in V_\bullet$, then

$$\begin{aligned}
T_i(F/(y_1 \dots y_k)) &= (T_i F)/(y_1 \dots y_k) \in W_\bullet, \\
d_+(F/(y_1 \dots y_k)) &= (T_1 T_2 \dots T_k y_{k+1} F[X + (q-1)y_{k+1}])/(y_1 \dots y_{k+1}) \in W_\bullet, \\
d_-(F/(y_1 \dots y_k)) &= -(y_1 \dots y_{k-1})^{-1} \operatorname{Res}_{y_k} F[X - (q-1)y_k] y_k^{-1} \operatorname{Exp}[-y_k^{-1} X] dy_k \in W_\bullet, \\
-qtd_+^*(F/(y_1 \dots y_k)) &= -y_1^{-1} \gamma \left(F[X + (q-1)y_{k+1}]/(y_1 \dots y_k)\right) \\
&= (y_1 \dots y_{k+1})^{-1} \gamma(F[X + (q-1)y_{k+1}] \in W_\bullet.
\end{aligned}$$

using the fact that T_i commutes with $y_1 \dots y_k$, and $T_i(1) = 1$. The verification of the commutation is identical to [4] and we leave it to the reader.

To prove that $\alpha(\mathbb{B}_{q,t})$ preserves $V_\bullet$, it is sufficient to prove that the commutator $[d_+^*, d_-]$ preserves $V_\bullet$ (then z_i preserve $V_\bullet$, and T_i, d_+, d_- preserve $V_\bullet$ by definition). For $F \in V_k$ we have:

$$\begin{aligned}
-qtd_+^* d_- F = -y_1^{-1} & F[X + (1-q)ty_1 - (q-1)u, y_2, \dots, y_k, u] \\
& \times \operatorname{Exp}[-u^{-1}X - u^{-1}(q-1)ty_1]|_{u^{-1}}, \\
& -qtd_- d_+^* = -y_1^{-1} F[X + (1-q)ty_1 - (q-1)u, y_2, \dots, y_k, u] \\
& \operatorname{Exp}[-u^{-1}X]|_{u^{-1}}.
\end{aligned}$$

Now

$$\frac{1-u^{-1}qty_1}{1-u^{-1}ty_1}-1=(1-q)\frac{u^{-1}ty_1}{1-u^{-1}ty_1},$$

so

$$[d_+^*, d_-]F = (1-q^{-1})F[X+(1-q)ty_1-(q-1)u, y_2, \ldots, y_k, u]\,\mathrm{Exp}[u^{-1}ty_1-u^{-1}X]|_{u^0}.$$

Finally, $\alpha\beta(d_+^*) = -qty_1d_+^* = d_{+,CM}^*$. □

This result is very useful in the proof of our main theorem. Namely, we will define a geometric representation of $\mathbb{B}_{q,t}$ and identify it with the space $V_\bullet$. Then, using the homomorphism β, we will define a representation of $\mathbb{A}_{q,t}$ which, by the above, is isomorphic to the representation from Proposition 2.11.

Finally, a key observation from [4] is that there is a symmetry in the relations of $\mathbb{A}_{q,t}$ which is antilinear with respect to the conjugation $(q,t)\mapsto(q^{-1},t^{-1})$, and is given on generators by

$$d_-\leftrightarrow d_-,\quad T_i\leftrightarrow T_i^{-1},\quad y_i\leftrightarrow z_i,\quad d_+\leftrightarrow d_+^* \tag{2.11}$$

Furthermore, this symmetry preserves the kernel of the map $\mathbb{A}_{q,t}\to\mathrm{End}(V_\bullet)$, and so determines a map

$$\mathcal{N}: V_\bullet \to V_\bullet \tag{2.12}$$

which is antilinear, and satisfies $\mathcal{N}^2=1$. On $V_0=\Lambda\otimes\mathbb{C}(q,t)$ the involution $\mathcal{N}$ is related to the celebrated operator ∇ on symmetric polynomials, see (6.5) below.

3 The spaces

3.1 Parabolic flag Hilbert schemes

Definition 3.1 The parabolic flag Hilbert scheme $\mathrm{PFH}_{n,n-k}$ of points on $\mathbb{C}^2$ is the moduli space of flags

$$I_n\subset I_{n-1}\subset\cdots\subset I_{n-k}$$

where I_{n-i} is an ideal in $\mathbb{C}[x,y]$ of codimension $(n-i)$ and $yI_{n-k}\subset I_n$.

Definition 3.2 The parabolic flag Hilbert scheme $\mathrm{PFH}_{n,n-k}$ of points on $\mathbb{C}^2$ is the GIT quotient of the space of triples (X,Y,v) by the group G, where $v\in\mathbb{C}^n$, X and Y are $(n-k,k)$ block lower-triangular matrices such that $k\times k$ block is lower-triangular in X and vanishes in Y:

$$X = \left(\begin{array}{c|ccc} * & & 0 & \\ \vdots & \vdots & & \\ \hline & * & 0 & \dots 0 \\ & * & * & \dots 0 \\ \vdots & \vdots & \vdots & \vdots \\ & * & \dots & * \end{array}\right), \quad Y = \left(\begin{array}{c|c} * & 0 \\ \hline & 0 \dots 0 \\ \vdots & \vdots \\ & 0 \dots 0 \end{array}\right). \tag{3.1}$$

We require that $[X, Y] = 0$ and the stability condition $\mathbb{C}\langle X, Y\rangle v = \mathbb{C}^n$ holds. The group G consists of $(n-k, k)$ invertible block lower-triangular matrices with lower-triangular $k \times k$ block, and acts by $g.(X, Y, v) = (gXg^{-1}, gYg^{-1}, gv)$.

Proposition 3.3 *The two definitions above of* $\mathrm{PFH}_{n,n-k}$ *are equivalent.*

Proof The proof is standard but we include it here for completeness. Given a flag of ideals $\{I_n \subset I_{n-1} \subset \cdots \subset I_{n-k} \subset \mathbb{C}[x, y]\}$, consider the sequence of vector spaces $W_s = \mathbb{C}[x, y]/I_s$. The multiplication by x and y induces an action of two commuting operators X and Y on each W_s. There is a sequence of surjective maps $W_n \twoheadrightarrow W_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow W_{n-k}$ which commute with the action of X and Y. Since $yI_{n-k} \subset I_n$, the operator Y annihilates

$$\mathrm{Ker}(W_n \twoheadrightarrow W_{n-k}) = I_{n-k}/I_n.$$

If one chooses a basis in all W_s compatible with the projections, then the operators X and Y in this basis would have the form (3.1). The vector v corresponds to the projection of $1 \in \mathbb{C}[x, y]$, and the matrix g corresponds to the change of basis.

Conversely, given a triple X, Y, v, let W_s be the vector space spanned by the first s coordinate vectors, and let X_s, Y_s, v_s denote the restrictions of X, Y and v to W_s. Let $I_s = \{f \in \mathbb{C}[x, y] : f(X_s, Y_s)(v_s) = 0\}$. Clearly, I_s is an ideal, $I_{s+1} \subset I_s$ and $yI_{n-k} \subset I_n$. □

Example 3.4 If $k = 0$ then clearly $\mathrm{PFH}_{n,n-k} = \mathrm{Hilb}^n(\mathbb{C}^2)$. If $k = n$ then $\mathrm{PFH}_{n,n-k} = \mathbb{C}^n$. Indeed, for $k = n$ the matrix Y vanishes, and the stability condition implies that X is determined up to conjugation by its eigenvalues (that is, all generalized eigenvectors with the same eigenvalue belong to a single Jordan block). Therefore the natural projection

$$\mathrm{PFH}_{n,0} \to \mathbb{C}^n, \ (X, Y, v) \mapsto (x_{11}, \dots, x_{nn})$$

is an isomorphism.

These examples indicate that $\mathrm{PFH}_{n,n-k}$ behaves better than the full flag Hilbert scheme which is very singular [11]. This is indeed true in general.

Theorem 3.5 *The space* $\mathrm{PFH}_{n,n-k}$ *is a smooth manifold of dimension* $2n - k$ *for all* n *and* k.

In the proof of this theorem we will use a version of the geometric construction of Biswas and Okounkov [1] (see also [6, Sect. 3.4], [22, Sect. 4.3] and references therein). Consider the map

$$\sigma : \mathbb{C}^2 \to \mathbb{C}^2,\ \sigma(x, y) = (x, y^{k+1}).$$

Also, consider an action of the group $\Gamma = \mathbb{Z}/(k+1)\mathbb{Z}$ on $\mathbb{C}^2$ given by $(x, y) \mapsto (x, \zeta y)$, where ζ is a primitive $(k+1)$st root of unity. Given a sequence of ideals $I_n, \ldots, I_{n-k}$, we can consider the space

$$J(I_n, \ldots, I_{n-k}) = \sigma^* I_n + y\sigma^* I_{n-1} + \cdots + y^k \sigma^* I_{n-k} \subset \mathbb{C}[x, y]$$

Lemma 3.6 *The space $J(I_n, \ldots, I_{n-k})$ is an ideal in $\mathbb{C}[x, y]$ if and only if $yI_{n-k} \subset I_n \subset I_{n-1} \subset \ldots \subset I_{n-k}$.*

Proof Clearly, multiplication by x preserves the space $J(I_n, \ldots, I_{n-k})$, so it is an ideal if and only if it is preserved by the multiplication by y. For $0 \le j < k$ one has

$$y \cdot y^j \sigma^* I_{n-j} = y^{j+1} \sigma^* I_{n-j}$$

which is contained in $y^{j+1}\sigma^* I_{n-j-1}$ if and only if $I_{n-j} \subset I_{n-j-1}$. Furthermore,

$$y \cdot y^k \sigma^* I_{n-k} = y^{k+1} \sigma^* I_{n-k} = \sigma^*(yI_{n-k}),$$

which is contained in $\sigma^* I_n$ if and only if yI_{n-k} is contained in I_n. □

Lemma 3.7 *An ideal $J \subset \mathbb{C}[x, y]$ is invariant under the action of Γ if and only if $J = J(I_n, \ldots, I_{n-k})$ for some ideals $I_n \subset \cdots \subset I_{n-k}$ with $yI_{n-k} \subset I_n$. In this case the ideals I_{n-j} are uniquely determined by J.*

Proof Clearly, $\sigma^*\mathbb{C}[x, y] = \mathbb{C}[x, y^{k+1}] \subset \mathbb{C}[x, y]$ is invariant under the action of Γ, so $J(I_n, \ldots, I_{n-k})$ is also invariant. Conversely, let J be a Γ-invariant ideal in $\mathbb{C}[x, y]$, we can decompose it according to the action of Γ:

$$J = \oplus_{s=0}^{k} J^{(s)},\ \zeta(f) = \zeta^s f \text{ for } f \in J^{(s)}.$$

Since $y^{k+1} J^{(s)} \subset J^{(s)}$, we can write $J^{(s)} = y^s \sigma^*(I_{n-s})$ for some ideal I_{n-s}. By Lemma 3.6, $I_{n-s} \subset I_{n-s-1}$ and $yI_{n-k} \subset I_n$. □

Proof of Theorem 3.5 By Lemma 3.7, the space $\mathrm{PFH}_{n,n-k}$ can be identified with a subset of the fixed point set of the action of a finite group Γ on the Hilbert scheme $\mathrm{Hilb}^n(\mathbb{C}^2)$. The codimensions of I_{n-s} are locally constant functions on the fixed point set. Therefore $\mathrm{PFH}_{n,n-k}$ can be identified with a union of several connected components of the fixed point set. Since $\mathrm{Hilb}^n(\mathbb{C}^2)$ is smooth, the fixed point set is also smooth. □

3.2 Torus action

The group $T = \mathbb{C}^* \times \mathbb{C}^*$ acts on $\mathbb{C}^2$ by scaling the coordinates: $(x, y) \to (q^{-1}x, t^{-1}y)$. This action can be lifted to the action on the Hilbert schemes Hilb^n and the spaces $\mathrm{PFH}_{n,n-k}$. The fixed points of this action on Hilb^n correspond to monomial ideals I_λ and are labeled by Young diagrams λ with $|\lambda| = n$. It is convenient to encode a single cell $\square$ by its monomial $\chi(\square) = q^c t^r$, where c resp. r is the column resp. row index of $\square$. It is well known (e.g. Lemma 5.4.5 in [12], see also [21]) that the equivariant character of the cotangent space at I_λ is given by

$$\begin{aligned} \mathrm{ch}\,\Omega_{I_\lambda}\,\mathrm{Hilb}^n &= \sum_{\square\in\lambda}(q^{a(\square)+1}t^{-l(\square)} + q^{-a(\square)}t^{l(\square)+1}) \\ &= qtB_\mu + B_\mu^* - (q-1)(t-1)B_\mu B_\mu^*, \end{aligned} \tag{3.2}$$

where $a(\square)$ and $l(\square)$ denote the lengths of the arm and the leg of $\square$ in λ, $B_\mu = \sum_{\square\in\mu}\chi(\square)$ and $*$ in B_μ^* denotes the substitution $q \to q^{-1}, t \to t^{-1}$.

The fixed points of $\mathrm{PFH}_{n,n-k}$ are labeled by sequences of monomial ideals $I_n \subset \cdots \subset I_{n-k}$ corresponding to Young diagrams $\lambda^{(n)} \supset \cdots \supset \lambda^{(n-k)}$. The condition $yI_{n-k} \subset I_n$ can be translated to $\lambda^{(i)}$ as follows: $\lambda^{(n)}\backslash\lambda^{(n-k)}$ is a (possibly disconnected) **horizontal strip**, that is, it contains at most one box in each column. Another useful reformulation of this condition is

$$\lambda_i^{(n-k)} \ge \lambda_{i+1}^{(n)}, \text{ where } \lambda^{(n-j)} = (\lambda_1^{(n-j)} \ge \lambda_2^{(n-j)} \ge \ldots). \tag{3.3}$$

Note that the difference $\lambda^{(n-j)}\backslash\lambda^{(n-j-1)}$ consists of a single box. Instead of keeping track of the sequence of partitions we prefer to remember only the first one, which we denote by $\lambda = \lambda^{(n)}$, and the successive differences $\square_j = \lambda^{(n-j+1)}\backslash\lambda^{(n-j)}$ ($j = 1, \ldots, k$). When drawing a picture we will display λ as a Young diagram, together with labeling of some of its cells by numbers from 1 to k where we put j in $\square_j$. Alternatively, we will form a vector $w = (w_1, \ldots, w_k)$ where $w_j = \chi(\square_j)$. A fixed point in $\mathrm{PFH}_{n,n-k}$ will be denoted by $I_{\lambda,w}$ when we specify a pair of a partition λ and a vector w, or by $I_{\lambda^{(\bullet)}}$ when we specify a decreasing sequence of partitions $\lambda^{(\bullet)}$.

Another way of encoding sequences of partitions $\lambda^{(n-j)}$ comes from the proof of Theorem 3.5. If all I_{n-j} are monomial ideals, so is $J(I_n, \ldots, I_{n-k})$. The corresponding Young diagram μ has rows:

$$\mu = \left(\lambda_1^{(n)}, \ldots, \lambda_1^{(n-k)}, \lambda_2^{(n)}, \ldots, \lambda_2^{(n-k)}, \lambda_3^{(n)}, \ldots\right),$$

which decrease by (3.3). Note that

$$B_\mu = B_{\lambda^{(n)}}(q, t^{k+1}) + tB_{\lambda^{(n-1)}}(q, t^{k+1}) + \cdots + t^k B_{\lambda^{(n-k)}}(q, t^{k+1}).$$

To calculate the character of $\Omega_{\lambda_\bullet}\,\mathrm{PFH}_{n,n-k}$ we need to extract the terms in $\mathrm{ch}\,\Omega_{I_\mu}\,\mathrm{Hilb}$ whose t-degree is divisible by $k+1$, and then replace each term $q^a t^{b(k+1)}$ by $q^a t^b$. Performing this with (3.2) we obtain:

$$qtB_{\lambda^{(n-k)}} + B^*_{\lambda^{(n)}} + (q-1)\left(\sum_{i=0}^{k} B_{\lambda^{(n-i)}}B^*_{\lambda^{(n-i)}} - tB_{\lambda^{(n-k)}}B^*_{\lambda^{(n)}} - \sum_{i=1}^{k} B_{\lambda^{(n-i+1)}}B^*_{\lambda^{(n-i)}}\right),$$

which can be rewritten as

$$qtB_{\lambda^{(n-k)}} + B^*_{\lambda^{(n)}} + (q-1)\left((B_{\lambda^{(n)}} - tB_{\lambda^{(n-k)}})B^*_{\lambda^{(n)}} - \sum_{i=1}^{k} w_i B^*_{\lambda^{(n-i)}}\right),$$

so we obtain

$$\begin{aligned}\operatorname{ch}\Omega_{\lambda^{(\bullet)}}\operatorname{PFH}_{n,n-k} &= qtB_{\lambda^{(n-k)}} + B^*_{\lambda^{(n)}} - (t-1)(q-1)B_{\lambda^{(n-k)}}B^*_{\lambda^{(n)}}\\ &\quad + (q-1)\sum_{k\geq i\geq j\geq 1} w_i w_j^{-1}.\end{aligned} \tag{3.4}$$

By using (3.4) and (3.2), one can check the following:

Proposition 3.8 *Let $a(\square, j)$ denote the arm length of $\square$ in $\lambda^{(n-k+j)}$, and let $l(\square)$ denote the leg length in $\lambda^{(n)}$. The equivariant character of the tangent space to* $\operatorname{PFH}_{n,n-k}$ *at a point $\lambda_\bullet = (\lambda^{(n)} \supset \cdots \supset \lambda^{(n-k)})$ equals*

$$\operatorname{ch} T_{\lambda_\bullet}(\operatorname{PFH}_{n,n-k}) = kq + \sum_{\square\in\lambda^{(n-k)}} \theta(\square)$$

where

$$\theta(\square) = q^{a(\square,0)+1}t^{-l(\square)} + q^{-a(\square,k)}t^{l(\square)+1}$$

if there are no boxes in $\lambda^{(n)}\backslash\lambda^{(n-k)}$ above $\square$, and

$$\theta(\square) = q^{a(\square,i)+1}t^{-l(\square)-1} + q^{-a(\square,i-1)}t^{l(\square)+1}$$

if there is a box labeled by i above $\square$.

4 Geometric operators

4.1 Equvariant K-theory

We recall the basic constructions in equivariant K-theory, referring the reader to Chriss and Ginzburg, as well as Okounkov's lectures [2,25]. If X is a complex algebraic variety with an action of a complex torus T, we have the equivariant K-theory of coherent sheaves on X, denoted $K_T(X)$. If $f : X \to Y$ is proper and equivariant, recall that we have a pushforward map $f_* : K_T(X) \to K_T(Y)$, determined by

$$f_*([\mathcal{F}]) = \sum_{i\geq 0}(-1)^i[R^i f_*(\mathcal{F})].$$

If X and Y are smooth, then the map $K_T^\circ(Y) \to K_T(Y)$ is an isomorphism, where $K_T^\circ(Y)$ is the Grothendieck group of locally free sheaves. In this case, we also have the pullback map $f^* : K_T(Y) \to K_T(X)$ given by

$$f^*([\mathcal{F}]) = \sum_i (-1)^i [\mathrm{Tor}_i^{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{F})],$$

even when f is not proper.

We denote by $\bar{K}(X)$ the localized K-theory

$$\bar{K}(X) = K(X) \otimes_R F$$

where $R = K_T(pt)$ is the ring of Laurent polynomials in the torus variables, and F is the field of fractions of R. If X is smooth with isolated fixed points, then the localization theorem [2,29] says that the pullback $i^* : K_T(X) \to K_T(X^T)$ becomes an isomorphism after localization,

$$\bar{K}_T(X) \cong \bigoplus_{p \in X^T} K_T(p) \otimes_R F.$$

Moreover, the pushforward and pullback maps may be uniquely extended to F-linear maps which are uniquely determined due to the injectivity of i^*.

These linear maps are given explicitly as follows: for a fixed point $x \in X$ we denote by $[x] = i_*(1) = [\mathcal{O}_x] \in K_T(X)$ where $i : x \hookrightarrow X$ is the inclusion. We denote by $[x]'$ the dual class

$$[x]' = \frac{[x]}{\Lambda^* \Omega_x} \in \bar{K}_T(X),$$

where

$$\Lambda^* \Omega_x = \sum_i (-1)^i \Lambda^i \Omega_x.$$

Then the extension of the pushforward and pullback are given by

$$f_*[x] = [f(x)], \quad f^*[y]' = \sum_{x \in X^T : f(x) = y} [x]'. \tag{4.1}$$

Since our spaces V_k are vector spaces over $\mathbb{Q}(q,t)$, we will define our spaces as localized K-theory, and our proofs will be based on (4.1). However, it is an important to note for future applications to categorification that in our new description, the algebra generators of $\mathbb{A}_{q,t}$ act on on actual, nonlocalized K-theory. We derive the relations in this algebra in localized K-theory, proving that these relations are satisfied up to torsion elements, i.e. the kernel of the localization map i^*, which we will study in future papers. We expect that there is an integral form of $\mathbb{A}_{q,t}$ and that the corresponding relations hold in non-localized K-theory. For instance, a similar result for the elliptic Hall algebra was recently proved by Neguț in [24].

For the rest of the paper, we will drop the symbol T from K_T, and simply write $K(X)$. The torus will always be $T = \mathbb{C}^* \times \mathbb{C}^*$, with coordinates (q, t).

4.2 The affine Hecke action

For $1 \leq m \leq k-1$ consider the space $\mathrm{PFH}^{(m)}_{n,n-k}$ consisting of partial flags $I_n \subset \cdots \subset I_{n-m+1} \subset I_{n-m-1} \subset \ldots I_{n-k}$ with the same condition $yI_{n-k} \subset I_n$. In complete parallel with Theorem 3.5, one can prove that this space is smooth. There is a natural projection $\pi : \mathrm{PFH}_{n,n-k} \to \mathrm{PFH}^{(m)}_{n,n-k}$, which is projective. For a fixed point $I_{\lambda^{(\bullet)}} \in \mathrm{PFH}_{n,n-k}$ we have that $\pi(I_{\lambda^{(\bullet)}}) = I_{\lambda'^{(\bullet)}}$ where the sequence of partitions $\lambda'^{(\bullet)}$ is obtained from $\lambda^{(\bullet)}$ by removing $\lambda^{(n-m)}$. There is at most one other fixed point that goes to $I_{\lambda'^{(\bullet)}}$, corresponding to a sequence which we denote by $s_m(\lambda^{(\bullet)})$. If $I_{\lambda^{(\bullet)}}$ is specified as $I_{\lambda,w}$ then $I_{s_m(\lambda^{(\bullet)})} = I_{\lambda,s_m(w)}$, where s_m swaps w_m and w_{m+1}. A formula similar to (3.4) can be proved for $I_{\lambda'^{(\bullet)}}$, we have

$$\begin{aligned}\mathrm{ch}\,\Omega_{\lambda'^{(\bullet)}}\,\mathrm{PFH}^{(m)}_{n,n-k} &= qtB_{\lambda^{(n-k)}} + B^*_{\lambda^{(n)}} - (t-1)(q-1)B_{\lambda^{(n-k)}}B^*_{\lambda^{(n)}}\\ &+(q-1)\sum_{k-1\geq i\geq j\geq 1} w'_i w'^*_j,\end{aligned}$$

where

$$w'_i = \begin{cases} w_i & (i < m),\\ w_m + w_{m+1} & (i = m),\\ w_{i+1} & (i > m).\end{cases}$$

Therefore we have

$$\begin{aligned}\mathrm{ch}\,\Omega_{\lambda'^{(\bullet)}} - \mathrm{ch}\,\Omega_{\lambda^{(\bullet)}} &= (q-1)w_m w_{m+1}^{-1},\\ \mathrm{ch}\,\Omega_{\lambda'^{(\bullet)}} - \mathrm{ch}\,\Omega_{s(\lambda^{(\bullet)})} &= (q-1)w_{m+1} w_m^{-1}.\end{aligned}$$

We obtain

$$\begin{aligned}\pi^*\pi_* I_{\lambda,w} &= \Lambda^*\left((q-1)w_m w_{m+1}^{-1}\right) I_{\lambda,w} + \Lambda^*\left((q-1)w_{m+1}w_m^{-1}\right) I_{\lambda,s_m(w)}\\ &= \frac{1-qw_m w_{m+1}^{-1}}{1-w_m w_{m+1}^{-1}} I_{\lambda,w} + \frac{1-qw_{m+1}w_m^{-1}}{1-w_{m+1}w_m^{-1}} I_{\lambda,s_m(w)}.\end{aligned}$$

Note that the second summand should be omitted if $I_{\lambda^{(\bullet)}}$ is the only fixed point that goes to $I_{\lambda'^{(\bullet)}}$. This happens precisely when $\lambda^{(n-m+1)}\backslash\lambda^{(n-m-1)}$ is a pair of horizontally adjacent cells, i.e. $w_m = qw_{m+1}$. In such situation the factor in front of $I_{\lambda,s_m(w)}$ vanishes anyway, so the formula still holds formally even though $I_{\lambda,s_m(w)}$ does not correspond to a point in $\mathrm{PFH}_{n,n-k}$.

We get the following lemma:

Lemma 4.1 *Let $T_m = \pi^*\pi_* - q$. Then*

$$T_m(I_{\lambda,w}) = \frac{(q-1)w_{m+1}}{w_m - w_{m+1}} I_{\lambda,w} + \frac{w_m - qw_{m+1}}{w_m - w_{m+1}} I_{\lambda,s_m(w)}. \tag{4.2}$$

The operators z_i are given by multiplication by line bundles $\mathcal{L}_j = I_{n-j}/I_{n-j+1}$. Note that we have

$$\mathcal{L}_j I_{\lambda,w} = w_j I_{\lambda,w}. \tag{4.3}$$

4.3 Creation and annihilation

There are natural projection maps forgetting the first and the last ideal respectively

$$f : \mathrm{PFH}_{n+1,n-k} \to \mathrm{PFH}_{n,n-k}, \;\; g : \mathrm{PFH}_{n,n-k} \to \mathrm{PFH}_{n,n-k+1}\,.$$

Here g is projective. We will denote

$$d_- = g_*, \quad d_+ = q^k(q-1)f^*.$$

Note that d_+ increases k and d_- decreases k.

Lemma 4.2 *We have*

$$d_- I_{\lambda,wx} = I_{\lambda,w},$$
$$d_+ I_{\lambda,w} = -q^k \sum_x x d_{\lambda+x,\lambda} \prod_{i=1}^{k} \frac{x - tw_i}{x - qtw_i} I_{\lambda+x,xw},$$

where $xw = (x, w_1, w_2, \ldots, w_k)$, and $d_{\lambda,\mu}$ is the Pieri coefficient

$$d_{\lambda,\mu}(q,t) = \prod_{s\in R_{\lambda,\mu}} \frac{q^{a_\mu(s)} - t^{l_\mu(s)+1}}{q^{a_\lambda(s)} - t^{l_\lambda(s)+1}} \prod_{s\in C_{\lambda,\mu}} \frac{q^{a_\mu(s)+1} - t^{l_\mu(s)}}{q^{a_\lambda(s)+1} - t^{l_\lambda(s)}}$$

for multiplication by e_1 in the modified Macdonald basis e.g. from [9] *formula 3.1, which satisfies*

$$e_1 \tilde{H}_\mu = \sum_\lambda d_{\lambda,\mu} \tilde{H}_\lambda.$$

Here $R_{\lambda,\mu}$ is the set of cells in the row of the unique box in $\lambda\backslash\mu$, and $C_{\lambda,\mu}$ is the set of cells in the column.

Proof The formula for d_- is immediate from the definition. For d_+ we calculate

$$\begin{aligned}&\operatorname{ch}\Omega_{\lambda,w}-\operatorname{ch}\Omega_{\lambda+x,xw}\\&\quad=-x^{-1}+(t-1)(q-1)B_{\lambda^{(n-k)}}x^{-1}-(q-1)x^{-1}\sum_{i=1}^{k}w_i-(q-1)\\&\quad=-x^{-1}+(t-1)(q-1)B_{\lambda}x^{-1}-(q-1)-t(q-1)x^{-1}\sum_{i=1}^{k}w_i.\end{aligned}$$

Below we will show

$$d_{\lambda+x,\lambda}=x^{-1}\Lambda^*(-x^{-1}+(t-1)(q-1)B_\lambda x^{-1}+1). \tag{4.4}$$

Assuming (4.4) we have

$$f^*I_{\lambda,w}=\sum_x x d_{\lambda+x,\lambda}\frac{1}{1-q}\prod_{i=1}^{k}\frac{x-tw_i}{x-qtw_i}I_{\lambda+x,xw}.$$

and we are done.

To prove (4.4) we will use the following summation formula for the Pieri coefficients, see e.g. Theorem 2.4 b) in [9]:

$$\sum_x d_{\lambda+x,\lambda}x^{i+1}=(-1)^i e_i[-1+(q-1)(t-1)B_\lambda]\quad (i\geq 0).$$

Let u be a formal variable. Multiplying both sides by u^k and summing over $k\geq 0$ produces the following identity of rational functions:

$$\sum_x d_{\lambda+x,\lambda}\frac{x}{1-ux}=\Lambda^*((-1+(q-1)(t-1)B_\lambda)u). \tag{4.5}$$

Note that the left hand side has simple pole at $u=x^{-1}$ and

$$xd_{\lambda+x,\lambda}=\left((1-ux)\Lambda^*((-1+(q-1)(t-1)B_\lambda)u)\right)\big|_{u=x^{-1}}.$$

Moving $1-ux$ inside Λ^* we obtain

$$xd_{\lambda+x,\lambda}=\Lambda^*((-1+(q-1)(t-1)B_\lambda)u+ux)\big|_{u=x^{-1}}.$$

Now we can substitute $u=x^{-1}$ before applying Λ^* and arrive at (4.4). □

Example 4.3 Let $k = 0$. We have $\mathrm{PFH}_{n,n} = \mathrm{Hilb}_n$. Let us identify the fixed point corresponding to a partition λ with the symmetric function

$$I_\lambda = \frac{\tilde{H}_\lambda}{\tilde{H}_\lambda[-1]} = (-1)^{|\lambda|} q^{-n(\lambda')} t^{-n(\lambda)} \tilde{H}_\lambda = \tilde{H}_\lambda \prod_{\square \in \lambda} (-\chi(\square)^{-1}),$$

where $\tilde{H}_\lambda$ is the modified Macdonald polynomial. Then we obtain

$$d_+ \tilde{H}_\lambda = -\tilde{H}_\lambda[-1] \sum_{w_1} w_1 d_{\lambda+w_1,\lambda} I_{\lambda+w_1,w_1},$$

and using $\tilde{H}_{\lambda+w_1}[-1] = -w_1 \tilde{H}_\lambda[-1]$

$$d_- d_+ \tilde{H}_\lambda = \sum_{w_1} d_{\lambda+w_1,\lambda} \tilde{H}_{\lambda+w_1},$$

therefore $d_- d_+$ acts like the operator of multiplication by e_1, which matches the action of $\mathbb{A}_{q,t}$ on $V_\bullet$.

5 Verification of relations

Let

$$U_k = \bigoplus_{n \geq k} \bar{K}(\mathrm{PFH}_{n,n-k}), \quad U_\bullet = \bigoplus_{k \geq 0} U_k.$$

In this section, we will prove the following theorem:

Theorem 5.1 *The geometric operators written as T_i, z_i, d_+ and d_- define a representation of the algebra $\mathbb{B}_{q,t}$ on $U_\bullet$, and therefore a representation of $\mathbb{A}_{q,t}$ via the map $\beta : \mathbb{A}_{q,t} \to \mathbb{B}_{q,t}$.*

We split the relations into several groups and prove them in the subsections below. We will denote $H_{\lambda,w} = (-1)^{|\lambda|} q^{n(\lambda')} t^{n(\lambda)} I_{\lambda,w}$, so that the $H_{\lambda,w}$ form a basis of $U_\bullet$. Note that the formulas for the action of T_m, $\mathcal{L}_j$, d_- in the H-basis are the same as for I-basis.

5.1 z_i, T_i

The following relations are easy to verify

Proposition 5.2 *The operators $z_i := \mathcal{L}_i$ and T_i satisfy relations of the (conjugate) affine Hecke algebra:*

$$(T_i-1)(T_i+q)=0,\quad T_iT_{i+1}T_i=T_{i+1}T_iT_{i+1},\quad T_iT_j=T_jT_i\quad (|i-j|>1),$$
$$T_iz_iT_i=qz_{i+1}\ (1\le i\le k-1),$$
$$z_iT_j=T_jz_i\ (i\notin\{j,j+1\}),\ z_iz_j=z_jz_i\ (1\le i,j\le k),$$

In fact, the construction of z_i and T_i is very similar to the classical construction of finite-dimensional representations of the affine Hecke algebra using "multisegments" (see e.g. [30]). The operators T_i and z_i do not change the biggest ideal I_n and the smallest ideal I_{n-k}. In terms of the fixed point basis, this means that we can fix two partitions $\lambda_{n-k}\subset\lambda_n$ such that the skew shape $\lambda_n\backslash\lambda_{n-k}$ consists of several horizontal strips. The choice of $\lambda_{n-k+1},\ldots,\lambda_{n-1}$ is equivalent to the choice of a standard tableau of this skew shape. Then (4.2) and (4.3) agree with the action of the affine Hecke algebra on such standard tableaux [26,30].

5.2 d_-, d_+, T_i

From Lemma 4.2 we obtain

$$d_+H_{\lambda,w}=q^k\sum_x d_{\lambda+x,\lambda}\prod_{i=1}^{k}\frac{x-tw_i}{x-qtw_i}H_{\lambda+x,xw},$$

for $wy=(w_1,\ldots,w_{k-1},y)$

$$d_-d_+H_{\lambda,wy}=q^k\sum_x d_{\lambda+x,\lambda}\frac{x-ty}{x-qty}\prod_{i=1}^{k-1}\frac{x-tw_i}{x-qtw_i}H_{\lambda+x,xw},$$
$$d_+d_-H_{\lambda,wy}=q^{k-1}\sum_x d_{\lambda+x,\lambda}\prod_{i=1}^{k-1}\frac{x-tw_i}{x-qtw_i}H_{\lambda+x,xw},$$
$$\frac{d_+d_--d_-d_+}{q-1}H_{\lambda,wy}=-q^{k-1}\sum_x d_{\lambda+x,\lambda}\frac{x}{x-qty}\prod_{i=1}^{k-1}\frac{x-tw_i}{x-qtw_i}H_{\lambda+x,xw},\tag{5.1}$$
$$\frac{qd_+d_--d_-d_+}{q-1}H_{\lambda,wy}=-q^kt\sum_x d_{\lambda+x,\lambda}\frac{y}{x-qty}\prod_{i=1}^{k-1}\frac{x-tw_i}{x-qtw_i}H_{\lambda+x,xw}.\tag{5.2}$$

We have

Proposition 5.3 *The operators d_+, d_-, T_i extend to a representation of $\mathbb{A}_q$ on $U_\bullet$.*

Proof The Hecke algebra relations for T_i were verified above. The relations $T_id_-=d_-T_i$, $d_+T_i=T_{i+1}d_+$ are straightforward. Then we need to check that

$$d_-^2T_k=d_-^2,\quad T_1d_+^2=d_+^2.$$

The first one is straightforward. To establish the second one write

$$d_+^2 H_{\lambda,w} = q^{2k+1} \sum_{x,y} d_{\lambda+x,\lambda} d_{\lambda+x+y,\lambda+x} \frac{y-tx}{y-qtx} \prod_{i=1}^{k-1} \frac{(x-tw_i)(y-tw_i)}{(x-qtw_i)(y-qtw_i)} H_{\lambda+x+y,yxw}.$$

Note that there are no terms with $y = tx$. All the terms with $y = qx$ are invariant under T_1. Suppose $y \neq qx$, $y \neq tx$, in other words the cells x, y are non-adjacent. Using (4.4) we have

$$d_{\lambda+x,\lambda} d_{\lambda+x+y,\lambda+x} = (xy)^{-1} \Lambda^* \left(((q-1)(t-1)B_\lambda - 1)(x^{-1}+y^{-1}) + (t-1)(q-1)xy^{-1} + 2 \right),$$

so

$$d_{\lambda+x,\lambda} d_{\lambda+x+y,\lambda+x} \frac{y-tx}{y-qtx} \prod_{i=1}^{k-1} \frac{(x-tw_i)(y-tw_i)}{(x-qtw_i)(y-qtw_i)} = C_{\lambda,w}(x,y) \frac{y-x}{y-qx}, \tag{5.3}$$

where the function $C_{\lambda,w}(x, y)$ is symmetric in x, y. So we have

$$(T_1 - 1) d_+^2 H_{\lambda,w} = \sum_{x,y \text{ non adjacent}} C_{\lambda,w}(x,y)(H_{\lambda+x+y,yxw} - H_{\lambda+x+y,xyw}) = 0.$$

Denote by φ the operator $\varphi = \frac{d_+d_- - d_-d_+}{q-1}$,

$$\varphi H_{\lambda,wy} = -q^{k-1} \sum_x d_{\lambda+x,\lambda} \frac{x}{x-qty} \prod_{i=1}^{k-1} \frac{x-tw_i}{x-qtw_i} H_{\lambda+x,xw}.$$

By Theorem 2.4 it is enough to show that the following identities hold:

$$q\varphi d_- = d_-\varphi T_{k-1}, \quad T_1\varphi d_+ = qd_+\varphi.$$

The first one is easier. Let

$$C_u = d_{\lambda+u,\lambda} \prod_{i=1}^{k-2} \frac{u-tw_i}{u-qtw_i}.$$

Then we have

$$
\begin{aligned}
&q\varphi d_- H_{\lambda,wxy} \\
&= -q^{k-1}\sum_u \frac{u}{u-qtx} C_u H_{\lambda+u,uw}, \quad d_-\varphi T_{k-1} \\
&= -q^{k-1}\sum_u \left(\frac{(q-1)y}{x-y}\frac{u(u-tx)}{(u-qty)(u-qtx)} + \frac{x-qy}{x-y}\frac{u(u-ty)}{(u-qtx)(u-qty)}\right) \\
&\quad C_u H_{\lambda+u,uw}.
\end{aligned}
$$

The rational function in parentheses equals $\frac{u}{u-qtx}$, so the identity holds. Finally we compare

$$
\begin{aligned}
A = q d_+ \varphi H_{\lambda,wu} &= -q^k \sum_{x,y} d_{\lambda+x,\lambda} d_{\lambda+x+y,\lambda+x} \frac{x(y-tx)}{(x-qtu)(y-qtx)} \\
&\times \prod_{i=1}^{k-1} \frac{(x-tw_i)(y-tw_i)}{(x-qtw_i)(y-qtw_i)} H_{\lambda+x+y,yxw}
\end{aligned}
$$

and

$$
\begin{aligned}
B = T_1\varphi d_+ H_{\lambda,wu} &= -q^k T_1 \sum_{x,y} d_{\lambda+x,\lambda} d_{\lambda+x+y,\lambda+x} \frac{y(y-tx)(x-tu)}{(y-qtu)(y-qtx)(x-qtu)} \\
&\times \prod_{i=1}^{k-1} \frac{(x-tw_i)(y-tw_i)}{(x-qtw_i)(y-qtw_i)} H_{\lambda+x+y,yxw}.
\end{aligned}
$$

Similar to the computations with d_+^2 we analyze two cases. If $y = qx$, i.e. x and y are adjacent, we have $T_1 H_{\lambda+x+y,yxw} = H_{\lambda+x+y,yxw}$ and coefficients of these terms coincide. Suppose x and y are not adjacent. Using (5.3) we write the coefficient of $H_{\lambda+x+y,yxw}$ in A as

$$
\frac{x(y-x)}{(x-qtu)(y-qx)} C_{\lambda,w}(x,y).
$$

Using symmetry of $C_{\lambda,w}(x,y)$, we see that the corresponding coefficient in B is

$$
\begin{aligned}
&\left(\frac{(q-1)xy(y-x)(x-tu)}{(y-x)(y-qtu)(y-qx)(x-qtu)} + \frac{(x-qy)x(x-y)(y-tu)}{(x-y)(x-qtu)(x-qy)(y-qtu)}\right) \\
&\times C_{\lambda,w}(x,y).
\end{aligned}
$$

Comparing the rational functions we see that the coefficients coincide. ◻

5.3 d_-, d_+, z_i

It remains to check the following relations:

$$z_i d_- = d_i z_i, \quad d_+ z_i = z_{i+1} d_+,$$
$$z_1(qd_+d_- - d_-d_+) = qt(d_+d_- - d_-d_+)z_k.$$

The proof of the first two is straightforward, and the last one immediately follows from (5.1) and (5.2). The proof of Theorem 5.1 is complete.

5.4 Serre duality

We have two additional involutions on $K(\mathrm{PFH}_{n,n-k})$ and $\bar{K}(\mathrm{PFH}_{n,n-k})$, given by Serre duality and dualization of vector bundles, respectively:

$$\mathrm{SD}\left(\sum_{\lambda,w} a_{\lambda,w}(q,t) I_{\lambda,w}\right) = \sum_{\lambda,w} a_{\lambda,w}(q^{-1},t^{-1}) I_{\lambda,w},$$
$$\left(\sum_{\lambda,w} a_{\lambda,w}(q,t) I'_{\lambda,w}\right)^* = \sum_{\lambda,w} a_{\lambda,w}(q^{-1},t^{-1}) I'_{\lambda,w}.$$

We have another involution $\mathcal{N} = \mathcal{L}\,\mathrm{SD}\,\mathcal{L}^{-1}$, where $\mathcal{L}$ is the pullback of the determinant of the tautlogical bundle from Hilb_n, satisfying $H_{\mu,w} = (-1)^{|\mu|}\mathcal{L} I_{\mu,w}$.

$$\mathcal{N}\left(\sum_{\lambda,w} a_{\lambda,w}(q,t) H_{\lambda,w}\right) = \sum_{\lambda,w} a_{\lambda,w}(q^{-1},t^{-1}) H_{\lambda,w}. \tag{5.4}$$

This operator has the commutation relations agreeing with (2.11), justifying calling it $\mathcal{N}$:

Proposition 5.4 *One has*

$$\mathcal{N} d_- \mathcal{N} = d_-, \; \mathcal{N} T_i \mathcal{N} = T_i^{-1}, \; \mathcal{N} d_+ \mathcal{N} = q^{-k} z_1 d_+ = \beta(d_+^*).$$

Proof The first equation is clear from Lemma 4.2. For the second, observe that the Hecke relations imply

$$T_m^{-1} = q^{-1} T_m + q^{-1}(q-1).$$

On the other hand, by (4.2) one has

$$\mathcal{N} T_m \mathcal{N}(H_{\lambda,w}) = \frac{(q^{-1}-1)w_{m+1}^{-1}}{w_m^{-1} - w_{m+1}^{-1}} H_{\lambda,w} + \frac{w_m^{-1} - q^{-1} w_{m+1}^{-1}}{w_m^{-1} - w_{m+1}^{-1}} H_{\lambda,s_m(w)}$$

$$= q^{-1}\left[\frac{(q-1)w_m}{w_m - w_{m+1}} H_{\lambda,w} + \frac{w_m - q w_{m+1}}{w_m - w_{m+1}} H_{\lambda,s_m(w)}\right]$$
$$= q^{-1}\left[(q-1) + T_m\right].$$

Finally,

$$\mathcal{N} d_+ \mathcal{N} = q^{-k} \sum_x d_{\lambda+x,\lambda}(q^{-1}, t^{-1}) \prod_{i=1}^{k} \frac{x^{-1} - t^{-1} w_i^{-1}}{x^{-1} - q^{-1} t^{-1} w_i^{-1}} H_{\lambda+x,xw}$$
$$= \sum_x x d_{\lambda+x,\lambda} \prod_{i=1}^{k} \frac{x - t w_i}{x - q t w_i} H_{\lambda+x,xw} = q^{-k} z_1 d_+.$$

Here we used the fact that $d_{\lambda+x,\lambda}(q^{-1}, t^{-1}) = x d_{\lambda+x,\lambda}(q, t)$. □

6 Comparison with the polynomial representation

Theorem 5.1 showed that there is an action of $\mathbb{A}_{q,t}$ on $U_\bullet$, and so in particular an action of the subalgebra $\mathbb{A}_q \subset \mathbb{A}_{q,t}$. It is an immediate consequence of Proposition 2.11 that there is a unique $\mathbb{A}_q$-equivariant sequence of maps $\Phi_k : V_k \to U_k$ sending $1 \in V_0$ to $H_\emptyset \in K(\mathrm{PFH}_{0,0})$. We denote by $\Phi : V_\bullet \to U_\bullet$ the resulting map.

In this section, we will prove:

Theorem 6.1 *The map Φ_k is an isomorphism. Moreover, we have that*

$$\Phi_0(H_\mu) = \tilde{H}_\mu,$$

where $\tilde{H}_\mu$ is the modified Macdonald polynomial, and that $\Phi_k \mathcal{N} = \mathcal{N} \Phi_k$, where the two operators denoted $\mathcal{N}$ are the involutions in Eqs. (2.12) and (5.4).

We now start proving this theorem, beginning with the statement that Φ_k is an isomorphism.

Let $V_{n,k}$ denote the degree $(n-k)$ part of V_k. Let $U_{n,k} = \bar{K}(\mathrm{PFH}_{n,n-k})$. It is clear that the bi-degrees of T_i, d_-, d_+ are $(0,0)$, $(0,-1)$, $(1,1)$ respectively both in $V_{n,k}$ and $U_{n,k}$, so that Φ preserves the bi-grading. We begin by showing that $V_{n,k}$ and $U_{n,k}$ have the same dimension.

Define two collections of sets by

$$A(n,k) = \left\{(\mu, a) \in \mathcal{P} \times \mathbb{Z}^k_{\geq 0} : |\mu| + |a| = n - k\right\},$$
$$M(n,k) = \left\{\lambda^{(n)} \supset \cdots \supset \lambda^{(n-k)} : \lambda^{(n-i)} \in \mathcal{P}_{n-i},\ \lambda^{(n)} \backslash \lambda^{(n-k)} \text{ is a horizontal strip}\right\}.$$

Then the elements of $M(n,k)$ are just the indices $\lambda^{(\bullet)}$ of the basis $H_{\lambda^{(\bullet)}}$ of $U_{n,k}$ and elements of $A(n,k)$ index elements

$$v_{\mu,a} = d_-^l y_1^{a_k} \ldots y_k^{a_1} y_{k+1}^{\mu_l} \ldots y_{k+l}^{\mu_1}, \tag{6.1}$$

which make up a basis of $V_{n,k}$, because the Hall-Littlewood polynomials make up a basis of symmetric functions. Define a function $A(n,k) \to M(n,k)$ by the following procedure: given μ, a we set

$$\lambda^{(n-i)} = \mathrm{sort}(\mu_1, \mu_2, \ldots, \mu_{l(\mu)}, a_1, \ldots, a_i, a_{i+1}+1, \ldots, a_k+1)' \quad (0 \le i \le k),$$

where sort transforms a sequence into a partition by sorting the entries and throwing away zeros, and $'$ takes the conjugate partition. For instance, we would have

$$[3,1],[1,0,1,2,3] \mapsto [[7,5,3,1],[7,4,3,1],[6,4,3,1],[6,3,3,1],[6,3,2,1],[6,3,2]].$$

It is straightforward to see that this is a bijection, proving that the two spaces have the same dimension.

We will prove our theorem by showing that Φ_k has a triangularity property with respect to a partial order on $A(n,k) \leftrightarrow M(n,k)$ that we now define: Given $(\mu, a) \in A(n,k)$, and some l greater than the length of μ, let

$$\alpha = (\mu; a+1)_l^{rev} = (a_k+1, ..., a_1+1; \mu_l, ..., \mu_1)$$

denote the reversed order of the concatenation of μ and $(a_1+1, ..., a_k+1)$, which always has at least one leading zeros included in the μ terms. For instance, if we took $(\mu, a) = ([2,1]; (1,0,2))$, and chose $l = 4$, we would have

$$\alpha = (\mu; a+1)_4^{rev} = (3,1,2,0,0,1,2).$$

We will describe the procedure for determining how to compare two elements in terms of these vectors.

For any (μ, a), we start by asserting the following moves produce an element that is larger in this order in $A(n,k)$. In our description, the operation "set $\alpha_i = c$ and sort" means to make the desired substitution, then sort the leading "partition terms" if $i \le l$, so as to obtain something that we may regard as an element of $A(n,k)$. In the example above, the operation "set $\alpha_4 = 2$ and sort" would yield $(3,1,2,0,1,2,2)$, corresponding to $\mu = [2,2,1]$, and $a = (1,0,2)$.

(1) If $\alpha_i > \alpha_j$ for $i < j$, set $(\alpha_i, \alpha_j) = (\alpha_j, \alpha_i)$, i.e. switch the labels and sort.
(2) If $\alpha_i < \alpha_j - 1$ for any i, j, set $(\alpha_i, \alpha_j) = (\alpha_j - 1, \alpha_i + 1)$ and sort.

We let $\le_{bru}$ denote the binary relation transitively generated by these moves, which we can see does not depend on l, provided it is large enough. This is in fact a partial order, which can be seen using an alternative description in terms of the Bruhat order on affine permutations for GL_{k+l}. To see this, fix some value of l, and let $\widehat{W} = \mathbb{Z}^{k+l} \ltimes W_0$ denote the affine Weyl group for GL_{k+l}. Now identify compositions α with sorted final l coordinates with elements of $S_l \backslash \widehat{W} / S_{k+l}$, by choosing a representative of minimal length from each coset, of which there is a unique one. Then $\le_{bru}$ is the order induced by the Bruhat order on $\widehat{W}$. Without the sorting condition from the second action of S_l, this also appears in [15]. Notice that for $k = 0$ it becomes the usual dominance order on partitions.

Proposition 6.2 *We have that*

$$\Phi_k(v_{\mu,a}) = \sum_{(\nu,b)\leq_{bru}(\mu,a)} c_{\nu,b}(q,t)H_{\nu,b} \tag{6.2}$$

with $c_{\mu,a}(q,t) \neq 0$.

Proof Given

$$f = \sum_{(a,\mu)} c_{a,\mu}(q,t)H_{\mu,a} \in U_{n,k},$$

let terms(f) denote the set of those $(a,\mu) \in A(n,k)$ such that $c_{\mu,a}(q,t) \neq 0$. Let us write Eq. (6.2) as

$$\mathrm{LT}(\Phi_k(v_{\mu,a})) = (\mu, a),$$

where the statement $\mathrm{LT}(f) = (\mu, a)$ asserts that $(\mu, a) \in$ terms(f), and is greater than all other elements with respect to $\leq_{bru}$. Note that not every f has a leading term because $\leq_{bru}$ is only a partial order.

Let $b = s_i(a)$, the result of switching the labels a_i, a_{i+1}. Then we use the following description of the terms of our operators:

$$\begin{aligned}\mathrm{terms}(T_{k-i}^{\pm 1}(H_{\mu,a})) &= \{(\mu, a), (\mu, b)\}\\ \mathrm{terms}(\varphi(H_{\mu,a})) &= \{(\mu \cup \{a_1+1\} - \{i\}, (a_2, ..., a_k, i))\}.\\ \mathrm{terms}(d_-(H_{\mu,a})) &= \{(\mu \cup \{a_1+1\}, (a_2, ..., a_k))\}.\end{aligned}$$

In the second to last line, $\nu - \{i\}$ means the result of removing one of the occurrences of i, where i ranges over all possible elements that can be removed. We include the case where i is zero, and make the sensible convention that $0 \in \nu$ for any ν, and that $\nu - \{0\} = \nu$.

From these statements, we can check that

$$\begin{aligned}\mathrm{LT}(T_{k-i}^{\pm 1}(H_{\mu,a})) &= \max\left((\mu, a), (\mu, s_i(a))\right),\\ \mathrm{LT}(\varphi(H_{\mu,a})) &= (\mu, (a_2, ..., a_k, a_1+1)),\\ \mathrm{LT}(d_-(H_{\mu,a})) &= (\mu \cup \{a_1+1\}, (a_2, ..., a_k)).\end{aligned} \tag{6.3}$$

It follows from the properties of the Bruhat order on $\widehat{W}$ that if $(\mu, a) \leq_{bru} (\nu, b)$, then

$$\begin{aligned}(\mu, s_i(a)), (\mu, a) &\leq_{bru} (\nu, s_i(b)), \quad \text{if } b \leq_{bru} s_i(b),\\ (\mu, (a_2, ..., a_k, a_1+1)) &\leq_{bru} (\nu, (b_2, ..., b_k, b_1+1)),\\ (\mu \cup \{a_1+1\}, (a_2, ..., a_k)) &\leq_{bru} (\nu \cup \{b_1+1\}, (b_2, ..., b_k)).\end{aligned} \tag{6.4}$$

The second set of equations gives conditions for when $A(f)$ has a leading term depending only on the leading term for f for each operator A, and the first set describes what that leading term is. These two sets of rules will be enough to prove the result.

By the statements about d_- in (6.3) and (6.4), it suffices to prove the proposition in the case when μ is the empty partition. We will prove this by induction on $|a|$. If $m = \max(a)$ is zero, then we are done. Otherwise, let i be the smallest index such that $a_i = m$. Let $g \in U_{n,k}$ be any element with a leading term given by $\mathrm{LT}(g) = (\emptyset, b)$, where b is the composition that agrees with a, except that $b_i = a_{i-1}$. It suffices to show that $\mathrm{LT}(y_{k-i}g) = (\emptyset, a)$, where y_i is the operator on $U_{n,k}$ defined in terms of T_i, T_i^{-1}, φ by Eq. (2.7). Note the reversal of the ordering of a in the definition (6.1) of the basis $v_{\mu,a}$, which is why we use y_{k-i} instead of y_i.

Consider the sequences of elements of $U_{n,k}$ given by

$$g^i = g, \quad g^j = T_{k-j}(g^{j+1}) \text{ for } 1 \le j \le i-1,$$
$$f^k = \varphi(g^1), \quad f^j = T_{k-j}^{-1}(f^{j+1}) \text{ for } i \le j \le k-1.$$

We also define a sequence of compositions by

$$b^j = s_j(b^{j+1}), \quad a^k = \left(b_2^1, \ldots, b_k^1, b_1^1 + 1\right), \quad a^j = s_j(a^{j+1}).$$

For instance, if $a = (2, 0, 3, 1, 3, 0, 3, 0, 1)$, then we would have $i = 3$, and

$$\begin{aligned}
&b^3, b^2, b^1, a^9, a^8, a^7, a^6, a^5, a^4, a^3 = (2, 0, 2, 1, 3, 0, 3, 0, 1), (2, 2, 0, 1, 3, 0, 3, 0, 1),\\
&(2, 2, 0, 1, 3, 0, 3, 0, 1), (2, 0, 1, 3, 0, 3, 0, 1, 3), (2, 0, 1, 3, 0, 3, 0, 3, 1),\\
&(2, 0, 1, 3, 0, 3, 3, 0, 1), (2, 0, 1, 3, 0, 3, 3, 0, 1), (2, 0, 1, 3, 3, 0, 3, 0, 1),\\
&(2, 0, 1, 3, 3, 0, 3, 0, 1), (2, 0, 3, 1, 3, 0, 3, 0, 1).
\end{aligned}$$

By (2.7), we have that $f = f^i$, and we clearly have that $a = a^i$. It therefore suffices to prove the the more general statement that

$$(\emptyset, a^j) = \mathrm{LT}(f^j), \quad (\emptyset, b^j) = \mathrm{LT}(g^j)$$

for all j.

To see this, notice that we have $a^j \le_{bru} a^{j-1}$, and $b^j \le_{bru} b^{j-1}$. The first statement follows simply because $a_i = m$ is the maximum entry, and so the order can only be increased by moving it to the left. The second statement follows because i is the leftmost occurrence of the maximum entry, so $b_i = m - 1$ greater than or equal to every term to its left. Therefore, the condition in the first part of (6.4) is satisfied, and the desired statement follows by induction from the first two parts of equations (6.3) and (6.4). □

To complete the proof of Theorem 6.1, we first see that $\Phi_k \mathcal{N} = \mathcal{N} \Phi_k$ by Proposition 5.4, so it only remains to show that the fixed points map to the modified Macdonald polynomials for $k = 0$. For $k = 0$, it was proved in [4] that $\mathcal{N}$ acts as ∇ composed with conjugation, i.e.

$$\mathcal{N}\left(\sum_{\lambda,w} a_{\lambda,w}(q,t)\tilde{H}_\lambda\right) = \sum_{\lambda,w} a_{\lambda,w}(q^{-1},t^{-1})\tilde{H}_\lambda. \tag{6.5}$$

In [8], it was shown that the ring of symmetric functions are generated by the multiplication operator e_1, and $\nabla e_1 \nabla^{-1}$, or equivalently, $\mathcal{N} e_1 \mathcal{N}$. It therefore suffices to show that $\mathcal{N}$, e_1 have the same representation in each basis. The involution $\mathcal{N}$ fixes both sets of basis by definition. To show that e_1 has the same coefficients, it suffices to notice that $e_1 = d_- d_+$ when restricted to V_0, and recall that the coefficients in Lemma 4.2 are just the coefficients in the Pieri rule for e_1. □

7 Examples

7.1 Simple Nakajima correspondences

An important collection of operators on the K-theory of Hilbert schemes can be defined as follows. Consider *nested Hilbert scheme* $\mathrm{Hilb}^{n,n+1} = \{J \subset I \subset \mathbb{C}[x, y]\}$, where J and I are ideals of codimensions $(n + 1)$ and n, respectively. The variety $\mathrm{Hilb}^{n,n+1}$ is well known to be smooth [5] and carries a natural line bundle $\mathcal{L} := I/J$. It has two projections $f : \mathrm{Hilb}^{n,n+1} \to \mathrm{Hilb}^n$ and $g : \mathrm{Hilb}^{n,n+1} \to \mathrm{Hilb}^{n+1}$ which send a pair $(J \subset I)$ to I and J, respectively. In the constructions of [7,27] a crucial role was played by the operators

$$P_{1,k} : K(\mathrm{Hilb}^n) \to K(\mathrm{Hilb}^{n+1}),\ \ P_{1,k} := g_*(\mathcal{L}^k \otimes f^*(-)).$$

Remark that the quotient I/J in the nested Hilbert scheme is supported at one point, which can be translated to the line $\{y = 0\}$. Thus, $\mathrm{Hilb}^{n,n+1} = \mathrm{PFH}_{n+1,n} \times \mathbb{C}_t$, and $K(\mathrm{Hilb}^{n,n+1}) \subset U_1$. Using the algebra $\mathbb{A}_{q,t}$, we can realize these operators as a composition of three:

$$(q-1)f^* = d_+ : U_0 \to U_1, \quad \mathcal{L}_k = z_1^k : U_1 \to U_1, \quad g_* = d_- : U_1 \to U_0,$$

so

$$P_{1,k} = \frac{1}{(q-1)(1-t)} d_- z_1^k d_+.$$

7.2 Generators of the elliptic Hall algebra

We next describe another proof of the formula from [23] for the generator $P_{m,n}$ of the elliptic Hall algebra (for coprime m and n). It was proved in [19] that

$$P_{m,n} = d_-(z_1^{S_n} y_1 z_1^{S_{n-1}} y_1 \dots z_1^{S_1} y_1) d_+, \;\; S_i = \left\lfloor \frac{mi}{n} \right\rfloor - \left\lfloor \frac{m(i-1)}{n} \right\rfloor.$$

Negut's formula is equivalent to the following proposition, after substituting these values of S_i into (7.1).

Proposition 7.1 *The following identity holds for all S_i:*

$$d_-(z_1^{S_n} y_1 z_1^{S_{n-1}} y_1 \dots z_1^{S_1} y_1) d_+ = (-1)^n \sum_T \prod_{i<j} \omega(w_i/w_j) \frac{w_i^{S_i+1}}{w_i - qtw_{i-1}} H_\lambda, \tag{7.1}$$

where T is a standard tableaux of shape λ and size n, w_i is the q,t-content of the box labeled by i in T, and

$$\omega(x) = \frac{(1-x)(1-qtx)}{(1-qx)(1-tx)}.$$

Proof First, we need an explicit formula for the action of y_1 on U_1. Since there are no T's and $k = 1$, by (5.1) we have

$$y_1(H_{\lambda,y}) = \frac{1}{q-1}[d_+, d_-] H_{\lambda,y} = -\sum_x d_{\lambda+x,\lambda} \frac{x}{x - qty} H_{\lambda+x,x}. \tag{7.2}$$

Next, it is sufficient to prove by induction that

$$(z_1^{S_n} y_1 z_1^{S_{n-1}} y_1 \dots z_1^{S_1} y_1) d_+ = (-1)^n \sum_T \prod_{i<j} \omega(w_i/w_j) \frac{w_i^{S_i+1}}{w_i - qtw_{i-1}} H_{\lambda,\square_n}.$$

If we apply y_1 to the right hand side, we need to sum over all possible ways to add a box w_{n+1} to a standard Young tableau T, that is, over all standard Young tableaux of size $(n+1)$. The additional factor is described by (7.2) with $x = w_{n+1}$ and $y = w_n$:

$$-d_{\lambda+w_{n+1},\lambda} \frac{w_{n+1}}{w_{n+1} - qtw_n} = -\prod_{i \le n} \omega(w_i/w_{n+1}) \frac{w_{n+1}}{w_{n+1} - qtw_n}.$$

The action of $z_1^{S_{n+1}}$ on the result just adds a factor $w_{n+1}^{S_{n+1}}$. □

7.3 Complete symmetric functions in y_i

We conclude with a result describing the complete symmetric functions in y_i in the fixed point basis.

Theorem 7.2 *Suppose that*

$$h_n(y_1,\ldots,y_k) = \sum_{\mu^0,\ldots,\mu^k} a_{\mu^0,\ldots,\mu^k}(q,t) H_{\mu^0,\ldots,\mu^k}.$$

Then the following identity holds:

$$a_{\mu^0,\ldots,\mu^k}(q,t) = \frac{(q-1)q^{k(k-1)/2}}{[k-1]_q!} \frac{\Lambda\left(B_{\mu^0}-1\right)}{\det(B_{\mu^0})\Lambda\left(T^*_{\tilde{\mu}}\,\mathrm{PFH}_{n,n-k}\right)}$$

Proof First we obtain a recursive relation for $h_n(y_1,\ldots,y_k)$ in terms of the operators d_+, d_- and φ. By definition of d_+ (see Proposition 2.11):

$$\begin{aligned} d_+(h_n(y_1,\ldots,y_k)) &= T_1\cdots T_k h_n(y_1,\ldots,y_k)\\ &= T_1\cdots T_k(h_n(y_1,\ldots,y_{k+1}) - y_{n+1}h_{n-1}(y_1,\ldots,y_{k+1}))\\ &= h_n(y_1,\ldots,y_{k+1}) - T_1\cdots T_k y_{n+1}h_{n-1}(y_1,\ldots,y_{k+1})\\ &= h_n(y_1,\ldots,y_{k+1}) - \varphi h_{n-1}(y_1,\ldots,y_{k+1}), \end{aligned}$$

so

$$h_n(y_1,\ldots,y_{k+1}) = d_+(h_n(y_1,\ldots,y_k)) + \varphi h_{n-1}(y_1,\ldots,y_{k+1}). \tag{7.3}$$

It is not hard to see that the right hand side satisfies (7.3) as well. Indeed, the actions of d_+ and φ in the fixed point basis are given by (5.1), and summing the terms and applying an analogue of (4.5) yields the desired equation.

It remains to check the base case $n = 0$. In this case $h_0(y_1,\ldots,y_k) = 1 \in V_k$. The corresponding moduli space is $\mathrm{PFH}_{k,0} = \mathbb{C}^k$ (see Example 3.4), which contains a unique fixed point corresponding to the partition $\mu^0 = (k)$. Now $B_{\mu^0} = 1+\ldots+q^{k-1}$, so $\det B_{\mu^0} = q^{k(k-1)/2}$ and $\Lambda\left(B_{\mu^0}-1\right) = [k-1]_q!$.

Finally, $\Lambda\left(T^*_{\tilde{\mu}}\,\mathrm{PFH}_{k,0}\right) = (1-q)^k$. □

Example 7.3 For $k=1$ we get $y_1^n = \sum a_{\mu,\nu}(q,t)H_{\mu,\nu}$ and

$$a_{\mu,\nu}(q,t) = \frac{(1-q)\lambda\left(B_\mu - 1\right)}{\det(B_\mu)\lambda\left(T^*_{\mu,\nu}\,\mathrm{PFH}_{n,n+1}\right)}$$

Observe that for $k=1$ we have $y_1 = \varphi$, so in this case Theorem 7.2 follows from the explicit formula for the action of φ in the Macdonald basis (5.1).

By comparing the numerator in Theorem 7.2 with the numerator in Haiman's description of the punctual Hilbert scheme [13], we expect that $h_n(y_1, ..., y_k)$ is the class of a compactly supported sheaf on $\mathrm{PFH}_{n,n-k}$. Combinatorially, after twisting by $\mathcal{O}(1)$, its contribution to the shuffle formula must consist of those Dyck paths with k touch points. We expect this to be part of a potential geometric proof of the shuffle theorem, which we leave for future papers.

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