CENTRAL STABILITY FOR THE HOMOLOGY OF CONGRUENCE SUBGROUPS AND THE SECOND HOMOLOGY OF TORELLI GROUPS

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ABSTRACT. We prove a representation stability result for the second homology groups of Torelli subgroups of mapping class groups and automorphism groups of free groups. This strengthens the results of Boldsen–Hauge Dollerup and Day–Putman. We also prove a new representation stability result for the homology of certain congruence subgroups, partially improving upon the work of Putman–Sam. These results follow from a general theorem on syzygies of certain modules with finite polynomial degree.

CONTENTS

1. Introduction

The purpose of this paper is to prove cases of a variant of two conjectures from Church and Farb [CF13, variations on Conjectures 6.3 and 6.5] as well as to answer a question posed by Putman [Put15, fifth Remark]. Specifically, we prove a central stability result for the second homology groups of the Torelli subgroups of mapping class groups and automorphism groups of free groups, and a central stability result for the homology groups in all degrees of certain congruence subgroups of general linear groups. We begin by recalling the definition of central stability.

1.1. **Central stability.** Let

$$G_0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow \cdots$$

be a sequence of groups and group homomorphisms, and let

$$A_0 \xrightarrow{\phi_0} A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} \cdots$$

be a sequence with A_n a G_n–representations and ϕ_n a G_n–equivariant map. In this paper, the groups G_n will be symmetric groups, general linear groups, subgroups of general linear groups with restricted determinant, or symplectic groups. Church and Farb [CF13] described a condition they called *representation stability* for these types of sequences $\{A_n\}$. In its original formulation, representation stability was only defined for semisimple representations A_n and for families of groups admitting natural identifications between irreducible G_n–representations for different n. A much more general and formal notion of representation stability called *central stability* was introduced by Putman [Put15].

For purposes of exposition, we will now specialize to the case that the groups G_n are either the family of general linear groups $\mathrm{GL}_n(\mathbb{Z})$ or the family of symmetric groups $\mathfrak{S}_n$. Let $\sigma_n \in G_n$ be the transposition $(n-1 \ n)$ in the case $G_n = \mathfrak{S}_n$ and the associated permutation matrix in the case $G_n = \mathrm{GL}_n(\mathbb{Z})$. We assume that σ_n fixes the map $(\phi_{n-1} \circ \phi_{n-2}) \colon A_{n-2} \to A_n$. There are two natural maps

$$\mathrm{Ind}_{G_{n-2}}^{G_n} A_{n-2} \rightrightarrows \mathrm{Ind}_{G_{n-1}}^{G_n} A_{n-1},$$

the first induced by ϕ_{n-2} and the second by postcomposing this induced map by σ_n. We say that the sequence $\{A_n\}$ has *central stability degree* $\leq d$ if for all $n > d$ the map

$$\mathrm{coeq}\left(\mathrm{Ind}_{G_{n-2}}^{G_n} A_{n-2} \rightrightarrows \mathrm{Ind}_{G_{n-1}}^{G_n} A_{n-1}\right) \to A_n$$

induced by ϕ_{n-1} is an isomorphism. We say that $\{A_n\}$ is *centrally stable* if it has finite central stability degree. If $\{A_n\}$ has central stability degree $\leq d$, then the entire sequence is determined by the finite sequence $A_0 \to A_1 \to \cdots \to A_d$. Analogous definitions exist for symplectic groups and subgroups of the general linear groups with restricted determinant, which we review in Section 3. The main result of this paper is to prove that certain homology groups of Torelli groups and congruence subgroups are centrally stable.

A cautionary remark for experts: there are different definitions of *central stability* in the literature that are not equivalent in all situations; see Remark 3.11.

1.2. **Central stability for Torelli groups and congruence subgroups.**

Automorphisms of free groups. Let F_n denote the free group on n letters. A group automorphism $f\colon F_n \to F_n$ induces a linear map $f^{\mathrm{ab}}\colon \mathbb{Z}^n \to \mathbb{Z}^n$ on the abelianization of F_n. This construction defines a surjective map

$$\mathrm{Aut}(F_n) \to \mathrm{GL}_n(\mathbb{Z}).$$

We write IA_n to denote the kernel of this map, which is often called the *Torelli subgroup* of $\mathrm{Aut}(F_n)$. If we fix an inclusion of $\mathrm{Aut}(F_n)$ into $\mathrm{Aut}(F_{n+1})$ as the stabilizer of a free generator, the Torelli group IA_n maps into IA_{n+1}. The induced maps $H_i(\mathrm{IA}_n) \to H_i(\mathrm{IA}_{n+1})$ are equivariant with respect to the induced actions of the groups $\mathrm{GL}_n(\mathbb{Z})$, and so we may ask whether the sequences of homology groups $\{H_i(\mathrm{IA}_n)\}$ exhibit central stability as $\mathrm{GL}_n(\mathbb{Z})$–representations.

The first homology group of IA_n was computed independently in work of Andreadakis [And65, see Section 6] for $n = 2, 3$, Cohen–Pakianathan (unpublished), Farb (unpublished) and Kawazumi [Kaw06, Theorem 6.1]. Very little is known about the higher homology groups of IA_n. Bestvina–Bux–Margalit [BBM07, Main Theorem, part (2)] proved that $H_2(\mathrm{IA}_3)$ is not finitely generated as an abelian group which implies that IA_3 is not finitely presented, a result originally due to Krstić–McCool [KM97, Theorem 1]. However, for $n > 3$, it is unknown if $H_2(\mathrm{IA}_n)$ is finitely generated or if IA_n is finitely presented.

Church and Farb [CF13, Conjecture 6.3] conjectured "mixed representation stability" for a certain summand of the rational homology of IA_n. Given Putman's subsequent work on central stability, it is natural to modify Church and Farb's Conjecture 6.3 to also ask if the integral homology groups of IA_n exhibit central stability as representations of the general linear groups. We prove this result for the second homology groups.

Theorem A. *The sequence $H_2(\mathrm{IA}_n)$ has central stability degree ≤ 38 as $\mathrm{GL}_n(\mathbb{Z})$–representations.*

This theorem partially improves upon a result of Day and Putman [DP17, Theorem B], which established surjectivity of the maps

$$\mathrm{coeq}\left(\mathrm{Ind}_{\mathrm{GL}_{n-2}(\mathbb{Z})}^{\mathrm{GL}_n(\mathbb{Z})} H_2(\mathrm{IA}_{n-2}) \rightrightarrows \mathrm{Ind}_{\mathrm{GL}_{n-1}(\mathbb{Z})}^{\mathrm{GL}_n(\mathbb{Z})} H_2(\mathrm{IA}_{n-1})\right) \to H_2(\mathrm{IA}_n)$$

for $n > 6$. Our techniques only show these maps surject for $n > 18$ (Theorem A'), but additionally prove the maps are injective for $n > 38$. We note that our method is substantially different from that of Day and Putman: our proof centers on properties of general linear groups, while their proof focused on properties of automorphism groups of free groups. Our proof strategy applies to Torelli subgroups of mapping class groups with little modification. In contrast, Day and Putman [DP17, Remark 1.3] noted that their techniques for Torelli subgroups of $\mathrm{Aut}(F_n)$ do not easily generalize to Torelli subgroups of mapping class groups, and that the techniques used by Boldsen and Hauge Dollerup [BHD12] for Torelli subgroups of mapping class groups do not easily generalize to prove results about IA_n.

Our proof uses a spectral sequence due to Putman–Sam [PS17], and relies on known computations of $H_1(\mathrm{IA}_n)$ (Kawazumi [Kaw06, Theorem 6.1]) and a high-connectivity result of Hatcher–Vogtmann [HV98, Proposition 6.4].

Mapping class groups. Let $\Sigma_{g,1}$ denote an oriented genus-g surface with one boundary component, and let Mod_g denote the group of connected components of the group of orientation-preserving diffeomorphisms of $\Sigma_{g,1}$ that fix the boundary pointwise. The induced action of these diffeomorphisms

on the first homology group of $\Sigma_{g,1}$ preserves the intersection form, which happens to be symplectic. This action therefore induces a map

$$\mathrm{Mod}_g \to \mathrm{Sp}_{2g}(\mathbb{Z}),$$

where $\mathrm{Sp}_{2g}(\mathbb{Z})$ denotes the group of linear automorphisms of $\mathbb{Z}^{2g}$ that preserve the standard symplectic form. This map is well known to be surjective. The kernel $\mathcal{I}_g$ of this map is called the *Torelli subgroup* of Mod_g. Just as with the groups IA_n, very little is known about the homology groups $H_i(\mathcal{I}_g)$ for $i > 1$. Church and Farb conjectured a form of representation stability for $H_i(\mathcal{I}_g)$ [CF13, Conjecture 6.1], motivated by their earlier work [CF12] constructing nontrivial classes in $H_i(\mathcal{I}_g; \mathbb{Q})$. In this paper we prove a central stability result for the homology groups $H_2(\mathcal{I}_g)$ as symplectic group representations.

Theorem B. *The sequence $H_2(\mathcal{I}_g)$ has central stability degree ≤ 69 as $\mathrm{Sp}_{2g}(\mathbb{Z})$–representations.*

This partially improves upon a result of Boldsen and Hauge Dollerup [BHD12, Theorem 1.0.1] proving that

$$\mathrm{coeq}\left(\mathrm{Ind}_{\mathrm{Sp}_{2g-4}(\mathbb{Z})}^{\mathrm{Sp}_{2g}(\mathbb{Z})} H_2(\mathcal{I}_{g-2}; \mathbb{Q}) \rightrightarrows \mathrm{Ind}_{\mathrm{Sp}_{2g-2}(\mathbb{Z})}^{\mathrm{Sp}_{2g}(\mathbb{Z})} H_2(\mathcal{I}_{g-1}; \mathbb{Q})\right) \to H_2(\mathcal{I}_g; \mathbb{Q})$$

is surjective for $g > 6$; our techniques only show the map is surjective for $g > 33$ (Theorem B') but also establish injectivity for $g > 69$, and our result holds with integral coefficients.

Our proof again uses the spectral sequence from Putman–Sam [PS17], Johnson's computations of $H_1(\mathcal{I}_g)$ ([Joh85b, Theorem 3]) and a high–connectivity result of Hatcher–Vogtmann [HV17, Main Theorem].

Prior work had established representation stability results for certain subrepresentations of the rational cohomology of Torelli groups. Hain [Hai97, Remark 14.2 and Proposition 5.1] computed the images of the cup product maps

$$H^1(\mathcal{I}_g; \mathbb{Q}) \otimes H^1(\mathcal{I}_g; \mathbb{Q}) \longrightarrow H^2(\mathcal{I}_g; \mathbb{Q})$$

as $\mathrm{Sp}_{2g}(\mathbb{Z})$-representations, and his results show in particular that the multiplicities stabilize for $g > 5$. Church–Ellenberg–Farb [CEF15, proof of Theorem 7.2.2] proved a representation stability result for the subalgebras of $H^*(\mathcal{I}_g; \mathbb{Q})$ generated by $H^1(\mathcal{I}_g; \mathbb{Q})$, viewed as S_n-representations. They also proved the analogous result for IA_n [CEF15, proof of Theorem 7.2.3]. Kupers–Randal-Williams [KRW19, Theorem 8.1] computed the algebraic part of $H^2(\mathcal{I}_g; \mathbb{Q})$ for $g \geq 4$.

Shortly after this paper was released, Kassabov–Putman [KP18] proved that $H_2(\mathcal{I}_g; \mathbb{Z})$ is finitely generated as an $\mathrm{Sp}_{2g}(\mathbb{Z})$–module for $g \geq 3$ [KP18, Theorem A], and that $\mathcal{I}_g$ has a *finite* Mod_g-*equivariant presentation* for $g \geq 3$ [KP18, Theorem C].

Congruence subgroups of $\mathrm{GL}_n(R)$. We also investigate representation stability for congruence subgroups of general linear groups. Homological and representation stability properties of congruence subgroups have had applications in homotopy theory – for example, in work of Charney [Cha84] on excision for algebraic K-theory – and applications in number theory; see Calegari–Emerton [CE16]. Let I be a two-sided ideal in a ring R, and let $\mathrm{GL}_n(R, I)$ denote the kernel of the "reduction modulo I" map

$$\mathrm{GL}_n(R) \to \mathrm{GL}_n(R/I).$$

We call $\mathrm{GL}_n(R, I)$ the *level I congruence subgroup* of $\mathrm{GL}_n(R)$.

For R commutative and H a subgroup of the group of units $R^\times$ of R, let $\mathrm{GL}_n^H(R)$ denote the subgroup of matrices with determinant in H. Let $\mathfrak{U}$ denote the image of $R^\times$ in R/I. In many cases, $\mathrm{GL}_n^{\mathfrak{U}}(R/I)$ is the image of $\mathrm{GL}_n(R)$ in $\mathrm{GL}_n(R/I)$. For example, this is the case when R is the ring of integers in a number field by strong approximation (see e.g [PR94, Chapter 7]) or when R/I is Euclidean since then $\mathrm{GL}_n^{\mathfrak{U}}(R/I)$ will be generated by elementary matrices. In these cases, the homology groups $H_i(\mathrm{GL}_n(R, I))$ will have natural linear $\mathrm{GL}_n^{\mathfrak{U}}(R/I)$–actions and there are equivariant maps $H_i(\mathrm{GL}_n(R, I)) \to H_i(\mathrm{GL}_{n+1}(R, I))$ induced by the inclusions $\mathrm{GL}_n(R, I) \hookrightarrow \mathrm{GL}_{n+1}(R, I)$.

When $I \subsetneq R$, the symmetric group $\mathfrak{S}_n$ is naturally a subgroup of $\mathrm{GL}_n^{\mathfrak{U}}(R/I)$, so we may view the the groups $H_i(\mathrm{GL}_n(R, I))$ as representations of the symmetric group. When $I = R$, we endow $H_i(\mathrm{GL}_n(R, I))$ with the trivial symmetric group action. Putman [Put15, Theorem B] proved that when R has finite stable rank (see Definition 2.18 and Bass [Bas64, Section 4, "Definition"]), these homology groups have central stability as $\mathfrak{S}_n$–representations. Putman's result gave explicit stable ranges for the homology groups $H_i(\mathrm{GL}_n(R, I); K)$ over certain fields K, and later Church–Ellenberg–Farb–Nagpal [CEFN14, Theorem D] proved a central stability result for the homology of certain congruence subgroups with coefficients in a general Noetherian ring K but without explicit bounds. Church and Ellenberg [CE17, Theorem D'] generalized both theorems with a result for integral homology with explicit stable ranges, and Church–Miller–Nagpal–Reinhold [CMNR18, Application B] and Gan–Li [GL17a, Theorem 11] further improved these ranges.

Putman [Put15, fifth Remark] comments that it would be ideal to understand stability properties of $\mathrm{GL}_n(R, I)$ as $\mathrm{GL}_n^{\mathfrak{U}}(R/I)$–representations instead of just $\mathfrak{S}_n$–representations. Using Gan–Li [GL17a, Theorem 11] (stated in our Theorem 4.3), we provide the following partial solution to this problem.

Theorem C. *Let I be a two-sided ideal in a ring R of stable rank r and let t be the minimal stable rank of all rings containing I as a two-sided ideal. Let $\mathfrak{U}$ denote the image of $R^\times$ in R/I. If R/I is a PID of stable rank s, and the natural map $\mathrm{GL}_n(R) \to \mathrm{GL}_n^{\mathfrak{U}}(R/I)$ is surjective, then the central stability degree of the sequence $H_i(\mathrm{GL}_n(R, I))$ is*

$$
\begin{aligned}
&\leq s + 1 && \text{for } i = 0 \\
&\leq \min(24i + 12t + s + 4, \max(5 + r, 5 + s)) && \text{for } i = 1 \\
&\leq 24i + 12t + s + 4 && \text{for } i > 1
\end{aligned}
$$

as $\mathbb{Z}[\mathrm{GL}_n^{\mathfrak{U}}(R/I)]$–modules.

In particular, Theorem C applies in the case that R is a ring of stable rank $t = r$, and R/I is a Euclidean domain. It follows from the work of Putman and Sam [PS17] that $H_i(\mathrm{GL}_n(R, I))$ has finite central stability degree when R is the ring of integers in an algebraic number field. However, their techniques do not give an explicit central stability range or address congruence subgroups with infinite quotients R/I.

Remark 1.1. Observe that the group $\mathrm{GL}(R, I)$ only depends on I as a non-unital ring. The only assumption we put on the non-unital ring I in Theorem C is that I is a two-sided ideal in a ring with finite Bass stable rank. We can always find a ring R containing I as a two sided ideal with $\mathrm{GL}_n(R) \to \mathrm{GL}_n^{\mathfrak{U}}(R/I)$ surjective. In particular, R can be taken to be the *unitalization* I_+, R/I can be taken to be $\mathbb{Z}$, and $\mathfrak{U}$ can be taken to be $\{\pm 1\}$. The unitalization of I is defined as the image of I under the left adjoint of the forgetful functor from the category of unital rings to the category of possibly

non-unital rings. As an abelian group, $I_+ \cong \mathbb{Z} \oplus I$. The unitalization is an important construction when studying excision for algebraic K-theory (see e.g. Suslin–Wodzicki [SW92]). In particular, for any non-unital ring I, $H_i(\mathrm{GL}_n(I_+, I))$ is a $\mathrm{GL}_n(\mathbb{Z})$–representation.

Because $\mathrm{GL}(R, I) \cong \mathrm{GL}_n(I_+, I)$ for any ring R and ideal I, and because $\mathbb{Z}$ has stable rank 2, we obtain the following corollary of Theorem C.

Corollary 1.2. *Let I be a two-sided ideal in a ring R and let t be the minimal stable rank of all rings containing I as a two-sided ideal. Then, as a sequence of $\mathbb{Z}[\mathrm{GL}_n(\mathbb{Z})]$–modules, the sequence $H_i(\mathrm{GL}_n(R, I))$ has central stability degree*

$$
\begin{aligned}
&\leq 3 && \text{for } i = 0 \\
&\leq \min(24i + 12t + 6,\, 7) && \text{for } i = 1 \\
&\leq 24i + 12t + 6 && \text{for } i > 1.
\end{aligned}
$$

Remark 1.3. Currently, the representation stability literature has a number of effective tools for establishing stability results, including central stability, for sequences of symmetric group representations (see for example Church–Ellenberg [CE17]) but often these tools cannot be applied to sequences of general linear (or symplectic) group representations. In Corollary 3.34, we prove that if a sequence of general linear representations happens to be centrally stable with respect to the underlying actions of the symmetric groups, then it is also centrally stable with respect to the action of the general linear groups (though the converse does not hold in general). Corollary 3.34 then allows us to use known or elementary results about central stability for certain sequences of symmetric group representations to prove our central stability results for general linear group representations, in particular Theorem A and Theorem C.

We prove Corollary 3.34 by comparing central stability with the notion of *polynomial degree* (see Definition 3.24). This polynomial condition and its variants have a long history in the literature; see e.g. [EML54, Pir88, Pir, BP99, Pir00, BDFP01, HV11, Dja12, DV10, Ves13, DV13a, DV13b, DPV16, Dja17, Dja16a, RWW17]. The definition used in this paper agrees with that of an arXiv version of Randal-Williams–Wahl [RWW15, Definition 4.10]. A convenient fact is that the polynomial degree of a sequence does not depend on the choice of automorphism group, which allows us to leverage results about symmetric group actions to prove results about general linear group actions. We use a similar strategy in the symplectic case as well in order to prove Theorem B.

We note that we do not prove that the second homology groups of these Torelli groups satisfy a polynomial condition. Instead, we use that the zeroth and first homology groups do statisfy a polynomial condition to prove that the second homology groups are centrally stable. This contrasts with the case of congruence subgroups where we use that all of the homology groups satisfy a polynomial condition.

1.3. **Outline.** In Section 2, we prove that some relevant semisimplicial sets are highly connected. We use these connectivity results in Section 3 to prove that certain modules with finite polynomial degree exhibit central stability. In Section 4, we apply our results on polynomial degree to prove Theorem A, Theorem B, and Theorem C.

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2. High connectivity results

As is common in stability arguments, our proofs will involve establishing high connectivity for certain spaces with actions of our families of groups. We start out in Section 2.1 with a review of simplicial complexes and semisimplicial sets, and an overview of techniques to prove that their geometric realizations are highly connected. Then in Section 2.2 we review elementary properties of free modules and symplectic structures over PIDs. In Section 2.3 we prove some new high connectivity results for spaces relevant to stability for general linear group representations. In Section 2.4, we review results from the literature concerning high connectivity of spaces relevant to stability for symplectic group representations.

2.1. **Review of simplicial techniques.** Recall that the data of a semisimplicial set is the same as the data of a simplicial set without degeneracy maps; see Weibel [Wei95, Definition 8.1.9]. We will also consider simplicial complexes in this paper. Simplicial complexes differ from semisimplicial sets in several ways. For example, each simplex of a semisimplicial set comes equipped with an order on its set of faces, while the faces of a simplicial complex are not ordered. Further, a collection of vertices can be the set of vertices of at most one simplex of a simplicial complex, but there are no such restrictions for semisimplicial sets. See Randal-Williams–Wahl [RWW17, Section 2.1] for a discussion of the differences between semisimplicial sets and simplicial complexes.

We say that a semisimplicial set or simplicial complex is n–connected if its geometric realization is n–connected. If σ is a simplex of a simplicial complex $X_\circ$, we write $\mathrm{Lk}^X(\sigma)$ to denote the link of σ in X, or simply $\mathrm{Lk}(\sigma)$ when the ambient complex is clear from context. We now recall the definition of weakly Cohen–Macaulay simplicial complexes.

Definition 2.1. A simplicial complex $X_\circ$ is called *weakly Cohen–Macaulay* (abbreviated wCM) of dimension n if it satisfies the following two conditions.

- $X_\circ$ is $(n-1)$–connected.
- If σ is a p–simplex of $X_\circ$, then $\mathrm{Lk}^X(\sigma)$ is $(n-2-p)$–connected.

Definition 2.2. If $X_\circ$ is a simplicial complex, let $X_\bullet = X_\circ^{\mathrm{ord}}$ be the associated semisimplicial set formed by taking all simplices of $X_\circ$ with all choices of orderings on their vertices.

Convention 2.3. We adopt the following convention on subscript notation: we always write $X_\bullet$ for a semisimplicial set and $X_\circ$ for a simplicial complex. If we denote a semisimplicial set and a simplicial complex by the same letter, then they are related by $X_\circ^{\mathrm{ord}} = X_\bullet$. By X_p we will always mean the set of p–simplices of the semisimplicial set $X_\bullet$, not the set of p–simplices of the simplicial complex $X_\circ$, which is then in bijection with $X_p/\mathfrak{S}_{p+1}$.

The following theorem is well known; see for example Kupers–Miller [KM16, Lemma 3.16] and Randal-Williams–Wahl [RWW17, Proposition 2.14].

Theorem 2.4.

(a) If $X_\bullet = X_\circ^{\mathrm{ord}}$ is n–connected, then so is $X_\circ$.

(b) If $X_\circ$ is weakly Cohen–Macaulay of dimension n, then $X_\bullet = X_\circ^{\mathrm{ord}}$ is $(n-1)$–connected.

These results will allow us to pass between high connectivity results for semisimplicial sets and simplicial complexes. The following definition and theorem is a useful tool for proving simplicial complexes are highly connected.

Definition 2.5. A map of simplicial complexes $\pi\colon Y_\circ \to X_\circ$ is said to exhibit $Y_\circ$ as a *join complex* over $X_\circ$ if it satisfies all of the following:

- π is surjective
- π is simplex-wise injective, that is, the induced map on the geometric realizations is injective when restricted to a (closed) simplex
- a collection of vertices $(y_0, \ldots, y_p)$ spans a simplex of Y whenever there exists simplices $\theta_0, \ldots, \theta_p$ such that for all i, y_i is a vertex of θ_i and the simplex $\pi(\theta_i)$ has vertices $\pi(y_0), \ldots, \pi(y_p)$.

The following result is due to Hatcher–Wahl [HW10].

Theorem 2.6 (Hatcher–Wahl [HW10, Theorem 3.6]). *Let $\pi\colon Y_\circ \to X_\circ$ be a map of simplicial complexes exhibiting $Y_\circ$ as a join complex over $X_\circ$. Assume $X_\circ$ is wCM of dimension n. Further assume that for all p-simplices τ of $Y_\circ$, the image of the link $\pi(\mathrm{Lk}^Y(\tau))$ is wCM of dimension $(n-p-2)$. Then $Y_\circ$ is $\left\lfloor \dfrac{n-2}{2} \right\rfloor$–connected.*

Definition 2.7. Let $X_\circ$ be a simplicial complex and $X_\bullet = X_\circ^{\mathrm{ord}}$. Let $\widetilde{\sigma} \in X_p$ and $\sigma \subseteq X_\circ$ the corresponding simplex. Then define the *link*

$$\mathrm{Lk}_\bullet^X \widetilde{\sigma} = (\mathrm{Lk}_\circ^X \sigma)^{\mathrm{ord}}$$

as a sub-semisimplicial set of $X_\bullet$.

2.2. **Algebraic preliminaries.** In this subsection, we recall some basic facts and definitions concerning free modules over PIDs as well as symplectic structures.

Let R be a PID. For S a set, throughout this paper we write $R[S]$ to denote the free R–module with basis S. Notably, this does *not* denote the polynomial algebra with variables S.

Given a submodule W of a free R–module V, we say W is *splittable* or *has a complement* if there exists a submodule U with $V = W \oplus U$. Given a submodule W of a free module V, let $\mathrm{sat}(W)$ denote the intersection of all splittable submodules of V which contain W, equivalently, $\mathrm{sat}(W)$ is the preimage of the torsion submodule of V/W under the quotient map $V \to V/W$. We call $\mathrm{sat}(W)$ the *saturation* of W. We write $\mathrm{rk}(W)$ for the rank of a free module. The following proposition collects some elementary facts concerning free modules over PIDs and their submodules; see for example Kaplansky [Kap54].

Proposition 2.8. *Let R be a PID and A, B, C submodules of a finitely generated free R–module V.*

- *i) If A and B have complements in V, then so does $A \cap B$.*
- *ii) Let B have a complement. Then A has a complement containing B if and only if there is a submodule D with $V = A \oplus B \oplus D$.*
- *iii) $\mathrm{rk}(A) = \mathrm{rk}(\mathrm{sat}(A))$.*
- *iv) $\left((A \cap C) \oplus (B \cap C)\right) \subseteq (A + B) \cap C$ but equality does not hold in general.*

    v) If $V = A \oplus B$ and $C \supseteq A$, then $C = A \oplus (B \cap C)$.

    vi) If $V = A \oplus B$ and $C \subseteq A$, then $C = A \cap (B \oplus C)$.

    vii) A has a complement in V if and only if V/A is torsion free.

Again let R be a PID and V a finitely generated free R–module. Recall that a *symplectic form* on V is a perfect bilinear pairing $\langle\,,\,\rangle\colon V \times V \to R$ such that $\langle v, v \rangle = 0$ for all $v \in V$. For a submodule $W \subseteq V$, we write $W^\perp$ to denote its symplectic complement

$$W^\perp = \{v \in V \mid \langle v, w \rangle = 0 \text{ for all } w \in W\}.$$

Recall that a submodule W of a symplectic R–module V is called *symplectic* if the form restricts to a symplectic form on W. We note that, when R is not a field, the condition that W is symplectic may be stronger than the condition that $W \cap W^\perp = 0$.

The following proposition is an elementary fact about free symplectic modules over PIDs and their submodules (see for example Knus [Knu91, 4.1.2]).

Proposition 2.9. *Let R be a PID and V a finitely generated free symplectic module over R. Then the rank of V is even and there is a basis $v_1, w_1, \ldots, v_n, w_n$ of V such that*

$$\langle v_i, v_j \rangle = \langle w_i, w_j \rangle = 0, \quad \langle v_i, w_j \rangle = \delta_{ij} \qquad \text{for all } i, j = 1, \ldots, n.$$

2.3. **Generalized partial basis complexes.** In this subsection, we recall the definition of several simplicial complexes and semisimplicial sets involving partial bases and complements. These connectivity results will imply that $\mathsf{VIC}(R)$–modules satisfying a polynomial condition exhibit central stability. Throughout the subsection we let R be a PID and V be a finite-rank free R–module.

Definition 2.10. A *partial basis* of a free module V is a linearly independent set $\{v_0, \ldots, v_p\} \subseteq V$ such that there is a free (possibly zero) submodule C with $\mathrm{span}(v_0, \ldots, v_p) \oplus C = V$. Such a set $\{v_0, \ldots, v_p\}$ is also called *unimodular*, and the submodule C is called a *complement* for the partial basis. An *ordered partial basis* is a partial basis with a choice of bijection with a set of the form $\{0, \ldots, p\}$.

Definition 2.11. For $p \geq 0$, let $\mathrm{PB}_p(V)$ be the set of ordered partial bases of size $p + 1$. For $0 \leq i \leq p$, there are maps $d_i\colon \mathrm{PB}_p(V) \to \mathrm{PB}_{p-1}(V)$ given by forgetting the ith basis element. With these maps, the sets $\mathrm{PB}_p(V)$ assemble into a semisimplicial set $\mathrm{PB}_\bullet(V)$. Let $\mathrm{PB}_\circ(V)$ be the unique simplicial complex with $\mathrm{PB}_\circ^{\mathrm{ord}}(V) = \mathrm{PB}_\bullet(V)$

The existence of $\mathrm{PB}_\circ(V)$ and all other simplicial complexes considered in this subsection follow easily from the work of Randal-Williams–Wahl [RWW17, Definition 2.8 and proof of Lemma 5.10].

The link of a simplex $\sigma = (v_0, v_1, \ldots, v_p) \in \mathrm{PB}_p(V)$ is the subcomplex of ordered partial bases $(u_0, u_1, \ldots, u_q)$ such that $\{v_0, v_1, \ldots, v_p, u_0, u_1, \ldots, u_q\}$ is a partial basis of V. Notably, this link only depends on the submodule $W = \mathrm{span}(v_0, \ldots, v_p)$. By abuse of notation, we will often denote $\mathrm{Lk}_\bullet(\sigma)$ by $\mathrm{Lk}_\bullet(W)$. By convention if $W = 0$, we let $\mathrm{Lk}_\bullet(W)$ be the entire complex of ordered partial bases $\mathrm{PB}_\bullet(V)$. If $U \subseteq V$ is a splittable submodule, then there is a canonical inclusion of $\mathrm{PB}_\bullet(U) \subseteq \mathrm{PB}_\bullet(V)$.

Definition 2.12. Let $U, W \subseteq V$ be splittable submodules. Define

$$\mathrm{PB}_\bullet(U, W) = \mathrm{PB}_\bullet(U) \cap \mathrm{Lk}_\bullet(W) \subseteq \mathrm{PB}_\bullet(V)$$

and let $\mathrm{PB}_\circ(U, W)$ be the unique simplicial complex with $\mathrm{PB}_\circ^{\mathrm{ord}}(U, W) = \mathrm{PB}_\bullet(U, W)$

Concretely, $\mathrm{PB}_\bullet(U, W)$ is the sub-semisimplicial set of $\mathrm{PB}_\bullet(U)$ consisting of ordered nonempty partial bases of U contained in a complement of W. The complex $\mathrm{PB}_\bullet(U, W)$ depends only on the submodule $\mathrm{sat}(U + W)$ and not on V. We note that

$$\mathrm{PB}_\bullet(U, W) = \mathrm{Lk}_\bullet^{\mathrm{PB}(U)}(W) \qquad \text{if } W \subseteq U,$$

and in particular that

$$\mathrm{PB}_\bullet(V, V) = \varnothing \qquad \text{and} \qquad \mathrm{PB}_\bullet(V, 0) = \mathrm{PB}_\bullet(V).$$

More generally, whenever W is contained in any complement of U, then $\mathrm{PB}(U, W) \cong \mathrm{PB}(U)$.

Remark 2.13. For vector spaces,

$$\mathrm{PB}_\bullet(U, W) = \mathrm{PB}_\bullet(U, W \cap U) = \mathrm{Lk}_\bullet^{\mathrm{PB}(U)}(W \cap U) \qquad (R \text{ a field}).$$

We caution, however, that this identification does not hold in general. For example, when $R = \mathbb{Z}$ and $V = \mathbb{Z}^3$, consider the submodules

$$U = \mathrm{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) \qquad \text{and} \qquad W = \mathrm{span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right).$$

Since the determinant of the matrix

$$\begin{bmatrix} 1 & 1 & a \\ 1 & -1 & b \\ 0 & 0 & c \end{bmatrix}$$

is a multiple of 2 for any $a, b, c \in \mathbb{Z}$, there is no basis for V that contains both a basis for U and a basis for W, and $\mathrm{PB}_\bullet(U, W)$ is empty. In contrast, the complex $\mathrm{PB}_\bullet(U, W \cap U) = \mathrm{PB}_\bullet(U, 0) = \mathrm{PB}_\bullet(U)$ is nonempty.

We next define a variation of the semisimplicial set $\mathrm{PB}_\bullet(V)$ consisting of ordered partial bases with distinguished choices of complements.

Definition 2.14. Let $\mathrm{PBC}_p(V)$ be the set of ordered partial bases $(v_0, \ldots, v_p)$ of V as well as a choice of complement C such that

$$C \oplus \mathrm{span}(v_0, \ldots, v_p) = V.$$

Let $d_i : \mathrm{PBC}_p(V) \to \mathrm{PBC}_{p-1}(V)$ be given by the formula

$$d_i(v_0, \ldots, v_p, C) = (v_0, \ldots, \hat{v}_i, \ldots, v_p, C \oplus R v_i).$$

Here the hat indicates omission. These sets assemble to form a semisimplicial set $\mathrm{PBC}_\bullet(V)$. Let $\mathrm{PBC}_\circ(V)$ be the unique simplicial complex with $\mathrm{PBC}_\circ^{\mathrm{ord}}(V) = \mathrm{PBC}_\bullet(V)$

Proposition 2.15. *Let* $\partial_i : \mathrm{PBC}_p(V) \to \mathrm{PBC}_0(V)$ *be the map induced by* $\{i\} \hookrightarrow \{0, \ldots, p\}$ *and let* $\partial : \mathrm{PBC}_p(V) \to \mathrm{PBC}_0(V)^{p+1}$ *be the product of these maps. The map* ∂ *is injective. If* R *is a PID, then*

$$\big((v_0, C_0), \ldots, (v_p, C_p) \big)$$

is in the image of ∂ *if and only if* $v_i \in C_j$ *for all* $i \neq j$.

Proof. Consider $(v_0, \ldots, v_p, C) \in \mathrm{PBC}_p(V)$. We have that

$$\partial_i(v_0, \ldots, v_p, C) = (v_j, C \oplus \mathrm{span}(v_0 \ldots, \hat{v}_j, \ldots, v_p)) \in \mathrm{PBC}_0(V).$$

The "only if" portion of the claim follows from this description.

Conversely, let $\big((v_0, C_0), \ldots, (v_p, C_p)\big) \in (\mathrm{PBC}_0(V))^{p+1}$ such that $v_i \in C_j$ for $i \neq j$. We will prove that

$$(v_0, \ldots, v_p, C) \in \mathrm{PBC}_p(V), \qquad \text{where} \qquad C = \bigcap_{i=0}^{p} C_i .$$

This will imply the "if" portion of the claim as then

$$\partial(v_0, \ldots, v_p, C) = \big((v_0, C_0), \ldots, (v_p, C_p)\big).$$

Define

$$D_j = \bigcap_{i=0}^{j} C_i,$$

and we will prove by induction that

$$\mathrm{span}(v_0, \ldots, v_j) \oplus D_j = V.$$

The base case is the statement that $\mathrm{span}(v_0) \oplus C_0 = V$. Since $\mathrm{span}(v_0, \ldots, v_{j-1}) \subseteq C_j$ by assumption, Proposition 2.8 Part v) and the inductive hypothesis imply that

$$\mathrm{span}(v_0, \ldots, v_{j-1}) \oplus (D_{j-1} \cap C_j) = C_j.$$

By taking the direct sum of both sides of this equation with $\mathrm{span}(v_j)$, we conclude the inductive step. Applying this result when $j = p$ yields the desired decomposition

$$\mathrm{span}(v_0, \ldots, v_p) \oplus C = \mathrm{span}(v_0, \ldots, v_p) \oplus D_p = V.$$

This establishes the "if" portion of the claim.

Injectivity of ∂ follows from the fact that if

$$\partial(v_0, \ldots, v_p, C) = \big((v_0, C_0), \ldots, (v_p, C_p)\big),$$

then

$$C = \bigcap_{i=0}^{p} C_i. \qquad \qquad \square$$

It will be convenient for us to realize the complexes $\mathrm{PBC}_\bullet(V)$ and their links as special cases of the following more general construction. In the following, we will often identify an ordered partial basis $(v_0, \ldots, v_p)$ of V with a split R–linear monomorphism $f \colon R^{p+1} \to V$.

Definition 2.16. Let $U, W \subseteq V$ be splittable submodules. Define the sub-semisimplicial set $\mathrm{PBC}_\bullet(V, U, W) \subseteq \mathrm{PBC}_\bullet(V)$ by

$$\mathrm{PBC}_p(V, U, W) = \{(f, C) \in \mathrm{PBC}_p(V) \mid \mathrm{im}\, f \subseteq U, \ W \subseteq C\}.$$

Let $\mathrm{PBC}_\circ(V, U, W)$ be the unique simplicial complex with $\mathrm{PBC}_\circ^{\mathrm{ord}}(V, U, W) = \mathrm{PB}_\bullet(V, U, W)$

Note that
$$\mathrm{PBC}_\bullet(V) \cong \mathrm{PBC}_\bullet(V, V, 0).$$
Given a simplex $\sigma = (f, C) \in \mathrm{PBC}_p(V)$, we can identify its link
$$\mathrm{Lk}_\bullet(\sigma) = \mathrm{PBC}_\bullet(V, C, \mathrm{im}\, f).$$
This link is isomorphic to $\mathrm{PBC}_\bullet(C)$, which we will see is a special case of Lemma 2.17 below. In general $\mathrm{PBC}_\bullet(V, U, W)$ is *not* isomorphic to $\mathrm{PBC}_\bullet(U)$, but the following lemma gives a more general picture.

Lemma 2.17. *Let R be a PID, $U, W \subseteq V$ be splittable, and $A \oplus B = V$ such that $U \subseteq A$ and $B \subseteq W$. Then*
$$\mathrm{PBC}_\bullet(V, U, W) \xrightarrow{\ \cong\ } \mathrm{PBC}_\bullet(A, U, W \cap A)$$
$$(f, C) \longmapsto (f, C \cap A)$$
is an isomorphism. In particular, if $U \oplus W = V$, then $\mathrm{PBC}_\bullet(V, U, W) \cong \mathrm{PBC}_\bullet(U)$.

Proof. Let $(f, C) \in \mathrm{PBC}_\bullet(V, U, W)$. Proposition 2.8 Part v) implies
$$\mathrm{im}\, f \oplus (C \cap A) = A$$
and so we conclude that $(f, C \cap A) \in \mathrm{PBC}_\bullet(A, U, W \cap A)$. If $(g, D) \in \mathrm{PBC}_\bullet(A, U, W \cap A)$, then $(g, D \oplus B) \in \mathrm{PBC}_\bullet(V, U, W)$. These two maps are inverses because
$$(C \cap A) \oplus B = C$$
by Proposition 2.8 Part v), and by Proposition 2.8 Part vi),
$$(D \oplus B) \cap A = D. \qquad \qquad \square$$

The main theorem of this subsection is Theorem 2.19. To state this theorem, we will use Bass' stable range condition for rings.

Definition 2.18 (Bass [Bas64, Section 4 "Definition"] and [Bas68, Definition 3.1])**.** Let s be a positive integer. A ring R has *stable rank s* if s is the smallest positive integer m for which the following *Condition (B_m)* holds: whenever
$$a_0 R + a_1 R + \cdots + a_m R = R, \qquad a_i \in R,$$
there exist elements $x_1, x_2, \ldots, x_m \in R$ such that
$$(a_1 + a_0 x_1) R + \ldots + (a_m + a_0 x_m) R = R.$$

Note that the indexing in Definition 2.18 differs from Bass' convention in his work [Bas68]; he called it the stable range condition SR_{s+1}. Bass ([Bas64, Section 4 "Examples"] and [Bas68, Proposition 3.4 (a) and Theorem 3.5]) states that semi-local rings have stable rank 1, and Dedekind domains have stable rank at most 2. So, for example, R has stable rank 1 if R is a field and R has stable rank at most 2 when R is a PID. Since the next result concerns PIDs, the stable rank s must be equal to 1 or 2.

The remainder of the subsection serves to prove the following theorem.

Theorem 2.19. *Let R be a PID. Let s be the stable rank of R. If U and W are splittable submodules of the finite-rank free R–module V, then $\mathrm{PBC}_\bullet(V, U, W)$ is $\left\lfloor \dfrac{\mathrm{rk}\, U - \mathrm{rk}\, W - s - 2}{2} \right\rfloor$–connected.*

Theorem 2.19 is a partial generalization of the result of [RWW17, Lemma 5.10] of Randal-Williams–Wahl that if R has stable rank s then $\mathrm{PBC}_\bullet(V)$ is $\left\lfloor \dfrac{\mathrm{rk}\, V - s - 2}{2} \right\rfloor$–connected. Following their proof, we will prove Theorem 2.19 by comparing $\mathrm{PBC}_\circ(V, U, W)$ with its image in $\mathrm{PB}_\circ(V)$. This will let us use a result of van der Kallen [vdK80, Theorem 2.6 (i)–(ii)]: Assume R is a ring with stable rank s. Let U and W be splittable subspaces of V. Then $\mathrm{PB}_\bullet(U, W)$ is $(\mathrm{rk}\, U - \mathrm{rk}\, W - 1 - s)$–connected. In particular, $\mathrm{PB}_\bullet(U)$ is $(\mathrm{rk}\, U - 1 - s)$–connected.

Note that van der Kallen's result is stated in terms of the poset whose underlying set is

$$\bigsqcup_p \mathrm{PB}_p(U, W)$$

and whose order is induced by the face maps. The geometric realization of this poset is isomorphic to the barycentric subdivision of the geometric realization of $\mathrm{PB}_\bullet(U, W)$ (see for example Randal-Williams–Wahl [RWW17, Proof of Lemma 5.10]) and hence the connectivity results apply. We will use this same identification in the proof of Proposition 3.35.

Proposition 2.20. *Let R be a ring with stable rank s, and let U and W be splittable subspaces of the finite-rank free R–module V. The simplicial complex $\mathrm{PB}_\circ(U, W)$ is wCM of dimension $(\mathrm{rk}\, U - \mathrm{rk}\, W - s)$. In particular, $\mathrm{PB}_\circ(U)$ is wCM of dimension $(\mathrm{rk}\, U - s)$.*

Proof. By [vdK80, Theorem 2.6 (ii)] and Theorem 2.4 (a), the complex $\mathrm{PB}_\circ(U, W)$ is $(\mathrm{rk}\, U - \mathrm{rk}\, W - 1 - s)$–connected. The link of $\{v_0, \ldots, v_p\}$ in $\mathrm{PB}_\circ(U, W) = \mathrm{PB}_\circ(U) \cap \mathrm{Lk}_\circ(W)$ is isomorphic to

$$\mathrm{PB}_\circ(U, W \oplus \mathrm{span}(v_0, \ldots, v_p)) = \mathrm{PB}_\circ(U) \cap \mathrm{Lk}_\circ(W \oplus \mathrm{span}(v_0, \ldots, v_p))$$

and so the links are $(\mathrm{rk}\, U - \mathrm{rk}\, W - s - 2 - p)$–connected as required. $\qquad\square$

To prove $\mathrm{PBC}_\bullet(V, U, W)$ is highly connected, we will show that $\mathrm{PBC}_\circ(V, U, W)$ is wCM. To do this, we need the following lemma.

Lemma 2.21. *Let R be a PID and U, W, X, Y, Z be splittable submodules of a finite-rank free R–module V.*

  i) $\mathrm{PBC}_\bullet(V, X, Y) \cap \mathrm{PBC}_\bullet(V, Z, W) = \mathrm{PBC}_\bullet(V, X \cap Z, \mathrm{sat}(Y + W))$.

  ii) Any simplex $\sigma = (f, C) \in \mathrm{PBC}_\bullet(V, U, W)$ has link $\mathrm{Lk}_\bullet(\sigma) = \mathrm{PBC}_\bullet(C, U \cap C, W)$.

Proof.   i) Both sides of the equation describe the following semisimplicial set:

$$\{(f, C) \in \mathrm{PBC}_\bullet(V) \mid \mathrm{im}\, f \subseteq X, \mathrm{im}\, f \subseteq Z, Y \subseteq C, W \subseteq C\}$$
$$= \{(f, C) \in \mathrm{PBC}_\bullet(V) \mid \mathrm{im}\, f \subseteq X \cap Z, \mathrm{sat}(Y + W) \subseteq C\}$$

  ii) Every simplex in $\mathrm{PBC}_\bullet(V, U, W)$ contains every simplex in $\mathrm{PBC}_\bullet(V)$ spanned by vertices in $\mathrm{PBC}_\bullet(V, U, W)$ (in other words, the inclusion $\mathrm{PBC}_\bullet(V, U, W) \hookrightarrow \mathrm{PBC}_\bullet(V)$ is *full*). Hence, for $\sigma \in \mathrm{PBC}_\bullet(V)$,

$$\mathrm{Lk}_\bullet^{\mathrm{PBC}_\bullet(V, U, W)}(\sigma) = \mathrm{Lk}_\bullet^{\mathrm{PBC}_\bullet(V)}(\sigma) \cap \mathrm{PBC}_\bullet(V, U, W)$$
$$= \mathrm{PBC}_\bullet(V, C, \mathrm{im}\, f) \cap \mathrm{PBC}_\bullet(V, U, W)$$
$$= \mathrm{PBC}_\bullet(V, U \cap C, W \oplus \mathrm{im}\, f).$$

The last step uses Part i) and the observation that, since W is contained in the complement C of $\operatorname{im} f$,

$$\operatorname{sat}(W + \operatorname{im} f) = W \oplus \operatorname{im} f.$$

By applying Lemma 2.17 with $A = C$ and $B = \operatorname{im} f$, we find

$$\operatorname{PBC}_\bullet(V, U \cap C, W \oplus \operatorname{im} f) = \operatorname{PBC}_\bullet\Big(C, U \cap C, (W \oplus \operatorname{im} f) \cap C\Big).$$

Then Proposition 2.8 Part vi) implies that

$$(W \oplus \operatorname{im} f) \cap C = W$$

and the result follows. $\qquad\qquad\square$

Proposition 2.22. *If R is a PID of stable rank s, the complex $\operatorname{PBC}_\circ(V, U, W)$ is wCM of dimension* $\left\lfloor \dfrac{\operatorname{rk} U - \operatorname{rk} W - s}{2} \right\rfloor$.

Proof. We begin by checking that the map

$$\theta \colon \operatorname{PBC}_\circ(V, U, W) \longrightarrow \operatorname{PB}_\circ(U, W)$$

$$\{(v_0, C_0), \ldots, (v_p, C_p)\} \longmapsto \{v_0, \ldots, v_p\}$$

exhibits $\operatorname{PBC}_\circ(V, U, W)$ as a join complex over $\operatorname{PB}_\circ(U, W)$. Simplex-wise injectivity is clear. We next check surjectivity. Let $\{v_0, \ldots, v_p\}$ be a p–simplex in $\operatorname{PB}_\circ(U, W)$. Because $\{v_0, \ldots, v_p\} \in \operatorname{Lk}_\circ(W)$, there is a submodule of $D \subseteq V$ such that

$$D \oplus \operatorname{span}(v_0, \ldots, v_p) \oplus W = V.$$

Then the simplex $\{(v_0, C_0), \ldots, (v_p, C_p)\}$ with $C_i = \operatorname{span}(v_0, \ldots, \hat{v}_i, \ldots, v_p) \oplus D \oplus W$ in $\operatorname{PBC}_\circ(V, U, W)$ is a preimage of $\{v_0, \ldots, v_p\}$ under θ, and θ is surjective.

It remains to verify the third condition of Definition 2.5. Let $(v_0, C_0), \ldots, (v_p, C_p)$ be vertices of $\operatorname{PBC}_\circ(V, U, W)$ and let $\sigma_0, \ldots, \sigma_p$ be simplices of $\operatorname{PBC}_\circ(V, U, W)$ such that for each $i = 0, \ldots, p$, the vertex (v_i, C_i) is a vertex of σ_i, and $\theta(\sigma_i)$ has vertices $\theta(v_0, C_0), \ldots, \theta(v_p, C_p)$, that is, vertices $\{v_0, \ldots, v_p\}$. We wish to show that the vertices $(v_0, C_0), \ldots, (v_p, C_p)$ span a simplex in $\operatorname{PBC}_\circ(V, U, W)$.

By Proposition 2.15, it suffices to check that $v_i \in C_j$ for all $i \neq j$. By assumption, (v_i, D) for some $D \subseteq R^n$ and (v_j, C_j) are vertices of σ_j. So a second application of Proposition 2.15 implies that $v_i \in C_j$ as required.

We now prove the simplicial complex $\operatorname{PBC}_\circ(V, U, W)$ is $\left\lfloor \frac{\operatorname{rk} U - \operatorname{rk} W - s - 2}{2} \right\rfloor$–connected. This follows by applying Theorem 2.6 to the map θ. Since θ exihibits $\operatorname{PBC}_\circ(V, U, W)$ as a join complex over $\operatorname{PB}_\circ(U, W)$ and $\operatorname{PB}_\circ(U, W)$ is wCM of dimension $(\operatorname{rk} U - \operatorname{rk} W - s)$ by Proposition 2.20, it only remains to check that the images of links of p–simplices are wCM of dimension $(\operatorname{rk} U - \operatorname{rk} W - s - p - 2)$.

Let $\sigma = \{(v_0, C_0), \ldots, (v_p, C_p)\}$ be a p–simplex of $\operatorname{PBC}_\circ(V, U, W)$ and $C = \bigcap_{i=0}^p C_i$. Lemma 2.21 Part ii) implies that

$$\operatorname{Lk}_\circ(\sigma) = \operatorname{PBC}_\circ(C, U \cap C, W).$$

Note that

$$\theta(\operatorname{Lk}_\circ(\sigma)) = \operatorname{PB}_\circ(U \cap C, W) \subseteq \operatorname{PB}_\circ(U, W).$$

Since $\mathrm{rk}(U \cap C) \geq \mathrm{rk}\,U - p - 1$, Proposition 2.20 implies that $\theta(\mathrm{Lk}_\circ(\sigma))$ is wCM of dimension $(\mathrm{rk}\,U - \mathrm{rk}\,W - (p+1) - s)$, in particular wCM of dimension $(\mathrm{rk}\,U - \mathrm{rk}\,W - s - p - 2)$. We conclude that $\mathrm{PBC}_\circ(V, U, W)$ is wCM of dimension $\left\lfloor \frac{\mathrm{rk}\,U - \mathrm{rk}\,W - s}{2} \right\rfloor$. $\qquad\square$

Proposition 2.22 and Theorem 2.4 (b) together establish Theorem 2.19.

2.4. **Symplectic partial bases complexes.** In this subsection, we consider the symplectic group analogues of the complexes from the previous section. Let R be a PID. The high connectivity results discussed here will later be used to show that $\mathsf{SI}(R)$–modules satisfying a polynomial condition exhibit central stability.

Definition 2.23. Let W be a free finite-rank R–module equipped with a skew-symmetric bilinear form (not necessarily alternating or perfect). A *symplectic partial basis* of W is a set of pairs

$$\{(v_0, w_0), \ldots, (v_p, w_p)\} \subseteq W \times W$$

such that $\{v_0, w_0, \ldots, v_p, w_p\}$ is a partial basis of W with

$$\langle v_i, v_j \rangle = \langle w_i, w_j \rangle = 0 \quad \text{and} \quad \langle v_i, w_j \rangle = \delta_{ij}.$$

An *ordered symplectic partial basis* is a symplectic partial basis with a choice of bijection between $\{(v_0, w_0), \ldots, (v_p, w_p)\}$ and $\{0, \ldots, p\}$.

When W is a symplectic R–module (that is, the form is alternating and perfect), then any symplectic partial basis may be extended to a symplectic basis for W. For general W, such a basis may not exist.

Definition 2.24. Let W be a free finite-rank R–module equipped with a skew-symmetric bilinear form. We call a submodule H of W *symplectic* if the bilinear form restricts to a perfect alternating form on H. Define the *Witt index* $g(W)$ of W to be

$$g(W) = \sup \left\{ \tfrac{1}{2} \, \mathrm{rk}(H) \; \middle| \; \begin{array}{l} \text{there is a submodule } P \subseteq W \text{ and a symplectic} \\ \text{submodule } H \subseteq W \text{ such that } W \cong P \oplus H \end{array} \right\}.$$

.

Definition 2.25. Let W be a free finite-rank R–module equipped with a skew-symmetric bilinear form. For $p \geq 0$, let $\mathrm{SPB}_p(V)$ be the set of ordered symplectic partial bases of size $p + 1$. There are maps $d_i \colon \mathrm{SPB}_p(V) \to \mathrm{SPB}_{p-1}(V)$ given by forgetting the ith pair. With these maps, the sets assemble into a semisimplicial set $\mathrm{SPB}_\bullet(V)$.

In the case that W is a symplectic R–module, these semisimplicial sets (or the corresponding posets) have been studied, for example, by Charney [Cha87], Panin [Pan87], and Mirzaii–van der Kallen [MvdK02]. This generalization to arbitrary W is due to Friedrich [Fri17]. In [MvdK02, Fri17] more generally unitary groups are considered. In their notation, we are considering the case that the anti-involution is the identity, $\varepsilon = -1$, $\Lambda = R$, $\mu = 0$, and λ the symplectic form is

$$R^{2m} \times R^{2m} \longrightarrow R,$$

$$\langle x, y \rangle = \sum_{i=1}^{m} (x_{2i-1} y_{2i} - x_{2i} y_{2i-1}).$$

Mirzaii–van der Kallen [MvdK02, Definition 6.3] defined a broad concept of the *unitary stable rank* of a ring R, which specializes to apply to the study of symplectic, orthogonal, and unitary groups over R. Results of Mirzaii–van der Kallen [MvdK02, Remark 6.4] and Magurn–van der Kallen–Vaserstein [MVdKV88, Theorems 1.3 and 2.4] together imply that semi-local rings (including fields) have symplectic unitary stable rank 1, and PIDs have symplectic unitary stable rank at most 2. Most relevant to this paper are the integers $R = \mathbb{Z}$, which have symplectic unitary stable rank 2.

Let R be a PID of (symplectic) unitary stable rank s. Mirzaii and van der Kallen [MvdK02, Theorem 7.4] proved that if $V \cong R^{2n}$ is a symplectic R–module, then $\mathrm{SPB}_\bullet(V)$ is $\lfloor \frac{1}{2} (n - s - 3) \rfloor$–connected. Analogously to the case of partial bases discussed in the previous subsection, Mirzaii and van der Kallen considered a simplicial complex which is homeomorphic but not isomorphic to the one considered here. Friedrich [Fri17, Theorem 3.4] proved that for a general finite-rank free R–module W with a skew-symmetric bilinear form of Witt index $g(W)$, then $\mathrm{SPB}_\bullet(W)$ is $\lfloor \frac{1}{2} (g(W) - s - 3) \rfloor$–connected.

3. Modules over stability categories

In Section 3.1, we recall the definition of the categories $\mathsf{VIC}^H(R)$, $\mathsf{SI}(R)$, and FI. Our approach to these topics uses the formalism of stability categories. We review the notions of central stability homology, degree of presentation, and polynomial degree for modules over these categories in Sections 3.2, 3.3, and 3.4. The main result of Section 3 is that modules over $\mathsf{VIC}^H(R)$ and $\mathsf{SI}(R)$ with finite polynomial degree exhibit central stability, and we prove this result in Section 3.5. In Section 3.6, we describe a spectral sequence introduced by Putman–Sam [PS17], which we use in Section 4 to prove our representation stability results.

3.1. **Preliminaries.** In this subsection, we recall the definition of the category FI of Church–Ellenberg–Farb [CEF15] and the categories $\mathsf{VIC}^H(R)$ and $\mathsf{SI}(R)$ of Putman–Sam [PS17, Section 1.2]. We will view these constructions as (equivalent to) stability categories (see Definition 3.4), following the second author [Pat17, Section 3]. Stability categories are *homogeneous categories* in the sense of Randal-Williams–Wahl [RWW17, Definition 1.3] and *weakly complemented categories* in the sense of Putman–Sam [PS17, Section 1.3]. We will state their definition using a related concept, stability groupoids.

Definition 3.1. Let $(\mathcal{G}, \oplus, 0)$ be a monoidal groupoid whose monoid of objects is the natural numbers $\mathbb{N}_0$. The automorphism group of the object $n \in \mathbb{N}_0$ is denoted $G_n = \mathrm{Aut}^{\mathcal{G}}(n)$. Then $\mathcal{G}$ is called a *stability groupoid* if it satisfies the following properties.

i) The monoidal structure

$$\oplus \colon G_m \times G_n \longhookrightarrow G_{m+n}$$

is injective for all $m, n \in \mathbb{N}_0$.

ii) The group G_0 is trivial.

iii) $(G_{l+m} \times 1) \cap (1 \times G_{m+n}) = 1 \times G_m \times 1 \subseteq G_{l+m+n}$ for all $l, m, n \in \mathbb{N}_0$.

Definition 3.2. A *homomorphism* $\mathcal{G} \to \mathcal{H}$ of stability groupoids is a strict monoidal functor sending 1 to 1.

The construction used to define stability categories is originally due to Quillen but the notation and content of the following proposition is adopted from Randal-Williams–Wahl [RWW17, Section 1.1].

Proposition 3.3 (Randal-Williams–Wahl [RWW17, Remark 1.4, Proposition 1.8])**.** *Let $\mathcal{G}$ be a stability groupoid. If $\mathcal{G}$ is braided, there is a monoidal category $U\mathcal{G}$ on the same objects, such that*

$$\mathrm{Hom}^{U\mathcal{G}}(m, n) \cong G_n/(1 \times G_{n-m}).$$

as a G_n–set.

If a stability groupoid $\mathcal{G}$ is symmetric monoidal, then so is $U\mathcal{G}$.

Definition 3.4 (Patzt [Pat17, Definition 3.5])**.** Let $\mathcal{G}$ be a stability groupoid. If $\mathcal{G}$ is braided, then $U\mathcal{G}$ is called its *stability category*.

The following properties of stability categories are implicitly used throughout this section.

Proposition 3.5 (Randal-Williams–Wahl [RWW17, Proposition 1.7, 1.8(i)])**.** *Let $\mathcal{G}$ be a braided stability groupoid and $U\mathcal{G}$ its stability category, then:*

   i) $\mathcal{G}$ is the underlying groupoid of $U\mathcal{G}$.
   ii) The object $0 \in U\mathcal{G}$ is initial. We will denote the unique map from 0 to n by ι_n.

Example 3.6. The following are examples of stability groupoids $\mathcal{G}$ and associated categories equivalent to $U\mathcal{G}$.

   i) **Symmetric groups:** Let $\mathfrak{S}$ be the symmetric stability groupoid of symmetric groups. Then $U\mathfrak{S}$ is equivalent to the category FI of finite sets and injection of Church–Ellenberg–Farb [CEF15].

   ii) **General linear groups:** Let R be a ring. Let $\mathrm{GL}(R)$ be the symmetric stability groupoid of the general linear groups over R. Then $U\,\mathrm{GL}(R)$ is equivalent to the category $\mathsf{VIC}(R)$ of Djament [Dja12] and Putman–Sam [PS17]. The objects of $\mathsf{VIC}(R)$ are finite rank free R–modules and the morphisms from V to W are given by the set of pairs (f, C) of an injective homomorphism $f \colon V \to W$ and a free submodule $C \subseteq W$ such that $\mathrm{im}(f) \oplus C = W$ and $\mathrm{rk}(C) + \mathrm{rk}(V) = \mathrm{rk}(W)$. The composition law is defined by

$$(f, C) \circ (g, D) = (f \circ g, C \oplus f(D)).$$

   A $\mathsf{VIC}(R)$–*module* is a functor from $\mathsf{VIC}(R)$ to the category of abelian groups.

   iii) **General linear groups with restricted determinant:** Let R be a commutative ring and H a subgroup of the group of units $R^\times$. Let $\mathrm{GL}^H(R)$ denote the stability groupoid of the subgroups

$$\mathrm{GL}_n^H(R) = \{A \in \mathrm{GL}_n(R) \mid \det A \in H\}.$$

   $\mathrm{GL}^H(R)$ is symmetric monoidal if $-1 \in H$. Note that determinant is well defined since the ring is commutative. Its stability category $U\,\mathrm{GL}^H(R)$ is equivalent to the category $\mathsf{VIC}^H(R)$, defined by Putman–Sam [PS17, Section 1.2]. Its objects are finite-rank free R–modules V with a choice of generator of $a \in \left(\bigwedge^{\mathrm{rk}(V)} V\right)/H$. The morphisms $\mathrm{Hom}^{\mathsf{VIC}^H(R)}((V, a), (W, b))$ are split injective linear maps $f : V \to W$ and a choice of complement of the image of f with the added requirement that $f_*(a) = b$ if f is an isomorphism. Here $f_* : \left(\bigwedge^{\mathrm{rk}(V)} V\right)/H \to \left(\bigwedge^{\mathrm{rk}(W)} W\right)/H$ is the induced map. In particular,

$$\mathrm{Hom}^{\mathsf{VIC}^H(R)}((V, a), (W, b)) \cong \mathrm{Hom}^{\mathsf{VIC}(R)}(V, W), \qquad \text{for } \mathrm{rk}(V) \neq \mathrm{rk}(W), \text{ and}$$

$$\mathrm{Hom}^{\mathsf{VIC}^H(R)}((V, a), (W, b)) \cong \mathrm{GL}_n^H(R) \qquad \text{if } V \cong W \cong R^n.$$

iv) **Symplectic groups:** Let R be a ring. Let $\mathrm{Sp}(R)$ be the symmetric stability groupoid of the symplectic groups over R. Then $U\,\mathrm{Sp}(R)$ is equivalent to the category $\mathsf{SI}(R)$ of free finite-rank symplectic R–modules and isometric embeddings. Details are given in Putman–Sam [PS17, Section 1.2].

v) **Automorphisms of free groups:** Let $\mathsf{Aut\,F}$ be the symmetric stability groupoid of the automorphism groups of free groups of finite rank. Then $U\mathsf{Aut\,F}$ is equivalent to the category of finite-rank free groups and monomorphisms together with a choice of free complement, that is,

$$\mathrm{Hom}(F,G) = \left\{ (f,C) \,\middle|\, \begin{array}{l} f \colon F \longhookrightarrow G \text{ an injective group homomorphism,} \\ C \subseteq G \text{ a free subgroup with } (\mathrm{im}\, f) * C = G \end{array} \right\}.$$

See Randal-Williams–Wahl [RWW17, Section 5.2.1].

vi) **Mapping class groups of compact oriented surfaces with one boundary component:** Let Mod be the braided stability groupoid of mapping class groups of compact oriented surfaces with one boundary component. Its monoidal structure is induced by boundary connect sum. See Randal-Williams–Wahl [RWW17, Section 5.6.3].

Definition 3.7. We call functors from a category $\mathcal{C}$ to the category of abelian groups $\mathcal{C}$–*modules* and denote the category of $\mathcal{C}$–modules by $\mathrm{Mod}_{\mathcal{C}}$. If $\mathcal{C}$ is a stability category or stability groupoid and A is a $\mathcal{C}$–module, we let A_n denote the value of A on the object $n \in \mathbb{N}_0$.

If a category $\mathcal{C}$ is equivalent to the category $U\mathcal{G}$, then the categories of $\mathcal{C}$–modules and $U\mathcal{G}$–modules are equivalent. We therefore use the terms of $U\mathfrak{S}$–module and FI–module roughly interchangeably, and similarly for other items in Example 3.6.

If $\mathcal{G}$ is a stability groupoid, then the data of $\mathcal{G}$–module is precisely the data of a $\mathbb{Z}[G_n]$–module for every n. For m fixed, we will often view a $\mathbb{Z}[G_m]$–module W as a $\mathcal{G}$–module by putting W in degree m and the module 0 in all other degrees. We now define free $U\mathcal{G}$–modules.

Definition 3.8. Let $\mathcal{G}$ be a stability groupoid and let $M : \mathrm{Mod}_{\mathcal{G}} \to \mathrm{Mod}_{U\mathcal{G}}$ be the left adjoint to the forgetful functor $\mathrm{Mod}_{U\mathcal{G}} \to \mathrm{Mod}_{\mathcal{G}}$. We say that $M(W)$ is the the *free $U\mathcal{G}$–module* on W. Concretely, given a $\mathbb{Z}[G_m]$–module W, the $\mathbb{Z}[G_n]$–module $M(W)_n$ is given by the formula

$$M(W)_n \cong \mathbb{Z}[\mathrm{Hom}^{U\mathcal{G}}(m,n)] \otimes_{\mathbb{Z}[G_m]} W.$$

For a general $\mathcal{G}$–module W,

$$M(W) = \bigoplus_{n=0}^{\infty} M(W_n).$$

We abbreviate $M(\mathbb{Z}[G_m])$ by $M(m)$.

3.2. **Central stability homology and resolutions.** We begin by defining degree of generation and presentation and discuss how these concepts relate to central stability degree. We then review central stability homology and how it relates to the degrees of higher syzygies of $U\mathcal{G}$–modules.

Definition 3.9. Let $\mathcal{G}$ be a stability groupoid. We say a $\mathcal{G}$–module W has *degree* $\leq d$ if $W_n \cong 0$ for $n > d$. A $U\mathcal{G}$–module A is *generated in degrees* $\leq d$ if there is an exact sequence of $U\mathcal{G}$–modules

$$M(W^0) \to A \to 0$$

with W^0 of degree $\leq d$. A $U\mathcal{G}$–module A is *related in degrees $\leq d$* if there is an exact sequence of $U\mathcal{G}$–modules

$$M(W^1) \to M(W^0) \to A \to 0$$

with W^1 of degree $\leq d$. We say A is *presented in degrees $\leq d$* if it is generated and related in degrees $\leq d$.

Definition 3.10. Let $\mathcal{G}$ be a braided stability groupoid with braiding b. Let A be a $U\mathcal{G}$–module, and let $\phi_{n-2}\colon A_{n-2} \to A_{n-1}$ denote the map induced by the morphism

$$\mathrm{id}_{n-2} \oplus \iota_1\colon (n-2) \oplus 0 \longrightarrow (n-2) \oplus 1$$

that is the sum of the identity map of $n-2$ and the unique morphism $0 \to 1$. We consider two maps

$$\mathrm{Ind}_{G_{n-2}\times 1}^{G_n} A_{n-2} \rightrightarrows \mathrm{Ind}_{G_{n-1}\times 1}^{G_n} A_{n-1}.$$

The first is induced by ϕ_{n-2} and the second is the map induced by ϕ_{n-2} postcomposed with $\mathrm{id}_{n-2}\oplus(b_{1,1})$. The $U\mathcal{G}$–module A has *central stability degree $\leq d$* if the map

$$\mathrm{coeq}(\mathrm{Ind}_{G_{n-2}}^{G_n} A_{n-2} \rightrightarrows \mathrm{Ind}_{G_{n-1}}^{G_n} A_{n-1}) \longrightarrow A_n$$

induced by ϕ_{n-1} is an isomorphism for all $n > d$.

Remark 3.11. The definition of central stability used by Putman [Put15, see paragraph "Central stability, definition"], the one used by Putman–Sam [PS17, see paragraph "Asymptotic structure"] and the one we use are all slightly different. These concepts turn out to coincide for the symmetric groups [PS17, Remark 1.2 & Theorem F]. The work of Djament [Dja16b, Proposition 2.14] implies that Putman–Sam's definition agrees with the definition of degree of presentation used in this paper; see also Gan–Li [GL17b, Theorem 3.2]. It is unclear how to generalize Putman's definitions to braided stability categories, but there is a straightforward generalization to symmetric stability categories, which is equivalent to our definition. See [Pat17, Proposition 6.1 and 6.2] for a comparison of these different forms of stability.

We now compare the notion of central stability degree to the notion of presentation degree. The following theorem will allow us to translate results on presentation degree quoted from other papers into statements about central stability. This theorem also allows us to translate the main theorems of this paper into results about presentation degree.

Theorem 3.12. *Let A be a module over FI, a module over $\mathsf{VIC}^H(R)$ for a ring R of finite stable rank and a subgroup $H \leq R^\times$ containing -1, or a module over $\mathsf{SI}(R)$ for a ring R of finite (symplectic) unitary stable rank. Then A is presented in finite degrees if and only if it has finite central stability degree. Specifically, we have the following bounds.*

- FI (Patzt [Pat17, Corollary 6.4(a)]; see also Church–Ellenberg [CE17, Proposition 4.2])**:** *For $d \geq 0$, let A be an FI–module generated in degrees $\leq d$. Then A is presented in degrees $\leq (d+1)$ if and only if A has central stability degree $\leq (d+1)$.*

- $\mathsf{VIC}^H(R)$ (Patzt [Pat17, Corollary 6.4(b)])**:** *Suppose R is a ring of stable rank s, $-1 \in H \leq R^\times$, and let A be a $\mathsf{VIC}^H(R)$–module generated in degrees $\leq g$. If A has central stability degree $\leq d$, then it is presented in degrees $\leq \max\left(d, g+s+1\right)$. If A is related in degrees $\leq d$, then it has central stability degree $\leq \max\left(d, g+s+1\right)$.*

$\mathsf{SI}(R)$ (Patzt [Pat17, Corollary 6.4(c)]): *Suppose R is a ring of (symplectic) unitary stable rank s and let A be a $\mathsf{SI}(R)$–module generated in degrees $\leq g$. If A has central stability degree $\leq d$, then it is presented in degrees $\leq \max\big(d, g+s+2\big)$. If A is related in degrees $\leq d$, then it has central stability degree $\leq \max\big(d, g+s+2\big)$.*

We note that for $H \neq R^{\times}$, we implicitly assume R is commutative when considering $\mathsf{VIC}^{H}(R)$ as we have not defined $\mathsf{VIC}^{H}(R)$ for noncommutative rings, only $\mathsf{VIC}(R)$.

Observe that the central stability degree of a $U\mathcal{G}$–module is always at least its degree of generation. Thus the central stability degree alone gives a bound on degree of presentation, though this bound may be improved with the data of both central stability degree and the degree of generation.

In fact, the following lemma states an easy equivalence that has been pointed out in many papers, for example [Pat17, Proposistion 5.4].

Lemma 3.13. *A $U\mathcal{G}$–module A is generated in degrees $\leq d$ if and only if*

$$\mathrm{Ind}_{G_{n-1} \times 1}^{G_n} A_{n-1} \longrightarrow A_n$$

is surjective for all $n > d$.

Given a $U\mathcal{G}$–module A, central stability can be rephrased as acyclicity of a chain complex

$$\mathrm{Ind}_{G_{n-2}}^{G_n} A_{n-2} \longrightarrow \mathrm{Ind}_{G_{n-1}}^{G_n} A_{n-1} \longrightarrow A_n \longrightarrow 0.$$

This chain complex is the tail of a longer chain complex:

Definition 3.14 (Compare [Pat17, Definition 2.4, Proposition 4.3]). Let A be a $U\mathcal{G}$–module, and let $\phi_n \colon A_n \to A_{n+1}$ denote the map induced by the morphism

$$\mathrm{id}_n \oplus \iota_1 \colon n \oplus 0 \longrightarrow n \oplus 1.$$

Define $\widetilde{C}_{\bullet}^{\mathcal{G}}(A)$ to be the augmented semisimplicial $U\mathcal{G}$–module with p–simplices given by

$$\widetilde{C}_{p}^{\mathcal{G}}(A)_n = \mathbb{Z}[G_n] \otimes_{\mathbb{Z}[G_{n-(p+1)} \times 1]} A_{n-(p+1)}$$

and the ith face map by

$$d_i \colon \mathbb{Z}[G_n] \otimes_{\mathbb{Z}[G_{n-(p+1)} \times 1]} A_{n-(p+1)} \longrightarrow \mathbb{Z}[G_n] \otimes_{\mathbb{Z}[G_{n-p} \times 1]} A_{n-p}$$
$$g \otimes a \longmapsto g \cdot (\mathrm{id}_{n-p} \oplus h_i) \otimes \phi_{n-p-1}(a)$$

where the coset $h_i G_1$ corresponds to the morphism

$$\mathrm{id}_i \oplus \iota_1 \oplus \mathrm{id}_{p-i} \colon i \oplus 0 \oplus (p-i) \longrightarrow i \oplus 1 \oplus (p-i) \quad \text{in } \mathrm{Hom}^{U\mathcal{G}}(p, p+1) \cong G_{p+1}/G_{(p+1)-p}.$$

Let $\widetilde{C}_{*}^{\mathcal{G}}(A)$ denote the chain complex associated to $\widetilde{C}_{\bullet}^{\mathcal{G}}(A)$; we will refer to it as the *central stability complex* of A. Let $\widetilde{H}_{*}^{\mathcal{G}}(A)$ denote the homology of this chain complex; we will call it the *central stability homology* of A.

The chain complex $\widetilde{C}_{*}^{\mathcal{G}}(A)$ and groups $H_{i}^{\mathcal{G}}(A)$ have an $\mathbb{N}_0$-grading coming from the fact that the set of objects of a stability groupoid is $\mathbb{N}_0$. We denote the piece in grading n by $\widetilde{C}_{*}^{\mathcal{G}}(A)_n$ or $\widetilde{H}_{i}^{\mathcal{G}}(A)_n$ respectively. We will drop the superscript $\mathcal{G}$ when the groupoid is clear from context.

Definition 3.15. We say that a $U\mathcal{G}$–module A has *higher central stability* if $\widetilde{H}_i(A)_n \cong 0$ for n sufficiently large compared to i.

Remark 3.16. Note that in particular $\widetilde{H}_{-1}(A)_n = 0$ and $\widetilde{H}_0(A)_n = 0$ is equivalent to the surjectivity and the injectivity, respectively, of the last map in Definition 3.10.

The central stability complex has appeared in the literature in varying degrees of generality. In the FI case, it is closely related to a complex introduced in [Put15, Section 4] whose homology is called FI-*homology* by Church-Ellenberg [CE17]. FI–homology has appeared in work of Church, Ellenberg, Farb, and Nagpal [CEF15, CEFN14, CE17] and Gan and Li [GL17b, GL16, Gan16, Li16]. For the category FI, the complex $\widetilde{C}_*(A)$ itself was denoted by $B_{*+1}(\mathcal{A})$ by Church–Ellenberg–Farb–Nagpal [CEFN14, Definition 2.16], by $\widetilde{C}_{*+1}^{F_A}$ by Church–Ellenberg [CE17, Section 5.1], and by $\mathrm{Inj}_*(A)$ by the first and third author [MW16, Section 2.2]. Putman–Sam [PS17, Section 3] defined the complex for modules A over general "cyclically generated" *complemented categories* and used the notation $\Sigma_{*+1}(A)$. In this paper, we adopt notation used by the second author [Pat17].

When $A = M(0)$, the central stability homology is the reduced homology of the following augmented semisimplicial set that was previously studied by Randal-Williams–Wahl [RWW17, Definition 2.2, Theorem 3.1, 4.20].

Definition 3.17. Let $K_\bullet\mathcal{G}$ be the semisimplicial $U\mathcal{G}$–set with set of p–simplices given by

$$(K_p\mathcal{G})_n = \mathrm{Hom}^{U\mathcal{G}}(p+1, n) \cong G_n/G_{n-(p+1)}$$

and the ith face map is induced by precomposition with the morphism

$$\mathrm{id}_i \oplus \iota_1 \oplus \mathrm{id}_{p-i} \colon i \oplus 0 \oplus (p-i) \longrightarrow i \oplus 1 \oplus (p-i).$$

By definition, the reduced chain complex $\widetilde{C}_*((K_\bullet\mathcal{G})_n)$ agrees with the central stability complex of $M(0)$. Here the term reduced chain complex of a semisimplicial set means the chain complex formed by first augmenting the semisimplicial set by inserting a point in degree -1, then composing with the free abelian group functor to obtain an augmented semisimplicial abelian group, and then forming a chain complex by taking the alternating sum of face maps as the differential.

The semisimplicial sets $(K_\bullet\mathcal{G})_n$ were first introduced by Randal-Williams–Wahl [RWW17] and are examples of a larger class of semisimplicial sets which they denote by $W_n(A, X)$. They describe these semisimplicial sets explicitly in many examples, often showing they are isomorphic to complexes previously considered in the literature. In particular, the semisimplicial set $(K_\bullet \mathrm{GL}(R))_n$ is isomorphic to $\mathrm{PBC}_\bullet(R^n)$ [RWW17, Section 5.3], $(K_\bullet \mathrm{Sp}(R))_n$ is isomorphic to $\mathrm{SPB}_\bullet(R^n)$ [RWW17, Section 5.4], and $(K_\bullet\mathfrak{S})_n$ is isomorphic to the *complex of injective words* [RWW17, Section 5.1], a complex introduced by Farmer [Far79]. Randal-Williams–Wahl [RWW17] proved that high connectivity of $(K_\bullet\mathcal{G})_n$ implies homological stability for the groups G_n. The name **H3** is related to their [RWW17, Definition 2.2].

Definition 3.18. Let $a, k \in \mathbb{N}_0$. Define the following condition for a stability category $U\mathcal{G}$.

H3(k,a): $\widetilde{H}_i(M(0))_n = 0$ for all $i \geq -1$ and all $n > k \cdot i + a$.

Remark 3.19. For all stability categories $U\mathcal{G}$, $\widetilde{H}_{-1}(M(0))_n = 0$ for all $n > 0$.

In the following proposition, we compile information from the literature about this condition for the stability categories appearing in Example 3.6.

Proposition 3.20. *These stability categories satisfy the condition* **H3***.*

- *i)* $U\mathfrak{S}$ *satisfies* **H3**$(1, 1)$.
- *ii)* $U\mathsf{Aut}\,\mathsf{F}$ *and* $U\,\mathrm{Mod}$ *satisfy* **H3**$(2, 2)$.
- *iii)* *If* R *is a ring with stable rank* s, *then* $U\,\mathrm{GL}(R)$ *and satisfies* **H3**$(2, s{+}1)$. *If* R *is a commutative ring with stable rank* s, *then* $U\,\mathrm{GL}^H(R)$ *and satisfies* **H3**$(2, s+1)$.
- *iv)* *If* R *is a ring with (symplectic) unitary stable rank* s, *then* $U\,\mathrm{Sp}(R)$ *satisfies* **H3**$(2, s+2)$.

Proof. Since $\widetilde{H}_i((K_\bullet\mathcal{G})_n) \cong \widetilde{H}_i(M(0))_n$, it suffices to show $(K_\bullet\mathcal{G})_n$ is highly connected. With the exception of $U\,\mathrm{GL}^H(R)$, these connectivity results appear in Randal-Williams–Wahl [RWW17].

- i) Randal-Williams–Wahl [RWW17, Section 5.1] proved that $(K_\bullet\mathfrak{S})_n$ is isomorphic to complex of injective words which which was shown to be $(n-2)$–connected by Farmer [Far79, Theorem 5].
- ii) Hatcher–Vogtmann [HV98, Proposition 6.4] showed a space closely related to $K_\bullet U\mathsf{Aut}\,\mathsf{F}$ is highly connected. Similarly, Hatcher–Vogtmann [HV17, Main Theorem] proved that a space closely related to $(K_\bullet U\,\mathrm{Mod})_n$ is highly connected. Using these results, Randal-Williams–Wahl [RWW17, Proposition 5.3 and Lemma 5.25] proved that $(K_\bullet\mathsf{Aut}\,\mathsf{F})_n$ and $(K_\bullet\,\mathrm{Mod})_n$ are $\left\lfloor\frac{1}{2}(n-3)\right\rfloor$–connected.
- iii) Randal-Williams–Wahl [RWW17, Lemma 5.10] proved that $(K_\bullet\,\mathrm{GL}(R))_n$ is $\left\lfloor\frac{1}{2}(n-s-2)\right\rfloor$–connected. This builds on the work of van der Kallen [vdK80, Theorem 2.6]. Also see Charney [Cha84, Theorem 3.5].
- iv) From the description of $(K_\bullet\,\mathrm{Sp}(R))_n$ in [RWW17, Section 5.4], it follows that it is isomorphic to the space appearing in Mirzaii–van der Kallen [MvdK02, Theorem 7.4] and hence is $\left\lfloor\frac{1}{2}(n-s-3)\right\rfloor$–connected.

We now consider the case $\mathcal{G} = \mathrm{GL}^H(R)$. Because

$$\mathrm{Hom}^{U\,\mathrm{GL}^H(R)}(m,n) = \mathrm{Hom}^{U\,\mathrm{GL}(R)}(m,n) \qquad \text{for } m < n,$$

it follows that

$$\widetilde{C}_p^{\mathrm{GL}^H(R)}(M(0))_n = \widetilde{C}_p^{\mathrm{GL}(R)}(M(0))_n \qquad \text{for } p \leq n-2.$$

We conclude that the map

$$\widetilde{C}_*^{\mathrm{GL}^H(R)}(M(0))_n \hookrightarrow \widetilde{C}_*^{\mathrm{GL}(R)}(M(0))_n$$

induces an isomorphism on homology groups for $* \leq (n-3)$. Thus for $i \geq 0$,

$$\widetilde{H}_i^{\mathrm{GL}^H(R)}(M(0))_n = \widetilde{H}_i^{\mathrm{GL}(R)}(M(0))_n = 0 \qquad \text{when } n > 2i + s + 1$$

since $U\,\mathrm{GL}(R)$ satisfies **H3**$(2, s+1)$ and necessarily $(n-3) \geq i$ in this range. Moreover, as in Remark 3.19,

$$\widetilde{H}_{-1}^{\mathrm{GL}^H(R)}(M(0))_n = 0 \qquad \text{for } n > 0$$

and we conclude **H3**$(2, s+1)$ for $U\,\mathrm{GL}^H$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The following theorem generalizes Theorem 3.12 to general stability categories and higher central stability homology groups.

Theorem 3.21 (Patzt [Pat17, Theorem 5.7])**.** *Assume* **H3**(k,a). *Let* A *be a* $U\mathcal{G}$–*module and* $\{d_n\}_{n\in\mathbb{N}_0}$ *a sequence of integers with* $d_{i+1} - d_i \geq \max(k,a)$, *then the following statements are equivalent.*

 i) *There is a resolution*

$$\cdots \to M(W^1) \to M(W^0) \to A \to 0$$

 with W^i a $\mathcal{G}$–module of degree $\leq d_i$.

 ii) *The homology groups*

$$\widetilde{H}_i(A)_n \cong 0 \qquad \text{for all } i \geq -1 \text{ and all } n > d_{i+1}.$$

3.3. **Polynomial degree.** To prove that the second homology of Torelli subgroups are centrally stable, will first prove that the first homology group of these Torelli groups exhibit higher central stability. To do this, we establish higher central stability for *polynomial* $\mathsf{VIC}(\mathbb{Z})$– and $\mathsf{SI}(\mathbb{Z})$–modules. We can apply these results to the Torelli groups because their first homology groups are known to be polynomial. There are several notions of polynomial functors—as mentioned in the introduction. We use the same definition as in the arXiv version [RWW15, Definition 4.10]; in the published version the definition was slightly changed.

Definition 3.22. Let $\mathcal{G}$ be a stability groupoid. Define the endofunctor

$$S \colon U\mathcal{G} \to U\mathcal{G}$$

$$\text{via the formula} \quad S = 1 \oplus -.$$

We will consider the natural transformation $\mathrm{id} \to S$ given by

$$\iota_1 \oplus \mathrm{id}_n : 0 \oplus n \longrightarrow 1 \oplus n.$$

By abuse of notation, we denote the endofunctor of $U\mathcal{G}$–modules given by precomposition by S also by S. Again, there is an induced natural transformation $\mathrm{id} \to S$ defined by precomposition with the above natural transformation.

Concretely, if $\mathcal{G}$ is braided and A is a module over its stability category $U\mathcal{G}$, then there are isomorphisms of G_n–representations

$$(SA)_n \cong \mathrm{Res}^{G_{n+1}}_{1 \times G_n} A_{n+1} \cong \mathrm{Res}^{G_{n+1}}_{G_n \times 1} A_{n+1}.$$

This last isomorphism uses the braiding.

Definition 3.23. Given a $U\mathcal{G}$–module A, define $U\mathcal{G}$–modules

$$\ker A := \ker(A \to SA) \qquad \text{and}$$

$$\mathrm{coker}\, A := \mathrm{coker}(A \to SA).$$

Definition 3.24. We say that A has *polynomial degree* $-\infty$ *in ranks* $> d$ if $A_n = 0$ for all $n > d$. We say A has *polynomial degree* ≤ 0 *in ranks* $> d$ if $\ker A_n = 0$ for all $n > d$ and $\mathrm{coker}\, A$ has polynomial degree $-\infty$ in ranks $> d$. For $r \geq 1$, we say A has *polynomial degree* $\leq r$ *in ranks* $> d$ if $\ker A_n = 0$ for all $n > d$ and $\mathrm{coker}\, A$ has polynomial degree $\leq r - 1$ in ranks $> d$.

We say A has *polynomial degree* $\leq r$ if it has polynomial degree $\leq r$ in all ranks > -1.

Polynomial modules have higher central stability when the category satisfies the following condition.

Definition 3.25. Let $b, \ell \geq 1$ be natural numbers. Define the following condition on a stability category $U\mathcal{G}$:

H4(ℓ,b): $\widetilde{H}_i(\operatorname{coker} M(m))_n = 0$ for all $m \geq 0$, all $i \geq -1$, and all $n > \ell \cdot (i + m) + b$.

Theorem 3.26 (Patzt [Pat17, Corollary 7.9]). *Let $a, b, k, \ell \geq 1$. Let $\mathcal{G}$ be a braided stability groupoid, and assume its stability category $U\mathcal{G}$ satisfies **H3(k,a)** and **H4(ℓ,b)** with $b \geq \max(k, a)$. If A is a $U\mathcal{G}$–module of polynomial degree ≤ 0 in ranks $> d$ for some $d \geq -1$, then*

$$\widetilde{H}_i(A)_n = 0 \qquad \text{for all} \quad i \geq -1 \quad \text{and} \quad n > \max(d + i + 2, ki + a).$$

If A is a $U\mathcal{G}$–module of polynomial degree $\leq r$ in ranks $> d$ for some $r \geq 1$ and $d \geq -1$, then

$$\widetilde{H}_i(A)_n = 0 \qquad \text{for all} \quad i \geq -1 \quad \text{and} \quad n > \ell^{i+1}(d + r) + (\ell^i + \ell^{i-1} + \cdots + \ell + 1)b + 1.$$

Note that for $i = -1$, the above sum is an empty sum and hence zero.

Proof of Theorem 3.26. [Pat17, Corollary 7.9] only considers the case $r \geq 1$. But the cases $r \leq 0$ are easy to see. First note that if $A_n = 0$ for all $n > d$, also $\widetilde{C}_i(A)_n \cong \operatorname{Ind}_{G_{n-i-1}}^{G_n} A_{n-i-1} = 0$ for all $n > i + 1 + d$ and thus $\widetilde{H}_i(A)_n = 0$ for all $n > i + 1 + d$.

Now let us assume A has polynomial degree ≤ 0 in ranks $> d$. From the definition of polynomial degree, it follows that for $n > d$, the maps $A_n \to (SA)_n$ are isomorphisms. Thus there is a short exact sequence of $U\mathcal{G}$–modules

$$0 \longrightarrow B \longrightarrow A \longrightarrow C \longrightarrow 0$$

such that $B_n = 0$ for all $n \leq d$, $B_n \to (SB)_n$ is an isomorphism for all $n > d$, and $C_n = 0$ for all $n > d$. Here B is obtained from A by replacing the groups in ranks $\leq d$ with 0 and C is the cokernel of the natural map $B \to A$.

We will show moreover that, for $n > d$, the group G_n acts trivially on A_n. Fix $n > d$. Consider the map $\rho : A_n \to A_{2n}$ induced by the morphism $\operatorname{id}_n \oplus \iota_n : (n \oplus 0) \to (n \oplus n)$. The subgroup $(1 \times G_n) \subseteq G_{2n}$ acts trivially on $\rho(A_n) = A_{2n}$ by functoriality. Now, the category $\mathcal{G}$ is braided, and so we have natural maps from the braid group on $2n$ strands to G_{2n}. Then the subgroup $(1 \times G_n) \subseteq G_{2n}$ is conjugate to $(G_n \times 1)$ via the image of a braid that wraps the first n strands past the second n strands. It follows that $(G_n \times 1)$ also acts trivially on A_{2n}. Since $A_n \cong \operatorname{Res}_{G_n \times 1}^{G_{2n}} A_{2n}$ as G_n–representations, we conclude that A_n has trivial G_n–action as claimed.

It follows that G_n also acts trivially on $B_n \cong A_n$ for $n > d$, and so we obtain another short exact sequence of $U\mathcal{G}$–modules

$$0 \longrightarrow B \longrightarrow M(B_{d+1}) \longrightarrow D \longrightarrow 0,$$

where B_{d+1} denotes the $\mathcal{G}$–module that is B_{d+1} in degree 0 and zero in all degrees > 0, and D is defined as the cokernel of $B \to M(B_{d+1})$. Note that $D_n = 0$ for all $n > d$. Condition **H3(k,a)** implies that $\widetilde{H}_i(M(B_{d+1}))_n = 0$ for all $n > ki + a$. The first paragraph implies that $\widetilde{H}_i(C)_n = \widetilde{H}_i(D)_n = 0$ for all $n > i + 1 + d$. Observe that $\widetilde{C}_i$ is functorial with respect to maps of $U\mathcal{G}$–modules, and it is an exact functor. Thus short exact sequences of $U\mathcal{G}$-modules give short exact sequences of central stability chains and hence a long exact sequences of central stability homology groups. Using the long exact sequences on homology associated to these two short exact sequences, we conclude that $\widetilde{H}_i(A)_n = 0$ for all $n > \max(d + i + 2, ki + a)$. $\qquad\square$

By specializing Theorem 3.26 to homological degrees $i = -1, 0$ and invoking Theorem 3.21, we obtain the following consequences for functors of finite polynomial degree.

Corollary 3.27. *Let $a, b, k, \ell \geq 1$. Let $\mathcal{G}$ be a braided stability groupoid, and assume its stability category $U\mathcal{G}$ satisfies $\mathbf{H3}(k,a)$ and $\mathbf{H4}(\ell,b)$ with $b \geq \max(k, a)$. Suppose A is a $U\mathcal{G}$–module of polynomial degree ≤ 0 in ranks $> d$ for some $d \geq -1$.*

    i) Then A has central stability degree $\leq \max(a, d + 2)$.

    ii) Then A is generated in degrees $\leq \max(a-k, d+1)$ and related in degrees $\leq \max(a-k+b, d+1+b)$.

Suppose A is a $U\mathcal{G}$–module of polynomial degree $\leq r$ in ranks $> d$ for some $r \geq 1$ and $d \geq -1$.

    i) Then A has central stability degree $\leq \ell(d + r) + b + 1$.

    ii) Then A is generated in degrees $\leq d + r + 1$ and A is related in degrees $\leq \ell(d + r) + b + 1$.

3.4. **FI–modules of finite polynomial degree.** It is easy to see that $U\mathfrak{S}$ is a subcategory of every symmetric stability category, in particular of $U\,\mathrm{GL}(R)$, $U\,\mathrm{Sp}(R)$, and $U\,\mathrm{GL}^H(R)$ for any H containing -1. Note that the definition of polynomial degree never mentions the groups G_n and only involves vanishing of kernels and cokernels of shift functors. In particular, it follows from the definition of polynomial degree that the polynomial degree of a $U\mathcal{G}$–module coincides with the polynomial degree of the underlying FI–module.

We now compute the polynomial degree of free FI–modules.

Proposition 3.28. *Let W be a $\mathbb{Z}[\mathfrak{S}_m]$–module, then $M(W)$ has polynomial degree $\leq m$.*

Proof. We prove the assertion by induction over m. If $m = 0$, then

$$M(W) = M(0) \otimes_{\mathbb{Z}} W$$

and the map $M(W) \to SM(W)$ is an isomorphism. This implies that $M(W)$ has polynomial degree ≤ 0. Now suppose $m > 0$. Church–Ellenberg [CE17, Lemma 4.4] showed that

$$\operatorname{coker} M(W) = M(\operatorname{Res}^{\mathfrak{S}_m}_{\mathfrak{S}_{m-1}} W).$$

Thus $\operatorname{coker} M(W)$ has polynomial degree $\leq m - 1$ by induction. The proof of [CE17, Lemma 4.4] also shows that $\ker M(W) = 0$. This completes the induction. $\qquad\square$

Proposition 3.29. *$U\mathfrak{S}$ satisfies $\mathbf{H4}(1,1)$.*

Proof. The isomorphisms

$$\operatorname{coker} M(m) \cong \operatorname{coker} M\left(\mathbb{Z}[\mathfrak{S}_m]\right) \cong M\left(\operatorname{Res}^{\mathfrak{S}_m}_{\mathfrak{S}_{m-1}} \mathbb{Z}[\mathfrak{S}_m]\right) \cong M(m-1)^{\oplus m}$$

show that we can apply Theorem 3.21, where W^0 is of degree $m - 1$ and $W^i = 0$ for $i > 0$. Invoking Proposition 3.20 and choosing $d_i = m - 1 + i$ implies that

$$\widetilde{H}_i(\operatorname{coker}(M(m)))_n = 0 \quad \text{for } n > d_{i+1} = m + i > 1 \cdot (m + i) + 1. \qquad\square$$

Theorem 3.30. *An FI–module A is presented in finite degree if and only if it has finite polynomial degree. Specifically, if A is generated in degrees $\leq r$ and presented in degrees $\leq d$, then A has*

$$\text{polynomial degree} \leq r \quad \text{in ranks } > d + \min(r, d) - 1.$$

If A has polynomial degree $\leq r$ in ranks $> d$ for some $d \geq -1$, it is

$$\text{generated in degrees } \leq d + r + 1, \text{ and}$$
$$\text{presented in degrees } \leq d + r + 2.$$

Proof. The first direction is given by Randal-Williams–Wahl [RWW15, Example and Proposition 4.18]. Assume A has polynomial degree $\leq r$ in ranks $> d$. Then by Proposition 3.20 and Proposition 3.29 we may apply Corollary 3.27 with the values $a = k = b = \ell = 1$. $\qquad\square$

3.5. $\mathsf{VIC}(R)$–, $\mathsf{VIC}^H(R)$–, $\mathsf{SI}(R)$–**modules of finite polynomial degree.** In this subsection, we use the connectivity results of Section 2 to prove a vanishing result for the central stability homology of $\mathsf{VIC}^H(R)$–modules and $\mathsf{SI}(R)$–modules with finite polynomial degree for a R a PID. More specifically, we will show that the connectivity results of Section 2 establish the hypotheses of Theorem 3.26. We use the following criterion for $\mathbf{H4}(\ell,b)$.

Proposition 3.31 (Patzt [Pat17, Proposition 7.10]). *Assume $\mathbf{H3}(k,a)$ and let $b \geq \max(k,a)$. Then the condition $\mathbf{H4}(\ell,b)$ holds if for every $(m-1)$–simplex $\tau \in (K_{m-1}\mathcal{G})_{n+1}$, the intersection*

$$\text{Lk}_\bullet^{(K_\bullet\mathcal{G})_{n+1}}(\tau) \cap (K_\bullet\mathcal{G})_n \subset (K_\bullet\mathcal{G})_{n+1}$$

$\left\lfloor \left(\dfrac{n-b}{\ell}\right) - m - 1 \right\rfloor$*–connected, where is $(K_\bullet\mathcal{G})_n \hookrightarrow (K_\bullet\mathcal{G})_{n+1}$ is induced by $0 \oplus n \to 1 \oplus n$.*

Proof. The statement is a reformulation of [Pat17, Proposition 7.10]. As noted in the paragraph above [Pat17, Proposition 7.10], $\widetilde{H}_*(\operatorname{coker} M(m))_n$ is a quotient of the associated homology of the augmented semisimplicial set $(K_\bullet T \operatorname{Hom}(m,-))_n$. Therefore when $(K_\bullet T \operatorname{Hom}(m,-))_n$ is $\left\lfloor \left(\frac{n-b}{\ell}\right) - m - 1 \right\rfloor$–connected, the central stability homology $\widetilde{H}_i(\operatorname{coker} M(m))_n$ vanishes for all $n > \ell(i+m) + b$. [Pat17, Proposition 7.10] states that $(K_\bullet T \operatorname{Hom}(m,-))_n$ is the disjoint union of the augmented semisimplicial sets

$$\text{Lk}_\bullet^{(K_\bullet\mathcal{G})_{n+1}}(\tau) \cap (K_\bullet\mathcal{G})_n$$

for $\tau \in \operatorname{Hom}^{U\mathcal{G}}(m, n+1) = (K_{m-1}\mathcal{G})_{n+1}$. These augmented semisimplicial sets have singleton sets in degree -1. $\qquad\square$

Proposition 3.32. *Let R be a PID of stable rank s and let $H \leq R^\times$ contain -1. Then $U\operatorname{GL}^H(R)$ satisfy $\mathbf{H4}(2,s+1)$.*

Proof. By Proposition 3.20, we can use Proposition 3.31 to prove this proposition.

Let us first check this for $U\operatorname{GL}(R)$. By Proposition 3.31, we must check that for every $(m-1)$–simplex $\tau \in (K_{m-1}\operatorname{GL}(R))_{n+1}$, the intersection

$$\text{Lk}_\bullet^{(K_\bullet \operatorname{GL}(R))_{n+1}}(\tau) \cap (K_\bullet \operatorname{GL}(R))_n$$

is $\left\lfloor \left(\dfrac{n-s-1}{2}\right) - m - 1 \right\rfloor$–connected. Recall that $\text{PBC}_\bullet(R^n) \cong (K_\bullet \operatorname{GL}(R))_n$. Lemma 2.21 applied to

$$\text{Lk}_\bullet^{(K_\bullet \operatorname{GL}(R))_{n+1}}(\tau) = \text{PBC}_\bullet(R^{n+1}, C, \operatorname{im}(f)) \quad \text{and} \quad (K_\bullet \operatorname{GL}(R))_n = \text{PBC}_\bullet(R^{n+1}, R^n, R),$$

where $(f,C) = \tau \in (K_{m-1}\operatorname{GL}(R))_{n+1} = \operatorname{Hom}^{U\operatorname{GL}(R)}(m, n+1)$ and $R \oplus R^n = R^{n+1}$, shows that the intersection appearing in Proposition 3.31 is isomorphic to

$$\text{PBC}_\bullet(R^{n+1}, C \cap R^n, \operatorname{sat}(\operatorname{im} f + R)).$$

Since

$$\mathrm{rk}(C \cap R^n) \geq n - m$$

and

$$\mathrm{rk}\,\mathrm{sat}(\mathrm{im}\, f + R) \leq m + 1,$$

Theorem 2.19 implies that the semisimplicial set is $\left\lfloor \frac{(n-m)-(m+1)-s-2}{2} \right\rfloor$–connected. But

$$\frac{(n-m)-(m+1)-s-2}{2} = \frac{n-s-1}{2} - m - 1,$$

and the result follows.

We will now check the assertion for $U\,\mathrm{GL}^H(R)$. We must check that for every m–simplex $\tau \in (K_{m-1}\,\mathrm{GL}^H(R))_{n+1}$, the intersection

$$\mathrm{Lk}_\bullet^{(K_\bullet\,\mathrm{GL}^H(R))_{n+1}}(\tau) \cap (K_\bullet\,\mathrm{GL}^H(R))_n$$

is $\left\lfloor \left(\frac{n-s-1}{2} \right) - m - 1 \right\rfloor$–connected. Because

$$(K_p\,\mathrm{GL}^H(R))_{n+1} = (K_p\,\mathrm{GL}(R))_{n+1} \qquad \text{for } p \leq n - 1,$$

$$\mathrm{Lk}_\bullet^{(K_\bullet\,\mathrm{GL}^H(R))_{n+1}}(\tau) \cap (K_\bullet\,\mathrm{GL}^H(R))_n \hookrightarrow \mathrm{Lk}_\bullet^{(K_\bullet\,\mathrm{GL}(R))_{n+1}}(\tau) \cap (K_\bullet\,\mathrm{GL}(R))_n$$

induces a bijection of the set of p–simplices for $m + p + 1 < n + 1$. Thus it induces an $(n - m - 1)$–connected map on geometric realizations and so the assertion follows from the connectivity of the codomain which was proved in Proposition 3.32. $\qquad\square$

Because $U\,\mathrm{GL}^H(R)$ satisfies **H3(2,$s+1$)** and **H4(2,$s+1$)**, Theorem 3.26 implies the following corollary.

Corollary 3.33. *Assume R is a PID of stable rank s and let $H \leq R^\times$ contain -1. If A is a $\mathsf{VIC}^H(R)$–module of polynomial degree ≤ 0 in ranks $> d$ for some $d \geq -1$, then*

$$\widetilde{H}_i(A)_n = 0 \qquad \text{for all} \quad i \geq -1 \quad \text{and} \quad n > \max(d+i+2, 2i+s+1).$$

If A is a $\mathsf{VIC}^H(R)$–module of polynomial degree $\leq r$ in ranks $> d$ for some $r \geq 1$ and $d \geq -1$, then

$$\widetilde{H}_i(A)_n = 0 \qquad \text{for all} \quad i \geq -1 \quad \text{and} \quad n > 2^{i+1}(d+r+s+1) - s.$$

In particular, A has central stability degree $\leq \max(d+2, s+1, 2d+2r+s+2)$.

Corollary 3.34. *Let R be a PID of stable rank s, and let $H \leq R^\times$ contain -1. Let A be a $\mathsf{VIC}^H(R)$–module such that the underlying FI–module is generated in degrees $\leq g$ and related in degrees $\leq r$. Then, as a $\mathsf{VIC}^H(R)$–module, A is generated in degrees $\leq g$ and presented in degrees*

$$\begin{aligned} &\leq \max(2s, r + \min(g,r) + s + 1) && \text{if } g = 0, \text{ and} \\ &\leq 2r + 2g + 2\min(g,r) + s && \text{if } g > 0. \end{aligned}$$

In particular, as a $\mathsf{VIC}^H(R)$–module, A has central stability degree

$$\begin{aligned} &\leq \max(s+1, r+1) && \text{if } g = 0, \text{ and} \\ &\leq 2r + 2g + 2\min(g,r) + s && \text{if } g > 0. \end{aligned}$$

Proof. Since $\mathsf{FI} \subseteq \mathsf{VIC}^H(R)$, generation in degree $\leq g$ over FI implies generation in degree $\leq g$ over the larger category $\mathsf{VIC}^H(R)$. By Theorem 3.30, the sequence A has polynomial degree $\leq g$ in ranks $> r + \min(g, r) - 1$, viewed either as a module over FI or over $\mathsf{VIC}^H(R)$. Because $\mathsf{VIC}^H(R)$ satisfies conditions $\mathbf{H3(2,}s+1\mathbf{)}$ and $\mathbf{H4(2,}s+1\mathbf{)}$, by Proposition 3.20 and Proposition 3.32, respectively, we can use Corollary 3.27 to conclude the result. $\qquad\square$

Proposition 3.35. *Let R be a PID of unitary stable rank s. Then $U\operatorname{Sp}(R)$ satisfies* $\mathbf{H4(2,\ }s+2\mathbf{)}$.

Proof. We can apply Proposition 3.31 because of Proposition 3.20. We need to check the connectivity of the intersection

$$\operatorname{Lk}_\bullet \tau \cap (K_\bullet \operatorname{Sp}(R))_n$$

for every $(m-1)$–simplex $\tau \in (K_{m-1}\operatorname{Sp}(R))_{n+1} \cong \operatorname{SPB}_{m-1}(R^{2n+2})$. Let U be the symplectic submodule of R^{2n+2} generated by τ. Then a simplex σ is contained in $\operatorname{Lk}_\bullet \tau \cap (K_\bullet \operatorname{Sp}(R))_n$ exactly when σ is an ordered symplectic partial basis for the R–module

$$U^\perp \cap R^{2n} = U^\perp \cap (R^2)^\perp = (U + R^2)^\perp.$$

Note that here we are viewing R^2 as the first two coordinates in R^{2n+2} and R^{2n} as the last $2n$ coordinates. We have that $\operatorname{Lk}_\bullet \tau \cap (K_\bullet \operatorname{Sp}(R))_n = \operatorname{SPB}_\bullet(W^\perp)$, where $W = U + R^2$.

We can decompose W as $W = U \oplus U'$, where U' is a submodule of rank at most 2 contained in $U^\perp$. By Mirzaii-van der Kallen [MvdK02, Lemma 6.6], we can embed U' in a symplectic submodule H of $U^\perp$ of rank 4 as long as $\frac{1}{2}\operatorname{rk}(U^\perp) = (n+1-m)$ is at least $(2+s)$. (If $(n+1-m) < (2+s)$ then our conclusion will be vacuous because then $\left(\frac{1}{2}(n-(s+2)) - m - 1\right) < -1$.) So $U \oplus H$ is a rank-$(2m+4)$ symplectic submodule of R^{2n+2} containing W, hence $(U \oplus H)^\perp$ is a symplectic submodule of rank $(2n+2) - (2m+4)$ contained in $W^\perp$. This implies that $W^\perp$ has Witt index

$$g(W^\perp) \geq n - m - 1.$$

Friedrich [Fri17, Theorem 3.4] then implies that $\operatorname{SPB}_\bullet(W^\perp)$ is $\left\lfloor \frac{1}{2}\left((n - m - 1) - s - 3\right)\right\rfloor$–connected. Since

$$\left(\frac{(n - m - 1) - s - 3}{2}\right) \ \geq \ \left(\frac{n - (s+2)}{2} - m - 1\right),$$

Proposition 3.31 implies that $U\operatorname{Sp}(R)$ satisfies $\mathbf{H4(2,\ }s+2\mathbf{)}$, as claimed. $\qquad\square$

Combining this result with Proposition 3.20 and Theorem 3.26 gives the following Corollary.

Corollary 3.36. *Assume R is a PID of (symplectic) unitary stable rank s. If A is a $\mathsf{SI}(R)$–module of polynomial degree ≤ 0 in ranks $> d$ for some $d \geq -1$, then*

$$\widetilde{H}_i(A)_n = 0 \qquad \text{for all} \quad i \geq -1 \quad \text{and} \quad n > \max(d + i + 2, 2i + s + 2).$$

If A is a $\mathsf{SI}(R)$–module of polynomial degree $\leq r$ in ranks $> d$ for some $r \geq 1$ and $d \geq -1$, then

$$\widetilde{H}_i(A)_n = 0 \qquad \text{for all} \quad i \geq -1 \quad \text{and} \quad n > 2^{i+1}(d + r + s + 2) - s - 1.$$

Remark 3.37. All of the results in this section and the previous section apply equally well to orthogonal groups. We chose not to include these results because we do not know of any applications.

3.6. **A spectral sequence.** In this subsection, we summarize results about a spectral sequence introduced by Putman and Sam. For notation that matches our paper we merge [Pat17, Proposition 8.2, Lemma 8.3] into the next proposition.

Proposition 3.38. *Let $\mathcal{G}, \mathcal{Q}$ be stability groupoids and $\mathcal{G} \to \mathcal{Q}$ a homomorphism such that $G_n \to Q_n$ is surjective for every $n \in \mathbb{N}_0$. Then there is a unique stability groupoid $\mathcal{N}$ and a homomorphism $\mathcal{N} \to \mathcal{G}$ such that*

$$N_n \cong \ker(G_n \longrightarrow Q_n),$$

where $N_n = \mathcal{N}(n)$.

Furthermore, for every $i \geq 0$ there is a $U\mathcal{Q}$–module that we denote by $H_i(\mathcal{N})$ with

$$H_i(\mathcal{N})_n \cong H_i(N_n).$$

The following spectral sequence is central to our proof.

Proposition 3.39 (Putman–Sam [PS17, Theorem 5.9], reformulated by Patzt [Pat17, Corollary 8.5])**.** *Let $\mathcal{G}, \mathcal{Q}$ be stability groupoids and $\mathcal{G} \to \mathcal{Q}$ a homomorphism such that $G_n \to Q_n$ is surjective for every $n \in \mathbb{N}_0$ and $\mathcal{N}$ as in Proposition 3.38. Assume that $\mathcal{G}$ and $\mathcal{Q}$ are braided and that $\mathcal{G} \to \mathcal{Q}$ is a map of braided monoidal groupoids. For each n, there is a spectral sequence with*

$$E^2_{p,q} \cong \widetilde{H}^{\mathcal{Q}}_p(H_q(\mathcal{N}))_n.$$

Notably, $E^2_{p,q}$ vanishes for $p < -1$ or $q < 0$. If $U\mathcal{G}$ satisfies $\mathbf{H3}(k,a)$ then the spectral sequence converges to zero for $p + q \leq \left(\dfrac{n - a - 1}{k} \right)$.

4. Applications

In this section, we prove our central stability results for the second homology groups of Torelli subgroups and for the homology of congruence subgroups.

4.1. $H_2(\mathrm{IA}_n)$**.** We will recall a computation of $H_1(\mathrm{IA}_n)$. Then we will use a spectral sequence argument and our results about polynomial $\mathsf{VIC}(R)$–modules to prove Theorem A, central stability for $H_2(\mathrm{IA}_n)$.

Recall that $\mathsf{Aut}\,\mathsf{F}$ is the stability groupoid given by $\mathsf{Aut}\,\mathsf{F}_n = \mathrm{Aut}(F_n)$. Abelianization induces a monoidal functor

$$\mathsf{Aut}\,\mathsf{F} \longrightarrow \mathrm{GL}$$

that is surjective in every degree. By Proposition 3.38, the kernels form a stability groupoid which we denote by IA.

Reinterpreting the work of Andreadakis [And65, Section 6], Cohen–Pakianathan (unpublished), Farb (unpublished) and Kawazumi [Kaw06, Theorem 6.1] in the language of FI–modules, Church, Ellenberg, and Farb [CEF15, Equation (25) and the proof of Theorem 7.2.3] proved the following. See also Day–Putman [DP17, Page 5].

Theorem 4.1 (Andreadakis, Farb, Kawazumi, Cohen–Pakianathan, Church–Ellenberg–Farb)**.** *Use the following notation for bases for the FI–module $M(1)$:*

$$M(1)_n \cong \mathbb{Z}[e_1, e_2, \ldots, e_n].$$

Then

$$H_1(\mathrm{IA}) \cong \left(\bigwedge\nolimits^2 M(1) \right) \otimes M(1)$$

$$\cong M\Big(\mathbb{Z}[e_1 \wedge e_2 \otimes e_1,\ e_1 \wedge e_2 \otimes e_2] \Big) \oplus M\Big(\mathbb{Z}[e_1 \wedge e_2 \otimes e_3,\ e_1 \wedge e_3 \otimes e_2,\ e_2 \wedge e_3 \otimes e_1] \Big)$$

as an FI–*module.*

We are now ready to prove Theorem A and the following variant Theorem A'.

Theorem A'. *The* $\mathsf{VIC}(\mathbb{Z})$–*module* $H_2(\mathrm{IA})$ *is generated in degrees* ≤ 18 *and presented in degrees* ≤ 38.

Proof of Theorem A and Theorem A'. $\mathsf{Aut}\,\mathsf{F}$ is symmetric monoidal and thus so is $U\mathsf{Aut}\,\mathsf{F}$. This follows from Proposition 3.3. We can therefore apply the spectral sequences of Proposition 3.39 with $\mathcal{N} = \mathrm{IA}$, $\mathcal{G} = \mathsf{Aut}\,\mathsf{F}$, $\mathcal{Q} = \mathrm{GL}(\mathbb{Z})$. Proposition 3.20 says that $U\mathsf{Aut}\,\mathsf{F}$ satisfies **H3(2,2)**, hence these spectral sequences converge to zero for $p + q \leq \frac{n-3}{2}$.

We will show that

$$E^2_{-1,2} \cong 0 \quad \text{for } n > 18 \qquad \text{and} \qquad E^2_{0,2} \cong \widetilde{H}_0^{\mathrm{GL}(\mathbb{Z})}(H_2(\mathrm{IA}))_n = 0 \quad \text{for } n > 38$$

by showing that there are no nontrivial differentials to or from the groups $E^2_{-1,2}$ and $E^2_{0,2}$ of the spectral sequence in this range, as in Figure 1. By Proposition 3.20, the central stability homology
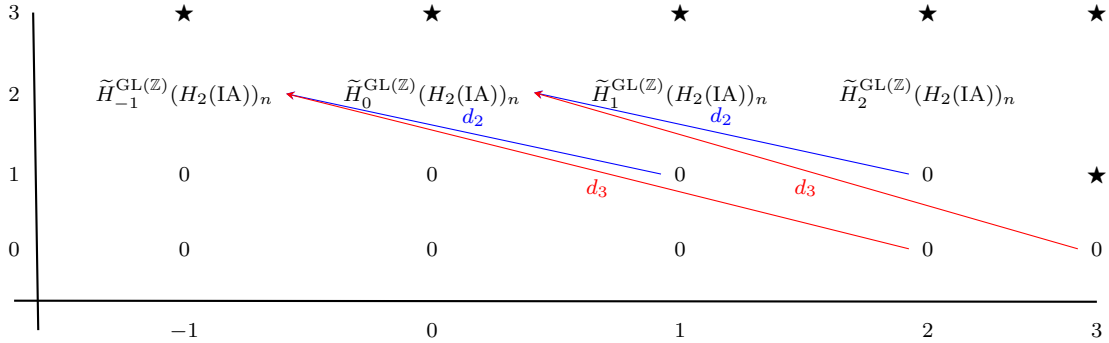

FIGURE 1. Page $E^2_{p,q}$ for $n \geq 39$.

of the $\mathsf{VIC}(\mathbb{Z})$–module $H_0(\mathrm{IA}) \cong M(0)$ is zero in degree q for $n > 2q + 3$. This implies the vanishing of the groups $E^2_{2,0} \cong \widetilde{H}_2^{\mathrm{GL}(\mathbb{Z})}(H_0(\mathrm{IA}))_n$ and $E^2_{3,0} \cong \widetilde{H}_3^{\mathrm{GL}(\mathbb{Z})}(H_0(\mathrm{IA}))_n$ for $n > 9$. Theorem 4.1 and Proposition 3.28 imply that $H_1(\mathrm{IA})$ has polynomial degree ≤ 3. Thus, by Corollary 3.33, its central stability homology as a $\mathsf{VIC}(\mathbb{Z})$–module $\widetilde{H}_q^{\mathrm{GL}(\mathbb{Z})}(H_1(\mathrm{IA}))_n$ vanishes for $n > 2^{q+1}(5) - 2$. It follows that the groups $E^2_{1,1} \cong \widetilde{H}_1^{\mathrm{GL}(\mathbb{Z})}(H_1(\mathrm{IA}))_n$ vanish for $n > 2^{1+1}(5) - 2 = 18$ and that the groups $E^2_{2,1} \cong \widetilde{H}_2^{\mathrm{GL}(\mathbb{Z})}(H_1(\mathrm{IA}))_n$ vanish for $n > 2^{2+1}(5) - 1 = 38$. Thus $E^2_{-1,2} \cong 0$ for $n > 18$ and $E^2_{0,2} \cong 0$ for $n > 38$.

Since $E^2_{-1,2} \cong \widetilde{H}_{-1}^{\mathrm{GL}(\mathbb{Z})}(H_2(\mathrm{IA}))_n$, Lemma 3.13 implies that $H_2(\mathrm{IA})$ is generated in degrees ≤ 18. Because $E^2_{0,2} \cong \widetilde{H}_0^{\mathrm{GL}(\mathbb{Z})}(H_2(\mathrm{IA}))_n$, we deduce that $H_2(\mathrm{IA})$ has centrally stability degree ≤ 38. Using Theorem 3.12, this implies that $H_2(\mathrm{IA})$ is presented in degrees ≤ 38. $\square$

4.2. $H_2(\mathcal{I}_g)$. Using a similar proof strategy to that of the previous subsection, we will prove Theorem B, central stability for $H_2(\mathcal{I})$.

Recall that Mod is the stability groupoid given by the mapping class groups of compact, connected, oriented surfaces with one boundary component. Its action on the first homology group of the surface (which preserves the intersection form) gives a monoidal functor

$$\mathrm{Mod} \longrightarrow \mathrm{Sp}$$

that is surjective in every degree. Let $\mathcal{I}$ denote the stability groupoid given by assembling the kernels of this map using Proposition 3.38.

Similar to Theorem 4.1, we have the following description of $H_1(\mathcal{I})$.

Theorem 4.2 (Johnson [Joh85b, Theorem 3]; see Brendle–Farb [BF07, Sections 1 and 2.2]). *As a module over $\mathsf{FI} \subseteq \mathsf{SI}(\mathbb{Z})$, the first homology groups $H_1(\mathcal{I})$ decompose as follows. Let*

$$H_g := H_1(\Sigma_{g,1}; \mathbb{Z}) \cong M(1)_g^{\oplus 2}, \qquad H_g \cong \mathbb{Z}[a_1, b_1, a_2, b_2, \ldots, a_g, b_g].$$

Then for all $g \geq 3$,

$$H_1(\mathcal{I}_g) \cong \ \bigwedge^3 H_g \oplus \left(\mathrm{Sym}^0(H_g) \oplus \mathrm{Sym}^1(H_g) \oplus \mathrm{Sym}^2(H_g)/H_g\right) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$$

and these isomorphisms respect the FI–module structure. In particular, in the range $g \geq 3$, the FI–module $H_1(\mathcal{I})$ coincides with FI–module

$$M\left(\mathbb{Z}[\ a_i \wedge b_i \wedge b_j,\ a_i \wedge a_j \wedge b_i \mid \{i,j\} = \{1,2\}]\right)$$
$$\oplus M\left(\mathbb{Z}[\ a_i \wedge a_j \wedge a_k,\ a_i \wedge a_j \wedge b_k,\ a_i \wedge b_j \wedge b_k,\ b_i \wedge b_j \wedge b_k \mid \{i,j,k\} = \{1,2,3\}]\right)$$
$$\oplus M\left(\mathbb{Z}/2\mathbb{Z}[\mathfrak{S}_0]\right) \oplus M\left(\mathbb{Z}/2\mathbb{Z}[a_1, b_1, a_1 b_1]\right) \oplus M\left(\mathbb{Z}/2\mathbb{Z}[a_1 a_2,\ a_1 b_2,\ a_2 b_1,\ b_1 b_2]\right).$$

This description of the abelianization $H_1(\mathcal{I}_{g,1}; \mathbb{Z})$ of the Torelli group $\mathcal{I}_{g,1}$ was first computed by Johnson [Joh85b, Theorem 3] in a series of papers [Joh80, Joh83, Joh85a, Joh85b] building on work of Birman–Craggs [BC78]. See Brendle–Farb [BF07, Sections 1 and 2.2] or van den Berg [vdB03, Theorem 3.5.6] for a more self-contained statement of the description of $H_1(\mathcal{I}_g)$. This isomorphism arises as follows. The *Johnson homomorphism* is the map

$$\tau_g : \mathcal{I}_g \twoheadrightarrow \bigwedge^3 H_g$$

which admits several algebraic and topological descriptions; see for example Farb–Margalit [FM11, Section 6.6]. The maps τ_g are equivariant with respect to the action of the mapping class group, and natural with respect to the maps $\mathcal{I}_k \to \mathcal{I}_g$ and $H_k \to H_g$ induced by the embedding of a subsurface $\Sigma_{k,1} \hookrightarrow \Sigma_{g,1}$. The torsion subgroup of the abelianization

$$B_g^2 := \left(\mathrm{Sym}^0(H_g) \oplus \mathrm{Sym}^1(H_g) \oplus \mathrm{Sym}^2(H_g)/H_g\right) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$$

is identified with the space of *Boolean polynomials* of degree at most 2 in the elements $\overline{a_1}, \overline{b_1}, \ldots, \overline{a_g}, \overline{b_g}$. These elements represent functions $\Omega \to \mathbb{Z}/2\mathbb{Z}$ from the space Ω of mod-2 self-linking forms on H_g associated to embeddings of $\Sigma_{g,1}$ into S^3. The map

$$\sigma_g : \mathcal{I}_g \twoheadrightarrow B_g^2$$

is the *Johnson–Birman–Craggs homomorphism* (after modding out by Boolean polynomials in homogeneous degree 3). The maps σ_g are also equivariant with respect to the action of the mapping class group and natural with respect to embeddings $\Sigma_{k,1} \hookrightarrow \Sigma_{g,1}$. We caution, however, that the action of the mapping class group (and induced action of $\mathrm{Sp}_{2g}(\mathbb{Z})$) on the elements $\overline{a_1}, \overline{b_1}, \ldots, \overline{a_g}, \overline{b_g}$ does not simply correspond to its action on H_g; see for example Johnson [Joh85b, Section 2] or Brendle–Farb [BF07, Section 2] for an explicit description of the space B_g^2 and map σ_g. The action of the symmetric group $\mathfrak{S}_g \subseteq \mathrm{Sp}_{2g}(\mathbb{Z})$ on B_g^2 is, however, induced by its standard permutation action on H_g, extended diagonally to monomials in the elements $\overline{a_i}, \overline{b_i}$. The FI action on the torsion subgroup of $H_1(\mathcal{I}_g)$ for $g \geq 3$ is induced by the usual action on $H = M(1)^{\oplus 2}$.

We can now prove Theorem B and the following variant Theorem B'.

Theorem B'. *The* $\mathsf{SI}(\mathbb{Z})$*–module* $H_2(\mathcal{I})$ *is generated in degrees* ≤ 33 *and is presented in degrees* ≤ 69.

Proof of Theorem B and Theorem B'. We can apply Proposition 3.39 with $\mathcal{N} = \mathcal{I}$, $\mathcal{G} = \mathrm{Mod}$, and $\mathcal{Q} = \mathrm{Sp}(\mathbb{Z})$.

Proposition 3.20 implies that U Mod satisfies **H3(2,2)**, so the spectral sequence converges to zero for $p + q \leq \frac{n-3}{2}$. By Theorem 4.2 and Proposition 3.28, $H_1(\mathcal{I})$ has polynomial degree ≤ 3 in ranks > 2, thus by Corollary 3.36 its central stability homology $\widetilde{H}_q^{\mathrm{Sp}(\mathbb{Z})}(H_1(\mathcal{I}))_n$ vanishes for $n > 2^{q+1} \cdot 9 - 3$. Moreover $\widetilde{H}_q^{\mathrm{Sp}(\mathbb{Z})}(H_0(\mathcal{I}))_n \cong \widetilde{H}_q^{\mathrm{Sp}(\mathbb{Z})}(M(0))_n$ vanishes for $n > 2q + 4$. This implies that $E_{-1,2}^2 = 0$ for $n > 2^{1+1} \cdot 9 - 3 = 33$ and $E_{0,2}^2 = 0$ for $n > 2^{2+1} \cdot 9 - 3 = 69$. This proves that $H_2(\mathcal{I})$ is generated in degrees ≤ 33 and has central stability degree ≤ 69. The bound on its degree of presentation follows from Theorem 3.12. $\qquad\square$

4.3. **Congruence subgroups.** Let R be a ring and $I \subseteq R$ be a two-sided ideal. Let $\mathfrak{U}$ denote the image $R^\times$ in R/I. The quotient map $R \longrightarrow R/I$ induces a homomorphisms

$$\mathrm{GL}_n(R) \to \mathrm{GL}_n^{\mathfrak{U}}(R/I).$$

We denote the kernel by $\mathrm{GL}_n(R, I)$, which is called the *level I congruence subgroup* of $\mathrm{GL}_n(R)$. These assemble to form an FI–module [CEFN14, Application 1]. Building the work of Putman [Put15, Theorems B and C], Church–Ellenberg–Farb–Nagpal [CEFN14, Theorem D], Church–Ellenberg [CE17, Theorem D'], Church–Miller–Nagpal–Reinhold [CMNR18, Application B], Gan and Li proved the following representation stability result for congruence subgroups viewed as FI–modules.

Theorem 4.3 (Gan–Li [GL17a, Theorem 11])**.** *Let R be a ring of stable rank t and let I be a two-sided ideal of R. Then for $i \geq 0$, $H_i(\mathrm{GL}(R, I))$ is generated in degrees*

$$\leq 4i + 2t - 1$$

and presented in degree

$$\leq 4i + 2t + 4$$

when viewed as an FI*–module.*

We have three remarks on the statement of Gan–Li [GL17a, Theorem 11]. First note that they use a different indexing convention for stable rank; their d is our $(t - 1)$. Secondly, they use a different definition of *presentation degree* than we do here, however, the two definitions are equivalent (see for example Church–Ellenberg [CE17, Proposition 4.2] for one direction; the other direction follows from

[CE17, Lemma 2.3] and a computation of the FI–homology groups using an acyclic resolution). Third, note that although [GL17a, Theorem 11] assumes I is proper, their theorem is also true in the case that $I = R$. Since $\mathrm{GL}_n(R, R) \cong \mathrm{GL}_n(R)$, van der Kallen's homological stability result [vdK80] implies representation stability (see [CMNR18, Proof of Application B]).

We will now restrict attention to the case that the quotient map $\mathrm{GL}_n(R) \to \mathrm{GL}_n^{\mathfrak{U}}(R/I)$ is surjective. This gives a surjective homomorphism of stability groupoids

$$\mathrm{GL}(R) \longrightarrow\!\!\!\rightarrow \mathrm{GL}^{\mathfrak{U}}(R/I).$$

By Proposition 3.38, these groups assemble to form a stability groupoid which we denote by $\mathrm{GL}(R, I)$. Theorem 4.3 and Corollary 3.34 together imply the following result which in turn implies Theorem C. Recall that the group $\mathrm{GL}_n(R, I)$ depends only on I as a nonunital ring, and not on the ambient ring R.

Theorem C'. *Let I be a two-sided ideal in a ring R of stable rank r and let t be the minimal stable rank of all rings containing I as a two-sided ideal. Let $\mathfrak{U}$ denote the image of $R^{\times}$ in R/I. If R/I is a PID of stable rank s, and the natural map $\mathrm{GL}_n(R) \to \mathrm{GL}_n^{\mathfrak{U}}(R/I)$ is surjective, then the $\mathsf{VIC}^{\mathfrak{U}}(R/I)$–module $H_i(\mathrm{GL}(R, I))$ is generated in degrees*

$$
\begin{aligned}
&\leq 0 && \text{for } i = 0 \\
&\leq \min(3 + 2t,\, \max(3 + r, 3 + s)) && \text{for } i = 1 \\
&\leq 4i + 2t - 1 && \text{for } i > 1
\end{aligned}
$$

and is presented in degrees

$$
\begin{aligned}
&\leq s + 1 && \text{for } i = 0 \\
&\leq \min(28 + 12t + s,\, \max(5 + r, 5 + s)) && \text{for } i = 1 \\
&\leq 24i + 12t + s + 4 && \text{for } i > 1.
\end{aligned}
$$

Proof of Theorem C and Theorem C'. Gan and Li's Theorem 4.3 states bounds on the generation and presentation degree of underlying FI–module structure on $H_i(\mathrm{GL}(R, I))$, and Corollary 3.34 allows us to deduce bounds on the generation and relation degree for the $\mathsf{VIC}^{\mathfrak{U}}(R/I)$–module structure. Then (because the FI–module structure on $H_i(\mathrm{GL}(R, I))$ does not depend on the choice of the ring R), the bounds for $i \geq 2$ and the first term in the bound on $i = 1$ follow. For $i = 0$, the $\mathsf{VIC}^{\mathfrak{U}}(R/I)$–module $H_0(\mathrm{GL}(R, I)) = M(0)$ is generated in degrees 0. It is presented in degrees $\leq (s + 1)$ by Proposition 3.20. For the second term in the case $i = 1$, we use the spectral sequence argument of the previous two subsections. Theorem 3.21 allows us to relate bounds on generation and presentation degree to bounds on the central stability degree of $H_i(\mathrm{GL}(R, I))$. $\qquad\square$

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