

Vertex degree sums for matchings in 3-uniform hypergraphs

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Abstract

Let n, s be positive integers such that n is sufficiently large and $s \leq n/3$. Suppose H is a 3-uniform hypergraph of order n without isolated vertices. If $\deg(u) + \deg(v) > 2(s-1)(n-1)$ for any two vertices u and v that are contained in some edge of H , then H contains a matching of size s . This degree sum condition is best possible and confirms a conjecture of the authors [Electron. J. Combin. 25 (3), 2018], who proved the case when $s = n/3$.

Mathematics Subject Classifications: 05C70, 05C65

1 Introduction

A k -uniform hypergraph H (in short, k -graph) is a pair (V, E) , where V is a finite set of vertices and E is a family of k -element subsets of V . Note that a 2-graph is simply a graph. Let $V(H)$ and $E(H)$ denote the vertex set and edge set of H , respectively. A *matching of size s* in H is a family of s pairwise disjoint edges of H . If the matching covers

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all the vertices of H , then we call it a *perfect matching*. Given a set $S \subseteq V$, the *degree* $\deg_H(S)$ of S is the number of the edges of H containing S . We simply write $\deg(S)$ when H is obvious from the context. Further, let $\delta_\ell(H) = \min\{\deg(S) : S \subseteq V(H), |S| = \ell\}$.

Given integers $\ell < k \leq n$ such that k divides n , let $m_\ell(k, n)$ denote the smallest integer m such that every k -graph H on n vertices with $\delta_\ell(H) \geq m$ contains a perfect matching. In recent years the problem of determining $m_\ell(k, n)$ has received much attention (see [2, 5, 6, 7, 8, 9, 10, 12, 14, 17, 16, 18, 20, 21, 22]). In particular, Rödl, Ruciński and Szemerédi [18] determined $m_{k-1}(k, n)$ for all $k \geq 3$ and sufficiently large n . Treglown and Zhao [20, 21] determined $m_\ell(k, n)$ for all $\ell \geq k/2$ and sufficiently large n . More Dirac-type results on hypergraphs can be found in surveys [15, 27].

A well-known result of Ore [13] extended Dirac's theorem by determining the smallest degree sum of two non-adjacent vertices that guarantees a Hamilton cycle in graphs. Ore-type problems for hypergraphs have been studied recently. For example, Tang and Yan [19] studied the degree sum of two $(k-1)$ -sets that guarantees a tight Hamilton cycle in k -graphs. Zhang and Lu [23] studied the degree sum of two $(k-1)$ -sets that guarantees a perfect matching in k -graphs. Zhang, Zhao and Lu [26] determined the minimum degree sum of two adjacent vertices that guarantees a perfect matching in 3-graphs without isolated vertices, see Theorem 2 (two vertices in a hypergraph are *adjacent* if there exists an edge containing both of them). Note that one may study the minimum degree sum of two arbitrary vertices and that of two non-adjacent vertices that guarantees a perfect matching instead. In fact, it was mentioned in [26] that the former equals to $2m_1(3, n) - 1$ while the latter does not exist.

Let us define (potential) extremal 3-graphs for the matching problem. For $1 \leq \ell \leq 3$, let $H_{n,s}^\ell$ denote the 3-graph of order n , whose vertex set is partitioned into two sets S and T of size $n - s\ell + 1$ and $s\ell - 1$, respectively, and whose edge set consists of all triples with at least ℓ vertices in T . A well-known conjecture of Erdős [3], recently verified for 3-graphs [4, 11], implies that $H_{n,s}^1$ or $H_{n,s}^3$ is the densest 3-graph on n vertices not containing a matching of size s . On the other hand, Kühn, Osthus and Treglown [10] showed that for sufficiently large n , $H_{n,s}^1$ has the largest minimum vertex degree among all 3-graphs on n vertices not containing a matching of size s .

Theorem 1. [10] *There exists $n_0 \in \mathbb{N}$ such that if H is a 3-graph of order $n \geq n_0$ with $\delta_1(H) > \delta_1(H_{n,s}^1) = \binom{n-1}{2} - \binom{n-s}{2}$ and $n \geq 3s$, then H contains a matching of size s .*

Given a 3-graph H , let $\sigma_2(H)$ denote the minimum $\deg(u) + \deg(v)$ among all adjacent vertices u and v . It is easy to see that

$$\begin{aligned}\sigma_2(H_{n,s}^3) &= 2\binom{3s-2}{2}, \quad \sigma_2(H_{n,s}^1) = 2\left(\binom{n-1}{2} - \binom{n-s}{2}\right), \text{ and} \\ \sigma_2(H_{n,s}^2) &= \binom{2s-2}{2} + (n-2s+1)\binom{2s-2}{1} + \binom{2s-1}{2} = (2s-2)(n-1).\end{aligned}$$

The following is [26, Theorem 1], which implies that, when n is divisible by 3 and sufficiently large, $H_{n,n/3}^2$ has the largest $\sigma_2(H)$ among all n -vertex 3-graphs H containing no isolated vertex or perfect matching.

Theorem 2. [26] *There exists $n_0 \in \mathbb{N}$ such that the following holds for all integers $n \geq n_0$ that are divisible by 3. Let H be a 3-graph of order n without an isolated vertex. If $\sigma_2(H) > \sigma_2(H_{n,n/3}^2) = \frac{2}{3}n^2 - \frac{8}{3}n + 2$, then H contains a perfect matching.*

Zhang, Zhao and Lu [26, Conjecture 12] further conjectured that for sufficiently large n and any $s < n/3$, $H_{n,s}^2$ has the largest $\sigma_2(H)$ among all n -vertex 3-graphs H containing no isolated vertex or matching of size s . In this paper we verify this conjecture.

Theorem 3. *There exists $n_1 \in \mathbb{N}$ such that the following holds for all integers $n \geq n_1$ and $s \leq n/3$. If H is a 3-graph of order n without an isolated vertex and $\sigma_2(H) > \sigma_2(H_{n,s}^2) = 2(s-1)(n-1)$, then H contains a matching of size s .*

Since the two theorems have different extremal hypergraphs, Theorem 3 does not imply Theorem 1 (analogously Theorem 1 does not imply Erdős' matching conjecture for 3-graphs). On the other hand, one may wonder why we assume that H contains no isolated vertex in Theorem 3 (especially when $s < n/3$). In fact, as shown in the concluding remarks of [26], Theorem 3 implies another conjecture [26, Conjecture 13], which determines the largest $\sigma_2(H)$ among all 3-graphs containing no matching of size s . Note that $\sigma_2(H_{n,s}^2) \geq \sigma_2(H_{n,s}^3)$ if and only if $s \leq (2n+4)/9$.

Corollary 4. *There exists $n_2 \in \mathbb{N}$ such that the following holds. Suppose that H is a 3-graph of order $n \geq n_2$ and $2 \leq s \leq n/3$. If $\sigma_2(H) > \max\{\sigma_2(H_{n,s}^2), \sigma_2(H_{n,s}^3)\}$, then H contains a matching of size s .*

Let us explain our approach towards Theorem 3. The case when $s \leq n/13$ was already solved by Zhang and Lu [24] in a stronger form. Note that $\sigma_2(H_{n,s}^2) > \sigma_2(H_{n,s}^1)$. The following theorem shows that, when $n \geq 13s$, not only is $H_{n,s}^2$ the (unique) 3-graph with the largest $\sigma_2(H)$ among all H containing no isolated vertex or a matching of size s , but also $H_{n,s}^1$ is the sub-extremal 3-graph for this problem. (In fact, Zhang and Lu [24] conjectured that Theorem 5 holds for all $n \geq 3s$. If true, this strengthens Theorem 1 and actually provides a link between Ore's and Dirac's problems.)

Theorem 5. [25] *Let n, s be positive integers and H be a 3-graph of order $n \geq 13s$ without an isolated vertex. If $\sigma_2(H) > \sigma_2(H_{n,s}^1) = 2\left(\binom{n-1}{2} - \binom{n-s}{2}\right)$, then either H contains a matching of size s or H is a subgraph of $H_{n,s}^2$.*

Therefore it suffices to prove Theorem 3 for reasonably large s . For such s , we actually prove a (stronger) stability theorem.

Theorem 6. *Given $0 < \varepsilon \ll \tau \ll 1$, let n be sufficiently large and $\tau n < s \leq n/3$. If H is a 3-graph of order n without an isolated vertex such that $\sigma_2(H) > 2sn - \varepsilon n^2$, then either H is a subgraph of $H_{n,s}^2$ or H contains a matching of size s .*

Theorem 3 follows from Theorem 6 immediately. Indeed, if $\sigma_2(H) > \sigma_2(H_{n,s}^2)$, then it is easy to see that H is not a subgraph of $H_{n,s}^2$.¹ Suppose instead, that $V(H)$ can be

¹Unfortunately σ_2 is not a monotone function: for example, adding an edge to $H_{n,s}^2$ indeed reduces the value of σ_2 because two vertices in S now become adjacent and their degree sum is smaller than $\sigma_2(H_{n,s}^2)$.

partitioned $S \cup T$ such that $|S| = n - 2s + 1$, $|T| = 2s - 1$, and every edge of H contains at least two vertices of T . Since H contains no isolated vertices, every vertex of S is adjacent to some vertex of T . Thus $\sigma_2(H) \leq \deg(u) + \deg(v)$ for some $u \in S$ and $v \in T$. Consequently $\sigma_2(H) \leq \sigma_2(H_{n,s}^2)$, a contradiction. We therefore apply Theorem 6 to derive that H contains a matching of size s . Furthermore, Theorem 6 implies that $H_{n,s}^2$ is the unique extremal 3-graph for Theorem 3 because all proper subgraphs H of $H_{n,s}^2$ satisfy $\sigma_2(H) < \sigma_2(H_{n,s}^2)$.

In order to prove Theorem 6, we follow the same approach as in [26]: using the condition on $\sigma_2(H)$, we greedily extend a matching of H until it has s edges. An important intermediate step is finding a matching that covers a certain number of low-degree vertices (see Lemma 7). Nevertheless, the proof of Theorem 6 does require new ideas: in particular, the meaning of an *optimal* matching is more complicated (see Definition 8); we proceed differently depending on whether the number of low-degree vertices in the optimal matching is at the threshold. In one case we reduce the problem to that of finding a perfect matching in a subgraph of H and apply the main result of [26] (see Theorem 9).

This paper is organized as follows. In Section 2, we give an outline of the proof along with some preliminary results. We prove Lemma 7 in Section 3 and complete the proof in Section 4.

Notation: Given a graph G and a vertex u in G , $N_G(u)$ is the set of neighbors of u in G . Suppose H is a 3-uniform hypergraph. For $u \neq v \in V(H)$, let $N_H(u, v) = \{w \in V(H) : \{u, v, w\} \in E(H)\}$ (the subscript is often omitted when H is clear from the context). Given three subsets V_1, V_2, V_3 of $V(H)$, we say that an edge $\{v_1, v_2, v_3\} \in E(H)$ is a type of $V_1V_2V_3$ if $v_i \in V_i$ for $1 \leq i \leq 3$. Given a vertex $v \in V(H)$ and a subset $A \subseteq V(H)$, we define the *link* $L_v(A) = \{uw : u, w \in A \text{ and } \{u, v, w\} \in E(H)\}$. When A and B are two disjoint subsets of $V(H)$, we let $L_v(A, B) = \{uw : u \in A, w \in B \text{ and } \{u, v, w\} \in E(H)\}$.

We write $0 < a_1 \ll a_2 \ll a_3$ if we can choose the constants a_1, a_2, a_3 from right to left. More precisely there are increasing functions f and g such that given a_3 , whenever we choose some $a_2 \leq f(a_3)$ and $a_1 \leq g(a_2)$, all calculations needed in our proof are valid.

2 Outline of the proof and preliminaries

Let n be sufficiently large and $\tau n < s \leq n/3$. Suppose H is a 3-graph of order n without an isolated vertex and $\sigma_2(H) > 2sn - \varepsilon n^2$. Let $U = \{u \in V(H) : \deg(u) > sn - \frac{\varepsilon}{2}n^2\}$ and $W = V \setminus U$. Then any two vertices of W are not adjacent – otherwise $\sigma_2(H) \leq 2sn - \varepsilon n^2$, a contradiction. If $|U| \leq 2s - 1$, then H is a subgraph of $H_{n,s}^2$ and we are done. We thus assume that $|U| \geq 2s$.

Throughout the proof we use small constants

$$0 < \varepsilon \ll \varepsilon' \ll \varepsilon'' \ll \eta_1 \ll \eta_2 \ll \gamma \ll \gamma' \ll \tau \ll 1. \quad (1)$$

We first prove the following lemma, which is an extension of [26, Lemma 4].

Lemma 7. *Given $0 < \varepsilon \ll \tau \ll 1$, let n be sufficiently large and $\tau n < s \leq n/3$. Suppose H is a 3-graph of order n without an isolated vertex and $\sigma_2(H) > 2sn - \varepsilon n^2$. Let*

$U = \{u \in V(H) : \deg(u) > sn - \varepsilon n^2/2\}$ and $W = V \setminus U$. If $2s \leq |U| \leq 3s$, then H contains a matching of size $3s - |U|$, each of which contains exactly one vertex of W .

Definition 8. We call a matching M *optimal* if (i) M contains a submatching $M_1 = \{e \in M : e \cap W \neq \emptyset\}$ of size at least $3s - |U|$; (ii) subject to (i), $|M|$ is as large as possible; (iii) subject to (i) and (ii), $|M_1|$ is as large as possible.

Lemma 7 shows that H contains an optimal matching M . We separate the cases when $|M_1| = 3s - |U|$ and when $|M_1| > 3s - |U|$. When $|M_1| = 3s - |U|$, we first consider the case when $s \leq n/3 - \eta_1 n$. If no vertex of $U_3 := U \setminus V(M)$ is adjacent to any vertex of $W_2 := W \setminus V(M)$, then the assumption $|M_1| = 3s - |U|$ forces $\sum_{i=1}^3 \deg(u_i)$ to be smaller than $3sn - \frac{3}{2}\varepsilon n^2$ for any three vertices $u_1, u_2, u_3 \in U_3$. If some vertex $u_1 \in U_3$ is adjacent to $v_1 \in W_2$, then the fact $v_1 \in W$ reduces $\sum_{i=1}^2 \deg(u_i) + \deg(v_1)$ to a number less than $3sn - \frac{3}{2}\varepsilon n^2$ (where u_2 is another vertex of U_3). When $s > n/3 - \eta_1 n$, we consider $H' = H[V \setminus W_2]$. Since $|W_2| = n - 3s$ is very small, we deduce that $\sigma_2(H')$ is greater than $2sn - \eta_2 n^2$. This allows us to apply the following theorem from [26] to obtain a perfect matching of H' , which is also a matching of size s of H .

Theorem 9. [26] *There exist $\eta_2 > 0$ and $n_0 \in \mathbb{N}$ such that the following holds for all integers $n \geq n_0$ that are divisible by 3. Suppose that H is a 3-graph of order n without an isolated vertex and $\sigma_2(H) > 2n^2/3 - \eta_2 n^2$, then either H is a subgraph of $H_{n,n/3}^2$ or H contains a perfect matching.*

Now consider the case when $|M_1| > 3s - |U|$. Let $W' := \{v \in W : \deg(v) \leq sn - s^2/2 + \gamma' n^2\}$. If $|W'|$ is very small, then we can find a matching of size s in $H[V \setminus W']$ by Theorem 1. When $|W'|$ is not small, we consider $u_1, u_2, u_3 \in U_3$. If one of u_1, u_2, u_3 is adjacent to one vertex from W' , then $\sum_{i=1}^3 \deg(u_i)$ becomes much larger than $3sn$; otherwise we show that $\sum_{i=1}^3 \deg(u_i) < 3sn - \frac{3}{2}\varepsilon n^2$ by proceeding with the cases when $|W' \cap W_1| > \gamma n/2$ and when $|W' \cap W_2| > \gamma n/2$ separately.

In the proof we need several (simple) extremal results on (hyper)graphs. Lemma 10 is Observation 1.8 of Aharoni and Howard [1]. Lemmas 11 and 12 are from [26]. A k -graph H is called *k -partite* if $V(H)$ can be partitioned into V_1, \dots, V_k , such that each edge of H meets every V_i in precisely one vertex. If all parts are of the same size n , we call H *n -balanced*.

Lemma 10. [1] *Let F be the edge set of an n -balanced k -partite k -graph. If F does not contain s disjoint edges, then $|F| \leq (s-1)n^{k-1}$.*

Lemma 11. [26] *Let G_1, G_2, G_3 be three graphs on the same set V of $n \geq 4$ vertices such that every edge of G_1 intersects every edge of G_i for both $i = 2, 3$. Then $\sum_{i=1}^3 \sum_{v \in A} \deg_{G_i}(v) \leq 6(n-1)$ for any set $A \subset V$ of size 3.*

Lemma 12. [26] *Let G_1, G_2, G_3 be three graphs on the same set V of $n \geq 5$ vertices such that for any $i \neq j$, every edge of G_i intersects every edge of G_j . Then $\sum_{i=1}^3 \sum_{v \in A} \deg_{G_i}(v) \leq 3(n+1)$ for any set $A \subset V$ of size 3.*

Following the same proof of Lemmas 11 and 12 from [26], we obtain another lemma and omit its proof.

Lemma 13. *Let G_1, \dots, G_k be k graphs on the same set V of $n \geq 4$ vertices such that for any $1 \leq i < j \leq k$, every edge of G_i intersects every edge of G_j . Then $\sum_{i=1}^k \sum_{v \in A} \deg_{G_i}(v) \leq kn$ for any set $A \subset V$ of size 2.* \square

The following lemma needs slightly more work so we include a proof.

Lemma 14. *Given two disjoint vertex sets $A = \{u_1, u_2, \dots, u_a\}$ and $B = \{v_1, v_2, \dots, v_b\}$ with $a \geq 3$ and $b \geq 1$. Let G_i , $i = 1, 2, 3$, be graphs on $A \cup B$ such that every vertex of B is an isolated vertex in G_1 , and every edge of G_i ($i = 2, 3$) contains at least one vertex of A . If there are no two disjoint edges (i) one from G_1 and the other from G_2 or G_3 ; or (ii) one from G_2 and the other from G_3 , and at least one of them contains a vertex from B , then*

$$\sum_{i=1}^3 \left(\sum_{j=1}^2 \deg_{G_i}(u_j) + \deg_{G_i}(v_1) \right) \leq \max\{4a + 7, 3a + 2b + 5\}.$$

Proof. For convenience, let $s_i = \sum_{j=1}^2 \deg_{G_i}(u_j) + \deg_{G_i}(v_1)$ for $i = 1, 2, 3$ and $y = s_1 + s_2 + s_3$. Below we show that $y \leq \max\{4a + 7, 3a + 2b + 5\}$.

We first observe that if $\deg_{G_i}(v_1) \geq 3$ for some $i \in \{2, 3\}$, then $E(G_1) = \emptyset$ and $G_{i'}$ is a star centered at v_1 , where $i' = 5 - i$. Indeed, if G_1 or $G_{i'}$ contains an edge e not incident to v_1 , then e is disjoint from some edge of G_i that is incident to v_1 – this contradicts our assumption. The observation implies that if $\deg_{G_i}(v_1) \geq 3$ for both $i = 2, 3$, then $E(G_1) = \emptyset$ and both G_2 and G_3 are stars centered at v_1 . In this case, $s_i \leq a + 2$ for $i = 2, 3$ and thus $y \leq 2(a + 2)$. If $\deg_{G_2}(v_1) \geq 3$ and $\deg_{G_3}(v_1) \leq 2$, then $E(G_1) = \emptyset$ and G_3 consists of at most two edges incident to v_1 . In this case, $s_1 \leq 2(a + b - 1) + a$, $s_2 \leq 4$ and thus $y \leq 3a + 2b + 2$. The case when $\deg_{G_2}(v_1) \leq 2$ and $\deg_{G_3}(v_1) \geq 3$ is analogous. We thus assume that

$$\deg_{G_i}(v_1) \leq 2 \quad \text{for } i = 2, 3 \tag{2}$$

for the rest of the proof.

Next, we observe that if $|N_{G_i}(u_j) \cap B| \geq 2$ for some $i \in \{2, 3\}$ and some $j \in \{1, 2\}$, then $G_{i'}$ is a star centered at u_j for $i' \in \{1, 2, 3\} \setminus \{i\}$. This is again due to our assumption on G_1, G_2 and G_3 . The observation implies that if $|N_{G_i}(u_j) \cap B| \geq 2$ for both $j = 1, 2$, then $E(G_{i'}) \subseteq \{u_1 u_2\}$ and consequently, $s_{i'} \leq 2$ for $i' \in \{1, 2, 3\} \setminus \{i\}$. By (2), we have $s_i \leq 2(a + b - 1) + 2$. Therefore, $y \leq 2(a + b - 1) + 2 + 4 = 2a + 2b + 4$. The observation also implies that if $|N_{G_i}(u_j) \cap B| \geq 2$ for both $i = 2, 3$, then G_1, G_2, G_3 are all stars centered at u_j . In this case, $s_1 \leq a$ and $s_i \leq a + b + 1$ for $i = 2, 3$, which implies that $y \leq a + 2(a + b + 1) = 3a + 2b + 2$. We now consider the case when $|N_{G_2}(u_1) \cap B| \geq 2$, $|N_{G_2}(u_2) \cap B| \leq 1$, and $|N_{G_3}(u_1) \cap B| \leq 1$. Thus G_3 is a star (centered at u_1) of size at most a , which yields $s_3 \leq a + 2$. Now suppose $N_{G_2}(u_2) \cap B \subseteq \{v_p\}$ for some p . Let $A' := A \cup \{v_p\}$ (note that $|A'| = a + 1 \geq 4$). Since every edge of G_1

intersects every edge of G_2 , we can apply Lemma 13 to $G_1[A']$ and $G_2[A']$ and obtain that $\sum_{i=1}^2 \sum_{j=1}^2 \deg_{G_i[A']}(u_j) \leq 2a+2$. Since $|N_{G_2}(u_1) \cap (B \setminus \{v_p\})| \leq b-1$ and $\deg_{G_2}(v_1) \leq 2$, it follows that $s_1 + s_2 \leq 2a+2+b-1+2 = 2a+b+3$ and $y \leq 2a+b+3+a+2 = 3a+b+5$.

We thus assume that $|N_{G_i}(u_j) \cap B| \leq 1$ for $i = 2, 3$ and $j = 1, 2$. Suppose $N_{G_2}(u_2) \cap B \subseteq \{v_p\}$ for some p and let $A' := A \cup \{v_p\}$. We apply Lemma 13 to $G_1[A']$ and $G_2[A']$ and obtain that $\sum_{i=1}^2 \sum_{j=1}^2 \deg_{G_i[A']}(u_j) \leq 2a+2$. Since $|N_{G_2}(u_1) \cap B| \leq 1$ and $\deg_{G_2}(v_1) \leq 2$, it follows that $s_1 + s_2 \leq 2a+2+1+2$. On the other hand, we have $s_3 \leq 2a+2$ because $\deg_{G_3}(u_j) \leq a$ for $j = 1, 2$ and $\deg_{G_3}(v_1) \leq 2$. Thus $y \leq 2a+5+2a+2 = 4a+7$. \square

3 Proof of Lemma 7

The proof is similar to that of [26, Lemma 4]. Let M be a largest matching of H such that each edge of M contains (exactly) one vertex of W . To the contrary, assume $|M| \leq 3s - |U| - 1$. Let $U_1 = V(M) \cap U$, $U_2 = U \setminus U_1$, $W_1 = V(M) \cap W$ and $W_2 = W \setminus W_1$. Since $|U| \geq 2s$, we have $|U_2| = |U| - 2|M| \geq 2$. Since $|W_2| = |W| - |M|$ and $|W| \geq 3s - |U|$, it follows that $W_2 \neq \emptyset$.

Below is a sketch of the proof. We first assume $|U| < 2s + \varepsilon'n$. In this case every vertex in U is adjacent to some vertex in W . If $|M|$ is not close to s , then we easily obtain a contradiction because U_2 is not small. When $|M|$ is close to s , we consider three vertices $u_1 \neq u_2 \in U_2$ and $v_0 \in W_2$, and derive a contradiction on $\deg(u_1) + \deg(u_2) + \deg(v_0)$. Next we assume $|U| \geq 2s + \varepsilon'n$. In this case U_2 is not small. If no vertex of W_2 is adjacent to any vertex of U_2 , then consider two adjacent vertices $v_0 \in W_2$ and $u_0 \in U_1$. We have $\deg(v_0) \leq \binom{2|M|}{2}$, which eventually yields that $\deg(v_0) + \deg(u_0) < 2sn - \varepsilon n^2$. Now assume $v_0 \in W_2$ is adjacent to some vertex $u_0 \in U_2$. In this case we define M' consisting of all $e \in M$ that contains a vertex $u' \in U$ such that $|N(v_0, u') \cap U_2| \geq 3$. We show that if $|M'|$ is small, then $\deg(v_0)$ is small; otherwise $\deg(u_0)$ is small. In either case we derive that $\deg(v_0) + \deg(u_0) < 2sn - \varepsilon n^2$.

We now give the details of the proof.

Case 1. $2s \leq |U| < 2s + \varepsilon'n$.

In this case we have the following two claims.

Claim 15. $|M| \geq s - \varepsilon''n$.

Proof. To the contrary, assume that $|M| < s - \varepsilon''n$. Fix $v_0 \in W_2$. Then $\deg(v_0) \leq \binom{|U|}{2} - \binom{|U_2|}{2}$ because there is no edge of type $U_2U_2W_2$. Since v_0 is not an isolated vertex, v_0 is adjacent to some vertex $u \in U$. Trivially $\deg(u) \leq \binom{|U|-1}{2} + (|U|-1)|W|$. Thus

$$\begin{aligned} \deg(v_0) + \deg(u) &\leq \binom{|U|-1}{2} + (|U|-1)|W| + \binom{|U|}{2} - \binom{|U_2|}{2} \\ &= (n-1)(|U|-1) - \binom{|U_2|}{2}. \end{aligned}$$

Since $|U| \geq 2s$ and $|M| < s - \varepsilon''n$, it follows that $|U_2| = |U| - 2|M| > 2\varepsilon''n$. As a result,

$$\deg(u) + \deg(v_0) \leq (n-1)(2s + \varepsilon'n - 1) - \binom{2\varepsilon''n}{2},$$

which contradicts the condition that $\deg(u) + \deg(v_0) > 2sn - \varepsilon n^2$ because $\varepsilon \ll \varepsilon' \ll \varepsilon''$. \square

Claim 16. *Every vertex in U is adjacent to one vertex in W .*

Proof. To the contrary, assume that $u \in U$ is not adjacent to any vertex in W . Then

$$\deg(u) \leq \binom{|U| - 1}{2} < \binom{2s + \varepsilon' n}{2},$$

which contradicts the condition that $\deg(u) > sn - \frac{1}{2}\varepsilon n^2$ because $\tau n < s \leq n/3$ and $\varepsilon \ll \varepsilon' \ll \tau$. \square

Fix $u_1 \neq u_2 \in U_2$ and $v_0 \in W_2$. Trivially $\deg(w) \leq \binom{|U|}{2}$ for any vertex $w \in W$ and $\deg(u) \leq \binom{|U|-1}{2} + |W|(|U| - 1)$ for any vertex $u \in U$. Furthermore, for any two distinct edges $e_1, e_2 \in M$, we observe that at least one triple of type UUW with one vertex in e_1 , one vertex in e_2 and one vertex in $\{u_1, u_2, v_0\}$ is *not* an edge by the choice of M . By Claim 15, $|M| \geq s - \varepsilon'' n$. Thus,

$$\deg(u_1) + \deg(u_2) + \deg(v_0) \leq 2 \left(\binom{|U| - 1}{2} + |W|(|U| - 1) \right) + \binom{|U|}{2} - \binom{s - \varepsilon'' n}{2}.$$

On the other hand, Claim 16 implies that u_i is adjacent to some vertex in W for $i = 1, 2$. We know that v_0 is adjacent to some vertex in U . Therefore, $\deg(u_i) > (2sn - \varepsilon n^2) - \binom{|U|}{2}$ for $i = 1, 2$, and $\deg(v_0) > (2sn - \varepsilon n^2) - \left(\binom{|U|-1}{2} + |W|(|U| - 1) \right)$. It follows that

$$\deg(u_1) + \deg(u_2) + \deg(v_0) > 3(2sn - \varepsilon n^2) - 2 \binom{|U|}{2} - \binom{|U| - 1}{2} - |W|(|U| - 1).$$

The upper and lower bounds for $\deg(u_1) + \deg(u_2) + \deg(v_0)$ together imply that

$$3 \left(\binom{|U| - 1}{2} + |W|(|U| - 1) + \binom{|U|}{2} \right) - \binom{s - \varepsilon'' n}{2} > 3(2sn - \varepsilon n^2),$$

$$\text{or } (|U| - 1)(n - 1) - \frac{1}{3} \binom{s - \varepsilon'' n}{2} > 2sn - \varepsilon n^2,$$

which is impossible because $|U| < 2s + \varepsilon' n$, $\tau n < s \leq n/3$, and $\varepsilon \ll \varepsilon' \ll \varepsilon'' \ll \tau$.

Case 2. $2s + \varepsilon' n \leq |U| \leq 3s$.

We consider the following two subcases.

Subcase 2.1. No vertex in U_2 is adjacent to any vertex in W_2 .

Fix $v_0 \in W_2$. Then $\deg(v_0) \leq \binom{|U_1|}{2} = \binom{2|M|}{2}$. Since v_0 is not an isolated vertex, v_0 is adjacent to some vertex $u_0 \in U_1$. We know that $\deg(u_0) \leq \binom{|U|-1}{2} + (|U| - 1)|W| - |U_2||W_2|$

because no vertex in U_2 is adjacent to any vertex in W_2 . Since $|W| = n - |U|$, $|U_2| = |U| - 2|M|$ and $|W_2| = n - |U| - |M|$, we derive that

$$\begin{aligned}\sigma_2(H) &\leq \deg(v_0) + \deg(u_0) \\ &\leq \binom{2|M|}{2} + \binom{|U| - 1}{2} + (|U| - 1)(n - |U|) - (|U| - 2|M|)(n - |U| - |M|) \\ &\leq (2n - |U|)|M| + \frac{|U|^2}{2}.\end{aligned}$$

Since $|M| < 3s - |U|$, it follows that

$$\sigma_2(H) < (2n - |U|)(3s - |U|) + \frac{|U|^2}{2} = 6sn - (3s + 2n)|U| + \frac{3}{2}|U|^2.$$

Note that the quadratic function $\frac{3}{2}x^2 - (3s + 2n)x$ is minimized at $x = s + \frac{2}{3}n$. Since $2s + \varepsilon'n \leq |U| \leq 3s \leq s + \frac{2}{3}n$, we derive that

$$\begin{aligned}\sigma_2(H) &\leq 6sn - (3s + 2n)(2s + \varepsilon'n) + \frac{3}{2}(2s + \varepsilon'n)^2 \\ &= 2sn - 2\varepsilon'n^2 + 3s\varepsilon'n + \frac{3}{2}\varepsilon'^2n^2 \leq 2sn - \varepsilon'n^2 + \frac{3}{2}\varepsilon'^2n^2\end{aligned}$$

because $s \leq n/3$. Since $\varepsilon \ll \varepsilon'$, this contradicts the assumption that $\sigma_2(H) > 2sn - \varepsilon n$.

Subcase 2.2. Two vertices $u_0 \in U_2$ and $v_0 \in W_2$ are adjacent.

Let $M' = \{e \in M : \exists u' \in e, |N(v_0, u') \cap U_2| \geq 3\}$. Assume $\{u_1, u_2, v_1\} \in M'$ such that $u_1, u_2 \in U_1$, $v_1 \in W_1$ and $|N(v_0, u_1) \cap U_2| \geq 3$. We claim that

$$N(u_0, v_1) \cap U_2 = \emptyset. \quad (3)$$

Indeed, if $\{u_0, v_1, u_3\} \in E(H)$ for some $u_3 \in U_2$, then we can find $u_4 \in U_2 \setminus \{u_0, u_3\}$ such that $\{v_0, u_1, u_4\} \in E(H)$. Replacing $\{u_1, u_2, v_1\}$ by $\{u_0, v_1, u_3\}$ and $\{v_0, u_1, u_4\}$ gives a larger matching than M , a contradiction.

By the definition of M' , we have

$$\deg(v_0) \leq \binom{|U_1|}{2} + 2|M'||U_2| + 2(|U_1| - 2|M'|) = \binom{|U_1|}{2} + 2|U_1| + |M'|(2|U_2| - 4).$$

By (3), we have

$$\deg(u_0) \leq \binom{|U| - 1}{2} + |U_1||W| + (|U_2| - 1)(|W_1| - |M'|)$$

and consequently

$$\deg(v_0) + \deg(u_0) \leq \binom{|U_1|}{2} + \binom{|U| - 1}{2} + |U_1|(|W| + 2) + (|U_2| - 1)|W_1| + |M'|(|U_2| - 3).$$

Since $|M'| \leq |M| = |W_1| = \frac{|U_1|}{2}$, it follows that

$$\begin{aligned} \deg(v_0) + \deg(u_0) &\leq \binom{|U_1|}{2} + \binom{|U| - 1}{2} + |U_1|(|W| + 2) + (|U_2| - 2)|U_1| \\ &= \binom{|U|}{2} - \binom{|U_2|}{2} + \binom{|U| - 1}{2} + |U_1||W| \\ &= (|U| - 1)^2 - \binom{|U_2|}{2} + 2|M|(n - |U|). \end{aligned}$$

Since $|M| \leq 3s - |U|$ and $|U_2| = |U| - 2|M| \geq 3|U| - 6s$, we have

$$\begin{aligned} \deg(v_0) + \deg(u_0) &\leq (|U| - 1)^2 - \binom{3|U| - 6s}{2} + 2(3s - |U|)(n - |U|) \\ &= -\frac{3}{2}|U|^2 + \left(12s - 2n - \frac{1}{2}\right)|U| + 6sn - 18s^2 - 3s + 1 \\ &\leq -\frac{3}{2}|U|^2 + (12s - 2n)|U| + 6sn - 18s^2. \end{aligned}$$

Note that the quadratic function $-\frac{3}{2}x^2 + (12s - 2n)x$ is maximized at $x = 4s - \frac{2}{3}n$. Since $3s \geq |U| \geq 2s + \varepsilon'n \geq 4s - \frac{2}{3}n$, we have

$$\begin{aligned} \sigma_2(H) \leq \deg(v_0) + \deg(u_0) &\leq -\frac{3}{2}(2s + \varepsilon'n)^2 + (12s - 2n)(2s + \varepsilon'n) + 6sn - 18s^2 \\ &= 2sn - 2\varepsilon'n^2 + 6\varepsilon'sn - \frac{3}{2}\varepsilon'^2n^2 \leq 2sn - \frac{3}{2}\varepsilon'^2n^2 \end{aligned}$$

because $s \leq n/3$. Since $\varepsilon \ll \varepsilon'$, this contradicts the assumption that $\sigma_2(H) > 2sn - \varepsilon n$.

4 Proof of Theorem 6

Suppose H is a 3-graph of order n without an isolated vertex and $\sigma_2(H) > 2sn - \varepsilon n^2$. Let $U = \{u \in V(H) : \deg(u) > sn - \varepsilon n^2/2\}$ and $W = V \setminus U$. We know that no two vertices in W are adjacent and $|U| \geq 2s$. Let M be an optimal matching as in Definition 8. By Lemma 7, such M exists. Let $M_2 = M \setminus M_1$, $U_1 = V(M_1) \cap U$, $U_2 = V(M_2)$, $U_3 = U \setminus V(M)$, $W_1 = V(M_1) \cap W$ and $W_2 = W \setminus W_1$. Since M is optimal, no edge of H is of type $W_2U_3U_3$ or $W_2U_2U_3$. In addition, for any $e \in M_1$, there are no two disjoint edges $e_1, e_2 \in e \cup W_2 \cup U_3$ such that $(e_1 \cup e_2) \cap W_2 \neq \emptyset$.

Suppose to the contrary, that $|M| \leq s - 1$. We know that $|U_3| = |U| + |M_1| - 3|M| \geq 3 + |M_1| - (3s - |U|) \geq 3$. Let $u_1, u_2, u_3 \in U_3$. Since $u_i \in U$ for $i = 1, 2, 3$, we have

$$\sum_{i=1}^3 \deg(u_i) > 3sn - \frac{3}{2}\varepsilon n^2. \quad (4)$$

On the other hand, if u_1 is adjacent to some $v_1 \in W_2$, then

$$\sum_{i=1}^2 \deg(u_i) + \deg(v_1) \geq \sigma_2(H) + \deg(u_2) > 3sn - \frac{3}{2}\varepsilon n^2. \quad (5)$$

Claim 17. For any two distinct edges e_1, e_2 from M , we have $\sum_{i=1}^3 |L_{u_i}(e_1, e_2)| \leq 18$ and $\sum_{i=1}^2 |L_{u_i}(e_1, e_2)| + |L_{v_1}(e_1, e_2)| \leq 18$.

Proof. Let H_1 be the 3-partite subgraph of H induced on three parts $\{u_1, u_2, u_3\}$, e_1 , and e_2 . We observe that H_1 does not contain a perfect matching by the choice of M . By Lemma 10, we have $|E(H_1)| = \sum_{i=1}^3 |L_{u_i}(e_1, e_2)| \leq 18$. The same argument shows that $\sum_{i=1}^2 |L_{u_i}(e_1, e_2)| + |L_{v_1}(e_1, e_2)| \leq 18$. \square

We proceed in two cases.

Case 1. $|M_1| = 3s - |U|$.

In this case, we have $|M_2| = |M| + |U| - 3s$, $|U_3| = 3s - 3|M|$ and $|W_2| = n - 3s$.

Claim 18. For any $e \in M_1$, we have

- (i) $\sum_{i=1}^2 |L_{u_i}(e, U_3 \cup W_2)| + |L_{v_1}(e, U_3 \cup W_2)| \leq \max\{4|U_3| + 7, 3|U_3| + 2|W_2| + 5\}$, where $v_1 \in W_2$;
- (ii) $\sum_{i=1}^3 |L_{u_i}(e, U_3)| \leq 6|U_3|$.

Proof. Assume $e = \{u'_1, u'_2, u'_3\} \in M_1$ with $u'_1 \in W_1$ and $u'_2, u'_3 \in U_1$.

(i) Let $A = U_3$, $B = W_2$, and $E(G_i) = L_{u'_i}(U_3 \cup W_2)$ for $i = 1, 2, 3$. By the choice of M , there are not two disjoint edges, one from G_1 and the other from G_2 or G_3 ; or one from G_2 and the other from G_3 , and at least one of them contains one vertex from B . Furthermore, it is easy to see that

$$\sum_{i=1}^2 |L_{u_i}(e, U_3 \cup W_2)| + |L_{v_1}(e, U_3 \cup W_2)| = \sum_{i=1}^3 \left(\sum_{j=1}^2 \deg_{G_i}(u_j) + \deg_{G_i}(v_1) \right).$$

The desired inequality thus follows from Lemma 14.

(ii) For $i = 1, 2, 3$, let G_i be the graph obtained from $L_{u'_i}(U_3)$ after adding an isolated vertex u^* . Then $|V(G_i)| = |U_3| + 1 \geq 4$. By the choice of M , every edge of G_1 intersects every edge of G_2 and G_3 . The desired inequality thus follows from Lemma 11. \square

Claim 19. For any $e \in M_2$, we have

- (i) $\sum_{i=1}^3 |L_{u_i}(e, U_3)| \leq 3(|U_3| + 3)$;
- (ii) $\sum_{i=1}^2 |L_{u_i}(e, U_3)| \leq 3(|U_3| + 1)$.

Proof. Assume $e = \{u'_1, u'_2, u'_3\} \in M_2$ with $u'_1, u'_2, u'_3 \in U_2$.

(i) For $i = 1, 2, 3$, let G_i be the graph obtained from $L_{u'_i}(U_3)$ after adding two isolated vertices u' and u'' . Then $|V(G_i)| = |U_3| + 2 \geq 5$. Since M is optimal, the desired inequality follows from Lemma 12.

(ii) For $i = 1, 2, 3$, let G_i be the graph obtained from $L_{u'_i}(U_3)$ after adding an isolated vertex u^* . Then $|V(G_i)| = |U_3| + 1 \geq 4$. Since M is optimal, the desired inequality follows from Lemma 13. \square

Claim 20. $s > n/3 - \eta_1 n$.

Proof. Suppose $s \leq n/3 - \eta_1 n$. We first consider the case that u_1, u_2, u_3 are not adjacent to any vertex of W_2 .

Following Claim 17, we have

$$\sum_{i=1}^3 \deg(u_i) \leq 18 \binom{|M|}{2} + 9|M| + \sum_{i=1}^3 |L_{u_i}(V(M_1), U_3)| + \sum_{i=1}^3 |L_{u_i}(V(M_2), U_3)|. \quad (6)$$

Furthermore, by Claims 18 (ii) and 19 (i), we obtain that

$$\begin{aligned} \sum_{i=1}^3 \deg(u_i) &\leq 18 \binom{|M|}{2} + 9|M| + 6|M_1||U_3| + 3|M_2|(|U_3| + 3) \\ &= 18 \binom{|M|}{2} + 9|M| + 6(3s - |U|)(3s - 3|M|) \\ &\quad + 3(|M| + |U| - 3s)(3s - 3|M| + 3) \\ &= (9|U| - 18s + 9)|M| + (3s - |U|)(9s - 9). \end{aligned}$$

Since $|M| \leq s - 1$, it follows that

$$\sum_{i=1}^3 \deg(u_i) \leq (9|U| - 18s + 9)(s - 1) + (3s - |U|)(9s - 9) = 9s^2 - 9.$$

Since $\tau n < s \leq n/3 - \eta_1 n$ and $\eta_1 < \tau$, we know that

$$3s^2 - sn = s(3s - n) \leq \max \{-\eta_1 n(n - 3\eta_1 n), -\tau n(n - 3\tau n)\} = -\eta_1 n(n - 3\eta_1 n). \quad (7)$$

Consequently, $\sum_{i=1}^3 \deg(u_i) < 9s^2 \leq 3sn - 3\eta_1 n(n - 3\eta_1 n)$. Since $\varepsilon \ll \eta_1$, this contradicts (4).

Now we assume, without loss of generality, that u_1 is adjacent to v_1 . The choice of M implies that $L_v(e, U_3) = L_u(e, W_2) = \emptyset$ for any $v \in W_2$, $u \in U_3$ and $e \in M_2$. By Claim 17, we have

$$\begin{aligned} \sum_{i=1}^2 \deg(u_i) + \deg(v_1) &\leq 18 \binom{|M|}{2} + 9|M| + \sum_{i=1}^2 |L_{u_i}(V(M_1), U_3 \cup W_2)| \\ &\quad + |L_{v_1}(V(M_1), U_3)| + \sum_{i=1}^2 |L_{u_i}(V(M_2), U_3)|. \end{aligned} \quad (8)$$

We know that $4|U_3|+7 \geq 3|U_3|+2|W_2|+5$ if and only if $|U_3| \geq 2|W_2|-2$. If $|U_3| \geq 2|W_2|-2$, then by (8), Claim 18 (i) and Claim 19 (ii), we have

$$\begin{aligned} \sum_{i=1}^2 \deg(u_i) + \deg(v_1) &\leq 18 \binom{|M|}{2} + 9|M| + |M_1|(4|U_3| + 7) + 3|M_2|(|U_3| + 1) \\ &= 18 \binom{|M|}{2} + 9|M| + (3s - |U|)(4(3s - 3|M|) + 7) \\ &\quad + 3(|M| + |U| - 3s)(3s - 3|M| + 1) \\ &= (3|U| + 3)|M| - 3s|U| - 4|U| + 9s^2 + 12s. \end{aligned}$$

Since $|M| \leq s - 1$ and $|U| \geq 2s$, it follows that

$$\begin{aligned} \sum_{i=1}^2 \deg(u_i) + \deg(v_1) &\leq (3|U| + 3)(s - 1) - 3s|U| - 4|U| + 9s^2 + 12s \\ &= -7|U| + 9s^2 + 15s - 3 \leq 9s^2 + s - 3. \end{aligned}$$

Following (7), we have $\sum_{i=1}^2 \deg(u_i) + \deg(v_1) < 3sn - 3\eta_1 n(n - 3\eta_1 n) + n/3 - 3$. Since $\varepsilon \ll \eta_1$ and n is sufficiently large, this contradicts (5).

If $|U_3| < 2|W_2| - 2$, by (8), Claim 18 (i) and Claim 19 (ii), we have

$$\begin{aligned} \sum_{i=1}^2 \deg(u_i) + \deg(v_1) &\leq 18 \binom{|M|}{2} + 9|M| + |M_1|(3|U_3| + 2|W_2| + 5) + 3|M_2|(|U_3| + 1) \\ &= (9s + 3)|M| + (-2n + 6s - 2)|U| + 6sn - 18s^2 + 6s. \end{aligned}$$

Since $|M| \leq s - 1$ and $|U| \geq 2s$, it follows that

$$\begin{aligned} \sum_{i=1}^2 \deg(u_i) + \deg(v_1) &\leq (9s + 3)(s - 1) + (-2n + 6s - 2)(2s) + 6sn - 18s^2 + 6s \\ &= 2sn + 3s^2 - 4s - 3. \end{aligned}$$

Applying (7), we have $\sum_{i=1}^2 \deg(u_i) + \deg(v_1) < 3sn - \eta_1 n(n - 3\eta_1 n)$, which contradicts (5) because $\varepsilon \ll \eta_1$. \square

By Claim 20, we have $|W_2| = n - 3s < 3\eta_1 n$. Let $H' = H[V \setminus W_2]$. We claim that $\sigma_2(H') > 2n^2/3 - \eta_2 n^2$. Indeed, recall that $\deg_H(u) + \deg_H(v) \geq 2n^2/3 - \varepsilon n^2$ for any two adjacent vertices u and v of H' . Since $|W_2| < 3\eta_1 n$ and $\varepsilon \ll \eta_1 \ll \eta_2$, it follows that

$$\deg_{H'}(u) + \deg_{H'}(v) \geq 2n^2/3 - \varepsilon n^2 - 2|W_2|n > 2n^2/3 - \eta_2 n^2.$$

Since $\eta_2 \ll 1$, we may apply Theorem 9 and conclude that either H' is a subgraph of $H_{3s,s}^2$ or H' contains a perfect matching. In the former case, there is a partition of $V(H')$ into two sets $|T| = 2s - 1$ and $|S| = s + 1$ such that for every vertex $u \in S$,

$$\deg_{H'}(u) \leq \binom{|T|}{2} = \binom{2s-1}{2} \leq \binom{2n/3-1}{2} < \frac{2}{9}n^2.$$

On the other hand, since $U \subseteq V(H')$ and $|U| \geq 2s$, there exists a vertex $u \in U \cap S$ such that

$$\begin{aligned} \deg_{H'}(u) &\geq \deg_H(u) - |W_2|n \geq sn - \frac{\varepsilon}{2}n^2 - |W_2|n \\ &\geq \left(\frac{n}{3} - \eta_1 n\right)n - \frac{\varepsilon}{2}n^2 - 3\eta_1 n^2 > \frac{2}{9}n^2, \end{aligned}$$

which is a contradiction. Therefore H' must contain a perfect matching, which is a matching of size s in H .

Case 2. $|M_1| > 3s - |U|$.

The difference from Case 1 is that, for any edge $e \in M$, we cannot find two disjoint edges e_1, e_2 from $e \cup U_3 \cup W_2$ – otherwise we can replace M by $M \setminus \{e\} \cup \{e_1, e_2\}$ contradicting the assumption that M is an optimal matching.

Note that $|U_3| = |U| + |M_1| - 3|M| \geq 3s + 1 - 3|M| \geq 4$.

Claim 21. For any $e \in M$, $\sum_{i=1}^3 |L_{u_i}(e, U_3 \cup W_2)| \leq 3(|U_3| + |W_2| + 2)$.

Proof. Assume $e = \{u'_1, u'_2, u'_3\} \in M$. For $i = 1, 2, 3$, let G_i be the graph obtained from $L_{u'_i}(U_3 \cup W_2)$ after adding an isolated vertex u^* . Then $|V(G_i)| = |U_3| + |W_2| + 1 \geq 5$. Since H contains no two disjoint edges e_1, e_2 from $e \cup U_3 \cup W_2$, we know that for any $i \neq j$, every edge of G_i intersects every edge of G_j . The desired inequality thus follows from Lemma 12. \square

By Claims 17 and 21, we obtain that

$$\begin{aligned} \sum_{i=1}^3 \deg(u_i) &\leq 18 \binom{|M|}{2} + 9|M| + \sum_{i=1}^3 |L_{u_i}(V(M), U_3 \cup W_2)| \\ &\leq 18 \binom{|M|}{2} + 9|M| + 3|M|(|U_3| + |W_2| + 2) \\ &= (3n + 6)|M| \leq 3sn + 6s. \end{aligned} \tag{9}$$

Let $W' = \{v \in W : \deg(v) \leq sn - s^2/2 + \gamma'n^2\}$. If $|W'| \leq \gamma n$, then we let $H' := H[V \setminus W']$. By the definition of W' , $\deg_H(u) > sn - s^2/2 + \gamma'n^2$ for every $u \in V(H') \cap W$. For any $u \in V(H') \cap U$, $\deg_H(u) > sn - \varepsilon n^2/2 > sn - s^2/2 + \gamma'n^2$ because $s > \tau n$ and $\varepsilon \ll \gamma' \ll \tau$. Therefore every vertex $u \in V(H')$ satisfies

$$\deg_{H'}(u) \geq \deg_H(u) - n|W'| > sn - \frac{s^2}{2} + \gamma'n^2 - \gamma n^2 > \binom{n-1}{2} - \binom{n-s}{2} + 1,$$

because $|W'| \leq \gamma n$, $\gamma \ll \gamma'$, and n is sufficiently large. By Theorem 1, H' contains a matching of size s .

We thus assume that $|W'| > \gamma n$ for the rest of the proof. If one of u_1, u_2, u_3 is adjacent to a vertex of W' , then

$$\sum_{i=1}^3 \deg(u_i) > 4 \left(sn - \frac{\varepsilon}{2}n^2 \right) - \left(sn - \frac{s^2}{2} + \gamma'n^2 \right) = 3sn + \frac{s^2}{2} - 2\varepsilon n^2 - \gamma'n^2,$$

which contradicts (9) because $s > \tau n$ is sufficiently large and $\varepsilon \ll \gamma' \ll \tau$.

If none of u_1, u_2, u_3 is adjacent to a vertex of W' , then we distinguish the following two subcases.

Subcase 2.1. $|W' \cap W_1| > \gamma n/2$.

Let $M' = \{e \in M : e \cap W' \neq \emptyset\}$, thus $|M'| > \gamma n/2$. Since u_1, u_2, u_3 are not adjacent to any vertex in $W' \cap W_1$, then for any distinct e_1, e_2 from M' , we have

$$\sum_{i=1}^3 |L_{u_i}(e_1, e_2)| \leq 12. \quad (10)$$

By Claims 17, 21 and (10), we have

$$\begin{aligned} \sum_{i=1}^3 \deg(u_i) &\leq \left(18 \binom{|M|}{2} - 6 \binom{|M'|}{2}\right) + 9|M| + 3|M|(n - 3|M| + 2) \\ &\leq (3n + 6)|M| - 6 \binom{|M'|}{2}. \end{aligned}$$

Since $|M'| > \gamma n/2$, it follows that

$$\sum_{i=1}^3 \deg(u_i) \leq (3n + 6)(s - 1) - 6 \binom{\gamma n/2}{2},$$

which contradicts (4) because $s \leq n/3$ and $\varepsilon \ll \gamma$.

Subcase 2.2. $|W' \cap W_1| \leq \gamma n/2$.

Since $|W'| > \gamma n$, we have $|W' \cap W_2| > \gamma n/2$. Let $W_2^* = W_2 \setminus W'$. Then $W_2 \setminus W_2^* = W' \cap W_2$. By Claim 21, we obtain that $\sum_{i=1}^3 |L_{u_i}(V(M), U_3 \cup W_2^*)| \leq 3|M|(|U_3| + |W_2^*| + 2)$. Therefore,

$$\begin{aligned} \sum_{i=1}^3 \deg(u_i) &\leq 18 \binom{|M|}{2} + 9|M| + \sum_{i=1}^3 |L_{u_i}(V(M), U_3 \cup W_2^*)| \\ &\leq 18 \binom{|M|}{2} + 9|M| + 3|M|(|U_3| + |W_2^*| + 2) \\ &= 18 \binom{|M|}{2} + 9|M| + 3|M|(|U_3| + |W_2| + 2) - 3|M||W_2 \setminus W_2^*| \\ &= \left(3n + 6 - \frac{3}{2}\gamma n\right)|M|, \end{aligned}$$

which contradicts (4) because $|M| \leq s$, $\tau n < s$, and $\varepsilon \ll \gamma \ll \tau$. This completes the proof of Theorem 6.

References

- [1] R. Aharoni and D. Howard, A rainbow r -partite version of the Erdős-Ko-Rado theorem, *Combin. Probab. Comput.* 26 (2017), 321–337.

- [2] N. Alon, P. Frankl, H. Huang, V. Rödl, A. Ruciński, and B. Sudakov. Large matchings in uniform hypergraphs and the conjectures of Erdős and Samuels, *J. Combin. Theory Ser. A* 119 (2012), 1200–1215.
- [3] P. Erdős, A problem on independent r -tuples, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* 8 (1965), 93–95.
- [4] P. Frankl, On the maximum number of edges in a hypergraph with given matching number, *Discrete Appl. Math.* 216 (2017), 562–581.
- [5] H. Hàn, Y. Person, and M. Schacht, On perfect matchings in uniform hypergraphs with large minimum vertex degree, *SIAM J. Discrete Math.* 23 (2009), 732–748.
- [6] J. Han, Perfect matchings in hypergraphs and the Erdős matching conjecture, *SIAM J. Discrete Math.* 30 (2016), 1351–1357.
- [7] I. Khan, Perfect matching in 3-uniform hypergraphs with large vertex degree, *SIAM J. Discrete Math.* 27 (2013), 1021–1039.
- [8] I. Khan, Perfect matchings in 4-uniform hypergraphs, *J. Combin. Theory Ser. B* 116 (2016), 333–366.
- [9] D. Kühn and D. Osthus, Matchings in hypergraphs of large minimum degree, *J. Graph Theory* 51 (2006), 269–280.
- [10] D. Kühn, D. Osthus and A. Treglown, Matchings in 3-uniform hypergraphs, *J. Combin. Theory Ser. B* 103 (2013), 291–305.
- [11] T. Łuczak and K. Mieczkowska, On Erdős’ extremal problem on matchings in hypergraphs, *J. Combin. Theory, Ser. A* 124 (2014), 178–194.
- [12] K. Markström and A. Ruciński, Perfect matchings (and Hamilton cycles) in hypergraphs with large degrees, *European J. Combin.* 32 (2011), 677–687.
- [13] O. Ore, Note on Hamilton circuits. *Amer. Math. Monthly* 67 (1960), 55.
- [14] O. Pikhurko, Perfect matchings and K_4^3 -tilings in hypergraphs of large codegree, *Graphs Combin.* 24 (2008), 391–404.
- [15] V. Rödl and A. Ruciński, Dirac-type questions for hypergraphs – a survey (or more problems for Endre to solve), *An Irregular Mind*, Bolyai Soc. Math. Studies 21 (2010), 561–590.
- [16] V. Rödl, A. Ruciński, and E. Szemerédi, Perfect matchings in uniform hypergraphs with large minimum degree, *European J. Combin.* 27 (2006), 1333–1349.
- [17] V. Rödl, A. Ruciński, and E. Szemerédi, An approximate Dirac-type theorem for k -uniform hypergraphs, *Combinatorica*, 28 (2008), 229–260.
- [18] V. Rödl, A. Ruciński, and E. Szemerédi, Perfect matchings in large uniform hypergraphs with large minimum collective degree, *J. Combin. Theory Ser. A* 116 (2009), 613–636.
- [19] Y. Tang and G. Yan, An approximate Ore-type result for tight hamilton cycles in uniform hypergraphs, *Discrete Math.* 340 (2017), 1528–1534.

- [20] A. Treglown and Y. Zhao, Exact minimum degree thresholds for perfect matchings in uniform hypergraphs, *J. Combin. Theory Ser. A* 119 (2012), 1500–1522.
- [21] A. Treglown and Y. Zhao, Exact minimum degree thresholds for perfect matchings in uniform hypergraphs II, *J. Combin. Theory Ser. A* 120 (2013), 1463–1482.
- [22] A. Treglown and Y. Zhao, A note on perfect matchings in uniform hypergraphs, *Electron. J. Combin.* 23 (2016), #P1.16.
- [23] Y. Zhang and M. Lu, Some Ore-type results for matching and perfect matching in k -uniform hypergraphs, *Acta. Math. Sin. – English Ser.* 34 (2018) 1795–1803.
- [24] Y. Zhang and M. Lu, d -matching in 3-uniform hypergraphs, *Discrete Math.* 341 (2018), 748–758.
- [25] Y. Zhang and M. Lu, Matching in 3-uniform hypergraphs, *Discrete Math.* 342 (2019), 1731–1737.
- [26] Y. Zhang, Y. Zhao and M. Lu, Vertex degree sums for perfect matchings in 3-uniform hypergraphs, *Electron. J. Combin.* 25 (2018), #P3.45.
- [27] Y. Zhao, Recent advances on dirac-type problems for hypergraphs. In *Recent Trends in Combinatorics*, volume 159 of the *IMA Volumes in Mathematics and its Applications*. Springer, New York, 2016.