

# Vertex degree sums for matchings in 3-uniform hypergraphs

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## Abstract

Let  $n, s$  be positive integers such that  $n$  is sufficiently large and  $s \leq n/3$ . Suppose  $H$  is a 3-uniform hypergraph of order  $n$  without isolated vertices. If  $\deg(u) + \deg(v) > 2(s-1)(n-1)$  for any two vertices  $u$  and  $v$  that are contained in some edge of  $H$ , then  $H$  contains a matching of size  $s$ . This degree sum condition is best possible and confirms a conjecture of the authors [Electron. J. Combin. 25 (3), 2018], who proved the case when  $s = n/3$ .

**Mathematics Subject Classifications:** 05C70,05C65

## 1 Introduction

A *k-uniform hypergraph*  $H$  (in short, *k-graph*) is a pair  $(V, E)$ , where  $V$  is a finite set of vertices and  $E$  is a family of  $k$ -element subsets of  $V$ . Note that a 2-graph is simply a graph. Let  $V(H)$  and  $E(H)$  denote the vertex set and edge set of  $H$ , respectively. A *matching of size  $s$*  in  $H$  is a family of  $s$  pairwise disjoint edges of  $H$ . If the matching covers

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all the vertices of  $H$ , then we call it a *perfect matching*. Given a set  $S \subseteq V$ , the *degree*  $\deg_H(S)$  of  $S$  is the number of the edges of  $H$  containing  $S$ . We simply write  $\deg(S)$  when  $H$  is obvious from the context. Further, let  $\delta_\ell(H) = \min\{\deg(S) : S \subseteq V(H), |S| = \ell\}$ .

Given integers  $\ell < k \leq n$  such that  $k$  divides  $n$ , let  $m_\ell(k, n)$  denote the smallest integer  $m$  such that every  $k$ -graph  $H$  on  $n$  vertices with  $\delta_\ell(H) \geq m$  contains a perfect matching. In recent years the problem of determining  $m_\ell(k, n)$  has received much attention (see [2, 5, 6, 7, 8, 9, 10, 12, 14, 17, 16, 18, 20, 21, 22]). In particular, Rödl, Ruciński and Szemerédi [18] determined  $m_{k-1}(k, n)$  for all  $k \geq 3$  and sufficiently large  $n$ . Treglown and Zhao [20, 21] determined  $m_\ell(k, n)$  for all  $\ell \geq k/2$  and sufficiently large  $n$ . More Dirac-type results on hypergraphs can be found in surveys [15, 27].

A well-known result of Ore [13] extended Dirac's theorem by determining the smallest degree sum of two non-adjacent vertices that guarantees a Hamilton cycle in graphs. Ore-type problems for hypergraphs have been studied recently. For example, Tang and Yan [19] studied the degree sum of two  $(k-1)$ -sets that guarantees a tight Hamilton cycle in  $k$ -graphs. Zhang and Lu [23] studied the degree sum of two  $(k-1)$ -sets that guarantees a perfect matching in  $k$ -graphs. Zhang, Zhao and Lu [26] determined the minimum degree sum of two adjacent vertices that guarantees a perfect matching in 3-graphs without isolated vertices, see Theorem 2 (two vertices in a hypergraph are *adjacent* if there exists an edge containing both of them). Note that one may study the minimum degree sum of two arbitrary vertices and that of two non-adjacent vertices that guarantees a perfect matching instead. In fact, it was mentioned in [26] that the former equals to  $2m_1(3, n) - 1$  while the latter does not exist.

Let us define (potential) extremal 3-graphs for the matching problem. For  $1 \leq \ell \leq 3$ , let  $H_{n,s}^\ell$  denote the 3-graph of order  $n$ , whose vertex set is partitioned into two sets  $S$  and  $T$  of size  $n - s\ell + 1$  and  $s\ell - 1$ , respectively, and whose edge set consists of all triples with at least  $\ell$  vertices in  $T$ . A well-known conjecture of Erdős [3], recently verified for 3-graphs [4, 11], implies that  $H_{n,s}^1$  or  $H_{n,s}^3$  is the densest 3-graph on  $n$  vertices not containing a matching of size  $s$ . On the other hand, Kühn, Osthus and Treglown [10] showed that for sufficiently large  $n$ ,  $H_{n,s}^1$  has the largest minimum vertex degree among all 3-graphs on  $n$  vertices not containing a matching of size  $s$ .

**Theorem 1.** [10] *There exists  $n_0 \in \mathbb{N}$  such that if  $H$  is a 3-graph of order  $n \geq n_0$  with  $\delta_1(H) > \delta_1(H_{n,s}^1) = \binom{n-1}{2} - \binom{n-s}{2}$  and  $n \geq 3s$ , then  $H$  contains a matching of size  $s$ .*

Given a 3-graph  $H$ , let  $\sigma_2(H)$  denote the minimum  $\deg(u) + \deg(v)$  among all adjacent vertices  $u$  and  $v$ . It is easy to see that

$$\begin{aligned} \sigma_2(H_{n,s}^3) &= 2 \binom{3s-2}{2}, \quad \sigma_2(H_{n,s}^1) = 2 \left( \binom{n-1}{2} - \binom{n-s}{2} \right), \text{ and} \\ \sigma_2(H_{n,s}^2) &= \binom{2s-2}{2} + (n-2s+1) \binom{2s-2}{1} + \binom{2s-1}{2} = (2s-2)(n-1). \end{aligned}$$

The following is [26, Theorem 1], which implies that, when  $n$  is divisible by 3 and sufficiently large,  $H_{n,n/3}^2$  has the largest  $\sigma_2(H)$  among all  $n$ -vertex 3-graphs  $H$  containing no isolated vertex or perfect matching.

**Theorem 2.** [26] *There exists  $n_0 \in \mathbb{N}$  such that the following holds for all integers  $n \geq n_0$  that are divisible by 3. Let  $H$  be a 3-graph of order  $n$  without an isolated vertex. If  $\sigma_2(H) > \sigma_2(H_{n,n/3}^2) = \frac{2}{3}n^2 - \frac{8}{3}n + 2$ , then  $H$  contains a perfect matching.*

Zhang, Zhao and Lu [26, Conjecture 12] further conjectured that for sufficiently large  $n$  and any  $s < n/3$ ,  $H_{n,s}^2$  has the largest  $\sigma_2(H)$  among all  $n$ -vertex 3-graphs  $H$  containing no isolated vertex or matching of size  $s$ . In this paper we verify this conjecture.

**Theorem 3.** *There exists  $n_1 \in \mathbb{N}$  such that the following holds for all integers  $n \geq n_1$  and  $s \leq n/3$ . If  $H$  is a 3-graph of order  $n$  without an isolated vertex and  $\sigma_2(H) > \sigma_2(H_{n,s}^2) = 2(s-1)(n-1)$ , then  $H$  contains a matching of size  $s$ .*

Since the two theorems have different extremal hypergraphs, Theorem 3 does not imply Theorem 1 (analogously Theorem 1 does not imply Erdős' matching conjecture for 3-graphs). On the other hand, one may wonder why we assume that  $H$  contains no isolated vertex in Theorem 3 (especially when  $s < n/3$ ). In fact, as shown in the concluding remarks of [26], Theorem 3 implies another conjecture [26, Conjecture 13], which determines the largest  $\sigma_2(H)$  among all 3-graphs containing no matching of size  $s$ . Note that  $\sigma_2(H_{n,s}^2) \geq \sigma_2(H_{n,s}^3)$  if and only if  $s \leq (2n+4)/9$ .

**Corollary 4.** *There exists  $n_2 \in \mathbb{N}$  such that the following holds. Suppose that  $H$  is a 3-graph of order  $n \geq n_2$  and  $2 \leq s \leq n/3$ . If  $\sigma_2(H) > \max\{\sigma_2(H_{n,s}^2), \sigma_2(H_{n,s}^3)\}$ , then  $H$  contains a matching of size  $s$ .*

Let us explain our approach towards Theorem 3. The case when  $s \leq n/13$  was already solved by Zhang and Lu [24] in a stronger form. Note that  $\sigma_2(H_{n,s}^2) > \sigma_2(H_{n,s}^1)$ . The following theorem shows that, when  $n \geq 13s$ , not only is  $H_{n,s}^2$  the (unique) 3-graph with the largest  $\sigma_2(H)$  among all  $H$  containing no isolated vertex or a matching of size  $s$ , but also  $H_{n,s}^1$  is the sub-extremal 3-graph for this problem. (In fact, Zhang and Lu [24] conjectured that Theorem 5 holds for all  $n \geq 3s$ . If true, this strengthens Theorem 1 and actually provides a link between Ore's and Dirac's problems.)

**Theorem 5.** [25] *Let  $n, s$  be positive integers and  $H$  be a 3-graph of order  $n \geq 13s$  without an isolated vertex. If  $\sigma_2(H) > \sigma_2(H_{n,s}^1) = 2\left(\binom{n-1}{2} - \binom{n-s}{2}\right)$ , then either  $H$  contains a matching of size  $s$  or  $H$  is a subgraph of  $H_{n,s}^2$ .*

Therefore it suffices to prove Theorem 3 for reasonably large  $s$ . For such  $s$ , we actually prove a (stronger) stability theorem.

**Theorem 6.** *Given  $0 < \varepsilon \ll \tau \ll 1$ , let  $n$  be sufficiently large and  $\tau n < s \leq n/3$ . If  $H$  is a 3-graph of order  $n$  without an isolated vertex such that  $\sigma_2(H) > 2sn - \varepsilon n^2$ , then either  $H$  is a subgraph of  $H_{n,s}^2$  or  $H$  contains a matching of size  $s$ .*

Theorem 3 follows from Theorem 6 immediately. Indeed, if  $\sigma_2(H) > \sigma_2(H_{n,s}^2)$ , then it is easy to see that  $H$  is not a subgraph of  $H_{n,s}^2$ .<sup>1</sup> Suppose instead, that  $V(H)$  can be

<sup>1</sup>Unfortunately  $\sigma_2$  is not a monotone function: for example, adding an edge to  $H_{n,s}^2$  indeed reduces the value of  $\sigma_2$  because two vertices in  $S$  now become adjacent and their degree sum is smaller than  $\sigma_2(H_{n,s}^2)$ .

partitioned  $S \cup T$  such that  $|S| = n - 2s + 1$ ,  $|T| = 2s - 1$ , and every edge of  $H$  contains at least two vertices of  $T$ . Since  $H$  contains no isolated vertices, every vertex of  $S$  is adjacent to some vertex of  $T$ . Thus  $\sigma_2(H) \leq \deg(u) + \deg(v)$  for some  $u \in S$  and  $v \in T$ . Consequently  $\sigma_2(H) \leq \sigma_2(H_{n,s}^2)$ , a contradiction. We therefore apply Theorem 6 to derive that  $H$  contains a matching of size  $s$ . Furthermore, Theorem 6 implies that  $H_{n,s}^2$  is the unique extremal 3-graph for Theorem 3 because all proper subgraphs  $H$  of  $H_{n,s}^2$  satisfy  $\sigma_2(H) < \sigma_2(H_{n,s}^2)$ .

In order to prove Theorem 6, we follow the same approach as in [26]: using the condition on  $\sigma_2(H)$ , we greedily extend a matching of  $H$  until it has  $s$  edges. An important intermediate step is finding a matching that covers a certain number of low-degree vertices (see Lemma 7). Nevertheless, the proof of Theorem 6 does require new ideas: in particular, the meaning of an *optimal* matching is more complicated (see Definition 8); we proceed differently depending on whether the number of low-degree vertices in the optimal matching is at the threshold. In one case we reduce the problem to that of finding a perfect matching in a subgraph of  $H$  and apply the main result of [26] (see Theorem 9).

This paper is organized as follows. In Section 2, we give an outline of the proof along with some preliminary results. We prove Lemma 7 in Section 3 and complete the proof in Section 4.

**Notation:** Given a graph  $G$  and a vertex  $u$  in  $G$ ,  $N_G(u)$  is the set of neighbors of  $u$  in  $G$ . Suppose  $H$  is a 3-uniform hypergraph. For  $u \neq v \in V(H)$ , let  $N_H(u, v) = \{w \in V(H) : \{u, v, w\} \in E(H)\}$  (the subscript is often omitted when  $H$  is clear from the context). Given three subsets  $V_1, V_2, V_3$  of  $V(H)$ , we say that an edge  $\{v_1, v_2, v_3\} \in E(H)$  is a type of  $V_1 V_2 V_3$  if  $v_i \in V_i$  for  $1 \leq i \leq 3$ . Given a vertex  $v \in V(H)$  and a subset  $A \subseteq V(H)$ , we define the *link*  $L_v(A) = \{uw : u, w \in A \text{ and } \{u, v, w\} \in E(H)\}$ . When  $A$  and  $B$  are two disjoint subsets of  $V(H)$ , we let  $L_v(A, B) = \{uw : u \in A, w \in B \text{ and } \{u, v, w\} \in E(H)\}$ .

We write  $0 < a_1 \ll a_2 \ll a_3$  if we can choose the constants  $a_1, a_2, a_3$  from right to left. More precisely there are increasing functions  $f$  and  $g$  such that given  $a_3$ , whenever we choose some  $a_2 \leq f(a_3)$  and  $a_1 \leq g(a_2)$ , all calculations needed in our proof are valid.

## 2 Outline of the proof and preliminaries

Let  $n$  be sufficiently large and  $\tau n < s \leq n/3$ . Suppose  $H$  is a 3-graph of order  $n$  without an isolated vertex and  $\sigma_2(H) > 2sn - \varepsilon n^2$ . Let  $U = \{u \in V(H) : \deg(u) > sn - \frac{\varepsilon}{2}n^2\}$  and  $W = V \setminus U$ . Then any two vertices of  $W$  are not adjacent – otherwise  $\sigma_2(H) \leq 2sn - \varepsilon n^2$ , a contradiction. If  $|U| \leq 2s - 1$ , then  $H$  is a subgraph of  $H_{n,s}^2$  and we are done. We thus assume that  $|U| \geq 2s$ .

Throughout the proof we use small constants

$$0 < \varepsilon \ll \varepsilon' \ll \varepsilon'' \ll \eta_1 \ll \eta_2 \ll \gamma \ll \gamma' \ll \tau \ll 1. \quad (1)$$

We first prove the following lemma, which is an extension of [26, Lemma 4].

**Lemma 7.** *Given  $0 < \varepsilon \ll \tau \ll 1$ , let  $n$  be sufficiently large and  $\tau n < s \leq n/3$ . Suppose  $H$  is a 3-graph of order  $n$  without an isolated vertex and  $\sigma_2(H) > 2sn - \varepsilon n^2$ . Let*

$U = \{u \in V(H) : \deg(u) > sn - \varepsilon n^2/2\}$  and  $W = V \setminus U$ . If  $2s \leq |U| \leq 3s$ , then  $H$  contains a matching of size  $3s - |U|$ , each of which contains exactly one vertex of  $W$ .

**Definition 8.** We call a matching  $M$  *optimal* if (i)  $M$  contains a submatching  $M_1 = \{e \in M : e \cap W \neq \emptyset\}$  of size at least  $3s - |U|$ ; (ii) subject to (i),  $|M|$  is as large as possible; (iii) subject to (i) and (ii),  $|M_1|$  is as large as possible.

Lemma 7 shows that  $H$  contains an optimal matching  $M$ . We separate the cases when  $|M_1| = 3s - |U|$  and when  $|M_1| > 3s - |U|$ . When  $|M_1| = 3s - |U|$ , we first consider the case when  $s \leq n/3 - \eta_1 n$ . If no vertex of  $U_3 := U \setminus V(M)$  is adjacent to any vertex of  $W_2 := W \setminus V(M)$ , then the assumption  $|M_1| = 3s - |U|$  forces  $\sum_{i=1}^3 \deg(u_i)$  to be smaller than  $3sn - \frac{3}{2}\varepsilon n^2$  for any three vertices  $u_1, u_2, u_3 \in U_3$ . If some vertex  $u_1 \in U_3$  is adjacent to  $v_1 \in W_2$ , then the fact  $v_1 \in W$  reduces  $\sum_{i=1}^2 \deg(u_i) + \deg(v_1)$  to a number less than  $3sn - \frac{3}{2}\varepsilon n^2$  (where  $u_2$  is another vertex of  $U_3$ ). When  $s > n/3 - \eta_1 n$ , we consider  $H' = H[V \setminus W_2]$ . Since  $|W_2| = n - 3s$  is very small, we deduce that  $\sigma_2(H')$  is greater than  $2sn - \eta_2 n^2$ . This allows us to apply the following theorem from [26] to obtain a perfect matching of  $H'$ , which is also a matching of size  $s$  of  $H$ .

**Theorem 9.** [26] *There exist  $\eta_2 > 0$  and  $n_0 \in \mathbb{N}$  such that the following holds for all integers  $n \geq n_0$  that are divisible by 3. Suppose that  $H$  is a 3-graph of order  $n$  without an isolated vertex and  $\sigma_2(H) > 2n^2/3 - \eta_2 n^2$ , then either  $H$  is a subgraph of  $H_{n,n/3}^2$  or  $H$  contains a perfect matching.*

Now consider the case when  $|M_1| > 3s - |U|$ . Let  $W' := \{v \in W : \deg(v) \leq sn - s^2/2 + \gamma'n^2\}$ . If  $|W'|$  is very small, then we can find a matching of size  $s$  in  $H[V \setminus W']$  by Theorem 1. When  $|W'|$  is not small, we consider  $u_1, u_2, u_3 \in U_3$ . If one of  $u_1, u_2, u_3$  is adjacent to one vertex from  $W'$ , then  $\sum_{i=1}^3 \deg(u_i)$  becomes much larger than  $3sn$ ; otherwise we show that  $\sum_{i=1}^3 \deg(u_i) < 3sn - \frac{3}{2}\varepsilon n^2$  by proceeding with the cases when  $|W' \cap W_1| > \gamma n/2$  and when  $|W' \cap W_2| > \gamma n/2$  separately.

In the proof we need several (simple) extremal results on (hyper)graphs. Lemma 10 is Observation 1.8 of Aharoni and Howard [1]. Lemmas 11 and 12 are from [26]. A  $k$ -graph  $H$  is called  *$k$ -partite* if  $V(H)$  can be partitioned into  $V_1, \dots, V_k$ , such that each edge of  $H$  meets every  $V_i$  in precisely one vertex. If all parts are of the same size  $n$ , we call  $H$   *$n$ -balanced*.

**Lemma 10.** [1] *Let  $F$  be the edge set of an  $n$ -balanced  $k$ -partite  $k$ -graph. If  $F$  does not contain  $s$  disjoint edges, then  $|F| \leq (s-1)n^{k-1}$ .*

**Lemma 11.** [26] *Let  $G_1, G_2, G_3$  be three graphs on the same set  $V$  of  $n \geq 4$  vertices such that every edge of  $G_1$  intersects every edge of  $G_i$  for both  $i = 2, 3$ . Then  $\sum_{i=1}^3 \sum_{v \in A} \deg_{G_i}(v) \leq 6(n-1)$  for any set  $A \subset V$  of size 3.*

**Lemma 12.** [26] *Let  $G_1, G_2, G_3$  be three graphs on the same set  $V$  of  $n \geq 5$  vertices such that for any  $i \neq j$ , every edge of  $G_i$  intersects every edge of  $G_j$ . Then  $\sum_{i=1}^3 \sum_{v \in A} \deg_{G_i}(v) \leq 3(n+1)$  for any set  $A \subset V$  of size 3.*

Following the same proof of Lemmas 11 and 12 from [26], we obtain another lemma and omit its proof.

**Lemma 13.** *Let  $G_1, \dots, G_k$  be  $k$  graphs on the same set  $V$  of  $n \geq 4$  vertices such that for any  $1 \leq i < j \leq k$ , every edge of  $G_i$  intersects every edge of  $G_j$ . Then  $\sum_{i=1}^k \sum_{v \in A} \deg_{G_i}(v) \leq kn$  for any set  $A \subset V$  of size 2.  $\square$*

The following lemma needs slightly more work so we include a proof.

**Lemma 14.** *Given two disjoint vertex sets  $A = \{u_1, u_2, \dots, u_a\}$  and  $B = \{v_1, v_2, \dots, v_b\}$  with  $a \geq 3$  and  $b \geq 1$ . Let  $G_i$ ,  $i = 1, 2, 3$ , be graphs on  $A \cup B$  such that every vertex of  $B$  is an isolated vertex in  $G_1$ , and every edge of  $G_i$  ( $i = 2, 3$ ) contains at least one vertex of  $A$ . If there are no two disjoint edges (i) one from  $G_1$  and the other from  $G_2$  or  $G_3$ ; or (ii) one from  $G_2$  and the other from  $G_3$ , and at least one of them contains a vertex from  $B$ , then*

$$\sum_{i=1}^3 \left( \sum_{j=1}^2 \deg_{G_i}(u_j) + \deg_{G_i}(v_1) \right) \leq \max\{4a + 7, 3a + 2b + 5\}.$$

*Proof.* For convenience, let  $s_i = \sum_{j=1}^2 \deg_{G_i}(u_j) + \deg_{G_i}(v_1)$  for  $i = 1, 2, 3$  and  $y = s_1 + s_2 + s_3$ . Below we show that  $y \leq \max\{4a + 7, 3a + 2b + 5\}$ .

We first observe that if  $\deg_{G_i}(v_1) \geq 3$  for some  $i \in \{2, 3\}$ , then  $E(G_1) = \emptyset$  and  $G_{i'}$  is a star centered at  $v_1$ , where  $i' = 5 - i$ . Indeed, if  $G_1$  or  $G_{i'}$  contains an edge  $e$  not incident to  $v_1$ , then  $e$  is disjoint from some edge of  $G_i$  that is incident to  $v_1$  – this contradicts our assumption. The observation implies that if  $\deg_{G_i}(v_1) \geq 3$  for both  $i = 2, 3$ , then  $E(G_1) = \emptyset$  and both  $G_2$  and  $G_3$  are stars centered at  $v_1$ . In this case,  $s_i \leq a + 2$  for  $i = 2, 3$  and thus  $y \leq 2(a + 2)$ . If  $\deg_{G_2}(v_1) \geq 3$  and  $\deg_{G_3}(v_1) \leq 2$ , then  $E(G_1) = \emptyset$  and  $G_3$  consists of at most two edges incident to  $v_1$ . In this case,  $s_1 \leq 2(a + b - 1) + a$ ,  $s_2 \leq 4$  and thus  $y \leq 3a + 2b + 2$ . The case when  $\deg_{G_2}(v_1) \leq 2$  and  $\deg_{G_3}(v_1) \geq 3$  is analogous. We thus assume that

$$\deg_{G_i}(v_1) \leq 2 \quad \text{for } i = 2, 3 \tag{2}$$

for the rest of the proof.

Next, we observe that if  $|N_{G_i}(u_j) \cap B| \geq 2$  for some  $i \in \{2, 3\}$  and some  $j \in \{1, 2\}$ , then  $G_{i'}$  is a star centered at  $u_j$  for  $i' \in \{1, 2, 3\} \setminus \{i\}$ . This is again due to our assumption on  $G_1, G_2$  and  $G_3$ . The observation implies that if  $|N_{G_i}(u_j) \cap B| \geq 2$  for both  $j = 1, 2$ , then  $E(G_{i'}) \subseteq \{u_1 u_2\}$  and consequently,  $s_{i'} \leq 2$  for  $i' \in \{1, 2, 3\} \setminus \{i\}$ . By (2), we have  $s_i \leq 2(a + b - 1) + 2$ . Therefore,  $y \leq 2(a + b - 1) + 2 + 4 = 2a + 2b + 4$ . The observation also implies that if  $|N_{G_i}(u_j) \cap B| \geq 2$  for both  $i = 2, 3$ , then  $G_1, G_2, G_3$  are all stars centered at  $u_j$ . In this case,  $s_1 \leq a$  and  $s_i \leq a + b + 1$  for  $i = 2, 3$ , which implies that  $y \leq a + 2(a + b + 1) = 3a + 2b + 2$ . We now consider the case when  $|N_{G_2}(u_1) \cap B| \geq 2$ ,  $|N_{G_2}(u_2) \cap B| \leq 1$ , and  $|N_{G_3}(u_1) \cap B| \leq 1$ . Thus  $G_3$  is a star (centered at  $u_1$ ) of size at most  $a$ , which yields  $s_3 \leq a + 2$ . Now suppose  $N_{G_2}(u_2) \cap B \subseteq \{v_p\}$  for some  $p$ . Let  $A' := A \cup \{v_p\}$  (note that  $|A'| = a + 1 \geq 4$ ). Since every edge of  $G_1$

intersects every edge of  $G_2$ , we can apply Lemma 13 to  $G_1[A']$  and  $G_2[A']$  and obtain that  $\sum_{i=1}^2 \sum_{j=1}^2 \deg_{G_i[A']}(u_j) \leq 2a+2$ . Since  $|N_{G_2}(u_1) \cap (B \setminus \{v_p\})| \leq b-1$  and  $\deg_{G_2}(v_1) \leq 2$ , it follows that  $s_1+s_2 \leq 2a+2+b-1+2 = 2a+b+3$  and  $y \leq 2a+b+3+a+2 = 3a+b+5$ .

We thus assume that  $|N_{G_i}(u_j) \cap B| \leq 1$  for  $i = 2, 3$  and  $j = 1, 2$ . Suppose  $N_{G_2}(u_2) \cap B \subseteq \{v_p\}$  for some  $p$  and let  $A' := A \cup \{v_p\}$ . We apply Lemma 13 to  $G_1[A']$  and  $G_2[A']$  and obtain that  $\sum_{i=1}^2 \sum_{j=1}^2 \deg_{G_i[A']}(u_j) \leq 2a+2$ . Since  $|N_{G_2}(u_1) \cap B| \leq 1$  and  $\deg_{G_2}(v_1) \leq 2$ , it follows that  $s_1+s_2 \leq 2a+2+1+2$ . On the other hand, we have  $s_3 \leq 2a+2$  because  $\deg_{G_3}(u_j) \leq a$  for  $j = 1, 2$  and  $\deg_{G_3}(v_1) \leq 2$ . Thus  $y \leq 2a+5+2a+2 = 4a+7$ .  $\square$

### 3 Proof of Lemma 7

The proof is similar to that of [26, Lemma 4]. Let  $M$  be a largest matching of  $H$  such that each edge of  $M$  contains (exactly) one vertex of  $W$ . To the contrary, assume  $|M| \leq 3s - |U| - 1$ . Let  $U_1 = V(M) \cap U$ ,  $U_2 = U \setminus U_1$ ,  $W_1 = V(M) \cap W$  and  $W_2 = W \setminus W_1$ . Since  $|U| \geq 2s$ , we have  $|U_2| = |U| - 2|M| \geq 2$ . Since  $|W_2| = |W| - |M|$  and  $|W| \geq 3s - |U|$ , it follows that  $W_2 \neq \emptyset$ .

Below is a sketch of the proof. We first assume  $|U| < 2s + \varepsilon'n$ . In this case every vertex in  $U$  is adjacent to some vertex in  $W$ . If  $|M|$  is not close to  $s$ , then we easily obtain a contradiction because  $U_2$  is not small. When  $|M|$  is close to  $s$ , we consider three vertices  $u_1 \neq u_2 \in U_2$  and  $v_0 \in W_2$ , and derive a contradiction on  $\deg(u_1) + \deg(u_2) + \deg(v_0)$ . Next we assume  $|U| \geq 2s + \varepsilon'n$ . In this case  $U_2$  is not small. If no vertex of  $W_2$  is adjacent to any vertex of  $U_2$ , then consider two adjacent vertices  $v_0 \in W_2$  and  $u_0 \in U_1$ . We have  $\deg(v_0) \leq \binom{2|M|}{2}$ , which eventually yields that  $\deg(v_0) + \deg(u_0) < 2sn - \varepsilon n^2$ . Now assume  $v_0 \in W_2$  is adjacent to some vertex  $u_0 \in U_2$ . In this case we define  $M'$  consisting of all  $e \in M$  that contains a vertex  $u' \in U$  such that  $|N(v_0, u') \cap U_2| \geq 3$ . We show that if  $|M'|$  is small, then  $\deg(v_0)$  is small; otherwise  $\deg(u_0)$  is small. In either case we derive that  $\deg(v_0) + \deg(u_0) < 2sn - \varepsilon n^2$ .

We now give the details of the proof.

**Case 1.**  $2s \leq |U| < 2s + \varepsilon'n$ .

In this case we have the following two claims.

**Claim 15.**  $|M| \geq s - \varepsilon''n$ .

*Proof.* To the contrary, assume that  $|M| < s - \varepsilon''n$ . Fix  $v_0 \in W_2$ . Then  $\deg(v_0) \leq \binom{|U|}{2} - \binom{|U_2|}{2}$  because there is no edge of type  $U_2U_2W_2$ . Since  $v_0$  is not an isolated vertex,  $v_0$  is adjacent to some vertex  $u \in U$ . Trivially  $\deg(u) \leq \binom{|U|-1}{2} + (|U|-1)|W|$ . Thus

$$\begin{aligned} \deg(v_0) + \deg(u) &\leq \binom{|U|-1}{2} + (|U|-1)|W| + \binom{|U|}{2} - \binom{|U_2|}{2} \\ &= (n-1)(|U|-1) - \binom{|U_2|}{2}. \end{aligned}$$

Since  $|U| \geq 2s$  and  $|M| < s - \varepsilon''n$ , it follows that  $|U_2| = |U| - 2|M| > 2\varepsilon''n$ . As a result,

$$\deg(u) + \deg(v_0) \leq (n-1)(2s + \varepsilon'n - 1) - \binom{2\varepsilon''n}{2},$$

which contradicts the condition that  $\deg(u) + \deg(v_0) > 2sn - \varepsilon n^2$  because  $\varepsilon \ll \varepsilon' \ll \varepsilon''$ .  $\square$

**Claim 16.** *Every vertex in  $U$  is adjacent to one vertex in  $W$ .*

*Proof.* To the contrary, assume that  $u \in U$  is not adjacent to any vertex in  $W$ . Then

$$\deg(u) \leq \binom{|U|-1}{2} < \binom{2s + \varepsilon' n}{2},$$

which contradicts the condition that  $\deg(u) > sn - \frac{1}{2}\varepsilon n^2$  because  $\tau n < s \leq n/3$  and  $\varepsilon \ll \varepsilon' \ll \tau$ .  $\square$

Fix  $u_1 \neq u_2 \in U_2$  and  $v_0 \in W_2$ . Trivially  $\deg(w) \leq \binom{|U|}{2}$  for any vertex  $w \in W$  and  $\deg(u) \leq \binom{|U|-1}{2} + |W|(|U|-1)$  for any vertex  $u \in U$ . Furthermore, for any two distinct edges  $e_1, e_2 \in M$ , we observe that at least one triple of type  $UUW$  with one vertex in  $e_1$ , one vertex in  $e_2$  and one vertex in  $\{u_1, u_2, v_0\}$  is *not* an edge by the choice of  $M$ . By Claim 15,  $|M| \geq s - \varepsilon'' n$ . Thus,

$$\deg(u_1) + \deg(u_2) + \deg(v_0) \leq 2 \left( \binom{|U|-1}{2} + |W|(|U|-1) \right) + \binom{|U|}{2} - \binom{s - \varepsilon'' n}{2}.$$

On the other hand, Claim 16 implies that  $u_i$  is adjacent to some vertex in  $W$  for  $i = 1, 2$ . We know that  $v_0$  is adjacent to some vertex in  $U$ . Therefore,  $\deg(u_i) > (2sn - \varepsilon n^2) - \binom{|U|}{2}$  for  $i = 1, 2$ , and  $\deg(v_0) > (2sn - \varepsilon n^2) - \left( \binom{|U|-1}{2} + |W|(|U|-1) \right)$ . It follows that

$$\deg(u_1) + \deg(u_2) + \deg(v_0) > 3(2sn - \varepsilon n^2) - 2 \left( \binom{|U|}{2} - \binom{|U|-1}{2} \right) - |W|(|U|-1).$$

The upper and lower bounds for  $\deg(u_1) + \deg(u_2) + \deg(v_0)$  together imply that

$$\begin{aligned} 3 \left( \binom{|U|-1}{2} + |W|(|U|-1) + \binom{|U|}{2} \right) - \binom{s - \varepsilon'' n}{2} &> 3(2sn - \varepsilon n^2), \\ \text{or } (|U|-1)(n-1) - \frac{1}{3} \binom{s - \varepsilon'' n}{2} &> 2sn - \varepsilon n^2, \end{aligned}$$

which is impossible because  $|U| < 2s + \varepsilon' n$ ,  $\tau n < s \leq n/3$ , and  $\varepsilon \ll \varepsilon' \ll \varepsilon'' \ll \tau$ .

**Case 2.**  $2s + \varepsilon' n \leq |U| \leq 3s$ .

We consider the following two subcases.

**Subcase 2.1.** No vertex in  $U_2$  is adjacent to any vertex in  $W_2$ .

Fix  $v_0 \in W_2$ . Then  $\deg(v_0) \leq \binom{|U_1|}{2} = \binom{2|M|}{2}$ . Since  $v_0$  is not an isolated vertex,  $v_0$  is adjacent to some vertex  $u_0 \in U_1$ . We know that  $\deg(u_0) \leq \binom{|U|-1}{2} + (|U|-1)|W| - |U_2||W_2|$

because no vertex in  $U_2$  is adjacent to any vertex in  $W_2$ . Since  $|W| = n - |U|$ ,  $|U_2| = |U| - 2|M|$  and  $|W_2| = n - |U| - |M|$ , we derive that

$$\begin{aligned}\sigma_2(H) &\leq \deg(v_0) + \deg(u_0) \\ &\leq \binom{2|M|}{2} + \binom{|U| - 1}{2} + (|U| - 1)(n - |U|) - (|U| - 2|M|)(n - |U| - |M|) \\ &\leq (2n - |U|)|M| + \frac{|U|^2}{2}.\end{aligned}$$

Since  $|M| < 3s - |U|$ , it follows that

$$\sigma_2(H) < (2n - |U|)(3s - |U|) + \frac{|U|^2}{2} = 6sn - (3s + 2n)|U| + \frac{3}{2}|U|^2.$$

Note that the quadratic function  $\frac{3}{2}x^2 - (3s + 2n)x$  is minimized at  $x = s + \frac{2}{3}n$ . Since  $2s + \varepsilon'n \leq |U| \leq 3s \leq s + \frac{2}{3}n$ , we derive that

$$\begin{aligned}\sigma_2(H) &\leq 6sn - (3s + 2n)(2s + \varepsilon'n) + \frac{3}{2}(2s + \varepsilon'n)^2 \\ &= 2sn - 2\varepsilon'n^2 + 3s\varepsilon'n + \frac{3}{2}\varepsilon'^2n^2 \leq 2sn - \varepsilon'n^2 + \frac{3}{2}\varepsilon'^2n^2\end{aligned}$$

because  $s \leq n/3$ . Since  $\varepsilon \ll \varepsilon'$ , this contradicts the assumption that  $\sigma_2(H) > 2sn - \varepsilon n$ .

**Subcase 2.2.** Two vertices  $u_0 \in U_2$  and  $v_0 \in W_2$  are adjacent.

Let  $M' = \{e \in M : \exists u' \in e, |N(v_0, u') \cap U_2| \geq 3\}$ . Assume  $\{u_1, u_2, v_1\} \in M'$  such that  $u_1, u_2 \in U_1$ ,  $v_1 \in W_1$  and  $|N(v_0, u_1) \cap U_2| \geq 3$ . We claim that

$$N(u_0, v_1) \cap U_2 = \emptyset. \quad (3)$$

Indeed, if  $\{u_0, v_1, u_3\} \in E(H)$  for some  $u_3 \in U_2$ , then we can find  $u_4 \in U_2 \setminus \{u_0, u_3\}$  such that  $\{v_0, u_1, u_4\} \in E(H)$ . Replacing  $\{u_1, u_2, v_1\}$  by  $\{u_0, v_1, u_3\}$  and  $\{v_0, u_1, u_4\}$  gives a larger matching than  $M$ , a contradiction.

By the definition of  $M'$ , we have

$$\deg(v_0) \leq \binom{|U_1|}{2} + 2|M'||U_2| + 2(|U_1| - 2|M'|) = \binom{|U_1|}{2} + 2|U_1| + |M'|(2|U_2| - 4).$$

By (3), we have

$$\deg(u_0) \leq \binom{|U| - 1}{2} + |U_1||W| + (|U_2| - 1)(|W_1| - |M'|)$$

and consequently

$$\deg(v_0) + \deg(u_0) \leq \binom{|U_1|}{2} + \binom{|U| - 1}{2} + |U_1|(|W| + 2) + (|U_2| - 1)|W_1| + |M'|(|U_2| - 3).$$

Since  $|M'| \leq |M| = |W_1| = \frac{|U_1|}{2}$ , it follows that

$$\begin{aligned}\deg(v_0) + \deg(u_0) &\leq \binom{|U_1|}{2} + \binom{|U| - 1}{2} + |U_1|(|W| + 2) + (|U_2| - 2)|U_1| \\ &= \binom{|U|}{2} - \binom{|U_2|}{2} + \binom{|U| - 1}{2} + |U_1||W| \\ &= (|U| - 1)^2 - \binom{|U_2|}{2} + 2|M|(n - |U|).\end{aligned}$$

Since  $|M| \leq 3s - |U|$  and  $|U_2| = |U| - 2|M| \geq 3|U| - 6s$ , we have

$$\begin{aligned}\deg(v_0) + \deg(u_0) &\leq (|U| - 1)^2 - \binom{3|U| - 6s}{2} + 2(3s - |U|)(n - |U|) \\ &= -\frac{3}{2}|U|^2 + \left(12s - 2n - \frac{1}{2}\right)|U| + 6sn - 18s^2 - 3s + 1 \\ &\leq -\frac{3}{2}|U|^2 + (12s - 2n)|U| + 6sn - 18s^2.\end{aligned}$$

Note that the quadratic function  $-\frac{3}{2}x^2 + (12s - 2n)x$  is maximized at  $x = 4s - \frac{2}{3}n$ . Since  $3s \geq |U| \geq 2s + \varepsilon'n \geq 4s - \frac{2}{3}n$ , we have

$$\begin{aligned}\sigma_2(H) \leq \deg(v_0) + \deg(u_0) &\leq -\frac{3}{2}(2s + \varepsilon'n)^2 + (12s - 2n)(2s + \varepsilon'n) + 6sn - 18s^2 \\ &= 2sn - 2\varepsilon'n^2 + 6\varepsilon'sn - \frac{3}{2}\varepsilon'^2n^2 \leq 2sn - \frac{3}{2}\varepsilon'^2n^2\end{aligned}$$

because  $s \leq n/3$ . Since  $\varepsilon \ll \varepsilon'$ , this contradicts the assumption that  $\sigma_2(H) > 2sn - \varepsilon n$ .

## 4 Proof of Theorem 6

Suppose  $H$  is a 3-graph of order  $n$  without an isolated vertex and  $\sigma_2(H) > 2sn - \varepsilon n^2$ . Let  $U = \{u \in V(H) : \deg(u) > sn - \varepsilon n^2/2\}$  and  $W = V \setminus U$ . We know that no two vertices in  $W$  are adjacent and  $|U| \geq 2s$ . Let  $M$  be an optimal matching as in Definition 8. By Lemma 7, such  $M$  exists. Let  $M_2 = M \setminus M_1$ ,  $U_1 = V(M_1) \cap U$ ,  $U_2 = V(M_2)$ ,  $U_3 = U \setminus V(M)$ ,  $W_1 = V(M_1) \cap W$  and  $W_2 = W \setminus W_1$ . Since  $M$  is optimal, no edge of  $H$  is of type  $W_2U_3U_3$  or  $W_2U_2U_3$ . In addition, for any  $e \in M_1$ , there are no two disjoint edges  $e_1, e_2 \in e \cup W_2 \cup U_3$  such that  $(e_1 \cup e_2) \cap W_2 \neq \emptyset$ .

Suppose to the contrary, that  $|M| \leq s - 1$ . We know that  $|U_3| = |U| + |M_1| - 3|M| \geq 3 + |M_1| - (3s - |U|) \geq 3$ . Let  $u_1, u_2, u_3 \in U_3$ . Since  $u_i \in U$  for  $i = 1, 2, 3$ , we have

$$\sum_{i=1}^3 \deg(u_i) > 3sn - \frac{3}{2}\varepsilon n^2. \tag{4}$$

On the other hand, if  $u_1$  is adjacent to some  $v_1 \in W_2$ , then

$$\sum_{i=1}^2 \deg(u_i) + \deg(v_1) \geq \sigma_2(H) + \deg(u_2) > 3sn - \frac{3}{2}\varepsilon n^2. \tag{5}$$

**Claim 17.** For any two distinct edges  $e_1, e_2$  from  $M$ , we have  $\sum_{i=1}^3 |L_{u_i}(e_1, e_2)| \leq 18$  and  $\sum_{i=1}^2 |L_{u_i}(e_1, e_2)| + |L_{v_1}(e_1, e_2)| \leq 18$ .

*Proof.* Let  $H_1$  be the 3-partite subgraph of  $H$  induced on three parts  $\{u_1, u_2, u_3\}$ ,  $e_1$ , and  $e_2$ . We observe that  $H_1$  does not contain a perfect matching by the choice of  $M$ . By Lemma 10, we have  $|E(H_1)| = \sum_{i=1}^3 |L_{u_i}(e_1, e_2)| \leq 18$ . The same argument shows that  $\sum_{i=1}^2 |L_{u_i}(e_1, e_2)| + |L_{v_1}(e_1, e_2)| \leq 18$ .  $\square$

We proceed in two cases.

**Case 1.**  $|M_1| = 3s - |U|$ .

In this case, we have  $|M_2| = |M| + |U| - 3s$ ,  $|U_3| = 3s - 3|M|$  and  $|W_2| = n - 3s$ .

**Claim 18.** For any  $e \in M_1$ , we have

- (i)  $\sum_{i=1}^2 |L_{u_i}(e, U_3 \cup W_2)| + |L_{v_1}(e, U_3 \cup W_2)| \leq \max\{4|U_3| + 7, 3|U_3| + 2|W_2| + 5\}$ , where  $v_1 \in W_2$ ;
- (ii)  $\sum_{i=1}^3 |L_{u_i}(e, U_3)| \leq 6|U_3|$ .

*Proof.* Assume  $e = \{u'_1, u'_2, u'_3\} \in M_1$  with  $u'_1 \in W_1$  and  $u'_2, u'_3 \in U_1$ .

(i) Let  $A = U_3$ ,  $B = W_2$ , and  $E(G_i) = L_{u'_i}(U_3 \cup W_2)$  for  $i = 1, 2, 3$ . By the choice of  $M$ , there are not two disjoint edges, one from  $G_1$  and the other from  $G_2$  or  $G_3$ ; or one from  $G_2$  and the other from  $G_3$ , and at least one of them contains one vertex from  $B$ . Furthermore, it is easy to see that

$$\sum_{i=1}^2 |L_{u_i}(e, U_3 \cup W_2)| + |L_{v_1}(e, U_3 \cup W_2)| = \sum_{i=1}^3 \left( \sum_{j=1}^2 \deg_{G_i}(u_j) + \deg_{G_i}(v_1) \right).$$

The desired inequality thus follows from Lemma 14.

(ii) For  $i = 1, 2, 3$ , let  $G_i$  be the graph obtained from  $L_{u'_i}(U_3)$  after adding an isolated vertex  $u^*$ . Then  $|V(G_i)| = |U_3| + 1 \geq 4$ . By the choice of  $M$ , every edge of  $G_1$  intersects every edge of  $G_2$  and  $G_3$ . The desired inequality thus follows from Lemma 11.  $\square$

**Claim 19.** For any  $e \in M_2$ , we have

- (i)  $\sum_{i=1}^3 |L_{u_i}(e, U_3)| \leq 3(|U_3| + 3)$ ;
- (ii)  $\sum_{i=1}^2 |L_{u_i}(e, U_3)| \leq 3(|U_3| + 1)$ .

*Proof.* Assume  $e = \{u'_1, u'_2, u'_3\} \in M_2$  with  $u'_1, u'_2, u'_3 \in U_2$ .

(i) For  $i = 1, 2, 3$ , let  $G_i$  be the graph obtained from  $L_{u'_i}(U_3)$  after adding two isolated vertices  $u'$  and  $u''$ . Then  $|V(G_i)| = |U_3| + 2 \geq 5$ . Since  $M$  is optimal, the desired inequality follows from Lemma 12.

(ii) For  $i = 1, 2, 3$ , let  $G_i$  be the graph obtained from  $L_{u'_i}(U_3)$  after adding an isolated vertex  $u^*$ . Then  $|V(G_i)| = |U_3| + 1 \geq 4$ . Since  $M$  is optimal, the desired inequality follows from Lemma 13.  $\square$

**Claim 20.**  $s > n/3 - \eta_1 n$ .

*Proof.* Suppose  $s \leq n/3 - \eta_1 n$ . We first consider the case that  $u_1, u_2, u_3$  are not adjacent to any vertex of  $W_2$ .

Following Claim 17, we have

$$\sum_{i=1}^3 \deg(u_i) \leq 18 \binom{|M|}{2} + 9|M| + \sum_{i=1}^3 |L_{u_i}(V(M_1), U_3)| + \sum_{i=1}^3 |L_{u_i}(V(M_2), U_3)|. \quad (6)$$

Furthermore, by Claims 18 (ii) and 19 (i), we obtain that

$$\begin{aligned} \sum_{i=1}^3 \deg(u_i) &\leq 18 \binom{|M|}{2} + 9|M| + 6|M_1||U_3| + 3|M_2|(|U_3| + 3) \\ &= 18 \binom{|M|}{2} + 9|M| + 6(3s - |U|)(3s - 3|M|) \\ &\quad + 3(|M| + |U| - 3s)(3s - 3|M| + 3) \\ &= (9|U| - 18s + 9)|M| + (3s - |U|)(9s - 9). \end{aligned}$$

Since  $|M| \leq s - 1$ , it follows that

$$\sum_{i=1}^3 \deg(u_i) \leq (9|U| - 18s + 9)(s - 1) + (3s - |U|)(9s - 9) = 9s^2 - 9.$$

Since  $\tau n < s \leq n/3 - \eta_1 n$  and  $\eta_1 < \tau$ , we know that

$$3s^2 - sn = s(3s - n) \leq \max \{-\eta_1 n(n - 3\eta_1 n), -\tau n(n - 3\tau n)\} = -\eta_1 n(n - 3\eta_1 n). \quad (7)$$

Consequently,  $\sum_{i=1}^3 \deg(u_i) < 9s^2 \leq 3sn - 3\eta_1 n(n - 3\eta_1 n)$ . Since  $\varepsilon \ll \eta_1$ , this contradicts (4).

Now we assume, without loss of generality, that  $u_1$  is adjacent to  $v_1$ . The choice of  $M$  implies that  $L_v(e, U_3) = L_u(e, W_2) = \emptyset$  for any  $v \in W_2$ ,  $u \in U_3$  and  $e \in M_2$ . By Claim 17, we have

$$\begin{aligned} \sum_{i=1}^2 \deg(u_i) + \deg(v_1) &\leq 18 \binom{|M|}{2} + 9|M| + \sum_{i=1}^2 |L_{u_i}(V(M_1), U_3 \cup W_2)| \\ &\quad + |L_{v_1}(V(M_1), U_3)| + \sum_{i=1}^2 |L_{u_i}(V(M_2), U_3)|. \end{aligned} \quad (8)$$

We know that  $4|U_3|+7 \geq 3|U_3|+2|W_2|+5$  if and only if  $|U_3| \geq 2|W_2|-2$ . If  $|U_3| \geq 2|W_2|-2$ , then by (8), Claim 18 (i) and Claim 19 (ii), we have

$$\begin{aligned} \sum_{i=1}^2 \deg(u_i) + \deg(v_1) &\leq 18 \binom{|M|}{2} + 9|M| + |M_1|(4|U_3| + 7) + 3|M_2|(|U_3| + 1) \\ &= 18 \binom{|M|}{2} + 9|M| + (3s - |U|)(4(3s - 3|M|) + 7) \\ &\quad + 3(|M| + |U| - 3s)(3s - 3|M| + 1) \\ &= (3|U| + 3)|M| - 3s|U| - 4|U| + 9s^2 + 12s. \end{aligned}$$

Since  $|M| \leq s - 1$  and  $|U| \geq 2s$ , it follows that

$$\begin{aligned} \sum_{i=1}^2 \deg(u_i) + \deg(v_1) &\leq (3|U| + 3)(s - 1) - 3s|U| - 4|U| + 9s^2 + 12s \\ &= -7|U| + 9s^2 + 15s - 3 \leq 9s^2 + s - 3. \end{aligned}$$

Following (7), we have  $\sum_{i=1}^2 \deg(u_i) + \deg(v_1) < 3sn - 3\eta_1 n(n - 3\eta_1 n) + n/3 - 3$ . Since  $\varepsilon \ll \eta_1$  and  $n$  is sufficiently large, this contradicts (5).

If  $|U_3| < 2|W_2| - 2$ , by (8), Claim 18 (i) and Claim 19 (ii), we have

$$\begin{aligned} \sum_{i=1}^2 \deg(u_i) + \deg(v_1) &\leq 18 \binom{|M|}{2} + 9|M| + |M_1|(3|U_3| + 2|W_2| + 5) + 3|M_2|(|U_3| + 1) \\ &= (9s + 3)|M| + (-2n + 6s - 2)|U| + 6sn - 18s^2 + 6s. \end{aligned}$$

Since  $|M| \leq s - 1$  and  $|U| \geq 2s$ , it follows that

$$\begin{aligned} \sum_{i=1}^2 \deg(u_i) + \deg(v_1) &\leq (9s + 3)(s - 1) + (-2n + 6s - 2)(2s) + 6sn - 18s^2 + 6s \\ &= 2sn + 3s^2 - 4s - 3. \end{aligned}$$

Applying (7), we have  $\sum_{i=1}^2 \deg(u_i) + \deg(v_1) < 3sn - \eta_1 n(n - 3\eta_1 n)$ , which contradicts (5) because  $\varepsilon \ll \eta_1$ .  $\square$

By Claim 20, we have  $|W_2| = n - 3s < 3\eta_1 n$ . Let  $H' = H[V \setminus W_2]$ . We claim that  $\sigma_2(H') > 2n^2/3 - \eta_2 n^2$ . Indeed, recall that  $\deg_H(u) + \deg_H(v) \geq 2n^2/3 - \varepsilon n^2$  for any two adjacent vertices  $u$  and  $v$  of  $H'$ . Since  $|W_2| < 3\eta_1 n$  and  $\varepsilon \ll \eta_1 \ll \eta_2$ , it follows that

$$\deg_{H'}(u) + \deg_{H'}(v) \geq 2n^2/3 - \varepsilon n^2 - 2|W_2|n > 2n^2/3 - \eta_2 n^2.$$

Since  $\eta_2 \ll 1$ , we may apply Theorem 9 and conclude that either  $H'$  is a subgraph of  $H_{3s,s}^2$  or  $H'$  contains a perfect matching. In the former case, there is a partition of  $V(H')$  into two sets  $|T| = 2s - 1$  and  $|S| = s + 1$  such that for every vertex  $u \in S$ ,

$$\deg_{H'}(u) \leq \binom{|T|}{2} = \binom{2s - 1}{2} \leq \binom{2n/3 - 1}{2} < \frac{2}{9}n^2.$$

On the other hand, since  $U \subseteq V(H')$  and  $|U| \geq 2s$ , there exists a vertex  $u \in U \cap S$  such that

$$\begin{aligned}\deg_{H'}(u) &\geq \deg_H(u) - |W_2|n \geq sn - \frac{\varepsilon}{2}n^2 - |W_2|n \\ &\geq \left(\frac{n}{3} - \eta_1 n\right)n - \frac{\varepsilon}{2}n^2 - 3\eta_1 n^2 > \frac{2}{9}n^2,\end{aligned}$$

which is a contradiction. Therefore  $H'$  must contain a perfect matching, which is a matching of size  $s$  in  $H$ .

**Case 2.**  $|M_1| > 3s - |U|$ .

The difference from Case 1 is that, for any edge  $e \in M$ , we cannot find two disjoint edges  $e_1, e_2$  from  $e \cup U_3 \cup W_2$  – otherwise we can replace  $M$  by  $M \setminus \{e\} \cup \{e_1, e_2\}$  contradicting the assumption that  $M$  is an optimal matching.

Note that  $|U_3| = |U| + |M_1| - 3|M| \geq 3s + 1 - 3|M| \geq 4$ .

**Claim 21.** *For any  $e \in M$ ,  $\sum_{i=1}^3 |L_{u_i}(e, U_3 \cup W_2)| \leq 3(|U_3| + |W_2| + 2)$ .*

*Proof.* Assume  $e = \{u'_1, u'_2, u'_3\} \in M$ . For  $i = 1, 2, 3$ , let  $G_i$  be the graph obtained from  $L_{u'_i}(U_3 \cup W_2)$  after adding an isolated vertex  $u^*$ . Then  $|V(G_i)| = |U_3| + |W_2| + 1 \geq 5$ . Since  $H$  contains no two disjoint edges  $e_1, e_2$  from  $e \cup U_3 \cup W_2$ , we know that for any  $i \neq j$ , every edge of  $G_i$  intersects every edge of  $G_j$ . The desired inequality thus follows from Lemma 12.  $\square$

By Claims 17 and 21, we obtain that

$$\begin{aligned}\sum_{i=1}^3 \deg(u_i) &\leq 18 \binom{|M|}{2} + 9|M| + \sum_{i=1}^3 |L_{u_i}(V(M), U_3 \cup W_2)| \\ &\leq 18 \binom{|M|}{2} + 9|M| + 3|M|(|U_3| + |W_2| + 2) \\ &= (3n + 6)|M| \leq 3sn + 6s.\end{aligned}\tag{9}$$

Let  $W' = \{v \in W : \deg(v) \leq sn - s^2/2 + \gamma'n^2\}$ . If  $|W'| \leq \gamma n$ , then we let  $H' := H[V \setminus W']$ . By the definition of  $W'$ ,  $\deg_H(u) > sn - s^2/2 + \gamma'n^2$  for every  $u \in V(H') \cap W$ . For any  $u \in V(H') \cap U$ ,  $\deg_H(u) > sn - \varepsilon n^2/2 > sn - s^2/2 + \gamma'n^2$  because  $s > \tau n$  and  $\varepsilon \ll \gamma' \ll \tau$ . Therefore every vertex  $u \in V(H')$  satisfies

$$\deg_{H'}(u) \geq \deg_H(u) - n|W'| > sn - \frac{s^2}{2} + \gamma'n^2 - \gamma n^2 > \binom{n-1}{2} - \binom{n-s}{2} + 1,$$

because  $|W'| \leq \gamma n$ ,  $\gamma \ll \gamma'$ , and  $n$  is sufficiently large. By Theorem 1,  $H'$  contains a matching of size  $s$ .

We thus assume that  $|W'| > \gamma n$  for the rest of the proof. If one of  $u_1, u_2, u_3$  is adjacent to a vertex of  $W'$ , then

$$\sum_{i=1}^3 \deg(u_i) > 4 \left( sn - \frac{\varepsilon}{2}n^2 \right) - \left( sn - \frac{s^2}{2} + \gamma'n^2 \right) = 3sn + \frac{s^2}{2} - 2\varepsilon n^2 - \gamma'n^2,$$

which contradicts (9) because  $s > \tau n$  is sufficiently large and  $\varepsilon \ll \gamma' \ll \tau$ .

If none of  $u_1, u_2, u_3$  is adjacent to a vertex of  $W'$ , then we distinguish the following two subcases.

**Subcase 2.1.**  $|W' \cap W_1| > \gamma n/2$ .

Let  $M' = \{e \in M : e \cap W' \neq \emptyset\}$ , thus  $|M'| > \gamma n/2$ . Since  $u_1, u_2, u_3$  are not adjacent to any vertex in  $W' \cap W_1$ , then for any distinct  $e_1, e_2$  from  $M'$ , we have

$$\sum_{i=1}^3 |L_{u_i}(e_1, e_2)| \leq 12. \quad (10)$$

By Claims 17, 21 and (10), we have

$$\begin{aligned} \sum_{i=1}^3 \deg(u_i) &\leq \left(18 \binom{|M|}{2} - 6 \binom{|M'|}{2}\right) + 9|M| + 3|M|(n - 3|M| + 2) \\ &\leq (3n + 6)|M| - 6 \binom{|M'|}{2}. \end{aligned}$$

Since  $|M'| > \gamma n/2$ , it follows that

$$\sum_{i=1}^3 \deg(u_i) \leq (3n + 6)(s - 1) - 6 \binom{\gamma n/2}{2},$$

which contradicts (4) because  $s \leq n/3$  and  $\varepsilon \ll \gamma$ .

**Subcase 2.2.**  $|W' \cap W_1| \leq \gamma n/2$ .

Since  $|W'| > \gamma n$ , we have  $|W' \cap W_2| > \gamma n/2$ . Let  $W_2^* = W_2 \setminus W'$ . Then  $W_2 \setminus W_2^* = W' \cap W_2$ . By Claim 21, we obtain that  $\sum_{i=1}^3 |L_{u_i}(V(M), U_3 \cup W_2^*)| \leq 3|M|(|U_3| + |W_2^*| + 2)$ . Therefore,

$$\begin{aligned} \sum_{i=1}^3 \deg(u_i) &\leq 18 \binom{|M|}{2} + 9|M| + \sum_{i=1}^3 |L_{u_i}(V(M), U_3 \cup W_2^*)| \\ &\leq 18 \binom{|M|}{2} + 9|M| + 3|M|(|U_3| + |W_2^*| + 2) \\ &= 18 \binom{|M|}{2} + 9|M| + 3|M|(|U_3| + |W_2| + 2) - 3|M||W_2 \setminus W_2^*| \\ &= \left(3n + 6 - \frac{3}{2}\gamma n\right)|M|, \end{aligned}$$

which contradicts (4) because  $|M| \leq s$ ,  $\tau n < s$ , and  $\varepsilon \ll \gamma \ll \tau$ . This completes the proof of Theorem 6.

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