

Boundary criticality of topological quantum phase transitions in two-dimensional systems

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We discuss the boundary critical behaviors of two-dimensional (2D) quantum phase transitions with fractionalized degrees of freedom in the bulk, motivated by the fact that usually it is the one-dimensional boundary that is exposed and can be conveniently probed in many experimental platforms. In particular, we mainly discuss boundary criticality of two examples: (i) the quantum phase transition between a 2D Z_2 topological order and an ordered phase with spontaneous symmetry breaking; (ii) the continuous quantum phase transition between metal and a particular type of Mott insulator [U(1) spin liquid]. In particular, we obtain the critical exponents and scaling laws of these exotic quantum phase transitions when the systems are probed from the boundary in proposed experimental setup. These critical exponents obtained are significantly different from those one would see through bulk measurements. This theoretical study could be relevant to many purely 2D systems, where recent experiments have found correlated insulator, superconductor, and metal in the same phase diagram.

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I. INTRODUCTION

Two-dimensional (2D) quantum many-body systems at zero temperature gave us a plethora of exotic phenomena beyond the classical wisdom of phases of matter. These phenomena include topological orders [1,2], symmetry protected topological orders [3,4] (generalization of topological insulators), and unconventional quantum phase transitions beyond the Landau's paradigm [5–9]. The unconventional quantum phase transitions usually have very distinct universal scalings compared with the ordinary $(2 + 1)d$ Landau's transitions. These unconventional quantum phase transitions, or unconventional quantum critical points (QCP), could happen between two ordinary Landau's phases with different patterns of spontaneous symmetry breaking [5,6], they can also happen between a topological order and an ordered phase [7,8,10]. Although many appealing numerical evidences of these unconventional QCPs have been found [11–14], direct clear experimental observation of these unconventional QCPs is still demanded.

To identify an unconventional QCP in an experimental system, we need to measure the correlation functions and scaling dimensions of various operators at this QCP, and compare the results with analytical predictions. In this work we do not attempt to propose a particular experimental system that realizes one of the unconventional QCPs, instead we try to address one general issue that many experimental platforms would face, platforms where potentially these unconventional QCPs can be found. In numerical simulations of a QCP, correlation functions and scalings in the bulk can be directly computed. But experimentally many purely 2D systems of interest are sandwiched between other auxiliary layers in a Van der Waals heterostructure [15]. Hence, the bulk of

the 2D system is often not exposed for probing for many experimental techniques. Instead, the 1D boundary of the 2D system is exposed and can often be probed directly. Based on the early studies of the boundary of Wilson-Fisher fixed points [16–19] and the boundary of 2D conformal field theories [20], we learned that the scaling of operators at the boundary of a system can be very different from the bulk, hence the previous calculations about unconventional QCPs in the bulk may not be so relevant to many experimental platforms. We need to restudy the critical exponents at the 1D boundary of the system to compare with future experimental observations.

II. BOUNDARY CRITICALITY OF Z_2 TOPOLOGICAL QUANTUM PHASE TRANSITIONS

In this section we discuss the boundary critical behaviors of a 2D topological quantum phase transition between a fully gapped Z_2 topological order, and an ordered phase which spontaneously breaks the global symmetry of the system and has no topological order. We assume that the “electric gauge particle” (the so-called e anyon) of the Z_2 topological order is an N -component complex boson b_a . This topological transition is described by the following field theory:

$$\mathcal{S} = \int d\tau d^2x \sum_{a=1}^N |\partial\phi_a|^2 + r|\phi_a|^2 + g \left(\sum_{a=1}^N |\phi_a|^2 \right)^2, \quad (1)$$

where the complex scalar ϕ_a is the low-energy field of anyon b_a , and it is coupled to a Z_2 gauge field which is not written explicitly. Because a Z_2 gauge field does not have gapless gauge boson, it does not contribute any infrared corrections to gauge invariant operators. This model has a QCP at $r = r_c$ which separates two different phases. The physical meaning

of the fields and parameters in this action depends on the specific realization of Eq. (1), and two potential examples of realizing Eq. (1) with different N will be discussed in the next few paragraphs. When $r > r_c$, ϕ_a is disordered and the system is a Z_2 topological order which is also the deconfined phase of the Z_2 gauge field; when $r < r_c$, ϕ_a condenses and destroys the Z_2 topological order through the Higgs mechanism, and the condensate of ϕ_a has ground-state manifold S^{2N-1}/Z_2 , where S^{2N-1} is a $2N - 1$ -dimensional sphere.

This theory Eq. (1) with different N can be realized in various scenarios. For $N = 1$, this theory can be realized as the transition between a 2D superconductor and a Z_2 spin liquid. Similar unconventional topological transitions have been observed in numerical simulations in lattice spin (or quantum boson) models [7,8], and theoretical predictions of the bulk critical exponents have been confirmed quantitatively. In this realization the boson b can be introduced by formally fractionalizing the electron operator on the lattice as

$$c_{j,\alpha} = f_{j,\alpha} b_j, \quad (2)$$

where b_j is a charge-carrying bosonic “rotor,” $f_{j,\alpha}$ is the fermionic parton that carries the spin quantum number. The fractionalized degrees of freedom introduced here are essentially the same as Ref. [21]: $f_{j,\alpha}$ and b_j share a $U(1)$ gauge symmetry, and the Z_2 topological order is constructed by assuming that b_j has a finite mass gap, while $f_{j,\alpha}$ forms a superconductor at the mean-field level, which breaks the $U(1)$ gauge symmetry down to Z_2 . The quantum phase transition between the superconductor and the Z_2 topological is described by Eq. (1) with $N = 1$. And in this particular realization of Eq. (1), r is the tuning parameter for the deconfined quantum phase transition between the superconductor and a Z_2 spin liquid. Similar Z_2 spin liquid to superconductor quantum phase transitions were discussed in the context of organic frustrated quantum magnets and cuprates materials [22,23]. The phases $r > r_c$ and $r < r_c$ correspond to the Z_2 spin liquid (quantum disordered phase) and the superconductor (ordered phase), respectively. In the condensate of ϕ ($r < r_c$), the physical pairing symmetry of the superconductor is inherited from the mean-field band structure of f_α . The long range Coulomb interaction between charge carriers is often screened by auxiliary layers such as metallic gates in experimental systems, hence in Eq. (1) there is only a short range interaction. Equation (1) with $N = 1$ is often referred to as the “ XY^* ” transition. In the dual picture, starting from the superconducting phase, the XY^* transition can also be viewed as the condensation of double vortices of the superconductor.

Equation (1) with even N and $N \geq 2$ can be realized in $Sp(N)$ spin systems, as the Z_2 spin liquid can be naturally constructed in $Sp(N)$ spin systems. $b_a \sim \phi_a$ is introduced as the fractionalized Schwinger boson of the spin system, and the Z_2 topological order emerges when a pair of b_a [which forms a $Sp(N)$ singlet] condenses on the lattice [24,25]. In particular, when $N = 2$, the theory Eq. (1) can be realized as the quantum phase transition between a Z_2 topological order and a non-collinear spin density wave of spin-1/2 systems on a frustrated lattice, for example the so-called 120° antiferromagnetic state on the triangular lattice [10,26]. Now the quantum critical point $r = r_c$ separates the quantum disordered Z_2 spin liquid, and the ordered 120° state. The order parameter of the non-

collinear spin order of a fully $SU(2)$ invariant Hamiltonian will form a ground-state manifold $SO(3)$, which is equivalent to $SU(2)/Z_2 = S^3/Z_2$, where the Z_2 is identified as the Z_2 gauge group, and also the center of the spin $SU(2)$ group. The gauge invariant order parameter can be constructed with the low-energy field ϕ_a as

$$\vec{N}_1 = \text{Re}[\phi^\dagger i \sigma^2 \bar{\sigma} \phi], \quad \vec{N}_2 = \text{Im}[\phi^\dagger i \sigma^2 \bar{\sigma} \phi], \quad \vec{N}_3 = \phi^\dagger \bar{\sigma} \phi, \quad (3)$$

and one can show that $\vec{N}_i$ are three orthogonal vectors. In this case theory Eq. (1) is referred to as the $O(4)^*$ transition, because there is an emergent $O(4)$ symmetry rotates $(\text{Re}[\phi_1], \text{Im}[\phi_1], \text{Re}[\phi_2], \text{Im}[\phi_2])$ as a four component real vector (for more detailed discussions about the emergent symmetry, we refer the readers to Refs. [10,26]). Other systems can potentially realize the theory with larger N , for instance, spin systems with $Sp(4)$ symmetry can be realized in spin-3/2 cold atom systems [27].

We are most interested in the composite operator $\sum_a \phi_a^2$, which is invariant under the Z_2 gauge symmetry, but transforms nontrivially under the physical symmetry, hence it is a physical order parameter. When $N = 1$, in the condensate of ϕ (or b_j), the electron operator has a finite overlap with the fermionic parton operator $c_{j,\alpha} \sim f_{j,\alpha} \langle \phi \rangle$, hence the superconductor order parameter $\Delta \sim \langle \phi^2 \rangle$. In the bulk the scaling dimension of ϕ^2 can be extracted through the standard ϵ expansion or numerical simulation [28]. Near the critical point the superconductor order parameter should scale as $\Delta \sim |r - r_c|^\beta$, where $\beta = [\phi^2]_\nu$ and $[\phi^2]$ is the scaling dimension of operator ϕ^2 . At the XY^* critical point the exponent $\nu \sim 2/3$. When $N = 2$, the composite operator $\sum_a \phi_a^2$ is one component of the spin order parameter of the noncollinear spin density wave.

All the results above are only valid in the 2D bulk. But in experiments on the boundary (as we discussed previously, it is the boundary that is exposed and hence can be probed conveniently), many of the critical exponents are modified. We now consider a system whose 2D bulk is in the semi-infinite xz plane with $z > 0$, with a 1D boundary at $z = 0$. For simplicity, let us tentatively ignore the Z_2 gauge field, and view ϕ_a as a physical order parameter. The most natural boundary condition is the Dirichlet boundary condition, i.e., the field vanishes at the boundary and also outside of the system $z \leq 0$. The boundary condition of the system can be imposed by turning on a large $c|\phi_a|^2$ term along the boundary, which fixes $\phi_a(\mathbf{x}, z = 0) = 0$, where $\mathbf{x} = (\tau, x)$.

At the mean-field level, the correlation function of the ϕ_a field near the boundary can be computed using the “image method” [16]:

$$\begin{aligned} G(\mathbf{x}_1 - \mathbf{x}_2, z_1, z_2) &= d \langle \phi_a(\mathbf{x}_1, z_1) \phi_a^*(\mathbf{x}_2, z_2) \rangle \\ &= G(\mathbf{x}_1 - \mathbf{x}_2, z_1 - z_2)_{\text{bulk}} \\ &\quad - G(\mathbf{x}_1 - \mathbf{x}_2, z_1 + z_2)_{\text{bulk}}, \end{aligned} \quad (4)$$

where $G_{\text{bulk}} = \langle \phi_a(\mathbf{x}_1, z_1) \phi_a^*(\mathbf{x}_2, z_2) \rangle_{\text{bulk}}$ is the bulk correlation function far from the boundary. Notice that the boundary breaks the translation symmetry along the z direction, hence the full expression of the correlation function near the boundary is no longer a function of $z_1 - z_2$. The

expression in Eq. (4) guarantees that the correlation function satisfies $G(\mathbf{x}_1 - \mathbf{x}_2, 0, z_2) = G(\mathbf{x}_1 - \mathbf{x}_2, z_1, 0) = 0$, which is consistent with the boundary condition. The fact that the correlation function of the ϕ_a field vanishes at the boundary means that ϕ_a itself is no longer the leading representation of the field at the boundary $z = 0$. Instead, another field with the same global symmetry and quantum number at the boundary,

$$\Phi_{1,a} = \partial_z \phi_a, \quad (5)$$

should be viewed as the leading representation of the field near the boundary [29]. The field $\Phi_{1,a}$ and ϕ_a have different spatial symmetry transformations in the bulk of the system, but they do have the same spatial symmetry at the boundary, as the boundary breaks part of the spatial symmetry. In fact, since $\Phi_{1,a}$ and ϕ_a have the same symmetry transformation near the boundary, an external field that couples to ϕ_a should also couple to $\partial_z \phi_a$. At the mean-field level, a typical configuration of ϕ_a scales as $\phi_a(\mathbf{x}, z) \sim z$ near the boundary, hence $\Phi_{1,a} = \partial_z \phi_a$ is not suppressed by the boundary condition. Also, the correlation function of $\Phi_{1,a}$ at the boundary does not vanish, and at the mean-field level it has scaling dimension $[\Phi_{1,a}] = [\phi_a] + 1 = D/2$, where D is the total space-time dimension of the bulk.

The gauge invariant order parameter $\sum_a \phi_a^2$ we are interested in reduces to $\Phi_2 = \sum_a \Phi_{1,a}^2$ at the boundary, and it has scaling dimension $[\Phi_2] = D$ at the mean-field level. If the Z_2 gauge field is ignored, then the correlation function of $\Phi_{1,a}$ at the boundary reads

$$\langle \Phi_{1,a}(\mathbf{x}_1) \Phi_{1,a}^*(\mathbf{x}_2) \rangle = \lim_{z_1, z_2 \rightarrow 0} \partial_{z_1} \partial_{z_2} G(\mathbf{x}_1 - \mathbf{x}_2, z_1, z_2), \quad (6)$$

where $G(\mathbf{x}_1 - \mathbf{x}_2, z_1, z_2)$ is still given by the image method Eq. (4). If we assume that G_{bulk} takes the standard form at the Gaussian fixed point

$$\langle \phi_a(\mathbf{x}_1, z_1) \phi_a^*(\mathbf{x}_2, z_2) \rangle_{\text{bulk}} = \frac{1}{[|\mathbf{x}_1 - \mathbf{x}_2|^2 + (z_1 - z_2)^2]^{\frac{D-2}{2}}}, \quad (7)$$

then the boundary correlation function of $\Phi_{1,a}$ at the mean-field level reads

$$\langle \Phi_{1,a}(\mathbf{x}_1) \Phi_{1,a}^*(\mathbf{x}_2) \rangle = \frac{2(D-2)}{|\mathbf{x}_1 - \mathbf{x}_2|^D}. \quad (8)$$

At the Gaussian fixed point, the correlation function of Φ_2 can be derived using the Wick theorem:

$$\begin{aligned} \langle \Phi_2(\mathbf{x}_1) \Phi_2^*(\mathbf{x}_2) \rangle &= \sum_a \langle \Phi_{1,a}(\mathbf{x}_1) \Phi_{1,a}^*(\mathbf{x}_2) \rangle^2 \\ &\sim \frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|^{2D}}. \end{aligned} \quad (9)$$

The scaling dimension of Φ_2 will acquire further correction from interaction, which can be computed through the $\epsilon = (4 - D)$ expansion. Interestingly, at the leading ϵ order, $[\Phi_2]$ will receive corrections from both wave-function renormalization and vertex corrections:

$$[\Phi_2] = D + 2\delta_{\text{wf}} + \delta_v. \quad (10)$$

The wave-function renormalization δ_{wf} can be extracted from the previously calculated ϵ expansion of the anomaly

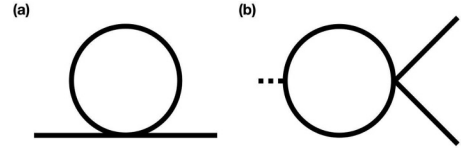


FIG. 1. The diagrams that renormalize Φ_2 at the first order of ϵ . In the bulk the first diagram only shifts the mass of ϕ_a , but at the boundary it makes a nontrivial contribution to the wave-function renormalization.

ous dimension at the boundary of the Wilson-Fisher fixed points, i.e.,

$$[\Phi_{1,a}] = \frac{D}{2} + \delta_{\text{wf}} = \frac{D}{2} - \frac{N+1}{2(N+4)}\epsilon. \quad (11)$$

In contrast, in the bulk renormalization group (RG) analysis of the Wilson-Fisher fixed point, the wave-function renormalization only appears at the second and higher order of ϵ expansion.

The vertex correction is most conveniently computed using the standard real-space RG, since now the momentum along the $\hat{z}$ direction is no longer conserved. We will use the following operator-product-expansion (OPE) between $\Phi_2(\mathbf{x}, 0)$ and the interaction term in Eq. (1) [Fig. 1(b)], where $\Phi_2(\mathbf{x}, 0)$ is defined as $\Phi_2(\mathbf{x}, 0) = \lim_{z \rightarrow 0} [\partial_z \phi(\mathbf{x}, z)]^2$:

$$\begin{aligned} \Phi_2(\mathbf{x}, 0) g \left[\sum_a \phi_a^*(\mathbf{x}', z') \phi_a(\mathbf{x}', z') \right]^2 \\ = 2g \lim_{z \rightarrow 0} [\partial_z G(\mathbf{x} - \mathbf{x}', z, z')]^2 \sum_a \phi_a^2(\mathbf{x}', z') \\ \sim \frac{32z'^4 g}{[(\mathbf{x} - \mathbf{x}')^2 + z'^2]^4} \lim_{z \rightarrow 0} [\partial_z \phi(\mathbf{x}, z)]^2. \end{aligned} \quad (12)$$

Notice that like all the $4 - \epsilon$ expansions, the OPE and loop integrals were performed by assuming the bulk system is in a four-dimensional space-time. Under rescaling $\mathbf{x} \rightarrow \mathbf{x}/b$, through the vertex correction the operator Φ_2 will acquire a correction

$$\begin{aligned} \delta \Phi_2 &= -\Phi_2 \int_{a/b}^a 4\pi r^2 dr \int_0^{+\infty} dz' \frac{32z'^4 g}{(r^2 + z'^2)^4} \\ &= -4g\pi^2 (\ln b) \Phi_2. \end{aligned} \quad (13)$$

The integral of z' is within the upper semi-infinite plane $z' > 0$.

Using ϵ expansion, g will flow from the noninteracting Gaussian fixed point to an interacting fixed point $g_* = \epsilon/[4(N+4)\pi^2]$. Plugging the fixed point value of g into Eq. (13), we obtain the vertex correction

$$\delta_v = \frac{\epsilon}{N+4}. \quad (14)$$

The wave-function renormalization δ_{wf} can be reproduced in the same way through OPE [Fig. 1(a)]. Eventually the scaling dimension of the gauge invariant order parameter Φ_2 at the

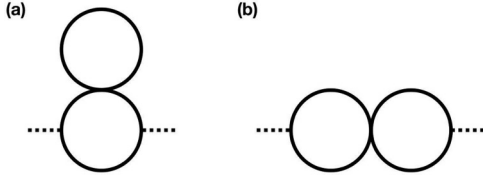


FIG. 2. The renormalization of operator Φ_2 at the leading order of ϵ can also be computed directly using the correlation functions in this figure.

boundary is

$$[\Phi_2] = D - \frac{N\epsilon}{N+4}. \quad (15)$$

We have also computed this scaling dimension through direct calculation of the correlation function of Φ_2 near the boundary (with diagrams in Fig. 2). These two calculations lead to the same results.

As we discussed before, the case with $N = 1$ can be realized as the transition between a Z_2 topological order and a superconductor. The QCP at $r = r_c$ separates the Z_2 topological order phase with $r > r_c$ and the superconducting phase with $r < r_c$, and potential experimental applications of this theory were discussed in Refs. [22,23]. In the bulk of the system, the gauge-invariant superconducting order parameter is given by $\sum_{a=1}^N \phi_a^2$ (with $N = 1$). However, if the system is probed from the boundary, then the leading gauge-invariant superconducting order parameter is given by Φ_2 instead. Therefore, in the ordered phase but close to the critical point, the superconductor order parameter probed from the boundary should scale with the tuning parameter r as

$$\Delta \sim |r - r_c|^{[\Phi_2]\nu} \sim |r - r_c|^{1.87}, \quad (16)$$

and we have taken $\nu \sim 2/3$ for the XY^* fixed point [28].

For $N = 2$, the Φ_2 operator is one component of the noncollinear spin order of a $SU(2)$ spin system, which scales as

$$\langle \vec{S} \rangle \sim \Phi_2 \sim |r - r_c|^{[\Phi_2]\nu} = |r - r_c|^{1.97}. \quad (17)$$

Again, we have taken $\nu = 0.74$ for the $O(4)^*$ fixed point [28]. As a comparison, in the 2D bulk Φ_2 should scale with r as $\Phi_2 \sim |r - r_c|^{0.82}$ ($N = 1$) and $\Phi_2 \sim |r - r_c|^{0.87}$ ($N = 2$), respectively, which is significantly different from the boundary scaling.

When $N = 1$, the action Eq. (1) may or may not allow an extra chemical potential term $\mu \phi^* \partial_\tau \phi$, depending on whether the system has a (emergent) particle-hole symmetry $\phi \rightarrow \phi^*$ or not. With nonzero μ the system has the same scaling as a mean-field transition (with logarithmic corrections) as the total space-time dimension is effectively $D = 2 + d = 4$, and g is marginally irrelevant. In this case the scaling dimension of the Cooper pair at the boundary becomes $[\Phi_2]_{\mu \neq 0} = D = 4$, and $\nu = 1/2$ as in the mean-field transition.

The boundary scaling is valid as long as we consider correlation function $G(\mathbf{x}_1 - \mathbf{x}_2, z_1, z_2)$ with $|\mathbf{x}_1 - \mathbf{x}_2| \gg z_1, z_2$. Right at the boundary of a 2D Z_2 topological order, the gauge field is confined, due to the condensation of the m anyons of the Z_2 topological order at the boundary (the boundary of a Z_2 topological order can also have e -anyon condensate,

but since in our case the e anyons carry nontrivial symmetry transformations, we assume our boundary always has m -anyon condensate). Near the boundary, the system still has a finite confinement length $\xi(z)$ as a function of z , i.e., the distance from the boundary, due to the “proximity effect” of the m condensation at the boundary. To guarantee that we can approximately assume a deconfined Z_2 gauge field near the boundary, we need $\xi(z) \gg z$.

The most convenient way to estimate the confinement length $\xi(z)$ close to the boundary is to evaluate the energy cost of two gauge charged particles separated with distance x near the boundary. This energy cost can be estimated in the “dual” Hamiltonian of a Z_2 gauge theory, which is a $(2+1)d$ quantum Ising model: $H_{\text{dual}} = \sum_{\vec{j}} -h\tau_{\vec{j}}^x - \sum_{\mu=x,y} J_{\vec{j},\mu} \tau_{\vec{j}}^z \tau_{\vec{j}+\mu}^z$, where $\tau_{\vec{j}}^x, \tau_{\vec{j}}^z$ are a pair of Pauli operators defined on the dual lattice sites $\vec{j}$. The dual Ising operator $\tau_{\vec{j}}^z$ is a creation and annihilation operator of the Z_2 gauge flux. A confined (and deconfined) phase of the Z_2 gauge field corresponds to the ordered (and disordered) phase of the dual quantum Ising model with nonzero (and zero) expectation value $\langle \tau^z \rangle$ [30]. If there is a pair of static e particles with Z_2 gauge charges separated with distance x , then this system is dual to a frustrated Ising model with $J_{\vec{j},\mu} = -J$ on the links along the branch-cut that connects the two particles, while $J_{\vec{j},\mu} = +J$ everywhere else. The energy cost of the two separated static particles corresponds to the energy difference between this frustrated Ising model nonuniform $J_{\vec{j},\mu}$, and the case with uniform $J_{\vec{j},\mu}$. Then if $\tau_{\vec{j}}^z$ has a nonzero expectation value $\langle \tau^z \rangle$, the pair of Z_2 gauge charges will approximately cost energy $E \sim J \langle \tau^z \rangle^2 x$, i.e., the system is in a confined phase with a linear confining potential between the two Z_2 gauge charges, and the confinement length is roughly $\xi \sim 1/(J \langle \tau^z \rangle^2)$. In our system with a boundary at $z = 0$, although $\langle \tau^z \rangle$ is nonzero at the boundary, its expectation value decays exponentially with z because the Z_2 gauge field is in a deconfined phase deep in the bulk with $\langle \tau^z \rangle = 0$. Hence, the confinement length $\xi(z)$ also increases with z exponentially, and we can safely assume that the Z_2 gauge field is still approximately deconfined near the boundary.

III. CONTINUOUS METAL-INSULATOR TRANSITION

Another unconventional quantum phase transition that can happen in 2D systems is the continuous metal-insulator transition, where the insulator is a $U(1)$ liquid phase with a fermi surface of the fermionic parton $f_{j,\alpha}$. Both $f_{j,\alpha}$ and b_j are coupled to an emergent $U(1)$ gauge field, which is presumably deconfined in the 2D bulk due to the existence of the Fermi surface and finite density of states of the matter fields. The critical behavior of this transition in the bulk was studied in Ref. [31], and it is again described by the condensation of b_j , but in this case b_j is coupled to a dynamic $U(1)$ gauge field a_μ .

Although there is a gapless gauge field a_μ in the bulk, the gauge field dynamics is over-damped by the fermi surface of f_α through a term $\mathcal{S}_{\text{damp}} \sim \frac{1}{e^2} \sum_{\omega, \vec{q}} |a'_{\omega, \vec{q}}|^2 \frac{|\omega|}{|q|}$ based on the standard Hertz-Millis formalism [32,33], where a' is the transverse mode of the gauge field. A simple power-counting would suggest that the gauge coupling e^2 becomes irrelevant

at the transition where b_j condenses, for both $\mu = 0$ and $\mu \neq 0$. Hence, the universality class of this transition does not receive relevant infrared corrections from the gauge field. Moreover, the direct density-density interaction between the bosonic and fermionic partons also does not lead to relevant effects [31]. Hence, the metal-insulator transition can still be described by Eq. (1). The quasiparticle residue is proportional to $|\langle b \rangle|$, and the electron Green's function is proportional to $|\langle b \rangle|^2$. Hence, if one probes from the boundary, then the local density of states of electrons at low energy, which is proportional to the electron Green's function, scales with the tuning parameter r as

$$\rho \sim |\langle \Phi_1 \rangle|^2 \sim |r - r_c|^{2[\Phi_1]\nu}, \quad (18)$$

where $r = r_c$ is the location of the metal-insulator transition point [31]. For $\mu = 0$, $[\Phi_1]$ is calculated in Eq. (11), and $\nu \sim 2/3$; for $\mu \neq 0$, $[\Phi_1] = 2$ and $\nu = 1/2$.

Again we need to address the question of confinement length near the boundary, and demonstrate that $\xi(z) \gg z$. A pure U(1) gauge field in $(2+1)d$ is dual to a scalar boson $\varphi \sim \exp(i\theta)$ which physically is the Dirac monopole operator, and the confined phase of a U(1) gauge field corresponds to a phase with a pinned nonzero expectation value of φ . A U(1) gauged particle becomes a vortex of θ in the dual formalism, and in a deconfined phase a vortex costs logarithmically divergent energy; but if φ has a pinned nonzero expectation value, then a vortex will cost linearly diverging energy and hence confined. Now suppose we consider a pair of gauge charged particles separated at distance x , the energy cost will be roughly $x\langle\varphi\rangle^2$. Hence, we need to evaluate $\langle\varphi(z)\rangle$ as a function of z away from the boundary, assuming a nonzero expectation value of φ at the boundary $\varphi_0 = \langle\varphi(z=0)\rangle$. $\langle\varphi(z)\rangle$ can be inferred from the correlation function $\langle\varphi(z)\rangle \sim \langle\varphi(z)\varphi(0)^*\rangle \sim \exp[\langle\theta(z)\theta(0)\rangle]$.

A $(2+1)d$ pure U(1) gauge field without the matter field is dual to a scalar boson model with an ordinary action $\mathcal{S} \sim \int d^2x d\tau \rho_s (\partial_\mu \theta)^2$, then θ has a positive scaling dimension $[\theta] = 1/2$. The correlation function of θ reads $\langle\theta(r)\theta(0)\rangle \sim 1/r$, which makes the correlation function of the monopole operator saturates to a nonzero value as $r \rightarrow \infty$. Hence, a positive scaling dimension of θ in the dual action renders the confinement of the compact gauge field in $(2+1)d$. If θ has a negative scaling dimension in its (dual) action, then the correlation function of φ will decay exponentially. Then the confinement length $\xi(z) \sim 1/\langle\varphi(z)\rangle^2 \sim 1/\langle\varphi(z)\varphi(0)^*\rangle^2$ will grow exponentially with z in the bulk away from the boundary. And since $\xi(z) \gg z$, the boundary scaling behavior calculated in this work can be applied under the assumption that the gauge field is sufficiently deconfined near the boundary since the confinement length is long enough in the vicinity of the boundary.

Now we need to derive the dual action for θ more carefully. Schematically the action for the transverse gauge field is

$$\mathcal{S} = \sum_{\omega, \vec{q}} \frac{1}{2} \left(\frac{|\omega|}{e^2 |\vec{q}|} + c^2 |\vec{q}|^2 \right) |a'|^2. \quad (19)$$

This action gives us an over-damped gauge boson $\omega \sim iec|\vec{q}|^3$. The canonical conjugate field of $\vec{a}$, i.e., the electric field of the gauge field is defined as $\vec{E} = \delta\mathcal{L}/\delta\vec{a}$, hence $\vec{E}_{\omega, \vec{q}} \sim$

$\vec{a}_{\omega, \vec{q}}/(e^2 |\vec{q}|)$, hence the action can also be written as

$$\mathcal{S} = \sum_{\omega, \vec{q}} \frac{e^2}{2} |\omega| |\vec{q}| |\vec{E}_{\omega, \vec{q}}|^2 + \frac{c^2}{2} |\vec{q}|^2 |a'_{\omega, \vec{q}}|^2. \quad (20)$$

Then we can use the standard canonical duality transformation that preserves the commutation relation between the canonical conjugate variables $\vec{E}$ and $\vec{a}$:

$$\vec{E} = \hat{z} \times \vec{\nabla} \theta, \quad \vec{\nabla} \times \vec{a} = \hat{n}, \quad (21)$$

where $\hat{n}$ is the flux density, or the particle density conjugate to θ (here the duality relation between gauge theory and scalar boson is identical to the duality used in the quantum dimer model, which was spelled out in, for example, Refs. [34,35]). Eventually the dual action reads

$$\mathcal{S}_d = \sum_{\omega, \vec{q}} \frac{1}{2} \left(e^2 |\omega| |\vec{q}|^3 + \frac{1}{c^2} \omega^2 \right) |\theta_{\omega, \vec{q}}|^2. \quad (22)$$

The dual action also gives us an over-damped mode $\omega \sim iec|\vec{q}|^3$ which is identical to the original gauge theory action. Indeed, in the dual action $\theta(\mathbf{x}, \tau)$ has a negative scaling dimension in this dual action, which is consistent with our expectation that $\langle\varphi(z)\rangle$ decays exponentially in the bulk, hence the gauge field is still approximately deconfined in the vicinity of the boundary.

IV. DISCUSSION

In this work we computed the boundary universal scaling behaviors of a class of deconfined quantum phase transitions, which is relevant to future realization of these exotic transitions in experimental systems. From the perspective of the pure Landau's paradigm, the cases we study correspond to the "ordinary transitions" of boundary CFT [16], meaning the bulk will enter an ordered phase before the boundary, which we believe is the most natural case in real systems. Measurement of the scaling laws we calculated depends on the specific realization of the theory Eq. (1). For example, if the $N = 1$ theory is realized (as we proposed in this work) as the transition between the Z_2 spin liquid to superconductor, then the amplitude of the Cooper pair at the boundary predicted in our calculation can be measured through the Josephson effect by building a junction between the boundary of the system and another ordinary bulk superconductor, as the Josephson current is proportional to the amplitude of the superconductor order parameter near the boundary. The Josephson current should follow the same scaling law as Eq. (16).

The studies in this work can be naturally generalized to higher dimensions. If there is a deconfined QCP between the Z_2 topological order and an ordered phase in the $(3+1)d$ bulk, then at its $(2+1)d$ boundary the gauge invariant order parameter Φ_2 has precise scaling dimension $[\Phi_2] = 4$, since in the bulk this transition is described by a mean-field theory and received no extra corrections.

The direct transition between the Néel and valence bond solid (VBS) order is another type of deconfined QCP that has attracted a great deal of attentions. The boundary effect of this deconfined QCP is more complex than the situations we have considered because the boundary breaks the lattice symmetry,

hence the boundary condition would couple to the VBS order parameter. Another interesting scenario worth studying is the boundary scaling of a bulk transition between a symmetry protected topological (SPT) states and an ordered phase which spontaneously breaks part of the defining symmetries of the SPT phase. Although the bulk transition should belong to the same universality class as the ordinary Ginzburg-Landau transition, its boundary is expected to be very different due to the existence of symmetry protected nontrivial boundary states even in the SPT phase. Efforts have been made along

this direction including numerical simulation [36] and construction of exactly soluble models [37]. We will leave these subjects to future studies.

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