ASYMPTOTIC ANALYSIS OF THE EXPECTED UTILITY MAXIMIZATION PROBLEM WITH RESPECT TO PERTURBATIONS OF THE NUMÉRAIRE

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Abstract. In an incomplete model, where under an appropriate numéraire, the stock price process is driven by a sigma-bounded semimartingale, we investigate the sensitivity of the expected utility maximization problem to small perturbations of the numéraire. We establish a second-order expansion of the value function and a first-order approximation of the terminal wealth. Relying on a description of the base return process in terms of its semimartingale characteristics, we also construct wealth processes and corrections to optimal strategies that match the indirect utility function up to the second order. We also relate the asymptotic expansions to the existence of the risk-tolerance wealth process and link perturbations of the numéraire to distortions of the finite-variation part and martingale part of the stock price return.

1. Introduction

In the settings of a complete financial market, it is proven in [GEKR95] that the choice of a numéraire affects neither arbitrage-free prices of the securities nor replicating strategies (see also a discussion in [HH09]). However, by an appropriate change of numéraire (sometimes combined with a change of measure), one can simplify a valuational framework, see, e.g., [GEKR95]. Possibly the most illuminating example corresponds to the LIBOR market interest rate model, which is based on a dynamic change of numéraire and which allows for pricing a wide class of interest rate derivatives.

In incomplete markets the situation is more delicate in general. As numéraire is a crucial ingredient in essentially all problems of mathematical finance, it is important

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to understand their sensitivity to misspecifications of the numéraire. In this paper in a general incomplete semimartingale model of a financial market, we investigate the response of the value function and the optimal solution to the expected utility maximization from terminal wealth problem to small perturbations of the numéraire. To the best of our knowledge sensitivity of the expected utility maximization problem to perturbations of numéraire has not been studied in the literature. We establish a second-order expansion of the value function, a first-order approximation of the terminal wealth, and construct wealth processes and corrections to optimal strategies that match the indirect utility function up to the second order. The latter development is conducted via a representation of base return process in terms of its semimartingale characteristics. In particular, we establish an envelope-type theorem for both primal and dual value functions. We also relate the asymptotic expansions to the existence of the risk-tolerance wealth process, which was introduced in [KS06b], and give a characterization of the correction terms in terms of a Kunita-Watanabe decomposition under certain changes of measure and numéraire.

Our results provide a way to estimate the effect of misspecification of the initial data on the expected utility maximization problem. This in particular applies to models, which allow for explicit solutions, see e.g., [Zar01], [GK03], [HIM05], [Liu07], [KS06b], [GR12], [HHI$^+$14], [ST14], and to so-called asymptotically complete models, see [Rob17, RSA17]. In many cases, a closed-form solution ceases to exist under perturbations of model parameters. Note that [KS06b], [HHI$^+$14], and [ST14] deal with a general utility function. This, in particular, emphasizes the importance of non confining oneself to power or logarithmic utilities.

In order to obtain the asymptotic expansions mentioned above, we introduce a *linear* parametrization of *returns* of a perturbed family of numéraires such that the corresponding numéraires are positive wealth processes for the values of the parameter being sufficiently close to 0. Note that positivity is a necessary condition for a process to be considered a numéraire. Even though, in principle, by a numéraire one can choose any strictly positive semimartingale, in this work we focus on tradable numéraires, in the terminology of [Bec01], i.e., the ones can be obtained as outcomes of trading strategies. Such a choice is standard in the mathematical finance literature, see for example [Bec01], [KS06a], [KS06b], [KK07].

In the case when the stock price process is one-dimensional and continuous, our structure of perturbations is closely connected to distortions of the finite-variation part of the return of the stock (as in [MS19]) and perturbations of the volatility (as in [HMKS17]), see the discussion in section 6.5 below. The proofs rely on the auxiliary minimization problems, which in turn are closely related to the ones in [CLP98], [PRS98], [LP99], [ČK07],

[CS13], [JMSS12], see also an overview of several approaches to quadratic problems in [Pha09]. Asymptotics analysis based on Malliavin calculus is implemented in [Mon13]. Simultaneous primal-dual asymptotic expansion method in mathematical finance has been (arguably) introduced in [Hen02] in the context of a utility-based pricing problem. Related analysis has been performed (at approximately the same time) in [HH02], [Kal02]. The first-order differentiability of the value functions with respect to the perturbations of the initial wealth and convergence of the optimizers are established in [KS99], whereas twice-differentiability is investigated in [KS06a].

As we expand the value function also in the initial wealth, analysis from [KS06a] turns out to be very helpful in the present work. On the other hand, Remark 4.3 below gives *corrections to the optimal trading strategy*, such that the corresponding wealth processes match the indirect utility up to the second order. This complements the results in [KS06a]. In this part, a representation of the base return process and in terms of its semimartingale characteristics is crucial.

The closest paper (to the best of our knowledge), mathematically, is [MS19], which deals with different perturbations, namely of the market price of risk, and where the underlying framework includes a continuous and one-dimensional stock price process. In the present paper, we impose neither one-dimensionality nor continuity of the stock (and the perturbations are different from the ones in [MS19]).

The remainder of this paper is organized as follows. In section 2 we present the model, in section 3 we formulate auxiliary minimization problems and state the expansion theorems; section 4 contains an explicit construction of nearly optimal wealth processes that match the primal value function up to the second order and corrections to the optimal strategies. In section 6.2 we give proofs of these results. In section 5 we relate the expansion theorems to the existence of a risk-tolerance wealth process, and we conclude the paper with section 7, where we show the necessity of Assumptions 2.3 and 2.8, under which the expansion theorems are proven.

2. Model

2.1. **Parametrized family of stock prices processes.** Let us consider a complete stochastic basis $\big(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}\big)$, where $T \in (0,\infty)$ is the time horizon, $\mathcal{F}$ satisfies the usual conditions, and $\mathcal{F}_0$ is a trivial σ-algebra. For the 0-model, we assume that there is a bank account with zero interest rate and d traded stocks, whose returns are modeled via a general d-dimensional semimartingale $(\rho^1, \dots, \rho^d)$. We set $R = (0, \rho^1, \dots, \rho^d)$ and suppose that (every component of) $R_0 = 0$.

The numéraire of 0-model is $N^0 \equiv 1$, equivalently the numéraire, whose return equals to zero and whose initial value equals 1. For perturbed models, we introduce linear perturbations of the returns of the numéraires, which are given by

$$\varepsilon\theta \cdot R, \quad \varepsilon \in (-\varepsilon_0, \varepsilon_0), \tag{2.1}$$

where θ is some predictable and R-integrable process that represents the *proportions of a wealth process* invested in the corresponding stocks for some portfolio (i.e., $\theta^0_t = 1 - \sum_{i=1}^{d} \theta^i_t$, $t \in [0,T]$) and that satisfies Assumptions 2.3 and 2.8 below, and ε_0 is a positive constant specified via Assumption 2.3. Equivalently, (2.1) can be restated in terms of the parametrized family of numéraires $(N^\varepsilon)_{\varepsilon\in(-\varepsilon_0,\varepsilon_0)}$, that satisfy

$$N^\varepsilon = \mathcal{E}\left((\varepsilon\theta)\cdot R\right), \quad \varepsilon \in (-\varepsilon_0, \varepsilon_0), \tag{2.2}$$

where $\mathcal{E}$ denotes the stochastic exponential. Thus, the family of stock price processes under numéraires N^ε is given by

$$S^\varepsilon := \left(\frac{1}{N^\varepsilon}, \frac{\mathcal{E}(\rho^1)}{N^\varepsilon}, \dots, \frac{\mathcal{E}\left(\rho^d\right)}{N^\varepsilon}\right), \quad \varepsilon \in (-\varepsilon_0, \varepsilon_0).$$

2.2. **Primal problem.** Let U be a utility function satisfying Assumption 2.1 below.

Assumption 2.1. The function $U\colon (0,\infty) \to \mathbb{R}$ is strictly increasing, strictly concave, two times continuously differentiable, and there exist positive constants c_1 and c_2, such that

$$c_1 \le A(x) := -\frac{U''(x)x}{U'(x)} \le c_2.$$

The family of primal feasible sets is defined as

$$\mathcal{X}(x,\varepsilon) := \left\{x + H\cdot S^\varepsilon \ge 0 :\ H \text{ is } S^\varepsilon - \text{integrable}\right\}, \quad (x,\varepsilon) \in [0,\infty)\times(-\varepsilon_0,\varepsilon_0),$$

where H is a predictable and S^ε-integrable process representing the amount invested in the stock. The corresponding family of the value functions is given by

$$u(x,\varepsilon) := \sup_{X\in\mathcal{X}(x,\varepsilon)} \mathbb{E}\left[U(X_T)\right], \quad (x,\varepsilon) \in (0,\infty)\times(-\varepsilon_0,\varepsilon_0). \tag{2.3}$$

We use the convention

$$\mathbb{E}\left[U(X_T)\right] := -\infty, \quad \text{if} \quad \mathbb{E}\left[U^-(X_T)\right] = \infty,$$

where U^- is the negative part of U.

2.3. **Dual problem.** The investigation of the primal problem (2.3) is conducted via the dual problem. First, let us define the dual domain for the 0-model as follows:

$$\begin{aligned}\mathcal{Y}(y,\varepsilon) := \{Y : \quad & Y \text{ is a nonnegative supermartingale, such that } Y_0 = y \\ & \text{and } XY = (X_t Y_t)_{t\geq 0} \text{ is a supermartingale} \\ & \text{for every } X \in \mathcal{X}(1,\varepsilon)\}, \quad (y,\varepsilon) \in [0,\infty)\times(-\varepsilon_0,\varepsilon_0).\end{aligned} \tag{2.4}$$

Remark 2.2. Definition (2.4) is an alternative version of a two-step natural definition of the dual domain, where in the first step one defines $\mathcal{Y}(y,0)$ as above and then sets $\mathcal{Y}(y,\varepsilon) = \mathcal{Y}(y,0)N^\varepsilon$. However, Lemma 6.1 asserts that both constructions are equivalent.

We set the convex conjugate to U as

$$V(y) := \sup_{x>0} \left(U(x) - xy\right), \quad y \in (0,\infty).$$

Note that for $y = U'(x)$, we have

$$V''(y) = -\frac{1}{U''(x)},$$

and

$$B(y) := -\frac{V''(y)y}{V'(y)} = \frac{1}{A(x)}.$$

Therefore, Assumption 2.1 implies that

$$\frac{1}{c_2} \leq B(y) \leq \frac{1}{c_1}, \quad y > 0.$$

The parametrized family of dual value functions is given by

$$v(y,\varepsilon) := \inf_{Y\in\mathcal{Y}(y,\varepsilon)} \mathbb{E}\left[V\left(Y_T\right)\right], \quad (y,\varepsilon) \in (0,\infty)\times(-\varepsilon_0,\varepsilon_0). \tag{2.5}$$

We use the convention

$$\mathbb{E}\left[V\left(Y_T\right)\right] := \infty, \quad \text{if} \quad \mathbb{E}\left[V^+\left(Y_T\right)\right] = \infty,$$

where V^+ is the positive part of V.

2.4. **Technical assumptions.** For nondegeneracy of 0-model, we suppose that

$$u(x,0) < \infty \quad \text{for some} \quad x > 0. \tag{2.6}$$

One needs to ensure that the perturbations of the form (2.1) (or equivalently in the form (2.2)) are such that the resulting processes N^ε are nonnegative at least for ε being sufficiently close to 0, as a necessary way of making N^ε's numéraires. This can be achieved via the following condition. Example 7.2 below demonstrates the necessity of a boundedness Assumption 2.3.

Assumption 2.3. We suppose that there exists $\varepsilon_0 > 0$ such that the jumps of the process $\bar{R} := -\theta \cdot R$ are bounded by $\frac{1}{2\varepsilon_0}$, i.e.,

$$|\Delta \bar{R}_t| \leq \tfrac{1}{2\varepsilon_0}, \quad t \in [0, T].$$

Note that Assumption 2.3 implies that N^ε in (2.2) is a strictly positive process $\mathbb{P}$–a.s., for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$.

2.5. **Absence of arbitrage.** The absence of arbitrage opportunities in the 0-model in the sense of no unbounded profit with bounded risk follows from condition (2.7), which by the results of [KK07] can equivalently be stated as

$$\mathcal{Y}(1,0) \text{ contains a strictly positive element.} \tag{2.7}$$

We refer to [KK07] for characterizations of no unbounded profit with bounded risk condition, which is also equivalent to the existence of a strict sigma-martingale density, see [TS14] for details.

Remark 2.4. Condition (2.7) and Lemma 6.1 imply no unbounded profit with bounded risk for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, thus

$$\mathcal{Y}(1,\varepsilon) \neq \emptyset, \quad \varepsilon \in (-\varepsilon_0, \varepsilon_0).$$

Remark 2.5. Assumption 2.1 implies that U satisfies the Inada conditions and that asymptotic elasticity of U (in the sense of [KS99]) is less than 1, see [KS06a, Lemma 3] for the proof. Therefore, under (2.1), (2.6), and (2.7), existence and uniqueness of a solution to (2.3) for every $x > 0$ and other standard assertions of the utility maximization theory follow from the abstract theorems in [KS99].

Remark 2.6. [KKS16, Theorem 2.1] gives a characterization of no unbounded profit with bounded risk condition in terms of the existence of *local martingale* deflators (as opposed to *supermartingale* deflators in [KK07]).

For every $x > 0$, under Assumption 2.1, (2.6), and (2.7) it follows from Remark 2.5, that $y = u_x(x, 0)$ exists and is unique and there exist unique solutions to (2.3) and (2.5), $\widehat{X}(x, 0)$ and $\widehat{Y}(y, 0)$, respectively, such that $\widehat{X}(x, 0)\widehat{Y}(y, 0)$ is a uniformly integrable martingale under $\mathbb{P}$. An important role will be played by the probability measures $\mathbb{R}(x)$, given by

$$\frac{d\mathbb{R}(x)}{d\mathbb{P}} := \frac{\widehat{X}_T(x, 0)\widehat{Y}_T(y, 0)}{xy}, \quad x > 0, \quad y = u_x(x, 0). \tag{2.8}$$

Note that, $\mathbb{R}(x)$ defined in (2.8) coincides with the measure $\mathbb{R}(x)$ in the notations of [KS06a], [KS06b] and with measure $\mathbb{R}(x, 0)$ in terminology of [MS19], and that $\mathbb{R}(x)$

naturally appears in the asymptotic analysis of optimal investment, see [KS06a], [KS06b], and [MS19].

Since we consider an expansion also in the initial wealth, in order for the value function u to be twice differentiable in the first argument (which corresponds to the initial wealth x), we need to impose the sigma-boundedness assumption, see [KS06a, Definition 1] for the definition, also [KS06b] and [BS12] contain discussions on this subject and applications of sigma-bounded processes to the problem of the expected utility maximization.

Assumption 2.7. Let $x > 0$ be fixed. We suppose that the process

$$S^{\widehat{X}(x,0)} := \left(\frac{x}{\widehat{X}(x,0)}, \frac{x\mathcal{E}(\rho^1)}{\widehat{X}(x,0)}, \dots, \frac{x\mathcal{E}(\rho^d)}{\widehat{X}(x,0)} \right)$$

is sigma-bounded.

When using $S^{\widehat{X}(x,0)}$, we discount the assets by the normalized primal optimizer for the 0-model. We also need the following integrability assumption on perturbations, whose necessity is demonstrated in Example 7.1 below.

Assumption 2.8. Let $x > 0$ be fixed. There exists $c > 0$, such that

$$\mathbb{E}^{\mathbb{R}(x)} \left[\exp \left\{ c \left(\left| \bar{R}_T \right| + [\bar{R}, \bar{R}]_T \right) \right\} \right] < \infty.$$

3. Expansion Theorems

We begin with an envelope theorem.

Theorem 3.1. *Let $x > 0$ be fixed, assume that (2.6) and (2.7) as well as Assumptions 2.1, 2.3, 2.7, and 2.8 hold, and let $y = u_x(x,0)$. Then there exists $\bar{\varepsilon} > 0$ such that for every $\varepsilon \in (-\bar{\varepsilon}, \bar{\varepsilon})$, $u(\cdot, \varepsilon)$ and $v(\cdot, \varepsilon)$ are finite-valued functions. The functions u and v are jointly differentiable (and, consequently, continuous) at $(x,0)$ and $(y,0)$, respectively. We also have*

$$\nabla u(x,0) = \begin{pmatrix} y \\ u_\varepsilon(x,0) \end{pmatrix} \quad \text{and} \quad \nabla v(y,0) = \begin{pmatrix} -x \\ v_\varepsilon(y,0) \end{pmatrix}, \tag{3.1}$$

where

$$u_\varepsilon(x,0) = v_\varepsilon(y,0) = xy\mathbb{E}^{\mathbb{R}(x)} \left[\bar{R}_T \right].$$

Note that, the key formula in Theorem 3.1 is the expression for $u_\varepsilon(x,0)$. In the case when $\hat{Y}(y,0)$ is a uniformly integrable martingale itself, this process is often used to define a new measure $\hat{\mathbb{Q}}(y)$ via $\frac{d\hat{\mathbb{Q}}(y)}{d\mathbb{P}} := \frac{\hat{Y}(y,0)}{y}$. Then, the first-order derivatives in ε can be restated as

$$u_\varepsilon(x,0) = v_\varepsilon(y,0) = y\mathbb{E}^{\hat{\mathbb{Q}}(y)} \left[\hat{X}_T(x,0)\bar{R}_T \right].$$

In order to characterize the second-order derivatives of the value functions, we will need the following notations. For every $x > 0$, let $\mathbf{H}_0^2(\mathbb{R}(x))$ denote the space of square integrable martingales under $\mathbb{R}(x)$ that start at 0. Let us recall that $S^{X(x,0)}$ was defined in Assumption 2.7 and set

$$\mathcal{M}^2(x,0) := \left\{ M \in \mathbf{H}_0^2(\mathbb{R}(x)) : M = H \cdot S^{\widehat{X}(x,0)} \right\},$$
$$\mathcal{N}^2(y,0) := \left\{ N \in \mathbf{H}_0^2(\mathbb{R}(x)) : MN \text{ is } \mathbb{R}(x) - \text{martingale for every } M \in \mathcal{M}^2(x,0) \right\},$$
$$\text{here } y = u_x(x,0).$$

Auxiliary minimization problems. As in [KS06a], for $x > 0$ let us consider

$$a(x,x) := \inf_{M \in \mathcal{M}^2(x,0)} \mathbb{E}^{\mathbb{R}(x)} \left[A(\widehat{X}_T(x,0))(1+M_T)^2 \right], \tag{3.2}$$

$$b(y,y) := \inf_{N \in \mathcal{N}^2(y,0)} \mathbb{E}^{\mathbb{R}(x)} \left[B(\widehat{Y}_T(y,0))(1+N_T)^2 \right], \quad y = u_x(x,0), \tag{3.3}$$

where A is the relative risk aversion and B is the relative risk tolerance of U, respectively. We refer to [KS06a] for the details behind the derivation of (3.2) and (3.3). Note that (3.2) and (3.3) govern the second-order derivatives of u in x and v in y, respectively.

Remark 3.2. Existence and uniqueness of a solution to every quadratic minimization problem in this paper follows from the closedness of its domain (in the appropriate sense), convexity of the objective, and Komlos' lemma, see [KS06a, Lemma 2].

Let us also set

$$F := \bar{R}_T \quad \text{and} \quad G := [\bar{R}, \bar{R}]_T. \tag{3.4}$$

We consider the following minimization problems:

$$a(\varepsilon,\varepsilon) := \inf_{M \in \mathcal{M}^2(x,0)} \mathbb{E}^{\mathbb{R}(x)} \left[A(\widehat{X}_T(x,0))(M_T + xF)^2 - 2xFM_T - x^2(F^2+G) \right], \tag{3.5}$$

$$b(\varepsilon,\varepsilon) := \inf_{N \in \mathcal{N}^2(y,0)} \mathbb{E}^{\mathbb{R}(x)} \left[B(\widehat{Y}_T(y,0))(N_T - yF)^2 + 2yFN_T - y^2(F^2-G) \right]. \tag{3.6}$$

Quadratic minimization problems (3.5) and (3.6) govern the second-order correction terms associated with perturbations in ε in the expansion for u and v, where the exact structure is given through Theorem 3.3. Denoting by $M^1(x,0)$ and $N^1(y,0)$ the unique solutions to (3.5) and (3.6) respectively, we also set

$$a(x,\varepsilon) := \mathbb{E}^{\mathbb{R}(x)} \left[A(\widehat{X}_T(x,0))(1+M_T^0(x,0))(xF + M_T^1(x,0)) - xF(1+M_T^0(x,0)) \right], \tag{3.7}$$

$$b(y,\varepsilon) := \mathbb{E}^{\mathbb{R}(x)} \left[B(\widehat{Y}_T(y,0))(1+N_T^0(y,0))(N_T^1(y,0) - yF) + yF(1+N_T^0(y,0)) \right]. \tag{3.8}$$

Theorems 3.3, 3.5, and 3.6 contain the second-order expansions of the value functions, derivatives of the optimizers, and properties of such derivatives.

Theorem 3.3. *Let $x > 0$ be fixed. Assume all conditions of Theorem 3.1 hold, with $y = u_x(x, 0)$. Let us define*

$$H_u(x,0) := -\frac{y}{x}\begin{pmatrix} a(x,x) & a(x,\varepsilon) \\ a(x,\varepsilon) & a(\varepsilon,\varepsilon)\end{pmatrix}, \tag{3.9}$$

where $a(x,x)$, $a(\varepsilon,\varepsilon)$, and $a(x,\varepsilon)$ are specified in (3.2), (3.5), *and* (3.7), *and, respectively,*

$$H_v(y,0) := \frac{x}{y}\begin{pmatrix} b(y,y) & b(y,\varepsilon) \\ b(y,\varepsilon) & b(\varepsilon,\varepsilon)\end{pmatrix},$$

where $b(y,y)$, $b(\varepsilon,\varepsilon)$, $b(y,\varepsilon)$ are specified in (3.3), (3.6), *and* (3.8). *Then, the value functions u and v admit the second-order expansions around $(x,0)$ and $(y,0)$, respectively,*

$$u(x+\Delta x,\varepsilon) = u(x,0) + (\Delta x \quad \varepsilon)\nabla u(x,0) + \tfrac{1}{2}(\Delta x \quad \varepsilon)H_u(x,0)\begin{pmatrix}\Delta x \\ \varepsilon\end{pmatrix} + o(\Delta x^2 + \varepsilon^2),$$

and

$$v(y+\Delta y,\varepsilon) = v(y,0) + (\Delta y \quad \varepsilon)\nabla v(y,0) + \tfrac{1}{2}(\Delta y \quad \varepsilon)H_v(y,0)\begin{pmatrix}\Delta y \\ \varepsilon\end{pmatrix} + o(\Delta y^2 + \varepsilon^2).$$

Remark 3.4. Similarly to [MS19], even though we only have second-order expansions, we may abuse the language and call $H_u(x,0)$ and $H_v(y,0)$ the Hessians of u and v, without having twice differentiability.

Theorem 3.5. *Let $x > 0$ be fixed, the assumptions of Theorem 3.1 hold, and $y = u_x(x,0)$. Then, the auxiliary value functions satisfy*

$$\begin{pmatrix} a(x,x) & 0 \\ a(x,\varepsilon) & -\frac{x}{y}\end{pmatrix}\begin{pmatrix} b(y,y) & 0 \\ b(y,\varepsilon) & -\frac{y}{x}\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}$$

and

$$\frac{y}{x}a(\varepsilon,\varepsilon) + \frac{x}{y}b(\varepsilon,\varepsilon) = a(x,\varepsilon)b(y,\varepsilon).$$

The optimizers to auxiliary minimization problems are related via the following formulas.

$$U''(\widehat{X}_T(x,0))\widehat{X}_T(x,0)\begin{pmatrix} M_T^0(x,0)+1 \\ M_T^1(x,0)+xF\end{pmatrix} = -\begin{pmatrix} a(x,x) & 0 \\ a(x,\varepsilon) & -\frac{x}{y}\end{pmatrix}\widehat{Y}_T(y,0)\begin{pmatrix} N_T^0(y,0)+1 \\ N_T^1(y,0)-yF\end{pmatrix},$$

$$V''(\widehat{Y}_T(y,0))\widehat{Y}_T(y,0)\begin{pmatrix} 1+N_T^0(y,0) \\ -yF+N_T^1(y,0)\end{pmatrix} = \begin{pmatrix} b(y,y) & 0 \\ b(y,\varepsilon) & -\frac{y}{x}\end{pmatrix}\widehat{X}_T(x,0)\begin{pmatrix} 1+M_T^0(x,0) \\ xF+M_T^1(x,0)\end{pmatrix}.$$

Moreover, the product of any of $\widehat{X}(x,0)$, $\widehat{X}(x,0)M^0(x,0)$, $\widehat{X}(x,0)M^1(x,0)$ and any of $\widehat{Y}(y,0)$, $\widehat{Y}(y,0)N^0(y,0)$, $\widehat{Y}(y,0)N^1(y,0)$ is a martingale under $\mathbb{P}$, where $M_T^0(x,0)$, $M_T^0(x,0)$, $N_T^0(y,0)$, and $N_T^1(y,0)$ are the solutions to (3.2), (3.5), (3.3), *and* (3.6), *correspondingly.*

Theorem 3.6. *Let $x > 0$ be fixed, the assumptions of Theorem 3.1 hold, and $y = u_x(x,0)$. Then, if we define*

$$X^x_T(x,0) := \frac{\widehat{X}_T(x,0)}{x}(1 + M^0_T(x,0)), \quad Y^y_T(y,0) := \frac{\widehat{Y}_T(y,0)}{y}(1 + N^0_T(y,0)),$$

and

$$X^\varepsilon_T(x,0) := \frac{\widehat{X}_T(x,0)}{x}(M^1_T(x,0) + xF), \quad Y^\varepsilon_T(y,0) := \frac{\widehat{Y}_T(y,0)}{y}(N^1_T(y,0) - yF),$$

we have

$$\lim_{|\Delta x|+|\varepsilon|\to 0} \frac{1}{|\Delta x| + |\varepsilon|} \left|\widehat{X}_T(x+\Delta x,\varepsilon) - \widehat{X}_T(x,0) - \Delta x X^x_T(x,0) - \varepsilon X^\varepsilon_T(x,0)\right| = 0,$$

$$\lim_{|\Delta y|+|\varepsilon|\to 0} \frac{1}{|\Delta y| + |\varepsilon|} \left|\widehat{Y}_T(y+\Delta y,\varepsilon) - \widehat{Y}_T(y,0) - \Delta y Y^y_T(y,0) - \varepsilon Y^\varepsilon_T(y,0)\right| = 0,$$

where the convergence takes place in $\mathbb{P}$-probability.

4. Construction of nearly optimal wealth processes

Here $x > 0$ will be fixed $\widehat{\pi}$ will denote the optimal proportion invested in stock for 0-model and initial wealth x, i.e., $\widehat{\pi}$ satisfies

$$\widehat{X}(x,0) = x\mathcal{E}\left(\widehat{\pi} \cdot R\right),$$

where $R = (\rho^0, \rho^1, \ldots, \rho^d)$, $\rho^0 \equiv 0$. For the results below, we will need a representation of R in terms of its *predictable characteristics*. Notation-wise here, we follow [JS03]. Thus, we fix the *truncation function* $h(x) : x \to x1_{\{|x|\le 1\}}$ and denote by R^c the *continuous martingale part* of R, by B the *predictable finite variation part* of R (corresponding to the truncation function h), by μ the *jump measure* of R, i.e., a random counting measure on $[0,T] \times \mathbb{R}^d$ defined by

$$\mu\left([0,t] \times E\right) := \sum_{0\le s\le t} 1_{\{E\setminus\{0\}\}}(\Delta R_s), \quad t \in [0,T], E \subseteq \mathbb{R}^d,$$

where 1_E is the indicator function of a set E, by ν we denote the *predictable compensator* of μ, i.e., a *predictable random measure* on $[0,T] \times \mathbb{R}^d$, such that, in particular, $\left(x1_{\{|x|\le 1\}}\right) * (\mu - \nu)$ is a *purely discontinuous local martingale*. Setting the *quadratic covariation process* $C := [R^c, R^c]$ of R^c, we call (B, C, η) the *triplet of predictable characteristics* of R (associated with the truncation function h).

It it well-known (see for example [JS03]), that semimartingale R can be represented in terms of (B, C, η) as

$$R = R^c + B + \left(x1_{\{|x|\le 1\}}\right) * (\mu - \nu) + \left(x1_{\{|x|>1\}}\right) * \mu.$$

Note that predictable characteristics (B, C, ν) are unique up to a $\mathbb{P}$-null set. Moreover, let us define a predictable scalar-valued locally integrable increasing process process A as

$$A := \sum_{i\leq d} Var(B^i) + \sum_{i\leq d} C^{i,i} + (\min(1, |x|^2)) * \nu,$$

where $Var(B^i)$ denotes the variation process of B^i, $i = 1, \dots, d$. Then B, C, and ν are absolutely continuous with respect to A, therefore

$$B = b \cdot A, \quad C = c \cdot A, \quad \text{and} \quad \nu = \eta \cdot A,$$

where b is a predictable $\mathbb{R}^d$-valued process, c is a predictable process with values in the set of nonnegative-definite matrices, and ν is a predictable Levy-measure-valued process. Let us define a vector-valued process $R^{\{\widehat{\pi}\}}$ as

$$R^{\{\widehat{\pi}\}} := R - (c\widehat{\pi}) \cdot A - \left(\tfrac{\widehat{\pi}^\top x}{1+\widehat{\pi}^\top x} x\right) * \mu. \tag{4.1}$$

Note that, the process $R^{\{\widehat{\pi}\}}$ governs the return of the traded assets under the numeraire $\frac{\widehat{X}(x,0)}{x} = \mathcal{E}(\widehat{\pi} \cdot R)$. Here end below superscript $\top$ denotes the transpose of a vector. Also note that $R^{\{\widehat{\pi}\}}$ is a semimartingale as

$$\sum_{i=0}^{d} \sum_{s\leq \cdot} \left(\tfrac{\widehat{\pi}_s^\top \Delta R_s}{1+\widehat{\pi}_s^\top \Delta R_s}\right)^2 (\Delta \rho_s^i)^2 < \infty.$$

Let $\mathcal{M}^\infty(x)$ denote the set of uniformly bounded elements of $\mathcal{M}^2(x)$.

Lemma 4.1. *Let us assume that the assumptions of Theorem 3.1 hold. Then every element of $\mathcal{M}^\infty(x)$ be represented as a stochastic integral with respect to $R^{\{\widehat{\pi}\}}$.*

Proof. Let $M \in \mathcal{M}^\infty(x)$. Then for a sufficiently large constant $C' > 0$, we have

$$0 < C' + M = C' + H \cdot S^X = \frac{C'\mathcal{E}(\tilde{\pi} \cdot R)}{\mathcal{E}(\widehat{\pi} \cdot R)}, \tag{4.2}$$

for some predictable and R-integrable process $\tilde{\pi}$. First, as $\Delta(\widehat{\pi} \cdot R) > -1$, we have

$$\frac{\mathcal{E}(\tilde{\pi} \cdot R)}{\mathcal{E}(\widehat{\pi} \cdot R)} = \mathcal{E}(D),$$

where

$$D = \tilde{\pi} \cdot R - \widehat{\pi} \cdot R - [(\tilde{\pi} \cdot R)^c - (\widehat{\pi} \cdot R)^c, (\widehat{\pi} \cdot R)^c] - \sum_{t\leq \cdot} \left(\Delta(\tilde{\pi} \cdot R_t - \widehat{\pi} \cdot R_t) \frac{\Delta\widehat{\pi} \cdot R_t}{1 + \Delta\widehat{\pi} \cdot R_t}\right),$$

which is a (well-defined) semimartingale in view of finiteness of $\sum_{t\leq \cdot}(\Delta\tilde{\pi} \cdot R_t)^2$ and $\sum_{t\leq \cdot}(\Delta\widehat{\pi} \cdot R_t)^2$, see [KK07, Lemma 3.4]. Therefore, we can restate $\frac{\mathcal{E}(\tilde{\pi}\cdot R)}{\mathcal{E}(\widehat{\pi}\cdot R)}$ as

$$\frac{\mathcal{E}(\tilde{\pi} \cdot R)}{\mathcal{E}(\widehat{\pi} \cdot R)} = \mathcal{E}\left((\tilde{\pi} - \widehat{\pi}) \cdot R^{\{\widehat{\pi}\}}\right). \tag{4.3}$$

Using representation (4.3), in (4.2) we obtain

$$C' + M = C'\mathcal{E}\left((\tilde{\pi} - \widehat{\pi}) \cdot R^{\{\widehat{\pi}\}}\right) = C' + C'\left\{\mathcal{E}\left((\tilde{\pi} - \widehat{\pi}) \cdot R^{\{\widehat{\pi}\}}\right)_{-} (\tilde{\pi} - \widehat{\pi})\right\} \cdot R^{\{\widehat{\pi}\}}.$$

Solving for M, we get

$$M = \left\{C'\mathcal{E}\left((\tilde{\pi} - \widehat{\pi}) \cdot R^{\{\widehat{\pi}\}}\right)_{-} (\tilde{\pi} - \widehat{\pi})\right\} \cdot R^{\{\widehat{\pi}\}},$$

which completes the proof. □

Let M^0 and M^1 denote the solutions to (3.2) and (3.5), respectively. It follows from [KS06a, Lemma 6] that there exist sequences $(\bar{M}^{0,n})_{n\geq 1}$ and $(\bar{M}^{1,n})_{n\geq 1}$ in $\mathcal{M}^{\infty}(x)$, such that

$$\lim_{n\to\infty} \bar{M}^{0,n}_T = M^0_T \quad \text{and} \quad \lim_{n\to\infty} \bar{M}^{1,n}_T = M^1_T, \quad \mathbb{P}\text{–a.s.}$$

Without loss of generality, we may assume that $\bar{M}^{0,n}$ is bounded by n, $n \geq 1$. Therefore, the jumps of $\bar{M}^{0,n}$ are bounded by $2n$ and the quadratic variation of $\bar{M}^{0,n}$ is locally bounded, where

$$T_k := \inf\left\{t \geq 0 : \ [\bar{M}^{0,n}]_t \geq k\right\}, \quad k \geq 1,$$

is a localizing sequence for $[\bar{M}^{0,n}]$. Note that $[\bar{M}^{0,n}]_{T_k} \leq k + 4n^2$. Let us define

$$\tilde{M}^{0,n}_t := \bar{M}^{0,n}_{\min(t,T_n)}, \quad t \in [0,T], n \geq 1.$$

Then $\tilde{M}^{0,n}$ is bounded by n, its quadratic variation is bounded $n + 4n^2$, and its jumps are bounded by $2n$. Moreover, by construction we have

$$\lim_{n\to\infty} \tilde{M}^{0,n}_T = M^0_T, \quad \mathbb{P}\text{–a.s.}$$

Analogously, we can construct a sequence $\tilde{M}^{1,n}$, $n \geq 1$, of martingales under $\mathbb{R}(x)$, such that $\tilde{M}^{1,n}$ is bounded by n, its quadratic variation is bounded by $n + 4n^2$, and its jumps are bounded by $2n$, $n \geq 1$, and such that

$$\lim_{n\to\infty} \tilde{M}^{1,n}_T = M^1_T, \quad \mathbb{P}\text{–a.s.}$$

Lemma 4.1 implies the existence of predictable $R^{\{\widehat{\pi}\}}$-integrable processes $\gamma^{0,n}$ and $\gamma^{1,n}$, $n \geq 1$, such that

$$\gamma^{0,n} \cdot R^{\{\widehat{\pi}\}} = \frac{\tilde{M}^{0,n}}{x}, \quad \gamma^{1,n} \cdot R^{\{\widehat{\pi}\}} = \frac{\tilde{M}^{1,n}}{x}, \quad n \geq 1. \tag{4.4}$$

We define the family of processes $(R^{\{\varepsilon\theta\}})_{\varepsilon\in(-\varepsilon_0,\varepsilon_0)}$ as

$$R^{\{\varepsilon\theta\}} \ := \ R - \varepsilon(c\theta) \cdot A - \varepsilon\left(\tfrac{\theta_s^{\top} x}{1+\varepsilon\theta_s^{\top} x} x\right) * \mu, \tag{4.5}$$

where $R^{\{\varepsilon\theta\}}$ governs the returns of the traded assets under N^ε, and similarly to the verification after (4.1), one can show that $R^{\{\varepsilon\theta\}}$ is a semimartingale for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. Finally, let us define the family $\left(\tilde{X}^{\Delta x,\varepsilon,n}\right)_{(\Delta x,\varepsilon,n)\in(-x,\infty)\times(-\varepsilon_0,\varepsilon_0)\times\mathbb{N}}$ as

$$\tilde{X}^{\Delta x,\varepsilon,n} := (x+\Delta x)\mathcal{E}\left(\left(\widehat{\pi} + \Delta x\gamma^{0,n} + \varepsilon(-\theta + \gamma^{1,n})\right)\cdot R^{\{\varepsilon\theta\}}\right). \tag{4.6}$$

Theorem 4.2. *Let $x > 0$ be fixed and the assumptions of Theorem 3.1 hold. Then we have.*

(1) *For every $n \in \mathbb{N}$, there exists $\delta = \delta(n) > 0$, such that,*

$$\tilde{X}^{\Delta x,\varepsilon,n} \in \mathcal{X}(x+\Delta x,\varepsilon), \quad (\Delta x,\varepsilon) \in B_{\delta(n)}(0,0),$$

where $B_{\delta(n)}(0,0)$ denotes a ball of radius $\delta(n)$ centered at $(0,0)$.

(2) *There exists a function $n = n(\Delta x,\varepsilon) : (-x,\infty)\times(-\varepsilon_0,\varepsilon_0) \to \mathbb{N}$, such that*

$$\mathbb{E}\left[U\left(\tilde{X}_T^{\Delta x,\varepsilon,n(\Delta x,\varepsilon)}\right)\right] = u(x+\Delta x,\varepsilon) - o(\Delta x^2+\varepsilon^2). \tag{4.7}$$

(3) *The processes $\tilde{X}^{\Delta x,\varepsilon,n(\Delta x,\varepsilon)}$'s from the previous item have the following proportions invested in the corresponding stocks:*

$$\left((1-\varepsilon)I + \varepsilon\theta\vec{1}^\top\right)\left(\widehat{\pi} + \Delta x\gamma^{0,n(\Delta x,\varepsilon)} + \varepsilon(-\theta + \gamma^{1,n(\Delta x,\varepsilon)})\right), \tag{4.8}$$

where I is $(d+1)\times(d+1)$ identity matrix and $\theta\vec{1}^\top$ is the outer product of θ and the vector, whose every component equals to 1.

Remark 4.3. By taking $\varepsilon = 0$, Theorem 4.2 theorem gives corrections to optimal proportions invested in stock with respect to perturbations of the initial wealth only. In this case the nearly optimal family of wealth processes is given by

$$\bar{X}^{\Delta x,n} := (x+\Delta x)\mathcal{E}\left(\left(\widehat{\pi} + \Delta x\gamma^{0,n}\right)\cdot R\right), \quad (\Delta x,n) \in (-x,\infty)\times\mathbb{N},$$

where $\gamma^{0,n}$ are given in (4.4). Theorem 4.2 asserts that there exists a function $n = n(\Delta x) : (-x,\infty) \to \mathbb{N}$, such that

$$\mathbb{E}\left[U\left(\bar{X}_T^{\Delta x,n(\Delta x)}\right)\right] = u(x+\Delta x,0) - o(\Delta x^2).$$

This allows to construct corrections to optimal trading strategies in the settings of [KS06a].

5. Relationship to the risk-tolerance wealth process

We recall here that for an initial wealth $x > 0$, the *risk-tolerance wealth process* is defined as a maximal wealth process $\mathcal{R}(x)$, such that

$$\mathcal{R}_T(x) = -\frac{U'(\widehat{X}_T(x,0))}{U''(\widehat{X}_T(x,0))}, \tag{5.1}$$

i.e. it is a replication process for the random payoff given by the right-hand side of (5.1). The term *risk-tolerance wealth process* was introduced in [KS06b] in the context of

asymptotic analysis of utility-based prices, in general it may not exist. As in [KS06b], for $x > 0$ and with $y = u_x(x, 0)$, let us define

$$\frac{d\widetilde{\mathbb{R}}(x)}{d\mathbb{P}} := \frac{\mathcal{R}_T(x)\widehat{Y}_T(y,0)}{\mathcal{R}_0(x)y},$$

and choose $\frac{\mathcal{R}(x)}{\mathcal{R}_0(x)}$ as a numéraire in the 0-model, i.e., let us set

$$S^{\mathcal{R}(x)} := \left(\frac{\mathcal{R}_0(x)}{\mathcal{R}(x)}, \frac{\mathcal{R}_0(x)\mathcal{E}(\rho^1)}{\mathcal{R}(x)}, \dots, \frac{\mathcal{R}_0(x)\mathcal{E}(\rho^d)}{\mathcal{R}(x)}\right).$$

We define the spaces of martingales

$$\widetilde{\mathcal{M}}^2(x,0) := \left\{M \in \mathbf{H}_0^2(\widetilde{\mathbb{R}}(x)) : \ M = H \cdot S^{\mathcal{R}(x)}\right\},$$

and denote by $\widetilde{\mathcal{N}}^2(y,0)$ the orthogonal complement of $\widetilde{\mathcal{M}}^2(x,0)$ in $\mathbf{H}_0^2(\widetilde{\mathbb{R}}(x))$. Theorem 5.1 below relates the structural properties of the approximations in Theorems 3.3, 3.5, and 3.6 to a Kunita-Watanabe decomposition (under the changes of measure and numéraire described above), under the assumption that the risk-tolerance process exists. Theorem 5.1 is stated without a proof, as line by line adaptation of the proof of [MS19, Theorem 8.3] applies here.

Theorem 5.1. *Let $x > 0$ be fixed, assume that (2.7), (2.6), and Assumption 2.1 hold, and denote $y = u_x(x, 0)$. Let us also assume that the risk-tolerance process $\mathcal{R}(x)$ exists. Consider the Kunita-Watanabe decomposition of the square integrable martingale*

$$P_t := \mathbb{E}^{\widetilde{\mathbb{R}}(x)}\left[\left(A(\widehat{X}_T(x,0)) - 1\right) xF|\mathcal{F}_t\right], \quad t \in [0,T]$$

given by

$$P = P_0 - \widetilde{M}^1 - \widetilde{N}^1, \quad \text{where} \quad \widetilde{M}^1 \in \widetilde{\mathcal{M}}^2(x,0), \quad \widetilde{N}^1 \in \widetilde{\mathcal{N}}^2(y,0), \quad P_0 \in \mathbb{R}. \tag{5.2}$$

Then, the optimal solutions $M^1(x,0)$ and $N^1(y,0)$ of the quadratic optimization problems (3.5) and (3.6) can be obtained from the Kunita-Watanabe decomposition (5.2) by reverting to the original numéraire through the identities

$$\widetilde{M}_t^1 = \frac{\widehat{X}_t(x,0)}{\mathcal{R}_t(x)} M_t^1(x,0), \quad \widetilde{N}_t^1 = \frac{x}{y} N_t^1(y,0), \quad t \in [0,T].$$

With

$$C_a := x^2 \mathbb{E}^{\mathbb{R}(x)}\left[F^2 \frac{A(\widehat{X}_T(x,0)) - 1}{A(\widehat{X}_T(x,0))} - G\right],$$

$$C_b := y^2 \mathbb{E}^{\mathbb{R}(x)}\left[G + F^2\left(1 - A\left(\widehat{X}_T(x,0)\right)\right)\right],$$

the Hessian terms in the quadratic expansion of u and v are given by

$$\begin{aligned} a(\varepsilon,\varepsilon) &= \frac{\mathcal{R}_0(x)}{x} \inf_{\widetilde{M}\in\widetilde{\mathcal{M}}^2(x,0)} \mathbb{E}^{\widetilde{\mathbb{R}}(x)}\left[\left(\widetilde{M}_T + xF\left(A\left(\widehat{X}_T(x,0)\right)-1\right)\right)^2\right] + C_a. \\ &= \frac{\mathcal{R}_0(x)}{x}\mathbb{E}^{\widetilde{\mathbb{R}}(x)}\left[\left(\widetilde{N}^1_T\right)^2\right] + \frac{\mathcal{R}_0(x)}{x}P_0^2 + C_a, \end{aligned}$$

and

$$\begin{aligned} b(\varepsilon,\varepsilon) &= \frac{\mathcal{R}_0(x)}{x} \inf_{\widetilde{N}\in\mathcal{N}^2(y,0)} \mathbb{E}^{\widetilde{\mathbb{R}}(x)}\left[\left(\widetilde{N}_T + yF\left(A\left(\widehat{X}_T(x,0)\right)-1\right)\right)^2\right] + C_b. \\ &= \frac{\mathcal{R}_0(x,0)}{x}\left(\frac{y}{x}\right)^2\mathbb{E}^{\widetilde{\mathbb{R}}(x)}\left[\left(\widetilde{M}^1_T\right)^2\right] + \frac{\mathcal{R}_0(x)}{x}\left(\frac{y}{x}\right)^2 P_0^2 + C_b. \end{aligned}$$

We also have

$$a(x,\varepsilon) = P_0 \quad \textit{and} \quad b(y,\varepsilon) = \frac{y}{x}\frac{P_0}{a(x,x)}.$$

With these notations, all the conclusions of Theorem 3.3 hold true.

Remark 5.2. In many references, in order to call (5.2) the Kunita-Watanabe decomposition of P, one additionally needs $\widetilde{N}^1$ to be orthogonal to $S^{\mathcal{R}(x)}$, which amounts to $\widetilde{N}^1 S^{\mathcal{R}(x)}$ being a martingale under $\widetilde{\mathbb{R}}(x)$. Some authors, see e.g., [KS06b, p. 2181], do not require this.

6. Proofs

6.1. **Characterization of primal and dual admissible sets.** The following lemma gives a useful characterization of the primal and dual admissible sets after perturbations.

Lemma 6.1. *Under Assumption* (2.7)*, for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, we have*

$$\mathcal{X}(1,\varepsilon) = \mathcal{X}(1,0)\frac{1}{N^\varepsilon}, \tag{6.1}$$

$$\mathcal{Y}(1,\varepsilon) = \mathcal{Y}(1,0)N^\varepsilon, \tag{6.2}$$

where we have used the following notations

$$\begin{aligned} \mathcal{X}(1,0)\frac{1}{N^\varepsilon} &= \left\{\frac{X}{N^\varepsilon} = \left(\frac{X_t}{N^\varepsilon_t}\right)_{t\in[0,T]} :\ X\in\mathcal{X}(1,0)\right\}, \\ \mathcal{Y}(1,0)N^\varepsilon &= \left\{YN^\varepsilon = (Y_tN^\varepsilon_t)_{t\in[0,T]} :\ Y\in\mathcal{Y}(1,0)\right\}. \end{aligned}$$

In particular, both $\mathcal{X}(1,\varepsilon)$ and $\mathcal{Y}(1,\varepsilon)$ are non-empty and no unbounded profit with bounded risk holds for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$.

Proof. Let us fix $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. Then, for an arbitrary predictable and S^ε-integrable process ψ, such that $\Delta(\psi \cdot S^\varepsilon) > -1$, let us set $X^\varepsilon := \mathcal{E}(\psi \cdot S^\varepsilon)$. Then $X^\varepsilon \in \mathcal{X}(1, \varepsilon)$. Let us consider $X^0 := X^\varepsilon \mathcal{E}\left(-\varepsilon \bar{R}\right)$. One can see that $X^0 \in \mathcal{X}(1, 0)$. This implies that

$$\mathcal{X}(1, \varepsilon) N^\varepsilon \subseteq \mathcal{X}(1, 0).$$

Similarly, one can show the reverse inclusion. Therefore, (6.1) is valid.

Let us fix $Y \in \mathcal{Y}(1, 0)$ and take an arbitrary $\tilde{X}^\varepsilon \in \mathcal{X}(1, \varepsilon)$. By (6.1), $\tilde{X}^\varepsilon N^\varepsilon \in \mathcal{X}(1, 0)$. Therefore, $Y \tilde{X}^\varepsilon N^\varepsilon$ is a supermartingale. We deduce that $Y N^\varepsilon \in \mathcal{Y}(1, \varepsilon)$. As a consequence, we have

$$\mathcal{Y}(1, 0) N^\varepsilon \subseteq \mathcal{Y}(1, \varepsilon).$$

In a similar manner, one can show that $\mathcal{Y}(1, 0) N^\varepsilon \supseteq \mathcal{Y}(1, \varepsilon)$. As a result, (6.2) holds. □

We will need the following lemma from [MS19].

Lemma 6.2 (Mostovyi, Sirbu, 2017)**.** *Under Assumption 2.1, for every $z > 0$ and $x > 0$, we have*

$$\begin{aligned} U'(zx) &\leq \max\left(z^{-c_2}, 1\right) U'(x) \leq \left(z^{-c_2} + 1\right) U'(x), \\ -V'(zx) &\leq \max\left(z^{-\frac{1}{c_1}}, 1\right) (-V'(x)) \leq \left(z^{-\frac{1}{c_1}} + 1\right) (-V'(x)). \end{aligned}$$

For brevity of notations in the proof of Lemma 6.3 below, we denote by G^c the continuous part of $[\bar{R}, \bar{R}]$ evaluated at T and let H_i, where H_i takes values in $\left[-\frac{1}{2\varepsilon_0}, \frac{1}{2\varepsilon_0}\right]$, $i \in \mathbb{N}$, are the jumps of $\bar{R}$ up to T. Note that, with G being defined in (3.4), we have

$$G^c + \sum_{i=1}^{\infty} H_i^2 = G, \quad \mathbb{P}\text{–a.s.} \tag{6.3}$$

We define

$$\tilde{N}^\varepsilon := \exp\left(-\varepsilon F - \tfrac{1}{2}\varepsilon^2 G^c + \sum_{i=1}^{\infty} \left(\log(1 - \varepsilon H_i) + \varepsilon H_i\right)\right), \quad \varepsilon \in (-\varepsilon_0, \varepsilon_0),$$

and observe that the series $\sum_{i=1}^{\infty} \left(\log(1 - \varepsilon H_i) + \varepsilon H_i\right)$ converges absolutely for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, $\mathbb{P}$–a.s., in view of (6.3) and since $|\log(1 + x) - x| \leq x^2$ for every $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$.

Lemma 6.3. *Let $x > 0$ be fixed and the conditions of Theorem 3.1 hold, and $y = u_x(x, 0)$. Let α^0 and α^1 be the terminal values of some elements of $\mathcal{M}^\infty(x)$. With $\xi := \widehat{X}_T(x, 0)$ denoting the solution to (2.3) corresponding to $x > 0$ and $\varepsilon = 0$, we define*

$$\begin{aligned} \psi(s, t) &:= \frac{1}{x}\left(x + s(1 + \alpha^0) + t\alpha^1\right) \frac{1}{\tilde{N}^t}, \\ w(s, t) &:= \mathbb{E}\left[U(\xi \psi(s, t))\right], \quad (s, t) \in \mathbb{R} \times (-\varepsilon_0, \varepsilon_0). \end{aligned}$$

Then w admits the following second-order expansion at $(0,0)$.

$$w(s,t) = w(0,0) + (s \quad t)\nabla w(0,0) + \tfrac{1}{2}(s \quad t)H_w \begin{pmatrix} s \\ t \end{pmatrix} + o(s^2+t^2),$$

where

$$w_s(0,0) = u_x(x,0),$$
$$w_t(0,0) = xy\mathbb{E}^{\mathbb{R}(x)}\left[F\right],$$

and

$$H_w := \begin{pmatrix} w_{ss}(0,0) & w_{st}(0,0) \\ w_{st}(0,0) & w_{tt}(0,0) \end{pmatrix},$$

where the second-order partial derivatives of w at $(0,0)$ are given by

$$w_{ss}(0,0) = -\frac{y}{x}\mathbb{E}^{\mathbb{R}(x)}\left[A(\xi)(1+\alpha^0)^2\right],$$
$$w_{st}(0,0) = -\frac{y}{x}\mathbb{E}^{\mathbb{R}(x)}\left[A(\xi)(1+\alpha^0)(xF+\alpha^1) - xF(1+\alpha^0)\right],$$
$$w_{tt}(0,0) = -\frac{y}{x}\mathbb{E}^{\mathbb{R}(x)}\left[A(\xi)(\alpha^1+xF)^2 - 2xF\alpha^1 - x^2(F^2+G)\right].$$

Proof. As α^0 and α^1 are bounded, there exists a positive constant $\varepsilon < \min(\varepsilon_0, 1)$, such that

$$\varepsilon\left(|\alpha^0+1| + |\alpha^1|\right) \leq \frac{x}{2}. \tag{6.4}$$

Let us fix an arbitrary $(s,t) \in B_\varepsilon(0,0)$ and define

$$\widetilde{\psi}(z) := \psi(zs, zt), \quad z \in (-1,1).$$

As by construction of $(H_k)_{k\in\mathbb{N}}$, see (6.3), we have that $\sum_{k\geq 1}\left(\log(1-tH_k)+tH_k\right)$ converges for every $t \in [-\varepsilon/2, \varepsilon/2]$, $\mathbb{P}$–a.s., and the series of term by term derivatives, $\sum_{k\geq 1}\frac{-tH_k^2}{1-tH_k}$, converges uniformly in $t \in [-\varepsilon/2, \varepsilon/2]$, where $\frac{tH_k^2}{1-tH_k}$ is continuous in t on $[-\varepsilon/2, \varepsilon/2]$ for every $k \geq 1$, we deduce that

$$-\frac{\partial}{\partial t}\sum_{k\geq 1}\left(\log(1-tH_k)+tH_k\right) = t\sum_{k\geq 1}\frac{H_k^2}{1-tH_k}, \quad t \in (-\varepsilon/2, \varepsilon/2),$$

and we get

$$\psi_t(s,t) = \frac{\alpha^1}{x\tilde{N}^t} + \psi(s,t)\left(F + tG^c + t\sum_{k\geq 1}\frac{H_k^2}{1-tH_k}\right) \quad \text{and} \quad \psi_s(s,t) = \frac{1+\alpha^0}{x\tilde{N}^t},$$

Consequently, we obtain

$$\begin{aligned} \widetilde{\psi}'(z) &= \psi_s(sz,tz)s + \psi_t(sz,tz)t \\ &= \frac{1+\alpha^0}{x\tilde{N}^{zt}}s + \left(\frac{\alpha^1}{x\tilde{N}^{zt}} + \widetilde{\psi}(z)\left(F + ztG^c + zt\sum_{k\geq 1}\frac{H_k^2}{1-ztH_k}\right)\right)t. \end{aligned} \tag{6.5}$$

Similarly, since $\mathbb{P}$–a.s., $\sum_{k\geq 1}\frac{tH_k^2}{1-tH_k}$ converges for every $t\in[-\varepsilon/2,\varepsilon/2]$, since the series of term by term partial derivatives, $\sum_{k\geq 1}\frac{H_k^2}{(1-tH_k)^2}$, converges uniformly in $t\in[-\varepsilon/2,\varepsilon/2]$, and from continuity of $\frac{H_k^2}{(1-tH_k)^2}$ in t on $[-\varepsilon/2,\varepsilon/2]$ for every $k\geq 1$, we deduce that

$$\frac{\partial}{\partial t}\left(\sum_{k\geq 1}\frac{tH_k^2}{1-tH_k}\right)=\sum_{k\geq 1}\frac{H_k^2}{(1-tH_k)^2},\quad t\in(-\varepsilon/2,\varepsilon/2),$$

and we get

$$\begin{aligned}\psi_{tt}(s,t)&=\frac{2\alpha^1}{xN^t}\left(F+tG^c+t\sum_{k\geq 1}\frac{H_k^2}{1-tH_k}\right)\\&\quad+\psi(s,t)\left(\left(F+tG^c+t\sum_{k\geq 1}\frac{H_k^2}{1-tH_k}\right)^2+G^c+\sum_{k\geq 1}\frac{H_k^2}{(1-tH_k)^2}\right),\\\psi_{st}(s,t)&=\frac{1+\alpha^0}{x\tilde{N}^t}\left(F+tG^c+t\sum_{k\geq 1}\frac{H_k^2}{1-tH_k}\right),\quad\text{and}\quad\psi_{ss}(s,t)=0.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\widetilde{\psi}''(z)&=\psi_{tt}(zs,zt)t^2+2\psi_{st}(zs,zt)ts+\psi_{ss}(zs,zt)s^2\\&=\left(\frac{2\alpha^1}{xN^{zt}}\left(F+ztG^c+zt\sum_{k\geq 1}\frac{H_k^2}{1-ztH_k}\right)\right.\\&\quad\left.+\tilde{\psi}(z)\left(\left(F+ztG^c+zt\sum_{k\geq 1}\frac{H_k^2}{1-ztH_k}\right)^2+G^c+\sum_{k\geq 1}\frac{H_k^2}{(1-ztH_k)^2}\right)\right)t^2\\&\quad+2\frac{1+\alpha^0}{x\tilde{N}^{zt}}\left(F+ztG^c+zt\sum_{k\geq 1}\frac{H_k^2}{1-ztH_k}\right)ts.\end{aligned}$$

Setting $W(z):=U(\xi\widetilde{\psi}(z))$, $z\in(-1,1)$, by direct computations, we get

$$\begin{aligned}W'(z)&=U'(\xi\widetilde{\psi}(z))\xi\widetilde{\psi}'(z),\\W''(z)&=U''(\xi\widetilde{\psi}(z))\left(\xi\widetilde{\psi}'(z)\right)^2+U'(\xi\widetilde{\psi}(z))\xi\widetilde{\psi}''(z).\end{aligned}\tag{6.6}$$

Let us define

$$J:=1+|F|+G.$$

As

$$\left|ztG^c+zt\sum_{k\geq 1}\frac{H_k^2}{1-ztH_k}\right|\leq 2|zt|G,\quad z\in(-1,1),$$

from (6.5) using (6.4) and since

$$\frac{1}{2}\leq\widetilde{\psi}(z)\tilde{N}^{zt}\leq\frac{3}{2},\quad z\in(-1,1),$$

we deduce the existence of a constant $b_1 > 0$, such that

$$|\widetilde{\psi}'(z)| \le b_1 J \exp(b_1 \varepsilon J), \quad \text{and} \quad \widetilde{\psi}(z)^{-c_2} + 1 \le b_1 \exp(b_1 \varepsilon J), \quad z \in (-1,1).$$

Therefore, from (6.19) using Lemma 6.2, we obtain

$$\begin{aligned} \sup_{z\in(-1,1)} |W'(z)| &\le \sup_{z\in(-1,1)} U'(\xi)\xi \left((\widetilde{\psi}(z))^{-c_2} + 1 \right) \left| \widetilde{\psi}'(z) \right| \\ &\le b_1^2 U'(\xi)\xi J \exp(2b_1 \varepsilon J). \end{aligned} \tag{6.7}$$

Similarly, from (6.19) applying Assumption 2.1 and Lemma 6.2, we deduce the existence of a constant $b_2 > 0$, such that

$$\sup_{z\in(-1,1)} |W''(z)| \;\; \le \;\; b_2 U'(\xi)\xi J^2 \exp(b_2 \varepsilon J). \tag{6.8}$$

Combining (6.7) and (6.13), we obtain

$$\sup_{z\in(-1,1)} \left(|W'(z)| + |W''(z)| \right) \le U'(\xi)\xi \left(b_1^2 J \exp(2b_1 \varepsilon J) + b_2 J^2 \exp(b_2 \varepsilon J) \right).$$

Consequently, as $1 \le J \le J^2$, one can find a constant $b > 0$ such that for every z_1 and z_2 in $(-1,1)$, we get

$$\left| \tfrac{W(z_1)-W(z_2)}{z_1-z_2} \right| + \left| \tfrac{W'(z_1)-W'(z_2)}{z_1-z_2} \right| \;\; \le \;\; b U'(\xi)\xi J^2 \exp(b\varepsilon J). \tag{6.9}$$

By passing to a smaller ε, if necessary, and by applying Hölder's inequality, we deduce from Assumption 2.8 that the right-hand side of (6.9) integrable. Since the bound in (6.9) is uniform in $(s,t) \in B_\varepsilon(0,0)$, applying the dominated convergence theorem we deduce the assertions of the lemma. □

6.2. **Proofs of Theorems 3.1, 3.3, 3.5, and 3.6.** From (2.7) it follows that the respective closures of the convex solid hulls of $\{X_T : \; X \in \mathcal{X}(1,0)\}$ and $\{Y_T : \; Y \in \mathcal{Y}(1,0)\}$ satisfy [MS19, Assumption 5.1]. Using Lemma 6.1, we get

$$\begin{aligned} \left\{ \frac{X_T}{N_T^\varepsilon} : \; X \in \mathcal{X}(1,0) \right\} &= \{X_T : \; X \in \mathcal{X}(1,\varepsilon)\}, \\ \{Y_T N_T^\varepsilon : \; Y \in \mathcal{Y}(1,0)\} &= \{Y_T : \; Y \in \mathcal{Y}(1,\varepsilon)\}, \quad \varepsilon \in (-\varepsilon_0, \varepsilon_0). \end{aligned}$$

Consequently, the respective closures of convex solid hulls of

$$\{X_T : \; X \in \mathcal{X}(1,\varepsilon)\} \quad \text{and} \quad \{Y_T : \; Y \in \mathcal{Y}(1,\varepsilon)\}$$

also satisfy [MS19, Assumption 5.1] for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$.

From Assumption 2.7 and [KS06a, Lemma 6], we deduce that the sets $\mathcal{M}^2(x)$ and $\mathcal{N}^2(x)$ satisfy [MS19, Assumption 5.3]. With the notations (3.4), using Assumption 2.3, we get

$$\max\left(N_T^\varepsilon, \frac{1}{N_T^\varepsilon} \right) \le \exp\left(|\varepsilon F| + \varepsilon^2 G \right), \quad \varepsilon \in (-\varepsilon_0, \varepsilon_0).$$

Therefore, Assumption 2.8 is analogous to [MS19, Assumption 5.2].

In view of Lemma 6.3, from which the greatest lower bound for the quadratic expansion of u can be obtained, the least upper bound for v can be obtained similarly. Moreover, even though in [MS19] and the present paper, the perturbations are different, the second-order expansions for the value functions, which stem from Lemma 6.3 and its consequences, coincide (here and in [MS19]). Now, in view of the structures of perturbations represented by N_T^ε here and by L^δ in [MS19, p.14], the assertions of Theorems 3.1, 3.3, 3.5, and 3.6 follow from the line by line adaptation of the proofs of [MS19, Theorem 5.4, Theorem 5.6, Theorem 5.7, and Theorem 5.8], respectively. Further details are not included for the brevity of the exposition.

6.3. **Proofs of the assertions from section 4.** For the proof of Theorem 4.2, we will need the following technical lemma. First, for $(\Delta x, \varepsilon, n) \in (-x, \infty) \times (-\varepsilon_0, \varepsilon_0) \times \mathbb{N}$, let us recall that $\nabla u(x,0)$, $H_u(x,0)$, and $\tilde{X}^{\Delta x,\varepsilon,n}$'s are defined in (3.1), (3.9), and (4.6), respectively, and set

(6.10)
$$f(\Delta x, \varepsilon, n) := \frac{u(x,0) + (\Delta x \quad \varepsilon)\nabla u(x,0) + \frac{1}{2}(\Delta x \quad \varepsilon)H_u(x,0)\begin{pmatrix}\Delta x\\ \varepsilon\end{pmatrix} - \mathbb{E}\left[U\left(\tilde{X}_T^{\Delta x,\varepsilon,n}\right)\right]}{\Delta x^2 + \varepsilon^2}.$$

Lemma 6.4. *Assume that $x > 0$ is fixed and the assumptions of Theorem 3.1 hold. Then, for f defined in* (6.10)*, there exists a monotone function g, such that*

$$g(n) \geq \lim_{|\Delta x|+|\varepsilon|\to 0} f(\Delta x, \varepsilon, n), \quad n \in \mathbb{N}, \tag{6.11}$$

and

$$\lim_{n\to\infty} g(n) = 0. \tag{6.12}$$

Proof. The proof goes along the lines of the proof of Lemma 6.3. We only outline the main steps for brevity. For a fixed $\varepsilon > 0$, let us define

$$\psi(\Delta x, \varepsilon) := \frac{x + \Delta x}{x} \frac{\mathcal{E}\left((\Delta x \gamma^{0,n} + \varepsilon \gamma^{1,n}) \cdot R^{\{\widehat{\pi}\}}\right)}{\mathcal{E}\left((\varepsilon\theta) \cdot R\right)},$$
$$w(\Delta x, \varepsilon) := \mathbb{E}\left[U(\widehat{X}_T(x,0)\psi(\Delta x, \delta))\right], \quad (\Delta x, \varepsilon) \in (-x, \infty) \times (-\varepsilon_0, \varepsilon_0),$$

Let us first fix $\varepsilon' > 0$, then fix $(\Delta x, \varepsilon) \in B_{\varepsilon'}(0,0)$, and set

$$\widetilde{\psi}(z) := \psi(z\Delta x, z\delta), \quad z \in (-1, 1).$$

Setting $W(z) := U(\widehat{X}_T(x,0)\widetilde{\psi}(z))$, $z \in (-1,1)$, by direct computations, we get

$$W'(z) = U'(\widehat{X}_T(x,0)\widetilde{\psi}(z))\widehat{X}_T(x,0)\widetilde{\psi}'(z),$$
$$W''(z) = U''(\widehat{X}_T(x,0)\widetilde{\psi}(z))\left(\widehat{X}_T(x,0)\widetilde{\psi}'(z)\right)^2 + U'(\widehat{X}_T(x,0)\widetilde{\psi}(z))\widehat{X}_T(x,0)\widetilde{\psi}''(z).$$

As in Lemma 6.3, from boundedness of $\gamma^{0,n} \cdot R^{\{\widehat{\pi}\}} = \tilde{M}^{0,n}$, $\gamma^{1,n} \cdot R^{\{\widehat{\pi}\}} = \tilde{M}^{1,n}$, their quadratic variations and jumps, via Lemma 6.2 and Assumption 2.8, one can show that

$$\left|\frac{W(z_1) - W(z_2)}{z_1 - z_2}\right| + \left|\frac{W'(z_1) - W'(z_2)}{z_1 - z_2}\right| \leq \eta,$$

for some random variable η, which depend on ε' and which is integrable for a sufficiently small ε'. The derivatives of W plugged inside the expectation result in the exact form of the gradient $\nabla u(x,0)$ and the Hessian $H_u^n(x,0)$, such that $\lim_{n\to\infty} H_u^n(x,0) = H_u(x,0)$. This results in the existence of a function g satisfying (6.11) and (6.12). Finally, g can be selected to be monotone. □

Proof of Theorem 4.2. Let us fix $n \in \mathbb{N}$ and consider

$$(\gamma^{0,n} + \gamma^{1,n}) \cdot R^{\{\widehat{\pi}\}} = \tilde{M}^{0,n} + \tilde{M}^{1,n} \in \mathcal{M}^{\infty}(x).$$

By construction, the jumps of this process process are bounded by $4n$. Therefore, setting $\delta(n) := \min\left(\varepsilon_0, \frac{1}{9n}\right)$, we obtain that for every $(\Delta x, \varepsilon) \in B_{\delta(n)}(0,0)$, the jumps of

$$\Delta x \tilde{M}^{0,n} + \varepsilon \tilde{M}^{1,n} \quad \text{and} \quad (\varepsilon\theta) \cdot R$$

take values in $(-1,1)$. Consequently, for every $(\Delta x, \varepsilon) \in B_{\delta(n)}(0,0)$, we get

$$\mathcal{E}\left((\Delta x \gamma^{0,n} + \varepsilon\gamma^{1,n}) \cdot R^{\{\widehat{\pi}\}}\right) > 0 \quad \text{and} \quad \mathcal{E}\left((\varepsilon\theta) \cdot R\right) > 0.$$

Therefore, via direct computations, we obtain

$$0 < \mathcal{E}\left(\widehat{\pi} \cdot R\right) \frac{\mathcal{E}\left((\Delta x \gamma^{0,n} + \varepsilon\gamma^{1,n}) \cdot R^{\{\widehat{\pi}\}}\right)}{\mathcal{E}\left((\varepsilon\theta) \cdot R\right)} = \frac{\mathcal{E}\left((\widehat{\pi} + \Delta x \gamma^{0,n} + \varepsilon\gamma^{1,n}) \cdot R\right)}{\mathcal{E}\left((\varepsilon\theta) \cdot R\right)} = \frac{\tilde{X}^{\Delta x, \varepsilon, n}}{x + \Delta x}.$$

In view of Lemma 6.1, this implies that

$$\tilde{X}^{\Delta x, \varepsilon, n} \in \mathcal{X}(x + \Delta x, \varepsilon), \quad (\Delta x, \varepsilon) \in B_{\delta(n)}(0,0). \tag{6.13}$$

This completes the proof of the first assertion of the theorem.

In order to prove the second assertion, we proceed as follows. First, for f defined in (6.10), via Lemma 6.4, we deduce the existence of a monotone function g, such that (6.11) and (6.12) hold. Let us define

$$\Phi(n) := \{(\Delta x, \varepsilon) : \ f(t\Delta x, t\varepsilon, n) \leq 2g(n), \text{ for every } t \in [0,1]\}, \quad n \in \mathbb{N},$$

$$m(n) := 2 \inf\left\{m \geq n : \ B_{1/m}(0,0) \subseteq \Phi(n)\right\}, \quad n \in \mathbb{N}.$$

Note that $m(n) < \infty$ for every $n \in \mathbb{N}$. With

$$n(\Delta x, \varepsilon) := \min\left\{n \in \mathbb{N} : \ m(n) \geq \frac{1}{\sqrt{\Delta x^2 + \varepsilon^2}}\right\}, \quad (\Delta x, \varepsilon) \in (-x, \infty) \times (-\varepsilon_0, \varepsilon_0),$$

we have

$$\lim_{|\Delta x| + |\varepsilon| \to 0} \frac{u(x + \Delta x, \varepsilon) - \mathbb{E}\left[U\left(\tilde{X}_T^{\Delta x, \varepsilon, n(\Delta x, \varepsilon)}\right)\right]}{\Delta x^2 + \varepsilon^2} = 0. \tag{6.14}$$

In order to prove the third assertion of this theorem, let us consider

$$S^\varepsilon = \left(\frac{1}{N^\varepsilon}, \frac{\mathcal{E}(\rho^1)}{N^\varepsilon}, \dots, \frac{\mathcal{E}(\rho^d)}{N^\varepsilon}\right),$$

the $(d+1)$-dimensional stock price process under N^ε. By direct computations, we get

$$\begin{aligned}\left(\tfrac{1}{N^\varepsilon}, \tfrac{\mathcal{E}(\rho^1)}{N^\varepsilon}, \dots, \tfrac{\mathcal{E}(\rho^d)}{N^\varepsilon}\right) &= \left(\tfrac{1}{\mathcal{E}((\varepsilon\theta)\cdot R)}, \tfrac{\mathcal{E}(\rho^1)}{\mathcal{E}((\varepsilon\theta)\cdot R)}, \dots, \tfrac{\mathcal{E}(\rho^d)}{\mathcal{E}((\varepsilon\theta)\cdot R)}\right) \\ &= \left(\mathcal{E}\left((e^0-\varepsilon\theta)\cdot R^{\{\varepsilon\theta\}}\right), \dots, \mathcal{E}\left((e^d-\varepsilon\theta)\cdot R^{\{\varepsilon\theta\}}\right)\right),\end{aligned} \tag{6.15}$$

where e^i is the constant-valued process whose i-th component equals to 1 and all other components equal to zero at all times and $\left(R^{\{\varepsilon\theta\}}\right)_{\varepsilon\in(-\varepsilon_0,\varepsilon_0)}$ defined in (4.5). Therefore, introducing the vector of returns under the numéraire N^ε, R^ε, from (6.15) we get

$$R^\varepsilon = \frac{1}{1-\varepsilon}\left((e^0-\varepsilon\theta)\cdot R^{\{\varepsilon\theta\}}, \dots, (e^d-\varepsilon\theta)\cdot R^{\{\varepsilon\theta\}}\right),$$

equivalently

$$R^\varepsilon = \frac{1}{1-\varepsilon}(I-\varepsilon\vec{1}\theta^\top)\cdot R^{\{\varepsilon\theta\}}, \tag{6.16}$$

where $\frac{1}{1-\varepsilon}$ is a normalization constant.

Following the construction above, see (6.13) and (6.14), for every $(\Delta x, \varepsilon)$ in a certain neighborhood of the origin, one can find $n(\Delta x,\varepsilon)$, such that $\tilde{X}^{\Delta x,\varepsilon,n(\Delta x,\varepsilon)}$'s form a family of wealth processes that match the indirect utility up to the second order. To show that the corrections to optimal proportions (invested in the corresponding stocks) are given by (4.8), for every ε being sufficiently close to 0 and every $\Delta x > -x$, we need to show that $\tilde{X}^{\Delta x,\varepsilon,n}$'s defined in (4.6) can be represented as

$$\tilde{X}^{\Delta x,\varepsilon,n} = (x+\Delta x)\mathcal{E}\left(\left(\left(\widehat{\pi}+\Delta x\gamma^{0,n}+\varepsilon(-\theta+\gamma^{1,n})\right)^\top\left((1-\varepsilon)I+\varepsilon\vec{1}\theta^\top\right)\right)^\top\cdot R^\varepsilon\right). \tag{6.17}$$

Here $\mathcal{E}\left(\left(\left(\widehat{\pi}+\Delta x\gamma^{0,n}+\varepsilon(-\theta+\gamma^{1,n})\right)^\top\left((1-\varepsilon)I+\varepsilon\vec{1}\theta^\top\right)\right)^\top\cdot R^\varepsilon\right)\in\mathcal{X}(1,\varepsilon)$, by the subsequent argument. We recall that $\theta^0_t = 1-\sum_{i=1}^d\theta^i_t$, $t\in[0,T]$, as the $\mathcal{E}(\theta\cdot R)$ is a wealth process of a self-financing portfolio, and therefore,

$$\vec{1}^\top\theta\equiv 1.$$

Consequently, we have

$$
\begin{aligned}
&\mathcal{E}\left(\left(\left(\widehat{\pi}+\Delta x\gamma^{0,n}+\varepsilon(-\theta+\gamma^{1,n})\right)^{\top}\left((1-\varepsilon)I+\varepsilon\vec{1}\theta^{\top}\right)\right)^{\top}\cdot R^{\varepsilon}\right)\\
=&\mathcal{E}\left(\left(\left(\widehat{\pi}+\Delta x\gamma^{0,n}+\varepsilon(-\theta+\gamma^{1,n})\right)^{\top}\left(I+\frac{\varepsilon\vec{1}\theta^{\top}}{1-\varepsilon\vec{1}^{\top}\theta}\right)\right)^{\top}\cdot((1-\varepsilon)R^{\varepsilon})\right)\\
=&\mathcal{E}\left(\left(\widehat{\pi}+\Delta x\gamma^{0,n}+\varepsilon(-\theta+\gamma^{1,n})\right)\cdot R^{\{\varepsilon\theta\}}\right)\\
=&\frac{\mathcal{E}\left(\left(\widehat{\pi}+\Delta x\gamma^{0,n}+\varepsilon\gamma^{1,n}\right)\cdot R\right)}{\mathcal{E}((\varepsilon\theta)\cdot R)}\in\mathcal{X}(1,\varepsilon),
\end{aligned}
\tag{6.18}
$$

by Lemma 6.1 and where the third line in (6.18) is exactly $\frac{\tilde{X}^{\Delta x,\varepsilon,n}}{x+\Delta x}$ from (4.6). Note that in (6.17), we used the Sherman-Morrison inversion formula, which asserts that

$$
\left(I-\varepsilon\vec{1}\theta^{\top}\right)^{-1}=I+\frac{\varepsilon\vec{1}\theta^{\top}}{1-\varepsilon\vec{1}^{\top}\theta}=I+\frac{\varepsilon}{1-\varepsilon}\vec{1}\theta^{\top},
$$

where in the last equality, we have used again $\vec{1}^{\top}\theta\equiv 1$. Therefore, the invertibility of $\left(I-\varepsilon\vec{1}\theta^{\top}\right)$ holds if and only if $\varepsilon\neq 1$. Thus, in view of (6.18), the processes in (6.17) match the indirect utility up to the second order in the sense (4.7). Now, in (6.17) the integrand can be rewritten as follows.

$$
\begin{aligned}
&\left(\left(\widehat{\pi}+\Delta x\gamma^{0,n}+\varepsilon(-\theta+\gamma^{1,n})\right)^{\top}\left((1-\varepsilon)I+\varepsilon\vec{1}\theta^{\top}\right)\right)^{\top}\\
=&\left((1-\varepsilon)I+\varepsilon\theta\vec{1}^{\top}\right)\left(\widehat{\pi}+\Delta x\gamma^{0,n}+\varepsilon(-\theta+\gamma^{1,n})\right).
\end{aligned}
$$

The latter expression coincides with the one in (4.8), and, in view of (6.17), these are the proportions invested in traded assets under the numéraire N^{ε}. □

6.4. **On perturbations of models that admit closed-form solutions.** There are many models that admit explicit solutions, see [Zar01], [GK03], [HIM05], [Liu07], [KS06b], [GR12], [HHI+14], and [ST14] for their constructions and characterizations. In most cases, these solutions depend heavily on the exact dynamics of the stock price, and such solutions cease to exist under perturbations of the model parameters. The results of this paper provide both a stability result (as Theorems 3.1 and 3.6 assert that the value functions and the optimizers of the perturbed models are close to the ones of the unperturbed models) and a constructive way of obtaining nearly optimal wealth processes and strategies.

In the preferences are given by power utilities, then closed-form solutions are obtained in [Liu07] and [GR12], among others. In the asymptotic analysis, the corrections associated with perturbations of the initial wealth are trivial, as we have

$$
\widehat{X}(x,0)=x\widehat{X}(1,0)=x\mathcal{E}\left(\widehat{\pi}\cdot R\right).
$$

Thus, for the power utility case, in (4.6), only $\gamma^{1,n}$'s have to be estimated as $\gamma^{0,n} \equiv 0$. The Kunita-Watanabe decomposition provides a characterization of $\gamma^{1,n}$, as the risk-tolerance wealth process exists for the power utility and it is equal to $\widehat{X}(1,0)$ up to a multiplicative constant. Therefore, the measures $\widetilde{\mathbb{R}}$ and $\mathbb{R}$ coincide. This, in particular, is implicitly used in [LMŽ18], in the context of perturbations of the market price of risk.

In the case of general utility functions satisfying Assumption 2.1, models that admit closed-form or fairly explicit solutions, are also studied, see, e.g., [KS06b] and [MS19]. By [KS06b, Theorem 6], a class of models that gives the existence of the risk-tolerance wealth process for every utility function satisfying Assumption 2.1 is the one, where the dual domain $\mathcal{Y}(1,0)$ admits a maximal element in the sense of the second-order stochastic dominance, i.e., an element $\widehat{Y} \in \mathcal{Y}(1,0)$, such that for every $Y \in \mathcal{Y}(1,0)$, we have

$$\int_0^z \mathbb{P}[\widehat{Y}_T \geq y]dy \geq \int_0^z \mathbb{P}[Y_T \geq y]dy, \quad z \geq 0.$$

For example, this holds in a market, where there is a bank account with 0 interest rate and only one traded stock, whose return is given by:

$$\rho^1_t = \mu t + \sigma B_t, \quad t \in [0,T],$$

for some constants μ and $\sigma > 0$, where the filtration is generated by (B,W) a two-dimensional Brownian motion. Let us consider a one-dimensional and ρ^1-integrable process θ^1, such that Assumption 2.8 holds for $\bar{R} = -\theta^1 \cdot \rho^1 = -\theta \cdot R$, where $\theta = \begin{pmatrix} 1-\theta^1 \\ \theta^1 \end{pmatrix}$. In this case, the corresponding family of numéraires is

$$N^\varepsilon = \mathcal{E}(\varepsilon\theta^1 \cdot \rho^1), \quad \varepsilon \in \mathbb{R}.$$

Here ε_0 from Assumption 2.3 can be set to ∞, as there are no jumps of the underlying process ρ^1. For a given $x > 0$, let us consider $\widehat{\pi}^1$ and $\pi^{\mathcal{R},1}$, such that $\widehat{X}(x,0) = x\mathcal{E}\left(\widehat{\pi}^1 \cdot \rho^1\right)$ and $\mathcal{R}(x) = \mathcal{R}_0(x)\mathcal{E}\left(\pi^{\mathcal{R},1} \cdot \rho^1\right)$. Here, both $\widehat{\pi}^1$ and $\pi^{\mathcal{R},1}$ can be written in terms of the solution to a heat equation. Using an $\widetilde{\mathbb{R}}(x)$ local martingale $R^{\mathcal{R},1} := \rho^1 - \pi^{\mathcal{R},1} \cdot [\rho^1]$ and following Theorem 5.1, one needs to consider (5.2), which gives the decomposition of the process P, and which in the present settings becomes

$$P = P_0 - \varphi \cdot R^{\mathcal{R},1} - \varphi^\perp \cdot W,$$

for some processes φ and $\varphi^{\perp}$. With

$$\begin{aligned}
\zeta_t^1 &:= \frac{\mathcal{R}_t(x)}{\widehat{X}_t(x,0)\mathcal{R}_0(x)}(\pi_t^{\mathcal{R},1} - \widehat{\pi}_t^1),\\
\zeta_t^0 &:= 1 - \zeta_t^1,\\
\upsilon_t^1 &:= \frac{\mathcal{R}_t(x)}{\widehat{X}_t(x,0)}\left(\left(\varphi \cdot R^{\mathcal{R},1}\right)_t (\pi_t^{\mathcal{R},1} - \widehat{\pi}_t^1) + \varphi_t\right)\frac{1}{x},\\
\upsilon_t^0 &:= 1 - \upsilon_t^0, \quad t \in [0,T],
\end{aligned}$$

and by defining

$$\gamma^0 := \begin{pmatrix} \zeta^0 \\ \zeta^1 \end{pmatrix} \quad \text{and} \quad \gamma^1 := \begin{pmatrix} \upsilon^0 \\ \upsilon^1 \end{pmatrix},$$

one can construct $\gamma^{i,n}$, $i = 0, 1$ and $n \in \mathbb{N}$, appearing in (4.6) via setting $\gamma^{0,n} = \gamma^0 1_{[0,\tau_n]}$ and $\gamma^{1,n} = \gamma^1 1_{[0,\sigma_n]}$, $n \in \mathbb{N}$, where τ_n, $n \in \mathbb{N}$, is a localizing sequence for both $M^0(x,0)$ and $[M^0(x,0)]$ and σ_n, $n \in \mathbb{N}$, is a localizing sequence for both $M^1(x,0)$ and $[M^1(x,0)]$. Note that to get further characterizations of $\gamma^{1,n}$, one typically needs θ to be chosen in a more explicit (and restrictive) form that admits a characterization of φ in terms of a system of ordinary differential equations in the spirit of [LMŽ18, Example 5.3]. Then, with such $\gamma^{i,n}$'s, the nearly optimal wealth processes are given by (4.6), which reads

$$\tilde{X}^{\Delta x,\varepsilon,n} = (x + \Delta x)\mathcal{E}\left(\left(\widehat{\pi} + \Delta x \gamma^{0,n} + \varepsilon(-\theta + \gamma^{1,n})\right) \cdot R^{\{\varepsilon\theta\}}\right),$$

and where $R^{\{\varepsilon\theta\}}$ is specified in (4.5) that in the current settings becomes

$$R^{\{\varepsilon\theta\}} = \begin{pmatrix} 0 \\ \rho^1 - \varepsilon\theta^1 \cdot [\rho^1] \end{pmatrix}.$$

Therefore, we can rewrite the expression for $\tilde{X}^{\Delta x,\varepsilon,n}$ as

$$(x + \Delta x)\mathcal{E}\left(\left(\widehat{\pi}^1 + \Delta x \zeta^1 1_{[0,\tau_n]} + \varepsilon(-\theta^1 + \upsilon^1 1_{[0,\sigma_n]})\right) \cdot \left(\rho^1 - \varepsilon\theta^1 \cdot [\rho^1]\right)\right),$$

where $\widehat{\pi}^1$ is the second component of $\widehat{\pi}$. Note that for the wealth process $\tilde{X}^{\Delta x,\varepsilon,n}$, the proportions of the capital invested in the bank account and stock under the numéraire N^{ε} are given by (4.8), which in the current settings reads

$$\begin{pmatrix} (1-\varepsilon)\left(1 - \left(\widehat{\pi}^1 + \Delta x \zeta^1 1_{[0,\tau_n]} + \varepsilon(-\theta^1 + \upsilon^1 1_{[0,\sigma_n]})\right)\right) + \varepsilon(1 - \theta^1) \\ (1-\varepsilon)\left(\widehat{\pi}^1 + \Delta x \zeta^1 1_{[0,\tau_n]} + \varepsilon(-\theta^1 + \upsilon^1 1_{[0,\sigma_n]})\right) + \varepsilon\theta^1 \end{pmatrix}. \tag{6.19}$$

Further, with

$$\widetilde{\pi}^{\{1,\Delta x,\varepsilon,n\}} := \widehat{\pi}^1 + \Delta x \zeta^1 1_{[0,\tau_n]} + \varepsilon(-\theta^1 + \upsilon^1 1_{[0,\sigma_n]}),$$

one can rewrite (6.19) as

$$\begin{pmatrix} (1-\varepsilon)\left(1 - \widetilde{\pi}^{\{1,\Delta x,\varepsilon,n\}}\right) + \varepsilon(1 - \theta^1) \\ (1-\varepsilon)\widetilde{\pi}^{\{1,\Delta x,\varepsilon,n\}} + \varepsilon\theta^1 \end{pmatrix}. \tag{6.20}$$

To recapitulate, in the context of the stochastically dominant model specified above, (6.20) gives proportions invested in the traded assets under the (perturbed) numéraires $N^\varepsilon = \mathcal{E}(\varepsilon\theta^1 \cdot \rho^1)$'s, such that the corresponding wealth processes $\tilde{X}^{\Delta x,\varepsilon,n}$'s match the indirect utility up to the second order in the sense of Theorem 4.2, see (4.7) in the statement of this theorem.

6.5. **On an alternative parametrization of perturbations and a relation to perturbations of the drift and/or volatility.** In view of the family $R^{\{\varepsilon\theta\}}$, $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ defined in (4.5), that drive the processes (4.6), a different type of parametrization of perturbations of the form (4.5) can be used. We will illustrate this in the settings, where R is *continuous*. In this case, if θ is of the form $-\psi e^i$, where ψ is a one-dimensional bounded and predictable, $\left(\sum_{j=0}^{d}[\rho^i,\rho^j]\right)$-integrable process, and e^i is a (constant-valued) vector whose i-th component equals to 1 and all other components equal to zero, we the following dynamics of the returns of the stocks for perturbed models:

$$\begin{aligned} &R^{\varepsilon,j} = \rho^j, \quad \text{if} \quad j \neq i, \\ &R^{\varepsilon,j} = \rho^j + \varepsilon\psi \cdot \left(\sum_{k=0}^{d}[\rho^k,\rho^j]\right), \quad \text{if} \quad j = i, \end{aligned} \tag{6.21}$$

which in turn corresponds to perturbations of the finite-variation part of the i-th asset return only. This allows to consider perturbations of the finite-variation part of the return process. Moreover, by a different choice of θ, we can achieve simultaneous perturbations of multiple returns.

The relationship between these parametrization and the one considered in the remaining part of the paper can be obtained following the argument in the proof of Theorem 4.2, see (6.16) there. Thus, for perturbations of the form (6.21), under appropriate regularity conditions (similar to the ones in Theorem 3.1), the expansions of the value functions, derivatives of the optimal wealth processes, and approximations of trading strategies of the form (4.6) follow from the results of the present paper.

Let us discuss the relation to the framework in [MS19], where there is one traded stock, whose return, ρ^1, follows

$$\rho^1 = M + \lambda \cdot \langle M \rangle,$$

where M is a continuous local martingale. In this case, (6.21) gives the following dynamics for the perturbed models

$$\begin{aligned} R^{\varepsilon,1} &= \rho^1 + \varepsilon\psi \cdot \langle M \rangle \\ &= M + (\lambda + \varepsilon\psi) \cdot \langle M \rangle, \end{aligned}$$

which is the parametrization of perturbations in [MS19]. Further, the prototypical wealth process for a perturbed model, for some π, is given by

$$X^{\varepsilon} \;=\; x\mathcal{E}\left(\pi\cdot(\rho^{1}+\varepsilon\psi\cdot\langle M\rangle)\right).$$

Under the appropriate boundedness of ψ, with $\bar{\pi}:=\pi(\lambda+\varepsilon\psi)$, the evolution of X^{ε} can be rewritten as

$$X^{\varepsilon}=x\mathcal{E}\left((\pi(\lambda+\varepsilon\psi))\cdot\left(\lambda\cdot\langle M\rangle+\frac{\lambda}{\lambda+\varepsilon\psi}\cdot M\right)\right)=x\mathcal{E}\left(\bar{\pi}\cdot\left(\lambda\cdot\langle M\rangle+\frac{\lambda}{\lambda+\varepsilon\psi}\cdot M\right)\right).$$

This corresponds to perturbations of the martingale part (or volatility) of the return, similar to the ones in [HMKS17].

7. Counterxamples

The following example demonstrates the necessity of Assumption 2.8.

Example 7.1. Let us assume that the market consists of a bond with zero interest rate and one stock with return B, where B is a Brownian motion on the filtered probability space $\left(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,1]},\mathbb{P}\right)$, where 1 is the time horizon and $(\mathcal{F}_t)_{t\in[0,1]}$ is the usual augmentation of the filtration generated by B. In this case $\mathbb{P}$ is the martingale measure. Let us also suppose that $U(x)=\frac{x^p}{p}$, $x\in(0,\infty)$, where $p\in(0,1)$. An application of Jensen's inequality implies that for every $y>0$, $v(y)=V(y)=\frac{y^{-q}}{q}$, where $q=\frac{p}{1-p}$, and (a constant-valued process) y is the dual minimizer.

For the perturbed models, where $\bar{R}=-\theta\cdot B$ is such that $\bar{R}_1=|B_1|^{2+\delta}\mathsf{sign}(B_1)$ for some $\delta>0$. Then, $\mathbb{R}(x)=\mathbb{P}$, $x>0$, and for every constant $c>0$, we have

$$\begin{aligned}\mathbb{E}^{\mathbb{R}(x)}\left[\exp\left(c(|\bar{R}_1|+[\bar{R},\bar{R}]_1)\right)\right]&\geq\mathbb{E}\left[\exp\left(c|B_1|^{2+\delta}\mathsf{sign}(B_1)\right)\right]\\&=\tfrac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\exp\left(c|y|^{2+\delta}\mathsf{sign}(y)-\tfrac{1}{2}y^2\right)dy\\&=\infty,\end{aligned}$$

i.e., Assumption 2.8 does not hold. Nevertheless, $N^\varepsilon = \mathcal{E}\left(-\varepsilon \bar{R}\right)$ is a strictly positive wealth process for every $\varepsilon \in \mathbb{R}$ and thus a numéraire. For every $x > 0$ and $\varepsilon \neq 0$, we have

$$\begin{aligned} u(x,\varepsilon) &\geq \mathbb{E}\left[U\left(\frac{x}{N_1^\varepsilon}\right)\right] \\ &= \mathbb{E}\left[U\left(x\exp\left(\varepsilon\bar{R}_1 + \tfrac{\varepsilon^2}{2}[\bar{R},\bar{R}]_1\right)\right)\right] \\ &\geq \frac{x^p}{p}\mathbb{E}\left[\exp\left(\varepsilon p\bar{R}_1\right)\right] \\ &= \frac{x^p}{p}\mathbb{E}\left[\exp\left(\varepsilon p|B_1|^{2+\delta}\mathsf{sign}(B_1)\right)\right] \\ &= \frac{x^p}{p\sqrt{2\pi}}\int_{\mathbb{R}}\exp\left(\varepsilon p|y|^{2+\delta}\mathsf{sign}(y) - \tfrac{1}{2}y^2\right)dy \\ &= \infty. \end{aligned}$$

The following example shows that without Assumption 2.3, we might have a family of processes $(N^\varepsilon)_{\varepsilon\in(-\varepsilon_0,\varepsilon_0)}$, such that for every $\varepsilon \neq 0$, $N_T^\varepsilon < 0$ with positive probability.

Example 7.2. Let us consider model, where there are three times: 0, 1, and 2, where the process R is a one-dimensional semimartingale such that

$$R_0 = R_1 = 1,\ \mathbb{P}\text{–a.s., and } R_2 \text{ equals to } 3/2 \text{ or } 1/2 \text{ with probability } 1/2 \text{ each.}$$

Let us also consider a predictable process θ, such that

$$\theta_1 = 0,\ \mathbb{P}\text{–a.s.},\ \theta_2 = n \text{ with probability } \tfrac{1}{2^n},\ n \in \mathbb{N}.$$

Then in (2.1), for every $\varepsilon \neq 0$,

$$\mathbb{P}\left[\Delta\left((\varepsilon\theta)\cdot R\right)_2 < -1\right] = \mathbb{P}\left[\varepsilon\theta_2(R_2 - R_1) < -1\right] > 0,$$

thus, $N_2^\varepsilon < 0$ with positive probability. Therefore, *for every $\varepsilon \neq 0$, N^ε is not a numéraire.*

On the necessity of the remaining assumptions.

(1) Conditions (2.6) and (2.7) are necessary for the expected utility maximization problem to admit standard conclusions of the utility maximization theory, see the abstract theorems in [KS99] and [KK07, Proposition 4.19]. Note that we only impose them for $\varepsilon = 0$.
(2) Modeling the evolution of stocks with semimartingales is necessary for the absence of arbitrage as above, see [Kar13, Theorem 1.3], see also [KP11, Theorem 1.3] for the case of the nonnegative stock price process.
(3) If sigma-boundedness in the sense of Assumption 2.7 does not hold, then the second-order expansion in the initial wealth might not exist, see [KS06a, Example 3].

(4) [KS06a, Example 1 and Example 2] show the necessity of Assumption 2.1 for two-times differentiability of the value function in x. Note that, by the concavity of the value function in the x variable, two-times differentiability in the x variable at $x > 0$ holds if and only if the value function admits a quadratic expansion at x (in the x variable), see [HUL96, Theorem 5.1.2].

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