THE INCOMPATIBILITY OF CROSSING NUMBER AND BRIDGE NUMBER FOR KNOT DIAGRAMS

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Abstract. We study methods for computing the bridge number of a knot from a knot diagram. We prove equivalence between a geometric and a combinatorial definition of the bridge number of a knot diagram. For each notion of diagrammatic bridge number considered, we find crossing number minimizing knot diagrams which fail to minimize bridge number. Furthermore, we construct a family of minimal crossing diagrams for which the difference between diagrammatic bridge number and the actual bridge number of the knot grows to infinity.

1. Introduction

Bridge number and crossing number are two classical measures of the complexity of a knot, and we are interested in the existence of knot diagrams which simultaneously minimize both of these complexities. We prove two negative results in this regard (Theorem 1.3 and Theorem 5.1).

In contrast to the crossing number of a diagram, which is simply the number of crossings, the bridge number of a knot diagram can be reasonably defined in several essentially different ways. In Theorem 1.2, we prove equivalence between a natural geometric definition of the bridge number of a knot diagram, the perpendicular bridge number $b_\perp$, and a combinatorial definition more amenable to computation, the Wirtinger number ω. We show that none of these diagrammatic notions of bridge number is realized in every crossing number minimizing diagram. However, we conjecture that every knot admits at least one diagram which realizes both bridge number and crossing number.

1.1. Compatibility of the crossing number and the bridge number.

Let K be a knot-type in $\mathbb{R}^3$ and let $\gamma \in K$ be a smooth embedding of the knot-type K. Let $p : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $p(x, y, z) = (x, y)$, $h : \mathbb{R}^3 \to \mathbb{R}$ by $h(x, y, z) = z$ and $h_{||} : \mathbb{R}^3 \to \mathbb{R}$ by $h_{||}(x, y, z) = y$.

We denote by $\mathcal{C}(K)$ the set of embeddings $\gamma \in K$ which are regular and have minimal crossing number with respect to p; namely, these are crossing number minimizing embeddings. Also we denote by $\mathcal{B}(K)$ the set of embeddings $\gamma \in K$

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which have the minimal number of maximal points with respect to h; namely, these are bridge number minimizing embeddings.

We have experimentally verified in [5] that $\mathcal{C}(K) \cap \mathcal{B}(K) \neq \emptyset$ for at least 450,000 prime knots of up to 16 crossings. Does an embedding simultaneously minimizing crossing number and bridge number exist for every knot? That is, we put forth:

Conjecture 1.1. For any knot K, $\mathcal{C}(K) \cap \mathcal{B}(K) \neq \emptyset$.

Conjecture 1.1 implies that the set of possible positions of any knot forms an infinite rectangle region of integer lattice in the Cartesian coordinate system of crossing number and bridge number. In the region, an embedding $\gamma \in \mathcal{C}(K) \cap \mathcal{B}(K)$ becomes a vertex of the region.

1.2. **Equality of the Wirtinger number and the perpendicular bridge number of a diagram.** In order to study this conjecture, we introduce several diagrammatic notions of bridge number. These are integers associated to knot diagrams with the property that the minimum of the bridge number of D over all diagrams D of a given knot K is equal to the bridge number of K.

Alternatively, the typical diagrammatic depiction of knots obtaining their bridge number suggests defining the "parallel" bridge number of a diagram D, $b_{||}(D)$, as the minimal number of maxima of $h_{||}|_\gamma$ for any embedding γ that projects to D. We remark that this definition of diagrammatic bridge number is not very effective at calculating bridge number of a knot.

We focus instead on the perpendicular bridge number of a knot diagram D, $b_\perp(D)$ (Definition 2.2), and the Wirtinger number of D, $\omega(D)$ (Definition 2.4). The Wirtinger number is defined combinatorially, and the fact that the minimum value of $\omega(D)$ over all diagrams D of a knot K equals the bridge number of K is non-trivial – see [3] for a proof. On the plus side, the Wirtinger number is algorthmically computable. This allowed the detection of bridge numbers for nearly half a million knots from minimal crossing diagrams, and the tabulation of bridge numbers for the majority of these knots for the first time. By contrast, $b_\perp(D)$ is defined geometrically, and it follows easily that taking the minimum of $b_\perp(D)$ over all diagrams of a knot gives the bridge number, but $b_\perp(D)$ can be challenging to compute directly. Additionally, it is a straight forward exercise to show that $b_\perp(D) \leq b_{||}(D)$ for all knot diagrams, making $b_\perp(D)$ more effective at calculating the bridge number of a knot from one of its diagrams. Our next result relates the geometric invariant $\omega(D)$ and the combinatorial invariant $b_\perp(D)$.

Theorem 1.2. *For a knot diagram D, $\omega(D) = b_\perp(D)$.*

1.3. **Incompatibility of the bridge number and the crossing number for a fixed diagram.** In light of Theorem 1.2, we will sometimes use $b(D)$ to denote either of these quantities for a knot diagram, that is, $b(D) := \omega(D) = b_\perp(D)$. Throughout, $\beta(K)$ denotes the bridge number of K. We leverage the above equality to prove the following.

Theorem 1.3. *For every positive integer n, there exists an alternating knot K and a crossing number minimizing diagram D of K, with the property that $b(D) - \beta(K) \geq n$.*

We see from this theorem that not all minimal diagrams of alternating knots have Wirtinger number equal to the bridge number. Note, however, that the examples given in the proof of Theorem 1.3 are composite knots. This fact is leveraged in order to show that the difference $b(D) - \beta(K)$ can be made arbitrarily large. A more difficult question is whether a prime, reduced alternating diagram of a knot K always realizes the Wirtinger number of K. More generally, since any two prime reduced alternating diagrams of the same knot are related by a sequence of flypes, it is interesting to know whether the Wirtinger number is invariant under flypes. In an appendix by the first author and Nathaniel Morrison, they answer these two questions in the negative. Namely, they exhibit a knot whose crossing number and Wirtinger number are both realized in a prime, reduced alternating diagram; after performing a flype on this diagram, the Wirtinger number is seen to strictly increase. In light of this construction, we expect that the result of Theorem 1.3 would also hold with the additional assumption that the alternating knot K is prime.

Question 1.4. For a minimal diagram D of a prime alternating knot K, then $b(D) = \beta(K)$.

2. Preliminaries

Let K, p, h and γ be as above. Among the many equivalent definitions of bridge number of a knot, we favor the following one.

Definition 2.1. The *bridge number of* K, $\beta(K)$, is the minimal number of maxima of $h|_\gamma$ over all $\gamma \in K$.

The first definition of diagrammatic bridge number we consider is the following. Let K, γ, p, h be as above. When $p|_\gamma$ is regular and $|p^{-1}(x, y)| \leq 2, \forall (x, y) \in \mathbb{R}^2$, we say $p(\gamma)$ is a *knot projection* for K. Hence, every knot projection is a finite, 4-valent graph in the plane. A *knot diagram* of K is a knot projection $p(\gamma)$ together with labels that indicate which strand is the over-strand and which is the under-strand at each double point. Given a diagram D and an embedding γ of a knot type K, we say that γ *presents* D if the following hold:

(1) $h|_\gamma$ is Morse;
(2) $p(\gamma)$ is a knot projection of K;
(3) $p(\gamma)$ together with crossing labels is equal to D.

Definition 2.2. Given a knot diagram D of knot K, define the *perpendicular bridge number of* D, $b_\perp(D)$, to be the minimal number of maxima of $h|_\gamma$ over all γ presenting D.

For any diagram D of a knot K, if γ presents D, by definition $\gamma \in K$, so $b_\perp(D) \geq \beta(K)$. Furthermore, since, after an arbitrarily small perturbation that preserves the number of maxima of $h|_\gamma$, any $\gamma \in K$ has a diagram, $\beta(K)$ is realized as $b_\perp(D)$ for some diagram D of K. Therefore, $b_\perp(D)$ has the desired properties of a diagrammatic bridge number. Our first result, Thorem 1.2, is to prove that the perpendicular bridge number of a diagram D equals its Wirtinger number, $\omega(D)$.

The Wirtinger number of a diagram D is calculated algorithmically and is closely related to the problem of finding the minimal number of Wirtinger generators in D which suffice to generate the group of K. The Wirtinger number was introduced in [3], and it follows from the main theorem therein that $\omega(D)$ constitutes a diagrammatic bridge number in the above sense. We recall the definition here.

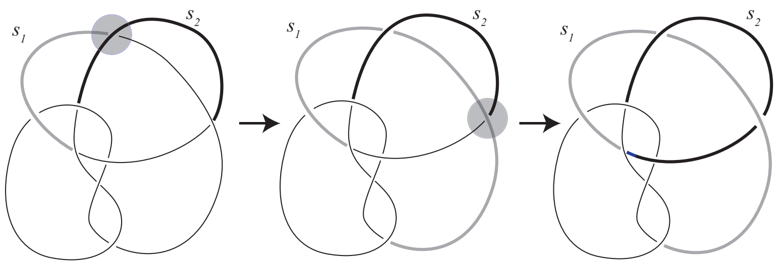

FIGURE 1. Two coloring moves on the knot 8_{17}, corresponding to the shaded crossings. The coloring process terminates at this stage. More generally, this diagram is not 2-colorable. 8_{17} is a three-bridge knot.

Let D be a knot diagram with n crossings and let $v(D)$ be the set of crossings c_1, c_2,..., c_n in the plane. Since we think of deleting a neighborhood of each under-strand from a knot projection to form the diagram D, then D consists of n disjoint closed arcs in the plane called *strands*. Denote by $s(D)$ the set of strands s_1, s_2,..., s_n for the diagram D. Two strands s_i and s_j of D are *adjacent* if s_i and s_j are the under-strands of some crossing in D. In what follows we assume that $n > 1$ so, in particular, no strand is adjacent to itself.

We call D *k-partially colored* if we have specified a subset A of the strands of D and a function $f : A \to \{1, 2, \dots, k\}$. We refer to this partial coloring by the tuple (A, f). Next, we define an operation that allows us to pass from one partial coloring on a diagram to another.

Definition 2.3. Given k-partial colorings (A_1, f_1) and (A_2, f_2) of D, we say (A_2, f_2) is the result of a *coloring move* on (A_1, f_1) if the following conditions hold:

(1) $A_1 \subset A_2$ and $A_2 \setminus A_1 = \{s_j\}$ for some strand s_j in D;
(2) $f_2|_{A_1} = f_1$;
(3) s_j is adjacent to s_i at some crossing $c \in v(D)$, and $s_i \in A_1$;
(4) the over-strand s_k at c is an element of A_1;
(5) $f_1(s_i) = f_2(s_j)$.

We denote the above coloring move by $(A_1, f_1) \to (A_2, f_2)$. Two consecutive coloring moves are illustrated in Figure 1, which we borrowed from [3]. We say a knot diagram D is *k-colorable* if there exists a k-partial coloring $(A_0, f_0) = (\{s_{i_1}, s_{i_2}, \dots, s_{i_k}\},\ f_0(s_{i_j}) = j)$ and a sequence of coloring moves which result in coloring the entire diagram.

Definition 2.4. Let D be a knot diagram. The smallest integer k such that D is k-colorable is the *Wirtinger number* of D, denoted $\omega(D)$.

For a proof that the minimum value of $\omega(D)$ over all diagrams D of a knot K equals the bridge number of K we again refer the reader to [3]. The Wirtinger number is our main tool for computing bridge numbers of minimal diagrams and comparing these to the bridge numbers of the corresponding knots.

In Section 3 we prove that $\omega(D) = b_\perp(D)$ for any knot diagram D. In Section 4 we analyze how Wirtinger number behaves with respect to diagrammatic connected sum and we use these results to show that given any knot K you can always find a diagram D such that the difference between $\omega(D)$ and $\beta(K)$ is arbitrarily large. Finally, we extend these results to show that the difference between $\omega(D)$ and $\beta(K)$ can be arbitrarily large even among crossing number minimizing diagrams if composite knots are allowed. We conclude by exhibiting a crossing number minimizing diagram of a *prime* knot which also fails to realize the Wirtinger number.

3. Equality of the Wirtinger number and the perpendicular bridge number of a diagram

Theorem 1.2 relates the Wirtinger number of a diagram to its perpendicular bridge number. It is proved in [3] that $\omega(D)$ can be calculated by computer from Gauss code for D, while the more geometric $b_\perp(D)$ tends to be rather elusive.

To prove this Theorem 1.2, some additional notation must be established. As above, given a diagram D, $s(D)$ denotes the set of strands s_1, s_2,..., s_n and $v(D)$ the set of crossings c_1, c_2,..., c_n. Furthermore, if γ presents D, we denote by $\overline{s}_1$, $\overline{s}_2$,..., $\overline{s}_n$ the subarcs of $p(\gamma)$ obtained by extending each of s_1, s_2,..., s_n slightly so that the endpoints of the arcs $\overline{s}_1$, $\overline{s}_2$,..., $\overline{s}_n$ are contained in $v(D)$. In particular, if we consider $p(\gamma)$ as a finite, 4-valent graph, then $\overline{s}_i$ is the union of all edges in $p(\gamma)$ whose interiors intersect s_i. Additionally, let $\{a_i^+, a_i^-\} := (p|_\gamma)^{-1}(v_i)$, where $h(a_i^+) > h(a_i^-)$ for all i. Finally, observe that $(p|_\gamma)^{-1}(\overline{s}_i)$ consists of a closed arc of positive length, possibly together with a collection of isolated points in γ, corresponding to crossings of D where s_i is the over-strand. We denote the closed arc component of $(p|_\gamma)^{-1}(\overline{s}_i)$ by $\hat{s}_i$.

Lemma 3.1. *Let D be a knot diagram and γ an embedding that presents D and minimizes $b_\perp(D)$. Then, after an isotopy of γ which fixes $p(\gamma)$ pointwise, we can assume the following: for every $s_i \in s(D)$, $\hat{s}_i$ contains at most three critical points of $h|_\gamma$; if $\hat{s}_i$ contains exactly three critical points of $h|_\gamma$, then two of these critical points are minima and one is a maximum; all critical points of $h|_\gamma$ corresponding to minima are contained in the set $\{a_1^-, a_2^-, ..., a_n^-\}$.*

Proof. Assume γ an embedding that presents D and minimizes $b_\perp(D)$. Consider a strand $s_i \in s(D)$. Let U denote the portion of the interior of $p^{-1}(\overline{s}_i)$ that lies above $\hat{s}_i$. By definition of strand, U is disjoint from γ. Hence, there is an ambient isotopy of γ supported in an arbitrarily small open neighborhood of U in $\mathbb{R}^3$, after which $\hat{s}_i$ is replaced by an arc with one maximum and at most two minima and $p(\gamma)$ is preserved pointwise. See Figure 2. This is a contradiction to the minimality of $h|_\gamma$, unless $\hat{s}_i$ contains at most three critical points of $h|_\gamma$. Moroever, this isotopy shows that we can assume that if $\hat{s}_i$ contains exactly three critical points of $h|_\gamma$, then two of these critical points are minima and one is a maximum.

As an intermediate next step, we arrange that the critical points of $h|_\gamma$ are contained in the interiors of the $\hat{s}_i$. Indeed, since $(p_\gamma)^{-1}(v(D))$ is a discrete set in γ, after an arbitrarily small ambient isotopy of γ that fixes $p(\gamma)$ pointwise and preserves that number of critical points, we can assume that no critical point of $h|_\gamma$ is contained in $p^{-1}(v(D))$.

We have already shown that, if $\hat{s}_i$ contains exactly three critical points of $h|_\gamma$, then two of these critical points are minima and one is a maximum. To conclude the proof, we need to isotope γ, preserving $p(\gamma)$ pointwise, so that all critical points

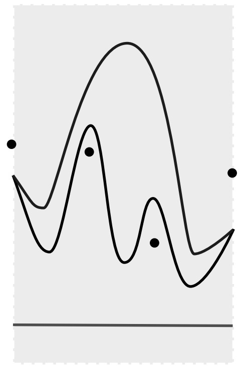

FIGURE 2. Isotopy decreasing the number of critical points for $\hat{s}_i$, if it contains more than three critical points of $h|_\gamma$. The red line is the projection, s_i, and the black dots represent the lifts of strands over which s_i passes.

of $h|_\gamma$ corresponding to minima are contained in the set $\{a_1^-, a_2^-, ..., a_n^-\}$. By the above, each arc $\overline{s}_i$ fits into one of three mutually disjoint categories:

(1) Type I: $\hat{s}_i$ contains no minima of $h|_\gamma$.
(2) Type II: $\hat{s}_i$ contains exactly one minimum and no maximum of $h|_\gamma$.
(3) Type III: $\hat{s}_i$ contains exactly one minimum and exactly one maximum of $h|_\gamma$.
(4) Type IV: $\hat{s}_i$ contains exactly two minima and exactly one maximum of $h|_\gamma$.

In each of these cases, we let U denote the portion of the interior of $p^{-1}(\overline{s}_i)$ that lies above $\hat{s}_i$. Recall that, by the definition of strand, U is disjoint from γ.

If $\overline{s}_i$ is an edge of Type II, let v_j and v_k be the endpoints of $\overline{s}_i$ and suppose $h(a_j^-) \leq h(a_k^-)$. Let U' be the portion of U below the plane $\{z = h^{-1}(h(a_j^-))\}$. Then there is an ambient isotopy of γ supported in an arbitrarily small open neighborhood of U' in $\mathbb{R}^3$ after which $\hat{s}_i$ is replaced by an arc with no critical points in its interior, $h|_\gamma$ has a minimum at a_j^-, and $p(\gamma)$ is preserved pointwise. See Figure 3.

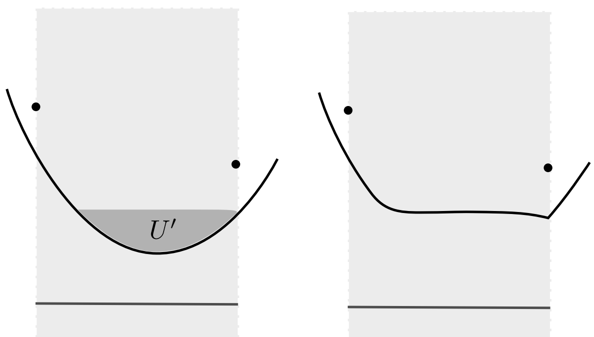

FIGURE 3. Isotopy for a strand of Type II.

If $\overline{s}_i$ is an edge of Type III, let M_i be the maximum of $h|_\gamma$ in $\hat{s}_i$ and let m_i be the minimum of $h|_\gamma$ in $\hat{s}_i$. Let v_j and v_k be the endpoints of $\overline{s}_i$ and suppose a_j^- is closer to m_i than it is to M_i, along $\hat{s}_i$. Let U^* be the portion of U below the plane $\{z = h^{-1}(h(a_j^-))\}$ and let U' be the component of U^* whose closure in U contains m_i and a_j^-. Note that M_i is not contained in the closure of U' as this would imply that there is an ambient isotopy of γ that eliminates a minimum and a maximum while preserving $p(\gamma)$, a contradiction to the minimality assumption for γ. Subsequently, there is an ambient isotopy of γ supported in an arbitrarily small open neighborhood of U' in $\mathbb{R}^3$ after which $\hat{s}_i$ is replaced by an arc with a single maximum critical point in its interior, $h|_\gamma$ has a minimum at a_j^-, and $p(\gamma)$ is preserved pointwise. See Figure 4.

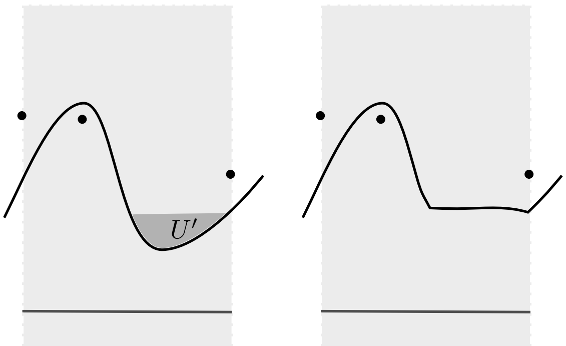

FIGURE 4. Isotopy for a strand of Type III.

If $\overline{s}_i$ is an edge of Type IV, let M_i be the maximum of $h|_\gamma$ in $\hat{s}_i$ and let m_i^1 and m_i^2 be the two minima of $h|_\gamma$ in $\hat{s}_i$. Let v_j and v_k be the endpoints of $\overline{s}_i$ and suppose a_j^- is closer to m_i^1 than it is to m_i^2, along $\hat{s}_i$. Let U^* be the portion of U below the plane $\{z = h^{-1}(h(a_j^-))\}$. Note that $h(M_i) > max\{h(a_j^-), h(a_k^-)\}$, since otherwise there would exists an ambient isotopy of γ which eliminates a minimum and a maximum while preserving $p(\gamma)$, a contradiction to the minimality assumption for γ. Hence, the portion of U below the plane $\{z = h^{-1}(h(a_j^-))\}$ contains a disk component U' that is incident to a_j^-. Similarly, the portion of U below the plane $\{z = h^{-1}(h(a_k^-))\}$ contains a disk component U'' that is incident to a_k^-. There is an ambient isotopy of γ supported in an arbitrarily small open neighborhood of $U' \cup U''$ in $\mathbb{R}^3$ after which $\hat{s}_i$ is replaced by an arc with a single maximum critical point in its interior, $h|_\gamma$ has a minimum at a_j^- and at a_k^-, and $p(\gamma)$ is preserved pointwise. See Figure 5.

Since each of the ambient isotopies described above are supported in pairwise disjoint regions in $\mathbb{R}^3$, we can simultaneously apply the above isotopies to produce an ambient isotopy of γ that fixes $p(\gamma)$ pointwise, results in an embedding of γ that minimizes $b_\perp(D)$, and results in all critical points of $h|_\gamma$ corresponding to minima being contained in $\{a_1^-, a_2^-, ..., a_n^-\}$. □

Proof of Theorem 1.2. Given a diagram D of a knot K, let $\mu := \omega(D)$. As in the proof of main result in [3], we can construct an embedding γ that presents D such

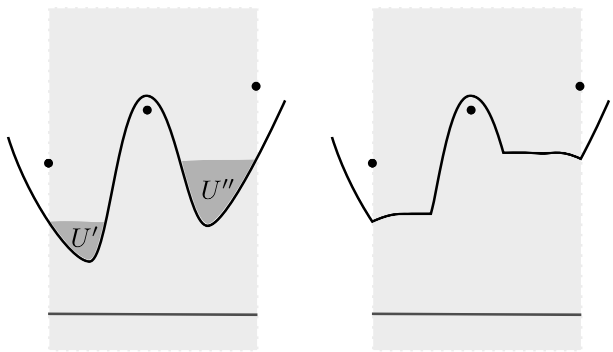

FIGURE 5. Isotopy for a strand of Type IV.

that $h|_\gamma$ has μ maxima. Roughly speaking, we can specify the height function of γ so that its only maximal points occur on the interiors of the strands s_j $(j = 1, \ldots, \mu)$ and there is exactly one maximum per strand s_j. This is achieved by first assigning the initial strands to a level plane; the strands subsequently colored via coloring moves are assigned to successively lower level planes whose height is determined by the order in which the coloring moves are performed. The complete collection of strands are then connected via monotone arcs. After a small perturbation, the result is an embedding of the knot with exactly μ maxima with respect to h. Hence, $b_\perp(D) \leq w(D)$. It remains to show that $b_\perp(D) \geq w(D)$.

Let D be a diagram of K and let γ be an embedding of K that presents D and realizes $b_\perp(D)$. Assume that we have isotoped γ so that the conclusions of Lemma 3.1 hold. That is, all critical points corresponding to minima of $h|_\gamma$ are contained in $\{a_1^-, a_2^-, ..., a_n^-\}$, all critical points corresponding to maxima of $h|_\gamma$ are contained in the interior of the arcs $\hat{s}_i$ for $i \in \{1, ..., n\}$, and each arc $\hat{s}_i$ contains at most one critical point corresponding to a maximum of $h|_\gamma$. Moreover, after a further isotopy of γ, fixing $p(\gamma)$ pointwise as always, we can additionally assume that all critical points of $h|_\gamma$ and all points in $(p|\gamma)^{-1}(v(D))$ take distinct values under h and that all critical points corresponding to maxima of $h|_\gamma$ lie above all of the points $(p|\gamma)^{-1}(v(D))$.

Define $g : s(D) \to \mathbb{R}$ by $g(s_i) = max(\{h(t)|t \in \hat{s}_i\})$. Note that this value is well-defined since $\hat{s}_i$ is compact and that g is one-to-one since all critical points of $h|_\gamma$ and all points in $(p|\gamma)^{-1}(v(D))$ take distinct values under h. Order the elements of $s(D)$ in order of decreasing value under the map g, and denote the resulting sequence by $s_{i_1}, ..., s_{i_n}$. Let $b_\perp(D) = m$ and observe that the above assumptions guarantee that $s_{i_1}, ..., s_{i_m}$ are exactly the strands which contain maxima of $h|_\gamma$. Set $A_k := \{s_{i_1}, ..., s_{i_{m+k}}\}$ for $0 \leq k \leq n - m$ and define $f_0 : A_0 \to \{1, ..., m\}$ by $f_0(s_{i_j}) := j$ for $1 \leq j \leq m$.

Let $M = \{d_1, ..., d_m\} \subset \{a_1^-, a_2^-, ..., a_n^-\}$ be the collection of critical points corresponding to minima for $h|_\gamma$. The closure of each component of $\gamma \setminus M$ is then a union of (consecutive) arcs $\hat{s}_j$ and contains a unique critical point corresponding to a maximum of $h|_\gamma$. Therefore, each component of $\gamma \setminus M$ contains the interior of a unique arc in the set $\{\hat{s}_{i_1}, ..., \hat{s}_{i_m}\}$. Extend f_0 to a function $f : s(D) \to \{1, ..., m\}$

by assigning $f(s_j) := f(s_{i_k}) = k$ where $\hat{s}_{i_k} \in \{\hat{s}_{i_1}, ..., \hat{s}_{i_m}\}$ is the unique element of this set with the property that $int(\hat{s}_j)$ and $int(\hat{s}_{i_k})$ are contained in the same component of $\gamma \setminus M$. Define $f_k = f|_{A_k}$ for all $1 \leq k \leq n-m$.

To show that D is m-colorable, it remains to be shown that $(A_k, f_k) \to (A_{k+1}, f_{k+1})$ is a valid coloring move for all $0 \leq k \leq n-m-1$. Note that $A_{k+1} \setminus A_k = \{s_{i_{m+k+1}}\}$ and let $p := f(s_{i_{m+k+1}}), p \in \{1, ..., m\}$. By the definition of f, we have that $\hat{s}_{i_{m+k+1}}$ is in the same component of $\gamma \setminus M$ as $\hat{s}_{i_p}$. But $\hat{s}_{i_{m+k+1}}$ does not contain a maximum of $h|_\gamma$, since $m+k+1 > m$. Therefore, $s_{i_{m+k+1}}$ is adjacent to a strand s_{i_r} such that $f(s_{i_r}) = p$ and $g(s_{i_r}) > g(s_{i_{m+k+1}})$. Since $g(s_{i_r}) > g(s_{i_{m+k+1}})$, we have that $s_{i_r} \in A_k$. Let s_{i_l} be the strand which passes over the crossing v_r at which $s_{i_{m+k+1}}$ and s_{i_r} are adjacent. Since $\hat{s}_{i_{m+k+1}}$ does not contain a maximum of $h|_\gamma$ and $g(s_{i_r}) > g(s_{i_{m+k+1}})$, then $g(s_{i_{m+k+1}}) = h(a_r^-)$. Since $h(a_r^+) > h(a_r^-)$ and $a_r^+ \in \hat{s}_{i_l}$, then $g(s_{i_l}) > g(s_{i_{m+k+1}})$ and, thus, $s_{i_l} \in A_k$. Hence, conditions (1)-(5) of Definition 2.3 are satisfied and $(A_k, f_k) \to (A_{k+1}, f_{k+1})$ is a valid coloring move for all $0 \leq k \leq n-m-1$. Therefore, D is m-colorable. This implies that $w(D) \leq b_\perp(D)$ and the theorem follows. □

In [3] the authors computed the Wirtinger number for crossing number minimizing diagrams of all knots of eleven or fewer crossings and verified that the Wirtinger number for these diagrams realized bridge number. By the same method, the authors calculated the bridge number of all knots with crossing number 12. Since then, the authors have also calculated the bridge number of more that 450,000 knots with 16 or fewer crossings. Moreover, all of these knots where found to have a minimal crossing diagram D such that $\omega(D) = \beta(K)$. These results motivate the search for the class of knots such that Wirtinger number is realized in a minimal crossing number diagram. We give some negative results in the next section as well as some ensuing conjectures regarding the compatibility of crossing number and Wirtinger number.

4. Incompatibility of the bridge number and the crossing number for a fixed diagram

We prove that the Wirtinger number of a knot diagram is super-additive with respect to the connected sum operation on knot diagrams, and we use this fact to construct minimal diagrams whose diagrammatic bridge number is arbitrarily large compared to the bridge number of the knot.

A diagram D of a knot K is *prime* if every simple closed curve in the plane of projection that is disjoint from the crossings of D and meets the edges of $p(K)$ transversally in two points bounds a disk in the plane of projection that is disjoint from the crossings of D. Otherwise, we say D is composite.

If D is composite, α is the simple closed curve in the plane of projection that illustrates that D is composite and $s \in \alpha \setminus D$, then the triple (D, α, s) induces a pair of diagrams D_1 and D_2 by surgering D along the arc of $\alpha \setminus D$ that contains D. See Figure 6. In this case, we write $D = D_1 \# D_2$ and say D is the *connected sum* of D_1 and D_2.

Proposition 4.1. Given a composite diagram $D := D_1 \# D_2$, the inequality $b_\perp(D) \geq b_\perp(D_1) + b_\perp(D_2) - 1$ holds.

Proof. Let (D, α, s) be the triple that gives rise to the decomposition $D = D_1 \# D_2$. Let $\gamma \in K$ such that γ presents D and $h|_\gamma$ has $b_\perp(D)$ maxima. After a small

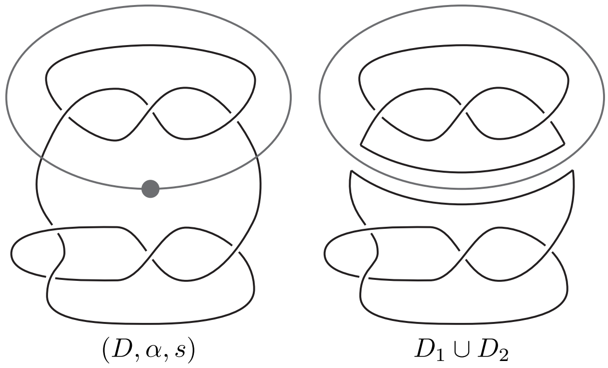

FIGURE 6. An example of the triple (D, α, s) inducing a pair of diagrams D_1 and D_2. The red dot specifies the arc along which the diagram is surgered.

perturbation of α that is transverse to $p(K)$, we can assume that $A = p^{-1}(\alpha)$ is disjoint from the critical points of $h|_\gamma$ and $A \cap \gamma = \{x, y\}$ such that $h(x) < h(y)$. Let $\hat{\alpha}$ be a monotone increasing (wrt the z coordinate) arc embedded in A connecting x to y that is mapped by p homeomorphically onto a sub arc of α. Surgering γ along $\hat{\alpha}$ with framing parallel to the plane of projection results in two knot embeddings γ_1 and γ_2 such that γ_1 presents D_1 and γ_2 presents D_2. Independent of the sign of the derivative of $h|_\gamma$ at x and y, $h|_{\gamma_1 \cup \gamma_2}$ has exactly one more maxima than $h|_\gamma$. See Figure 7.

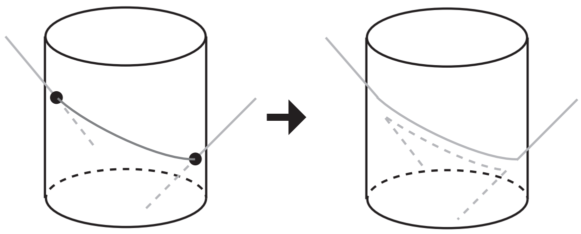

FIGURE 7. Surgering γ along $\hat{\alpha}$

Since the number of maxima of $h|_{\gamma_1 \cup \gamma_2}$ is bounded below by $b_\perp(D_1) + b_\perp(D_2)$, then $b_\perp(D) \geq b_\perp(D_1) + b_\perp(D_2) - 1$. □

Corollary 4.2. Given a composite diagram $D := D_1 \# D_2$, the inequality $\omega(D) \geq \omega(D_1) + \omega(D_2) - 1$ holds.

Proof. By Theorem 1.2, for any knot diagram D, $\omega(D) = b_\perp(D)$. □

Corollary 4.3. Given any integer n, there exists a diagram D_n of the unknot such that $b_\perp(D) = \omega(D) > n$.

Proof. Let D denote the Thistlethwaite diagram of the unknot, pictured in Figure 8. By direct computation, we find that $\omega(D) = 3$. (This can be done by hand. We show $\omega(D) > 2$ by verifying that, if we begin by coloring the over-strand and one under-strand at any crossing, the coloring process terminates before all strands are colored, regardless of the choice of crossing. By contrast, it is not hard to find three strands which, when colored, allow us to extend the coloring to the entire diagram, by iterating the coloring move.) Let $D_0 = D_1 = D$ and let D_n be a connected sum of n copies of D. By repeated application of Proposition 4.1, we have $\omega(D_n) = b_\perp(D_n) \geq 3 + 2(n-1) = 2n+1 > n$. □

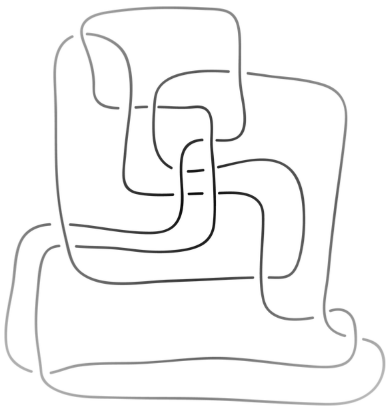

FIGURE 8. The Thistlethwaite unknot, whose Wirtinger number is 3.

This allows us to construct knot diagrams for which the gap between the Wirtinger number and the bridge number of the knot is arbitrarily large.

Corollary 4.4. For every positive integer m and every knot K, there exists a diagram D of K such that $b_\perp(D) = \omega(D) > \beta(K) + m$.

Proof. Let D_1 be any diagram of K, and D_n a diagram of the unknot satisfying the conclusion of Corollary 4.3 for $n = m+1$. Let $D := D_1 \# D_n$. By Proposition 4.1, D is a diagram of K such that

$$\omega(D) \geq \omega(D_1) + \omega(D_n) - 1 \geq \omega(K) + \omega(D_n) - 1 > \omega(K) + n - 1$$
$$= \omega(K) + m = \beta(K) + m.$$

□

Note, however, that the diagram constructed in the proof of Corollary 4.4 has a very large number of crossings, exceeding the crossing number of K by at least $15n$. A more subtle question is how big the gap between $\omega(D)$ and $\omega(K)$ can be for D a *minimal* diagram of the knot K. We demonstrate that this gap can be arbitrarily large as well.

Proof of Theorem 1.3. In Figure 9 we give an example of a knot K with reduced alternating diagram D such that $\omega(K) = \beta(K) = 5$ and $\omega(D) = 6$. The fact that $\omega(D) = 6$ was established by computer computation using the algorithm introduced in [3]. By Schubert's equality for bridge number, $\omega(\#_{i=1}^n K) = 5n - (n-1)$. Similarly, by Proposition 4.1, $\omega(\#_{i=1}^n D) \geq 6n - (n-1)$. Hence, $\omega(\#_{i=1}^n D) - \omega(\#_{i=1}^n K) \geq n$. □

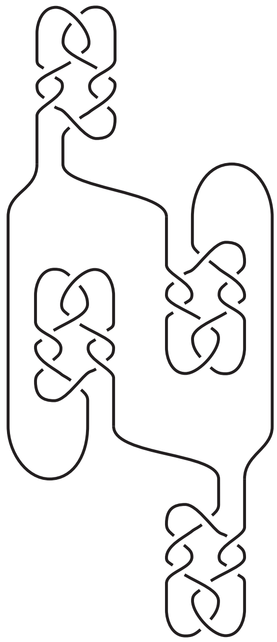

FIGURE 9. A minimal crossing diagram D of a composite knot K such that $\omega(D) = 6$ and $\beta(K) = 5$.

Having seen that the gap between the bridge number of a knot K and diagrammatic bridge number in a minimal diagram of K can be arbitrarily large for composite knots, we naturally ask whether minimal diagrams of *prime* knots always realize bridge number. We answer this question in the negative.

Theorem 4.5. *There exist a prime knot K and a minimal crossing diagram D of K such that $b_\perp(D) > \beta(K)$.*

Proof. Let K_0 be the knot in Figure 10, and denote the diagram depicted by D_0. Using the algorithm described in [3], we calculated the Wirtinger number of this diagram and found that $\omega(D_0) = 6$. Moreover, it is straight-forward to verify that the diagram D_0 is adequate and thus crossing number minimizing by [10]. Such a knot is known to be prime, see for example Lemma 6.0.29 of [1]. We note that K_0 is a satellite knot with a pattern consisting of a Montesinos knot (a knot constructed by a series of rational tangles). Such a knot is known to be prime, see for example Lemma 6.0.29 of [1]. It is similarly straight-forward to show that K_0 is an index two cable of the trefoil knot. Hence, by [9], $\beta(K_0) \geq 4$. However, Figure 11, illustrates that $\beta(K_0) \leq 4$. □

We believe the method used to construct the knot K_0 can be generalized to find other minimal diagrams D_i of prime non-alternating knots K_i, such that $b(D_i) > \beta(K_i)$ and the gap between $b(D_i)$ and $\beta(K_i)$ grows with the crossing number of D_i. It is interesting to note that there exists a minimal crossing diagram for K_0, given in Figure 11, that realizes bridge number.

By the Thurston's geometrization theorem, all knots are classified into mutually exclusive four classes; the trivial knot, torus knots, satellite knots and hyperbolic

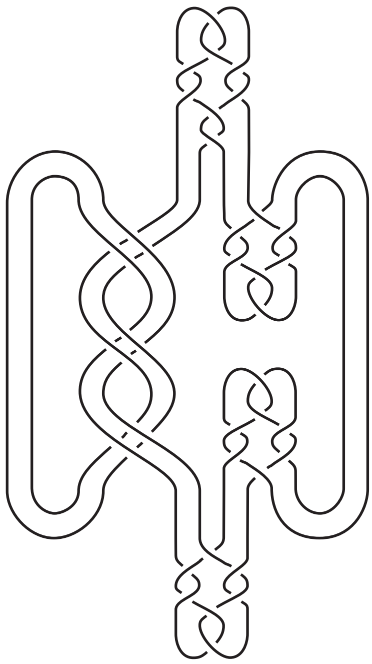

FIGURE 10. A minimal crossing diagram D of a prime knot K such that $\omega(D) = 6$ and $\beta(K) = 4$.

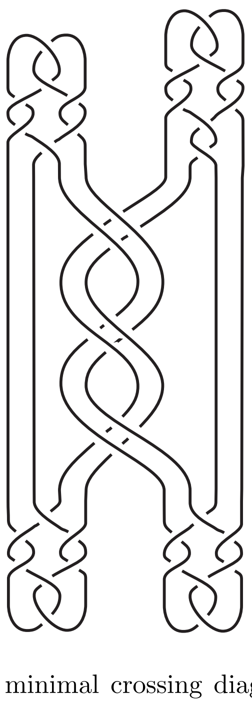

FIGURE 11. Another minimal crossing diagram D' of the prime knot K pictured in Figure 10. This time, $\omega(D') = \beta(K) = 4$.

knots. A knot is said to be *hyperbolic* if its complement has a complete hyperbolic structure of finite volume, or equivalently, its exterior does not contain essential surfaces of non-negative Euler characteristic. This implies that hyperbolic knots are "generic" among all knots. Our knots given in Figures 9 and 10 are classified into satellite knots. Then, the next question naturally arises.

Question 4.6. Does there exist a hyperbolic knot K and a minimal crossing diagram D of K such that $b_\perp(D) > \beta(K)$?

5. Incompatibility of the overpass bridge number and the crossing number

A classical notion of diagrammatic bridge number is that of "overpass" bridge number. Call an arc in knot diagram a *bridge* if it includes at least one overcrossing. The overpass bridge number of the diagram is the number of bridges in the diagram. It is well-known that this definition of diagrammatic bridge number is not well behaved with respect to minimal crossing number diagrams. For instance, the trefoil knot has bridge number two and crossing number three. However, it is a straight-forward exercise to show that every diagram of the trefoil with overpass bridge number two contains at least four crossings. More generally, we show that the minimal overpass bridge number and the minimal crossing number are wholly incompatible.

Theorem 5.1. *Let K be a non-trivial knot. K does not admit a diagram which realizes both the minimal overpass bridge number and the minimal crossing number of K.*

Theorem 5.1 follows from the next two lemmas. We recall from [9] the definition of an overpass (resp. underpass) in a knot diagram. For a diagram $D = p(\gamma)$ of a knot type K, an *overpass* (resp. *underpass*) is a subarc $p(\alpha)$, where α is a subarc of γ and $p|_\alpha$ is an injection, which contains only over-crossings (resp. under-crossings). Any knot diagram D can be decomposed into an alternative sequence of over- and under-passes $\alpha_1^+, \alpha_1^-, \ldots, \alpha_n^+, \alpha_n^-$. Let us rephrase the definition of overpass bridge number, which we recalled in the introduction, in this language.

Definition 5.2. The *overpass bridge number* of a knot diagram D as the minimal number of n over all alternative sequences of over/underpasses $\alpha_1^+, \alpha_1^-, \ldots, \alpha_n^+, \alpha_n^-$.

It is well-known that the *overpass bridge number* of a knot type K, namely, the minimal overpass bridge number over all diagrams of K, is simply the bridge number of K.

Lemma 5.3. *If a knot diagram has the minimal overpass bridge number, then any pair of consecutive over/underpasses intersect in at least one crossing.*

Proof. Suppose that there exists a pair of consecutive over/underpasses which has no crossing. Then we can obtain a knot diagram with a smaller overpass bridge number, by applying a move as shown in Figure 12. □

Lemma 5.4. *Let D be a crossing number minimizing knot diagram. Any pair of consecutive over/underpasses in D do not intersect at a crossing.*

Proof. Consider a knot diagram which contains a pair of consecutive over/underpasses intersecting at at least one crossings. Apply a move as shown

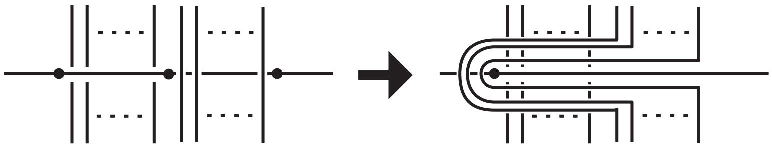

FIGURE 12. Reducing the overpass bridge number.

in Figure 13. This results in a knot diagram for the same knot whose crossing number is smaller. □

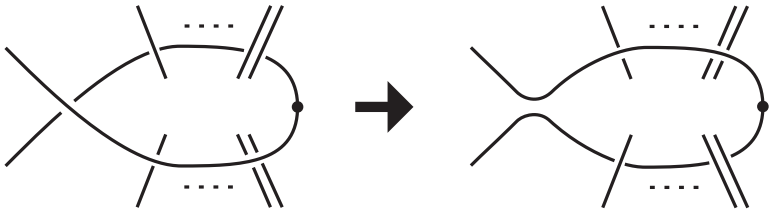

FIGURE 13. Reducing the crossing number.

APPENDIX: A NOTE ON WIRTINGER NUMBER AND FLYPES BY RYAN BLAIR AND NATHANIEL MORRISON

We provide an example demonstrating that the Wirtinger number of a prime, reduced, alternating diagram is not necessarily invariant under flype operations. This example also implies that a minimal diagram of a prime alternating knot does not necessarily admit a minimal Wirtinger number. This work resolves questions posed by the first author, Kjuchukova and Ozawa.

Introduction. A diagrammatic bridge number is an integer complexity associated to a knot diagram with the property that the minimum of this integer over all diagrams D of a given knot K is equal to the bridge number of K. The Wirtinger number of a knot diagram D, denoted $w(D)$, is an integer complexity of D defined in terms of "coloring moves" performed on D. It has previously been shown that the Wirtinger number is a diagrammatic bridge number [3]. The authors of [2] originally thought that Wirtinger number is invariant under flype operations and furthermore, that for any minimal diagram $D$of a prime alternating knot K, it holds that $w(D) = \beta(K)$, where $\beta(K)$ is the bridge number of the knot K with diagram D. However, this is not the case. In this note we present a prime, reduced, alternating diagram of a Montesinos knot with the property that the Wirtinger number of this diagram is distinct from the Wirtinger number of this diagram after a flype.

Counterexample. Consider the knot diagram D pictured in Figure 14. This diagram is given by the Gauss code

[1, -2, 3, -4, 5, -6, 7, -8, 9, -10, 11, -12, 13, -14, 15, -16, 17, -7, 2, -3, 6, -5, 4, -1, 8, -9, 18, -19, 26, -23, 22, -21, 24, -25, 20, -11, 16, -15, 12, -13, 14, -17, 10, -18, 19, -20, 21, -22, 23, -24, 25, -26].

Note that this diagram is prime, reduced and alternating. Hence, D has minimal crossing number [6] [8] [11]. Moreover, since D is a diagram for a Motesinos knot on three rational tanlges, then $\beta(K) = 3$ where K is the knot with diagram D [4].

Using the strands labeled α, β, γ, and δ as initial or "seed" strands, perform a sequence of coloring moves as in the definition of the Wirtinger number described in [3] until no further operations of this kind can be performed. Note that this process terminates with the diagram D being fully colored, and so $w(D) \leq 4$ by definition of the Wirtinger number.

Next we use a computer-aided calculation to perform this same process for all possible combinations of three strands as seed strands, and note that for no set of three seed strands is the diagram completely colorable. Note that this computation was performed using a version of code by P. Villanueva[12] modified by the second author. Thus, $3 < w(D) \leq 4$, and so $w(D) = 4$.

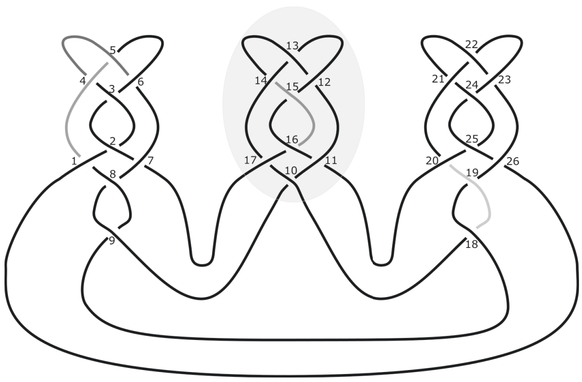

FIGURE 14. Diagram D of the counterexample knot. The seed strands α, β, γ, and δ are the overstrands at crossings 5, 4, 15 and 19, respectively. Region R, the region that will be flyped, is shaded.

Perform a flype on region R, defined by the set of crossings {10, 11, 12, 13, 14, 15, 16, 17}, by uncrossing crossing 10 and defining the new crossing to be crossing 10. The flyped diagram D' can be see in Figure 15, and is given by the Gauss code

[1, -2, 3, -4, 5, -6, 7, -8, 9, -11, 12, -13, 14, -15, 16, -17, 10, -7, 2, -3, 6, -5, 4, -1, 8, -9, 18, -19, 26, -23, 22, -21, 24, -25, 20, -10, 11, -16, 15, -12, 13, -14, 17, -18, 19, -20, 21, -22, 23, -24, 25, -26].

Using the strands labeled E, F, and G in D' as seed strands, perform a sequence of coloring moves until no further operations of this kind can be performed. Note that this process results in D' being fully colored, and so $w(D') \leq 3$ by definition of the Wirtinger number. Thus, the Wirtinger number is not preserved under flypes, and since $\beta(K) = 3 < 4 = w(D)$, it follows that the minimal, prime, alternating diagram D has a Wirtinger number strictly greater than the bridge number of the knot.

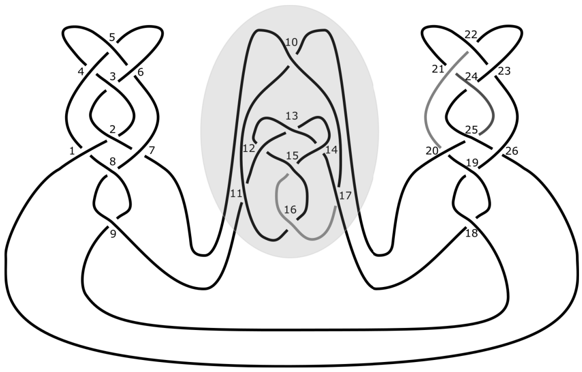

FIGURE 15. Diagram D' of the counterexample knot after a flype has been performed on region R. The seed strands E, F, and G of this new diagram are the overstrands at crossings 16, 21 and 24, respectively. Region R, the region that was flyped, is shaded.

REFERENCES

1. Ryan Blair, *Bridge number and Conway products*, PhD thesis, University of California, Santa Barbara, 2010.
2. Ryan Blair, Alexandra A. Kjuchukova, and Makoto Ozawa, *The Incompatiblity of Crossing Number and Bridge Number for Knot Diagrams*, arXiv:1710.11327v1 [math.GT]
3. Ryan Blair, Alexandra Kjuchukova, Roman Velazquez, and Paul Villanueva, *Wirtinger systems of generators of knot groups*, arXiv:1705.03108, to appear in Communications in Analysis and Geometry.
4. Michel Boileau and Heiner Zieschang, *Nombre de ponts et générateurs méridiens des entrelacs de Montesinos*, Comment. Math. Helv., **60** (1985), no. 2, 270–279.
5. Bridge Numbers, https://sites.google.com/a/wisc.edu/alexandra-a-kjuchukova/bridge-numbers, Accessed: 2017-10-31.
6. Louis H. Kauffman, *State models and the Jones polynomial*, Topology **26** (1987), no. 3, 395–407.
7. Markus Rost and Heiner Zieschang, *Meridional generators and plat presentations of torus links*, J. London Math. Soc. (2), 35(3):551–562, 1987.
8. Kunio Murasugi, *Jones polynomials and classical conjectures in knot theory*, Topology **26** (1987), no. 2, 187–194.
9. Horst Schubert. *Über eine numerische Knoteninvariante*, Math. Z., 61:245–288, 1954.
10. Morwen B. Thistlethwaite, *On the Kauffman polynomial of an adequate link*, Invent. Math., 93(2):285–296, 1988.
11. Morwen B. Thistlethwaite, *A spanning tree expansion of the Jones polynomial*, Topology **26** (1987), no. 3, 297–309.
12. Paul Villanueva (2018), calc_wirt (Version 1.0) [Python Code]. https://github.com/pommevilla/calc_wirt/blob/master/calc_wirt.py.

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