IDEMPOTENTS IN TANGLE CATEGORIES SPLIT

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ABSTRACT. In this paper we use 3-manifold techniques to illuminate the structure of the category of tangles. In particular, we show that every idempotent morphism A in such a category naturally splits as $A = B \circ C$ such that $C \circ B$ is an identity morphism.

1. Introduction

An *idempotent* of a category is a morphism that is idempotent with respect to composition, i.e. a morphism f such that $f = f \circ f$. Idempotents have significance to quantum observations or measurements [8], can reflect self-replication in biology (such as DNA) [6], and can form building blocks for numerous algebraic structures [5]. An idempotent f *splits* if there are morphisms g and h such that $f = g \circ h$, but $h \circ g$ is an identity morphism. By direct inspection, one can see that any morphism f with such a property is idempotent (if $h \circ g$ is an identity, then $(g \circ h) \circ (g \circ h) = g \circ (h \circ g) \circ h = g \circ h$); but in many categories, not all idempotents split. A category where every idempotent splits is called *Karoubi complete*. Idempotent splitting may adopt significance from various interpretations of the categories involved. For example, Selinger studied idempotents of dagger categories, and described in [8, Remark 3.5] how the splitting of idempotents may clarify data types. In [6], Kauffman related idempotents to DNA replication, and saw the idempotents in a Karoubi complete category as appealing models for self-replicators.

We show that the category of unoriented tangles up to isotopy is Karoubi complete. Objects of this category are points in the disc D^2, morphisms are properly embedded 1-manifolds in $D^2 \times I$ (these are the tangles), and the morphism composition is achieved via a stacking operation. Categories of tangles were studied in [9] to understand the combinatorial structure of tangle composition, and various categories of tangles are classified in [4] as certain types of braided pivotal categories. A related category, called the Temperley-Lieb category, can be described similarly, but with D^2 replaced by I, and hence the category of tangles we consider is a natural generalization of the Temperley-Lieb category. It was shown in [1] that the Temperley-Lieb category is Karoubi complete.

Although the main result of this paper extends that in [1], the techniques in this paper are different and are rather inspired by the proof of the prime decomposition theorem for string links provided in [2]. The main technical tool in the current paper, as well as in [2], is a bound, established in [3], on the number of non-parallel essential surfaces in a compact 3-manifold. The paper is organized as follows. In Section 3 we precisely define the tangle category. In Section 4 we review incompressible punctured surfaces and their properties. In Section 5 we adapt to tangles the notion of braid-equivalence for string links in [2] and apply this idea to factoring morphisms as the composition of two morphisms. In Section 6 we prove that all idempotents in the category of tangles split.

2. Acknowledgments

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3. Idempotents in the Category of Tangles

Let $\mathcal{T}$ be the category of smooth tangles. The definition of $\mathcal{T}$ that we give here is essentially equivalent to the definition of the *category of unoriented tangles up to isotopy*, denoted $\mathbb{TANG}$, in [4]. The objects of $\mathcal{T}$ are the natural numbers. Each natural number n is identified with n distinct fixed points $\{x_1, x_2, ..., x_n\}$ in the disk[1]. The morphisms of $\mathcal{T}$ are *tangles*. A tangle is a pair $(D^2 \times I, A)$ such that A is a properly embedded compact 1-manifold in $D^2 \times I$ with the following conditions: the boundary of each arc (connected component with non-trivial boundary) of A is contained in $(D^2 \times \{0\}) \cup (D^2 \times \{1\})$; the intersection of A with the lower and upper boundaries of $D^2 \times I$ are $A \cap (D^2 \times \{0\}) = \{(x_1, 0), (x_2, 0), ..., (x_n, 0)\}$ and $A \cap (D^2 \times \{1\}) = \{(x_1, 1), (x_2, 1), ..., (x_m, 1)\}$; and for each boundary point $(x_i, 0)$ or $(x_i, 1)$ of A and each sufficiently small neighborhood of that point, the derivatives of all orders of the embedding agree with the maps $t \mapsto (x_i, t)$. For simplicity, we will occasionally refer to the tangle $(D^2 \times I, A)$ as the morphism A. We denote $D^2 \times \{1\}$ by $\partial_+(D^2 \times I)$ and $D^2 \times \{0\}$ by $\partial_-(D^2 \times I)$.

[1]Without loss of generality, if $n < m$, then $x_1, ..., x_n$ associated with n is the same as the first n elements of $x_1, ..., x_m$ associated with m.

Definition 3.1 (Tangle equivalence)**.** *Tangles* $(D^2 \times I, A)$ *and* $(D^2 \times I, B)$ *are equivalent if there is an isotopy of* $D^2 \times I$ *fixing* $\partial(D^2 \times I)$ *that takes* A *to* B*. In this case, we will write* $A = B$*.*

Let $h : D^2 \times I \to I$ be the projection map onto the second coordinate. A *braid* is a tangle that is equivalent to a tangle $(D^2 \times I, A)$ with the property that for every component α of A, the restriction of h to α is a smooth, one-to-one and onto function with no critical points in its domain. Given tangle (M, A) from k to l and tangle (N, B) from l to m, with $M = N = D^2 \times I$, denote the *composition* of these morphisms by $(D^2 \times I, A \circ B)$ which is the quotient of $M \cup N$ achieved by gluing $\partial_+(M)$ to $\partial_-(N)$ via the map $(x, 1) \mapsto (x, 0)$, and $A \circ B$ is the properly embedded 1-manifold in the quotient which is the image of $A \cup B$ under this identification. The resulting quotient of $M \cup N$ is again homeomorphic to $D^2 \times I$ and we choose to identify the image of M under the quotient map with $D^2 \times [0, 1/2]$ and the image of N with $D^2 \times [1/2, 1]$ in the obvious ways. We will write the resulting tangle as the morphism $A \circ B$ and consider it the composition of morphisms A and B. See Figure 1.

We illustrate in Figure 2 an *idempotent* in the category of tangles, i.e. a morphism A of $\mathcal{T}$ such that $A \circ A = A$. Note that if A is a braid and an idempotent then A is an identity morphism, since braids are invertible morphisms. Recall that a category is *Karoubi complete* if all of its idempotents split, where an idempotent A *splits* if there exist morphisms C and B such that $A = C \circ B$ and $B \circ C = Id_n$ is an identity morphism. Our main theorem is the following.

Theorem 3.2. *The category* $\mathcal{T}$ *of tangles is Karoubi complete.*

4. Incompressible punctured surfaces

Our primary tool in the classification of idempotents in $\mathcal{T}$ will be the study of punctured surfaces up to transverse isotopy. Unless otherwise stated all manifolds are compact. Suppose α is a 1-manifold properly embedded in a 3-manifold M. If F is a properly embedded surface in M which meets α transversely in k points, we say F is *k-punctured.* An isotopy ϕ_t of F in M is *proper* if its restriction to $\partial F \times I$ is an isotopy of ∂F in ∂M and $\phi_t(F)$ is transverse to the boundary for all t. Moreover, the isotopy ϕ_t is *transverse* to α if the embedding ϕ_t is transverse to α for all fixed values of t. Isotopies of surfaces in this paper will always be proper isotopies that are transverse to the relevant 1-manifolds.

If α is a 1-manifold properly embedded in $M \cong D^2 \times I$ and F is a properly embedded k-punctured surface in M, F is *boundary-parallel*

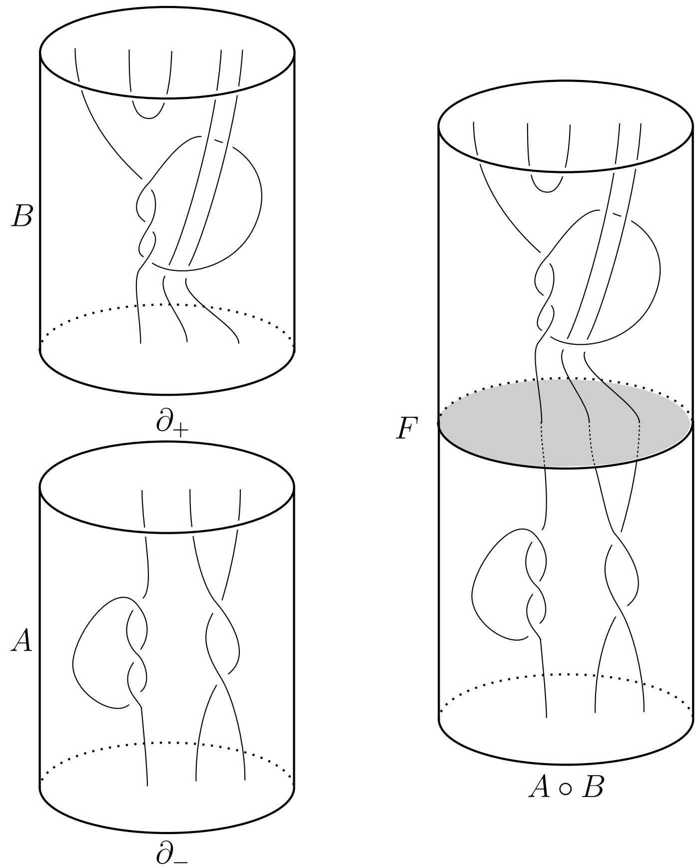

FIGURE 1. The composition of tangles

either if F is a 2-punctured 2-sphere bounding a 3-ball that meets α in an unknotted arc or if there is a transverse isotopy of F in M which fixes ∂F and takes F to a punctured subsurface contained in ∂M. Otherwise, we say F is *non-boundary parallel*. A loop γ embedded in F is *essential* if it does not bound a 0-punctured disk in F. The k-punctured surface F is *compressible* in (M, α) (or just *compressible* when context is understood) if F is a 0-punctured 2-sphere bounding a 3-ball or if there exists a disk D embedded in M such that $D \cap F = \partial D$, ∂D is essential in F and D is disjoint from α. Such a disk is called a *compressing disk*. Otherwise, we say F is *incompressible*. A punctured surface F is *essential* if F is incompressible and non-boundary parallel.

Given a 1-manifold α properly embedded in $M \cong D^2 \times I$ and F a properly embedded k-punctured surface with compressing disk D, we can *compress* F along D to form a new embedded k-punctured surface F^*. See Figure 3. Let $D^2 \times I$ be a fibered submanifold of M containing D such that $D = D^2 \times \{\frac{1}{2}\}$, $D^2 \times I$ is disjoint from α,

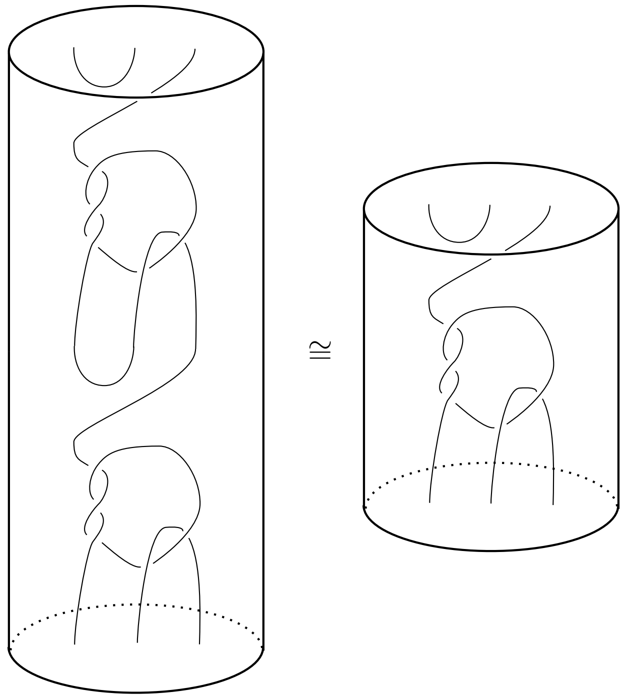

FIGURE 2. An idempotent morphism in the category of tangles

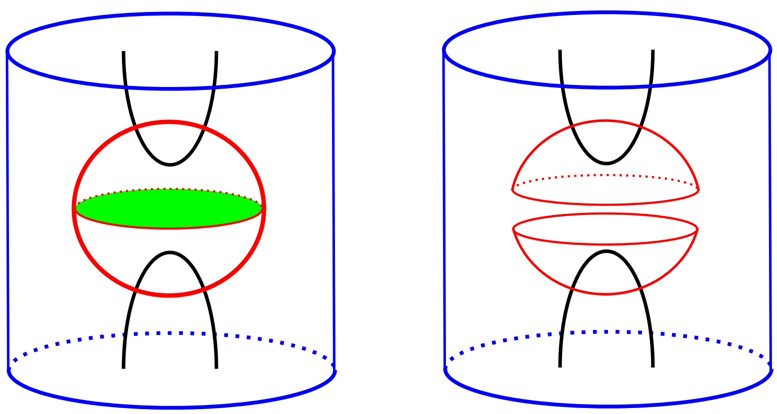

FIGURE 3. An example of compressing a 4-punctured sphere to obtain two boundary-parallel 2-punctured spheres.

and $\partial(D^2) \times I$ is an embedded annulus in F that is disjoint from the punctures of F. Then we define F^* to be a surface transversely isotopic to $(F \setminus (\partial(D^2) \times I)) \cup (D^2 \times \{0,1\})$.

Although we take the point of view of incompressible and non-boundary parallel punctured surfaces in this paper, we could have equivalently adopted the point of view of studying incompressible and non-boundary parallel surfaces properly embedded in the exterior of α in M. In particular, if F is a properly embedded non-boundary parallel punctured surface in $(D^2 \times I, \alpha)$ and $\eta(\alpha)$ is a small open regular neighborhood of α in $D^2 \times I$, then $F \setminus \eta(\alpha)$ is non-boundary parallel in $(D^2 \times I) \setminus \eta(\alpha)$. Similarly, if F is a properly embedded incompressible punctured surface in $(D^2 \times I, \alpha)$ and $\eta(\alpha)$ is a small open regular neighborhood of α in $D^2 \times I$, then $F \setminus \eta(\alpha)$ is incompressible in $(D^2 \times I) \setminus \eta(\alpha)$. In particular, we will make use of the following Theorem of Freedman and Freedman.

Theorem 4.1. [3] *Let M be a compact 3-manifold with boundary and b an integer greater than zero. There is a constant $c(M, b)$ so that if $F_1, ..., F_k$, $k > c$, is a collection of incompressible surfaces such that all the Betti numbers $b_1(F_i) < b$, $1 \leq i \leq k$, and no F_i, $1 \leq i \leq k$, is a boundary parallel annulus or a boundary parallel disk, then at least two members F_i and F_j are parallel.*

Note that Freedman and Freedman define two disjoint properly embedded surfaces F_i and F_j in a compact 3-manifold M to be *parallel* if $F_i \cup F_j$ cobound a product $F \times I$ in M such that $\partial F \times I \subset \partial M$, $F_i = F \times \{0\}$ and $F_j = F \times \{1\}$.

5. Decomposing Disks and Braid Equivalence

Decomposing a morphism in $\mathcal{T}$ as a composition of two morphisms involves some amount of choice. This choice can be captured via the notion of braid-equivalence.

Definition 5.1. *Two tangles $(D^2 \times I, A)$ and $(D^2 \times I, B)$ are* braid-equivalent *if there exist braids C_1 and C_2 such that $A = C_1 \circ B \circ C_2$.*

Proposition 5.2. *Tangles $(D^2 \times I, T_1)$ and $(D^2 \times I, T_2)$ are braid-equivalent if and only if there is an isotopy of $D^2 \times I$ which fixes $(\partial D^2) \times I$ and which takes T_1 to T_2.*

Proof. This follows from a nearly identical adaptation of the proof of Proposition 3.6 of [2]. □

Definition 5.3. *A* decomposing disk *for a tangle $(D^2 \times I, A)$ is a punctured disk which is properly embedded in $D^2 \times I$, whose boundary is isotopic in $\partial(D^2 \times I)$ to $\partial(\partial_+(D^2 \times I))$. See the disk F in Figure 1.*

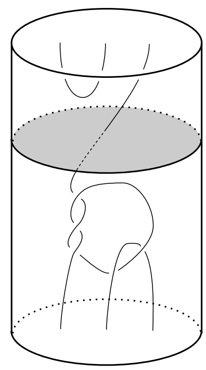

FIGURE 4. A minimal decomposing disk for an idempotent tangle.

A decomposing disk F for a tangle $(D^2 \times I, A)$ separates $D^2 \times I$ into two connected components, one containing $\partial_-(D^2 \times I)$ and the other containing $\partial_+(D^2 \times I)$. The closure of each component is homeomorphic to $D^2 \times I$, so F decomposes $(D^2 \times I, A)$ into two tangles, each of which is well-defined up to composition with braids (cf. *braid-equivalence* in Definition 5.1). If $(D^2 \times I, B)$ is the tangle resulting from restricting A to the side of F in $D^2 \times I$ that contains $\partial_-(D^2 \times I)$ and $(D^2 \times I, C)$ is the tangle resulting from restricting A to the side of F in $D^2 \times I$ that contains $\partial_+(D^2 \times I)$, we say that F *decomposes* $(D^2 \times I, A)$ *as* $(D^2 \times I, B \circ C)$, or, more simply, $A = B \circ C$. Note that any tangle A can be decomposed as $A \circ I$, where I here is an appropriate identity tangle, thus every tangle contains a decomposing disk.

6. The Proof

Definition 6.1. *A decomposing disk F for a tangle $(D^2 \times I, A)$ is* minimal *if there is no decomposing disk G such that $|F \cap A| > |G \cap A|$. See Figure 4.*

Lemma 6.2. *If $(D^2 \times I, A)$ is an idempotent, then either $(D^2 \times I, A)$ is an identity morphism or any minimal decomposing disk is essential.*

Proof. Assume that $(D^2 \times I, A)$ is an idempotent. If $(D^2 \times I, A)$ is a braid, then, as braids are invertible morphisms, A is an identity morphism. So, we may assume that $(D^2 \times I, A)$ is not a braid. If A contains l distinct closed loops, then, since A is idempotent, A must contain $2l$ distinct closed loops, a contradiction unless $l = 0$. Hence, we can assume that A contains no closed loops.

Since $(D^2 \times I, A)$ is an idempotent, then $(D^2 \times I, A)$ is a morphism from n points to n points for some n. Since A is non-empty, $n \geq 1$. By

Theorem 4.1, there is an integer c such that if $F_1, ..., F_k$ is a collection of disjoint incompressible decomposing disks in $D^2 \times I$ that each meet A in at most n points (hence the first Betti number of each F_i is bounded above by n) and $k > c$, then at least two members F_i and F_j are parallel or one of the surfaces $F_1, ..., F_k$ is a boundary parallel 0-punctured disk or a boundary parallel 1-punctured disk.

Claim: $\partial_+(D^2 \times I)$ or $\partial_-(D^2 \times I)$ is compressible in $(D^2 \times I, A)$.[2]

Proof of claim: Since $(D^2 \times I, A)$ is equivalent to $(D^2 \times I, A \circ A)$, then $(D^2 \times I, A)$ is equivalent to $(D^2 \times I, A^{c+2})$. Hence, we can find $c+1$ pairwise decomposing disks, $F_1, ..., F_{c+1}$, for $(D^2 \times I, A)$ that decompose $(D^2 \times I, A)$ into $c+2$ copies of $(D^2 \times I, A)$. If both $\partial_+(D^2 \times I)$ and $\partial_-(D^2 \times I)$ are incompressible in $(D^2 \times I, A)$, then each of $F_1, ..., F_{c+1}$ are incompressible in $(D^2 \times I, A)$. Since $n \geq 1$, then none of the surfaces $F_1, ..., F_{c+1}$ is a 0-punctured disk. Moreover, if any of the surfaces was a boundary parallel once punctured disk (i.e. a boundary parallel annulus in the exterior of A), then, by the isotopy extension theorem [7], A would be a trivial braid on one strand. Thus, the collection $F_1, ..., F_{c+1}$ meets the hypothesis of Theorem 4.1 and there exist two members F_i and F_j that are parallel. The tangle between F_i and F_j in $D^2 \times I$ is braid-equivalent to $(D^2 \times I, A^l)$ for some $l \geq 1$; however, since F_i is parallel to F_j, then A^l and, thus, A is a braid, a contradiction. Hence, one of $\partial_+(D^2 \times I)$ or $\partial_-(D^2 \times I)$ is compressible in $(D^2 \times I, A)$. □

Without loss of generality, suppose that $\partial_+(D^2 \times I)$ is compressible in $(D^2 \times I, A)$. Compressing $\partial_+(D^2 \times I)$ once results in a surface with two connected components. One component is a decomposing disk that meets A in strictly fewer than n points and the other component is a punctured sphere. Note that any boundary parallel decomposing disk would be properly, transversely isotopic to $\partial_+(D^2 \times I)$ or $\partial_-(D^2 \times I)$ in $(D^2 \times I, A)$, and hence meets A in n points. Since we have found a decomposing disk that meets A in strictly fewer than n points, then a minimal decomposing disk cannot be boundary parallel.

Let F be a minimal decomposing disk for $(D^2 \times I, A)$. By the above argument, F is non-boundary parallel. Next, we show that F is incompressible. If F is compressible, then compressing F once results in a surface with two connected components. One component is a decomposing disk that meets A in fewer points than F does. This is a contradiction to F being a minimal decomposing disk. Hence, F is incompressible. Since F is both incompressible and non-boundary parallel, then F is essential. □

[2]Observe that in Figure 2 $\partial_+(D^2 \times I)$ is compresible.

We now restate and prove our main theorem (Theorem 3.2).

Theorem 6.3. *If* $(D^2 \times I, A)$ *is an idempotent, then there exist* B *and* C*, such that* $A = B \circ C$ *and* $C \circ B$ *is an identity morphism.*

Proof. Let F be a minimal decomposing disk for $(D^2 \times I, A)$. By Lemma 6.2, F is essential. Denote the tangles that F decomposes $(D^2 \times I, A)$ into by $(D^2 \times I, B)$ and $(D^2 \times I, C)$ so that $A = B \circ C$. Note that since A is an idempotent, then A is a morphism from n points to n points for some n. Moreover, since $\partial_+(D^2 \times I)$ is a decomposing disk for every $(D^2 \times I, A)$, then F meets A in at most n points (in fact, F meets A in strictly fewer that n points by the claim in the proof of the previous theorem).

By Theorem 4.1, there is an integer c such that if $F_1, ..., F_k$ is a collection of disjoint essential decomposing disks in $D^2 \times I$ that each meet A in at most n points (hence the first Betti number of each F_i is bounded above by n) and $k > c$, then at least two members F_i and F_j are parallel (note that none of the F_i are boundary parallel since each is essential).

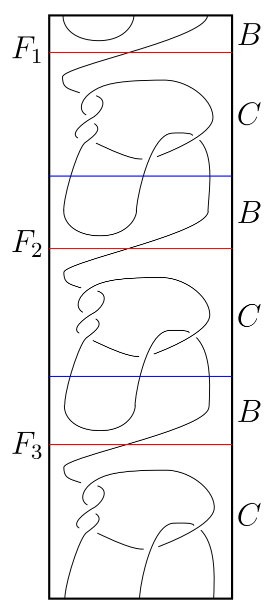

FIGURE 5. An example of how three copies of an idempotent $A = B \circ C$ can be decomposed into one copy of B, two copies of $C \circ B$, and one copy of C.

Since we have established that F is an essential punctured surface in $(D^2 \times I, A)$ and since $(D^2 \times I, A)$ is equivalent to $(D^2 \times I, A^{c+1})$, then we can find $c+1$ disjoint minimal decomposing disks, $F_1, ..., F_{c+1}$, for $(D^2 \times I, A)$ each representing the copy of F in each copy of A in A^{c+1}. Each of $F_1, ..., F_{c+1}$ is essential in $(D^2 \times I, A)$ and together they decompose $(D^2 \times I, A)$ into one copy of $(D^2 \times I, B)$, c copies of $(D^2 \times I, C \circ B)$, and one copy of $(D^2 \times I, C)$. See Figure 5. By Theorem 4.1, there exist two members F_i and F_j that are parallel. The tangle between F_i and F_j in $D^2 \times I$ is equivalent to $(D^2 \times I, (C \circ B)^l)$ for some $l \geq 1$, however, since F_i is parallel to F_j, then $(C \circ B)^l$ and, thus, $C \circ B$ is a braid.

Since $A^2 = A$, then $B \circ C \circ B \circ C = B \circ C$. We can compose on the left by C and the right by B to obtain $(C \circ B)^3 = (C \circ B)^2$. However, since braids on n strands are invertible morphisms, $(C \circ B)^3 = (C \circ B)^2$ implies $C \circ B$ is an identity morphism. □

7. Acknowledgements

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