



Design of Risk-Sharing Mechanism Related to Extreme Events

L. Tomaselli and M. Pozzi

Dept. of Civil and Environmental Engineering, Carnegie Mellon University, Pittsburgh, Pennsylvania, USA

B. Sinopoli

Dept. of Electrical and Computer Engineering, Carnegie Mellon University, Pittsburgh, Pennsylvania, USA

ABSTRACT

The occurrence of extreme events, either natural or man-made, puts stress on both the physical infrastructure, causing damages and failures, and the financial system. The following recovery process requires a large amount of resources from financial agents, such as insurance companies. If the demand for funds overpasses their capacity, these financial agents cannot fulfill their obligations, thus defaulting, without being able to deliver the requested funds. However, agents can share risk among each other, according to specific agreements. Our goal is to investigate the relationship between these agreements and the overall response of the physical/financial systems to extreme events and to identify the optimal set of agreements, according to some risk-based metrics.

We model the system as a directed and weighted graph, where nodes represent financial agents and links agreements among these. Each node faces an external demand of funds coming from the physical assets, modeled as a random variable, that can be transferred to other nodes, via the directed edges. For a given probabilistic model of demands and structure of the graph, we evaluate metrics such as the expected number of defaults, and we identify the graph configuration which optimizes the metric. The identified graph suggests to the agents a set of agreements to minimize global risk.

1 INTRODUCTION

When a disruptive event, such as earthquakes or hurricanes, strikes urban communities, insurance companies are required to pay the claims of the policyholders related to the particular disruptive event. These extreme scenarios may cause serious imbalances in the finances of the insurance companies, at the point that some of them may fail to pay the whole amount of claims submitted by the policyholders. According to the severity of the event, the default of many companies can produce significant losses for the involved communities, thus affecting their ability to recover after the disaster. The occurrence of extreme events and the consequential losses suffered by a physical infrastructure network and urban community can be modeled probabilistically (Malings and Pozzi (2016); Pozzi and Memarzadeh (2017)). A ready and fast recovery process after the occurrence of extreme events is an indicator of *resilience* of urban communities to natural or man-made disasters. The lack of financial resources to the amount needed during the recovery phase causes large costs for the affected communities [Cimellaro (2016)]. As a consequence, the important goal of creating resilient communities turns to be closely linked to the design of some mechanism that helps the financial agents, i.e. insurance companies, to share their demands for funds, in order to lower the probability that the financial system fails in providing the amount of funds required for a fast and effective recovery after the extreme events.

In our research, we are investigating if it is convenient for the insurance companies to enter in a risk-sharing mechanism that allows them to share a portion of the total claims owed to the policyholders

after the occurrence of a disruptive event. The goal of this research is to design an optimal risk-sharing mechanism that minimizes the global losses in the recovery process. This problem is mathematically analogous with the following one. Consider a set of power stations, that have to serve a population. If they operate in isolation, each of them covers the demand coming from a subset of that population. However, they can integrate their capacities, so that the whole set of stations jointly covers the demand of the entire population. By sharing the demands, an exceptionally high demand to one station can be distributed and absorbed also by others that were facing usual demands at that time. However, connecting the stations can expose them also to risks coming from the indirect demands posed by others, that are not present if they operate in isolation. Indeed, it is conceivable, for example, that a single extreme demand targeting one station is able to cause the failure of the entire set of stations, if they are connected, while this is impossible in the isolated scheme. To what extent should the power stations share their individual demands in order to avoid failures in serving their costumers? Or analogously, should we opt for a risk-sharing mechanism of mutually supportive insurance companies or should they rely on their own assets alone, in order to face the increased demands for funds caused by the occurrence of the extreme event? These are the main questions addressed in our work.

In this paper, we model the risk-sharing mechanisms through a network representation. Many forces act on a network of interconnected agents. The level of interconnectedness plays a central role in these particular problems, because it determines if by connecting more agents the mechanism helps them in successfully facing the shared external demands, or it weakens the network leading to an increased marginal probability of default of the nodes participating in it, as pointed out in [Elliott and Jackson \(2014\)](#). The first effect is what is generally called *diversification* of the risk and is related to the reduction of the marginal probability of default of a node achieved through the participation in the mechanism. The possibility to transfer part of the external demand of a node towards the other agents lowers its probability of default. Indeed, consider the scenario where an agent is subjected to a significant request for funds, that would determine its default if the node faced alone the high external demand. If the agent participates in the risk-sharing mechanism it may survive, because other nodes with positive excess capacities take on part of the external demand of the node. This translates in a lower probability of default of the node when it takes part in the mechanism. The second, negative, effect is related to *contagion*, a well studied phenomenon in financial and epidemics literature that refers to the spread of a disease through a network of connected agents. An exhaustive survey on the literature on the topic of contagion in financial networks is presented in the paper by [Glasserman and Young \(2016\)](#).

In our model, the higher the level of interconnectedness among the nodes that participate to the mechanism, the more each agent relies on the capacity of the other nodes to fulfill their in-network liabilities produced by the risk-sharing mechanism, namely the amount of effective demand that the other companies owe the agent. When the mechanism makes the assets of an agent overly dependent on the claims towards the other nodes, the consequence is that the likelihood of contagion increases. With contagion we refer to the spread of the default state throughout the set of agents starting from a scenario when only few of them are unable to afford their liabilities.

From the perspective of agent i , the participation in the mechanism produces both claims and liabilities. Indeed, the set of other agents involved in the mechanism owe the agent some money, namely the external demand that node i transfers to them, and the agent, in its turn, owes liabilities to the other agents equal to the demands that the latter transfer to the former. Referring to our model, every node bases its ability to fulfill its liabilities on the money received by the other nodes in the network. As a consequence, when some debtors of a node default, the likelihood that the node fails to pay its liabilities increases. A detailed analysis of the mechanism through which the losses caused by the default of an institution spread in the network of connected financial institutions is presented in the seminal works by [Allen and Gale \(2000\)](#) and [Eisenberg and Noe \(2001\)](#). The increase in the probability of default due to this network effect represents a negative aspect of the mechanism. The existence of conflicting effects related to the presence of a network structure among the agents, implies that we should carefully take into account the two above effects when we design an optimal risk-sharing mechanism.

2 MODELS OF PROBABILISTIC DEMAND AND OF THE RISK-SHARING MECHANISM

2.1 General settings

Consider a set of agents $N = \{1, 2, \dots, n\}$, each corresponding to a node in a graph. Every agent i faces a demand which we call the *external demand*, s_i , and has an individual capacity, c_i . The external demands corresponds to the claims of the policyholders (and, in the analogy with an electric system, to the demands for electricity from the users of a power station). Moreover, the individual capacity represents the upper bound on the total demand that the agent can afford. The effective demand, d_i , corresponds to the total demand that agent i faces under the risk-sharing mechanism, i.e. the set of agreements for the redistribution of the external demands among the agents. The effective demands are transformations of the external ones, through some functions modeling the mechanism of redistribution. To illustrate a simple example, consider the case when $n = 2$, i.e. when there are only two insurance companies. Suppose that an earthquake occurs, so that the insurance companies owe liabilities towards their policyholders, namely s_1 and s_2 . Therefore, s_1 and s_2 represent the total liabilities of node 1 and 2 respectively. Suppose that the insurance companies adopt a mechanism according to which each of the two companies shares with the other 20% of their individual claims. Their marginal effective demands are equal to $d_1 = 0.8s_1 + 0.2s_2$ and to $d_2 = 0.8s_2 + 0.2s_1$, respectively.

In the continuation of this work we will refer to random variables using capital letters, while to their realizations and to deterministic values using lower-case letters. We state the model in the following terms. Let $\mathbf{s} \in \mathbb{R}^n$, be a realization of the vector of *external demands* $\mathbf{S} \sim p_{\mathbf{S}}$, where $p_{\mathbf{S}}$ is its assumed distribution. Let $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the *redistribution function* that defines the mechanism.

Let $\mathbf{D}$ be the random vector of *effective demands*, obtained from the redistribution of the external demands among the agents, so that $\mathbf{D} = \mathbf{f}(\mathbf{S})$. Therefore, after observing the realization of $\mathbf{S}$, the vector $\mathbf{d} = \mathbf{f}(\mathbf{s})$ takes values in $\mathbb{R}^n$.

Let us define Y_i the binary state of node i , i.e. defaulted or not defaulted, and $\mathbf{Y}$ the vector listing all states. Let $\mathbf{h} : \mathbb{R}^n \rightarrow \mathbb{B}^n$ be the *default state function*, so that we have $\mathbf{Y} = \mathbf{h}(\mathbf{D})$. After the realization of the external demands, for all nodes in the system this function maps the actual effective demands to the space of Boolean variables. Therefore, for every agent i , function h_i returns 0 if the actual effective demand for the node is less than its capacity, or 1 if the effective demand is greater than the intrinsic capacity, thus meaning that the node is in a state of default.

Let $L : \mathbb{B}^n \rightarrow \mathbb{R}$ be the cost function that returns the total cost for the society given the set of defaults of the agents in the network. For example, L can be the number of defaulted nodes, the amount of losses in value of the agents due to their defaults in the system, the cost incurred by communities due to a slow recovery process caused by the default of some financial agents and so on. Note that for any possible choice of the redistribution function $\mathbf{f}$, the cost function L is a function of the random vector $\mathbf{S}$, thus being a random variable itself.

The goal of our work is to find $\mathbf{f}$, such that:

$$\mathbf{f}^* = \arg \min_{\mathbf{f} \in \mathbf{F}} \mathbb{E}_{\mathbf{S}} [L(\mathbf{h}(\mathbf{f}(\mathbf{S})))] \quad (1)$$

where $\mathbb{E}_{\mathbf{S}} [L(\cdot)]$ is the expectation of the losses in the system with respect to the probability distribution of $\mathbf{S}$.

In this paper we focus on a specific cost function, namely the number of defaulted agents. Our goal is, hence, to minimize the expected number of defaulted nodes after the extreme event. If the cost implied by the default of an agent is uniform, the choice of this particular cost function is related to the desire of minimizing the expected cost due to the insolvency of the companies. We may also choose this cost function when the functioning of the system only depends on the number of operative agents. Let the random variable X represent the number of defaulted nodes in the system, so that we have $X = \sum_i Y_i$. The cost function to which we refer is, hence, $L := X$.

Furthermore, a general assumption throughout this paper is that we look for the optimal risk-sharing mechanism under which the effective demands, $\mathbf{D}$, may be represented as a linear transformation of the external demands, $\mathbf{S}$. Therefore, in the optimization problem (1) we are looking for the optimal function $\mathbf{f}^*$ in the class of linear functions $\mathbf{F}$. Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be the linear transformation matrix, also called *redistribution matrix*. This matrix entirely describes the linear function $\mathbf{f}$ and, by using it, we can express the effective demands for all agents as $\mathbf{D} = \mathbf{A}^\top \mathbf{S}$. Entry ij of the matrix $\mathbf{A}$, namely α_{ij} , represents the portion of the external demand that agent i transfers to agent j . An equivalent representation of the effective demand for each node is $D_i = \sum_{j=1}^n \alpha_{ji} S_j$, a linear combination of the external demands of the agents through the columns of redistribution matrix $\mathbf{A}$.

In our settings, we put some conditions on the matrix $\mathbf{A}$, namely the entries on each row should sum to one, because the external demands should be completely allocated among the set of agents, considering also self-reallocations, and no demand is created nor erased during the reallocation process. The second condition is that the elements on the columns should sum to one, in order to get a "fair" allocation and avoid situations in which the largest part of the external demands is allocated to only one node, or a small subset of nodes. According to these conditions, the matrix $\mathbf{A}$ is a doubly stochastic one. Given these assumptions on the matrix $\mathbf{A}$, we can work with the following configuration of the redistribution matrix:

$$\mathbf{A} = \begin{bmatrix} 1 - \alpha & \frac{\alpha}{n-1} & \cdots & \frac{\alpha}{n-1} \\ \frac{\alpha}{n-1} & 1 - \alpha & \cdots & \frac{\alpha}{n-1} \\ \vdots & \cdots & \ddots & \vdots \\ \frac{\alpha}{n-1} & \frac{\alpha}{n-1} & \cdots & 1 - \alpha \end{bmatrix}$$

According to this model of matrix $\mathbf{A}$, each agent holds a portion of its external demand, $1 - \alpha$, and transfers the remaining portion, α to the other agents in the system in equal parts, $\frac{\alpha}{n-1}$. We define α as the so-called *transferred portion*, where $\alpha \in (0, 1)$, being it the same for every agent. So our optimization problem translates to finding α^* that defines a redistribution mechanism that minimizes the expected loss for our system.

Let the vector of intrinsic capacities, $\mathbf{c}$, be deterministically defined. Let $\mathbf{G}$ be the vector of the excess capacities of each node: its entries are obtained by subtracting the effective demands from the capacities of each agent, namely $G_i := c_i - D_i$, $\forall i \in N$. The result is the following:

$$\mathbf{G} = \mathbf{c} - \mathbf{D}$$

If the excess capacity of node i is less than zero, namely $G_i < 0$, the node fails on its obligations and enters in a state of default, because the intrinsic capacity of an agent, c_i , is not high enough to face the total effective demand arrived to the node, D_i . Function $\mathbf{h}$ is applied on the vector $\mathbf{g}$ in order to get the vector of agents' states in the system after the realization of the random vector $\mathbf{S}$. We study the case when the intrinsic capacity of each node is greater than the expected effective demands that the agent receives, namely $\mathbb{E}[D_i] < c_i$ for every node. The choice of the optimal linear function $\mathbf{f}^*$, or equivalently of the matrix $\mathbf{A}^*$, is deeply affected by the assumptions made on the distribution of the marginal external demands, p_S .

It is important to note that we can associate a directed and weighted graph $\mathcal{G}$ with the risk-sharing mechanism defined above. The vertices of the resulting graph correspond to the agents in the set N , while the arcs are defined by the redistribution matrix $\mathbf{A}$.

2.2 Two alternative probabilistic models of the external demands

2.2.1 Multivariate normal external demands

Suppose that the vector of external demands is distributed as a Multivariate normal random variable.

$$\mathbf{S} = [S_1, S_2, \dots, S_n]^\top \sim p_S = \mathcal{N}(\boldsymbol{\mu}_S, \boldsymbol{\Sigma}_S) \quad (2)$$

Where $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ is the multivariate normal with $\boldsymbol{\mu}$ as vector of the means and $\boldsymbol{\Sigma}$ as covariance matrix. The marginal distribution of the external demand, S_i , is equal to $S_i \sim \mathcal{N}(\mu_{S_i}, \sigma_{S_i}^2)$, $\forall i = \{1, \dots, n\}$. According to these assumptions, we have the following:

$$\boldsymbol{\mu}_S = [\mu_{S_1}, \mu_{S_2}, \dots, \mu_{S_n}]^\top, \boldsymbol{\Sigma}_S = \begin{bmatrix} \sigma_{S_1}^2 & \sigma_{12} & \dots & \sigma_{1n} \\ \sigma_{21} & \sigma_{S_2}^2 & \dots & \sigma_{2n} \\ \vdots & \dots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \dots & \sigma_{S_n}^2 \end{bmatrix} \quad (3)$$

In Eq. (3), σ_{ij} represents the covariance between the external demands of agents i and j .

Furthermore, assume that the nodes are homogeneous in terms of the distribution of external demands and in terms of intrinsic capacities. This means that the following is true:

$$S_i \sim \mathcal{N}(\mu_S, \sigma_S^2), c_i = c \quad \forall i = \{1, \dots, n\}$$

In addition, let the correlation coefficient be equal to ρ for all the pairs of external demands. Then, $\sigma_{ij} = \rho \cdot \sigma_S^2$, $\forall i \neq j$, where $i, j \in N$. Given the assumption of multivariate normality of the vector $\mathbf{S}$, vector $\mathbf{G}$ will be a multivariate normal random variable as well, with mean $\boldsymbol{\mu}_G = \mathbf{c} - \boldsymbol{\mu}_S$ and covariance matrix equal to $\boldsymbol{\Sigma}_D$. The marginal probability of failure for node i is equal to:

$$\mathbb{P}\{G_i < 0\} = \mathbb{P}\left\{\frac{G_i - \mu_{G_i}}{\sigma_{G_i}} < -\frac{\mu_{G_i}}{\sigma_{G_i}}\right\} = \Phi\left(-\frac{\mu_{G_i}}{\sigma_{G_i}}\right) = \Phi\left(-\frac{\mu_G}{\sigma_G}\right) \quad (4)$$

where Φ is the standard normal cumulative distribution function.

Minimizing the expected number of nodes in default is equivalent to minimizing the individual probabilities of default, with respect to the redistribution function. Moreover, the ratio between the mean and standard deviation of the excess capacity of a node, corresponding to the inverse of its *reliability index*, is the same for every node, because we assume homogeneity, same correlation coefficients for pairs of external demands and that matrix $\mathbf{A}$ is a doubly stochastic one. Indeed, we have:

$$\begin{aligned} \mu_{G_i} &= c_i - \sum_j \alpha_{ji} \mu_{S_i} = c - \mu_S = \mu_G \\ \sigma_{G_i} &= \sqrt{(1-\alpha)^2 \sigma_{S_i}^2 + 2 \frac{\alpha}{n-1} \sum_{i < j} \sigma_{S_{ij}}} = \sqrt{(1-\alpha)^2 \sigma_S^2 + 2 \frac{\alpha}{n-1} \sum_{i < j} \rho \sigma_S^2} = \sigma_G \end{aligned}$$

Then, the optimization problem becomes:

$$\min_{f \in F} \mathbb{P}\{G_i < 0\} = \min_{f \in F} \Phi\left(-\frac{\mu_G}{\sigma_G}\right) = \max_{f \in F} \frac{c - \mu_S}{\sigma_G} = \min_{f \in F} \sigma_G \quad (5)$$

We are minimizing the standard normal cumulative distribution function evaluated in the negative ratio of the mean and standard deviation of the individual excess capacities, with respect to the matrix $\mathbf{A}$. This is equivalent to maximize the ratio of the mean and standard deviation of the excess capacity of each node, so that the negative of this value is far from the origin, thus implying a low standard normal cumulative distribution function evaluated in that ratio. Moreover, since we are dealing with homogeneous nodes,

each node is equivalent a priori, so minimizing the standard deviation for one agent is equivalent to minimize (4) for all the nodes. In these settings, solving the problem stated in (1) leads to a quite intuitive result, namely that we should prefer a redistribution mechanism under which each node faces an effective demand equal to the average of the realized total external demand. The resulting optimal redistribution matrix is equal to the following:

$$\mathbf{A}^* = \frac{1}{n} \mathbf{1} \quad (6)$$

where $\mathbf{1}$ is the $n \times n$ matrix with all the entries equal to 1.

The optimal matrix in Eq.(6) defines a risk-sharing mechanism in such a way that any node faces the average of the realized external demands. We call this mechanism with the name of *perfect team*. In the multivariate normal case, the act of pooling the capacity of the nodes allows for a more efficient use of the total positive excess capacity, namely the sum of all the positive excess capacities, in order to offset the shortfalls in the excess capacity of the defaulted nodes. This way of reasoning is correct to the extent that, on expectation, the pooled capacity of the nodes is higher than the sum of the external demands.

The intuitive result may be proved by solving the optimization problem in (5) via matrix algebra, by minimizing one entry on the main diagonal of the covariance matrix of the effective demands. For every dimension of the set of agents and any positive correlation coefficient such that $\rho < 1$, the optimal redistribution matrix $\mathbf{A}^* = \frac{1}{n} \mathbf{1}$ minimizes the problem presented in Eq.(5).

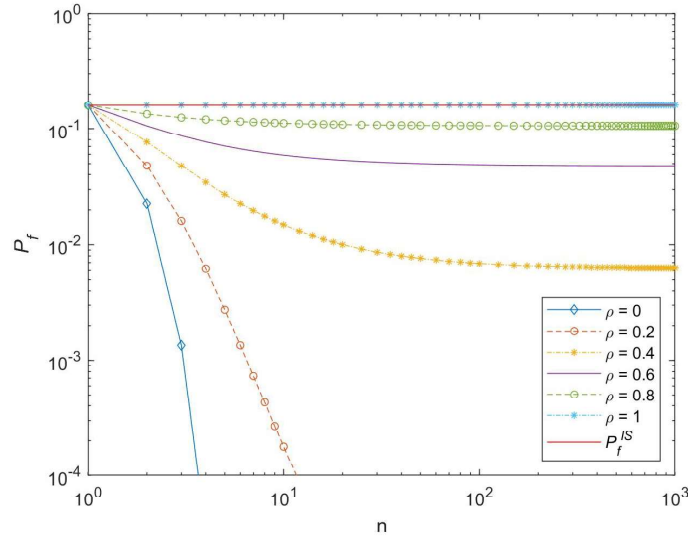


Figure 1: Probability of default of a node in the *perfect team* as the group dimension, n , increases, for different correlation coefficients, ρ compared to that of an *isolated agent*. The vector of external demands is multivariate normal, for each agent $S_i \sim \mathcal{N}(1, 1)$ and $c_i = 2$.

In Fig(1) we plot the relation between the probability of failure of the agents in the *perfect team* and the number of agents participating in the risk-sharing mechanism in the multivariate normal case, for different levels of correlation coefficients between pairs of external demands, represented by the colored lines. We compare these with the red line corresponding to the probability of default of an agent not taking part in the mechanism, i.e. $P_f^{IS} = \Phi\left(\frac{c - \mu_S}{\sigma_S}\right)$, that is constant because the effective demand of an isolated agent is not directly affected by the variation of the parameters ρ and n . We notice that the lines representing the probability of failure of the nodes in the *perfect team* are below the red line, corresponding to the probability of failure of the node when left alone, for any ρ but the case in which

$\rho = 1$. This means that, when the marginal external demands are distributed as $S_i \sim \mathcal{N}(1, 1)$ and the capacities of each node are $c_i = 2$, we always prefer the *perfect team* for any team size, n , because it minimizes the probability of default of the nodes, no matter of the structure of correlations between pairs of external demands. This result holds for any choice of μ_{S_i} , σ_{S_i} and c_i , as long as $\mu_{S_i} < c_i$ for any node in the system. The parameters in Fig.(1) are selected in order to facilitate graphical representation. In the extreme case of $\rho = 1$, we are indifferent to the choice between the *perfect team* and the isolated nodes, because in the case of perfect positive correlation all the nodes face the same realized external demands and any form of redistribution of the demands is ineffective.

In summary, in the multivariate normal case we should always prefer the *perfect team* for any dimension of the set of agents, because it minimizes the probability of failure of each node and, as a consequence, the expected number of defaulted nodes.

2.2.2 Binary external demands

In this section, we model the external demands as binary random variables instead of Gaussian random variables. The homogeneity assumption in terms of intrinsic capacities and identically distributed external demands holds also in this example, but we assume that the marginal external demands are distributed according to the following probability distribution, p_S :

$$S_i = \begin{cases} 1 & w.p. \quad p \\ 0 & w.p. \quad 1 - p \end{cases} \quad (7)$$

We set the intrinsic capacities for each node to be $c < 1$ and the external demands to have very low likelihood, low p , so that in this section we focus on a particular category of external demands, namely rare but high external demands, as those following an extreme event. In Eq.(7) the original monetary unit may be any currency, e.g. dollars, we normalized it using a reference monetary amount so that the value of the shock is unitary. We assume that the external demands are also independent, thus being i.i.d. Bernoulli random variables, with positive probability of occurrence of the demand, p . Moreover, since intrinsic capacities are less than one, $c < 1$, each node defaults whenever the realized external demand is equal to 1, with probability p .

Given these assumptions, it turns out that the *perfect team* is not always the optimal solution. Indeed, for some team sizes we should prefer the completely isolated nodes configuration to any active risk-sharing mechanism, in order to minimize the marginal probability of default and, as a consequence, the expected number of defaulted nodes. For this purpose, in the binary external demands case, we compare two particular redistribution mechanisms, namely the set of isolated nodes, corresponding to $\alpha = 0$, and the total risk-sharing mechanism, namely the *perfect team*, corresponding to $\alpha = \frac{n-1}{n}$. Our goal is to choose the mechanism, among the above two, that minimizes the cost function represented by the expected number of defaulted nodes, namely $L = X$. More formally, let $\Lambda := \{0, \frac{n-1}{n}\}$:

$$\alpha^* = \arg \min_{\alpha \in \Lambda} \mathbb{E}[L] \quad (8)$$

The sum of external demands for funds is distributed as a binomial random variable, with parameters n , as the number of Bernoulli trials, and p , as probability of request for funds. In the case of the *perfect team*, each node faces the same amount of effective demand of the other nodes, a fraction equal to $\frac{1}{n}$ of the total realized external demand. Therefore, when we consider the *perfect team*, the effective demand that each node receives is equal to:

$$D = \frac{1}{n} \sum_{i=1}^n S_i \quad (9)$$

where $(\sum_{i=1}^n S_i) \sim \text{Bin}(n, p)$. Given the value of the intrinsic capacities, in the case of the *perfect team* the number of expected defaulted nodes is equal to $n \cdot P_f$, where $P_f := \mathbb{P}\{D > c\}$ is the complementary

cumulative distribution function of the random variable D evaluated at the value of the intrinsic capacity and it represents the probability of failure of each node. Note that:

$$P_f := \mathbb{P} \{ D > c \} = \mathbb{P} \left\{ \frac{1}{n} \sum_{i=1}^n S_i > c \right\} = \mathbb{P} \left\{ \sum_{i=1}^n S_i > \lfloor n \cdot c \rfloor \right\} = 1 - F_{Bin(n,p)} \lfloor n \cdot c \rfloor \quad (10)$$

where $F_{Bin(n,p)}$ is the cumulative distribution function of a binomial with parameters n and p , while $\lfloor \cdot \rfloor$ is the floor function that returns the integer part of any number to which it is applied. In the case of isolated nodes, instead, each node defaults with probability p , because $c < 1$, and the expected number of defaulted nodes is equal to $n \cdot p$, as $P_f = p$ in this case.

For given levels of capacity, c , and probability of external demand, p , we computed the probability of failure of the *perfect team*, P_f , namely $1 - F_{Bin(n,p)} \lfloor n \cdot c \rfloor$, by varying the number of agents involved in the risk-sharing mechanism. Once computed P_f for each node under the *perfect team*, we compared it to the probability of failure of each node when left alone, p . For every dimension of the set of agents, we chose $\alpha \in \Lambda$ that minimizes the marginal probability of default of the nodes.

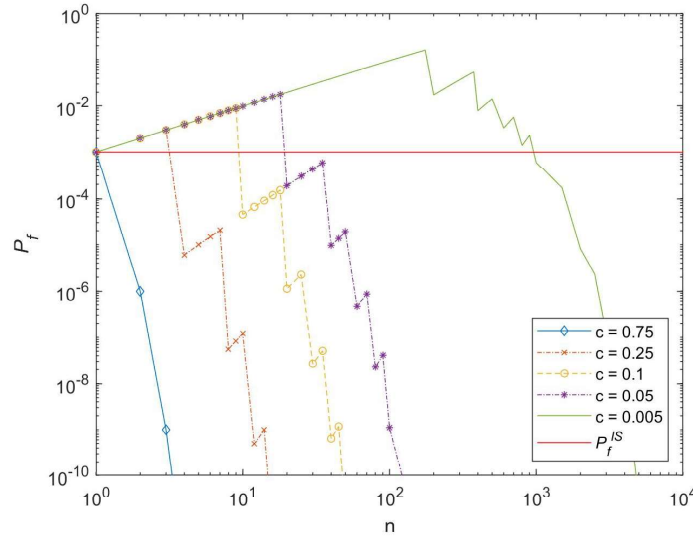


Figure 2: Probability of default of a node participating in the *perfect team* as the group dimension, n , increases, for different levels of capacity of the nodes, c , compared to that of an isolated agent. The probability of request for funds is $p = 10^{-3}$.

In Fig.(2) we study how the probability of failure of the agents in the *perfect team* varies for different levels of intrinsic capacity, represented by the colored lines, as the number of agents participating in the risk-sharing mechanism increases. The horizontal red line represents the probability of default of an agent not taking part in the mechanism, whose probability of default is a constant as n increases and is equal to probability of request for funds, $p = 10^{-3}$, because in any analyzed case $c < 1$. For those n for which the colored line is above the red line, we prefer the set of isolated nodes to the *perfect team*, because the probability of default of the nodes in the *perfect team* is greater than their probability of default if left isolated. For example, for the level of capacity corresponding to $c = 0.1$, yellow line, it is better to stay alone rather than being a part of a *perfect team* composed by other 8 agents, because 9 agents are not able to absorb the occurrence of a single external demand, while if we increase the number of participants to 10 there is an increase in the probability that the group will experience more than one external demand but this effect is totally offset by the increased pooled capacity that makes the team able to overcome to the occurrence of a single external demand. In this case, the final effect of increasing

the number of participants is that the achieved decrease in the probability of default of every node in the team leads to a shift in the preference from the isolated nodes to the *perfect team*.

If the capacity is not high enough with respect to the probability of demand, p , for finite dimensions of the *perfect team* we should prefer the set of isolated nodes to any active redistribution mechanism. Moreover, whenever the capacity of the agents decreases we need a large number of agents in the *perfect team* in order to achieve a level of diversification of the risk that allows us to prefer the *perfect team* to the isolated nodes case. Therefore, the results in terms of preferred mechanism in the binary case and in the normal case are different.

We see that for some levels of capacities, the default probability of the nodes in the *perfect team* as a function of the number of team members follows a particular pattern, it increases and decreases several times before reaching its minimum that approaches 0, attained when the number of agents goes to infinity. This particular shape of the colored lines is due to the presence of the floor function in Eq.(10). We observe that if we have an upper bound on the number of nodes allowed to enter in the *perfect team*, namely n_{max} , we should opt for teams of dimension equal to the following:

$$n^* = \arg \min_{n < n_{max}} 1 - F_{Bin(n,p)} \lfloor n \cdot c \rfloor = \arg \max_{n < n_{max}} F_{Bin(n,p)} \lfloor n \cdot c \rfloor \quad (11)$$

The solution to the above optimization problem is attained where the following is true:

$$\lfloor n^* \cdot c \rfloor = \lceil n^* \cdot c \rceil \quad (12)$$

Eq.(12) states that a solution to the problem (11), n^* , guarantees that the product $n^* \cdot c$ is an integer, namely a multiple of the external demand. Indeed, it is not convenient to form *perfect teams* of dimension n_j , namely group dimensions for which the following is true:

$$n_i < n_j < n_{i+1}, \lfloor n_i \cdot c \rfloor = \lceil n_i \cdot c \rceil, \lfloor n_{i+1} \cdot c \rfloor = \lceil n_{i+1} \cdot c \rceil, \lfloor n_{i+1} \cdot c \rfloor = \lfloor n_i \cdot c \rfloor + 1$$

Because for groups of dimension n_j , the additional $n_i - n_j$ agents do contribute to increasing the probability of the occurrence of more external demands, but do not increase the total capacity of the *perfect team* to the point that the pooled capacity is able to face an additional external demand. Therefore, the act of increasing the number of agents from n_i to any $n_j < n_{i+1}$ weakens the *perfect team* as a risk-sharing mechanism, by increasing the probability of default of the single nodes participating to the team.

It is interesting to notice how the result is different from the case of the multivariate normal external demands, even if the underlying assumptions are the same, except for the distribution of the external demands. Indeed, in the multivariate normal case, whenever the mean of the capacities is above the mean of the external demands the *perfect team* should always be preferred to any other linear redistribution mechanism, even to the set of isolated nodes, for any group dimension n . In the binary case, instead, we observe *perfect teams* of large dimension that are not preferred to the set of isolated nodes, because the marginal probability of default of their participants is not lower than the probability of default that the same agents would achieve if the redistribution mechanism was not active among them. As the probability of request for funds, p , approaches the intrinsic capacity of the nodes, the minimum group dimension for which we prefer the *perfect team* increases up to the point that we do not opt for the *perfect team* for any value of the group dimension, n , when p is equal or above the intrinsic capacity of the nodes.

For any level of capacity, we see a threshold for the team size, n , for which we pass from preferring the isolated nodes to choosing the *perfect team*. The threshold value of team size, n , is strictly related to c , because of the definition of the probability of failure in Eq. (10). The reason why in the *perfect team* the threshold is only a function of the pooled capacity relies in the fact that all the nodes of the team default on their liabilities whenever the sum of their realized external demands overpasses the sum of their individual capacities. As we increase the number of agents in a team, the distribution of the

effective demands, $\frac{1}{n} \sum_{i=1}^n S_i$, converges to a normal random variable, and its tail on the right of the value of c converges to a normal complementary distribution function evaluated at the value of capacity. At this point, if the value of the tail of the limiting distribution on the right of c is larger than p , then we prefer the isolated nodes to the *perfect team*, the *perfect team* otherwise.

3 CONCLUSIONS

In general, the idea of pooling the capacities of all agents together seems attractive. Increasing the number of agents that share a particular risk, is associated with more strength in facing uncertainties, such as external demands for funds or demand for electricity. In reality this is not always the case, especially when the external demands have low likelihood and very high value if compared to the capacity of the nodes. This is the case of the so-called *extreme events*, on which our work focuses its attention.

We saw that in the normal case the optimal result is in line with the intuition. Indeed, in this case it is possible to demonstrate that for any dimension of the team, whenever the capacities of the nodes are larger than the mean of external demands, the risk-sharing mechanism that minimizes the expected number of defaulted is the so-called *perfect team*, according to which each agent faces the average realized external demand. On the other hand, we saw that in the binary case the intuition may be misleading, because there are some finite team sizes, n , for which we prefer the set of isolated nodes to the *perfect team* as optimal risk-sharing mechanism.

Literature on financial networks is concerned about understanding when interconnectedness among financial institutions turns to determine a vulnerability to the system instead of guaranteeing a positive diversification of risk among the agents in the network. Similarly, we want to figure out what are the characteristics of the external demands for which we should prefer a risk-mechanism with the highest level of interconnectedness, as in the complete graph and maximum transferred portion of the *perfect team*, or we should opt for the mechanism with no interconnections as in the isolated nodes case.

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