



# Integrable $\mathfrak{sl}(\infty)$ -modules and category $\mathcal{O}$ for $\mathfrak{gl}(m|n)$

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## ABSTRACT

We introduce and study new categories  $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$  of integrable  $\mathfrak{g} = \mathfrak{sl}(\infty)$ -modules which depend on the choice of a certain reductive in  $\mathfrak{g}$  subalgebra  $\mathfrak{k} \subset \mathfrak{g}$ . The simple objects of  $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$  are tensor modules as in the previously studied category  $\mathbb{T}_{\mathfrak{g}}$  [Dan-Cohen, Penkov and Serganova, *Adv. Math.* 289 (2016) 250–278]; however, the choice of  $\mathfrak{k}$  provides for more flexibility of nonsimple modules in  $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$  compared to  $\mathbb{T}_{\mathfrak{g}}$ . We then choose  $\mathfrak{k}$  to have two infinite-dimensional diagonal blocks, and show that a certain injective object  $\mathbf{K}_{m|n}$  in  $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$  realizes a categorical  $\mathfrak{sl}(\infty)$ -action on the category  $\mathcal{O}_{m|n}^{\mathbb{Z}}$ , the integral category  $\mathcal{O}$  of the Lie superalgebra  $\mathfrak{gl}(m|n)$ . We show that the socle of  $\mathbf{K}_{m|n}$  is generated by the projective modules in  $\mathcal{O}_{m|n}^{\mathbb{Z}}$ , and compute the socle filtration of  $\mathbf{K}_{m|n}$  explicitly. We conjecture that the socle filtration of  $\mathbf{K}_{m|n}$  reflects a ‘degree of atypicality filtration’ on the category  $\mathcal{O}_{m|n}^{\mathbb{Z}}$ . We also conjecture that a natural tensor filtration on  $\mathbf{K}_{m|n}$  arises via the Duflo–Serganova functor sending the category  $\mathcal{O}_{m|n}^{\mathbb{Z}}$  to  $\mathcal{O}_{m-1|n-1}^{\mathbb{Z}}$ . We prove a weaker version of this latter conjecture for the direct summand of  $\mathbf{K}_{m|n}$  corresponding to finite-dimensional  $\mathfrak{gl}(m|n)$ -modules.

## 1. Introduction

Categorification has set a trend in mathematics in the last two decades and has proved important and useful. The opposite process of studying a given category via a combinatorial or algebraic object such as a single module has also borne ample fruit. An example is Brundan’s idea from 2003 to study the category  $\mathcal{F}_{m|n}^{\mathbb{Z}}$  of finite-dimensional integral modules over the Lie superalgebra  $\mathfrak{gl}(m|n)$  via the weight structure of the  $\mathfrak{sl}(\infty)$ -module  $\Lambda^m \mathbf{V} \otimes \Lambda^n \mathbf{V}_*$ , where  $\mathbf{V}$  and  $\mathbf{V}_*$  are the two nonisomorphic defining (natural) representations of  $\mathfrak{sl}(\infty)$ . Using this approach Brundan computes decomposition numbers in  $\mathcal{F}_{m|n}^{\mathbb{Z}}$  [1]. An extension of Brundan’s approach was proposed in the work of Brundan, Losev and Webster in [2], where a new proof of the Brundan–Kazhdan–Lusztig conjecture for the category  $\mathcal{O}$  over the Lie superalgebra  $\mathfrak{gl}(m|n)$  is given. (The first proof of the Brundan–Kazhdan–Lusztig conjecture for the category  $\mathcal{O}$  over the Lie superalgebra  $\mathfrak{gl}(m|n)$  was given by Cheng, Lam and Wang in [4].) The same approach was also used by Brundan and Stroppel in [3], where the algebra of endomorphisms of a projective generator in  $\mathcal{F}_{m|n}^{\mathbb{Z}}$  is described as a certain diagram algebra and the Koszulity of  $\mathcal{F}_{m|n}^{\mathbb{Z}}$  is established.

The representation theory of the Lie algebra  $\mathfrak{sl}(\infty)$  is of independent interest and has been developing actively also for about two decades. In particular, several categories of  $\mathfrak{sl}(\infty)$ -modules have been singled out and studied in detail, see [8, 9, 15–17].

The category  $\mathbb{T}_{\mathfrak{sl}(\infty)}$  from [8] has been playing a prominent role: its objects are finite-length submodules of a direct sum of several copies of the tensor algebra  $T(\mathbf{V} \oplus \mathbf{V}_*)$ . In [8] it is proved that  $\mathbb{T}_{\mathfrak{sl}(\infty)}$  is a self-dual Koszul category, in [18] it has been shown that  $\mathbb{T}_{\mathfrak{sl}(\infty)}$  has a universality property, and in [11]  $\mathbb{T}_{\mathfrak{sl}(\infty)}$  has been used to categorify the Boson–Fermion correspondence.

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Our goal in the present paper is to find an appropriate category of  $\mathfrak{sl}(\infty)$ -modules which contains modules relevant to the representation theory of the Lie superalgebras  $\mathfrak{gl}(m|n)$ . For this purpose, we introduce and study the categories  $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ , where  $\mathfrak{g} = \mathfrak{sl}(\infty)$  and  $\mathfrak{k}$  is a reductive subalgebra of  $\mathfrak{g}$  containing the diagonal subalgebra and consisting of finitely many blocks along the diagonal. The Lie algebra  $\mathfrak{k}$  is infinite dimensional and is itself isomorphic to the commutator subalgebra of a finite direct sum of copies of  $\mathfrak{gl}(n)$  (for varying  $n$ ) and copies of  $\mathfrak{gl}(\infty)$ . When  $\mathfrak{k} = \mathfrak{g}$ , this new category coincides with  $\mathbb{T}_{\mathfrak{g}}$ . A well-known property of the category  $\mathbb{T}_{\mathfrak{g}}$  states that for every  $\mathbf{M} \in \mathbb{T}_{\mathfrak{g}}$ , any vector  $m \in \mathbf{M}$  is annihilated by a ‘large’ subalgebra  $\mathfrak{g}' \subset \mathfrak{g}$ , that is, by an algebra which contains the commutator subalgebra of the centralizer of a finite-dimensional subalgebra  $\mathfrak{s} \subset \mathfrak{g}$ . For a general  $\mathfrak{k}$  as above, the category  $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$  has the same simple objects as  $\mathbb{T}_{\mathfrak{g}}$  but requires the following for a nonsimple module  $\mathbf{M}$ : the annihilator in  $\mathfrak{k}$  of every  $m \in \mathbf{M}$  is a large subalgebra of  $\mathfrak{k}$ . This makes the nonsimple objects of  $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$  more ‘flexible’ than in those of  $\mathbb{T}_{\mathfrak{g}}$ , the degree of flexibility being governed by  $\mathfrak{k}$ .

In Section 2, we study the category  $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$  in detail, one of our main results being an explicit computation of the socle filtration of an indecomposable injective object  $\mathbf{I}^{\lambda,\mu}$  of  $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$  (where  $\lambda$  and  $\mu$  are two Young diagrams), see Theorem 20. An effect which can be observed here is that with a sufficient increase in the number of infinite blocks of  $\mathfrak{k}$ , the layers of the socle filtration of  $\mathbf{I}^{\lambda,\mu}$  grow in a ‘self-similar’ manner. This shows that  $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$  is an intricate extension of the category  $\mathbb{T}_{\mathfrak{g}}$  within the category of all integrable  $\mathfrak{g}$ -modules.

In Section 3, we show that studying the category  $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$  achieves our goal of improving the understanding of the integral category  $\mathcal{O}_{m|n}^{\mathbb{Z}}$  for the Lie superalgebra  $\mathfrak{gl}(m|n)$ . More precisely, we choose  $\mathfrak{k}$  to have two blocks, both of them infinite. Then we show that the category  $\mathcal{O}_{m|n}^{\mathbb{Z}}$  is a categorification of an injective object  $\mathbf{K}_{m|n}$  in the category  $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ . In order to accomplish this, we exploit the properties of  $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$  as a category, and not just as a collection of modules. The object  $\mathbf{K}_{m|n}$  of  $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$  can be defined as the complexified reduced Grothendieck group of the category  $\mathcal{O}_{m|n}^{\mathbb{Z}}$ , endowed with an  $\mathfrak{sl}(\infty)$ -module structure (categorical action of  $\mathfrak{sl}(\infty)$ ). For  $m, n \geq 1$ ,  $\mathbf{K}_{m|n}$  is an object of  $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ , but not of  $\mathbb{T}_{\mathfrak{g}}$ . We prove that the socle of  $\mathbf{K}_{m|n}$  as an  $\mathfrak{sl}(\infty)$ -module is the submodule generated by classes of projective  $\mathfrak{gl}(m|n)$ -modules in  $\mathcal{O}_{m|n}^{\mathbb{Z}}$ . Moreover, we conjecture that the socle filtration of  $\mathbf{K}_{m|n}$  (which we already know from Section 2) arises from filtering the category  $\mathcal{O}_{m|n}^{\mathbb{Z}}$  according to the degree of atypicality of  $\mathfrak{gl}(m|n)$ -modules. We provide some partial evidence toward this conjecture.

We also show that the category  $\mathcal{F}_{m|n}^{\mathbb{Z}}$  of finite-dimensional integral  $\mathfrak{gl}(m|n)$ -modules categorifies a direct summand  $\mathbf{J}_{m|n}$  of  $\mathbf{K}_{m|n}$  which is nothing but an injective hull in  $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$  of Brundan’s module  $\Lambda^m \mathbf{V} \otimes \Lambda^n \mathbf{V}_*$ , see Corollary 28. (Note that the module  $\Lambda^m \mathbf{V} \otimes \Lambda^n \mathbf{V}_*$  is an injective object of  $\mathbb{T}_{\mathfrak{g}}$ , but is not injective in  $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$  when  $\mathfrak{k}$  has two (or more) infinite blocks.)

Finally, we conjecture that a natural filtration on the category  $\mathcal{O}_{m|n}^{\mathbb{Z}}$  defined via the Duflo–Serganova functor  $DS : \mathcal{O}_{m|n}^{\mathbb{Z}} \rightarrow \mathcal{O}_{m-1|n-1}^{\mathbb{Z}}$  categorifies the tensor filtration of  $\mathbf{K}_{m|n}$ , that is, the coarsest filtration of  $\mathbf{K}_{m|n}$  whose successive quotients are objects of  $\mathbb{T}_{\mathfrak{g}}$ . We have a similar conjecture for the direct summand  $\mathbf{J}_{m|n}$  of  $\mathbf{K}_{m|n}$ , and we provide evidence for this conjecture in Proposition 42.

## 2. New categories of integrable $\mathfrak{sl}(\infty)$ -modules

### 2.1. Preliminaries

Let  $\mathbf{V}$  and  $\mathbf{V}_*$  be countable-dimensional vector spaces with fixed bases  $\{v_i\}_{i \in \mathbb{Z}}$  and  $\{v_j^*\}_{j \in \mathbb{Z}}$ , together with a nondegenerate pairing  $\langle \cdot, \cdot \rangle : \mathbf{V} \otimes \mathbf{V}_* \rightarrow \mathbb{C}$  defined by  $\langle v_i, v_j^* \rangle = \delta_{ij}$ . Then  $\mathfrak{gl}(\infty) := \mathbf{V} \otimes \mathbf{V}_*$  has a Lie algebra structure such that

$$[v_i \otimes v_j^*, v_k \otimes v_l^*] = \langle v_k, v_j^* \rangle v_i \otimes v_l^* - \langle v_i, v_l^* \rangle v_k \otimes v_j^*.$$

We can identify  $\mathfrak{gl}(\infty)$  with the space of infinite matrices  $(a_{ij})_{i,j \in \mathbb{Z}}$  with finitely many nonzero entries, where the vector  $v_i \otimes v_j^*$  corresponds to the matrix  $E_{ij}$  with 1 in the  $i, j$ -position and zeros elsewhere. Then  $\langle \cdot, \cdot \rangle$  corresponds to the trace map, and its kernel is the Lie algebra  $\mathfrak{sl}(\infty)$ , which is generated by  $e_i := E_{i,i+1}, f_i := E_{i+1,i}$  with  $i \in \mathbb{Z}$ . One can also realize  $\mathfrak{sl}(\infty)$  as a direct limit of finite-dimensional Lie algebras  $\mathfrak{sl}(\infty) = \varinjlim \mathfrak{sl}(n)$ . In contrast to the finite-dimensional setting, the exact sequence

$$0 \rightarrow \mathfrak{sl}(\infty) \rightarrow \mathfrak{gl}(\infty) \rightarrow \mathbb{C} \rightarrow 0$$

does not split, and the center of  $\mathfrak{gl}(\infty)$  is trivial.

Let  $\mathfrak{g} = \mathfrak{sl}(\infty)$ . The representations  $\mathbf{V}$  and  $\mathbf{V}_*$  are the defining representations of  $\mathfrak{g}$ . The tensor representations  $\mathbf{V}^{\otimes p} \otimes \mathbf{V}_*^{\otimes q}$ ,  $p, q \in \mathbb{Z}_{\geq 0}$  have been studied in [17]. They are not semisimple when  $p, q > 0$ ; however, each simple subquotient of  $\mathbf{V}^{\otimes p} \otimes \mathbf{V}_*^{\otimes q}$  occurs as a submodule of  $\mathbf{V}^{\otimes p'} \otimes \mathbf{V}_*^{\otimes q'}$  for some  $p', q'$ . The simple submodules of  $\mathbf{V}^{\otimes p} \otimes \mathbf{V}_*^{\otimes q}$  can be parameterized by two Young diagrams  $\lambda, \mu$ , and we denote them  $\mathbf{V}^{\lambda, \mu}$ .

Recall that the *socle* of a module  $\mathbf{M}$ , denoted  $\text{soc } \mathbf{M}$ , is the largest semisimple submodule of  $\mathbf{M}$ . The *socle filtration* of  $\mathbf{M}$  is defined inductively by  $\text{soc}^0 \mathbf{M} := \text{soc } \mathbf{M}$  and  $\text{soc}^i \mathbf{M} := p_i^{-1}(\text{soc}(\mathbf{M}/(\text{soc}^{i-1} \mathbf{M})))$ , where  $p_i : \mathbf{M} \rightarrow \mathbf{M}/(\text{soc}^{i-1} \mathbf{M})$  is the natural projection. We also use the notation  $\overline{\text{soc}}^i \mathbf{M} := \text{soc}^i \mathbf{M} / \text{soc}^{i-1} \mathbf{M}$  for the layers of the socle filtration.

Schur–Weyl duality for  $\mathfrak{sl}(\infty)$  implies that the module  $\mathbf{V}^{\otimes p} \otimes \mathbf{V}_*^{\otimes q}$  decomposes as

$$\mathbf{V}^{\otimes p} \otimes \mathbf{V}_*^{\otimes q} = \bigoplus_{|\lambda|=p, |\mu|=q} (\mathbb{S}_\lambda(\mathbf{V}) \otimes \mathbb{S}_\mu(\mathbf{V}_*)) \otimes (Y_\lambda \otimes Y_\mu), \tag{2.1}$$

where  $Y_\lambda$  and  $Y_\mu$  are irreducible  $S_p$ - and  $S_q$ -modules, and  $\mathbb{S}_\lambda$  denotes the Schur functor corresponding to the Young diagram (equivalently, partition)  $\lambda$ . Each module  $\mathbb{S}_\lambda(\mathbf{V}) \otimes \mathbb{S}_\mu(\mathbf{V}_*)$  is indecomposable and its socle filtration is described in [17]. Moreover, [17, Theorem 2.3] claims that

$$\overline{\text{soc}}^k(\mathbb{S}_\lambda(\mathbf{V}) \otimes \mathbb{S}_\mu(\mathbf{V}_*)) \cong \bigoplus_{\lambda', \mu', |\gamma|=k} N_{\lambda', \gamma}^\lambda N_{\mu', \gamma}^\mu \mathbf{V}^{\lambda', \mu'} \tag{2.2}$$

where  $N_{\lambda', \gamma}^\lambda$  are the standard Littlewood–Richardson coefficients. In particular,  $\mathbb{S}_\lambda(\mathbf{V}) \otimes \mathbb{S}_\mu(\mathbf{V}_*)$  has simple socle  $\mathbf{V}^{\lambda, \mu}$ . It was also shown in [17, Theorem 2.2] that the socle of  $\mathbf{V}^{\otimes p} \otimes \mathbf{V}_*^{\otimes q}$  equals the intersection of the kernels of all contraction maps

$$\Phi_{ij} : \mathbf{V}^{\otimes p} \otimes \mathbf{V}_*^{\otimes q} \rightarrow \mathbf{V}^{\otimes(p-1)} \otimes \mathbf{V}_*^{\otimes(q-1)} \tag{2.3}$$

$$v_1 \otimes \cdots \otimes v_p \otimes v_1^* \otimes \cdots \otimes v_q^* \mapsto \langle v_j^*, v_i \rangle v_1 \otimes \cdots \otimes \widehat{v_i} \otimes \cdots \otimes v_p \otimes v_1^* \otimes \cdots \otimes \widehat{v_j^*} \otimes \cdots \otimes v_q^*.$$

A  $\mathfrak{g}$ -module is called a *tensor module* if it is isomorphic to a submodule of a finite direct sum of  $\mathfrak{sl}(\infty)$ -modules of the form  $\mathbf{V}^{\otimes p_i} \otimes \mathbf{V}_*^{\otimes q_i}$  for  $p_i, q_i \in \mathbb{Z}_{\geq 0}$ . The category of tensor modules  $\mathbb{T}_{\mathfrak{g}}$  is by definition the full subcategory of  $\mathfrak{g}\text{-mod}$  consisting of tensor modules [8]. A finite-length  $\mathfrak{g}$ -module  $\mathbf{M}$  lies in  $\mathbb{T}_{\mathfrak{g}}$  if and only if  $\mathbf{M}$  is integrable and satisfies the large annihilator condition [8]. Recall that a  $\mathfrak{g}$ -module  $\mathbf{M}$  is called *integrable* if  $\dim\{m, x \cdot m, x^2 \cdot m, \dots\} < \infty$  for any  $x \in \mathfrak{g}, m \in \mathbf{M}$ . A  $\mathfrak{g}$ -module is said to satisfy the *large annihilator condition* if for each  $m \in \mathbf{M}$ , the annihilator  $\text{Ann}_{\mathfrak{g}} m$  contains the commutator subalgebra of the centralizer of a finite-dimensional subalgebra of  $\mathfrak{g}$ .

The modules  $\mathbf{V}^{\otimes p} \otimes \mathbf{V}_*^{\otimes q}$ ,  $p, q \in \mathbb{Z}_{\geq 0}$  are injective in the category  $\mathbb{T}_{\mathfrak{g}}$ . Moreover, every indecomposable injective object of  $\mathbb{T}_{\mathfrak{g}}$  is isomorphic to an indecomposable direct summand of  $\mathbf{V}^{\otimes p} \otimes \mathbf{V}_*^{\otimes q}$  for some  $p, q \in \mathbb{Z}_{\geq 0}$  [8]. Consequently, by (2.1), an indecomposable injective in  $\mathbb{T}_{\mathfrak{g}}$  is isomorphic to  $\mathbb{S}_\lambda(\mathbf{V}) \otimes \mathbb{S}_\mu(\mathbf{V}_*)$  for some  $\lambda, \mu$ .

The category  $\mathbb{T}_{\mathfrak{g}}$  is a subcategory of the category  $\widetilde{\text{Tens}}_{\mathfrak{g}}$ , which was introduced in [16] as the full subcategory of  $\mathfrak{g}\text{-mod}$  whose objects  $\mathbf{M}$  are defined to be the integrable  $\mathfrak{g}$ -modules of finite

Loewy length such that the algebraic dual  $\mathbf{M}^* = \text{Hom}_{\mathbb{C}}(\mathbf{M}, \mathbb{C})$  is also integrable and of finite Loewy length. The categories  $\mathbb{T}_{\mathfrak{g}}$  and  $\widetilde{Tens}_{\mathfrak{g}}$  have the same simple objects  $\mathbf{V}^{\lambda, \mu}$  [8, 16]. The indecomposable injective objects of  $\widetilde{Tens}_{\mathfrak{g}}$  are (up to isomorphism) the modules  $(\mathbf{V}^{\mu, \lambda})^*$ , and  $\text{soc}(\mathbf{V}^{\mu, \lambda})^* \cong \mathbf{V}^{\lambda, \mu}$  [16]. A recent result of [6] shows that the Grothendieck envelope  $\overline{Tens}_{\mathfrak{g}}$  of  $\widetilde{Tens}_{\mathfrak{g}}$  is an ordered tensor category, and that any injective object in  $\overline{Tens}_{\mathfrak{g}}$  is a direct sum of indecomposable injectives from  $\widetilde{Tens}_{\mathfrak{g}}$ .

2.2. *The categories  $\mathbb{T}_{\mathfrak{g}, \mathfrak{k}}$*

In this section, we introduce new categories of integrable  $\mathfrak{sl}(\infty)$ -modules. This is motivated in part by the applications to the representation theory of the Lie superalgebras  $\mathfrak{gl}(m|n)$ .

Let  $\mathfrak{g} = \mathfrak{sl}(\infty)$  with the natural representation denoted  $\mathbf{V}$ . Consider a decomposition

$$\mathbf{V} = \mathbf{V}_1 \oplus \cdots \oplus \mathbf{V}_r, \tag{2.4}$$

for some vector subspaces  $\mathbf{V}_i$  of  $\mathbf{V}$ . Let  $\mathfrak{l}$  be the Lie subalgebra of  $\mathfrak{g}$  preserving this decomposition. Then  $\mathfrak{k} := [\mathfrak{l}, \mathfrak{l}]$  is isomorphic to  $\mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_r$ , where each  $\mathfrak{k}_i$  is isomorphic to  $\mathfrak{sl}(n_i)$  or  $\mathfrak{sl}(\infty)$ .

DEFINITION 1. Denote by  $\widetilde{\mathbb{T}}_{\mathfrak{g}, \mathfrak{k}}$  the full subcategory of  $\widetilde{Tens}_{\mathfrak{g}}$  consisting of modules  $\mathbf{M}$  satisfying the large annihilator condition as a module over  $\mathfrak{k}_i$  for all  $i = 1, \dots, r$ . By  $\mathbb{T}_{\mathfrak{g}, \mathfrak{k}}$  we denote the full subcategory of  $\widetilde{\mathbb{T}}_{\mathfrak{g}, \mathfrak{k}}$  consisting of finite-length modules.

Both categories  $\mathbb{T}_{\mathfrak{g}, \mathfrak{k}}$  and  $\widetilde{\mathbb{T}}_{\mathfrak{g}, \mathfrak{k}}$  are abelian symmetric monoidal categories with respect to the usual tensor product of  $\mathfrak{g}$ -modules. Two categories  $\widetilde{\mathbb{T}}_{\mathfrak{g}, \mathfrak{k}}$  and  $\widetilde{\mathbb{T}}_{\mathfrak{g}, \bar{\mathfrak{k}}}$  are equal if  $\mathfrak{k}$  and  $\bar{\mathfrak{k}}$  have finite corank in  $\mathfrak{k} + \bar{\mathfrak{k}}$ , so we will henceforth assume without loss of generality that each  $\mathbf{V}_i$  in decomposition (2.4) is infinite dimensional. Note that  $\mathbb{T}_{\mathfrak{g}, \mathfrak{g}} = \mathbb{T}_{\mathfrak{g}}$ .

We define the functor  $\Gamma_{\mathfrak{g}, \mathfrak{k}} : \widetilde{Tens}_{\mathfrak{g}} \rightarrow \widetilde{\mathbb{T}}_{\mathfrak{g}, \mathfrak{k}}$  by taking the maximal submodule lying in  $\widetilde{\mathbb{T}}_{\mathfrak{g}, \mathfrak{k}}$ . Then

$$\Gamma_{\mathfrak{g}, \mathfrak{k}}(\mathbf{M}) = \bigcup \mathbf{M}^{\mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_r}, \tag{2.5}$$

where the union is taken over all finite corank subalgebras  $\mathfrak{s}_1 \subset \mathfrak{k}_1, \dots, \mathfrak{s}_r \subset \mathfrak{k}_r$ .

LEMMA 2. *Let  $\mathbb{T}_{\mathfrak{g}, \mathfrak{k}}$  be as in Definition 1.*

- (1) *The simple objects of  $\mathbb{T}_{\mathfrak{g}, \mathfrak{k}}$  and of  $\widetilde{\mathbb{T}}_{\mathfrak{g}, \mathfrak{k}}$  are isomorphic to  $\mathbf{V}^{\lambda, \mu}$ .*
- (2) *The functor  $\Gamma_{\mathfrak{g}, \mathfrak{k}}$  sends injective modules in  $\widetilde{Tens}_{\mathfrak{g}}$  to injective modules in  $\widetilde{\mathbb{T}}_{\mathfrak{g}, \mathfrak{k}}$ .*
- (3) *The category  $\widetilde{\mathbb{T}}_{\mathfrak{g}, \mathfrak{k}}$  has enough injective modules.*
- (4) *The indecomposable injective objects of  $\widetilde{\mathbb{T}}_{\mathfrak{g}, \mathfrak{k}}$  are isomorphic to  $\Gamma_{\mathfrak{g}, \mathfrak{k}}((\mathbf{V}^{\mu, \lambda})^*)$ .*

*Proof.* (1) The category  $\mathbb{T}_{\mathfrak{g}}$  is a full subcategory of  $\mathbb{T}_{\mathfrak{g}, \mathfrak{k}}$  and of  $\widetilde{\mathbb{T}}_{\mathfrak{g}, \mathfrak{k}}$ , which are both full subcategories of  $\widetilde{Tens}_{\mathfrak{g}}$ . Since the categories  $\mathbb{T}_{\mathfrak{g}}$  and  $\widetilde{Tens}_{\mathfrak{g}}$  have the same simple objects  $\mathbf{V}^{\lambda, \mu}$ , the claim follows.

(2) This follows from the definition of  $\Gamma_{\mathfrak{g}, \mathfrak{k}}$ , since  $\text{Hom}_{\mathbb{T}_{\mathfrak{g}, \mathfrak{k}}}(X, \Gamma_{\mathfrak{g}, \mathfrak{k}}(Y)) = \text{Hom}_{\widetilde{Tens}_{\mathfrak{g}}}(X, Y)$  for all  $X \in \mathbb{T}_{\mathfrak{g}, \mathfrak{k}}$  and  $Y \in \widetilde{Tens}_{\mathfrak{g}}$ .

(3) Every module  $\mathbf{M}$  in  $\widetilde{\mathbb{T}}_{\mathfrak{g}, \mathfrak{k}}$  can be embedded into  $\Gamma_{\mathfrak{g}, \mathfrak{k}}(\mathbf{M}^{**})$ , which is injective in  $\widetilde{\mathbb{T}}_{\mathfrak{g}, \mathfrak{k}}$ , since  $\mathbf{M}^{**}$  is injective in  $\widetilde{Tens}_{\mathfrak{g}}$  [16].

(4) This follows from (1) and (2), since  $(\mathbf{V}^{\mu,\lambda})^*$  is an indecomposable injective object of  $\widetilde{Tens}_{\mathfrak{g}}$ , and consequently  $\Gamma_{\mathfrak{g},\mathfrak{k}}((\mathbf{V}^{\mu,\lambda})^*)$  is an indecomposable injective object of  $\widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}}$  with  $\text{soc } \Gamma_{\mathfrak{g},\mathfrak{k}}((\mathbf{V}^{\mu,\lambda})^*) \cong \mathbf{V}^{\lambda,\mu}$ .  $\square$

REMARK 3. It will follow from Corollary 12 that the indecomposable injective objects  $\Gamma_{\mathfrak{g},\mathfrak{k}}((\mathbf{V}^{\mu,\lambda})^*)$  are objects of  $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ . Consequently,  $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$  and  $\widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}}$  have the same indecomposable injectives.

2.3. *The functor R and Jordan–Hölder multiplicities*

In this section, we calculate the Jordan–Hölder multiplicities of the indecomposable injective objects of the categories  $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ . One of the main tools we use for this computation is the functor R, which we will now introduce.

Let

$$\mathbf{V}' = \mathbf{V}_1 \oplus \cdots \oplus \mathbf{V}_{r-1}, \quad \mathfrak{g}' = \mathfrak{g} \cap \mathfrak{gl}(\mathbf{V}'), \quad \mathfrak{k}' = \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_{r-1}. \tag{2.6}$$

Let  $(\mathbf{V}_r)_* \subset \mathbf{V}_*$  be the annihilator of  $\mathbf{V}' = \mathbf{V}_1 \oplus \cdots \oplus \mathbf{V}_{r-1}$  with respect to the pairing  $\langle \cdot, \cdot \rangle$ . We have  $\mathfrak{g}' \cong \mathfrak{sl}(\infty)$  and  $\mathfrak{k}' \subset \mathfrak{k}'$ .

Define a functor R from the category  $\mathfrak{g}\text{-mod}$  of all  $\mathfrak{g}$ -modules to the category  $\mathfrak{g}'\text{-mod}$  by setting

$$R(\mathbf{M}) = \mathbf{M}^{\mathfrak{k}'}$$

It follows from the definition that after restricting to  $\widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}}$  we have a functor  $R : \widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}} \rightarrow \widetilde{\mathbb{T}}_{\mathfrak{g}',\mathfrak{k}'}$ .

LEMMA 4. *The following diagram of functors is commutative:*

$$\begin{array}{ccc} \mathfrak{g}\text{-mod} & \xrightarrow{R} & \mathfrak{g}'\text{-mod} \\ \Gamma_{\mathfrak{g},\mathfrak{k}} \downarrow & & \Gamma_{\mathfrak{g}',\mathfrak{k}'} \downarrow \\ \widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}} & \xrightarrow{R} & \widetilde{\mathbb{T}}_{\mathfrak{g}',\mathfrak{k}'} \end{array} .$$

*Proof.* By (2.5) we have

$$\Gamma_{\mathfrak{g},\mathfrak{k}}(\mathbf{M}) = \bigcup \mathbf{M}^{\mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_r}$$

for any  $\mathfrak{g}$ -module  $\mathbf{M}$ . Then

$$R(\Gamma_{\mathfrak{g},\mathfrak{k}}(\mathbf{M})) = (\bigcup \mathbf{M}^{\mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_r})^{\mathfrak{k}'} = \bigcup \mathbf{M}^{\mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_{r-1} \oplus \mathfrak{k}'} = \bigcup (R(\mathbf{M}))^{\mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_{r-1}} = \Gamma_{\mathfrak{g}',\mathfrak{k}'}(R(\mathbf{M})).$$

$\square$

LEMMA 5. *If  $\lambda, \mu$  are Young diagrams, then*

$$R((\mathbb{S}_{\lambda}(\mathbf{V}) \otimes \mathbb{S}_{\mu}(\mathbf{V}_*))^*) = \bigoplus_{\lambda', \mu', \gamma} N_{\lambda', \gamma}^{\lambda} N_{\mu', \gamma}^{\mu} (\mathbb{S}_{\lambda'}(R(\mathbf{V})) \otimes \mathbb{S}_{\mu'}(R(\mathbf{V}_*)))^*.$$

*Proof.* Since  $R(\mathbf{V}) = \mathbf{V}'$ , we have the decompositions

$$\mathbf{V} = R(\mathbf{V}) \oplus \mathbf{V}_r, \quad \mathbf{V}_* = R(\mathbf{V}_*) \oplus (\mathbf{V}_r)_*.$$

We also have the identity

$$\mathbb{S}_{\lambda}(V \oplus W) = \bigoplus N_{\mu, \nu}^{\lambda} \mathbb{S}_{\mu}(V) \otimes \mathbb{S}_{\nu}(W), \tag{2.7}$$

which holds for all vector spaces  $V$  and  $W$ . These imply

$$\mathbb{S}_\lambda(\mathbf{V}) \otimes \mathbb{S}_\mu(\mathbf{V}_*) = \bigoplus_{\lambda', \mu', \gamma, \gamma'} N_{\lambda', \gamma}^\lambda N_{\mu', \gamma'}^\mu \mathbb{S}_{\lambda'}(\mathbf{R}(\mathbf{V})) \otimes \mathbb{S}_\gamma(\mathbf{V}_r) \otimes \mathbb{S}_{\mu'}(\mathbf{R}(\mathbf{V}_*)) \otimes \mathbb{S}_{\gamma'}((\mathbf{V}_r)_*).$$

By definition

$$\mathbf{R}((\mathbb{S}_\lambda(\mathbf{V}) \otimes \mathbb{S}_\mu(\mathbf{V}_*))^*) = \text{Hom}_{\mathfrak{g}'}(\mathbb{S}_\lambda(\mathbf{V}) \otimes \mathbb{S}_\mu(\mathbf{V}_*), \mathbb{C}),$$

and it follows from (2.2) that

$$\dim \text{Hom}_{\mathfrak{g}'}(\mathbb{S}_\gamma(\mathbf{V}_r) \otimes \mathbb{S}_{\gamma'}((\mathbf{V}_r)_*), \mathbb{C}) = \delta_{\gamma, \gamma'},$$

$\delta_{\gamma, \gamma'}$  being Kronecker's delta. Therefore,

$$\text{Hom}_{\mathfrak{g}'}(\mathbb{S}_\lambda(\mathbf{V}) \otimes \mathbb{S}_\mu(\mathbf{V}_*), \mathbb{C}) = \bigoplus_{\lambda', \mu', \gamma} N_{\lambda', \gamma}^\lambda N_{\mu', \gamma}^\mu (\mathbb{S}_{\lambda'}(\mathbf{R}(\mathbf{V})) \otimes \mathbb{S}_{\mu'}(\mathbf{R}(\mathbf{V}_*)))^*. \quad \square$$

LEMMA 6. *If  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  is an exact sequence of modules in  $\widetilde{\text{Tens}}_{\mathfrak{g}}$ , then the dual exact sequence  $0 \rightarrow C^* \rightarrow B^* \rightarrow A^* \rightarrow 0$  splits.*

*Proof.* This follows from the fact that  $C^*$  is injective in  $\widetilde{\text{Tens}}_{\mathfrak{g}}$ . □

LEMMA 7. *The functor  $\mathbf{R} : \widetilde{\mathbb{T}}_{\mathfrak{g}, \mathfrak{k}} \rightarrow \widetilde{\mathbb{T}}_{\mathfrak{g}', \mathfrak{k}'}$  sends an indecomposable injective object to an injective object.*

*Proof.* Let  $\mathbf{P}^{\lambda, \mu} = \Gamma_{\mathfrak{g}, \mathfrak{k}}((\mathbb{S}_\lambda(\mathbf{V}) \otimes \mathbb{S}_\mu(\mathbf{V}_*))^*)$ . Then by Lemma 4 we have

$$\mathbf{R}(\mathbf{P}^{\lambda, \mu}) = \Gamma_{\mathfrak{g}', \mathfrak{k}'}(\mathbf{R}((\mathbb{S}_\lambda(\mathbf{V}) \otimes \mathbb{S}_\mu(\mathbf{V}_*))^*)),$$

and hence by Lemma 5

$$\mathbf{R}(\mathbf{P}^{\lambda, \mu}) = \bigoplus_{\lambda', \mu', \gamma} N_{\lambda', \gamma}^\lambda N_{\mu', \gamma}^\mu \Gamma_{\mathfrak{g}', \mathfrak{k}'}((\mathbb{S}_{\lambda'}(\mathbf{R}(\mathbf{V})) \otimes \mathbb{S}_{\mu'}(\mathbf{R}(\mathbf{V}_*)))^*). \quad (2.8)$$

Therefore,  $\mathbf{R}(\mathbf{P}^{\lambda, \mu})$  is injective in  $\widetilde{\mathbb{T}}_{\mathfrak{g}', \mathfrak{k}'}$ . Every indecomposable injective object in  $\widetilde{\mathbb{T}}_{\mathfrak{g}, \mathfrak{k}}$  is isomorphic to  $\Gamma_{\mathfrak{g}, \mathfrak{k}}(\mathbf{L}^*)$  for some simple object  $\mathbf{L} = \mathbf{V}^{\lambda, \mu}$ , and by Lemma 6,  $\Gamma_{\mathfrak{g}, \mathfrak{k}}(\mathbf{L}^*)$  is a direct summand of  $\mathbf{P}^{\lambda, \mu} = \Gamma_{\mathfrak{g}, \mathfrak{k}}((\mathbb{S}_\lambda(\mathbf{V}) \otimes \mathbb{S}_\mu(\mathbf{V}_*))^*)$ . Since the functor  $\mathbf{R}$  is left exact,  $\mathbf{R}(\Gamma_{\mathfrak{g}, \mathfrak{k}}(\mathbf{L}^*))$  is a direct summand of  $\mathbf{R}(\mathbf{P}^{\lambda, \mu})$ . Hence,  $\mathbf{R}(\Gamma_{\mathfrak{g}, \mathfrak{k}}(\mathbf{L}^*))$  is injective in  $\widetilde{\mathbb{T}}_{\mathfrak{g}', \mathfrak{k}'}$ . □

LEMMA 8. *Let  $\mathbf{V} = V_n \oplus \mathbf{W}$  and  $\mathbf{V}_* = V_n^* \oplus \mathbf{W}_*$  be decompositions with  $\dim V_n = n$ ,  $\mathbf{W}^\perp = V_n^*$ , and  $\mathbf{W}_*^\perp = V_n$ . Let  $\mathfrak{s}$  be the commutator subalgebra of  $\mathbf{W} \otimes \mathbf{W}_*$ . Let  $\mathbf{M} \in \mathbb{T}_{\mathfrak{g}}$  be a module such that all its simple constituents are of the form  $\mathbf{V}^{\lambda, \mu}$  with  $|\lambda| + |\mu| \leq n$ . Then the length of  $\mathbf{M}^\mathfrak{s}$  in the category of  $\mathfrak{sl}(n)$ -modules equals the length of  $\mathbf{M}$  in  $\mathbb{T}_{\mathfrak{g}}$ .*

*Proof.* It follows from (2.7) and the fact that  $\mathbb{S}_\lambda(V_n)$  and  $\mathbb{S}_\mu(V_n^*)$  are nonzero (since  $\dim V_n \geq |\lambda|, |\mu|$ ) that

$$(\mathbb{S}_\lambda(\mathbf{V}) \otimes \mathbb{S}_\mu(\mathbf{V}_*))^\mathfrak{s} = \mathbb{S}_\lambda(V_n) \otimes \mathbb{S}_\mu(V_n^*).$$

The description of the layers of the socle filtration of  $\mathbb{S}_\lambda(\mathbf{V}) \otimes \mathbb{S}_\mu(\mathbf{V}_*)$  in (2.2) shows that the length of  $\mathbb{S}_\lambda(\mathbf{V}) \otimes \mathbb{S}_\mu(\mathbf{V}_*)$  equals the length of  $\mathbb{S}_\lambda(V_n) \otimes \mathbb{S}_\mu(V_n^*)$ . Furthermore, since the socle  $\mathbf{V}^{\lambda, \mu}$  of  $\mathbb{S}_\lambda(\mathbf{V}) \otimes \mathbb{S}_\mu(\mathbf{V}_*)$  coincides with the set of vectors annihilated by all contraction maps (see (2.3)), and the set of vectors in  $\mathbb{S}_\lambda(V_n) \otimes \mathbb{S}_\mu(V_n^*)$  annihilated by all contraction maps is the simple  $\mathfrak{sl}(n)$ -module  $V_n^{\lambda, \mu}$ , we obtain  $(\mathbf{V}^{\lambda, \mu})^\mathfrak{s} = V_n^{\lambda, \mu}$ . It then follows from left exactness that the functor  $(\cdot)^\mathfrak{s}$  does not increase the length.

Let  $\mathbf{M} \in \mathbb{T}_{\mathfrak{g}}$ , and let  $k(\mathbf{M})$  be the maximum of  $|\lambda| + |\mu|$  over all simple constituents  $\mathbf{V}^{\lambda, \mu}$  of  $\mathbf{M}$ . We proceed by proving the statement by induction on  $k(\mathbf{M})$  with the obvious base case  $k(\mathbf{M}) = 0$ . Consider an exact sequence

$$0 \rightarrow \mathbf{M} \rightarrow \mathbf{I} \rightarrow \mathbf{N} \rightarrow 0,$$

where  $\mathbf{I}$  is an injective hull of  $\mathbf{M}$  in  $\mathbb{T}_{\mathfrak{g}}$ . From the description of the socle filtration of an injective module in  $\mathbb{T}_{\mathfrak{g}}$  (see (2.2)), we have  $k(\mathbf{N}) < k(\mathbf{M})$ . Therefore, the length  $l(\mathbf{N})$  of  $\mathbf{N}$  equals the length  $l(\mathbf{N}^{\mathfrak{s}})$  of  $\mathbf{N}^{\mathfrak{s}}$  by the induction assumption. On the other hand, since  $\mathbf{I}$  is injective and hence isomorphic to a direct sum of  $\mathbb{S}_{\lambda}(\mathbf{V}) \otimes \mathbb{S}_{\mu}(\mathbf{V}_*)$  with  $|\lambda| + |\mu| \leq n$ , the length of  $\mathbf{I}$  equals the length of  $\mathbf{I}^{\mathfrak{s}}$ . Now if  $l(\mathbf{M}^{\mathfrak{s}}) < l(\mathbf{M})$ , then

$$l(\mathbf{N}^{\mathfrak{s}}) \geq l(\mathbf{I}^{\mathfrak{s}}) - l(\mathbf{M}^{\mathfrak{s}}) > l(\mathbf{I}) - l(\mathbf{M}) = l(\mathbf{N}),$$

which is a contradiction. □

**COROLLARY 9.** *Let  $\mathfrak{s}$  be a subalgebra of  $\mathfrak{g}$  as in Lemma 8, and let  $\mathbf{M} \in \widetilde{\mathbb{T}}_{\mathfrak{g}}$  be a module such that all its simple constituents are of the form  $\mathbf{V}^{\lambda, \mu}$  with  $|\lambda| + |\mu| \leq n$ . Then  $\mathbf{M} = U(\mathfrak{g})\mathbf{M}^{\mathfrak{s}}$ .*

*Proof.* Since  $\mathbf{M}$  is a direct limit of modules of finite length it suffices to prove the statement for  $\mathbf{M} \in \mathbb{T}_{\mathfrak{g}}$ . This can be easily done by induction on the length of  $\mathbf{M}$ . Indeed, consider an exact sequence  $0 \rightarrow \mathbf{N} \rightarrow \mathbf{M} \rightarrow \mathbf{L} \rightarrow 0$  with simple  $\mathbf{L}$ . Lemma 8 implies that  $0 \rightarrow \mathbf{N}^{\mathfrak{s}} \rightarrow \mathbf{M}^{\mathfrak{s}} \rightarrow \mathbf{L}^{\mathfrak{s}} \rightarrow 0$  is also exact, because the functor  $(\cdot)^{\mathfrak{s}}$  is left exact and  $l(\mathbf{L}^{\mathfrak{s}}) = l(\mathbf{M}^{\mathfrak{s}}) - l(\mathbf{N}^{\mathfrak{s}})$ . Now if  $U(\mathfrak{g})\mathbf{M}^{\mathfrak{s}} \neq \mathbf{M}$  then, since  $U(\mathfrak{g})\mathbf{N}^{\mathfrak{s}} = \mathbf{N}$  by the induction assumption, we obtain  $U(\mathfrak{g})\mathbf{M}^{\mathfrak{s}} = \mathbf{N}$ . This implies  $\mathbf{M}^{\mathfrak{s}} = \mathbf{N}^{\mathfrak{s}}$ , and hence  $l(\mathbf{L}^{\mathfrak{s}}) = 0$ , which contradicts Lemma 8. □

**LEMMA 10.** *For any  $\mathbf{M} \in \mathbb{T}_{\mathfrak{g}, \mathfrak{k}}$  we have  $U(\mathfrak{g})R(\mathbf{M}) = \mathbf{M}$ .*

*Proof.* Recall the definition of  $k(\mathbf{M})$  from the proof of Lemma 8, and recall the decomposition (2.4). Let  $\mathbf{U}$  be a subspace of  $\mathbf{V}$ , and  $\mathbf{U}_*$  be a subspace of  $\mathbf{V}_*$  such that  $\mathbf{V}_r \subset \mathbf{U}$  and  $(\mathbf{V}_r)_* \subset \mathbf{U}_*$ , each of codimension  $k(\mathbf{M})$ .

Denote by  $\mathfrak{l} \subset \mathfrak{g}$  the commutator subalgebra of  $\mathbf{U} \otimes \mathbf{U}_*$ , and by  $\text{Res}_{\mathfrak{l}}$  the restriction functor from  $\mathbb{T}_{\mathfrak{g}, \mathfrak{k}}$  to  $\widetilde{\mathbb{T}}_{\mathfrak{l}}$ . The identity (2.7) implies that  $k(\text{Res}_{\mathfrak{l}} \mathbf{M}) = k(\mathbf{M})$ . By Corollary 9 with  $\mathfrak{g} = \mathfrak{l}$  and  $\mathfrak{s} = \mathfrak{k}_r$ , we get  $\mathbf{M} = U(\mathfrak{l})R(\mathbf{M})$ . The statement follows. □

**LEMMA 11.** *The functor  $R : \mathbb{T}_{\mathfrak{g}, \mathfrak{k}} \rightarrow \mathbb{T}_{\mathfrak{g}', \mathfrak{k}'}$  is exact and sends a simple module  $\mathbf{V}^{\lambda, \mu} \in \mathbb{T}_{\mathfrak{g}, \mathfrak{k}}$  to the corresponding simple module  $\mathbf{V}^{\lambda, \mu} \in \mathbb{T}_{\mathfrak{g}', \mathfrak{k}'}$ , and hence induces an isomorphism between the Grothendieck groups of  $\mathbb{T}_{\mathfrak{g}, \mathfrak{k}}$  and  $\mathbb{T}_{\mathfrak{g}', \mathfrak{k}'}$ .*

*Proof.* Since  $\mathbf{V}^{\lambda, \mu}$  is in fact an object of  $\mathbb{T}_{\mathfrak{g}}$ , the statement about simple modules follows by the argument concerning contraction maps from the proof of Lemma 8.

Since  $R$  is left exact, we have the inequality

$$l(R(\mathbf{M})) \leq l(\mathbf{M}). \tag{2.9}$$

Thus, to prove exactness of  $R$  it suffices to show that  $R$  preserves the length, that is,  $l(\mathbf{M}) = l(R(\mathbf{M}))$ . We prove this by induction on  $l(\mathbf{M})$ . Consider an exact sequence of  $\mathfrak{g}$ -modules

$$0 \rightarrow \mathbf{N} \rightarrow \mathbf{M} \rightarrow \mathbf{L} \rightarrow 0,$$

such that  $\mathbf{L}$  is simple. By the induction hypothesis we have  $l(R(\mathbf{N})) = l(\mathbf{N})$ . If we assume that  $l(R(\mathbf{M})) < l(\mathbf{M})$ , then  $l(R(\mathbf{M})) = l(\mathbf{N})$  and so  $R(\mathbf{N}) = R(\mathbf{M})$ . But then by Lemma 10, we have  $\mathbf{N} = \mathbf{M}$ , which is a contradiction. □

COROLLARY 12. For any  $\lambda, \mu$ , the module  $\Gamma_{\mathfrak{g}, \mathfrak{t}}((\mathbb{S}_\lambda(\mathbf{V}) \otimes \mathbb{S}_\mu(\mathbf{V}_*))^*)$  has finite length. Hence, the module  $\mathbf{I}^{\lambda, \mu} := \Gamma_{\mathfrak{g}, \mathfrak{t}}((\mathbf{V}^{\mu, \lambda})^*)$  has finite length and is an object of the category  $\mathbb{T}_{\mathfrak{g}, \mathfrak{t}}$ .

*Proof.* It was proven in [8] that  $\Gamma_{\mathfrak{g}, \mathfrak{g}}((\mathbb{S}_\lambda(\mathbf{V}) \otimes \mathbb{S}_\mu(\mathbf{V}_*))^*)$  has finite length in  $\widetilde{\mathbb{T}}_{\mathfrak{g}}$  (see the proof of [8, Proposition 4.5] and note that the functor  $\Gamma_{\mathfrak{g}, \mathfrak{g}}$  is denoted by  $\mathcal{B}$  in [8]). Using (2.8), the first claim follows by induction on the number  $r$  of components in the decomposition of  $\mathbf{V}$ . For the second claim, observe that Lemma 6 implies  $\mathbf{I}^{\lambda, \mu}$  is isomorphic to a direct summand of the module  $\Gamma_{\mathfrak{g}, \mathfrak{t}}((\mathbb{S}_\mu(\mathbf{V}) \otimes \mathbb{S}_\lambda(\mathbf{V}_*))^*)$ .  $\square$

LEMMA 13. Let  $\mathbf{I}^{\lambda, \mu}$  denote an injective hull of the simple module  $\mathbf{V}^{\lambda, \mu}$  in  $\mathbb{T}_{\mathfrak{g}, \mathfrak{t}}$ , and let  $\mathbf{J}^{\lambda, \mu}$  denote an injective hull of  $\mathbf{R}(\mathbf{V}^{\lambda, \mu})$  in  $\mathbb{T}_{\mathfrak{g}', \mathfrak{t}'}$ . Then

$$\mathbf{R}(\mathbf{I}^{\lambda, \mu}) = \bigoplus_{\lambda', \mu', \gamma} N_{\lambda', \gamma}^\lambda N_{\mu', \gamma}^\mu \mathbf{J}^{\lambda', \mu'}.$$

*Proof.* We have  $\mathbf{I}^{\lambda, \mu} \cong \Gamma_{\mathfrak{g}, \mathfrak{t}}((\mathbf{V}^{\mu, \lambda})^*)$  and  $\mathbf{J}^{\lambda, \mu} \cong \Gamma_{\mathfrak{g}', \mathfrak{t}'}((\mathbf{V}^{\mu, \lambda})^*)$ . Let

$$\mathbf{P}^{\lambda, \mu} = \Gamma_{\mathfrak{g}, \mathfrak{t}}((\mathbb{S}_\mu(\mathbf{V}) \otimes \mathbb{S}_\lambda(\mathbf{V}_*))^*), \quad \mathbf{Q}^{\lambda, \mu} = \Gamma_{\mathfrak{g}', \mathfrak{t}'}((\mathbb{S}_\mu(\mathbf{R}(\mathbf{V})) \otimes \mathbb{S}_\lambda(\mathbf{R}(\mathbf{V}_*))^*).$$

Then we have

$$\mathbf{P}^{\lambda, \mu} \cong \bigoplus_{\lambda', \mu', \gamma} N_{\lambda', \gamma}^\lambda N_{\mu', \gamma}^\mu \mathbf{I}^{\lambda', \mu'}, \quad \mathbf{Q}^{\lambda, \mu} \cong \bigoplus_{\lambda', \mu', \gamma} N_{\lambda', \gamma}^\lambda N_{\mu', \gamma}^\mu \mathbf{J}^{\lambda', \mu'}. \quad (2.10)$$

Indeed, using Lemma 6, we can deduce from (2.2) that

$$(\mathbb{S}_\mu(\mathbf{V}) \otimes \mathbb{S}_\lambda(\mathbf{V}_*))^* = \bigoplus_{\lambda', \mu', \gamma} N_{\lambda', \gamma}^\lambda N_{\mu', \gamma}^\mu (\mathbf{V}^{\lambda', \mu'})^*,$$

and then by applying  $\Gamma_{\mathfrak{g}, \mathfrak{t}}$  to both sides we obtain (2.10).

By (2.8), we have

$$\mathbf{R}(\mathbf{P}^{\lambda, \mu}) = \bigoplus_{\lambda', \mu', \gamma} N_{\lambda', \gamma}^\lambda N_{\mu', \gamma}^\mu \mathbf{Q}^{\lambda', \mu'}.$$

Let  $\mathfrak{J}_{\mathfrak{g}, \mathfrak{t}}$  denote the complexified Grothendieck group of the additive subcategory of  $\mathbb{T}_{\mathfrak{g}, \mathfrak{t}}$  generated by indecomposable injective modules. Then  $\{[\mathbf{I}^{\lambda, \mu}]\}$  and  $\{[\mathbf{P}^{\lambda, \mu}]\}$  both form a basis for  $\mathfrak{J}_{\mathfrak{g}, \mathfrak{t}}$ . Let  $A = (A_{\lambda', \mu'}^{\lambda, \mu})$  be the change of basis matrix on  $\mathfrak{J}_{\mathfrak{g}, \mathfrak{t}}$  given by (2.10) which expresses  $\mathbf{P}^{\lambda, \mu}$  in terms of  $\mathbf{I}^{\lambda, \mu}$ . The same matrix  $A$  expresses  $\mathbf{Q}^{\lambda, \mu}$  in terms of  $\mathbf{J}^{\lambda, \mu}$  by (2.10).

The functor  $\mathbf{R}$  induces a linear operator from  $\mathfrak{J}_{\mathfrak{g}, \mathfrak{t}}$  to  $\mathfrak{J}_{\mathfrak{g}', \mathfrak{t}'}$  which is represented by the matrix  $A$  with respect to both bases  $\{[\mathbf{P}^{\lambda, \mu}]\}$  and  $\{[\mathbf{Q}^{\lambda, \mu}]\}$ . Hence, the matrix which represents  $\mathbf{R}$  with respect to the bases  $\{[\mathbf{I}^{\lambda, \mu}]\}$  and  $\{[\mathbf{J}^{\lambda, \mu}]\}$  is again  $A$  as  $A = AA(A^{-1})$ .  $\square$

COROLLARY 14. The Jordan–Hölder multiplicities of the indecomposable injective modules  $\mathbf{I}^{\lambda, \mu}$  are given by

$$[\mathbf{I}^{\lambda, \mu} : \mathbf{V}^{\lambda', \mu'}] = \sum_{\lambda', \mu', \gamma_1, \dots, \gamma_r} N_{\gamma_1, \dots, \gamma_r, \lambda'}^\lambda N_{\gamma_1, \dots, \gamma_r, \mu'}^\mu.$$

*Proof.* After applying the functor  $\mathbf{R}$  to the module  $\mathbf{I}^{\lambda, \mu}$  ( $r-1$ ) times, we obtain a direct sum of injective modules in the category  $\mathbb{T}_{\mathfrak{g}}$ . The multiplicity of each indecomposable injective in this sum is thus determined by applying the matrix  $A^{r-1}$  to  $[\mathbf{I}^{\lambda, \mu}]$ . The Jordan–Hölder multiplicities of an indecomposable injective module in  $\mathbb{T}_{\mathfrak{g}}$  are also given by the matrix  $A$  (see 2.2). Therefore,

$$[\mathbf{I}^{\lambda, \mu}] = \sum (A^r)_{\lambda', \mu'}^{\lambda, \mu} [\mathbf{V}^{\lambda', \mu'}]. \quad \square$$

2.4. *The socle filtration of indecomposable injective objects in  $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$*

In this section, we describe the socle filtration of the injective objects  $\mathbf{I}^{\lambda,\mu}$  in  $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ .

We consider the restriction functor

$$\text{Res}_{\mathfrak{k}} : \mathbb{T}_{\mathfrak{g},\mathfrak{k}} \rightarrow \mathbb{T}_{\mathfrak{k}},$$

where  $\mathbb{T}_{\mathfrak{k}}$  denotes the category of integrable  $\mathfrak{k}$ -modules of finite length which satisfy the large annihilator condition for each  $\mathfrak{k}_i$  (recall (2.4)). Note that simple objects of  $\mathbb{T}_{\mathfrak{k}}$  are outer tensor products of simple objects of the categories  $\mathbb{T}_{\mathfrak{k}_i}$  for each  $\mathfrak{k}_i$ ,  $i = 1, \dots, r$ , (recall that  $\mathfrak{k}_i \cong \mathfrak{sl}(\infty)$ ); we will use the notation

$$\mathbf{V}^{\lambda_1, \dots, \lambda_r, \mu_1, \dots, \mu_r} := \mathbf{V}_1^{\lambda_1, \mu_1} \boxtimes \dots \boxtimes \mathbf{V}_r^{\lambda_r, \mu_r}.$$

Injective hulls of simple objects in  $\mathbb{T}_{\mathfrak{k}}$  will be denoted by  $\mathbf{I}_{\mathfrak{k}}^{\lambda_1, \dots, \lambda_r, \mu_1, \dots, \mu_r}$ , and they are also outer tensor products of injective  $\mathfrak{k}_i$ -modules:

$$\mathbf{I}_{\mathfrak{k}}^{\lambda_1, \dots, \lambda_r, \mu_1, \dots, \mu_r} := (\mathbb{S}_{\lambda_1}(\mathbf{V}_1) \otimes \mathbb{S}_{\mu_1}(\mathbf{V}_1)_*) \boxtimes \dots \boxtimes (\mathbb{S}_{\lambda_r}(\mathbf{V}_r) \otimes \mathbb{S}_{\mu_r}(\mathbf{V}_r)_*).$$

Recall that for every object  $\mathbf{M}$  in  $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$  we denote by  $k(\mathbf{M})$  the maximum of  $|\lambda| + |\mu|$  for all simple constituents  $\mathbf{V}^{\lambda,\mu}$  of  $\mathbf{M}$ . Similarly for every object  $\mathbf{X}$  in  $\mathbb{T}_{\mathfrak{k}}$  we denote by  $c(\mathbf{X})$  the maximum of  $|\lambda_1| + \dots + |\lambda_r| + |\mu_1| + \dots + |\mu_r|$  for all simple constituents  $\mathbf{V}^{\lambda_1, \dots, \lambda_r, \mu_1, \dots, \mu_r}$  of  $\mathbf{X}$ . It follows from Corollary 14 that

$$k(\mathbf{M}) = k(\text{soc } \mathbf{M}), \quad c(\mathbf{X}) = c(\text{soc } \mathbf{X}). \tag{2.11}$$

The identities

$$k(\mathbf{M} \otimes \mathbf{N}) = k(\mathbf{M}) + k(\mathbf{N}), \quad c(\mathbf{X} \otimes \mathbf{Y}) = c(\mathbf{X}) + c(\mathbf{Y}). \tag{2.12}$$

follow easily from the Littlewood–Richardson rule, and we leave their proof to the reader.

LEMMA 15. *The restriction functor  $\text{Res}_{\mathfrak{k}}$  maps the category  $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$  to the category  $\mathbb{T}_{\mathfrak{k}}$ , and it maps  $\mathbb{S}_{\lambda}(\mathbf{V}) \otimes \mathbb{S}_{\mu}(\mathbf{V}_*)$  to an injective module. Furthermore, we have the identity*

$$c(\text{Res}_{\mathfrak{k}} \mathbf{M}) = k(\mathbf{M}).$$

*Proof.* After applying identity (2.7)  $r$ -times to  $\mathbb{S}_{\lambda}(\mathbf{V}) \otimes \mathbb{S}_{\mu}(\mathbf{V}_*)$ , we get

$$\text{Res}_{\mathfrak{k}}(\mathbb{S}_{\lambda}(\mathbf{V}) \otimes \mathbb{S}_{\mu}(\mathbf{V}_*)) \cong \bigoplus N_{\lambda_1, \dots, \lambda_r}^{\lambda} N_{\mu_1, \dots, \mu_r}^{\mu} \mathbf{I}_{\mathfrak{k}}^{\lambda_1, \dots, \lambda_r, \mu_1, \dots, \mu_r}.$$

This implies the first and the second assertions of the lemma. Identity (2.11) implies that it is sufficient to prove the last assertion for  $\mathbf{M} = \mathbb{S}_{\lambda}(\mathbf{V}) \otimes \mathbb{S}_{\mu}(\mathbf{V}_*)$ . Hence, this assertion follows from the above computation.  $\square$

CONJECTURE 16. Suppose  $\text{Ext}_{\mathbb{T}_{\mathfrak{g},\mathfrak{k}}}^k(\mathbf{V}^{\lambda',\mu'}, \mathbf{V}^{\lambda,\mu}) \neq 0$ . Then  $|\lambda| - |\lambda'| = |\mu| - |\mu'| = k$ .

REMARK 17. For  $\mathfrak{k} = \mathfrak{g}$ , this was proven in [8]. Proving this conjecture would imply that the category  $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$  is Koszul. We prove the case  $k = 1$ .

PROPOSITION 18. Suppose  $\text{Ext}_{\mathbb{T}_{\mathfrak{g},\mathfrak{k}}}^1(\mathbf{V}^{\lambda',\mu'}, \mathbf{V}^{\lambda,\mu}) \neq 0$ . Then  $|\lambda| - |\lambda'| = |\mu| - |\mu'| = 1$ .

*Proof.* Since  $\mathbf{V}^{\lambda',\mu'}$  is isomorphic to a simple constituent of  $\mathbf{I}^{\lambda,\mu}$ , we know by Corollary 14 that  $|\lambda| - |\lambda'| = |\mu| - |\mu'| = s \geq 1$ . It remains to show that  $s = 1$ . We will do this in two steps.

First, we show that  $\text{Ext}_{\mathbb{T}_{\mathfrak{g},\mathfrak{k}}}^1(\mathbf{V}^{\lambda',\mu'}, \mathbb{S}_{\lambda}(\mathbf{V}) \otimes \mathbb{S}_{\mu}(\mathbf{V}_*)) \neq 0$  implies  $s = 1$ . Consider a nonsplit short exact sequence in  $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$

$$0 \rightarrow \mathbb{S}_{\lambda}(\mathbf{V}) \otimes \mathbb{S}_{\mu}(\mathbf{V}_*) \rightarrow \mathbf{M} \rightarrow \mathbf{V}^{\lambda',\mu'} \rightarrow 0. \tag{2.13}$$

Let  $\varphi : \mathbf{V}^{\lambda', \mu'} \otimes \mathfrak{g} \rightarrow \mathbb{S}_\lambda(\mathbf{V}) \otimes \mathbb{S}_\mu(\mathbf{V}_*)$  be a cocycle which defines this extension. By Lemma 15, the module  $\text{Res}_{\mathfrak{k}}(\mathbb{S}_\lambda(\mathbf{V}) \otimes \mathbb{S}_\mu(\mathbf{V}_*))$  is injective in  $\mathbb{T}_{\mathfrak{k}}$ , and therefore the sequence (2.13) splits over  $\mathfrak{k}$ . Without loss of generality we may assume that  $\varphi(\mathbf{V}^{\lambda', \mu'} \otimes \mathfrak{k}) = 0$ . Then the cocycle condition implies that  $\varphi : \mathbf{V}^{\lambda', \mu'} \otimes (\mathfrak{g}/\mathfrak{k}) \rightarrow \mathbb{S}_\lambda(\mathbf{V}) \otimes \mathbb{S}_\mu(\mathbf{V}_*)$  is a nonzero homomorphism of  $\mathfrak{k}$ -modules. Consequently, the image of  $\varphi$  contains a simple submodule in the socle of  $\text{Res}_{\mathfrak{k}}(\mathbb{S}_\lambda(\mathbf{V}) \otimes \mathbb{S}_\mu(\mathbf{V}_*))$ . By Lemma 15, we have

$$\text{soc Res}_{\mathfrak{k}}(\mathbb{S}_\lambda(\mathbf{V}) \otimes \mathbb{S}_\mu(\mathbf{V}_*)) = \bigoplus N_{\lambda_1, \dots, \lambda_r}^\lambda N_{\mu_1, \dots, \mu_r}^\mu V^{\lambda_1, \dots, \lambda_r, \mu_1, \dots, \mu_r}.$$

In particular,

$$c(V^{\lambda_1, \dots, \lambda_r, \mu_1, \dots, \mu_r}) = |\lambda_1| + \dots + |\lambda_r| + |\mu_1| + \dots + |\mu_r| = |\lambda| + |\mu|$$

for every simple submodule  $V^{\lambda_1, \dots, \lambda_r, \mu_1, \dots, \mu_r}$  of  $\text{soc Res}_{\mathfrak{k}}(\mathbb{S}_\lambda(\mathbf{V}) \otimes \mathbb{S}_\mu(\mathbf{V}_*))$ . Therefore,

$$c(\mathbf{V}^{\lambda', \mu'} \otimes (\mathfrak{g}/\mathfrak{k})) \geq |\lambda| + |\mu|,$$

and so (2.12) implies

$$c(\mathbf{V}^{\lambda', \mu'}) + c(\mathfrak{g}/\mathfrak{k}) \geq |\lambda| + |\mu|.$$

Since  $\mathfrak{g}/\mathfrak{k} \cong \bigoplus_{i \neq j} (\mathbf{V}_i \otimes (\mathbf{V}_j)_*)$ , we have

$$c(\mathbf{V}^{\lambda', \mu'}) = |\lambda'| + |\mu'|, \quad c(\mathfrak{g}/\mathfrak{k}) = 2,$$

and thus  $|\lambda| - |\lambda'| + |\mu| - |\mu'| = 2s \leq 2$ . This yields  $s = 1$ .

Assume now to the contrary that  $s \geq 2$ . Set

$$\mathbf{X} = (\mathbb{S}_\lambda(\mathbf{V}) \otimes \mathbb{S}_\mu(\mathbf{V}_*)) / \mathbf{V}^{\lambda, \mu}$$

and consider the long exact sequence of Ext

$$\dots \rightarrow \text{Hom}_{\mathfrak{g}}(\mathbf{V}^{\lambda', \mu'}, \mathbf{X}) \rightarrow \text{Ext}_{\mathbb{T}_{\mathfrak{g}, \mathfrak{k}}}^1(\mathbf{V}^{\lambda', \mu'}, \mathbf{V}^{\lambda, \mu}) \rightarrow \text{Ext}_{\mathbb{T}_{\mathfrak{g}, \mathfrak{k}}}^1(\mathbf{V}^{\lambda', \mu'}, \mathbb{S}_\lambda(\mathbf{V}) \otimes \mathbb{S}_\mu(\mathbf{V}_*)) \rightarrow \dots$$

Since  $s \geq 2$ ,  $\mathbf{V}^{\lambda', \mu'}$  is not isomorphic to a submodule of  $\text{soc } \mathbf{X}$ , so  $\text{Hom}_{\mathfrak{g}}(\mathbf{V}^{\lambda', \mu'}, \mathbf{X}) = 0$ , and by the already considered case when  $s = 1$ , we have

$$\text{Ext}_{\mathbb{T}_{\mathfrak{g}, \mathfrak{k}}}^1(\mathbf{V}^{\lambda', \mu'}, \mathbb{S}_\lambda(\mathbf{V}) \otimes \mathbb{S}_\mu(\mathbf{V}_*)) = 0.$$

Hence,  $\text{Ext}_{\mathbb{T}_{\mathfrak{g}, \mathfrak{k}}}^1(\mathbf{V}^{\lambda', \mu'}, \mathbf{V}^{\lambda, \mu}) = 0$ , which is a contradiction. □

**COROLLARY 19.** *Suppose that  $\mathbf{M} \in \mathbb{T}_{\mathfrak{g}, \mathfrak{k}}$  has a simple socle  $\mathbf{V}^{\lambda, \mu}$  and the multiplicity of  $\mathbf{V}^{\lambda', \mu'}$  in  $\overline{\text{soc}}^k \mathbf{M}$  is nonzero. Then  $|\lambda| - |\lambda'| = |\mu| - |\mu'| = k$ .*

*Proof.* This follows by induction on  $|\lambda| + |\mu|$ . By Proposition 18, the module  $\mathbf{M}/\text{soc } \mathbf{M}$  embeds into a direct sum of injective indecomposable modules  $\bigoplus \mathbf{I}^{\gamma, \nu}$  with simple socles  $\mathbf{V}^{\gamma, \nu}$  satisfying  $|\lambda| - |\gamma| = |\mu| - |\nu| = 1$ , and by induction each  $\mathbf{I}^{\gamma, \nu}$  satisfies our claim. If the multiplicity of  $\mathbf{V}^{\lambda', \mu'}$  is nonzero in  $\overline{\text{soc}}^k \mathbf{M} = \overline{\text{soc}}^{k-1}(\mathbf{M}/\text{soc } \mathbf{M}) \subset \overline{\text{soc}}^{k-1}(\bigoplus \mathbf{I}^{\gamma, \nu})$ , then  $|\gamma| - |\lambda'| = |\nu| - |\mu'| = k - 1$ . The result follows. □

Finally, by combining Corollaries 14 and 19 we obtain the following.

**THEOREM 20.** *The layers of the socle filtration of an indecomposable injective  $\mathbf{I}^{\lambda, \mu}$  in  $\mathbb{T}_{\mathfrak{g}, \mathfrak{k}}$  satisfy*

$$\overline{\text{soc}}^k \mathbf{I}^{\lambda, \mu} \cong \bigoplus_{\lambda', \mu' \mid |\gamma_1| + \dots + |\gamma_r| = k} \bigoplus N_{\gamma_1, \dots, \gamma_r, \lambda'}^{\lambda} N_{\gamma_1, \dots, \gamma_r, \mu'}^{\mu} \mathbf{V}^{\lambda', \mu'},$$

where  $r$  is the number of (infinite) blocks in  $\mathfrak{k}$  (see (2.4)).

**EXAMPLE 21.** Consider an injective hull of the adjoint representation of  $\mathfrak{sl}(\infty)$  in the category  $\mathbb{T}_{\mathfrak{g}, \mathfrak{k}}$  in the case that  $\mathfrak{k}$  has  $k$  (infinite) blocks. Then  $\lambda$  and  $\mu$  each consist of one box, and  $\text{soc } V^{\lambda, \mu} = \mathfrak{sl}(\infty)$  and  $\overline{\text{soc}}^1 V^{\lambda, \mu} = \mathbb{C}^k$ , the trivial representation of dimension  $k$ . The self-similarity effect mentioned in the introduction amounts here to the increase of the dimension of  $\overline{\text{soc}}^1$  by 1 when the number of blocks of  $\mathfrak{k}$  increases by 1.

**REMARK 22.** Let us observe that the category  $\mathbb{T}_{\mathfrak{g}, \mathfrak{k}}$  is another example of an ordered tensor category as defined in [5]. Indeed, the set  $I$  in the notation of [5] can be chosen as the set of pairs of Young diagrams  $(\lambda, \mu)$ , and then the object  $X_i$  for  $i = (\lambda, \mu)$  equals  $\mathbf{I}^{\lambda, \mu}$ .

### 3. $\mathfrak{sl}(\infty)$ -modules arising from category $\mathcal{O}$ for $\mathfrak{gl}(m|n)$

For the remainder of this paper, we let  $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$  be the commutator subalgebra of the Lie algebra preserving a fixed decomposition  $\mathbf{V} = \mathbf{V}_1 \oplus \mathbf{V}_2$  such that both  $\mathfrak{k}_1$  and  $\mathfrak{k}_2$  are isomorphic to  $\mathfrak{sl}(\infty)$  ( $r = 2$  in (2.4)).

#### 3.1. Category $\mathcal{O}$ for the Lie superalgebra $\mathfrak{gl}(m|n)$

Let  $\mathcal{O}_{m|n}$  denote the category of  $\mathbb{Z}_2$ -graded modules over  $\mathfrak{gl}(m|n)$  which when restricted to  $\mathfrak{gl}(m|n)_{\bar{0}}$ , belong to the BGG category  $\mathcal{O}_{\mathfrak{gl}(m|n)_{\bar{0}}}$  [14, Section 8.2.3]. This category depends only on a choice of simple roots for the Lie algebra  $\mathfrak{gl}(m|n)_{\bar{0}}$ , and not for all of  $\mathfrak{gl}(m|n)$ . We denote by  $\mathcal{O}_{m|n}^{\mathbb{Z}}$  the Serre subcategory of  $\mathcal{O}_{m|n}$  consisting of modules with integral weights. Any simple object in  $\mathcal{O}_{m|n}^{\mathbb{Z}}$  is isomorphic to  $L(\lambda)$  (the unique simple quotient of the Verma module  $M(\lambda)$ ) for some  $\lambda \in \Phi$ , where  $\Phi$  denotes the set of integral weights. Any object in the category  $\mathcal{O}_{m|n}^{\mathbb{Z}}$  has finite length.

We denote by  $\mathcal{F}_{m|n}^{\mathbb{Z}}$  the Serre subcategory of  $\mathcal{O}_{m|n}^{\mathbb{Z}}$  consisting of finite-dimensional modules. Let  $\Pi : \mathcal{O}_{m|n}^{\mathbb{Z}} \rightarrow \mathcal{O}_{m|n}^{\mathbb{Z}}$  be the parity reversing functor. We define the *reduced Grothendieck group*  $K_{m|n}$  (respectively,  $J_{m|n}$ ) to be the quotient of the Grothendieck group of  $\mathcal{O}_{m|n}^{\mathbb{Z}}$  (respectively,  $\mathcal{F}_{m|n}^{\mathbb{Z}}$ ) by the relation  $[\Pi M] = -[M]$ . The elements  $[L(\lambda)]$  with  $\lambda \in \Phi$  (respectively,  $\lambda \in \Phi^+$ ) form a basis for  $K_{m|n}$  (respectively,  $J_{m|n}$ ).

We introduce an action of  $\mathfrak{sl}(\infty)$  on  $\mathbf{K}_{m|n} := K_{m|n} \otimes_{\mathbb{Z}} \mathbb{C}$  following Brundan [1]. Our starting point is to define the translation functors  $E_i$  and  $F_i$  on the category  $\mathcal{O}_{m|n}^{\mathbb{Z}}$ . Consider the invariant form  $\text{str}(XY)$  on  $\mathfrak{gl}(m|n)$  and let  $X_j, Y_j$  be a pair of  $\mathbb{Z}_2$ -homogeneous dual bases of  $\mathfrak{gl}(m|n)$  with respect to this form. Then for two  $\mathfrak{gl}(m|n)$ -modules  $V$  and  $W$  we define the operator

$$\begin{aligned} \Omega : V \otimes W &\rightarrow V \otimes W, \\ \Omega(v \otimes w) &:= \sum_j (-1)^{p(X_j)(p(v)+1)} X_j v \otimes Y_j w, \end{aligned}$$

where  $p(X_j)$  denotes the parity of the  $\mathbb{Z}_2$ -homogeneous element  $X_j$ . It is easy to check that  $\Omega \in \text{End}_{\mathfrak{gl}(m|n)}(V \otimes W)$ . Let  $U$  and  $U^*$  denote the natural and conatural  $\mathfrak{gl}(m|n)$ -modules. For every  $M \in \mathcal{O}_{m|n}^{\mathbb{Z}}$  we let  $E_i(M)$  (respectively,  $F_i(M)$ ) be the generalized eigenspace of  $\Omega$  in  $M \otimes U^*$  (respectively,  $M \otimes U$ ) with eigenvalue  $i$ . Then, as it follows from [2], the functor  $\cdot \otimes U^*$  (respectively,  $\cdot \otimes U$ ) decomposes into the direct sum of functors  $\bigoplus_{i \in \mathbb{Z}} E_i(\cdot)$  (respectively,  $\bigoplus_{i \in \mathbb{Z}} F_i(\cdot)$ ). Moreover, the functors  $E_i$  and  $F_i$  are mutually adjoint functors on  $\mathcal{O}_{m|n}^{\mathbb{Z}}$ . We will denote by  $e_i$  and  $f_i$  the linear operators which the functors  $E_i$  and  $F_i$  induce on  $\mathbf{K}_{m|n}$ .

If we identify  $e_i$  and  $f_i$  with the Chevalley generators  $E_{i,i+1}$  and  $F_{i+1,i}$  of  $\mathfrak{sl}(\infty)$ , then  $\mathbf{K}_{m|n}$  inherits the natural structure of a  $\mathfrak{sl}(\infty)$ -module. This follows from [1, 2]. Another proof can be obtained by using [7, Theorem 3.11] and (3.2). Weight spaces with respect to the diagonal subalgebra  $\mathfrak{h} \subset \mathfrak{sl}(\infty)$  correspond to the complexified reduced Grothendieck groups of the blocks of  $\mathcal{O}_{m|n}^{\mathbb{Z}}$ .

Let  $\mathbf{J}_{m|n} := J_{m|n} \otimes_{\mathbb{Z}} \mathbb{C}$ , and let  $\mathbf{T}_{m|n} \subset \mathbf{K}_{m|n}$  denote the subspace generated by the classes  $[M(\lambda)]$  of all Verma modules  $M(\lambda)$  for  $\lambda \in \Phi$ . Let furthermore  $\mathbf{\Lambda}_{m|n} \subset \mathbf{J}_{m|n}$  denote the subspace generated by the classes  $[K(\lambda)]$  of all Kac modules  $K(\lambda)$  for  $\lambda \in \Phi^+$  (for the definition of a Kac module see for example [1]). Then  $\mathbf{T}_{m|n}$  is an  $\mathfrak{sl}(\infty)$ -submodule isomorphic to  $\mathbf{V}^{\otimes m} \otimes \mathbf{V}_*^{\otimes n}$  and  $\mathbf{\Lambda}_{m|n}$  is a submodule of  $\mathbf{T}_{m|n}$  isomorphic to  $\Lambda^m \mathbf{V} \otimes \Lambda^n \mathbf{V}_*$  [1]. To see this, let  $\{v_i\}_{i \in \mathbb{Z}}$  and  $\{w_i\}_{i \in \mathbb{Z}}$  be the standard dual bases in  $\mathbf{V}$  and  $\mathbf{V}_*$  (that is,  $\mathfrak{h}$ -eigenbases in  $\mathbf{V}$  and  $\mathbf{V}_*$ ), and let  $\bar{\lambda} := \lambda + (m-1, \dots, 1, 0|0, -1, \dots, 1-n)$ ,

$$m_{\lambda} := v_{\bar{\lambda}_1} \otimes \cdots \otimes v_{\bar{\lambda}_m} \otimes v_{-\bar{\lambda}_{m+1}}^* \otimes \cdots \otimes v_{-\bar{\lambda}_{m+n}}^*.$$

The map  $[M(\lambda)] \mapsto m_{\lambda}$  establishes an isomorphism  $\mathbf{T}_{m|n} \cong \mathbf{V}^{\otimes m} \otimes \mathbf{V}_*^{\otimes n}$ , and restricts to an isomorphism

$$\mathbf{\Lambda}_{m|n} \cong \Lambda^m \mathbf{V} \otimes \Lambda^n \mathbf{V}_*$$

$$[K(\lambda)] \mapsto k_{\lambda} := v_{\bar{\lambda}_1} \wedge \cdots \wedge v_{\bar{\lambda}_m} \otimes v_{-\bar{\lambda}_{m+1}}^* \wedge \cdots \wedge v_{-\bar{\lambda}_{m+n}}^*.$$

**LEMMA 23.** *The  $\mathfrak{sl}(\infty)$ -module  $\mathbf{K}_{m|n}$  satisfies the large annihilator condition as a module over  $\mathfrak{k}_1$  and  $\mathfrak{k}_2$ , that is,  $\Gamma_{\mathfrak{g}, \mathfrak{k}}(\mathbf{K}_{m|n}) = \mathbf{K}_{m|n}$ .*

*Proof.* Note that an  $\mathfrak{sl}(\infty)$ -module  $\mathbf{M}$  satisfies the large annihilator condition over  $\mathfrak{k}_1$  and  $\mathfrak{k}_2$  if and only if for each  $x \in \mathbf{M}$ , we have  $e_i x = f_i x = 0$  for all but finitely many  $i \in \mathbb{Z}$ . Indeed, if  $e_i x = f_i x = 0$  for all but finitely many  $i \in \mathbb{Z}$ , then the subalgebra generated by the  $e_i, f_i$  that annihilate  $x$  contains the commutator subalgebra of the centralizer of a finite-dimensional subalgebra. The other direction is also clear.

Since the classes of simple  $\mathfrak{gl}(m|n)$ -modules  $[L(\lambda)]$  form a basis of  $\mathbf{K}_{m|n}$ , we just need to show that for each  $L(\lambda)$  we have  $E_i(L(\lambda)) = F_i(L(\lambda)) = 0$  for almost all  $i \in \mathbb{Z}$ . However, since  $\mathbf{T}_{m|n}$  satisfies the large annihilator condition, we know that the analogous statement is true for  $M(\lambda)$ . Therefore, since  $L(\lambda)$  is a quotient of  $M(\lambda)$ , the exactness of the functors  $E_i$  and  $F_i$  implies the desired statement for  $L(\lambda)$ .  $\square$

If we consider the Cartan involution  $\sigma$  of  $\mathfrak{sl}(\infty)$ ,  $\sigma(e_i) = -f_i$ ,  $\sigma(f_i) = -e_i$ , we obtain

$$\langle gx, y \rangle = -\langle x, \sigma(g)y \rangle \tag{3.1}$$

for all  $g \in \mathfrak{sl}(\infty)$ . If  $\mathbf{X}$  is a  $\mathfrak{sl}(\infty)$ -module, we denote by  $\mathbf{X}^{\vee}$  the twist of the algebraic dual  $\mathbf{X}^*$  by  $\sigma$ . Note that  $(\mathbf{V}^{\lambda, \mu})^{\vee} = \mathbf{V}^{\mu, \lambda}$ . Hence, if  $\mathbf{X}$  is a semisimple object of finite length in  $\widetilde{Tens}_{\mathfrak{g}}$ , then  $\mathbf{X}^{\vee}$  is an injective hull of  $\mathbf{X}$  in  $\widetilde{Tens}_{\mathfrak{g}}$ .

Let  $\mathcal{P}_{m|n}$  denote the semisimple subcategory of  $\mathcal{O}_{m|n}^{\mathbb{Z}}$  which consists of projective  $\mathfrak{gl}(m|n)$ -modules, and let  $P_{m|n}$  denote the reduced Grothendieck group of  $\mathcal{P}_{m|n}$ . The  $\mathfrak{sl}(\infty)$ -module  $\mathbf{P}_{m,n} := P_{m|n} \otimes_{\mathbb{Z}} \mathbb{C}$  is the socle of  $\mathbf{T}_{m,n}$  [7, Theorem 3.11]. Note that for any projective module

$P \in \mathcal{P}_{m|n}$  the functor  $\text{Hom}_{\mathfrak{gl}(m|n)}(P, \cdot)$  on  $\mathcal{O}_{m|n}^{\mathbb{Z}}$  is exact, and for any module  $M \in \mathcal{F}_{m|n}$  the functor  $\text{Hom}_{\mathfrak{gl}(m|n)}(\cdot, M)$  on  $\mathcal{P}_{m|n}$  is exact. Moreover, we have the dual bases in  $\mathbf{K}_{m|n}$  and  $\mathbf{P}_{m|n}$  given by the classes of irreducible modules and indecomposable projective modules, respectively.

Consider the pairing  $\mathbf{K}_{m|n} \times \mathbf{P}_{m|n} \rightarrow \mathbb{C}$  defined by

$$\langle [M], [P] \rangle := \dim \text{Hom}_{\mathfrak{gl}(m|n)}(P, M).$$

Since the functors  $E_i$  and  $F_i$  are adjoint, we have

$$\langle e_i x, y \rangle = \langle x, f_i y \rangle$$

and

$$\langle f_i x, y \rangle = \langle x, e_i y \rangle,$$

for all  $i \in \mathbb{Z}$ ,  $x \in \mathbf{K}_{m|n}$ ,  $y \in \mathbf{P}_{m|n}$ . Thus, there is an embedding of  $\mathfrak{sl}(\infty)$ -modules

$$\Psi : \mathbf{K}_{m|n} \hookrightarrow \mathbf{P}_{m|n}^{\vee} \tag{3.2}$$

given by  $[M] \mapsto \langle [M], \cdot \rangle$ .

**THEOREM 24.** *The  $\mathfrak{sl}(\infty)$ -module  $\mathbf{K}_{m|n}$  is an injective hull in the category  $\mathbb{T}_{\mathfrak{g}, \mathfrak{t}}$  of the semisimple module  $\mathbf{P}_{m|n}$ . Furthermore, there is an isomorphism*

$$\mathbf{K}_{m|n} \cong \bigoplus_{|\lambda|=m, |\mu|=n} \mathbf{I}^{\lambda, \mu} \otimes (Y_{\lambda} \otimes Y_{\mu})$$

where  $Y_{\lambda}, Y_{\mu}$  are irreducible modules over  $S_m$  and  $S_n$  respectively, and  $\mathbf{I}^{\lambda, \mu}$  is an injective hull of the simple module  $\mathbf{V}^{\lambda, \mu}$  in  $\mathbb{T}_{\mathfrak{g}, \mathfrak{t}}$ . Consequently, the layers of the socle filtration of  $\mathbf{K}_{m|n}$  are given by

$$\overline{\text{soc}}^k \mathbf{K}_{m|n} \cong \bigoplus_{|\lambda|=m, |\mu|=n} (\overline{\text{soc}}^k \mathbf{I}^{\lambda, \mu})^{\oplus (\dim Y_{\lambda} \dim Y_{\mu})}$$

where

$$\overline{\text{soc}}^k \mathbf{I}^{\lambda, \mu} \cong \bigoplus_{\lambda', \mu' \mid |\gamma_1| + |\gamma_2| = k} \bigoplus N_{\gamma_1, \gamma_2, \lambda'}^{\lambda} N_{\gamma_1, \gamma_2, \mu'}^{\mu} \mathbf{V}^{\lambda', \mu'}.$$

*Proof.* The module  $\Gamma_{\mathfrak{g}, \mathfrak{t}}(\mathbf{P}_{m|n}^{\vee})$  is an injective hull of the semisimple module  $\mathbf{P}_{m|n}$  in the category  $\mathbb{T}_{\mathfrak{g}, \mathfrak{t}}$ , so it suffices to show that the image of  $\mathbf{K}_{m|n}$  under the embedding (3.2) equals  $\Gamma_{\mathfrak{g}, \mathfrak{t}}(\mathbf{P}_{m|n}^{\vee})$ . The fact that  $\Psi(\mathbf{K}_{m|n}) \subset \Gamma_{\mathfrak{g}, \mathfrak{t}}(\mathbf{P}_{m|n}^{\vee})$  follows from Lemma 23. Herein, we will identify  $\mathbf{K}_{m|n}$  with its image  $\Psi(\mathbf{K}_{m|n}) = \text{span}\{\langle l_{\lambda}, \cdot \rangle \mid \lambda \in \Phi\}$ , where  $l_{\lambda} := [L(\lambda)]$ .

Now  $\text{soc}(\Gamma_{\mathfrak{g}, \mathfrak{t}}(\mathbf{P}_{m|n}^{\vee})) = \mathbf{P}_{m|n}$ , since  $\mathbf{P}_{m|n}$  is semisimple, and  $\text{soc} \mathbf{T}_{m|n} = \mathbf{P}_{m|n}$  by [7, Theorem 3.11]. Therefore, since  $\mathbf{T}_{m|n} \subset \mathbf{K}_{m|n} \subset \Gamma_{\mathfrak{g}, \mathfrak{t}}(\mathbf{P}_{m|n}^{\vee})$ , we have  $\text{soc} \mathbf{K}_{m|n} = \mathbf{P}_{m|n}$ .

We will show that  $\mathbf{K}_{m|n} = \Gamma_{\mathfrak{g}, \mathfrak{t}}(\mathbf{P}_{m|n}^{\vee})$ . To accomplish this, we use the existence of the dual bases  $p_{\lambda} := [P(\lambda)] \in \mathbf{P}_{m|n}$  and  $l_{\lambda} \in \mathbf{K}_{m|n}$ , where  $L(\lambda)$  denotes the irreducible  $\mathfrak{gl}(m|n)$ -module with highest weight  $\lambda \in \Phi$  and  $P(\lambda)$  is a projective cover of  $L(\lambda)$ .

Fix  $\omega \in \Gamma_{\mathfrak{g}, \mathfrak{t}}(\mathbf{P}_{m|n}^{\vee})$ . To prove that  $\omega \in \mathbf{K}_{m|n} = \text{span}\{\langle l_{\lambda}, \cdot \rangle \mid \lambda \in \Phi\}$ , it suffices to show that  $\omega(p_{\lambda}) = 0$  for almost all  $\lambda \in \Phi$ . For each  $q, r \in \mathbb{Z}$ , with  $q < r$ , we let  $\mathfrak{g}_{q,r} := \mathfrak{g}_q^- \oplus \mathfrak{g}_r^+$ , where  $\mathfrak{g}_q^-$  is the subalgebra of  $\mathfrak{g}$  generated by  $e_i, f_i$  for  $i < q$  and  $\mathfrak{g}_r^+$  is the subalgebra of  $\mathfrak{g}$  generated by  $e_i, f_i$  for  $i > r$ . By the annihilator condition,  $\omega$  is  $\mathfrak{g}_{q,r}$ -invariant for suitable  $q$  and  $r$ . Fix such  $q$  and  $r$ . Then since  $\omega$  is  $\mathfrak{g}_{q,r}$ -invariant, it suffices to show that  $p_{\lambda} \in \mathfrak{g}_{q,r} \mathbf{P}_{m|n}$  for almost all  $\lambda \in \Phi$ .

If  $p_\lambda \in \mathbf{P}_{m|n} \cap (\mathfrak{g}_{q,r} \mathbf{T}_{m|n})$ , then  $p_\lambda \in \mathfrak{g}_{q,r} \mathbf{P}_{m|n}$ . Indeed, for any  $\mathfrak{g}_{q,r}$ -module  $\mathbf{M}$  we have

$$\mathfrak{g}_{q,r} \mathbf{M} = \bigcap_{\varphi \in \text{Hom}_{\mathfrak{g}_{q,r}}(\mathbf{M}, \mathbb{C})} \ker \varphi.$$

Now any  $\mathfrak{g}_{q,r}$ -module homomorphism  $\varphi : \mathbf{P}_{m|n} \rightarrow \mathbb{C}$  lifts to a  $\mathfrak{g}_{q,r}$ -module homomorphism  $\varphi : \mathbf{K}_{m|n} \rightarrow \mathbb{C}$ , since the trivial module  $\mathbb{C}$  is injective in the full subcategory of  $\mathfrak{g}_{q,r}$ -mod consisting of integrable finite-length  $\mathfrak{g}_{q,r}$ -modules satisfying the large annihilator condition [8]. Hence, the claim follows.

For each  $\lambda \in \Phi$  we define  $\text{supp}(\bar{\lambda})$  to be the multiset  $\{\bar{\lambda}_1, \dots, \bar{\lambda}_m, -\bar{\lambda}_{m+1}, \dots, -\bar{\lambda}_{m+n}\}$ , where

$$\bar{\lambda} := \lambda + (m-1, \dots, 1, 0|0, -1, \dots, 1-n).$$

The set of  $\lambda \in \Phi$  such that  $\text{supp}(\bar{\lambda}) \cap (\mathbb{Z}_{<(q-m-n)} \cup \mathbb{Z}_{>(r+m+n)}) = \emptyset$  is finite. Hence, to finish the proof of the theorem, it suffices to show the following. □

**LEMMA 25.** *If  $\text{supp}(\bar{\lambda}) \cap \mathbb{Z}_{<(q-m-n)} \neq \emptyset$ , then  $p_\lambda \in \mathfrak{g}_q^- \mathbf{T}_{m|n}$ . Similarly, if  $\text{supp}(\bar{\lambda}) \cap \mathbb{Z}_{>(r+m+n)} \neq \emptyset$ , then  $p_\lambda \in \mathfrak{g}_r^+ \mathbf{T}_{m|n}$ .*

*Proof.* We will prove the first statement; the proof of the second statement is similar. We can write  $p_\lambda = \sum_\nu c_\nu m_\nu$ , where each  $c_\nu \in \mathbb{Z}_{>0}$  and  $m_\nu = [M(\nu)]$  is the class of the Verma module  $M(\nu)$  over  $\mathfrak{gl}(m|n)$  of highest weight  $\nu \in \Phi$ .

We claim that  $\text{supp}(\bar{\nu}) \cap \mathbb{Z}_{<q} \neq \emptyset$  for every  $m_\nu$  which occurs in the decomposition of  $p_\lambda$ . Indeed, recall that  $P(\lambda)$  is a direct summand in the induced module  $\text{Ind}_{\mathfrak{gl}(m|n)_0}^{\mathfrak{gl}(m|n)} P^0(\lambda)$ , where  $P^0(\lambda)$  is a projective cover of the simple  $\mathfrak{gl}(m|n)_0$ -module with highest weight  $\lambda$ . Now

$$[P^0(\lambda)] = \sum_{w \in \mathcal{W}} b_{w,\lambda} [M^0(w \cdot \lambda)], \tag{3.3}$$

where  $M^0(\mu)$  denotes the Verma module over  $\mathfrak{gl}(m|n)_0$  with highest weight  $\mu$ ,  $\mathcal{W}$  denotes the Weyl group of  $\mathfrak{gl}(m|n)_0$ , and  $w \cdot \lambda$  denotes the  $\rho_0$ -shifted action of  $\mathcal{W}$ . The isomorphism of  $\mathfrak{gl}(m|n)$ -modules

$$M(\mu) \cong \text{Ind}_{\mathfrak{gl}(m|n)_0 \oplus \mathfrak{gl}(m|n)_1}^{\mathfrak{gl}(m|n)} M^0(\mu)$$

implies that

$$\text{Ind}_{\mathfrak{gl}(m|n)_0}^{\mathfrak{gl}(m|n)} M^0(\mu) \cong \text{Ind}_{\mathfrak{gl}(m|n)_0 \oplus \mathfrak{gl}(m|n)_1}^{\mathfrak{gl}(m|n)} (M^0(\mu) \otimes U(\mathfrak{gl}(m|n)_1)).$$

Therefore,  $\text{Ind}_{\mathfrak{gl}(m|n)_0}^{\mathfrak{gl}(m|n)} M^0(\mu)$  admits a filtration by Verma modules  $M(\mu + \gamma)$  where  $\gamma$  runs over the set of weights of  $U(\mathfrak{gl}(m|n)_1)$ . Since  $\text{supp}(\gamma) \subset \{-m-n, \dots, m+n\}$  for every  $\gamma$ , we have

$$|(\overline{\mu + \gamma})_i - \bar{\mu}_i| \leq m + n.$$

Combining this with (3.3) we obtain that for each  $i \leq m+n$ ,  $|\bar{\nu}_i - \bar{\lambda}_{w(i)}| < m+n$ , for some  $w \in \mathcal{W}$ . The claim follows.

Following the notations of Lemma A.1 from the Appendix, we set

$$\mathbf{W}_1 = \text{span}\{v_i, |i < q\}, \quad \mathbf{W}_2 = \text{span}\{v_j, |j \geq q\}.$$

Then  $\mathfrak{g}_q^- = \mathfrak{sl}(\mathbf{W}_1) = \mathfrak{s}$ . By above, every  $m_\nu$  occurring in the decomposition of  $p_\lambda$  is contained in  $\mathbf{Y}_{m|n}$ . Hence  $p_\lambda \in \mathbf{Y}_{m|n}$ . Since we also have  $p_\lambda \in \text{soc } \mathbf{T}_{m|n}$ , Lemma A.1 implies  $p_\lambda \in \mathfrak{g}_q^- \mathbf{T}_{m|n}$ . □

Hence,  $\mathbf{K}_{m|n} = \Gamma_{\mathfrak{g}, \mathfrak{k}}(\mathbf{P}_{m|n}^\vee)$ , and the description of the socle filtration now follows from Theorem 20.

### 3.2. The symmetric group action on $\mathbf{K}_{m|n}$

Recall that we have a natural action of the product of symmetric groups  $S_m \times S_n$  on  $\mathbf{T}_{m|n}$ , which commutes with the  $\mathfrak{sl}(\infty)$ -module structure on  $\mathbf{T}_{m|n}$ . Moreover, it follows from [8, Section 6] that

$$\mathrm{End}_{\mathfrak{sl}(\infty)}(\mathbf{T}_{m|n}) = \mathrm{End}_{\mathfrak{sl}(\infty)}(\mathbf{P}_{m|n}) = \mathbb{C}[S_m \times S_n]. \quad (3.4)$$

A similar result is true for  $\mathbf{K}_{m|n}$ :

PROPOSITION 26.

$$\mathrm{End}_{\mathfrak{sl}(\infty)}(\mathbf{K}_{m|n}) = \mathrm{End}_{\mathfrak{sl}(\infty)}(\mathbf{P}_{m|n}) = \mathbb{C}[S_m \times S_n].$$

*Proof.* Recall that  $\mathbf{P}_{m|n}$  is the socle of  $\mathbf{K}_{m|n}$  by Theorem 24. Every  $\varphi \in \mathrm{End}_{\mathfrak{sl}(\infty)}(\mathbf{K}_{m|n})$  maps the socle to the socle, hence we have a homomorphism

$$\mathrm{End}_{\mathfrak{sl}(\infty)}(\mathbf{K}_{m|n}) \rightarrow \mathrm{End}_{\mathfrak{sl}(\infty)}(\mathbf{P}_{m|n}). \quad (3.5)$$

Let  $\mathbf{K}'_{m|n} = \mathbf{K}_{m|n}/\mathbf{P}_{m|n}$ . By Theorem 20, for every simple module  $\mathbf{V}^{\lambda, \mu}$  we have

$$[\mathbf{K}'_{m|n} : \mathbf{V}^{\lambda, \mu}][\mathbf{P}_{m|n} : \mathbf{V}^{\lambda, \mu}] = 0.$$

Therefore, every  $\varphi \in \mathrm{End}_{\mathfrak{sl}(\infty)}(\mathbf{K}_{m|n})$  such that  $\varphi(\mathbf{P}_{m|n}) = 0$  is identically zero, since for such  $\varphi$  the socle of  $\mathrm{im} \varphi$  is zero. In other words, homomorphism (3.5) is injective. The surjectivity follows from the fact that every  $\varphi : \mathbf{P}_{m|n} \rightarrow \mathbf{P}_{m|n} \hookrightarrow \mathbf{K}_{m|n}$  extends to  $\tilde{\varphi} : \mathbf{K}_{m|n} \rightarrow \mathbf{K}_{m|n}$  by the injectivity of  $\mathbf{K}_{m|n}$ .  $\square$

### 3.3. The Zuckerman functor $\Gamma_{\mathfrak{gl}(m|n)}$ and the category $\mathcal{F}_{m|n}^{\mathbb{Z}}$

Let us recall the definition of the derived Zuckerman functor. A systematic treatment of the Zuckerman functor for Lie superalgebras can be found in [19]. Assume that  $M$  is a finitely generated  $\mathfrak{gl}(m|n)$ -module which is semisimple over the Cartan subalgebra of  $\mathfrak{gl}(m|n)$ . Let  $\Gamma_{\mathfrak{gl}(m|n)}(M)$  denote the subspace of  $\mathfrak{gl}(m|n)_0$ -finite vectors. Then  $\Gamma_{\mathfrak{gl}(m|n)}(M)$  is a finite-dimensional  $\mathfrak{gl}(m|n)$ -module, and hence  $\Gamma_{\mathfrak{gl}(m|n)}$  is a left exact functor from the category of finitely generated  $\mathfrak{gl}(m|n)$ -modules, semisimple over the Cartan subalgebra, to the category  $\mathcal{F}_{m|n}$  of finite-dimensional modules. The corresponding right derived functor  $\Gamma_{\mathfrak{gl}(m|n)}^i$  is called the  $i$ th derived Zuckerman functor. Note that  $\Gamma_{\mathfrak{gl}(m|n)}^i(X) = 0$  for  $i > \dim \mathfrak{gl}(m|n)_0 - (m+n)$ . We are interested in the restriction of this functor

$$\Gamma_{\mathfrak{gl}(m|n)}^i : \mathcal{O}_{m|n}^{\mathbb{Z}} \rightarrow \mathcal{F}_{m|n}^{\mathbb{Z}}.$$

Let us consider the linear operator  $\gamma : \mathbf{K}_{m|n} \rightarrow \mathbf{J}_{m|n}$  given by

$$\gamma([M]) = \sum_i (-1)^i \left[ \Gamma_{\mathfrak{gl}(m|n)}^i M \right].$$

This operator is well defined as for any short exact sequence of  $\mathfrak{gl}(m|n)$ -modules

$$0 \rightarrow N \rightarrow M \rightarrow L \rightarrow 0,$$

we have the Euler characteristic identity

$$\gamma([M]) = \gamma([N]) + \gamma([L]).$$

It is well known that  $\Gamma_{\mathfrak{gl}(m|n)}^i$  commutes with the functors  $\cdot \otimes U$  and  $\cdot \otimes U^*$ , and with the projection to the block  $(\mathcal{O}_{m|n}^{\mathbb{Z}})_{\chi}$  with a fixed central character  $\chi$ . Therefore,  $\gamma$  is a homomorphism of  $\mathfrak{sl}(\infty)$ -modules.

PROPOSITION 27. *The homomorphism  $\gamma$  is given by the formula*

$$\gamma = \sum_{s \in S_m \times S_n} \text{sgn}(s)s, \tag{3.6}$$

where the action of  $s$  on  $\mathbf{K}_{m|n}$  is defined in Proposition 26.

*Proof.* By Proposition 26, it suffices to check the equality (3.6) on vectors in  $\mathbf{T}_{m|n}$ , which amounts to checking that for all Verma modules  $M(\lambda)$

$$\gamma([M(\lambda)]) = \sum_{s \in S_m \times S_n} \text{sgn}(s)[M(s \cdot \lambda)], \tag{3.7}$$

where  $s \cdot \lambda = s(\lambda + \rho) - \rho$  and  $\rho = (m - 1, \dots, 0|0, -1, \dots, 1 - n)$ .

Consider the functor  $\text{Res}_0$  of restriction to  $\mathfrak{gl}(m|n)_0$ . This is an exact functor from the category of finitely generated  $\mathfrak{gl}(m|n)$ -modules, semisimple over the Cartan subalgebra, to the similar category of  $\mathfrak{gl}(m|n)_0$ -modules. It is clear from the definition of  $\Gamma_{\mathfrak{gl}(m|n)}^i$  that

$$\text{Res}_0 \Gamma_{\mathfrak{gl}(m|n)}^i = \Gamma_{\mathfrak{gl}(m|n)_0}^i \text{Res}_0. \tag{3.8}$$

Recall that every Verma module  $M(\lambda)$  over  $\mathfrak{gl}(m|n)$  has a finite filtration with successive quotients isomorphic to Verma modules  $M^0(\mu)$  over  $\mathfrak{gl}(m|n)_0$ . Hence by (3.8) it suffices to check the analogue of (3.7) for even Verma modules:

$$\gamma^0([M^0(\lambda)]) = \sum_{s \in S_m \times S_n} \text{sgn}(s)[M^0(s \cdot \lambda)], \tag{3.9}$$

where  $\gamma^0$  is the obvious analogue of  $\gamma$ . To prove (3.9) we observe that  $[M^0(\lambda)] = [M^0(\lambda)^{\vee}]$  where  $X^{\vee}$  stands for the contragredient dual of  $X$ .

It is easy to compute  $\Gamma_{\mathfrak{gl}(m|n)_0}^i M^0(\lambda)^{\vee}$ . Let  $\mathfrak{t}$  denote the Cartan subalgebra of  $\mathfrak{gl}(m|n)$ , and let  $\mathfrak{n}_0^+$ ,  $\mathfrak{n}_0^-$  be the maximal nilpotent ideals of the Borel and opposite Borel subalgebras of  $\mathfrak{gl}(m|n)_0$ , respectively. From the definition of the derived Zuckerman functor, the following holds for any  $\mu \in \Phi^+$

$$\text{Hom}_{\mathfrak{gl}(m|n)_0}(L^0(\mu), \Gamma_{\mathfrak{gl}(m|n)_0}^i M) \simeq \text{Ext}^i(L^0(\mu), M),$$

where the extension is taken in the category of modules semisimple over  $\mathfrak{t}$ . If  $M = M^0(\lambda)^{\vee}$ , then  $M$  is cofree over  $U(\mathfrak{n}_0^+)$  and therefore

$$\text{Ext}^i(L^0(\mu), M^0(\lambda)^{\vee}) \simeq \text{Hom}_{\mathfrak{t}}(H_i(\mathfrak{n}_0^-, L^0(\mu)), \mathbb{C}_{\lambda}).$$

Now we apply Kostant's theorem to conclude that

$$\Gamma_{\mathfrak{gl}(m|n)_0}^i M^0(\lambda)^{\vee} = \begin{cases} L^0(\mu) & \text{if } \mu = s \cdot \lambda \text{ for } s \in S_m \times S_n, l(s) = i, \\ 0 & \text{otherwise.} \end{cases}$$

Here  $\mu$  is the only dominant weight in  $(S_m \times S_n) \cdot \lambda$  and hence  $s$  is unique. Moreover, if  $\lambda + \rho$  is a singular weight then  $\Gamma_{\mathfrak{gl}(m|n)_0}^i M^0(\lambda)^{\vee} = 0$  for all  $i$ . Combining this with the Weyl character formula

$$[L^0(\mu)] = \sum_{s \in S_m \times S_n} \text{sgn}(s)[M^0(s \cdot \mu)]$$

we obtain (3.9), and hence the proposition. □

COROLLARY 28. We have  $\mathbf{J}_{m|n} = \gamma(\mathbf{K}_{m|n})$  and  $\mathbf{K}_{m|n} = \mathbf{J}_{m|n} \oplus \ker \gamma$ . In particular,  $\mathbf{J}_{m|n}$  is an injective hull of  $\mathbf{\Lambda}_{m|n} \cong \Lambda^m \mathbf{V} \otimes \Lambda^n \mathbf{V}_*$ .

Recall that  $\mathbf{\Lambda}_{m|n} \subset \mathbf{J}_{m|n}$  denotes the subspace generated by the classes of all Kac modules. Let  $\mathcal{Q}_{m|n}$  denote the additive subcategory of  $\mathcal{F}_{m|n}^{\mathbb{Z}}$  which consists of projective finite-dimensional  $\mathfrak{gl}(m|n)$ -modules, and let  $Q_{m|n}$  denote the reduced Grothendieck group of  $\mathcal{Q}_{m|n}$ . It was proven in [7, Theorem 3.11] that  $\mathbf{Q}_{m|n} := Q_{m|n} \otimes_{\mathbb{Z}} \mathbb{C}$  is the socle of the module  $\mathbf{\Lambda}_{m|n}$ , implying that  $\mathbf{Q}_{m|n} \cong \mathbf{V}^{(m)^{\perp}, (n)^{\perp}}$ , where  $\perp$  indicates the conjugate partition. Corollary 28 implies the following.

COROLLARY 29.  $\mathbf{J}_{m|n}$  is an injective hull of  $\mathbf{Q}_{m|n}$ , and the socle filtration of  $\mathbf{J}_{m|n}$  is

$$\overline{\text{soc}}^i \mathbf{J}_{m|n} \cong \left( \mathbf{V}^{(m-i)^{\perp}, (n-i)^{\perp}} \right)^{\oplus(i+1)}.$$

### 3.4. The Duflo–Serganova functor and the tensor filtration

In this section, we discuss the relationship between the Duflo–Serganova functor and submodules of the  $\mathfrak{sl}(\infty)$ -modules  $\mathbf{K}_{m|n}$  and  $\mathbf{J}_{m|n}$ .

Let  $\mathfrak{a} = \mathfrak{a}_{\bar{0}} \oplus \mathfrak{a}_{\bar{1}}$  be a finite-dimensional contragredient Lie superalgebra. For any odd element  $x \in \mathfrak{a}_{\bar{1}}$  which satisfies  $[x, x] = 0$ , the Duflo–Serganova functor  $DS_x$  is defined by

$$DS_x : \mathfrak{a}\text{-mod} \rightarrow \mathfrak{a}_x\text{-mod}$$

$$M \mapsto \ker_M x / xM,$$

where  $\ker_M x / xM$  is a module over the Lie superalgebra  $\mathfrak{a}_x := \mathfrak{a}^x / [x, \mathfrak{a}]$  (here  $\mathfrak{a}^x$  denotes the centralizer of  $x$  in  $\mathfrak{a}$ ) [10]. In what follows we set

$$M_x := DS_x(M).$$

The Duflo–Serganova functor  $DS_x$  is a symmetric monoidal functor, [10], see also [20, Proposition 5].

It is known that the functor  $DS$  is not exact, nevertheless it induces a homomorphism  $ds_x$  between the reduced Grothendieck groups of the categories  $\mathfrak{a}\text{-mod}$  and  $\mathfrak{a}_x\text{-mod}$  defined by  $ds_x([M]) = [M_x]$ . (Recall that ‘reduced’ indicates passage to the quotient by the relation  $[\Pi M] = -[M]$ , where  $\Pi$  is the parity reversing functor.) This follows from the following statement, see [12, Section 1.1].

LEMMA 30. For every exact sequence of  $\mathfrak{a}$ -modules

$$0 \rightarrow M_1 \xrightarrow{\psi} M_2 \xrightarrow{\varphi} M_3 \rightarrow 0$$

there exists an exact sequence of  $\mathfrak{a}_x$ -modules

$$0 \rightarrow E \rightarrow DS_x(M_1) \xrightarrow{DS_x(\psi)} DS_x(M_2) \xrightarrow{DS_x(\varphi)} DS_x(M_3) \rightarrow \Pi E \rightarrow 0,$$

for an appropriate  $\mathfrak{a}_x$ -module  $E$ .

*Proof.* Set  $E := \text{Ker}(DS_x(\psi))$ ,  $E' := \text{Coker}(DS_x(\varphi))$ , and consider the exact sequence

$$0 \rightarrow E \rightarrow DS_x(M_1) \rightarrow DS_x(M_2) \rightarrow DS_x(M_3) \rightarrow E' \rightarrow 0.$$

The odd morphism  $\psi^{-1}x\varphi^{-1} : DS_x(M_3) \rightarrow DS_x(M_1)$  induces an isomorphism  $E' \rightarrow \Pi E$ .  $\square$

In [13] the existence of the homomorphism  $ds_x$  was proven for finite-dimensional modules.

REMARK 31. If  $0 \rightarrow C_1 \rightarrow \dots \rightarrow C_k \rightarrow 0$  is a complex of  $\mathfrak{a}$ -modules with odd differentials, the Euler characteristic of this complex is defined as the element  $\sum_{i=1}^k [C_i]$  in the reduced Grothendieck group. If  $H_i$  denotes the  $i$ th cohomology group, then

$$\sum_{i=1}^k [C_i] = \sum_{i=1}^k [H_i].$$

The absence of the usual sign follows from the relation  $[\Pi M] = -[M]$  and the fact that the differentials are odd. For example, for an acyclic complex  $0 \rightarrow X \rightarrow \Pi X \rightarrow 0$  the Euler characteristic is zero.

Let  $\mathfrak{a} = \mathfrak{gl}(m|n)$  and suppose  $\text{rank } x = k$ . Then  $\mathfrak{a}_x \cong \mathfrak{gl}(m-k|n-k)$ . Let  $\mathcal{O}_{m|n}^{ind}$  be the category whose objects are direct limits of objects in  $\mathcal{O}_{m|n}$ . Then by [7, Lemma 5.2] the restriction of  $DS_x$  to  $\mathcal{O}_{m|n}$  is a well-defined functor

$$DS_x : \mathcal{O}_{m|n} \rightarrow \mathcal{O}_{m-k|n-k}^{ind}.$$

LEMMA 32. *The functor  $DS_x : \mathcal{O}_{m|n}^{\mathbb{Z}} \rightarrow (\mathcal{O}_{m-k|n-k}^{\mathbb{Z}})^{ind}$  commutes with translation functors.*

*Proof.* Recall that  $U$  is the natural  $\mathfrak{gl}(m|n)$ -module. Since  $DS$  is a monoidal functor, we have a canonical isomorphism

$$(M \otimes U)_x \simeq M_x \otimes U_x.$$

Moreover, a direct computation shows that  $U_x$  is isomorphic to the natural  $\mathfrak{gl}(m-k|n-k)$ -module. We will use these observations to show that there is a canonical isomorphism

$$E_i(M_x) \simeq (E_i(M))_x. \tag{3.10}$$

Recall the notations of Section 3.1. Define the homomorphism of  $\mathfrak{gl}(m|n)$ -modules

$$\omega_{m|n} : \mathbb{C} \rightarrow \mathfrak{gl}(m|n) \otimes \mathfrak{gl}(m|n), \quad 1 \mapsto \sum (-1)^{p(X_j)} X_j \otimes Y_j.$$

We have  $DS_x(\omega_{m|n}) = \omega_{m-k|n-k}$ . Consider the composition

$$\Omega : M \otimes U \xrightarrow{1 \otimes \omega_{m|n} \otimes 1} M \otimes \mathfrak{gl}(m|n) \otimes \mathfrak{gl}(m|n) \otimes U \xrightarrow{r_M \otimes l_U} M \otimes U,$$

where  $r_M : M \otimes \mathfrak{gl}(m|n) \rightarrow M$  is the morphism of right action, and  $l_U : \mathfrak{gl}(m|n) \otimes U \rightarrow U$  is the morphism of left action. The morphism  $DS_x(\Omega) : M_x \otimes U_x \rightarrow M_x \otimes U_x$  is defined in a similar manner in the category of  $\mathfrak{gl}(m-k|n-k)$ -modules. Recall that

$$E_i(M) = \{v \in M \otimes U \mid (\Omega - i)^N v = 0 \text{ for some } N > 0\};$$

similarly

$$E_i(M_x) = \{v \in M_x \otimes U_x \mid (DS_x(\Omega) - i)^N v = 0 \text{ for some } N > 0\}.$$

This implies the existence of the isomorphism (3.10) as desired.

The proof for  $F_i$  is similar. □

We are going to strengthen the result of [7] by proving the following proposition.

PROPOSITION 33. *The restriction of  $DS_x$  to  $\mathcal{O}_{m|n}$  is a well-defined functor*

$$DS_x : \mathcal{O}_{m|n} \rightarrow \mathcal{O}_{m-k|n-k}.$$

To prove the proposition we first consider the case when  $k = 1$ .

LEMMA 34. *If  $k = 1$ , then the restriction of  $DS_x$  to  $\mathcal{O}_{m|n}$  is a well-defined functor*

$$DS_x : \mathcal{O}_{m|n} \rightarrow \mathcal{O}_{m-1|n-1}.$$

*Proof.* By [7, Theorem 5.1] we may assume without loss of generality that  $x$  is a generator of the root space  $\mathfrak{gl}(m|n)_\alpha$  for some  $\alpha = \pm(\varepsilon_i - \delta_j)$ . Moreover, we can choose a Borel subalgebra  $\mathfrak{b} \subset \mathfrak{gl}(m|n)$  so that  $\alpha$  is a simple root. Let  $M$  be an object in the category  $\mathcal{O}_{m|n}$  and  $M^\mu$  denote the weight space of weight  $\mu$ . The set of all weights of  $M$  is denoted by  $\text{supp } M$ . Let  $x_\mu : M^\mu \rightarrow M^{\mu+\alpha}$  be the restriction of  $x$  as an operator on  $M$ . Then

$$M_x = \bigoplus_{\mu \in \text{supp } M} M_x^\mu \quad \text{where} \quad M_x^\mu = \ker x_\mu / x_{\mu-\alpha}(M^{\mu-\alpha}).$$

Let us first check that all weight multiplicities of  $M_x$  are finite with respect to the Cartan subalgebra  $\mathfrak{h}_x := \ker \varepsilon_i \cap \ker \delta_j$  of  $\mathfrak{g}_x$ . We have to show that for any  $\nu \in \mathfrak{h}_x^*$

$$\sum_{\mu \in \text{supp } M, \mu|_{\mathfrak{h}_x} = \nu} \dim M_x^\mu < \infty. \tag{3.11}$$

Note that  $\dim M_x^\mu \neq 0$  implies  $(\mu, \alpha) = 0$ , by  $\mathfrak{sl}(1|1)$ -representation theory. If  $(\mu', \alpha') = 0$  and  $\mu|_{\mathfrak{h}_x} = \mu'|_{\mathfrak{h}_x}$ , then  $\mu - \mu' \in \mathbb{C}\alpha$ . Denote by  $\Delta_s$  the set of simple roots of  $\mathfrak{b}$ . Since  $M$  is an object of  $\mathcal{O}_{m|n}$ ,  $M$  has a finite filtration by highest weight modules. Therefore it suffices to consider the case when  $M$  is a highest weight module. Let  $\lambda$  be the highest weight of  $M$ . Then every  $\mu \in \text{supp } M$  has the form  $\lambda - \sum_{\beta \in \Delta_s} k_\beta \beta$  for some  $k_\beta \in \mathbb{Z}_{\geq 0}$  satisfying  $k_\alpha \leq 1 + \sum_{\beta \in \Delta_s \setminus \alpha} k_\beta$ . Therefore, for any  $\mu \in \text{supp } M$  the set  $(\mu + \mathbb{C}\alpha) \cap \text{supp } M$  is finite. Hence, for any  $\nu \in \mathfrak{h}_x^*$  the set of  $\mu \in \text{supp } M$  such that  $\mu|_{\mathfrak{h}_x} = \nu$  and  $(\mu, \alpha) = 0$  is finite. Since all weight spaces of  $M$  are finite dimensional, this implies (3.11).

To finish the proof we observe that Lemma 32 implies  $E_i(M_x) = F_i(M_x) = 0$  for almost all  $i \in \mathbb{Z}$ . Now for each  $i \in \text{supp}(\bar{\lambda})$ , at least one of the  $E_i, E_{i+1}, F_i, F_{i+1}$  does not annihilate  $L_{\mathfrak{g}_x}(\lambda)$ . Together this implies that the set  $S_M$  of all weights  $\lambda$  satisfying  $[M_x : L_{\mathfrak{g}_x}(\lambda)] \neq 0$  is a finite set. On the other hand, since  $M_x$  has finite weight multiplicities, every simple constituent occurs in  $M_x$  with finite multiplicity. Hence  $M_x$  has finite length.  $\square$

*Proof.* Now we prove Proposition 33 by induction on  $\text{rank}(x) = k$ . By [7, Theorem 5.1],  $x$  is  $B_0$ -conjugate to  $x_1 + \dots + x_k$ , where  $x_i \in \mathfrak{gl}(m|n)_{\alpha_i}$  for some linearly independent set of mutually orthogonal odd roots  $\beta_1, \dots, \beta_k$ . So without loss of generality we may suppose that  $x = x_1 + \dots + x_k$ . Let  $y = x_1 + \dots + x_{k-1}$ . Choose  $h_y \in \mathfrak{h}_{x_k}$  and  $h_{x_k} \in \mathfrak{h}_y$  such that  $\alpha(h_y), \alpha(h_{x_k}) \in \mathbb{Z}$  for all roots  $\alpha$  of  $\mathfrak{gl}(m|n)$ ,  $[h_y, y] = y$ , and  $[h_{x_k}, x_k] = x_k$ . Assume  $M \in \mathcal{O}_{m|n}$  and  $\text{supp } M \in \lambda + Q$ , where  $Q$  is the root lattice. Then  $\text{ad } h_y - \lambda(h_y)$  and  $\text{ad } h_{x_k} - \lambda(h_{x_k})$  define a  $\mathbb{Z} \times \mathbb{Z}$ -grading on  $M$  and the differentials  $y$  and  $x_k$  form a bicomplex. Moreover,  $M_x$  is nothing but the cohomology  $\bigoplus_r H^r(y + x_k, M)$  of the total complex.

Consider the second term

$$E_2^{p,q}(M) = H^p(x_k, H^q(y, M))$$

of the spectral sequence of this bicomplex. By the induction assumption  $M_y \in \mathcal{O}_{m-k+1|n-k+1}$ , and in particular,  $H^q(y, M) \neq 0$  for finitely many  $q$ . The induction assumption implies that  $H^p(x_k, H^q(y, M)) \in \mathcal{O}_{m-k|n-k}$  does not vanish for finitely many  $p$ . This yields  $\bigoplus_{p,q} E_2^{p,q}(M) \in \mathcal{O}_{m-k|n-k}$ . Since  $\bigoplus_r H^r(y + x_k, M)$  is a subquotient of  $\bigoplus_{p,q} E_2^{p,q}(M)$ , we obtain

$$M_x = \bigoplus_r H^r(y + x_k, M) \in \mathcal{O}_{m-k|n-k}. \quad \square$$

Next note that the restriction of  $DS_x$  to  $\mathcal{O}_{m|n}^{\mathbb{Z}}$  is a well-defined functor

$$\mathcal{O}_{m|n}^{\mathbb{Z}} \rightarrow \mathcal{O}_{m-k|n-k}^{\mathbb{Z}}.$$

Since  $DS_x$  is a well-defined functor from  $\mathcal{O}_{m|n}^{\mathbb{Z}}$  to  $\mathcal{O}_{m-k|n-k}^{\mathbb{Z}}$  we see that  $ds_x : K_{m|n} \rightarrow K_{m-k|n-k}$  is a well-defined group homomorphism.

LEMMA 35. *If  $x = x_1 + \dots + x_k$  with commuting  $x_1, \dots, x_k$  of rank 1, then on  $K_{m|n}$  we have the identity*

$$ds_x = ds_{x_k} \circ \dots \circ ds_{x_1}.$$

*Proof.* We retain the notation of the proof of Proposition 33. Clearly, it suffices to check that

$$ds_x = ds_{x_k} \circ ds_y,$$

where  $y = x_1 + \dots + x_{k-1}$ . The Euler characteristic of the  $E_s$ -terms of the spectral sequence from the proof of Proposition 33 remains unchanged for  $s \geq 2$ :

$$\left[ \bigoplus_{p,q} E_2^{p,q}(M) \right] = \left[ \bigoplus_{p,q} E_s^{p,q}(M) \right].$$

As the spectral sequence converges to  $[M_x]$ , we obtain

$$ds_{x_k} \circ ds_y([M]) = \left[ \bigoplus_{p,q} E_2^{p,q}(M) \right] = [M_x] = ds_x([M]). \quad \square$$

For the category of finite-dimensional modules the above statement is proven in [13].

PROPOSITION 36. *The complexification  $ds_x : \mathbf{K}_{m|n} \rightarrow \mathbf{K}_{m-k|n-k}$  is a homomorphism of  $\mathfrak{sl}(\infty)$ -modules, as is its restriction  $ds_x : \mathbf{J}_{m|n} \rightarrow \mathbf{J}_{m-k|n-k}$  to the  $\mathfrak{sl}(\infty)$ -submodule  $\mathbf{J}_{m|n} := J_{m|n} \otimes_{\mathbb{Z}} \mathbb{C}$ .*

*Proof.* This follows from the fact that the Duflo–Serganova functor commutes with translation functors, see Lemma 32. □

REMARK 37. Note that in [13] the ring  $J_{m|n}$  is denoted by  $\mathcal{J}_G$  where  $G = GL(m|n)$ .

Let  $X_{\mathfrak{a}} = \{x \in \mathfrak{a}_{\bar{1}} : [x, x] = 0\}$ , and let

$$\mathcal{B}_{\mathfrak{a}} = \{B \subset \Delta_{i_{so}} \mid B = \{\beta_1, \dots, \beta_k \mid (\beta_i, \beta_j) = 0, \beta_i \neq \pm\beta_j\}\} \tag{3.12}$$

be the set of subsets of linearly independent mutually orthogonal isotropic roots of  $\mathfrak{a}$ . Then the orbits of the action of the adjoint group  $G_{\bar{0}}$  of  $\mathfrak{a}_{\bar{0}}$  on  $X_{\mathfrak{a}}$  are in one-to-one correspondence with the orbits of the Weyl group  $\mathcal{W}$  of  $\mathfrak{a}_{\bar{0}}$  on  $\mathcal{B}_{\mathfrak{a}}$  via the correspondence

$$B = \{\beta_1, \dots, \beta_k\} \mapsto x = x_{\beta_1} + \dots + x_{\beta_k} \in X_{\mathfrak{a}}, \tag{3.13}$$

where each  $x_{\beta_i} \in \mathfrak{a}_{\beta_i}$  is chosen to be nonzero [10, Theorem 4.2].

LEMMA 38. *Let  $\mathfrak{a} = \mathfrak{gl}(m|n)$ . Fix  $x \in X_{\mathfrak{a}}$  and set  $k = |B_x|$ , where  $B_x \in \mathcal{B}_{\mathfrak{a}}$  corresponds to  $x$ . The homomorphism  $ds_x : J_{m|n} \rightarrow J_{m-k|n-k}$  depends only on  $k$ , and not on  $x$ .*

*Proof.* This follows from the description of  $ds_x$  given in [13, Theorem 10], using the fact that supercharacters of finite-dimensional modules are invariant under the Weyl group  $\mathcal{W} = S_m \times S_n$  of  $\mathfrak{gl}(m|n)$ . If  $B_1, B_2 \in \mathcal{B}$  with  $|B_1| = |B_2|$ , then there exists  $w \in \mathcal{W}$  satisfying:  $\pm\beta \in w(B_1)$  if and only if  $\pm\beta \in B_2$ . So if  $f \in J_{m|n}$  we have that

$$ds_{x_1}(f) = f|_{\beta_1^1, \dots, \beta_k^1=0} = w(f)|_{w(\beta_1^1), \dots, w(\beta_k^1)=0} = w(f)|_{\beta_1^2, \dots, \beta_k^2=0} = f|_{\beta_1^2, \dots, \beta_k^2=0} = ds_{x_2}(f).$$

□

Note that Lemma 38 does not hold if we replace  $J_{m|n}$  with  $K_{m|n}$ .

REMARK 39. Since the homomorphism  $ds_x : \mathbf{J}_{m|n} \rightarrow \mathbf{J}_{m-k|n-k}$  does not depend on  $x$ , we denote it by  $ds^k$ , where  $|B_x| = k$ , and we let  $ds := ds^1$ .

Now we introduce a filtration of an  $\mathfrak{sl}(\infty)$ -module  $\mathbf{M}$ , whose layers are tensor modules.

DEFINITION 40. The *tensor filtration* of an  $\mathfrak{sl}(\infty)$ -module  $\mathbf{M}$  is defined inductively by

$$\text{tens}^0 \mathbf{M} := \text{tens } \mathbf{M} := \Gamma_{\mathfrak{g}, \mathfrak{g}}(\mathbf{M}), \quad \text{tens}^i \mathbf{M} := p_i^{-1}(\text{tens}(\mathbf{M}/(\text{tens}^{i-1} \mathbf{M}))),$$

where  $p_i : \mathbf{M} \rightarrow \mathbf{M}/(\text{tens}^{i-1} \mathbf{M})$  is the natural projection.

We also use the notation  $\overline{\text{tens}}^i \mathbf{M} = \text{tens}^i \mathbf{M} / \text{tens}^{i-1} \mathbf{M}$ .

Note that  $\text{tens } \mathbf{M}$  is the maximal tensor submodule of  $\mathbf{M}$ .

EXAMPLE 41. The socle of  $\mathbf{J}_{1|1}$  is isomorphic to the adjoint module of  $\mathfrak{sl}(\infty)$ , and  $\overline{\text{soc}}^1 \mathbf{J}_{1|1} = \mathbb{C} \oplus \mathbb{C}$ . Note that this is a special case of Example 21 in the case that  $\mathfrak{k}$  has two infinite blocks.

Consider now the tensor filtration of  $\mathbf{J}_{1|1}$ . This filtration also has length 2,  $\text{tens } \mathbf{J}_{1|1} = \mathbf{\Lambda}_{1|1} \cong \mathbf{V} \otimes \mathbf{V}_*$  and  $\overline{\text{tens}}^1 \mathbf{J}_{1|1} \cong \mathbb{C}$ . The module  $\mathbf{J}_{1|1}$  admits a nice matrix realization. Indeed, we can identify the  $\mathfrak{sl}(\infty)$ -module  $\mathbf{\Lambda}_{1|1}$  with the matrix realization of  $\mathfrak{gl}(\infty)$  (see Section 2.1), and then extend it by the diagonal matrix  $D$  which has entries  $D_{ii} = 1$  for  $i \geq 1$  and 0 elsewhere. The action of  $\mathfrak{sl}(\infty)$  in this realization of  $\mathbf{J}_{1|1}$  is the adjoint action.

PROPOSITION 42. For each  $k$ , let  $ds^k : \mathbf{J}_{m|n} \rightarrow \mathbf{J}_{m-k|n-k}$  be the homomorphism induced by the Duflo–Serganova functor (see Remark 39). Set  $t := 1 + \min\{m, n\}$  and let  $\mathbf{M}_k^t := \ker ds^k$ . Consider the filtration of  $\mathfrak{sl}(\infty)$ -modules

$$\mathbf{M}_1^t \subset \mathbf{M}_2^t \subset \dots \subset \mathbf{M}_t^t = \mathbf{J}_{m|n}.$$

Then  $\mathbf{M}_1^t = \mathbf{\Lambda}_{m|n}$  and  $\mathbf{M}_{k+1}^t / \mathbf{M}_k^t \cong \Lambda^{m-k} \mathbf{V} \otimes \Lambda^{n-k} \mathbf{V}_*$ . This filtration is the tensor filtration of  $\mathbf{J}_{m|n}$ , that is,  $\text{tens}^{k-1} \mathbf{J}_{m|n} = \ker ds^k$ .

*Proof.* In the proof we let  $m$  and  $n$  vary. It follows from [13, Theorems 17 and 20] that for every  $m, n \in \mathbb{Z}_{>0}$  the map  $ds : \mathbf{J}_{m|n} \rightarrow \mathbf{J}_{m-1|n-1}$  is surjective and the kernel is spanned by the classes of Kac modules. So we have an exact sequence of  $\mathfrak{sl}(\infty)$ -modules

$$0 \rightarrow \mathbf{\Lambda}_{m|n} \rightarrow \mathbf{J}_{m|n} \xrightarrow{ds} \mathbf{J}_{m-1|n-1} \rightarrow 0.$$

Thus, we obtain the following diagram of  $\mathfrak{sl}(\infty)$ -modules for each  $l = |m - n|$ , in which the horizontal arrows represent the map  $ds$ .

$$\begin{array}{cccccc} \rightarrow & \mathbf{M}_5^5 & \rightarrow & \mathbf{M}_4^4 & \rightarrow & \mathbf{M}_3^3 & \rightarrow & \mathbf{M}_2^2 & \rightarrow & \mathbf{M}_1^1 \\ & \cup & & \cup & & \cup & & \cup & & \cup \\ \rightarrow & \mathbf{M}_4^5 & \rightarrow & \mathbf{M}_3^4 & \rightarrow & \mathbf{M}_2^3 & \rightarrow & \mathbf{M}_1^2 & \rightarrow & 0 \\ & \cup & & \cup & & \cup & & \cup & & \\ \rightarrow & \mathbf{M}_3^5 & \rightarrow & \mathbf{M}_2^4 & \rightarrow & \mathbf{M}_1^3 & \rightarrow & 0 & & \\ & \cup & & \cup & & \cup & & & & \\ \rightarrow & \mathbf{M}_2^5 & \rightarrow & \mathbf{M}_1^4 & \rightarrow & 0 & & & & \\ & \cup & & \cup & & & & & & \\ \rightarrow & \mathbf{M}_1^5 & \rightarrow & 0 & & & & & & \end{array}$$

By induction we get  $\mathbf{M}_{k+1}^t / \mathbf{M}_k^t \cong \mathbf{M}_1^{t-k} = \mathbf{\Lambda}_{m-k|n-k}$ , and by [1],  $\mathbf{\Lambda}_{m-k|n-k} \cong \Lambda^{m-k} \mathbf{V} \otimes \Lambda^{n-k} \mathbf{V}_*$ . Hence, the first claim follows.

For the second claim, suppose for sake of contradiction that for some  $k$ , the module  $\mathbf{M}_{k+1}^t/\mathbf{M}_k^t$  is not the maximal tensor submodule of  $\mathbf{J}_{m|n}/\mathbf{M}_k^t$ . By projecting to  $\mathbf{J}_{m-k|n-k}$ , we obtain that  $\mathbf{M}_1^t$  is not the maximal tensor submodule of  $\mathbf{J}_{m|n}$ , for some  $m, n$ . Since  $\mathbf{M}_1^t = \Lambda_{m|n} \cong \Lambda^m \mathbf{V} \otimes \Lambda^n \mathbf{V}_*$  is injective in the category  $\mathbb{T}_{\mathfrak{g}}$  [8], this implies that  $\text{soc } \mathbf{J}_{m|n}$  is larger than  $\text{soc } \mathbf{M}_1^t$ , which is a contradiction since  $\text{soc } \mathbf{J}_{m|n} = \text{soc } \Lambda_{m|n} = \mathbf{P}_{m|n}$ .  $\square$

In the rest of this subsection, we fix  $x$  to be a generator of the root space corresponding to  $\delta_j - \varepsilon_i$ . We denote by  $ds_{ij} : \mathbf{K}_{m|n} \rightarrow \mathbf{K}_{m-1|n-1}$  the  $\mathfrak{sl}(\infty)$ -module homomorphism  $ds_x$ .

PROPOSITION 43. *We have*

$$\bigcap_{i,j} \ker ds_{ij} = \mathbf{T}_{m|n}.$$

*Proof.* It follows from [13] that  $ds_{ij}[M] = 0$  if and only if  $e^{\varepsilon_i} - e^{\delta_j}$  divides the supercharacter  $\text{sch } M$  of  $M$ . Hence,  $[M]$  lies in the intersection of kernels of all  $ds_{ij}$  if and only if  $\prod_{i,j} (e^{\varepsilon_i} - e^{\delta_j})$  divides  $\text{sch } M$ . This means that  $\text{sch } M$  is a linear combination of supercharacters induced from the parabolic subalgebra  $\mathfrak{gl}(m|n)_{\bar{0}} \oplus \mathfrak{gl}(m|n)_1$ . Therefore,  $\text{sch } M$  is a linear combination of supercharacters of Verma modules.  $\square$

PROPOSITION 44. *We have  $\text{tens } \mathbf{K}_{m|n} = \mathbf{T}_{m|n}$ . Moreover,  $\mathbf{K}_{m|n}$  has an exhausting tensor filtration of length  $\min(m, n) + 1$ .*

*Proof.* Obviously  $\text{tens } \mathbf{K}_{m|n} \supset \mathbf{T}_{m|n}$ . Assume  $\text{tens } \mathbf{K}_{m|n} \neq \mathbf{T}_{m|n}$ . Then since  $\mathbf{T}_{m|n}$  is injective in  $\mathbb{T}_{\mathfrak{g}}$  the socle of  $\text{tens } \mathbf{K}_{m|n}$  is larger than the socle of  $\mathbf{T}_{m|n}$ , but this is a contradiction since  $\text{soc } \mathbf{T}_{m|n} = \text{soc } \mathbf{K}_{m|n}$ . The second claim can be proven by induction on  $\min(m, n)$ , since  $\mathbf{K}_{m|n}/\mathbf{T}_{m|n}$  is isomorphic to a submodule of  $\mathbf{K}_{m-1|n-1}^{\oplus mn}$  via the map  $\oplus_{i,j} ds_{ij}$ .  $\square$

### 3.5. Meaning of the socle filtration

Now we will define a filtration on the category  $\mathcal{O}_{m|n}^{\mathbb{Z}}$ . For a  $\mathfrak{gl}(m|n)$ -module  $M$ , let

$$X_M = \{x \in X_{\mathfrak{gl}(m|n)} \mid DS_x(M) \neq 0\},$$

and let  $X_{\mathfrak{gl}(m|n)}^k$  be the subset of all elements in  $X_{\mathfrak{gl}(m|n)}$  of rank less than or equal to  $k$ . We define  $[\mathcal{O}_{m|n}^{\mathbb{Z}}]^k$  to be the full subcategory of  $\mathcal{O}_{m|n}^{\mathbb{Z}}$  consisting of all modules  $M$  such that  $X_M \subset X_{\mathfrak{gl}(m|n)}^k$ . Note that  $[\mathcal{O}_{m|n}^{\mathbb{Z}}]^k$  is not an abelian category. Furthermore, we define  $[\mathcal{O}_{m|n}^{\mathbb{Z}}]_{-}^k$  to be the full subcategory of  $\mathcal{O}_{m|n}^{\mathbb{Z}}$  consisting of all modules  $M$  such that

$$X_M \cap \mathfrak{gl}(m|n)_{-1} \subset X_{\mathfrak{gl}(m|n)}^k.$$

Let  $\mathbf{K}_{m|n}^k$  denote the complexification of the subgroup in  $\mathbf{K}_{m|n}$  generated by the classes of modules lying in  $[\mathcal{O}_{m|n}^{\mathbb{Z}}]^k$ , and let  $(\mathbf{K}_{m|n}^k)_{-}$  be defined similarly for the category  $[\mathcal{O}_{m|n}^{\mathbb{Z}}]_{-}^k$ . Since both categories are invariant under the functors  $E_i$  and  $F_i$ , both  $\mathbf{K}_{m|n}^k$  and  $(\mathbf{K}_{m|n}^k)_{-}$  are  $\mathfrak{sl}(\infty)$ -submodules of  $\mathbf{K}_{m|n}$ .

CONJECTURE 45.  $\mathbf{K}_{m|n}^k = \text{soc}^{k+1} \mathbf{K}_{m|n}$  and  $(\mathbf{K}_{m|n}^k)_{-} = \text{tens}^{k+1} \mathbf{K}_{m|n}$ .

Here we prove a weaker statement. Recall that  $\mathcal{O}_{m|n}^{\mathbb{Z}}$  has block decomposition:

$$\mathcal{O}_{m|n}^{\mathbb{Z}} = \bigoplus (\mathcal{O}_{m|n}^{\mathbb{Z}})_{\chi},$$

where  $(\mathcal{O}_{m|n}^{\mathbb{Z}})_{\chi}$  is the subcategory of modules admitting generalized central character  $\chi$ . The complexified reduced Grothendieck group of  $(\mathcal{O}_{m|n}^{\mathbb{Z}})_{\chi}$  coincides with the weight subspace  $(\mathbf{K}_{m|n})_{\chi}$ . The degree of atypicality of  $\chi$  is defined in [10]. In [7] it is proven that  $(\mathcal{O}_{m|n}^{\mathbb{Z}})_{\chi} \subset [\mathcal{O}_{m|n}^{\mathbb{Z}}]^k$  if the degree of atypicality of  $\chi$  is not greater than  $k$ . Note that the degree of atypicality of the highest weight  $\chi$  of the irreducible  $\mathfrak{sl}_{\infty}$ -module  $\mathbf{V}^{\lambda, \mu}$  is equal to  $m - |\lambda| = n - |\mu|$  and the degree of atypicality of any weight of  $\mathbf{V}^{\lambda, \mu}$  is not less than the degree of atypicality of the highest weight. Combining this observation with the description of the socle filtration of  $\mathbf{K}_{m|n}$  we obtain the following.

PROPOSITION 46.  *$\text{soc}^{k+1} \mathbf{K}_{m|n}$  is the submodule in  $\mathbf{K}_{m|n}$  generated by weight vectors of weights with degree of atypicality less or equal to  $k$ . Therefore we have  $\text{soc}^{k+1} \mathbf{K}_{m|n} \subset \mathbf{K}_{m|n}^k$ .*

Appendix

In this section, we prove the technical lemma used in Lemma 25, which in turn is needed for the proof of Theorem 24.

Consider decompositions  $\mathbf{V} = \mathbf{W}_1 \oplus \mathbf{W}_2$  and  $(\mathbf{V})_* = (\mathbf{W}_1)_* \oplus (\mathbf{W}_2)_*$  such that  $\mathbf{W}_1^{\perp} = (\mathbf{W}_2)_*$  and  $\mathbf{W}_2^{\perp} = (\mathbf{W}_1)_*$ . Denote by  $\mathfrak{s}$  the subalgebra  $\mathfrak{sl}(\mathbf{W}_1)$  of  $\mathfrak{g}$ . Let  $\mathbf{T}_{m|n} = \mathbf{V}^{\otimes m} \otimes \mathbf{V}_*^{\otimes n}$ , and let  $\mathbf{Y}_{m|n}$  be the intersection with  $\mathbf{T}_{m|n}$  of the ideal generated by  $\mathbf{W}_1 \oplus (\mathbf{W}_1)_*$  in the tensor algebra  $T(\mathbf{V} \oplus \mathbf{V}_*)$ . Then  $\mathbf{T}_{m|n}$  considered as an  $\mathfrak{s}$ -module admits the decomposition

$$\text{Res}_{\mathfrak{s}} \mathbf{T}_{m|n} = (\mathbf{W}_2^{\otimes m} \otimes (\mathbf{W}_2)_*^{\otimes n}) \oplus \mathbf{Y}_{m|n}.$$

LEMMA A.1. *We have*

$$(\text{soc } \mathbf{T}_{m|n}) \cap \mathbf{Y}_{m|n} \subset \mathfrak{s} \mathbf{Y}_{m|n}.$$

*Proof.* Note that  $\mathbf{Y}_{m|n}$  is an object of  $\tilde{\mathbb{T}}_{\mathfrak{s}}$  and

$$\mathfrak{s} \mathbf{Y}_{m|n} = \bigcap_{\varphi \in \text{Hom}_{\mathfrak{s}}(\mathbf{Y}_{m|n}, \mathbb{C})} \ker \varphi. \tag{A.1}$$

Let  $\tau$  denote a map from  $\{1, \dots, m+n\}$  to  $\{1, 2\}$ . Denote by  $\mathbf{T}_{m|n}^{\tau}$  the subspace of  $\mathbf{T}_{m|n}$  spanned by  $v_1 \otimes \dots \otimes v_m \otimes u_{m+1} \otimes \dots \otimes u_{m+n}$  with  $v_i \in \mathbf{W}_{\tau(i)}$  and  $u_j \in (\mathbf{W}_{\tau(j)})_*$ . Clearly,

$$\text{Res}_{\mathfrak{s}} \mathbf{T}_{m|n} = \bigoplus_{\tau} \mathbf{T}_{m|n}^{\tau},$$

and we have an  $\mathfrak{s}$ -module isomorphism

$$\mathbf{T}_{m|n}^{\tau} \cong \mathbf{W}_1^{\otimes p(\tau)} \otimes \mathbf{W}_2^{\otimes (m-p(\tau))} \otimes (\mathbf{W}_1)_*^{\otimes q(\tau)} \otimes (\mathbf{W}_2)_*^{\otimes (n-q(\tau))},$$

where

$$p(\tau) := |\tau^{-1}(1) \cap \{1, \dots, m\}|, \quad q(\tau) := |\tau^{-1}(1) \cap \{m+1, \dots, m+n\}|.$$

Furthermore,

$$\mathbf{Y}_{m|n} = \bigoplus_{p(\tau)+q(\tau)>0} \mathbf{T}_{m|n}^{\tau}.$$

Recall from [17, Theorem 2.1] that

$$\text{soc } \mathbf{T}_{m|n} = \bigcap_{1 \leq i \leq m, m < j \leq m+n} \ker \Phi_{ij},$$

where  $\Phi_{ij}$  is defined in (2.3). For  $r = 1, 2$ , let  $\Phi_{ij}^{\mathbf{W}^r} : \mathbf{T}_{m|n} \rightarrow \mathbf{T}_{m-1|n-1}$  be defined by

$$v_1 \otimes \cdots \otimes v_m \otimes u_{m+1} \otimes \cdots \otimes u_{m+n} \mapsto \langle u_j, v_i \rangle^{\mathbf{W}^r} v_1 \otimes \cdots \otimes \widehat{v}_i \otimes \cdots \otimes v_m \otimes u_{m+1} \\ \otimes \cdots \otimes \widehat{u}_j \otimes \cdots \otimes u_{m+n},$$

where  $\langle \cdot, \cdot \rangle^{\mathbf{W}^r}$  is defined on homogeneous elements by

$$\langle u_j, v_i \rangle^{\mathbf{W}^r} := \begin{cases} \langle u_j, v_i \rangle & \text{if } u_j, v_i \in \mathbf{W}^r \\ 0 & \text{otherwise.} \end{cases}$$

Next, recall from [8] that  $\text{Hom}_{\mathfrak{s}}(\mathbf{W}_1^{\otimes p} \otimes (\mathbf{W}_1)_*^{\otimes q}, \mathbb{C}) = 0$  if  $p \neq q$ , and if  $p = q$ , is spanned by compositions of contractions  $\Phi_{1,j_1}^{\mathbf{W}_1} \dots \Phi_{p,j_p}^{\mathbf{W}_1}$  for all possible permutations  $j_1, \dots, j_p$ . Using (A.1) we can conclude that  $\mathfrak{s}\mathbf{Y}_{m|n}^\tau = \mathbf{Y}_{m|n}^\tau$  if  $p(\tau) \neq q(\tau)$ , whereas if  $p = p(\tau) = q(\tau)$  we have

$$\mathfrak{s}\mathbf{Y}_{m|n}^\tau = \bigcap_{i_1, \dots, i_p, j_1, \dots, j_p \in \tau^{-1}(1)} \ker \Phi_{i_1, j_1}^{\mathbf{W}_1} \dots \Phi_{i_p, j_p}^{\mathbf{W}_1}.$$

Observe that

$$\Phi_{ij} = \Phi_{ij}^{\mathbf{W}_1} + \Phi_{ij}^{\mathbf{W}_2}. \tag{A.2}$$

We claim that if  $y = \sum_{\tau} y_{\tau} \in \mathbf{Y}_{m|n}$  and  $\Phi_{ij}(y) = 0$  for all  $i, j$ , then  $y_{\tau} \in \mathfrak{s}\mathbf{T}_{m|n}^\tau$  for all  $\tau$ . The statement is trivial for every  $\tau$  such that  $p(\tau) \neq q(\tau)$ . Now we proceed to prove the claim in the case  $p(\tau) = q(\tau) = p$  by induction on  $p$ .

Let  $p = 1$  and consider  $\tau'$  with  $p(\tau') = 1 = q(\tau')$ . Let  $i \leq m$  and  $j > m$  be such that  $\tau'(i) = \tau'(j) = 1$ . Note that  $\Phi_{i,j}(y_{\tau'}) \in (\mathbf{W}_2^{\otimes m-1} \otimes (\mathbf{W}_2)_*^{\otimes n-1})$  and for  $\tau \neq \tau'$  we have  $\Phi_{i,j}(y_{\tau}) \in \mathbf{Y}_{m-1|n-1}$ . Therefore,  $\Phi_{i,j}(y_{\tau'}) = \Phi_{i,j}^{\mathbf{W}_1}(y_{\tau'}) = 0$  and hence  $y_{\tau'} \in \mathfrak{s}\mathbf{T}_{m|n}^{\tau'}$ .

Now consider  $y_{\tau'}$  such that  $p(\tau') = p = q(\tau')$ . Let  $i_1, \dots, i_p \leq m$  and  $j_1, \dots, j_p > m$  such that  $\tau'(i) = \tau'(j) = 1$ . We would like to show that

$$\Phi_{i_1, j_1}^{\mathbf{W}_1} \dots \Phi_{i_p, j_p}^{\mathbf{W}_1}(y_{\tau'}) = \Phi_{i_1, j_1} \dots \Phi_{i_p, j_p}(y_{\tau'}) = 0. \tag{A.3}$$

Note that  $\tau'$  has the property

$$\Phi_{i_1, j_1} \dots \Phi_{i_p, j_p}(y_{\tau'}) \in \mathbf{W}_2^{\otimes m-p} \otimes (\mathbf{W}_2)_*^{\otimes n-p}. \tag{A.4}$$

Suppose that  $\tau''$  also has property (A.4). Then  $(\tau'')^{-1}(1) \subset (\tau')^{-1}(1)$ , and if  $\Phi_{i_1, j_1} \dots \Phi_{i_p, j_p}(y_{\tau''}) \neq 0$ , then  $\tau''(i_r) = \tau''(j_r)$  for all  $r = 1, \dots, p$ . For every such  $\tau'' \neq \tau'$  we have  $p(\tau'') = q(\tau'') := l < p$ . Let  $\{i_{r_1}, \dots, i_{r_l}, j_{r_1}, \dots, j_{r_l}\} = (\tau'')^{-1}(1)$ . Then by induction assumption  $y_{\tau''} \in \mathfrak{s}\mathbf{T}_{m|n}^{\tau''}$  and hence

$$\Phi_{i_{r_1}, j_{r_1}}^{\mathbf{W}_1} \dots \Phi_{i_{r_l}, j_{r_l}}^{\mathbf{W}_1}(y_{\tau''}) = \Phi_{i_{r_1}, j_{r_1}} \dots \Phi_{i_{r_l}, j_{r_l}}(y_{\tau''}) = 0.$$

But then

$$\Phi_{i_1, j_1} \dots \Phi_{i_p, j_p}(y_{\tau''}) = 0,$$

which implies

$$\Phi_{i_1, j_1} \dots \Phi_{i_p, j_p}(y_{\tau'}) = 0.$$

Now (A.3) follows, and this implies  $y_{\tau'} \in \mathfrak{s}\mathbf{T}_{m|n}^\tau$ . □

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