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Representations of a central extension of the simple Lie superalgebra $\mathfrak{p}(3)$

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Abstract
We classify irreducible finite-dimensional representations of a non-trivial central extension of the Lie superalgebra $\mathfrak{p}(3)$, and compute their characters.

Keywords Lie superalgebra · Central extension · Simple module

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1 Introduction

Finite-dimensional simple Lie superalgebras over $\mathbb{C}$ have been classified in [8]. Some of these superalgebras have non-trivial central extensions: $\mathfrak{sl}(n|n)$ is a central extension of $\mathfrak{psl}(n|n)$, $\mathfrak{sq}(n)$ is a central extension of $\mathfrak{psq}(n)$, the Poisson Lie superalgebra $\mathcal{SP}(n)$ is a central extension of the special Hamiltonian superalgebra $SH(n)$, and finally the Lie superalgebra $\hat{\mathfrak{p}}(3)$ is the unique (up to isomorphism) non-trivial central extension of the Lie superalgebra $\mathfrak{p}(3)$.

It is natural to study finite-dimensional representations of these central extensions as well as representations of the corresponding simple superalgebras. For $\mathfrak{sl}(n|n)$ and $\mathfrak{sq}(n)$ it is convenient to study representations of their respective Lie algebras of derivations $\mathfrak{gl}(n|n)$ and $\mathfrak{q}(n)$. Finite-dimensional representations of these superalgebras are studied in detail in [2–4,10,11]. For the Lie superalgebra $\mathfrak{psl}(2|2)$ there is a three-dimensional space of central extensions, irreducible finite-dimensional representations over those central extensions are studied in [7]. Irreducible finite-dimensional representations of the Poisson superalgebra are described in [12].

Dedicated to Joe Wolf.

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As far as we know, the representations of $\hat{\mathfrak{p}}(3)$ with nonzero central charge have not been previously studied. In the present paper we try to fill this gap: we compute characters of irreducible finite-dimensional irreducible representations of $\hat{\mathfrak{p}}(3)$ with nonzero central charge. The case of zero central charge follows from more general results for the Lie superalgebra $\mathfrak{p}(n)$ for arbitrary n in [1], see also [5,6].

We plan to use the results of the present paper to compute the Cartan matrix of the category of finite-dimensional representations of $\hat{\mathfrak{p}(3)}$ and classify the blocks of this category.

2 The Lie superalgebra $\hat{\mathfrak{p}}(3)$

2.1 Central extension of $\mathfrak{p}(3)$

Consider the superspace $V = \mathbb{C}^{m|m}$ equipped with a non-degenerate odd symmetric form $\beta : V \times V \to \mathbb{C}$. By $\tilde{\mathfrak{p}}(m-1)$ we denote the subalgebra of all endomorphisms $X \in \mathrm{End}_{\mathbb{C}}(V)$ such that

$$\beta(X\xi, \eta) + (-1)^{\bar{X}\bar{\xi}}\beta(\xi, X\eta) = 0, \text{ for all } \xi, \eta \in V.$$

This Lie superalgebra has a codimension-1 ideal $\mathfrak{p}(m-1)$ consisting of matrices with zero supertrace. For $m \geq 3$ the Lie superalgebra $\mathfrak{p}(m-1)$ is simple and $\tilde{\mathfrak{p}}(m-1)$ coincides with the Lie superalgebra of all derivations of $\mathfrak{p}(m)$, see [8].

The case $m = 4$ is exceptional since $\mathfrak{p}(3)$ has a nontrivial central extension which we will denote by $\mathfrak{g} = \hat{\mathfrak{p}}(3)$. To describe this extension, note that in matrix form $\mathfrak{p}(3)$ is given by block matrices $\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}$ where A is a traceless 4×4 matrix, B is symmetric and C is skew-symmetric. In particular, the even part of $\mathfrak{p}(3)$ is the Lie algebra $\mathfrak{sl}(4) \simeq \mathfrak{so}(6)$, and the odd part is the sum $\Lambda^2 W^* \oplus S^2 W$ where W denotes the natural four dimensional $\mathfrak{sl}(4)$-module. It is clear that $\mathfrak{p}(3)$ has a $\mathbb{Z}$-grading

$$\mathfrak{p}(3) = \mathfrak{p}(3)_{-1} \oplus \mathfrak{p}(3)_0 \oplus \mathfrak{p}(3)_1, \quad \mathfrak{p}(3)_{-1} = \Lambda^2 W^*, \quad \mathfrak{p}(3)_0 = \mathfrak{sl}(4), \quad \mathfrak{p}(3)_1 = S^2 W.$$

Note that $\mathfrak{p}(3)_{-1} = \Lambda^2 W^*$ is isomorphic to the standard $\mathfrak{so}(6)$-module E with the scalar product $(\cdot, \cdot)$. Define the cocycle $\varphi : \Lambda^2(\mathfrak{p}(3)) \to \mathbb{C}$ as follows: for any $x \in \mathfrak{p}(3)_i$, $x \in \mathfrak{p}(3)_j$, set

$$\varphi(x, y) := \begin{cases} (x, y) & \text{if } \ i = j = -1 \\ 0 & \text{otherwise} \end{cases}.$$

We denote by $\hat{\mathfrak{p}}(3)$ the central extension of $\mathfrak{p}(3)$ defined by this cocycle.

2.2 The category of finite-dimensional representations

The goal of this paper is to understand finite-dimensional representation theory of $\mathfrak{g}$ in the case of nonzero central charge. We denote by $\mathrm{Rep}_{\mathfrak{g}}$ the category of finite-dimensional $\mathfrak{g}$-modules semisimple over $\mathfrak{g}_{\bar{0}}$, and by $\mathrm{Rep}^t_{\mathfrak{g}}$ the full subcategory of modules with central charge t, and concentrate on the case $t \neq 0$. We assume that all our categories are enriched over superspaces and we allow odd morphisms. In particular, we assume that $M \simeq \Pi M$, where Π is the change of parity functor. Working this way we obtain, generally speaking, a non-abelian category. However, if the morphism φ between two objects is graded (even or odd) then $\mathrm{Ker}\varphi$ and $\mathrm{Coker}\,\varphi$ are well defined.

Note that $\mathfrak{g}$ has a grading $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with $\mathfrak{g}_{-2} = \mathbb{C}z$, where z is the central element. Moreover, $\mathfrak{h} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ is isomorphic to the superextension of $\mathfrak{so}(6)$ which we study in the next section. We will show that the category of finite-dimensional $\mathfrak{h}$-modules with nonzero central charge is semisimple. It is useful to consider the restriction functor $\mathrm{Res} : \mathrm{Rep}^t_{\mathfrak{g}} \to \mathrm{Rep}^t_{\mathfrak{h}}$.

Proposition 1 *The category* $\mathrm{Rep}_{\mathfrak{g}}$ *decomposes into direct sum* $\bigoplus_t \mathrm{Rep}^t_{\mathfrak{g}}$. *If* $ts \neq 0$, *then* $\mathrm{Rep}^t_{\mathfrak{g}}$ *and* $\mathrm{Rep}^s_{\mathfrak{g}}$ *are equivalent.*

Proof The first assertion is obvious. Let us prove the second assertion. For any $u \in \mathbb{C}^*$ denote by τ_u the automorphism which acts by u^{-i} on the graded component $\mathfrak{g}_i$. The equivalence $\mathrm{Rep}^t_{\mathfrak{g}} \to \mathrm{Rep}^s_{\mathfrak{g}}$ is given by the twist with τ_u: $M \mapsto M^{\tau_u}$ for u such that $u^2 = ts^{-1}$. □

2.3 The standard representation

Let $V = \mathbb{C}^{4|4}$, and define a representation $\rho_t : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(V)$ by

$$\rho_t \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} := \begin{pmatrix} A & B + tC^* \\ C & -A^t \end{pmatrix}, \quad \rho(z) := t,$$

where $c^*_{ij} = (-1)^{\sigma} c_{kl}$ for the permutation $\sigma = \{1, 2, 3, 4\} \to \{i, j, k, l\}$. We denote the corresponding $\mathfrak{g}$-module by V_t. It is clear that V_t is a simple object of $\mathrm{Rep}^t_{\mathfrak{g}}$. When $t = 0$ this module coincides with the standard $\mathfrak{p}(3)$-module V.

2.4 Root decomposition

The Cartan subalgebra of $\mathfrak{t} \subset \mathfrak{g}$ is the direct sum of $\mathbb{C}z$ and the Cartan subalgebra in $\mathfrak{so}(6)$. The Lie superalgebra $\mathfrak{g}$ has a roots decomposition with even roots

$$\Delta_{\bar{0}} = \{(\pm(\varepsilon_i \pm \varepsilon_j) \mid 1 \leq i < j \leq 3\},$$

and odd roots

$$\Delta_{\bar{1}} = \{\pm\varepsilon_1, \pm\varepsilon_2, \pm\varepsilon_3, \varepsilon_1 + \varepsilon_2 + \varepsilon_3, \varepsilon_1 - \varepsilon_2 - \varepsilon_3, -\varepsilon_1 - \varepsilon_2 + \varepsilon_3, -\varepsilon_1 + \varepsilon_2 - \varepsilon_3\}.$$

Note that the odd roots $\pm\varepsilon_i$ have multiplicity 2 and the roots $\varepsilon_1 + \varepsilon_2 + \varepsilon_3, \varepsilon_1 - \varepsilon_2 - \varepsilon_3, -\varepsilon_1 - \varepsilon_2 + \varepsilon_3, -\varepsilon_1 + \varepsilon_2 - \varepsilon_3$ are not invertible.

Let us choose simple roots of $\mathfrak{g}_0 = \mathfrak{sl}(3) = \mathfrak{so}(6)$ by setting

$$\alpha_1 = \varepsilon_2 - \varepsilon_3, \quad \alpha_2 = \varepsilon_1 - \varepsilon_2, \quad \alpha_3 = \varepsilon_2 + \varepsilon_3.$$

We denote by $e_i, f_i, h_i,\ i = 1, 2, 3$ the Chevalley generators of $\mathfrak{g}_0$. For a weight $\lambda \in \mathfrak{t}^*$ we use the notation $\lambda = (a, b, c)$ if $(\lambda, \alpha_1) = a$, $(\lambda, \alpha_2) = b$ and $(\lambda, \alpha_3) = c$. We denote by $L(a, b, c)$ the simple $\mathfrak{g}_0$-module with highest weight $\lambda = (a, b, c)$.

Example 1 With this labeling, $L(0, 1, 0)$ is the natural 6-dimensional representation of $\mathfrak{so}(6)$, while $L(1, 0, 0)$ and $L(0, 0, 1)$ are spinor 4-dimensional representations dual to each other.

3 Finite dimensional representations of the superextension of $\mathfrak{so}(2n)$

3.1 Lie superalgebra $\mathfrak{h}(n)$

Consider the Lie superalgebra $\mathfrak{h}(n)$ such that $\mathfrak{h}(n)_{\bar{0}} = \mathfrak{so}(2n) \oplus \mathbb{C}z$, $\mathfrak{h}(n)_{\bar{1}} = E$ is the standard $\mathfrak{so}(2n)$-module with trivial action of z and the bracket $S^2E \to \mathfrak{h}(n)_{\bar{0}}$ is defined by

$$[v, w] := (v, w)z$$

where $v, w \in E$ and $(\cdot, \cdot)$ denotes a symmetric form on E invariant under the $\mathfrak{so}(2n)$-action. Note that $\mathfrak{h}(n)$ has a $\mathbb{Z}$-grading

$$\mathfrak{h}(n) = \mathfrak{h}(n)_{-2} \oplus \mathfrak{h}(n)_{-1} \oplus \mathfrak{h}(n)_0,$$

where $\mathfrak{h}_{-2} = \mathbb{C}z$, $\mathfrak{h}(n)_{-1} = E$ and $\mathfrak{h}(n)_0 = \mathfrak{so}(2n)$. For every $s \in \mathbb{C}^*$ we define the automorphism τ_s of $\mathfrak{h}(n)$ by the formula

$$\tau_s(x) = s^i x, \quad \text{for all } x \in \mathfrak{h}(n)_i.$$

Recall that one can identify $\mathfrak{so}(2n)$ with $\Lambda^2 E$ as follows. Define $T_{v\wedge w} \in \mathrm{End}_{\mathbb{C}}(E)$ by

$$T_{v\wedge w}(u) = (v, u)w - (w, u)v \tag{1}$$

for $v, w, u \in E$. Then $T_{v\wedge w} \in \mathfrak{so}(2n)$ and the span of $T_{v\wedge w}$ for all $v, w \in E$ coincides with $\mathfrak{so}(2n)$.

Let $\mathrm{Rep}_{\mathfrak{h}(n)}$ denote the category of finite-dimensional $\mathfrak{h}(n)$-modules semisimple over $\mathfrak{h}(n)_{\bar{0}}$, and by $\mathrm{Rep}^t_{\mathfrak{h}(n)}$ we denote the full subcategory of modules on which z acts by the scalar $t \in \mathbb{C}$. As in Proposition 1, $\mathrm{Rep}_{\mathfrak{h}(n)}$ decomposes into direct sum of $\mathrm{Rep}^t_{\mathfrak{h}(n)}$ and if $t_1, t_2 \neq 0$, then $\mathrm{Rep}^{t_1}_{\mathfrak{h}(n)}$ and $\mathrm{Rep}^{t_2}_{\mathfrak{h}(n)}$ are equivalent.

3.2 Spinor representation

Let us define the spinor representation V_t of $\mathfrak{h}(n)$. Fix the decomposition $E = E^+ \oplus E^-$ for two maximal isotropic subspaces $E^\pm \subset E$ and set $V_t := \Lambda(E^+)$. Assume that $t = s^2 \neq 0$. Define the $\mathfrak{h}(n)$-module structure on V_t (denoted by $\cdot$) by setting for any $\xi \in V_t$, $v \in E^+$ and $w \in E^-$

$$\begin{aligned} &v \cdot \xi := sv \wedge x, \quad w \cdot 1 = 0, \quad w \cdot (v \wedge x) = s(v, w)x - v \wedge w \cdot x, \\ &T_{v \wedge w} \cdot \xi = v \cdot (w \cdot \xi), \quad z \cdot \xi = t\xi. \end{aligned}$$

It is easy to see that V_t is obtained from V_1 by twisting with the automorphism τ_s. The following statement is straightforward.

Proposition 2 *If $t \neq 0$, then V_t is a simple $\mathfrak{h}(n)$-module. For $n = 3$, V_t is the restriction to $\mathfrak{h}(3)$ of the standard $\mathfrak{g}$-module.*

Lemma 1 *Let $\mathbb{C}_t$ denote the one-dimensional $\mathfrak{h}(n)_{\bar{0}}$-module with central charge $t \neq 0$. Then*

$$\mathrm{Ind}_{\mathfrak{h}(n)_{\bar{0}}}^{\mathfrak{h}(n)} \mathbb{C}_t \simeq V_t^{\oplus 2^n}.$$

Proof Let $\mathrm{Cliff}(2n)$ denote the Clifford algebra with $2n$ generators. Consider the Lie superalgebra map $\psi_t : \mathfrak{h}(n) \hookrightarrow \mathrm{Cliff}(2n)$ such that $\psi_t(z) = t$, and the corresponding surjective homomorphism $\varphi_t : U(\mathfrak{h}(n)) \to \mathrm{Cliff}(2n)$ of associative superalgebras. Then V_t is the pull back of the unique simple Cliff_n-module and $\mathrm{Ind}_{\mathfrak{h}_{\bar{0}}}^{\mathfrak{h}(n)} \mathbb{C}_t$ is the pullback of the free $\mathrm{Cliff}(2n)$-module of rank 1. From the structure theory of Clifford algebras we have

$$\mathrm{Cliff}(2n) \simeq V_t^{\oplus 2^n}.$$

This implies the lemma. □

3.3 Simple objects in $\mathrm{Rep}^t_{\mathfrak{h}(n)}$

For every simple $\mathfrak{so}(2n)$-module L, we set $\tilde{L}_t := L \otimes V_t$ where we assume that the action of $\mathfrak{h}(n)_{-1} \oplus \mathfrak{h}(n)_{-2}$ on L is trivial.

Theorem 1 *Let $t \neq 0$.*

(a) *The module $\tilde{L}_t$ is simple and every simple module in $\mathrm{Rep}^t_{\mathfrak{h}(n)}$ is isomorphic to $\tilde{L}_t$ for some, unique up to isomorphism, simple $\mathfrak{so}(2n)$-module L.*
(b) *The category $\mathrm{Rep}^t_{\mathfrak{h}(n)}$ is semisimple.*

Proof First, let us prove that $\tilde{L}_t$ is simple. Indeed, the restriction of $\tilde{L}_t$ to $\mathfrak{h}(n)_{-1} \oplus \mathfrak{h}(n)_{-2}$ is isomorphic to the direct sum of $\dim L$ copies of V_t. If M is a non-trivial submodule of $\tilde{L}_t$ then $M' := M \cap (L \otimes 1) \neq 0$, and hence by $\mathfrak{h}(n)_0$-invariance of M we obtain $M' = M \otimes 1$. Therefore $M = \tilde{L}_t$.

Every simple $\mathfrak{h}(n)_{\bar{0}}$-module with central charge t is isomorphic to $L \boxtimes \mathbb{C}_t$ for some simple $\mathfrak{so}(2n)$-module L. By Lemma 1 we have

$$\mathrm{Ind}_{\mathfrak{h}(n)_{\bar{0}}}^{\mathfrak{h}(n)}(L \boxtimes \mathbb{C}_t) \simeq L \otimes \mathrm{Ind}_{\mathfrak{h}(n)_{\bar{0}}}^{\mathfrak{h}(n)} \mathbb{C}_t \simeq \tilde{L}_t^{\oplus 2^n}.$$

Every simple object of $\mathrm{Rep}^t_{\mathfrak{h}(n)}$ is a quotient of $\mathrm{Ind}_{\mathfrak{h}(n)_{\bar{0}}}^{\mathfrak{h}(n)}(L \boxtimes \mathbb{C}_t)$, hence it is isomorphic to $\tilde{L}_t$ for some L. Hence (a).

Since $\mathrm{Ind}_{\mathfrak{h}(n)_{\bar{0}}}^{\mathfrak{h}(n)}(L \boxtimes \mathbb{C}_t)$ is projective in $\mathrm{Rep}^t_{\mathfrak{h}(n)}$ and $\tilde{L}_t$ is a direct summand, we obtain that every simple object in $\mathrm{Rep}^t_{\mathfrak{h}(n)}$ is projective. This implies (b). □

4 Kac modules in the category $\mathrm{Rep}^t_{\mathfrak{g}}$

We turn now to the representation theory of $\mathfrak{g} = \hat{\mathfrak{p}}(3)$.

4.1 Kac modules

Let $\mathfrak{p} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, $\lambda = (a, b, c)$ be a dominant $\mathfrak{g}_0$-weight, and $L_t(\lambda)$ denote the irreducible $\mathfrak{p}$-module with central charge t, $\mathfrak{g}_0$ highest weight λ, and trivial action of $\mathfrak{g}_1$. We define a Kac module $K_t(\lambda)$ by

$$K_t(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} L_t(\lambda).$$

Proposition 3 *Every simple module in* $\mathrm{Rep}^t_{\mathfrak{g}}$ *is a quotient of some Kac module* $K_t(\lambda)$.

Proof Let S be a simple $\mathfrak{g}$-module. Since $\mathfrak{g}_1$ is an abelian odd Lie superalgebra, we have $S^{\mathfrak{g}_1} \neq 0$. Then $S^{\mathfrak{g}_1}$ contains a $\mathfrak{g}_0$-submodule isomorphic to $L_t(\lambda)$ and we have a nonzero homomorphism $K_t(\lambda) \to S$ by Frobenius reciprocity. □

Remark 1 If $t \neq 0$, the cosocle of $K_t(\lambda)$ may be not simple and a simple module S may appear in a cosocle of several Kac modules.

In what follows we use the following fact about Lie superalgebra.

Lemma 2 *Let* $\mathfrak{g}$ *be a finite-dimensional Lie superalgebra and* $\mathfrak{p}$ *be a subalgebra which contains* $\mathfrak{g}_0$ *and such that* $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{m}$ *as a* $\mathfrak{p}$*-module. Then for every* $\mathfrak{p}$*-module* M *there is an isomorphism between induced and coinduced modules*

$$\mathrm{Hom}_{U(\mathfrak{p})}(U(\mathfrak{g}), M) \simeq U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} (M \otimes \Lambda^{top}(\mathfrak{m}^*)).$$

Proof Use the isomorphism of $\mathfrak{p}$-modules

$$\mathrm{Hom}_{U(\mathfrak{p})}(U(\mathfrak{g}), M) \simeq M \otimes \Lambda(\mathfrak{m}^*)$$

to construct an injective homomorphism

$$M \otimes \Lambda^{top}(\mathfrak{m}^*) \to \mathrm{Hom}_{U(\mathfrak{p})}(U(\mathfrak{g}), M)$$

of $\mathfrak{p}$-modules. By Frobenius reciprocity, the latter homomorphism induces an isomorphism

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} (M \otimes \Lambda^{top}(\mathfrak{m}^*)) \to \mathrm{Hom}_{U(\mathfrak{p})}(U(\mathfrak{g}), M).$$

□

Lemma 3 $K_t(a, b, c)^* \simeq K_{-t}(c, b, a)$.

Proof Indeed, we have

$$K_t(a, b, c)^* \simeq \mathrm{Hom}_{U(\mathfrak{p})}(U(\mathfrak{g}), L_t(a, b, c)^*) \simeq U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \left(L_t(a, b, c)^* \otimes \Lambda^{top}(\mathfrak{g}^*_{-1})\right).$$

Hence the statement follows from the following isomorphisms of $\mathfrak{g}_{\bar{0}}$-modules

$$\Lambda^{top}(\mathfrak{g}^*_{-1}) \simeq \mathbb{C}, \quad L_t(a, b, c)^* \simeq L_{-t}(c, b, a).$$

□

Lemma 4 *If* $t \neq 0$, *then the modules* $\Lambda^2 V_{t/2}$ *and* $S^2 V_{t/2}$ *are simple, and the structure of* $K_t(0)$ *can be described by the non-splitting exact sequence*

$$0 \to S^2 V_{t/2} \to K_t(0) \to \Lambda^2 V_{t/2} \to 0.$$

Proof The simplicity of $\Lambda^2 V_{t/2}$ and $S^2 V_{t/2}$ follows from the isomorphisms

$$\mathrm{Res}\Lambda^2 V_{t/2} \simeq \tilde{L}_t(0, 0, 1), \quad \mathrm{Res}S^2 V_{t/2} \simeq \tilde{L}_t(1, 0, 0).$$

Note that the modules V_t, considered as $\mathfrak{g}_0 \oplus \mathfrak{g}_1$-modules, are isomorphic for all t. In particular, $\Lambda^2 V_{t/2}$ is isomorphic to $\Lambda^2 V$ and hence

$$\mathrm{Hom}_{\mathfrak{p}}(L_t(0), \Lambda^2 V_{t/2}) = \mathrm{Hom}_{\mathfrak{g}_0 \oplus \mathfrak{g}_1}(\mathbb{C}, \Lambda^2 V) = \mathbb{C}.$$

Using duality $\Lambda^2 V^*_{-t/2} \simeq S^2 V_{t/2}$ we get

$$\mathrm{Hom}_{\mathfrak{p}}(S^2(V_t), L_t(0)) = \mathrm{Hom}_{\mathfrak{g}_0 \oplus \mathfrak{g}_1}(S^2 V, \mathbb{C}) = \mathbb{C}.$$

That gives us morphisms $K_t(0) \to \Lambda^2 V_{t/2}$ and $S^2 V_{t/2} \to K_t(0)$. Hence we have the required exact sequence. The sequence does not split since

$$\mathrm{Hom}_{\mathfrak{g}}(K_t(0), S^2 V_{t/2}) = \mathrm{Hom}_{\mathfrak{p}}(L_t(0), S^2 V_{t/2}) = \mathrm{Hom}_{\mathfrak{g}_0 \oplus \mathfrak{g}_1}(\mathbb{C}, S^2 V) = 0.$$

□

Corollary 1 *If* $t \neq 0$ *then*

$$\mathrm{Res} K_t(0) \simeq V_t \otimes L(0,0,1) \oplus V_t \otimes L(1,0,0).$$

Proof Follows from Lemma 4 since

$$\mathrm{Res} K_t(0) = \mathrm{Res} \Lambda^2 V_{t/2} \oplus \mathrm{Res} S^2 V_{t/2}.$$

□

4.2 The restriction functor

Recall that $\mathrm{Res} : \mathrm{Rep}^t_{\mathfrak{g}} \to \mathrm{Rep}^t_{\mathfrak{h}}$ denotes the restriction functor. For every $\mathfrak{g}$-module M we denote by $d(M)$ the length of $\mathrm{Res} M$.

Lemma 5 *Let* $t \neq 0$ *then*

$$\begin{aligned}\mathrm{Res} K_t(a,b,c) = {} & \tilde{L}_t(a+1,b,c) \oplus \tilde{L}_t(a-1,b+1,c) \oplus \tilde{L}_t(a,b-1,c+1)\\ & \oplus \tilde{L}_t(a,b,c-1)\\ & \oplus \tilde{L}_t(a-1,b,c) \oplus \tilde{L}_t(a+1,b-1,c) \oplus \tilde{L}_t(a,b+1,c-1)\\ & \oplus \tilde{L}_t(a,b,c+1),\end{aligned}$$

where we assume $\tilde{L}_t(a',b',c') = 0$ *whenever* a', b' *or* c' *is negative.*

Proof Use the isomorphisms

$$\mathrm{Res} K_t(a,b,c) \simeq K_t(0) \otimes L(a,b,c), \quad \mathrm{Res} K_t(0) \simeq V_t \otimes L(0,0,1) \oplus V_t \otimes L(1,0,0).$$

The statement follows by application of the Pieri rule for computing $L(0,0,1) \otimes L(a,b,c)$ and $L(1,0,0) \otimes L(a,b,c)$. □

Corollary 2 *Let* $m(a,b,c)$ *denote the number of zeros in* (a,b,c)*. Then* $d(K_t(a,b,c)) = 8 - 2m(a,b,c)$.

For a $\mathfrak{g}$-module M in $\mathrm{Rep}^t_{\mathfrak{g}}$, we introduce the generating function $G(M) \in \mathbb{Z}[x,y,z]$ by setting

$$G(M) = \sum m_{a,b,c} x^a y^b z^c,$$

where $m_{a,b,c}$ equals the multiplicity of $\tilde{L}_t(a,b,c)$ in $\mathrm{Res} M$. In particular,

$$G(K_t(a,b,c)) = [x^a y^b z^c (x + yx^{-1} + zy^{-1} + z^{-1} + x^{-1} + xy^{-1} + yz^{-1} + z)]^+,$$

where $[\cdot]^+$ denotes the polynomial part of a Laurent polynomial.

Corollary 3 *If* $t \neq 0$*, then* $\mathrm{End}_{\mathfrak{g}}(K_t(\lambda))$ *is a semisimple commutative algebra.*

Proof $\mathrm{Res} K_t(\lambda)$ is multiplicity free semisimple $\mathfrak{h}$-module and $\mathrm{End}_{\mathfrak{g}}(K_t(\lambda))$ is a subalgebra of $\mathrm{End}_{\mathfrak{h}}(K_t(\lambda))$. □

4.3 The case of zero central charge

Assume now that $t = 0$. Here we combine the results about the structure of Kac modules from [1,9]. Note that the superalgebra $\mathfrak{g}$ is an ideal of codimension 1 in its algebra of derivations $\tilde{\mathfrak{g}}$ and in the case $t = 0$, one can define Kac modules over $\tilde{\mathfrak{g}}$. This leads to an additional parameter in the definition of the Kac modules: $K_0(a, b, c)$ is isomorphic to the small Kac module $\nabla(\mu)$ in [1], the corresponding weight diagram μ has four black nodes and a, b, c stand for the number of white nodes between the first and the second, the second and the third and the third and the fourth black nodes respectively (counting from right left to right). Two diagrams obtained from each other by a shift encode the same weight (a, b, c).

Theorem 2 [9]

1. *$K_0(\lambda)$ is an indecomposable $\mathfrak{g}$-module with unique simple quotient $V_0(\lambda)$.*
2. *$K_0(\lambda)$ is simple if and only if $abc \neq 0$.*

Using Theorem 6.3.3 of [1] we can compute the Jordan–Hoelder multiplicities of simple modules in Kac modules. (Note that since we consider induced modules instead of coinduced as in [1], our arrows go from left to right.) The following identities are in the Grothendieck group of $\mathrm{Rep}^t_{\mathfrak{g}}$:

1. $[K_0(a, b, c)] = [V_0(a, b, c)]$ iff $abc \neq 0$;
2. If $b \geq 2, a \geq 1$, then $[K_0(a, b, 0)] = [V_0(a, b, 0)] + [V_0(a, b - 1, 0)]$ and $[K_0(0, b, a)] = [V_0(0, b, a)] + [V_0(0, b + 1, a)]$;
3. $[K_0(a, 1, 0)] = [V_0(a, 1, 0)] + [V_0(a, 0, 0)] + [V_0(a - 1, 0, 1)]$ if $a \geq 2$;
4. $[K_0(0, 1, a)] = [V_0(0, 1, a)] + [V_0(0, 0, a + 2)] + [V_0(0, 2, a)]$ if $a \geq 2$;
5. $[K_0(1, 1, 0)] = [V_0(1, 1, 0)] + [V_0(1, 0, 0)] + [V_0(0, 0, 1)] + [V_0(0, 1, 1)]$;
6. $[K_0(0, 1, 1)] = [V_0(0, 1, 1)] + [V_0(0, 0, 1)] + [V_0(0, 2, 1)] + [V_0(0, 0, 3)]$;
7. If $c \geq 2, a \geq 2$, then $[K_0(a, 0, c)] = [V_0(a, 0, c)] + [V_0(a - 1, 0, c + 1)]$;
8. If $a \geq 2$, then $[K_0(a, 0, 1)] = [V_0(a, 0, 1)]+[V_0(a-1, 0, 2)]+[V_0(a-1, 0, 0)]$;
9. If $a \geq 2$, then $[K_0(1, 0, a)] = [V_0(1, 0, a)]+[V_0(0, 1, a+1)]+[V_0(0, 0, a+1)]$;
10. $[K_0(1, 0, 1)] = [V_0(1, 0, 1)] + 2[V_0(0, 0, 0)] + [V_0(0, 1, 0)] + [V_0(0, 0, 2)] + [V_0(0, 1, 2)]$;
11. If $a \geq 3$, then $[K_0(a, 0, 0)] = [V_0(a, 0, 0)]+[V_0(a-2, 0, 0)]+[V_0(a-1, 0, 1)]$ and $[K_0(0, 0, a)] = [V_0(0, 0, a)] + [V_0(0, 0, a + 2)] + [V_0(0, 1, a)]$;
12. $[K_0(2, 0, 0)] = [V_0(2, 0, 0)] + [V_0(0, 0, 0)] + [V_0(1, 0, 1)] + [V_0(0, 1, 0)]$;
13. $[K_0(0, 0, 2)] = [V_0(0, 0, 2)] + [V_0(0, 0, 0)] + [V_0(0, 0, 4)] + [V_0(0, 1, 2)]$;
14. $[K_0(1, 0, 0)] = [V_0(1, 0, 0)] + [V_0(0, 0, 1)] + [V_0(0, 1, 1)]$;
15. $[K_0(0, 0, 1)] = [V_0(0, 0, 1)] + [V_0(0, 1, 1)] + [V_0(0, 0, 3)]$;
16. If $b \geq 3$ then $[K_0(0, b, 0)] = 2[V_0(0, b, 0)]+[V_0(0, b+1, 0)]+[V_0(0, b-1, 0)]$;
17. $[K_0(0, 2, 0)] = 2[V_0(0, 2, 0)] + [V_0(0, 1, 0)] + [V_0(0, 3, 0)] + [V_0(0, 0, 2)]$;
18. $[K_0(0, 1, 0)] = 2[V_0(0, 1, 0)] + [V_0(0, 2, 0)] + [V_0(0, 0, 0)] + 2[V_0(0, 0, 2)]$;
19. $[K_0(0, 0, 0)] = 2[V_0(0, 0, 0)] + [V_0(0, 1, 0)] + [V_0(0, 0, 2)]$.

The following trivial statement will be used later.

Lemma 6 $\dim\mathrm{Hom}_{\mathfrak{g}}(K_0(\mu), K_0(\nu)) \leq [K_0(\mu) : V_0(\mu)]$.

4.4 Deformation

We consider $K_t(\lambda)$ as a polynomial one-parameter deformation of $K_0(\lambda)$.

Lemma 7

$$\dim \mathrm{Hom}_{\mathfrak{g}}(K_t(\lambda), K_t(\mu)) \leq \dim \mathrm{Hom}_{\mathfrak{g}}(K_0(\lambda), K_0(\mu)). \tag{2}$$

Proof For any $t, s \neq 0$, λ and μ, we have

$$\mathrm{Hom}_{\mathfrak{g}}(K_t(\lambda), K_t(\mu)) \simeq \mathrm{Hom}_{\mathfrak{g}}(K_s(\lambda), K_s(\mu)). \tag{3}$$

On the other hand,

$$\mathrm{Hom}_{\mathfrak{g}}(K_t(\lambda), K_t(\mu)) = \mathrm{Hom}_{\mathfrak{p}}(L_t(\lambda), K_t(\mu)) = [H^0(\mathfrak{g}_1, K_t(\mu)) : L(\lambda)]. \tag{4}$$

We fix an isomorphism $K_t(\mu) \simeq K_0(\mu)$ of $\mathfrak{g}_0$-modules. Then the isomorphisms (3) and (4) yield

$$[H^0(\mathfrak{g}_1, K_t(\mu)) : L(\lambda)] = [H^0(\mathfrak{g}_1, K_s(\mu)) : L(\lambda)],$$

for all $s, t \neq 0$. Finally, the semicontinuity of invariants implies

$$[H^0(\mathfrak{g}_1, K_t(\mu)) : L(\lambda)] \leq [H^0(\mathfrak{g}_1, K_0(\mu)) : L(\lambda)].$$

□

Proposition 4 *Let $t \neq 0$. Then*

1. *The module $K_t(a, b, c)$ is simple whenever $abc \neq 0$;*
2. *The module $K_t(a, b, c)$ is indecomposable unless $a = c = 0$ and $b \neq 0$.*

Proof For (1) note that, if $\mathrm{Hom}_{\mathfrak{g}}(K_t(\mu), K_t(a, b, c)) = 0$ for all $\mu \neq (a, b, c)$, then $K_t(a, b, c)$ is simple. By Theorem 2 (2) we have $\mathrm{Hom}_{\mathfrak{g}}(K_0(\mu), K_0(a, b, c)) = 0$ for all $\mu \neq (a, b, c)$. Hence (1) follows from Lemma 7.

Let us prove (2). We see from formulas (1)–(19) and Lemma 6 that $\mathrm{End}_{\mathfrak{g}}(K_0(a, b, c)) = \mathbb{C}$ unless $a = c = 0$. Hence the statement follows from Lemma 7. □

4.5 Complexes

Proposition 5 *For every $t \in \mathbb{C}$ we have the following nonzero morphisms:*

1. $\theta_{b,a} : K_t(0, b, a) \to K_t(0, b - 1, a)$, *for all* $b \geq 1$, $a \geq 0$;
2. $\xi_{a,b} : K_t(a, b, 0) \to K_t(a, b + 1, 0)$, *for all* $b \geq 0$, $a \geq 0$;
3. $\eta_{a,c} : K_t(a, 0, c) \to K_t(a + 1, 0, c - 1)$, *for all* $a \geq 0$, $c \geq 1$.

Proof Let us choose nonzero vectors

$$Y_i \in \mathfrak{g}_{\varepsilon_i} \cap \mathfrak{g}_{-1}, \quad i = 1, 2, 3,$$

such that $[e_1, Y_3] = Y_2$, $[e_2, Y_2] = Y_1$. To construct $\theta_{b,a}$, we have to show that $K_t(0, b-1, a)$ contains a $\mathfrak{g}_1$-invariant vector of weight $(0, b, a)$, invariant under the action of the maximal nilpotent subalgebra $[\mathfrak{b}_0, \mathfrak{b}_0]$ of $\mathfrak{g}_0$. Let $v \in K_t(0, b-1, a)$ be the highest weight vector of weight $(0, b-1, a)$. Let $u = Y_1 v$. Note that $[e_i, Y_1] = 0$ for $i = 1, 2, 3$. Hence $e_i u = 0$. Let $Z \in \mathfrak{g}_{\varepsilon_3-\varepsilon_1-\varepsilon_2}$. Then Z is a $\mathfrak{b}_0$-lowest weight vector in $\mathfrak{g}_1$. Hence it suffices to check that $Zu = 0$. Indeed, $Zu = [Z, Y_1]v = f_1 v = 0$ as $h_1 v = 0$.

Using Lemma 3, we define $\xi_{a,b} = \theta^*_{b+1,a}$.

Finally, let us construct $\eta_{a,c}$. Similarly to above we have to show that $K_t(a+1, 0, c-1)$ contains a $\mathfrak{g}_1$-invariant vector u of weight $(a, 0, c)$, invariant under the action of $[\mathfrak{b}_0, \mathfrak{b}_0]$. Let $v \in K_t(a+1, 0, c-1)$ be a highest weight vector. Set

$$u = Y_3 v + \frac{1}{a+1}(-Y_2 f_1 + Y_1 f_2 f_1)v.$$

First, let us check that $e_i u = 0$ for $i = 1, 2, 3$. Note that e_3 commutes with Y_i and f_1, f_2. Therefore $e_3 u = 0$. Furthermore, we have

$$e_1 u = Y_2 v + \frac{1}{a+1}(-Y_2 e_1 f_1 v + Y_1 e_1 f_2 f_1)v = 0$$

since $f_2 v = 0$ and $e_1 f_1 v = (a+1)v$. We also get

$$e_2 u = \frac{1}{a+1}(-Y_1 f_1 v + Y_1 e_2 f_2 f_1 v) = 0$$

since $e_2 f_2 f_1 v = h_2 f_1 v = f_1 v$.

Now let us check that $Zu = 0$. We use the following relations: $[Z, Y_3] = 0$, $[Z, Y_2] = [f_2, f_1]$ and $[Z, Y_1] = f_1$. Therefore

$$Zu = \frac{1}{a+1}(-[f_2, f_1]f_1 + f_2 f_1 f_1)v,$$

again using $f_2 v = 0$ we get

$$Zu = \frac{1}{a+1}(-[f_2, f_1]f_1 + f_1[f_2, f_1])v = \frac{1}{a+1}[f_1, [f_2, f_1]]v = 0.$$

□

Lemma 8 *We have*

$$\theta_{b-1,a}\theta_{b,a} = \xi_{a,b}\xi_{a-1,b} = \eta_{a+1,c-1}\eta_{a,c} = 0.$$

Proof The identity $\theta_{b-1,a}\theta_{b,a} = 0$ follows from the identity $Y_1^2 = 0$. The identity $\xi_{a,b}\xi_{a,b-1} = 0$ follows by duality.

Let us show that $\eta_{a+1,c-1}\eta_{a,c} = 0$. Assume the contrary. Then $K_t(a+2, 0, c-2)$ has a $\mathfrak{b}_0$-semi-invariant vector of weight $(a, 0, b)$. But $[K_t(a+2, 0, c-2) : L(a, 0, c)] = 0$, hence a contradiction. □

The above lemma implies that we have the following complexes:

$$\cdots \to K_t(0,b,a) \to K_t(0,b-1,a) \to \cdots \to K_t(0,0,a) \to 0,$$
$$0 \to K_t(a,0,0) \to K_t(a,1,0) \to \cdots \to K_t(a,b-1,0) \to K_t(a,b,0) \to \dots,$$

and

$$0 \to K_t(0,0,a) \to K_t(1,0,a-1) \to \cdots \to K_t(a-1,0,1) \to K_t(a,0,0) \to 0.$$

We denote these complexes by $\mathcal{C}_{t,a}$, $\mathcal{D}_{t,a}$ and $\mathcal{B}_{t,a}$ respectively. Note that $\mathcal{D}_{-t,a} \simeq \mathcal{C}^*_{t,a}$ and $\mathcal{B}_{-t,a} \simeq \mathcal{B}^*_{t,a}$.

Lemma 9 *Let $a \geq 2$. Then*

$$H_i(\mathcal{C}_{t,a}) = 0 \quad \textit{for } i > 0$$

and

$$H^i(\mathcal{D}_{t,a}) = 0 \quad \textit{for } i > 0.$$

If $a \geq 1$, then

$$H_i(\mathcal{C}_{t,a}) = 0 \quad \textit{for } i > 1$$

and

$$H^i(\mathcal{D}_{t,a}) = 0 \quad \textit{for } i > 1.$$

Proof Note that if $ts \neq 0$, then $H_i(\mathcal{C}_{t,a}) = H_i(\mathcal{C}_{s,a})$. By semicontinuity of homology it suffices to check that $H_i(\mathcal{C}_{0,a}) = 0$ for $i > 0$. Note that $K_0(a,i,0)$ has length 2 for $i \geq 2$. Since both $\mathrm{Ker}\theta_{i,a}$ and $\mathrm{Im}\theta_{i+1,a}$ are proper nonzero submodules of $K_0(0,i,a)$ and $\mathrm{Im}\theta_{i+1,a} \subset \mathrm{Ker}\theta_{i,a}$, we have $\mathrm{Im}\theta_{i+1,a} = \mathrm{Ker}\theta_{i-1,a}$. Hence $H_i(\mathcal{C}_{0,a}) = 0$ for $i > 1$, (formula (2)). In the case $i = 1$ we still have that $\mathrm{Im}\theta_{2,a}$ is a simple $\mathfrak{g}$-module. Using formula (4) we have the following nonsplit exact sequence

$$0 \to V_0(0,0,a+2) \to \mathrm{Coker}\,\theta_{2,a} \to V_0(0,1,a) \to 0.$$

We claim that the socle of $K_0(0,0,a)$ is isomorphic to $V_0(0,0,a+2)$. This follows from

$$\mathrm{soc}\,K_0(0,0,a) \simeq (\mathrm{cosoc}\,K_0(a,0,0))^* \simeq V_0(a,0,0)^* \simeq V_0(0,0,a+2),$$

where the last equality is a consequence of Proposition 5.3.1 in [1]. Therefore $\mathrm{Im}\theta_{1,a}$ contains $V_0(0,0,a+2)$ and $\mathrm{Im}\theta_{1,a} \simeq \mathrm{Coker}\,\theta_{2,a}$, which implies $H_1(\mathcal{C}_{t,a}) = 0$.

The statement about $\mathcal{D}_{t,a}$ follows by duality. □

Lemma 10 *Let $a \geq 1$. The kernel of $\theta_{i,a} : K_t(0,i,a) \to K_t(0,i-1,a)$ for $i > 1$ and the kernel of $\xi_{a,i} : K_t(a,i,0) \to K_t(a,i+1,0)$ for $i > 2$ are simple $\mathfrak{g}$-modules. Moreover, if $a \geq 2$ then $H_0(\mathcal{C}_{t,a})$ and $H^0(\mathcal{D}_{t,a})$ are also simple $\mathfrak{g}$-modules.*

Proof We prove both statements for $\mathcal{C}_{t,a}$, the statements for $\mathcal{D}_{t,a}$ follow by duality. Assume that $M = \mathrm{Ker}\theta_{i,a}$ is not simple. Then $H^0(\mathfrak{g}_1, M)$ has at least two $\mathfrak{g}_0$-irreducible components. Therefore there are $\mu_1, \mu_2 \neq (a,i,0)$ such that $\mathrm{Hom}_{\mathfrak{g}}(K_t(\mu_j), K_t(0,i,a)) \neq 0$. However, this is false for $t = 0$, hence it is false for $t \neq 0$ by Lemma 7.

By direct computation we have

$$\begin{aligned}\mathrm{Res}K_t(0,1,a) &= \tilde{L}(0,1,a+1) \oplus \tilde{L}(0,1,a-1) \oplus \tilde{L}(1,1,a) \oplus \tilde{L}(0,0,a+1)\\ &\quad \oplus \tilde{L}(0,2,a-1) \oplus \tilde{L}(1,0,a),\\ \mathrm{ResIm}\theta_{1,a} &= \tilde{L}(0,0,a+1) \oplus \oplus \tilde{L}(0,1,a-1) \oplus \tilde{L}(1,0,a).\end{aligned}$$

and

$$\mathrm{Res}K_t(0,0,a) = \tilde{L}(0,0,a+1) \oplus \tilde{L}(0,0,a-1) \oplus \tilde{L}(0,1,a-1) \oplus \tilde{L}(1,0,a).$$

This implies $\mathrm{Res}H_0(\mathcal{C}_{t,a}) = \tilde{L}(0,0,a-1)$. Therefore $\mathrm{Res}H_0(\mathcal{C}_{t,a})$ is simple. □

Lemma 11 *Let $a \geq 2$. Then*

$$H^i(\mathcal{B}_{t,a}) = 0 \quad \textit{for } 0 < i < a+1.$$

Furthermore, if $a \geq 4$, then $\mathrm{Ker}\eta_{i,a-i}$ is simple for $2 \leq i \leq a-1$ and $H^0(\mathcal{B}_{t,a})$, $H^{a+1}(\mathcal{B}_{t,a})$ are simple $\mathfrak{g}$-modules.

Proof As in Lemma 9 we will prove the statement for $\mathcal{B}_{0,a}$. First assume $a \geq 3$. For $1 < i < a$ the proof goes exactly as the proof of Lemma 9 with use of formula (7) and we leave it to the reader. To check that $H^1(\mathcal{B}_{0,a}) = 0$ one can show that the socle of $K_0(0,0,a)$ is $V_0(0,0,a+2)$, and the socle of $K_0(1,0,a-1)$ is $V_0(0,1,a)$. Then formulas (9) and (11) imply $\mathrm{Im}\eta_{0,a} = \mathrm{Ker}\eta_{1,a-1}$. The identity $H^a(\mathcal{B}_{0,a}) = 0$ follows via duality.

Now consider the case $a = 2$. We have to show that the complex

$$0 \to K_0(0,0,2) \to K_0(1,0,1) \to K_0(2,0,0) \to 0$$

is exact in the middle. The socle of $K_0(1,0,1)$ is $V_0(0,1,2)$ and the socle of $K_0(2,0,0)$ is $V_0(0,1,0)$. Formula (10) ensures the exactness.

The last assertion can be proven as in Lemma 10. □

4.6 The structure of some degenerate Kac modules

Note that the complexes $\mathcal{C}_{t,0}$ and $\mathcal{D}_{t,0}$ have the same terms. Therefore we have the following diagram

$$0 \leftrightarrows K_t(0,0,0) \leftrightarrows K_t(0,1,0) \leftrightarrows \cdots \leftrightarrows K_t(0,b,0) \leftrightarrows K_t(0,b+1,0) \leftrightarrows \cdots$$

It is easy to see that $\theta_{b+1,0}\xi_{0,b} \neq 0$ and $\xi_{0,b-1}\theta_{b,0} \neq 0$.

Lemma 12 *Let $b \geq 1$. Then*

$$K_t(0,b,0) = \mathrm{Im}\theta_{b+1,0}\xi_{0,b} \oplus \mathrm{Im}\xi_{0,b-1}\theta_{b,0}.$$

For $b \geq 2$ the image of $\theta_{b+1,0}\xi_{0,b}$ and the image of $\xi_{0,b-1}\theta_{b,0}$ are simple $\mathfrak{g}$-modules.

Proof By Corollary 3 $\theta_{b+1,0}\xi_{0,b}$ and $\xi_{0,b-1}\theta_{b,0}$ are orthogonal idempotents in $\mathrm{End}_{\mathfrak{g}}(K_t(0,b,0))$. Furthermore, by a straightforward computation we have

$$\mathrm{Res}(\mathrm{Im}\theta_{b+1,0}\xi_{0,b}) = \tilde{L}_t(1,b,0) \oplus \tilde{L}_t(0,b,1)$$

and

$$\mathrm{Res}(\mathrm{Im}\xi_{0,b-1}\theta_{b,0}) = \tilde{L}_t(1,b-1,0) \oplus \tilde{L}_t(0,b-1,1).$$

Assume that $\mathrm{Im}\theta_{b+1,0}\xi_{0,b}$ is not simple. Then $\mathrm{Im}\theta_{b+1,0}\xi_{0,b}$ has simple socle M, and hence there exists a Kac module $K_t(\mu)$ and a morphism $K_t(\mu) \to K_t(0,b,0)$ with image equal to M. However, if $b \geq 3$, by formula (16) and Lemma 7, $\theta_{b+1,0}$ and $\xi_{0,b-1,0}$ exhaust the list of such morphisms. Moreover, the same is true for $b = 2$, since an additional morphism may only exist for $\mu = (0,0,2)$. But $\mathrm{Hom}_{\mathfrak{g}}(K_0(0,0,2), K_0(0,2,0) = 0$ as the socle $V_0(0,2,0)$ of $K_0(0,2,0)$ does not appear among simple constituents of $K_0(0,0,2)$. Hence we obtain a contradiction. □

5 Simple modules with nonzero central charge

5.1 Classification

Using the results of the previous section we will obtain the classification of simple objects in $\mathrm{Rep}^t_{\mathfrak{g}}$ for $t \neq 0$. By Proposition 4(a), every $K_t(a,b,c)$ with $abc \neq 0$ is simple, $d(K_t(a,b,c)) = 8$ and

$$G(K_t(a,b,c)) = x^a y^b z^c (x + yx^{-1} + zy^{-1} + z^{-1} + x^{-1} + xy^{-1} + yz^{-1} + z).$$

We will call such simple modules typical and all others atypical.

By Proposition 3 every atypical simple module is isomorphic to a quotient of $K_t(a,b,c)$ with $abc = 0$, hence is isomorphic to a quotient of some term of one of complexes $\mathcal{C}_{t,a}, \mathcal{D}_{t,a}, \mathcal{B}_{t,a}$. Therefore we obtain the following.

Corollary 4 *If S is an atypical simple $\mathfrak{g}$-module. Then $d(S)$ is* 1*,* 2 *or* 3*.*

Next we use Lemmas 9, 10 and 11 to finish the classification. The first step is to find all S with $d(S) = 3$.

Proposition 6 *Assume $d(S) = 3$. Then S isomorphic to one of the following:*

1. $V_t(0, b, a) := \mathrm{Im}\theta_{b,a}$, *where* $\theta_{b,a} : K_t(0, b, a) \to K_t(0, b-1, a)$ *with* $b \geq 2, a \geq 1$*;*
2. $V_t(0, b, a) := \mathrm{Ker}\xi_{a,b}$*, where* $\xi_{a,b-1} : K_t(a, b-1, 0) \to K_t(a, b+1, 0)$ *with* $b \geq 2, a \geq 1$*;*
3. $V_t(a, 0, c) := \mathrm{Coker}\eta_{a,c+1}$*, where* $\eta_{a,c+1} : K_t(a, 0, c+1) \to K_t(a+1, 0, c)$ *with* $a, c \geq 1$.

Moreover,

$$
\begin{aligned}
G(V_t(0, b, a)) &= xy^{b-1}z^a + y^{b-1}z^{a+1} + y^b z^{a-1} \text{ in } (1);\\
G(V_t(a, b, 0)) &= x^a y^{b-1} z + x^{a+1} y^{b-1} + x^{a-1} y^b \text{ in } (2);\\
G(V_t(a, 0, c)) &= x^a y z^c + x^{a+1} z^{c+1} + x^a z^c \text{ in } (3).
\end{aligned}
$$

Proof Apply the functor Res to the complexes $\mathcal{C}_{t,a}, \mathcal{D}_{t,a}, \mathcal{B}_{t,a}$. For example, if S appears in the complex $\mathcal{C}_{t,a}$, then $\mathrm{ResIm}\theta_{b,a} = \mathrm{ResKer}\theta_{b-1,a}$ consists of all components common for $K_t(0, b, a)$ and $K_t(0, b-1, a)$. The other cases are similar. □

Consider the sequence

$$K_t(0, 1, a) \xrightarrow{\theta_{1,a}} K_t(0, 0, a) \xrightarrow{\eta_{0,a}} K(1, 0, a),$$

and set $V_t(0, 0, a-1) := \mathrm{Coker}\theta_{1,a}$ and $V_t(0, 0, a+1) := \mathrm{Ker}\eta_{0,a}$. Similarly, for the sequence

$$K_t(a, 0, 1) \xrightarrow{\eta_{a,1}} K_t(a, 0, 0) \xrightarrow{\xi_{a,0}} K(a, 1, 0),$$

set $V_t(a+1, 0, 0) := \mathrm{Ker}\xi_{a,0}$ and $V_t(a-1, 0, 0) := \mathrm{Coker}\eta_{a,1}$.

Lemma 13 Res $V_t(a, 0, 0) \simeq \tilde{L}_t(a, 0, 0)$ *and* Res $V_t(0, 0, a) \simeq \tilde{L}_t(0, 0, a)$.

Proof Straightforward by computing the functor Res for the corresponding sequences. □

Proposition 7 *Assume $d(S) = 1$. Then* $\mathrm{Res}S \simeq \tilde{L}_t(a, 0, 0)$ *or* $\mathrm{Res}S \simeq \tilde{L}_t(0, 0, a)$*, and therefore S is isomorphic to $V_t(a, 0, 0)$ or $V_t(0, 0, a)$.*

Proof Assume $\mathrm{Res}S = \tilde{L}_t(a, b, c)$. First, we will prove that $b = 0$. We use the root decomposition of $\mathfrak{g}$. Fix the set of positive roots

$$\Delta^+ := \{\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq 3\} \cup \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_1 + \varepsilon_2 + \varepsilon_3, \varepsilon_1 - \varepsilon_2 - \varepsilon_3\},$$

and let

$$\mathfrak{b} := \mathfrak{t} \oplus \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha .$$

Then S has a unique, up to proportionality, highest weight vector v with respect to the Borel subalgebra $\mathfrak{b}$. It is clear that the weight of v equals $(a, b, c) + \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) = (a, b, c + 1)$. It is easy to see that $\dim \mathfrak{g}_{\pm\varepsilon_3} = 2$, and one can choose $y^{\pm} \in \mathfrak{g}_{-\varepsilon_3} \cap \mathfrak{g}_{\pm 1}$ and $x^{\pm} \in \mathfrak{g}_{\varepsilon_3} \cap \mathfrak{g}_{\pm 1}$ so that

$$[x^-, y^-] = z, \quad [x^+, y^+] = 0, \quad , [x^+, y^-] = [x^-, y^+] = h_2.$$

We claim that if $b > 0$ then $u := y^+y^-v \neq 0$. Indeed,

$$x^-x^+y^+y^-v = -x^-y^+x^+y^-v = -x^-y^+h_2v = -h_2v = -b^2v.$$

On the other hand, it is easy to check that $e_i u = 0$ for $i = 1, 2, 3$. Thus, we obtain that S must have a simple $\mathfrak{g}_0$-component isomorphic to $L(a+2, b, c-1)$. But $\tilde{L}_t(a, b, c)$ does not have such a component.

Now we assume that $b = 0$. Then S appears as a subquotient in one of the following Kac modules:

$$K_t(a \pm 1, 0, c), K_t(a-1, 1, c), K_t(a, 1, c-1), K_t(a, 0, c \pm 1.)$$

If $a, c > 1$, then all these Kac modules have two simple constituents with $d = 3$ by Proposition 6. Hence $a = 1$ or $c = 1$. Assume for example that $a = 1$. Then S is a subquotient of $K_t(0, 0, c)$. Since

$$\mathrm{Res} K_t(0, 0, c) \simeq \tilde{L}_t(0, 0, c-1) \oplus \tilde{L}_t(0, 0, c+1) \oplus \tilde{L}_t(1, 0, c) \oplus \tilde{L}_t(0, 1, c-1),$$

by Proposition 13 we have that $K_t(0, 0, c)$ has length 4 and, in particular, there is a simple constituent S' such that $\mathrm{Res} S' \simeq \tilde{L}_t(0, 1, c-1)$. However, this contradicts the assertion we just proved above. The other cases are similar. □

The remaining case $d(S) = 2$ is now easy to deal with.

Proposition 8 *Assume that $d(S) = 2$. Then S is isomorphic to one of the following:*

1. $V_t(0, 1, a) := \mathrm{Im}\theta_{1,a}/\mathrm{Ker}\eta_{0,a}$ *and* $G(V_t(0, 1, a)) = yz^{a-1} + xz^a$*, where* $a \geq 1$*;*
2. $V_t(a, 1, 0) := \mathrm{Im}\eta_{a-1,1}/\mathrm{Ker}\xi_{a,0}$ *and* $G(V_t(0, 1, a)) = x^{a-1}y + x^a z$*, where* $a \geq 1$*;*
3. $V_t(0, b, 0) := \mathrm{Im}\theta_{b+1,0} = \mathrm{Ker}\xi_{0,b}$ *and* $G(V_t(0, b, 0)) = xy^b + y^b z$*, where* $b \geq 1$.

Proof We just list all simple subquotients appearing in the complexes $\mathcal{C}_{t,a}, \mathcal{D}_{t,a}, \mathcal{B}_{t,a}$ which do not appear in Propositions 13 and 6. □

5.2 Characters

It is easy now to find the characters of simple $\mathfrak{g}$-modules with nonzero central charge. It suffices to use the following.

Proposition 9 *Let $M \in \mathrm{Rep}^t_{\mathfrak{g}}$ with $t \neq 0$. Then*

$$\mathrm{ch}\, M = R\frac{\sum_{w\in W}(-1)^w w(xyzG(M))}{\sum_{w\in W}(-1)^w w(xyz)},$$

where W is the Weyl group of $\mathfrak{g}_0$ and

$$R = x + x^{-1} + z + z^{-1} + xy^{-1} + x^{-1}y + yz^{-1} + y^{-1}z.$$

Proof The claim follows immediately from the formula

$$\mathrm{ch}\, \tilde{L}_t(a, b, c) = R\frac{\sum_{w\in W}(-1)^w w(x^{a+1}y^{b+1}z^{c+1})}{\sum_{w\in W}(-1)^w w(xyz)}.$$

□

6 Some remarks on projective modules

Lemma 14 *Let $t \neq 0$, then $U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} \tilde{L}_t(a, b, c)$ is projective in $\mathrm{Rep}^t_{\mathfrak{g}}$.*

Proof The induction functor $\mathrm{Rep}^t_{\mathfrak{h}} \to \mathrm{Rep}^t_{\mathfrak{g}}$ is left adjoint to Res. Hence it maps projective objects to projective objects. By Theorem 1(b), $\tilde{L}_t(a, b, c)$ is projective in $\mathrm{Rep}^t_{\mathfrak{h}}$. Therefore $U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} \tilde{L}_t(a, b, c)$ is projective in $\mathrm{Rep}^t_{\mathfrak{g}}$. □

Proposition 10 *The category $\mathrm{Rep}^t_{\mathfrak{g}}$ has enough projective objects, and every indecomposable projective object is a direct summand in $U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} \tilde{L}_t(a, b, c)$ for some (a, b, c). If we denote by $P(S)$ a projective cover of a simple module S, then*

$$[U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} \tilde{L}_t(a, b, c) : P(S)] = [\mathrm{Res} S : \tilde{L}_t].$$

In particular, $[U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} \tilde{L}_t(a, b, c) : P(S)] \leq 1$.

Remark 2 Since $\mathrm{Rep}^t_{\mathfrak{g}}$ is a Frobenius category, a similar statement holds for indecomposable injective modules in $\mathrm{Rep}^t_{\mathfrak{g}}$.

Proof Everything follows from the Frobenius reciprocity isomorphism

$$\mathrm{Hom}_{\mathfrak{g}}(U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} \tilde{L}_t(a, b, c), S) = \mathrm{Hom}_{\mathfrak{h}}(\tilde{L}_t(a, b, c), S),$$

and from the fact that $\mathrm{Res} S$ is multiplicity free (see Lemma 5). □

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