# Packing potential index for binary mixtures of granular soil 

Ching S. Chang*, Yibing Deng<br>Department of Civil and Environmental Engineering, University of Massachusetts, Amherst, MA 01003, United States of America

## A R T I C L E I N F O

## Article history:

Received 26 March 2020
Received in revised form 20 May 2020
Accepted 1 June 2020
Available online 06 June 2020

## Keywords:

Particle packing model
Minimum and maximum void ratios
Sand-silt mixture
Particle shape
Packing procedure


#### Abstract

Packing procedure is the mechanical process of forming a packing of soil particles, such as funnel pouring, tamping, rodding, pluviation, compaction, vibration, compression, etc. For a sand-silt mixture, packing procedure and particle shape have significant effects on the density of the binary mixture. However, these two factors have not been considered in most of the existing particle packing density models. Thus, the existing particle packing density models are not applicable to sand-silt mixtures. In this paper, we aim to study the packing procedure and particle shape effects on density of binary mixtures. We firstly define a packing potential index, which is a measure of volume reduction potential due to mixing of two components of a binary mixture system under a packing procedure. To understand the nature of packing potential index, we compare the packing potential indices of 24 different types of mixtures collected from the literature; the 24 types of mixtures were formed by two different types of packing procedure (i.e., for achieving minimum and maximum void ratios). It is found that the packing potential index is nearly independent of packing procedure but significantly dependent on the compound particle shapes of a mixture. Then, we mathematically link the packing potential index to the particle interaction parameters used in the particle packing density models. And we analyze the data to discuss the effect of packing procedure on the void ratios of sand-silt mixtures. We then propose an approach within the framework of particle packing density model to predict the void ratios of sand-silt mixtures under different packing procedures with the consideration of particle shape effect.


© 2020 Elsevier B.V. All rights reserved.

## 1. Introduction

The paper is motivated by problems of silty sand from soil mechanics, in which, the variable of void ratio, instead of packing density, is commonly used. The void ratio $e$ can be related to the packing density $\phi$ by: $\phi=\frac{1}{1+e}$ or $e=(1 / \phi)-1$

There are several analytical models to study the void ratios of binary particle mixtures in many branches of industry, such as ceramics processing [1], powder metallurgy [2], and concrete mixes [3]. Among these models, the most popular ones are based on the hypothesis of two mechanisms of particle arrangements [4,5]: (i) the filling mechanism of the fine particles filling into voids among coarse particles; (ii) the embedment mechanism of coarse particles occupying solid volumes in place of porous bulk volumes of the fine particles. The filling mechanism occurs for mixtures with low fines contents; and the embedment mechanism occurs for mixtures with high fines contents. In these two mechanisms, the models did not consider particle interactions that cause packing disturbance; thus, the models only provide good estimates of lower bound solutions.

[^0]These models were then evolved to consider the effect of particle interaction. During filling of fine particles, loosening of the coarse particle network may occur when fine particles are pushed into the voids among coarse particles. On the other hand, during embedment of coarse particles, disrupting the packing of fine particles may occur at the walllike boundaries of coarse particles. The packing model introduced by Powers [3] considers the loosening effect. The packing models developed by Aïm and Goff [6] and Toufar et al. [7] account for the wall effect. The packing models developed by Yu et al. [8], Goltermann et al. [9], Stovall et al. [10], De Larrard [11], Dewar [12], and Kwan et al. [13] take into account of both the loosening and wall effects.

The loosening and wall effects have been found to be significantly affected by particle size ratio $r$ (i.e., ratio of fine to coarse particle sizes) [14]. Thus, the effects are expressed as particle interaction functions dependent on the particle size ratio. The two parameters ( $a$ and $b$ ) in the particle interaction functions were obtained by regression analysis of experimental results on packing densities for mixtures with different size ratios. The interaction functions derived in the models by Yu et al. [8], de Larrard [11], and Kwan [13] have different forms.

It is obvious that the loosening and wall effects can be affected by other factors of particle morphology, such as particle shape, roundness and surface texture roughness. Among several aspects of morphology, the particle shape has been considered in the model by Yu et al. [8], however, the particle shape considered was simple idealized
nonspherical shape (e.g., cylinders, disks). For most material, the particle morphology is complex and difficult to be measured quantitatively. Hence, from either a theoretical or a practical point of view, the complex particle morphology cannot be considered in the model in a fully satisfied manner. And most currently available models do not consider factors of particle morphology. Because of this limitation in models, the evaluation of several models by Jones [15] indicated that each of these models is applicable only to a certain type of industrial material. Also indicated by Chang et al. [16], due to a large span of size and shape differences, the parameters used in these models are not suitable for sand-silt mixtures.

Furthermore, the complexity involves not only particle morphology, but also the packing procedure (i.e., method of mixing, placement and compaction), by which the binary packing is physically formed. The factor of packing procedure is not addressed in most binary packing models. de Larrard [11] developed the "compressible packing model" (CPM) by introducing the compaction index $K$, which is assumed to be related to the applied compaction effort, thus is dependent on the packing procedure. The value of $K$ is an empirical parameter varying from 4 to 9 suggested for pouring, rodding, vibration and compression, and is also varying with grain shape (round \& crushed) [11]. He proposed a method to convert the virtual packing density to the actual packing density through the compaction index. But the CPM is not suitable for geotechnical materials such as silty sand [16].

In geotechnical engineering, minimum and maximum void ratios ( $e_{\text {max }}$ and $e_{\text {min }}$ ), which represent the densest and loosest states of a soil mixture, are widely applied in earthwork design and planning. The packing procedures of achieving $e_{\max }$ and $e_{\min }$ are very different. Thus, to understand the effect of the packing procedures on the density of mixtures is important in geotechnical engineering.

In this paper, we aim to study the effects of particle shape and packing procedure on densities of binary mixtures. We firstly define a packing potential index, the value of which is a number between 0 and 1. This index is a measure of void reduction potential due to mixing of two components of a binary mixture under a packing procedure. To study how the packing potential index may vary with the type of mixtures and with the type of packing procedure, we compare the packing potential indices for 24 systems of soil mixtures collected from the literature; the 24 systems of mixtures were formed under two different types of packing procedure (i.e., for achieving minimum and maximum void ratios).

Then, we mathematically derive the relationship between the packing potential index and the particle interaction parameters, and analyze the data to discuss the effect of packing procedure on void ratio of mixtures. Finally, we discuss an approach, under the framework of particle packing model, for predicting void ratio ( $e_{\max }$ and $e_{\min }$ ) of a mixture under different packing procedures with the consideration of particle shape effect.

## 2. Packing potential index

Consider a binary packing mixture composed of 2 groups of particles. The particle sizes for the two groups are denoted by $d_{1}$ and $d_{2}$, respectively (note that $d_{1} \geq d_{2}$ ). The solid volume fractions of the two groups of particles are denoted by $y_{1}$ and $y_{2}$, respectively (note that $y_{1}+y_{2}=1$ ).

We aim to determine the void ratio $e$ of a binary soil mixture based on the monodisperse void ratios of the two components ( $e_{1}$ and $e_{2}$ ). Note that the void ratios, $e, e_{1}$ and $e_{2}$, are measured from three packings formed by the same packing procedure.

The void ratio $e$ of a binary soil mixture is between the upper bound and lower bound void ratios, which can be constructed by the monodisperse void ratios of the two components ( $e_{1}$ and $e_{2}$ ). The upper bound void ratio, $e^{U}$, is hypothesized to be the Voigt average of the monodisperse void ratios, given by
$e^{U}=e_{1} y_{1}+e_{2} y_{2}$

The upper bound is plotted as line $A B$ in Fig. 1, in which the fines content, $f_{c}=y_{2}$.

In contrast, the lower bound void ratio, $e^{L}$, is derived by assuming that the two groups of particles in the mixture have no interactions [4,5]. There are two scenarios: (1) every fine particle exists only in the void space between coarse particles (i.e., in the coarse particle dominant region), which is shown as line AC in Fig. 1, or (2) every coarse particle is fully dispersed in the matrix of fine particles (i.e., in the fine particle dominant region), which is shown as line CB in Fig. 1. The equations for $A C$ and $C B$ are given respectively by.
$e^{L}=e_{1}-\left(1+e_{1}\right) \mathrm{y}_{2} ; e^{L}=e_{2} y_{2}$

The slope of AC is $-\left(1+e_{1}\right)$ and the slope of CB is $e_{2}$ as shown in Fig. 1. The line ACB is the lower bound.

For a system of mixtures (i.e., mixtures with the same two components of various combinations), the void ratios of the binary mixtures with various $f_{c}$ are between the upper and the lower bounds as the curve ADB shown in Fig. 1. For convenient, we define a packing potential index $\Omega$ as the ratio of area ADB to area ACB , which is a material descriptor for a system of mixtures. This index is a measure of volume reduction potential due to mixing of two components of a binary mixture system under a packing procedure, which is a simple scaler and can be directly obtained from experimental data. Thus, it is convenient to be used for studying the effect of particle shape and packing procedure. The value of packing potential index $\Omega$ is between 0 and 1 . The higher value of $\Omega$ indicates a higher potential of volume reduction of the mixtures.

For the case of $e_{1}=e_{2}$, the monodisperse void ratio is same for both components and is the upper bound. Under the same packing procedure, the binary mixtures, for all fines content, can generally be packed to a denser packing than the monodisperse packing. The packing potential index $\Omega$ indicates roughly how much denser the binary mixtures can be effectively achieved compared to the monodisperse packing.


Fig. 1. Upper and lower bounds of void ratios of a system of mixtures under a packing procedure.


Fig. 2. (a) Test results for steel shots reported by McGeary [14], (b) Test results for glass beads reported by Kwan et al. [13], (c) the effect of particle size ratio on packing potential index obtained from test results on steel shots and glass beads.

The purpose of a packing density model is to predict the void ratio $e$ of a binary mixture based on the values $e_{1}$ and $e_{2}$ of the monodisperse packings. Thus, it is important to study various factors that affect the packing potential index.

## 3. Factors affect packing potential index

### 3.1. Effect of particle morphology

The packing potential index is significantly affected by particle size ratio $r$ (i.e., $r=d_{2} / d_{1}$, the particle size of fine particles divided by the particle size of coarse particles) as indicated in the test results of spherical steel shots by McGeary [14] and in the test results of spherical glass beads by Kwan et al. [13], as shown in Fig. 2a and Fig. 2b. Steel shots and glass beads are round particles. The particle size range is from 0.16 mm to 3.14 mm for steel shots, and from 1.43 mm to 15.73 mm for glass beads. The packing potential indices are plotted for mixtures with various size ratio in Fig. 2c. The size ratio of fine particles to coarse particles ranges from 0.05 to 0.75 . For binary mixtures with small size ratios ( $\mathrm{d}_{1}>$ $>\mathrm{d}_{2}$ ), the packing potential indices are nearly 1 , meaning that the mixture is more capable of specific volume reduction and can be packed approaching the lower bound solution. Whereas, for mixtures with large size ratios ( $\mathrm{d}_{1} \approx \mathrm{~d}_{2}$ ), the packing potential indices are nearly zero, and the void ratios of the mixtures can be achieved approaching to the upper bound.

It is noted that, for a binary mixture of steel shots or of glass beads, the mixture is composed of two groups of mono-sized particles. However, for silty-sand, the two groups of particles are not mono-sized. The sand particles are relatively uniform, but the silt particles usually have a wider range of sizes. Thus, the sand-silt mixture is a deviation from the standard meaning of binary mixtures defined in particle
packing models. However, in this study, we have neglected this factor because the grain size distribution of silt is seldom measured in geotechnical practice, and the information on measured void ratios of mono- sized silt is not available in the literature. The mean particle size for sand or silt is referred to as the particle size denoted by $d_{1}$ or $d_{2}$ in this study.

Besides particle size ratio, it is reasonable to expect that particle morphology is also a crucial factor that influences the packing potential index. There are many aspects of particle morphology, which can be generally expressed in terms of elongation ratio (i.e., aspect ratio), roundness, sphericity, angularity and surface roughness [17,18]. At a larger scale, the term "sphericity" is used to characterize the overall shape of the granular particle by a measure of the degree of conformity of particle shape to that of sphere circumscribing the particle [19]. At a smaller scale, the term "roundness" defined by Wadell [20] is used to describe the degree of sharpness of particle edges/corners. At an even smaller scale, surface roughness [21] is used to describe the surface texture. There is no consensus on which descriptor is better to characterize the particle morphology, for example of the overall particle shape alone, there are three measures: aspect ratio, sphericity, angularity, etc.

Although digital image analysis and computed tomography techniques have been employed to quantitatively characterize the aggregate morphology [22], in general practice, the morphology parameters are often not measured in experiments. In most test results presented in the literature, only qualitative descriptors of particle shapes (such as round, angular, sub-angular, etc.) are provided.

Fig. 3 illustrates the measured void ratios and the calculated upper and lower bounds for 3 systems of mixtures under the same packing condition. The compound particle shapes are denoted as coarse particle shape/ fine particle shape for the following 3 systems of mixtures: Steel shots (round/round), Silica sand-silt (subangular/subangular), and


Fig. 3. The packing potential index for three systems of mixtures.

Table 1
List of material properties for 24 sets of binary soil mixtures.

| Sand/silt mixture | Ref. | $d_{1}(\mathrm{~mm})$ | $d_{2}(\mathrm{~mm})$ | $e_{\text {min }}$ |  | $e_{\text {max }}$ |  | Particle shape (coarse) | Particle shape (fine) | $p$ | $s$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  |  | $e_{1}$ | $e_{2}$ | $e_{1}$ | $e_{2}$ |  |  |  |  |
| Ottawa 50/200-Nevada fines | [23] | 0.2 | 0.05 | 0.548 | 0.754 | 0.806 | 1.181 | Angular | Angular | 2.8 | 1.8 |
| Ottawa F95-Nevada fines | [23] | 0.16 | 0.05 | 0.58 | 0.754 | 0.868 | 1.179 | Subround | Angular | 3.6 | 2.3 |
| Nevada 50/200-Nevada fines | [23] | 0.14 | 0.04 | 0.57 | 0.754 | 0.878 | 1.181 | Subangular | Angular | 3.5 | 2.5 |
| Nevada 50/80-Nevada fines | [24] | 0.21 | 0.05 | 0.581 | 0.754 | 0.855 | 1.183 | Subangular to Angular | Subangular to Angular | 3.5 | 1.6 |
| Nevada 80/200-Nevada fines | [24] | 0.12 | 0.05 | 0.617 | 0.754 | 0.938 | 1.169 | Subangular to angular | Subangular to angular | 3.8 | 2.2 |
| Nevada 50/80-Nevada80/200 + fines | [24] | 0.17 | 0.05 | 0.581 | 0.754 | 0.876 | 1.180 | Subangular to angular | Subangular to angular | 3.3 | 2.3 |
| Hokksund | [25] | 0.45 | 0.04 | 0.57 | 0.76 | 0.949 | 1.413 | Sharp edges, cubical | Angular, subangular | 3.8 | 3.8 |
| MGM | [26] | 0.12 | 0.01 | 0.755 | 1 | 1.247 | 2.740 | Highly angular to subround | Thin and plate-like | 13 | 1.2 |
| Vietnam | [27] | 0.37 | 0.16 | 0.552 | 0.583 | 0.703 | 0.755 | Subangular | Subangular | 4 | 1.4 |
| Cambria-Nevada fines | [24] | 1.5 | 0.05 | 0.538 | 0.754 | 0.765 | 1.176 | Round | Angular | 10 | 3 |
| Cambria-Nevada 50/80 | [24] | 1.5 | 0.21 | 0.538 | 0.581 | 0.765 | 0.854 | Round | Subangular | 3 | 3 |
| Cambria-Nevada 80/200 | [24] | 1.5 | 0.12 | 0.538 | 0.624 | 0.768 | 0.937 | Round | Angular | 6.5 | 2.5 |
| Nevada 50/80- Nevada 80/200 | [24] | 0.21 | 0.12 | 0.581 | 0.617 | 0.854 | 0.938 | Subangular to angular | Subangular to angular | 2.4 | 1.9 |
| Foundry sand/crushed silica fines | [28] | 0.25 | 0.01 | 0.608 | 0.627 | 0.8 | 2.1 | Round to subround | Angular | 5.5 | 6 |
| Silica\#16-\#18 \#30-\#50 | [29] | 1.08 | 0.4 | 0.633 | 0.644 | 0.970 | 1.048 | Subangular | Subangular | 1.75 | 2.2 |
| Silica\#16-\#18 \#30-\#80 | [29] | 1.08 | 0.42 | 0.633 | 0.59 | 0.970 | 0.996 | Subangular | Subangular | 1.9 | 1.9 |
| Silica\#16-\#18 \#50-\#80 | [29] | 1.08 | 0.26 | 0.633 | 0.696 | 0.970 | 1.114 | Subangular | Subangular | 2.2 | 2 |
| Silica\#16-\#18 \#80-\#100 | [29] | 1.08 | 0.17 | 0.633 | 0.682 | 0.97 | 1.121 | Subangular | Subangular | 2.6 | 1.8 |
| Silica\#16-\#18 \#80-\#120 | [29] | 1.08 | 0.14 | 0.633 | 0.697 | 0.97 | 1.124 | Subangular | Subangular | 2.9 | 1.8 |
| Silica\#16-\#18 \#80-\#200 | [29] | 1.08 | 0.1 | 0.633 | 0.651 | 0.97 | 1.084 | Subangular | Subangular | 3.1 | 2.5 |
| Silica\#16-\#18 \#100-\#120 | [29] | 1.08 | 0.14 | 0.633 | 0.697 | 0.97 | 1.125 | Subangular | Subangular | 2.3 | 2 |
| Silica\#16-\#18 \#100-\#200 | [29] | 1.08 | 0.1 | 0.633 | 0.668 | 0.97 | 1.084 | Subangular | Subangular | 2.7 | 2 |
| Silica\#16-\#18 \#120-\#200 | [29] | 1.08 | 0.1 | 0.633 | 0.682 | 0.970 | 1.115 | Subangular | Subangular | 4.3 | 2.2 |
| Silica\#16-\#18 \#200-\#400 | [29] | 1.08 | 0.06 | 0.633 | 0.7 | 0.97 | 1.091 | Subangular | Subangular | 4 | 2 |

Cambria-Nevada sand-silt (round/angular). Note that the particle size ratios for these 3 systems of mixtures are nearly same, but the three packing potential indices are different as shown in Fig. 3. The shapes of coarse particles and fine particles are similar for steel shots and silica sand-silt, whereas different for Cambria-Nevada sand-silt. Because of the effect of particle shape the three packing potential indices do not follow the curves of sphere particles previously shown in Fig. 2c. Hence, not only the particle shape of a single component but also the compound particle shapes of two components have significant effect on the packing potential index.

To further investigate the effect of particle shape, 13 sets of spherical particles binary mixtures and 24 sets of sand-silt mixtures from the literature are collected (see Table 1) for studying the packing potential index as a function of particle shape in a qualitative way. The compound particle shapes of the 37 systems of mixtures are classified into three groups: round/round ( $\mathrm{R} / \mathrm{R}$ ), angular/angular or subangular/subangular (A/A, SA/SA), and round/ angular or round/subangular (R/A, R/SA).


Fig. 4. The packing potential index versus particle size ratio for mixtures with 3 groups of compound particle shapes.

The computed packing potential indices versus particle size ratio are shown in Fig. 4 for the three groups of compound particle shapes. Fig. 4 shows that the effect of particle shape is significant on the values of packing potential index. As shown in Fig. 4, given a particle size ratio, for binary mixtures composed of two similar shape components, the packing potential index $\Omega$ of a R/R mixture is greater than that of a $A /$ A or SA/SA mixture. The packing potential index $\Omega$ of a mixture composed of two different shape components is usual smaller than that of a mixture composed of two similar shape components.

### 3.2. Effect of packing procedure

In geotechnical engineering, the loosest and densest density states (i.e., maximum and minimum void ratios) of a soil are of interest. Several packing procedures have been used for the two limiting void ratios, such as moist tamping, vibratory table, customized sample preparation method, or a combination of these methods. Among these processes, two standards process specifications (ASTM) $[30,31]$ are most commonly used, in which the loosest state (maximum void ratio) is achieved by a process of funnel pouring, in which a funnel is used to pour the dry material into a mould, and slowly turn the mould upside down. Whereas, the densest state is achieved by vibration method with a static weight. It is noted that, before either packing procedure is applied, the particles are thoroughly mixed for all the fractions [32].

The three systems of mixtures (in Fig. 5) are used to examine the effect of packing procedure. Fig. 5a shows the void ratios of mixtures achieved by "minimum void ratio" packing procedure, and Fig. 5b shows the void ratios of mixtures achieved by "maximum void ratio" packing procedures. The two different packing procedures have significant effect not only on the void ratios of monodisperse packings but also on the void ratios of binary mixture packings.

The packing potential indices are different for different systems of mixtures as shown in Fig. 5: the packing potential index is high for the silica sand-silt mixture, medium for the Ottawa sand-Nevada silt mixture, and low for the Vietnam mixture. However, it is interesting to observe that for each system of mixtures, the packing potential index is nearly same between the two different procedures. Thus, the dependence of the packing potential index on the packing procedure may be very weak.


Fig. 5. (a) The void ratios achieved by "minimum void ratio" packing procedure: (a-1) Silica \#50-\#80 mixture (a-2) Ottawa F95-Nevada fines mixture (a-3) Vietnam sand-silt mixture; (b) The void ratios achieved by "maximum void ratio" packing procedure: (b-1) Silica \#50-\#80 mixture (b-2) Ottawa F95-Nevada fines mixture (b-3) Vietnam sand-silt mixture.

To further examine the influence of packing procedure on the packing potential index, the 24 systems of mixtures listed in Table 1 were analyzed. The packing potential indices obtained from the "minimum void ratio" packing procedure $\left(\Omega_{\min }\right)$ are plotted in Fig. 6 versus the packing potential index obtained from the "maximum void ratio" packing procedure $\left(\Omega_{\max }\right)$. The correlation between the two packing potential indices is very strong with a coefficient of determination $\mathrm{R}^{2}=0.91$. Thus, based on the results of the 24 systems of mixtures, packing potential index has a very weak dependence on packing procedure. Consequently, the packing potential index can be treated as a material property, dependent only on the system of mixtures.

It is interesting to note that packing procedure has a significant influence on the packing density, but very weak influence on the packing potential index. That means, for a system of mixtures, the upper and lower bound densities are affected by the packing procedure, but the mixture densities relative to the upper and lower bounds are not affected by the
packing procedure. This characteristic is helpful for modelling mixture densities under different packing procedures.

## 4. Role of packing potential in particle packing model

Most particle packing density models available in the literature [ $8,10,13,28$ ] have the similar approach, which is a two-step process: (1) develop upper bound and lower bound void ratios based on the given monodisperse void ratios $e_{1}$ and $e_{2}$, for packings of coarse and fine particles, and (2) determine the void ratio $e$ of the mixture based on the upper and lower bounds, using the particle interaction parameters. Note that the packing potential is a measure that represents the position of the void ratio $e$ relative to the upper bound and lower bound. Thus, the packing potential parameters and particle interaction parameters have the same physical meaning and the same purpose. In the


Fig. 6. The effect of packing procedure on packing potential index for 24 systems of binary soil mixtures.
following, we aim to find the relationship between the packing potential parameters and the particle interaction parameters.

### 4.1. Linear particle packing model

As defined previously in Fig. 1, the packing potential index $\Omega$ is a material property for a system of mixtures (i.e., mixtures with the same two components of various combinations). In order to reveal the effect of fines content of each mixture, we define a packing potential parameter $\omega$ for a mixture with specific fines content $f_{c}$. Fig. 7a shows a data point D , which represents a mixture with fines content $f_{c}$, the packing potential parameter $\omega$ is defined by
$\omega=\left(e^{U}-e\right) /\left(e^{U}-e^{L}\right)$

The value of $\omega$ represents the position of the void ratio $e$ relative to the upper bound and lower bound void ratios (i.e. $e^{U}$ and $e^{L}$ ). The parameter $\omega$ is dependent on $e^{U}$ and $e^{L}$, which are functions of fines content, given in Eqs. (1) and (2). The area ratio $\Omega$ as shown in Fig. 1 can be treated as the average of $\omega$ over the all range of fines content $f_{c}$.

$$
\begin{equation*}
\Omega=\int_{0}^{1} \omega\left(f_{c}\right) d f_{c} \tag{4}
\end{equation*}
$$

The void ratio $e$ for the mixture in Eq. (3) can be expressed as

$$
\begin{equation*}
e=e^{U}-\omega\left(e^{U}-e^{L}\right) \tag{5a}
\end{equation*}
$$

It is noted that, in Fig. 7a, the upper bound is a straight line, but the lower bound has two segments separated by the transitional fines content $f_{c 1}$ and the measured test results has two segments separated by the transitional fines content $f_{c 2}$. The coarse particle dominant region can be defined by $f_{c}<f_{c 1}$ and the fine particle dominant region can be defined by $f_{c}>f_{c 2}$. The region in $f_{c 1}<f_{c}<f_{c 2}$ is considered as transition zone. Thus, from Eqs. (1) and (2), the value of $\left(e^{U}-e^{L}\right)$ is different for the two dominant regions:
$e^{U}-e^{L}=\left(1+e_{2}\right) y_{2} \quad$ for $f_{c}<f_{c 1}$
$e^{U}-e^{L}=e_{1} y_{1} \quad$ for $\quad f_{c}>f_{c 2}$
The value of $\omega$ for a mixture in the coarse particle dominant region ( $f_{c}<f_{c 1}$ ) is different from for a mixture in the fine particle dominant region ( $f_{c}>f_{c 2}$ ). They are termed as $\omega_{a}$ and $\omega_{b}$, respectively. Thus Eq. (5a) becomes.
$e=\left(e_{1} y_{1}+e_{2} y_{2}\right)-\omega_{a}\left(1+e_{2}\right) y_{2} \quad$ for $f_{c}<f_{c 1}$
$e=\left(e_{1} y_{1}+e_{2} y_{2}\right)-\omega_{b} e_{1} y_{1} \quad$ for $f_{c}>f_{c 2}$
If $\omega_{a}$ and $\omega_{b}$ are two constants, Eqs. (6a) and (6b) represent two linear lines. Therefore, if we approximate the data by a bilinear line as shown in Fig. 7a, then in the range of $f_{c}<f_{c 1}, \omega_{a}$ is a constant, and in the range of $f_{c}>f_{c 2}, \omega_{b}$ is another constant as shown in Fig. 7b. In the range of $f_{c 2}>f>f_{c 1}, \omega$ is a transition value from $\omega_{a}$ to $\omega_{b}$.

The $\omega$ calculated directly from experimental data reported by Lade et al. [24] is shown as the symbol of circles in Fig. 7b. The $\omega$ calculated from the bilinear line, is shown as the solid curve in Fig. 7b. In the coarse particle dominant region, $\omega_{a}$ is a constant of 0.39. In the fine particle dominant region, $\omega_{b}$ is a different constant of 0.65 . In the transition zone, $\omega$ varies from 0.39 to 0.65 .

It is noted that Eqs. (6a) and (6b) have the similar form as the linear particle packing density models (LPDM) [8-12,16]. These models consider interaction parameters (i.e. the loosening parameter $a$ and wall parameter $b$ ), which always predicts a bilinear packing void ratio curve for a binary mixture as shown in Fig. 7a. Compared with these linear packing models, the packing potential parameter $\omega_{a}$ corresponds to


Fig. 7. (a) Definition of packing potential parameter $\omega$ for a given mixture with specific fines content, (b) The packing potential parameter $\omega$ as a function of fines content.
the loosening parameter, and parameter $\omega_{b}$ corresponds to the wall parameter. Precisely, the packing potential parameters can be related to the particle interaction parameters by: $\omega_{a}=1-a$, and $\omega_{b}=1-b$.

These two parameters, as shown in the test results on glass beads and steel shots (see Fig. 2a), are significantly dependent on the particle size ratio $r$ (i.e., ratio of fine to coarse particle sizes), which can be obtained by regression analysis of the packing density experimental results for different mixtures.

The interaction functions derived by Yu et al. [8] for their two interaction parameters:
$a=1-(1-r)^{3.3}-2.8 r(1-r)^{2.7}$
$b=1-(1-r)^{2}-0.4 r(1-r)^{3.7}$
The interaction functions derived by de Larrard [11] for the interaction parameters in CPM:
$a=\sqrt{1-(1-r)^{1.02}}$
$b=1-(1-r)^{1.5}$

### 4.2. Nonlinear particle packing model

Models that consider the loosening parameter $a$ and wall parameter $b$ can be collectively classified as a 2-parameter model, which includes most of the linear particle packing density models (LPDM) [8-12,16]. As shown in Fig. 7a, the bilinear curve does not fit well the data in the range of $f_{c}$ between $25 \%$ and $40 \%$. To correct this situation, three methods have been developed.

One of the three methods, proposed by Chang and Deng [34], can be linked to the concept of packing potential. Chang and Deng [34,35] believed that the predicted bilinear line is caused by the oversimplified assumption adopted in the LPDM. In LPDM, a packing of binary mixture is assumed to be built by one of the following two mechanisms: for lower content (coarse particle dominant region), fine particles are filled into the pores between coarse particles; for higher fines content (fine particle dominant region), the coarse particles are embedded into the fine particle matrix. In either case, only one type of mechanism (either filling or embedment) can occur for the binary mixture (see Eqs. 6a and 6b).

Opposite to the assumption adopted by LPDM, Chang and Deng [34] assumed that both mechanisms can occur simultaneously in a packing of mixture. Thus, the potential parameter $\omega$ is divided into two parts: the filling potential parameter $\omega_{1}$ and the embedment potential parameter $\omega_{2}$. Consequently, the term $\omega\left(e^{U}-e^{L}\right)$ in Eq. (5a) can be viewed as
a combination of two parts: $\omega_{1}\left(1+e_{2}\right) y_{2}$ due to fine particles filled into the packing mixture, and $\omega_{2} e_{1} y_{1}$ due to coarse particles embedded into the packing mixture, thus
$\omega\left(e^{U}-e^{L}\right)=\omega_{1}\left(1+e_{2}\right) y_{2}+\omega_{2} e_{1} y_{1}$
Using Eqs. (9a) and (1), (5a) becomes
$e=\left(e_{1} y_{1}+e_{2} y_{2}\right)-\omega_{1}\left(1+e_{2}\right) y_{2}-\omega_{2} e_{1} y_{1}$
Eq. (9b) is in the same form as that proposed by Chang and Deng [34], except $\omega_{1}$ and $\omega_{2}$ were expressed in symbols $a$ and $b$. To facilitate the notion of combined mechanism, they introduced a state parameter $x$, and both the filling potential parameter $\omega_{1}$ and the embedment potential parameter $\omega_{2}$ are functions of the state parameter $x$. Thus, the void ratio of mixture is a function of the state parameter $x$ :

$$
\begin{equation*}
e(x)=\left(e_{1} y_{1}+e_{2} y_{2}\right)-\omega_{1}(x)\left(1+e_{2}\right) y_{2}-\omega_{2}(x) e_{1} y_{1} \tag{10a}
\end{equation*}
$$

The state parameter $x$ can be regarded as the controlling size of the packing ( $d_{1} \geq x \geq d_{2}$ ), which governs the magnitudes of packing potential parameters $\omega_{1}$ and $\omega_{2}$. Chang and Deng [34] showed that these two parameters are functions of two size ratios (between particle sizes and packing controlling size $x$ ), given by
$\omega_{1}(x)=\left(1-d_{2} / x\right)^{p}$
$\omega_{2}(x)=\left(1-x / d_{1}\right)^{s}$
The size ratio $d_{2} / x$ governs the packing potential due to filling mechanism and the size ratio $x / d_{1}$ governs the packing potential due to embedment mechanism. The exponents $p$ and $s$ are two parameters corresponding to $\omega_{1}$ and $\omega_{2}$, respectively.

The state parameter $x$ does not need to be known priori. According to the second law of thermodynamics, for a system reaches equilibrium at constant temperature and pressure, there is a natural tendency to achieve a minimum of the Gibbs free energy (i.e., the thermodynamic potential). Gibbs energy is proportional to the specific volume (related to void ratio by $(1+e) / \rho_{s}, \rho_{s}$ is density of solid), which is an important parameter for describing the system's thermodynamic equilibrium state. By varying $x$, the specific volume alternates. The system reaches equilibrium when the following equation is satisfied.

$$
\begin{equation*}
\frac{d e(x)}{d x}=0 \tag{11}
\end{equation*}
$$

Fig. 8. (a) The predicted void ratios for mixtures with different fines content, and (b) the calculated value of state parameter $x$.

Thus, the solution of $e(x)$ can be solved from the set of governing Eqs. (10) and (11). This model requires only two parameters, $p$ and $s$, which can be calibrated from experimental results [34].

The second method proposed by Kwan et al. [13] introduced an additional parameter (i.e. wedging effect parameter $c$ ). The wedging effect becomes significant when fine particles are enough to fill voids among coarse particles (e.g. $f_{c}=25-40 \%$ ). The wedge effect is assumed to be related to the wedge parameter and proportional to the square of fines content (i.e., $c f_{c}^{2}$ ). This assumption allows the predicted relationship between $e$ and $f_{c}$ to be nonlinear so that the model is capable of modelling the nonlinear nature of data points as observed in Fig. 7a.

The interaction functions introduced by Kwan et al. [13] for the 3parameter packing model are as follows:
$a=1-(1-r)^{3.3}-2.6 r(1-r)^{3.6}$
$b=1-(1-r)^{1.9}-2 r(1-r)^{6}$
$c=0.322 \tanh (11.9 r)$
where $a, b$, and $c$ refer to the loosening, wall, and wedging effects, respectively.

The third method was proposed by de Larrard [11] who considered that if a specimen is perfect compacted, a bilinear line would be achieved, and the measured curved line is due to insufficient compaction. To this end, he introduced a parameter (i.e., compaction index $K$ ). As the value of $K$ approaches to infinity, the mixture is considered as a virtual packing being perfectly compacted. However, in real situations, the value of $K$ usually ranges in 4.5-15. He proposed a method of converting from the density of a virtual packing to the density of a real packing. Thus, the real packing void ratios converted from the bilinear line through the variable K would give a curved shape. The model is called the Compressible Packing Model (CPM). Recently, Roquier [33] introduced a 4th parameter (i.e., critical cavity size) within the framework of CPM.

Among the three methods for modelling the nonlinear nature of data points proposed by Chang and Deng [34], Kwan et al. [13] and de Larrard [11], both methods by Kwan et al. [13] and de Larrard [11] introduced a third parameter (i.e. the wedging effect parameter or the compaction index) in addition to the loosening parameter and the wall parameter. The physical meaning of the two added parameters are not related to the physical meaning of packing potential. Thus, these two methods cannot be linked to the concept of packing potential. It is noted that the method proposed by Chang and Deng [34] utilized the concept of


Fig. 9. Comparison of the parameters $p$ and $s$ obtained from "minimum void ratio" packing procedure and from "maximum void ratio" packing procedure.
packing potential to model the nonlinear nature, thus it remains to be a 2-parameter model, without the need to introduce a third parameter.

As an example, the nonlinear model by Chang and Deng [34] (i.e., Eqs. $(10-11)$ ) is now applied to model the experimental results by Lade [24], $e_{1}=0.58, e_{2}=0.72, d_{1}=0.5 \mathrm{~mm}, d_{2}=0.7 \mathrm{~mm}$. The two parameters $p$ and $s$ were determined using the method described in the reference $[34](p=3.3$, and $s=2.3)$. The value of $x$ computed for the data in Fig. 8a is a function of fines content as shown in Fig. 8b. The predicted curve of void ratio is shown by the solid curve in Fig. 8a, which is nonlinear with respect to fines content.

Note that the packing potential parameters $\omega_{1}$ to $\omega_{2}$ are independent of packing procedure as described in a previous section. And these two parameters are directly related to the parameters $p$ and $s$ as shown in Eqs. (10b) and (10c). Thus, we expect that the parameters $p$ and $s$, like the packing potential parameters, are independent of packing procedure.

## 5. The independence of packing procedure on the parameters $p$ and $s$

To verify this hypothesis that the parameters, $p$ and $s$, are independent of packing procedure, the 24 sets of sand-silt mixtures listed in Table 1 were used. The two parameters $p$ and $s$, determined from experimental results under both packing procedures of achieving minimum void ratio and maximum void ratios, are compared in Fig. 9 for the 24 sets of sand-silt mixtures. For the 45-degree line, the coefficient of determination $R^{2}$ is 0.97 , which indicates that the parameters are nearly independent of the packing procedure.

Since the parameters $p$ and $s$ obtained for the "minimum void ratio" packing procedure are nearly the same as those obtained from the "maximum void ratio" packing procedure, only the value of $p$ and $s$ obtained for the "minimum void ratio" packing procedure are listed in Table 1.

The values of $p$ and $s$ obtained from the "minimum void ratio" packing procedure are used for the prediction of both minimum and maximum void ratios using Eqs. (10) and (11). The predicted results are plotted in Fig. 10 for the 24 sets of sand-silt mixtures. Due to the good correlation of $p$ and $s$ shown in Fig. 9, it is not surprised to see the good agreement between the predicted and measured results for both minimum and maximum void ratios as shown in Fig. 10.

## 6. Values of parameters $\boldsymbol{p}$ and $\boldsymbol{s}$ for sand-silt mixtures

The values of $p$ and $s$ depend on complex factors of particle morphology such as particle shapes and surface textures. To study the range of values of $p$ and $s$ due to the effect of particle shapes, the 13 sets of spherical particle mixtures and the 24 sets of sand-silt mixtures listed in Table 1 are classified into 3 groups of compound particle shapes: (1) $R / R$, (2) $A / A$ or $S A / S A$, and (3) R/A or R/SA. (R-round, A-angular, SA- subangular). For the first two groups, coarse particles and fine particles have similar shapes. For the third group, coarse particles and fine particles have different shapes.

The box and whiskers plot was utilized to compare the values of $p$ and $s$ for the three groups of compound particle shapes as shown in Fig. 11. A box and whiskers plot is composed of a box and a set of whiskers. The upper whisker of the plot is the maximum of the data set and the lower whisker of the plot is the minimum of the data set. The box is drawn from the first quartile to third quartile with a horizontal line drawn in the box to denote the median. For the first two groups $(R / R$, $\mathrm{A} / \mathrm{A}$ or $\mathrm{SA} / \mathrm{SA}$ ), the value range of $p$ and $s$ are small compared to that of the third group ( $\mathrm{R} / \mathrm{A}$ or $\mathrm{R} / \mathrm{SA}$ ). For all three groups of compound particle shapes, the range of $p$ is greater than the range of $s$. The length of box also shows the same trend. The median value of $p$ is smallest for $R / R$, larger for $A / A$ or $S A / S A$, and largest for $R / A$ or $R / S A$. The median value of $s$ has the same trend.

From an engineering point of view, when experimental results are not available for calibration, the values of $p$ and $s$ can be approximately


Fig. 10. Comparison of the measured and predicted maximum void ratios using the values $p$ and $s$ obtained from the "minimum void ratio" packing procedure.


Fig. 11. Variation of parameters pand sfor 24 soil mixtures listed in Table 1 and for spherical particle mixtures used to produce Fig. 2.
estimated from Fig. 11 based on the rough descriptions of particle shapes of sand and silt. To assess the accuracy for this type of estimation, we classify the values of $p$ and $s$ into three groups. In each group, the median values are:
(1) $\mathrm{R} / \mathrm{R}: p=2.8$ and $s=1.75$;
(2) A/A or SA/SA: $p=2.9$ and $s=2.0$;
(3) $\mathrm{R} / \mathrm{A}$ or R/SA: $p=4.65$ and $s=3.0$.

The three sets of value are used for the prediction of the 24 sets of tests on sand-silt mixtures (Table 1), plus the two sets of tests on glass beads and steel shots mixtures (Fig. 3). The comparisons of meausred and predicted results are shown in Fig. 12 for mixtures with $R / R$ particles shapes (glass beads and steel shots mixtures), shown in Fig. 12 for sand-silt mixtures with A/A or SA/SA particles shapes, and shown in Fig. 12 for sand-ssilt mixtures with R/A or R/SA particles shapes.

The comparisons of measured and predicted results are plotted on Fig. 13a to show the degree of accuracy of the predicted values compared to the measured results. Fig. 13b show the distribution of $\Delta e$ (predicted $e$-measured $e$ ). The one-standar deviation is 0.02 for mixtures with $R / R$ particle shapes, is 0.03 for mixtures with $A / A$ or SA/SA particle shapes, and is 0.054 for mixtures with $\mathrm{R} / \mathrm{A}$ or R/SA particle shapes. This can be interpreted that, at least $68 \%$ of probability, the predicted error is within $\pm 0.02$ for mixture with $\mathrm{R} /$ R paticle shapes, within $\pm 0.03$ for mixture with $\mathrm{A} / \mathrm{A}$ or $\mathrm{SA} / \mathrm{SA}$ paticle shapes, and within $\pm 0.054$ for mixture with $\mathrm{R} / \mathrm{A}$ or $\mathrm{R} / \mathrm{SA}$ paticle shapes. In Fig. 13b, the shaded zone is the one-standard diviation band for all 3 cases.

## 7. Conclusion

In this paper, we aim to study the packing procedure effect on density of mixtures. We have defined a packing potential index, which is a measure of volume reduction potential due to mixing of two components of a binary mixtures under a packing procedure. Based on 24 sets of experiments on sand-silt mixtures collected from the literature, we found that the packing potential index is significantly influenced by particle size ratio $\left(d_{2} / d_{1}\right)$ and the particle morphology of the mixture, such as particle shape, particle surface texture. However, the packing potential index is nearly independent of packing procedure. Thus, packing potential index can be treated as a material property of the mixture system.

The packing potential for a mixture of given $f_{c}$ can be mathematically linked to the particle interaction parameters, which are used in the particle packing models to calculate the void ratio of a binary mixture based on the upper and lower bound void ratios. Thus, we found the parameters $p$ and $s$, similar to the packing potential index, are also independent of packing procedure, from the analyses of 24 sets of tests results on sand-silt mixtures.

The particle packing model approach is a two-step process: (1) develop upper and lower bounds based on the given monodisperse void ratios $e_{1}$ and $e_{2}$, for packings of coarse and fine particles, and (2) determine the $e$ of the mixture based on the bounds, using the particle interaction function.

The two-step process approach has two advantages. The first advantage is to account the complex factors of particle morphology (surface roughness, texture, sphericity), and the grain size distribution of silt or sand by using the values of $e_{1}$ and $e_{2}$ as input data, which are obtained


 $\mathrm{R} / \mathrm{A}$ or $\mathrm{R} / \mathrm{SA}$ compound particle shapes.


Fig. 12 (continued).
directly from experiments, and the complex factors of particle morphology and packing procedure are manifested in these two values. Thus, the model can at least capture some influence of these complex factors, which are usually not quantitatively measured, and no analytical method can include these factors in a satisfactory manner.

The second advantage is to use the particle interaction parameters, which are largely dependent only on the system of mixtures but
independent of the packing procedure. This characteristic is useful for the packing density model, because the same parameters and the same modelling methodology can be conveniently applied to predict void ratios of mixtures under different packing procedures (e.g. the maximum and minimum void ratios produced by two different processes).

For predicting void ratio of sand-silt mixtures, we proposed a set values for the particle interaction parameters, $p$ and $s$, (to be used in


Fig. 12 (continued).

 compound particle shapes.
the nonlinear packing density model proposed by Chang and Deng [34]). The values are suggested for mixtures with three types of compound particle shapes: $R / R, A / A$ ( or $S A / S A$ ), and $R / A$ ( or $R / S A$ ). The comparisons between measured and predicted results show that: the error of predicted values have a standard deviation of 0.02-0.03 for mixtures with compound particle shapes $R / R$ and A/A (or SA/SA), whereas, the error of predicted values have a standard deviation of 0.054 for mixtures with compound particle shapes $\mathrm{R} / \mathrm{A}$ ( or $\mathrm{R} / \mathrm{SA}$ ). Thus, it is more difficult to achieve accurate predicted
results for the binary mixtures composed of two components with different particle shapes.

## Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Acknowledgement

This paper was derived from research sponsored by National Science Foundation of the United States under a research grant (CMMI1917238). The support is greatly acknowledged.

## References

[1] J.S. Reed, Principles of Ceramics Processing, 2nd ed. John Wiley \& Sons, New York, 1995.
[2] L. Smith, A Knowledge-Based System for Powder Metallurgy Technology, John Wiley \& Sons, 2003.
[3] T.C. Powers, The Properties of Fresh Concrete, John Wiley \& Sons, New York, 1968.
[4] C.C. Furnas, Grading aggregates I: mathematical relations for beds of broken solids of maximum density, Ind. Eng. Chem. 23 (1931) 1052-1058, https://doi.org/10.1021/ ie50261a017.
[5] A.E.R. Westman, H.R. Hugill, The packing of particles, J. Am. Ceram. Soc. 13 (1930) 767-779, https://doi.org/10.1111/j.1151-2916.1930.tb16222.x.
[6] R.B. Aïm, P.L. Goff, Effet de paroi dans les empilements désordonnés de sphères et application à la porosité de mélanges binaires, Powder Technol. 1 (1968) 281-290 (doi:10.1016/0032-5910(68)80006-3).
[7] W. Toufar, E. Klose, M. Born, Berechnung der packungsdichte von korngemischen, Aufbereitung-Technik, 11, 1977 603-608.
[8] A.B. Yu, R.P. Zou, N. Standish, Modifying the linear packing model for predicting the porosity of nonspherical particle mixtures, Ind. Eng. Chem. Res. 35 (1996) 3730-3741, https://doi.org/10.1021/ie950616a.
[9] P. Goltermann, V. Johansen, L. Palbøl, Packing of aggregates: an alternative tool to determine the optimal aggregate mix, ACI Mater. J. 94 (1997) 435-443.
[10] T. Stovall, F. de Larrard, M. Buil, Linear packing density model of grain mixtures, Powder Technol. 48 (1986) 1-12, https://doi.org/10.1016/0032-5910(86)80058-4.
[11] F. de Larrard, Concrete Mixture Proportioning: A Scientifil Approach, E \& FN Spon, London, 1999.
[12] J.D. Dewar, Computer Modelling of Concrete Mixtures, E \& FN Spon, London, 1999.
[13] A.K.H. Kwan, K.W. Chan, V. Wong, A 3-parameter particle packing model incorporating the wedging effect, Powder Technol. 237 (2013) 172-179, https://doi.org/10. 1016/j.powtec.2013.01.043.
[14] R.K. McGeary, Mechanical packing of spherical particles, J. Am. Ceram. Soc. 44 (1961) 513-522, https://doi.org/10.1111/j.1151-2916.1961.tb13716.x.
[15] M.R. Jones, L. Zheng, M.D. Newlands, Comparison of particle packing models for proportioning concrete constituents for minimum voids ratio, Mater. Struct. 35 (2002) 301-309.
[16] C.S. Chang, J.-Y. Wang, L. Ge, Modeling of minimum void ratio for sand-silt mixtures, Eng. Geol. 196 (2015) 293-304, https://doi.org/10.1016/j.enggeo.2015.07.015.
[17] C.R.I. Clayton, C.O.R. Abbireddy, R. Schiebel, A method of estimating the form of coarse particulates, Géotechnique. 59 (2009) 493-501, https://doi.org/10.1680/ geot.2008.P.009.
[18] S.J. Blott, K. Pye, Particle shape: a review and new methods of characterization and classification, Sedimentology. 55 (2008) 31-63, https://doi.org/10.1111/j.13653091.2007.00892.x.
[19] K.A. Alshibli, M.I. Alsaleh, Characterizing surface roughness and shape of sands using digital microscopy, J. Comput. Civ. Eng. 18 (2004) 36-45, https://doi.org/10.1061/ (ASCE)0887-3801(2004)18:1(36).
[20] H. Wadell, Sphericity and roundness of rock particles, J. Geol. 41 (1933) 310-331.
[21] C. Hettiarachchi, W.K. Mampearachchi, Effect of surface texture , size ratio and large particle volume fraction on packing density of binary spherical mixtures, Granul. Matter 8 (2020) 1-13, https://doi.org/10.1007/s10035-019-0978-3.
[22] J. Zheng, R.D. Hryciw, Roundness and Sphericity of soil particles in assemblies by computational geometry, J. Comput. Civ. Eng. 30 (2016) 1-13, https://doi.org/10. 1061/(ASCE)CP.1943-5487.0000578.
[23] P.V. Lade, J.A. Yamamuro, Effects of nonplastic fines on static liquefaction of sands, Can. Geotech. J. 34 (1997) 918-928, https://doi.org/10.1139/t97-052.
[24] P.V. Lade, C.D. Liggio, J.A. Yamamuro, Effects of non-plastic fines on minimum and maximum void ratios of sand, Geotech. Test. J. 21 (1998) 336-347, https://doi.org/ 10.1520/GTJ11373J.
[25] S.L. Yang, Characterization of the Properties of Sand-Silt Mixtures, Ph.D. Thesis Norwegian University of Science and Technology, Trondheim, Norway, 2004.
[26] A.B. Fourie, G. Papageorigou, Defining an appropriate steady state line for Merriespruit gold tailings, Can. Geotech. J. 38 (2001) 695-706, https://doi.org/10. 1139/cgj-38-4-695.
[27] Y.T. Cho, The Study of GCTS Triaxial Apparatus Function and Mixing Sand Void Ratio, Master thesis National Taiwan University, Taiwan, 2014.
[28] S. Thevanayagam, T. Shenthan, S. Mohan, J. Liang, Undrained fragility of clean sands, silty sands, and sandy silts, J. Geotech. Geoenviron. Eng. 128 (2002) 849-859, https://doi.org/10.1061/(ASCE)1090-0241(2002)128:10(849).
[29] Y. Yilmaz, A study on the limit void ratio characteristics of medium to fine mixed graded sands, Eng. Geol. 104 (2009) 290-294, https://doi.org/10.1016/j.enggeo. 2008.11.009.
[30] ASTM D4253-00, Standard Test Methods for Maximum Index Density and Unit Weight of Soils Using a Vibratory Table, Annual Book of ASTM Standards, ASTM International, West Conshohocken, PA 2006, pp. 1-15.
[31] ASTM D4254-00, Standard Test Methods for Minimum Index Density and Unit Weight of Soils and Calculation of Relative Density, Annual Book of ASTM Standards. , ASTM International, West Conshohocken, PA, 2006 1-9.
[32] AASHTO, Correction for Coarse Particles in the Soil Compaction Test, AASHTO Designation: T224-86, Washington, D.C., 1986
[33] G. Roquier, The 4-parameter compressible packing model (CPM) including a new theory about wall effect and loosening effect for spheres, Powder Technol. 302 (2016) 247-253, https://doi.org/10.1016/j.powtec.2016.08.031.
[34] C.S. Chang, Y. Deng, A particle packing model for sand-silt mixtures with the effect of dual-skeleton, Granul. Matter 19 (2017), 80. https://doi.org/10.1007/s10035-017-0762-1.
[35] C. Chang S., Y. Deng, A nonlinear packing model for multi-sized particle mixtures, Powder Technol. 336 (2018) 449-464, https://doi.org/10.1016/j.powtec.2018. 06.008 .


[^0]:    * Corresponding author.

    E-mail addresses: cchang@engin.umass.edu (C.S. Chang), yibingdeng@engin.umass.edu (Y. Deng).

