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Near-isometric duality of Hardy norms with applications to harmonic mappings

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A B S T R A C T

Hardy spaces in the complex plane and in higher dimensions have natural finite-dimensional subspaces formed by polynomials or by linear maps. We use the restriction of Hardy norms to such subspaces to describe the set of possible derivatives of harmonic self-maps of a ball, providing a version of the Schwarz lemma for harmonic maps. These restricted Hardy norms display unexpected near-isometric duality between the exponents 1 and 4, which we use to give an explicit form of harmonic Schwarz lemma.

1. Introduction

This paper connects two seemingly distant subjects: the geometry of Hardy norms on finite-dimensional spaces and the gradient of a harmonic map of the unit ball. Specifically, writing H^1_* for the dual of the Hardy norm H^1 on complex-linear functions (defined in §2), we obtain the following description of the possible gradients of harmonic maps of the unit disk $\mathbb{D}$.

Theorem 1.1. *A vector $(\alpha, \beta) \in \mathbb{C}^2$ is the Wirtinger derivative at 0 of some harmonic map $f : \mathbb{D} \to \mathbb{D}$ if and only if $\|(\alpha, \beta)\|_{H^1_*} \le 1$.*

Theorem 1.1 can be compared to the behavior of holomorphic maps $f : \mathbb{D} \to \mathbb{D}$ for which the set of all possible values of $f'(0)$ is simply $\overline{\mathbb{D}}$. The appearance of H^1_* norm here leads one to look for a concrete description of this norm. It is well known that the duality of holomorphic Hardy spaces H^p is not isometric, and in particular the dual of H^1 norm is quite different from H^∞ norm even on finite dimensional subspaces (see (3.4)). However, it has a striking similarity to H^4 norm.

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Theorem 1.2. *For all* $\xi \in \mathbb{C}^2 \setminus \{(0,0)\}$, $1 \le \|\xi\|_{H^1_*}/\|\xi\|_{H^4} \le 1.01$.

Since the H^4 norm can be expressed as $\|(\xi_1,\xi_2)\|_4 = (|\xi_1|^4 + 4|\xi_1\xi_2|^2 + |\xi_4|^4)^{1/4}$, Theorem 1.2 supplements Theorem 1.1 with an explicit estimate.

In general, Hardy norms are merely quasinorms when $p < 1$, as the triangle inequality fails. However, their restrictions to the subspaces of degree 1 complex polynomials or of 2×2 real matrices are actual norms (Theorem 2.1 and Corollary 5.2). We do not know if this property holds for $n \times n$ matrices with $n > 2$.

The paper is organized as follows. Section 2 introduces Hardy norms on polynomials. In Section 3 we prove Theorem 1.2. Section 4 concerns the Schwarz lemma for planar harmonic maps, Theorem 1.1. In section 5 we consider higher dimensional analogues of these results.

2. Hardy norms on polynomials

For a polynomial $f \in \mathbb{C}[z]$, the Hardy space (H^p) quasinorm is defined by

$$\|f\|_{H^p} = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})|^p \, dt \right)^{1/p}$$

where $0 < p < \infty$. There are two limiting cases: $p \to \infty$ yields the supremum norm

$$\|f\|_{H^\infty} = \max_{t \in \mathbb{R}} |f(e^{it})|$$

and the limit $p \to 0$ yields the Mahler measure of f:

$$\|f\|_{H^0} = \exp \left(\frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{it})| \, dt \right).$$

An overview of the properties of these quasinorms can be found in [12, Chapter 13] and in [11]. In general they satisfy the definition of a norm only when $p \ge 1$.

The Hardy quasinorms on vector spaces $\mathbb{C}^n$ are defined by

$$\|(a_1, \dots, a_n)\|_{H^p} = \|f\|_{H^p}, \quad f(z) = \sum_{k=1}^n a_k z^{k-1}.$$

We will focus on the case $n = 2$, which corresponds to the H^p quasinorm of degree 1 polynomials $a_1 + a_2 z$. These quantities appear as multiplicative constants in sharp inequalities for polynomials of general degree: see Theorems 13.2.12 and 14.6.5 in [12], or Theorem 5 in [11]. In general, H^p quasinorms cannot be expressed in elementary functions even on $\mathbb{C}^2$. Notable exceptions include

$$\begin{aligned} \|(a_1,a_2)\|_{H^0} &= \max(|a_1|, |a_2|), \\ \|(a_1,a_2)\|_{H^2} &= \left(|a_1|^2 + |a_2|^2\right)^{1/2}, \\ \|(a_1,a_2)\|_{H^4} &= \left(|a_1|^4 + 4|a_1|^2|a_2|^2 + |a_2|^4\right)^{1/4}, \\ \|(a_1,a_2)\|_{H^\infty} &= |a_1| + |a_2|. \end{aligned} \tag{2.1}$$

Another easy evaluation is

$$\|(1,1)\|_{H^1} = \frac{1}{2\pi}\int_0^{2\pi} |1+e^{it}|\,dt = \frac{1}{2\pi}\int_0^{2\pi} 2|\cos(t/2)|\,dt = \frac{4}{\pi}. \tag{2.2}$$

However, the general formula for the H^1 norm on $\mathbb{C}^2$ involves the complete elliptic integral of the second kind E. Indeed, writing $k = |a_2/a_1|$, we have

$$\begin{aligned} \|(a_1,a_2)\|_{H^1} = |a_1|\,\|(1,k)\|_{H^1} &= \frac{|a_1|}{2\pi}\int_0^{2\pi} |1+ke^{2it}|\,dt \\ &= |a_1|\frac{2(k+1)}{\pi}\int_0^{\pi/2}\sqrt{1-\left(\frac{2\sqrt{k}}{k+1}\right)^2 \sin^2 t}\,dt \\ &= |a_1|\frac{2(k+1)}{\pi}E\left(\frac{2\sqrt{k}}{k+1}\right). \end{aligned} \tag{2.3}$$

Perhaps surprisingly, the Hardy quasinorm on $\mathbb{C}^2$ is a norm (i.e., it satisfies the triangle inequality) even when $p < 1$.

Theorem 2.1. *The Hardy quasinorm on $\mathbb{C}^2$ is a norm for all $0 \le p \le \infty$. In addition, it has the symmetry properties*

$$\|(a_1,a_2)\|_{H^p} = \|(a_2,a_1)\|_{H^p} = \|(|a_1|,|a_2|)\|_{H^p}. \tag{2.4}$$

Proof. For $p = 0, \infty$ all these statements follow from (2.1), so we assume $0 < p < \infty$. The identities

$$\int_0^{2\pi} |a_1 + a_2 e^{it}|^p\,dt = \int_0^{2\pi} |a_1 e^{-it} + a_2|^p\,dt = \int_0^{2\pi} |a_2 + a_1 e^{it}|^p\,dt \tag{2.5}$$

imply the first part of (2.4). Furthermore, the first integral in (2.5) is independent of the argument of a_2 while the last integral is independent of the argument of a_1. This completes the proof of (2.4).

It remains to prove the triangle inequality in the case $0 < p < 1$. To this end, consider the following function of $\lambda \in \mathbb{R}$.

$$G(\lambda) := \|(1,\lambda)\|_{H^p} = \left(\frac{1}{2\pi}\int_0^{2\pi} |1+\lambda e^{it}|^p\,dt\right)^{1/p}. \tag{2.6}$$

We claim that G is convex on $\mathbb{R}$. If $|\lambda| < 1$, the binomial series

$$(1+\lambda e^{it})^{p/2} = \sum_{n=0}^{\infty} \binom{p/2}{n} \lambda^n e^{nit}$$

together with Parseval's identity imply

$$G(\lambda) = \left(\sum_{n=0}^{\infty} \binom{p/2}{n}^2 \lambda^{2n}\right)^{1/p}. \tag{2.7}$$

Since every term of the series is a convex function of λ, it follows that G is convex on $[-1,1]$. The power series also shows that G is C^∞ smooth on $(0,1)$. For $\lambda > 1$ the symmetry property (2.4) yields $G(\lambda) = \lambda G(1/\lambda)$ which is a convex function by virtue of the identity $G''(\lambda) = \lambda^{-3} G''(1/\lambda)$. The piecewise convexity of G on $[0,1]$ and $[1,\infty)$ will imply its convexity on $[0,\infty)$ (hence on $\mathbb{R}$) as soon as we show that G is differentiable at $\lambda = 1$. Note that $|1+\lambda e^{it}|^p$ is differentiable with respect to λ when $e^{it} \neq -1$ and that for λ close to 1,

$$\frac{\partial}{\partial\lambda}|1+\lambda e^{it}|^p \le p|1+\lambda e^{it}|^{p-1}) \le C|t-\pi|^{p-1} \tag{2.8}$$

for all $t \in [0,2\pi] \setminus \{\pi\}$, with C independent of λ, t. The integrability of the right hand side of (2.8) justifies differentiation under the integral sign:

$$\frac{d}{d\lambda} G(\lambda)^p = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial}{\partial\lambda}|1+\lambda e^{it}|^p \, dt.$$

Thus $G'(1)$ exists.

Now that G is known to be convex, the convexity of the function $F(x,y) := \|(x,y)\|_{H^p} = xG(y/x)$ on the halfplane $(x,y) \in \mathbb{R}^2$, $x > 0$, follows by computing its Hessian, which exists when $|y| \neq x$:

$$H_F = G''(y/x)\begin{pmatrix} x^{-3}y^2 & -x^{-2}y \\ -x^{-2}y & x^{-1} \end{pmatrix}.$$

Since H_F is positive semidefinite, and F is C^1 smooth even on the lines $|y| = |x|$, the function F is convex on the halfplane $x > 0$. By symmetry, convexity holds on other coordinate halfplanes as well, and thus on all of $\mathbb{R}^2$. The fact that G is an increasing function on $[0,\infty)$ also shows that F is an increasing function of each of its variables in the first quadrant $x, y \ge 0$.

Finally, for any two points (a_1, a_2) and (b_1, b_2) in $\mathbb{C}^2$ we have

$$\begin{aligned}\|(a_1+b_1, a_2+b_2)\|_{H^p} &= F(|a_1+b_1|, |a_2+b_2|) \le F(|a_1|+|b_1|, |a_2|+|b_2|) \\ &\le F(|a_1|,|a_2|) + F(|b_1|,|b_2|) = \|(a_1,a_2)\|_{H^p} + \|(b_1,b_2)\|_{H^p}\end{aligned}$$

using (2.4) and the monotonicity and convexity of F. □

Remark 2.2. In view of Theorem 2.1 one might guess that the restriction of H^p quasinorm to the polynomials of degree at most n should satisfy the triangle inequality provided that $p > p_n$ for some $p_n < 1$. This is not so: the triangle inequality fails for any $p < 1$ even when the quasinorm is restricted to quadratic polynomials. Indeed, for small $\lambda \in \mathbb{R}$ we have

$$\begin{aligned}\|(\lambda, 1, \lambda)\|_{H^p}^p &= \frac{1}{2\pi}\int_0^{2\pi} (1+2\lambda\cos t)^p \, dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left(1 + 2\lambda p\cos t + 2\lambda^2 p(p-1)\cos^2 t + O(\lambda^3)\right) dt \\ &= 1 + \lambda^2 p(p-1) + O(\lambda^3)\end{aligned}$$

and this quantity has a strict local maximum at $\lambda = 0$ provided that $0 < p < 1$.

3. Dual Hardy norms on polynomials

The space $\mathbb{C}^n$ is equipped with the inner product $\langle \xi, \eta \rangle = \sum_{k=1}^n \xi_k \overline{\eta_k}$. Let H^p_* be the norm on $\mathbb{C}^n$ dual to H^p, that is

$$\|\xi\|_{H^p_*} = \sup \{|\langle \xi, \eta \rangle| \colon \|\eta\|_{H^p} \le 1\} = \sup_{\eta \in \mathbb{C}^n \setminus \{0\}} \frac{|\langle \xi, \eta \rangle|}{\|\eta\|_{H^p}}. \tag{3.1}$$

One cannot expect the H^p_* norm to agree with H^q for $q = p/(p-1)$ (unless $p = 2$), as the duality of Hardy spaces is not isometric [4, Section 7.2]. However, on the space $\mathbb{C}^2$ the H^1_* norm turns out to be surprisingly close to H^4, indicating that H^1 and H^4 have nearly isometric duality in this setting. The following is a restatement of Theorem 1.2 in the form that is convenient for the proof.

Theorem 3.1. *For all $\xi \in \mathbb{C}^2$ we have*

$$\|\xi\|_{H^1} \le \|\xi\|_{H^4_*} \le 1.01 \|\xi\|_{H^1} \tag{3.2}$$

and consequently

$$\|\xi\|_{H^4} \le \|\xi\|_{H^1_*} \le 1.01 \|\xi\|_{H^4}. \tag{3.3}$$

It should be noted that while the H^1 norm on $\mathbb{C}^2$ is a non-elementary function (2.3), the H^4 norm has a simple algebraic form (2.1). To see that having the exponent $p = 4$, rather than the expected $p = \infty$, is essential in Theorem 3.1, compare the following:

$$\begin{aligned} \|(1,1)\|_{H^1_*} &= \frac{2}{\|(1,1)\|_{H^1}} = \frac{\pi}{2} \approx 1.57, \\ \|(1,1)\|_{H^\infty} &= 2, \\ \|(1,1)\|_{H^4} &= 6^{1/4} \approx 1.57. \end{aligned} \tag{3.4}$$

The proof of Theorem 3.1 requires an elementary lemma from analytic geometry.

Lemma 3.2. *If $0 < r < a$ and $b \in \mathbb{R}$, then*

$$\sup_{\theta \in \mathbb{R}} \frac{b - r\sin\theta}{a - r\cos\theta} = \frac{ab + r\sqrt{a^2 + b^2 - r^2}}{a^2 - r^2}. \tag{3.5}$$

Proof. The quantity being maximized is the slope of a line through (a, b) and a point on the circle $x^2 + y^2 = r^2$. The slope is maximized by one of two tangent lines to the circle passing through (a, b). Let $\tan\alpha = b/a$ be the slope of the line L through $(0, 0)$ and (a, b). This line makes angle β with the tangents, where $\tan\beta = r/\sqrt{a^2 + b^2 - r^2}$. Thus, the slope of the tangent of interest is

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{b\sqrt{a^2 + b^2 - r^2} + ar}{a\sqrt{a^2 + b^2 - r^2} - br}$$

which simplifies to (3.5). □

Proof of Theorem 3.1. Because of the symmetry properties (2.4) and the homogeneity of norms, it suffices to consider $\xi = (1, \lambda)$ with $0 \le \lambda \le 1$. This restriction on λ will remain in force throughout this proof.

The function

$$G(\lambda) := \|(1,\lambda)\|_{H^1} = \frac{1}{2\pi}\int_0^{2\pi} |1+\lambda e^{it}|\, dt$$

has been intensely studied due to its relation with the arclength of the ellipse and the complete elliptic integral [1,3]. It can be written as

$$G(\lambda) = \frac{L(x,y)}{\pi(x+y)} = {}_2F_1(-1/2,-1/2;1;\lambda^2) = \sum_{n=0}^{\infty} \left(\frac{(-1/2)_n}{n!}\right)^2 \lambda^{2n} \tag{3.6}$$

where L is the length of the ellipse with semi-axes x, y and $\lambda = (x-y)/(x+y)$. The Pochhammer symbol $(z)_n = z(z+1)\cdots(z+n-1)$ and the hypergeometric function ${}_2F_1$ are involved in (3.6) as well. A direct way to obtain the Taylor series (3.6) for G is to use the binomial series as in (2.7).

As noted in (2.1), the H^4 norm of $(1,\lambda)$ is an elementary function:

$$F(\lambda) := \|(1,\lambda)\|_{H^4} = (1+4\lambda^2+\lambda^4)^{1/4}.$$

The dual norm H^4_* can be expressed as

$$F^*(\lambda) := \|(1,\lambda)\|_{H^4_*} = \sup_{t\in\mathbb{R}} \frac{1+\lambda t}{(1+4t^2+t^4)^{1/4}} \tag{3.7}$$

where the second equality follows from (3.1) by letting $b = (1,t)$. Similarly, the H^1_* norm of $(1,\lambda)$ is

$$G^*(\lambda) := \|(1,\lambda)\|_{H^1_*} = \sup_{t\in\mathbb{R}} \frac{1+\lambda t}{G(t)}. \tag{3.8}$$

Our first goal is to prove that

$$G^*(\lambda) \le 1.01 F(\lambda). \tag{3.9}$$

The proof of (3.9) is based on Ramanujan's approximation $G(\lambda) \approx 3 - \sqrt{4-\lambda^2}$ which originally appeared in [13]; see [1] for a discussion of the history of this and several other approximations to G. Barnard, Pearce, and Richards [3, Proposition 2.3] proved that Ramanujan's approximation gives a lower bound for G:

$$G(\lambda) \ge 3 - \sqrt{4-\lambda^2}. \tag{3.10}$$

We will use this estimate to obtain an upper bound for G^*.

The supremum in (3.8) only needs to be taken over $t \ge 0$ since the denominator is an even function. Furthermore, it can be restricted to $t \in [0,1]$ because for $t > 1$ the homogeneity and symmetry properties of H^1 norm imply

$$\frac{1+\lambda t}{\|(1,t)\|_{H^1}} = \frac{t^{-1}+\lambda}{\|(1,t^{-1})\|_{H^1}} < \frac{1+\lambda t^{-1}}{\|(1,t^{-1})\|_{H^1}}.$$

Restricting t to $[0,1]$ in (3.8) allows us to use inequality (3.10):

$$G^*(\lambda) \le \sup_{t\in[0,1]} \frac{1+\lambda t}{3-\sqrt{4-t^2}}. \tag{3.11}$$

Writing $t = -2\sin\theta$ and applying Lemma (3.5) we obtain

$$\begin{aligned} G^*(\lambda) &\le \lambda \sup_{\theta\in[-\pi/6,0]} \frac{\lambda^{-1} - 2\sin\theta}{3 - 2\cos\theta} \le \lambda\,\frac{3\lambda^{-1} + 2\sqrt{5+\lambda^{-2}}}{5} \\ &= \frac{3 + 2\sqrt{1+5\lambda^2}}{5}. \end{aligned} \tag{3.12}$$

The function

$$f(s) := \frac{3 + 2\sqrt{1+5s}}{(1+4s+s^2)^{1/4}}$$

is increasing on $[0,1]$. Indeed,

$$f'(s) = \frac{3(6s + 2 - (s+2)\sqrt{1+5s})}{2\sqrt{1+5s}(1+4s+s^2)^{5/4}}$$

which is positive on $(0,1)$ because

$$(6s+2)^2 - (s+2)^2(1+5s) = 5s^2(3-s) > 0.$$

Since f is increasing, the estimate (3.12) implies

$$\frac{G^*(\lambda)}{F(\lambda)} \le \frac{1}{5} f(\lambda^2) \le \frac{1}{5} f(1) = \frac{3+2\sqrt{6}}{5\cdot 6^{1/4}} < 1.01.$$

This completes the proof of (3.9).

Our second goal is the following comparison of F^* and G with a polynomial:

$$G(\lambda) \le 1 + \frac{1}{4}\lambda^2 + \frac{1}{64}\lambda^4 + \frac{1}{128}\lambda^6 \le F^*(\lambda). \tag{3.13}$$

To prove the left hand side of (3.13), let $T_4(\lambda) = 1 + \lambda^2/4 + \lambda^4/64$ be the Taylor polynomial of G of degree 4. Since all Taylor coefficients of G are nonnegative (3.6), the function

$$\phi(\lambda) := \frac{G(\lambda) - T_4(\lambda)}{\lambda^6} - \frac{1}{128}$$

is increasing on $(0,1]$. At $\lambda = 1$, in view of (2.2), it evaluates to

$$G(1) - 1 - \frac{1}{4} - \frac{1}{64} - \frac{1}{128} = \frac{4}{\pi} - \frac{163}{128}$$

which is negative because $512/163 = 3.1411\ldots < \pi$. Thus $\phi(\lambda) < 0$ for $0 < \lambda \le 1$, proving the left hand side of (3.13).

The right hand side of (3.13) amounts to the claim that for every λ there exists $t \in \mathbb{R}$ such that

$$\frac{1+\lambda t}{(1+4t^2+t^4)^{1/4}} \ge 1 + \frac{1}{4}\lambda^2 + \frac{1}{64}\lambda^4 + \frac{1}{128}\lambda^6.$$

This is equivalent to proving that the polynomial

$$\Phi(\lambda, t) := (1 + \lambda t)^4 - (1 + 4t^2 + t^4)\left(1 + \frac{1}{4}\lambda^2 + \frac{1}{64}\lambda^4 + \frac{1}{128}\lambda^6\right)^4$$

satisfies $\Phi(\lambda, t) \geq 0$ for some t depending on λ. We will do so by choosing $t = 4\lambda/(8 - 3\lambda^2)$. The function

$$\Psi(\lambda) := (8 - 3\lambda^2)^4 \Phi(\lambda, 4\lambda/(8 - 3\lambda^2))$$

is a polynomial in λ with rational coefficients. Specifically,

$$\begin{aligned}\frac{\Psi(\lambda)}{\lambda^8} &= 50 + \lambda^2 - \frac{149}{2^4}\lambda^4 - \frac{209}{2^6}\lambda^6 - \frac{5375}{2^{12}}\lambda^8 - \frac{3069}{2^{13}}\lambda^{10} - \frac{8963}{2^{17}}\lambda^{12} \\ &\quad - \frac{7837}{2^{19}}\lambda^{14} - \frac{36209}{2^{24}}\lambda^{16} - \frac{2049}{2^{23}}\lambda^{18} - \frac{1331}{2^{25}}\lambda^{20} - \frac{45}{2^{25}}\lambda^{22} - \frac{81}{2^{28}}\lambda^{24}\end{aligned} \tag{3.14}$$

which any computer algebra system will readily confirm. On the right hand side of (3.14), the coefficients of $\lambda^4, \lambda^6, \lambda^8$ are less than 10 in absolute value, while the coefficients of higher powers are less than 1 in absolute value. Thanks to the constant term of 50, the expression (3.14) is positive as long as $0 < \lambda \leq 1$. This completes the proof of (3.13).

In conclusion, we have $G(\lambda) \leq F^*(\lambda)$ from (3.13) and $G^*(\lambda) \leq 1.01F(\lambda)$ from (3.9). This proves the first half of (3.2) and the second half of (3.3). The other parts of (3.2)–(3.3) follow by duality. □

4. Schwarz lemma for harmonic maps

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in the complex plane. The classical Schwarz lemma concerns holomorphic maps $f : \mathbb{D} \to \mathbb{D}$ normalized by $f(0) = 0$. It asserts in part that $|f'(0)| \leq 1$ for such maps. This inequality is best possible in the sense that for any complex number α such that $|\alpha| \leq 1$ there exists f as above with $f'(0) = \alpha$. Indeed, $f(z) = \alpha z$ works.

The story of the Schwarz lemma for harmonic maps $f : \mathbb{D} \to \mathbb{D}$, still normalized by $f(0) = 0$, is more complicated. Such maps satisfy the Laplace equation $\partial\bar{\partial}f = 0$ written here in terms of Wirtinger's derivatives

$$\partial f = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \quad \bar{\partial} f = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

The estimate $|f(z)| \leq \frac{4}{\pi}\tan^{-1}|z|$ (see [6] or [5, p. 77]) implies that

$$|\partial f(0)| + |\bar{\partial} f(0)| \leq \frac{4}{\pi}. \tag{4.1}$$

Numerous generalizations and refinements of the harmonic Schwarz lemma appeared in recent years [8,10]. An important difference with the holomorphic case is that (4.1) does not completely describe the possible values of the derivative $(\partial f(0), \bar{\partial} f(0))$. Indeed, an application of Parseval's identity shows that

$$|\partial f(0)|^2 + |\bar{\partial} f(0)|^2 \leq 1 \tag{4.2}$$

and neither of (4.1) and (4.2) imply each other. It turns out that the complete description of possible derivatives at 0 requires the dual Hardy norm from (3.1). The following is a refined form of Theorem 1.1 from the introduction.

Theorem 4.1. *For a vector $(\alpha, \beta) \in \mathbb{C}^2$ the following are equivalent:*

(i) there exists a harmonic map $f\colon \mathbb{D} \to \mathbb{D}$ *with* $f(0) = 0$, $\partial f(0) = \alpha$, *and* $\bar\partial f(0) = \beta$*;*
(ii) there exists a harmonic map $f\colon \mathbb{D} \to \mathbb{D}$ *with* $\partial f(0) = \alpha$ *and* $\bar\partial f(0) = \beta$*;*
(iii) $\|(\alpha,\beta)\|_{H^1_*} \le 1$.

Remark 4.2. Both (4.1) and (4.2) easily follow from Theorem 4.1. To obtain (4.1), use the definition of H^1_* together with the fact that $\|(a_1,a_2)\|_{H^1} = 4/\pi$ whenever $|a_1| = |a_2| = 1$ (see (2.2), (2.4)). To obtain (4.2), use the comparison of Hardy norms: $\|\cdot\|_{H^1} \le \|\cdot\|_{H^2}$, hence $\|\cdot\|_{H^1_*} \ge \|\cdot\|_{H^2_*} = \|\cdot\|_{H^2}$.

Remark 4.3. Combining Theorem 4.1 with Theorem 3.1 we obtain

$$\|(\partial f(0), \bar\partial f(0))\|_{H^4} \le 1 \tag{4.3}$$

for any harmonic map $f\colon \mathbb{D} \to \mathbb{D}$. In view of (2.1) this means $|\partial f(0)|^4 + 4|\partial f(0)\bar\partial f(0)|^2 + |\bar\partial f(0)|^4 \le 1$.

Proof of Theorem 4.1. (i) $\implies$ (ii) is trivial. Suppose that (ii) holds. To prove (iii), we must show that

$$|\alpha\bar\gamma + \beta\bar\delta| \le \|(\gamma,\delta)\|_{H^1} \tag{4.4}$$

for every vector $(\gamma,\delta) \in \mathbb{C}^2$. Let $g(z) = \gamma z + \delta\bar z$. Expanding f into the Taylor series $f(z) = f(0) + \alpha z + \beta\bar z + \dots$ and using the orthogonality of monomials on every circle $|z| = r$, $0 < r < 1$, we obtain

$$|\alpha\bar\gamma + \beta\bar\delta| = \frac{1}{2\pi r^2}\left|\int_0^{2\pi} f(re^{it})\overline{g(re^{it})}\,dt\right| \le \frac{1}{2\pi r^2}\int_0^{2\pi} |g(re^{it})|\,dt. \tag{4.5}$$

Letting $r \to 1$ and observing that

$$\frac{1}{2\pi}\int_0^{2\pi} |\gamma e^{it} + \delta e^{-it}|\,dt = \frac{1}{2\pi}\int_0^{2\pi} |\gamma + \delta e^{-2it}|\,dt = \frac{1}{2\pi}\int_0^{2\pi} |\gamma + \delta e^{it}|\,dt = \|(\gamma,\delta)\|_{H^1} \tag{4.6}$$

completes the proof of (4.4).

It remains to prove the implication (iii) $\implies$ (i). Let $\mathcal{F}_0$ be the set of harmonic maps $f\colon \mathbb{D} \to \mathbb{D}$ such that $f(0) = 0$, and let $\mathcal{D} = \{(\partial f(0), \bar\partial f(0))\colon f \in \mathcal{F}_0\}$. Since $\mathcal{F}_0$ is closed under convex combinations, the set $\mathcal{D}$ is convex. Since the function $f(z) = \alpha z + \beta\bar z$ belongs to $\mathcal{F}_0$ when $|\alpha| + |\beta| \le 1$, the point $(0,0)$ is an interior point of $\mathcal{D}$. The estimate (4.2) shows that $\mathcal{D}$ is bounded. Furthermore, $c\mathcal{D} \subset \mathcal{D}$ for any complex number c with $|c| \le 1$, because $\mathcal{F}_0$ has the same property. We claim that $\mathcal{D}$ is also a closed subset of $\mathbb{C}^2$. Indeed, suppose that a sequence of vectors $(\alpha_n,\beta_n) \in \mathcal{D}$ converges to $(\alpha,\beta) \in \mathbb{C}^2$. Pick a corresponding sequence of maps $f_n \in \mathcal{F}_0$. Being uniformly bounded, the maps $\{f_n\}$ form a normal family [2, Theorem 2.6]. Hence there exists a subsequence $\{f_{n_k}\}$ which converges uniformly on compact subsets of $\mathbb{D}$. The limit of this subsequence is a map $f \in \mathcal{F}_0$ with $\partial f(0) = \alpha$ and $\bar\partial f(0) = \beta$.

The preceding paragraph shows that $\mathcal{D}$ is the closed unit ball for some norm $\|\cdot\|_{\mathcal{D}}$ on $\mathbb{C}^2$. The implication (iii) $\implies$ (i) amounts to the statement that $\|\cdot\|_{\mathcal{D}} \le \|\cdot\|_{H^1_*}$. We will prove it in the dual form

$$\sup\{|\gamma\bar\alpha + \delta\bar\beta|\colon (\alpha,\beta) \in \mathcal{D}\} \ge \|(\gamma,\delta)\|_{H^1} \quad \text{for all } (\gamma,\delta) \in \mathbb{C}^2. \tag{4.7}$$

Since norms are continuous functions, it suffices to consider $(\gamma,\delta) \in \mathbb{C}^2$ with $|\gamma| \ne |\delta|$. Let $g\colon \mathbb{D} \to \mathbb{D}$ be the harmonic map with boundary values

$$g(z) = \frac{\gamma z + \delta\bar z}{|\gamma z + \delta\bar z|}, \quad |z| = 1.$$

Note that $g(-z) = -g(z)$ on the boundary, and therefore everywhere in $\mathbb{D}$. In particular, $g(0) = 0$, which shows $g \in \mathcal{F}_0$. Let $(\alpha, \beta) = (\partial g(0), \bar{\partial} g(0)) \in \mathcal{D}$. A computation similar to (4.5) shows that

$$\begin{aligned}\gamma\bar{\alpha} + \delta\bar{\beta} &= \frac{1}{2\pi}\int_0^{2\pi} (\gamma e^{it} + \delta e^{-it})\overline{g(e^{it})}\, dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} (\gamma e^{it} + \delta e^{-it})\frac{\overline{\gamma e^{it} + \delta e^{-it}}}{|\gamma e^{it} + \delta e^{-it}|}\, dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} |\gamma e^{it} + \delta e^{-it}|\, dt = \|(\gamma, \delta)\|_{H^1}\end{aligned}$$

where the last step uses (4.6). This proves (4.7) and completes the proof of Theorem 4.1. □

5. Higher dimensions

A version of the Schwarz lemma is also available for harmonic maps of the (Euclidean) unit ball $\mathbb{B}$ in $\mathbb{R}^n$. Let $\mathbb{S} = \partial\mathbb{B}$. For a square matrix $A \in \mathbb{R}^{n\times n}$, define its Hardy quasinorm by

$$\|A\|_{H^p} = \left(\int_{\mathbb{S}} \|Ax\|^p\, d\mu(x)\right)^{1/p} \tag{5.1}$$

where the integral is taken with respect to normalized surface measure μ on $\mathbb{S}$ and the vector norm $\|Ax\|$ is the Euclidean norm. In the limit $p \to \infty$ we recover the spectral norm of A, while the special case $p = 2$ yields the Frobenius norm of A divided by $\sqrt{n}$. The case $p = 1$ corresponds to "expected value norms" studied by Howe and Johnson in [7]. Also, letting $p \to 0$ leads to

$$\|A\|_{H^0} = \exp\left(\int_{\mathbb{S}} \log\|Ax\|\, d\mu(x)\right) \tag{5.2}$$

In general, H^p quasinorms on matrices are not submultiplicative. However, they have another desirable feature, which follows directly from (5.1): $\|UAV\|_{H^p} = \|A\|_{H^p}$ for any orthogonal matrices U, V. The singular value decomposition shows that $\|A\|_{H^p} = \|D\|_{H^p}$ where D is the diagonal matrix with the singular values of A on its diagonal.

Let us consider the matrix inner product $\langle A, B\rangle = \frac{1}{n}\operatorname{tr}(B^T A)$, which is normalized so that $\langle I, I\rangle = 1$. This inner product can be expressed by an integral involving the standard inner product on $\mathbb{R}^n$ as follows:

$$\langle A, B\rangle = \int_{\mathbb{S}} \langle Ax, Bx\rangle\, d\mu(x). \tag{5.3}$$

Indeed, the right hand side of (5.3) is the average of the numerical values $\langle B^T Ax, x\rangle$, which is known to be the normalized trace of $B^T A$, see [9].

The dual norms H^p_* are defined on $\mathbb{R}^{n\times n}$ by

$$\|A\|_{H^p_*} = \sup\{\langle A, B\rangle \colon \|B\|_{H^p} \le 1\} = \sup_{B\in\mathbb{R}^{n\times n}\setminus\{0\}} \frac{\langle A, B\rangle}{\|B\|_{H^p}}. \tag{5.4}$$

Applying Hölder's inequality to (5.3) yields $\langle A, B\rangle \le \|A\|_{H^q}\|B\|_{H^p}$ when $p^{-1}+q^{-1}=1$. Hence $\|A\|_{H^p_*} \le \|A\|_{H^q}$ but in general the inequality is strict. As an exception, we have $\|A\|_{H^2_*} = \|A\|_{H^2}$ because $\langle A, A\rangle = \|A\|^2_{H^2}$. As in the case of polynomials, our interest in dual Hardy norms is driven by their relation to harmonic maps.

Theorem 5.1. *For a matrix $A \in \mathbb{R}^{n\times n}$ the following are equivalent:*

(i) there exists a harmonic map $f\colon \mathbb{B} \to \mathbb{B}$ with $f(0) = 0$ and $Df(0) = A$;
(ii) there exists a harmonic map $f\colon \mathbb{B} \to \mathbb{B}$ with $Df(0) = A$;
(iii) $\|A\|_{H^1_*} \le 1$.

Proof. Since the proof is essentially the same as of Theorem 4.1, we only highlight some notational differences. Suppose (ii) holds. Expand f into a series of spherical harmonics, $f(x) = \sum_{d=0}^\infty p_d(x)$ where $p_d\colon \mathbb{R}^n \to \mathbb{R}^n$ is a harmonic polynomial map that is homogeneous of degree d. Note that $p_1(x) = Ax$. For any $n \times n$ matrix B the orthogonality of spherical harmonics [2, Proposition 5.9] yields

$$\langle A, B\rangle = \lim_{r\nearrow 1}\int_{\mathbb{S}} \langle f(rx), Bx\rangle\, d\mu(x) \le \|B\|_1$$

which proves (iii).

The proof of (iii) $\Longrightarrow$ (i) is based on considering, for any nonsingular matrix B, a harmonic map $g\colon \mathbb{B} \to \mathbb{B}$ with boundary values $g(x) = (Bx)/\|Bx\|$. Its derivative $A = Dg(0)$ satisfies

$$\langle B, A\rangle = \int_{\mathbb{S}} \langle Bx, g(x)\rangle\, d\mu(x) = \int_{\mathbb{S}} \frac{\langle Bx, Bx\rangle}{\|Bx\|}\, d\mu(x) = \|B\|_{H^1}$$

and (i) follows by the same duality argument as in Theorem 4.1. □

As an indication that the near-isometric duality of H^1 and H^4 norms (Theorem 3.1) may also hold in higher dimensions, we compute the relevant norms of P_k, the matrix of an orthogonal projection of rank k in $\mathbb{R}^3$. For rank 1 projection

$$P_1 = \begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

the norms are

$$\|P_1\|_{H^1} = \int_0^1 r\, dr = \frac{1}{2},$$

$$\|P_1\|_{H^4} = \left(\int_0^1 r^4\, dr\right)^{1/4} = \frac{1}{5^{1/4}} \approx 0.67,$$

$$\|P_1\|_{H^1_*} = \frac{\langle P_1, P_1\rangle}{\|P_1\|_1} = \frac{1/3}{1/2} = \frac{2}{3} \approx 0.67.$$

For rank 2 projection

$$P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

they are

$$\|P_2\|_{H^1} = \int_0^1 \sqrt{1-r^2}\, dr = \frac{\pi}{4},$$

$$\|P_2\|_{H^4} = \left(\int_0^1 (1-r^2)^2\, dr \right)^{1/4} = \left(\frac{8}{15} \right)^{1/4} \approx 0.85,$$

$$\|P_2\|_{H^1_*} = \frac{\langle P_2, P_2 \rangle}{\|P_2\|_1} = \frac{2/3}{\pi/4} = \frac{8}{3\pi} \approx 0.85.$$

This numerical agreement does not appear to be merely a coincidence, as numerical experiments with random 3×3 indicate that the ratio $\|A\|_{H^1_*}/\|A\|_{H^4}$ is always near 1. However, we do not have a proof of this.

As in the case of polynomials, there is an explicit formula for the H^4 norm of matrices. Writing $\sigma_1, \dots, \sigma_n$ for the singular values of A, we find

$$\|A\|_{H^4}^4 = \alpha \sum_{k=1}^{n} \sigma_k^4 + 2\beta \sum_{k<l} \sigma_k^2 \sigma_l^2 \tag{5.5}$$

where $\alpha = \int_{\mathbb{S}} x_1^4\, d\mu(x)$ and $\beta = \int_{\mathbb{S}} x_1^2 x_2^2\, d\mu(x)$. For example, if $n = 3$, the expression (5.5) evaluates to

$$\|A\|_{H^4}^4 = \frac{1}{5} \sum_{k=1}^{3} \sigma_k^4 + \frac{2}{15} \sum_{k<l} \sigma_k^2 \sigma_l^2.$$

Theorem 2.1 has a corollary for 2×2 matrices.

Corollary 5.2. *The H^p quasinorm on the space of 2×2 matrices satisfies the triangle inequality even when $0 \le p < 1$.*

Proof. A real linear map $x \mapsto Ax$ in $\mathbb{R}^2$ can be written in complex notation as $z \mapsto az + b\bar{z}$ for some $(a, b) \in \mathbb{C}^2$. A change of variable yields

$$\int_{|z|=1} |az + b\bar{z}|^p = \int_{|z|=1} |a + bz|^p$$

which implies $\|A\|_{H^p} = \|(a, b)\|_{H^p}$ for $p > 0$. The latter is a norm on $\mathbb{C}^2$ by Theorem 2.1. The case $p = 0$ is treated in the same way. $\square$

The aforementioned relation between a 2×2 matrix A and a complex vector (a, b) also shows that the singular values of A are $\sigma_1 = |a| + |b|$ and $\sigma_2 = ||a| - |b||$. It then follows from (2.1) that

$$\|A\|_{H^0} = \max(|a|, |b|) = \frac{\sigma_1 + \sigma_2}{2},$$

which is, up to scaling, the trace norm of A. Unfortunately, this relation breaks down in dimensions $n > 2$: for example, rank 1 projection P_1 in $\mathbb{R}^3$ has $\|P_1\|_{H^0} = 1/e$ while the average of its singular values is $1/3$.

We do not know whether H^p quasinorms with $0 \leq p < 1$ satisfy the triangle inequality for $n \times n$ matrices when $n \geq 3$.

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