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STOCHASTIC PARTIAL DIFFERENTIAL EQUATION MODELS FOR SPATIALLY DEPENDENT PREDATOR-PREY EQUATIONS

Nhu N. Nguyen and George Yin

Department of Mathematics
Wayne State University
Detroit, MI 48202, USA

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Abstract. Stemming from the stochastic Lotka-Volterra or predator-prey equations, this work aims to model the spatial inhomogeneity by using stochastic partial differential equations (SPDEs). Compared to the classical models, the SPDE models are more versatile. To incorporate more qualitative features of the ratio-dependent models, the Beddington-DeAngelis functional response is also used. To analyze the systems under consideration, first existence and uniqueness of solutions of the SPDEs are obtained using the notion of mild solutions. Then sufficient conditions for permanence and extinction are derived.

1. Introduction.

The predator-prey models or Lotka-Volterra equations have a long history and have been widely studied because of their importance in ecology. Such models have also been used in for example, statistical mechanics and other related fields. In 1925, the model was first introduced in [22] as follows

$$\begin{cases} \dfrac{dU(t)}{dt} = \big[U(t)\big(a - bV(t)\big)\big], \\ \dfrac{dV(t)}{dt} = \big[V(t)\big(-c + fU(t)\big)\big]. \end{cases}$$

To improve the model, the prey and predator self-competition terms have been added to the original model while different types of functional responses such as Holling types I-III [17], ratio-dependence type [4], and Beddington-DeAngelis type [5, 13], etc., have also been considered. Recently, Li et al. studied a predator-prey system with Beddington-DeAngelis functional response in [20], in which the density functions are spatially homogeneous. The model is represented by

$$\begin{cases} \dfrac{dU(t)}{dt} = \Big[U(t)\big(a_1 - b_1U(t)\big) - \dfrac{c_1U(t)V(t)}{m_1 + m_2U(t) + m_3V(t)}\Big],\ t \geq 0, \\ \dfrac{dV(t)}{dt} = \Big[V(t)\big(-a_2 - b_2V(t)\big) + \dfrac{c_2U(t)V(t)}{m_1 + m_2U(t) + m_3V(t)}\Big],\ t \geq 0, \end{cases}$$

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where a_i, b_i, c_i, and m_i are positive constants. Although significant progress has been made, it is well recognized that noise effect often needs to be taken into consideration and that allowing spacial inhomogeneous variation could improve the model further. To take environment noise into consideration, one considers a stochastic differential equation model as follows

$$\begin{cases} dU(t)=\Big[U(t)\big(a_1-b_1U(t)\big)-\dfrac{c_1U(t)V(t)}{m_1+m_2U(t)+m_3V(t)}\Big]dt+\sigma_1U(t)dB_1(t), t\geq 0,\\ dV(t)=\Big[V(t)\big(-a_2-b_2V(t)\big)+\dfrac{c_2U(t)V(t)}{m_1+m_2U(t)+m_3V(t)}\Big]dt+\sigma_2V(t)dB_2(t), t\geq 0,\end{cases}$$

where $B_1(t)$ and $B_2(t)$ are independent and real-valued Brownian motions, and σ_1 and $\sigma_2\neq 0$ are intensities of the noises. Such a problem has been studied in [15]. In fact, the study is related to what is known as Kolmogorov systems, which has a wide range of applications in ecology [14], epidemiology, as well as other fields such as social networks. The long-time behaviors have been characterized by providing a threshold between extinction and permanence. To make the model more suitable for a wider class of systems, it is natural to include spatial dependence. In the deterministic setup, it has been shown that not only is the spatial inhomogeneity mathematically interesting, but also it is crucially important for practical concerns. Taking the spatially inhomogeneous case into consideration, a predator-prey reaction-diffusion system takes the form

$$\begin{cases} \dfrac{\partial}{\partial t}U(t,x)=d_1\Delta U(t,x)+U(t,x)\big(a_1(x)-b_1(x)U(t,x)\big)\\ \qquad\qquad -\dfrac{c_1(x)U(t,x)V(t,x)}{m_1(x)+m_2(x)U(t,x)+m_3(x)V(t,x)} \text{ in } \mathbb{R}^+\times\mathcal{O},\\ \dfrac{\partial}{\partial t}V(t,x)=d_2\Delta V(t,x)+V(t,x)\big(-a_2(x)-b_2(x)V(t,x)\big)\\ \qquad\qquad +\dfrac{c_2(x)U(t,x)V(t,x)}{m_1(x)+m_2(x)U(t,x)+m_3(x)V(t,x)} \text{ in } \mathbb{R}^+\times\mathcal{O},\\ \partial_\nu U(t,x)=\partial_\nu V(t,x)=0 \qquad \text{on} \quad \mathbb{R}^+\times\partial\mathcal{O},\\ U(x,0)=U_0(x), V(x,0)=V_0(x) \quad \text{in} \quad \mathcal{O},\end{cases}$$

where Δ is the Laplacian with respect to the spatial variable, $\mathcal{O}$ is a bounded smooth domain of $\mathbb{R}^l$ ($l\geq 1$), ∂_ν denotes the directional derivative with the ν being the outer normal direction on $\partial\mathcal{O}$, and d_1 and d_2 are positive constants representing the diffusion rates of the prey and predator population densities, respectively. In contrast to the previous cases, in lieu of constant values, $a_i(x)$, $b_i(x)$, $c_i(x)$, and $m_i(x)\in C^2(\overline{\mathcal{O}},\mathbb{R})$ are allowed to be positive functions. Recently, spatially heterogeneous systems have been widely studied; see [2, 16, 19, 18, 23, 29] and reference therein. It has been demonstrated that including spatial inhomogeneity has provided better models with high fidelity. As argued in [24], a fundamental problem faced by ecologists is that the spatial and temporal scales at which measurements are practical. Much evidence demonstrates the importance of interactions and dispersal, and the importance of including spatial dependence in the formulation. In the aforementioned paper, the authors proposed a specific spatially dependent model.

This work presents our initial effort in treating random environmental noise, as well as taking into consideration of spatial inhomogeneity. In view of the progress to date, this paper proposes and analyzes a predator-prey model under stochastic influence and spatial inhomogeneity. We consider a stochastic partial differential equation model with initial and boundary data as follows

$$\begin{cases} dU(t,x) = \Big[d_1\Delta U(t,x) + U(t,x)\big(a_1(x) - b_1(x)U(t,x)\big) \\ \qquad -\dfrac{c_1(x)U(t,x)V(t,x)}{m_1(x)+m_2(x)U(t,x)+m_3(x)V(t,x)}\Big]dt + U(t,x)dW_1(t,x), \quad \text{in } \mathbb{R}^+\times\mathcal{O}, \\ dV(t,x) = \Big[d_2\Delta V(t,x) + V(t,x)\big(-a_2(x) - b_2(x)U(t,x)\big) \\ \qquad +\dfrac{c_2(x)U(t,x)V(t,x)}{m_1(x)+m_2(x)U(t,x)+m_3(x)V(t,x)}\Big]dt + V(t,x)dW_2(t,x), \quad \text{in } \mathbb{R}^+\times\mathcal{O}, \\ \partial_\nu U(t,x) = \partial_\nu V(t,x) = 0, \quad \text{on} \quad \mathbb{R}^+\times\partial\mathcal{O}, \\ U(x,0) = U_0(x), V(x,0) = V_0(x), \quad \text{for} \ \ x\in\mathcal{O}, \end{cases} \tag{1.1}$$

where $W_1(t,x)$ and $W_2(t,x)$ are $L^2(\mathcal{O},\mathbb{R})$-valued Wiener processes, which represent the noises in both time and space. We refer the readers to [11] for more details on the $L^2(\mathcal{O},\mathbb{R})$-valued Winner process.

The rest of the paper is arranged as follows. Section 2 gives some preliminary results and also formulates the problem to be studied precisely. Section 3 establishes the existence and uniqueness of the solution of the associated stochastic partial differential equations as well as its positivity and its continuous dependence on initial data. Section 4 introduces a sufficient condition for the extinction and permanence. Finally, Section 5 provides an example.

2. **Formulation and preliminaries.** Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^l$ $(l\geq 1)$ having a regular boundary and $L^2(\mathcal{O},\mathbb{R})$ be the separable Hilbert spaces, endowed with the scalar product

$$\langle u_1, v_1\rangle_{L^2(\mathcal{O},\mathbb{R})} := \int_{\mathcal{O}} u_1(x)v_1(x)dx,$$

with the corresponding norm $\sqrt{\langle\cdot,\cdot\rangle}$. We say $u_1\geq 0$ if $u_1(x)\geq 0$ almost everywhere in $\mathcal{O}$. Moreover, we denote by $L^2(\mathcal{O},\mathbb{R}^2)$ the space of all functions $u(x) = \big(u_1(x),u_2(x)\big)$ where $u_1,u_2\in L^2(\mathcal{O},\mathbb{R})$ endowed with the inner product

$$\begin{aligned}\langle u,v\rangle_{L^2(\mathcal{O},\mathbb{R}^2)} &:= \int_{\mathcal{O}}\big\langle u(x),v(x)\big\rangle_{\mathbb{R}^2}dx = \int_{\mathcal{O}}\big(u_1(x)v_1(x)+u_2(x)v_2(x)\big)dx \\ &= \langle u_1,v_1\rangle_{L^2(\mathcal{O},\mathbb{R})} + \langle u_2,v_2\rangle_{L^2(\mathcal{O},\mathbb{R})},\end{aligned}$$

where $u(x) = (u_1(x),u_2(x))$ and $v(x) = (v_1(x),v_2(x))$. Then $L^2(\mathcal{O},\mathbb{R}^2)$ is also a separable Hilbert spaces. In addition, for $\varepsilon>0, p\geq 1$, denote by $W^{\varepsilon,p}(\mathcal{O},\mathbb{R}^2)$ (as well as $W^{\varepsilon,p}(\mathcal{O},\mathbb{R})$) the Sobolev-Slobodeckij space (the Sobolev space with non-integer exponent).

Let $\big\{\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P}\big\}$ be a complete filtered probability space, $L^p(\Omega; C([0,t], L^2(\mathcal{O},\mathbb{R}^2)))$ be the space of predictable processes u that takes values in $C([0,t], L^2(\mathcal{O},\mathbb{R}^2))$, $\mathbb{P}$-a.s. with the norm

$$|u|^p_{L_{t,p}} := \mathbb{E}\sup_{s\in[0,t]}|u(s)|^p_{L^2(\mathcal{O},\mathbb{R}^2)}.$$

Assume that $\{B_{k,1}(t)\}_{k=1}^\infty$ and $\{B_{k,2}(t)\}_{k=1}^\infty$ are independent sequences of $\{\mathcal{F}_t\}_{t\geq 0}$-adapted one-dimensional Wiener processes. Fix an orthonormal basis $\{e_k(x)\}_{k=1}^\infty$

in $L^2(\mathcal{O},\mathbb{R})$ and assume that it is uniformly bounded in $L^\infty(\mathcal{O},\mathbb{R})$, i.e.,

$$C_0 := \sup_{k\in\mathbb{N}} \operatorname*{ess\,sup}_{x\in\mathcal{O}} |e_k(x)| < \infty.$$

We define the infinite dimensional Winner processes $W_i(t)$, the driving noise in equation (1.1) as follows

$$W_i(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_{k,i}} B_{k,i}(t) e_k, \quad i = 1, 2,$$

where $\{\lambda_{k,i}\}_{k=1}^{\infty}, (i = 1, 2)$ are sequences of non-negative real numbers satisfying

$$\lambda_i := \sum_{k=1}^{\infty} \lambda_{k,i} < \infty, \quad i = 1, 2. \tag{2.1}$$

To proceed, let A_1 and A_2 be Neumann realizations of $d_1\Delta$ and $d_2\Delta$ in $L^2(\mathcal{O},\mathbb{R})$, respectively, where the Laplace operator is understood in the distribution sense; see [11, Appendix A]. Then, A_1 and A_2 are infinitesimal generators of analytic semi-groups e^{tA_1} and e^{tA_2}, respectively. In addition, if we denote $A = (A_1, A_2)$, then it generates an analytic semigroup $e^{tA} = (e^{tA_1}, e^{tA_2})$. In [12, Theorem 1.4.1], it is proved that the space $L^1(\mathcal{O},\mathbb{R}^2) \cap L^\infty(\mathcal{O},\mathbb{R}^2)$ is invariant under e^{tA}, so that e^{tA} may be extended to a non-negative one-parameter semigroup $e^{tA(p)}$ on $L^p(\mathcal{O};\mathbb{R}^2)$, for all $1 \le p \le \infty$. All these semi-groups are strongly continuous and consistent in the sense that $e^{tA(p)}u = e^{tA(q)}u$ for any $u \in L^p(\mathcal{O},\mathbb{R}^2) \cap L^q(\mathcal{O},\mathbb{R}^2)$; see [7]. Henceforth, we suppress the parameter p and denote $e^{tA(p)}$ as e^{tA} whenever there is no confusion. Finally, we recall some well-known properties of operators A_i and analytic semi-groups e^{tA_i} for $i = 1, 2$ as follows

- $\forall u \in L^2(\mathcal{O},\mathbb{R})$ then $\int_0^t e^{sA_i} u ds \in D(A_i)$ and $A_i(\int_0^t e^{sA_i} u ds) = e^{tA_i}u - u$.
- By Green's identity, it is possible to obtain $\forall u \in D(A_i)$, $\int_{\mathcal{O}} A_i u(x) dx = 0$.
- The semigroup e^{tA} satisfies the fowling properties

$$\left|e^{tA}u\right|_{L^\infty(\mathcal{O},\mathbb{R}^2)} \le c\,|u|_{L^\infty(\mathcal{O},\mathbb{R}^2)} \text{ and } \left|e^{tA}u\right|_{L^2(\mathcal{O},\mathbb{R}^2)} \le c\,|u|_{L^2(\mathcal{O},\mathbb{R}^2)}. \tag{2.2}$$

- For any $t, \varepsilon > 0$, $p \ge 1$, the semigroup e^{tA} maps $L^p(\mathcal{O},\mathbb{R}^2)$ into $W^{\varepsilon,p}(\mathcal{O},\mathbb{R}^2)$ and $\forall u \in L^p(\mathcal{O},\mathbb{R}^2)$

$$\left|e^{tA}u\right|_{\varepsilon,p} \le c(t\wedge 1)^{-\varepsilon/2}\,|u|_{L^p(\mathcal{O},\mathbb{R}^2)}, \tag{2.3}$$

for some constant c independent of u, t.

For further details, we refer the reader to the monographs [3, 12, 26] and references therein.

We rewrite (1.1) as a stochastic differential equation in infinite dimension

$$\begin{cases} dU(t) = \Big[A_1U(t) + U(t)\big(a_1 - b_1U(t)\big) - \dfrac{c_1U(t)V(t)}{m_1 + m_2U(t) + m_3V(t)}\Big]dt + U(t)dW_1(t), \\ dV(t) = \Big[A_2V(t) + V(t)\big(-a_2 - b_2V(t)\big) + \dfrac{c_2U(t)V(t)}{m_1 + m_2U(t) + m_3V(t)}\Big]dt + V(t)dW_2(t), \\ U(0) = U_0, \;\; V(0) = V_0. \end{cases} \tag{2.4}$$

As usual, we follow Walsh [28] to say that $(U(t), V(t))$ is a mild solution of (2.4) if

$$\begin{cases} U(t) = e^{tA_1}U_0 + \displaystyle\int_0^t e^{(t-s)A_1}\Big[U(s)\big(a_1 - b_1U(s)\big) \\ \qquad\qquad\qquad\qquad\qquad - \dfrac{c_1U(s)V(s)}{m_1 + m_2U(s) + m_3V(s)}\Big]ds + W_U(t), \\ V(t) = e^{tA_2}V_0 + \displaystyle\int_0^t e^{(t-s)A_2}\Big[V(s)\big(-a_2 - b_2V(s)\big) \\ \qquad\qquad\qquad\qquad\qquad + \dfrac{c_2U(s)V(s)}{m_1 + m_2U(s) + m_3V(s)}\Big]ds + W_V(t), \end{cases} \tag{2.5}$$

where

$$W_U(t) = \int_0^t e^{(t-s)A_1}U(s)dW_1(s) \quad \text{and} \quad W_V(t) = \int_0^t e^{(t-s)A_2}V(s)dW_2(s),$$

or in the vector form

$$Z(t) = e^{tA}Z_0 + \int_0^t e^{(t-s)A}F(Z(s))ds + W_Z(t), \quad Z_0 = (U_0, V_0), \tag{2.6}$$

where $Z(t) = (U(t), V(t))$, $e^{tA}Z_0 := (e^{tA_1}U_0, e^{tA_2}V_0)$, $W_Z(t) = (W_U(t), W_V(t))$ and $F(Z) := \big(F_1(Z), F_2(Z)\big)$, $e^{(t-s)A}F(Z) := (e^{(t-s)A_1}F_1(Z), e^{(t-s)A_2}F_2(Z))$ where

$$F_1(Z) := U(a_1 - b_1U) - \frac{c_1UV}{m_1 + m_2U + m_3V},$$

$$F_2(Z) := V(-a_2 - b_2V) + \frac{c_2UV}{m_1 + m_2U + m_3V}.$$

Remark 1. The first integrals on the right-hand sides of (2.5) are understood as Bochner integrals while $W_U(t), W_V(t)$ are the stochastic integrals (stochastic convolutions); see [11]. Moreover, $U(s)$ and $V(s)$ in the stochastic integrals are understood as multiplication operators. The calculations involving vectors are understood as in the usual sense.

For many problems in population dynamics or ecology, an important question is whether an individual will die out in the long time. That is, the consideration of extinction or permanence. Since the mild solution is used, let us modify some definitions in [25] as follows.

Definition 2.1. A population with density $u(t, x)$ is said to be extinct in the mean if

$$\limsup_{t\to\infty} \mathbb{E}\int_{\mathcal{O}} u(t, x)dx = 0,$$

and that is said to be permanent in the mean if there exist a positive number $\widehat{\delta}$, is independent of initial conditions of population, such that

$$\limsup_{t\to\infty} \mathbb{E}\int_{\mathcal{O}} u^2(t, x)dx \geq \widehat{\delta}.$$

In what follows, for convenience, we often suppress the "in the mean" when we refer to extinction and permanence in the mean, because we are mainly working with mild solutions.

3. **Existence, uniqueness and positivity of the mild solution.** Since the coefficients are non-Lipschitz and faster than linear growth, the existence and uniqueness of the mild solutions are not obvious. Although the existence of the mild solution of reaction-diffusion equations with non-Lipschitz term were treated in [8], we cannot apply directly the result in this paper since our coefficients do not satisfy the conditions in [8]. However, we can follow the method in [8] by considering the coefficients in each compact set so that they are Lipschitz continuous and therefore we will define the solution using these solutions. In what follows, without loss of the generality we can assume $|\mathcal{O}|=1$ for simplicity. Moreover, we also assume that the initial values are non-random.

Theorem 3.1. *For any initial data $0\le U_0,V_0\in L^\infty(\mathcal{O},\mathbb{R})$, there exists a unique mild solution $(U(t),V(t))$ of (2.4) belongs to $L^p(\Omega;C([0,T],L^2(\mathcal{O},\mathbb{R}^2)))$ for any $T>0,p\ge 1$. Moreover, the solution is positive, i.e., $U(t),V(t)\ge 0$ for any t and depends continuously on initial data.*

Proof. In this proof, the letter c denotes positive constants whose values may change in different occurrences. We will write the dependence of constant on parameters explicitly if it is essential. First, we rewrite the coefficients by defining

$$f_1(x,u,v)=u\big(a_1(x)-b_1(x)u\big)-\frac{c_1(x)uv}{m_1(x)+m_2(x)u+m_3(x)v},$$

$$f_2(x,u,v)=v\big(-a_2(x)-b_2(x)v\big)+\frac{c_2(x)uv}{m_1(x)+m_2(x)u+m_3(x)v},$$

where $f_i:\mathcal{O}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$. For each $n\in\mathbb{N}$, we define

$$f_{n,i}:=\begin{cases} f_i(x,u,v) \quad \text{if} \quad |(u,v)|_{\mathbb{R}^2}\le n,\\ f_i\Big(x,\dfrac{nu}{|(u,v)|_{\mathbb{R}^2}},\dfrac{nv}{|(u,v)|_{\mathbb{R}^2}}\Big) \quad \text{if} \quad |(u,v)|_{\mathbb{R}^2}>n.\end{cases}$$

For each n, $f_n(x,\cdot,\cdot)=\big(f_{n,1}(x,\cdot,\cdot),f_{n,2}(x,\cdot,\cdot)\big):\mathbb{R}^2\to\mathbb{R}^2$ is Lipschitz continuous, uniformly with respect to $x\in\mathcal{O}$, so that the composition operator $F_n(z)$ associated to f_n (with $z(x)=(u(x),v(x))$),

$$F_n(z)(x)=\big(F_{n,1}(z)(x),F_{n,2}(z)(x)\big):=\big(f_{n,1}(x,z(x)),f_{n,2}(x,z(x))\big),x\in\mathcal{O},$$

is Lipschitz continuous in both $L^2(\mathcal{O},\mathbb{R}^2)$ and $L^\infty(\mathcal{O},\mathbb{R}^2)$.

We proceed to consider the following problem

$$dZ_n(t)=\big[AZ_n(t)+F_n(Z_n(t))\big]dt+Z_n(t)dW(t),\quad Z_n(0)=(U_0,V_0),\tag{3.1}$$

where $Z_n(t)=\big(U_n(t),V_n(t)\big)$, $AZ_n(t):=\big(A_1U_n(t),A_2V_n(t)\big)$ and

$$Z_n(t)dW(t):=\big(U_n(t)dW_1(t),V_n(t)dW_2(t)\big).$$

Since the coefficient in (3.1) is Lipschitz continuous, by contraction mapping argument (see [25, Proof of Theorem 3.1] or [11]), we obtain that the equation (3.1) admits a unique mild solution $Z_n(t)=(U_n(t),V_n(t))\in L^p(\Omega;C([0,T_0],L^2(\mathcal{O},\mathbb{R}^2)))$ for some sufficiently small T_0. Therefore, for any finite $T>0$, there is a unique mild solution of (3.1) in $L^p(\Omega;C([0,T],L^2(\mathcal{O},\mathbb{R}^2)))$. To proceed, we will prove the positivity of $U_n(t),V_n(t)$.

Lemma 3.1. *For any initial condition $0\le U_0,V_0\in L^\infty(\mathcal{O},\mathbb{R})$, $U_n(t),V_n(t)\ge 0$, $\forall t\in[0,T]$.*

Proof. Let $(U_n^*(t), V_n^*(t))$ be the mild solution of the equation

$$\begin{cases} dU_n^*(t) = \Big[A_1 U_n^*(t) + F_{n,1}\big(U_n^*(t)\vee 0, V_n^*(t)\vee 0\big)\Big]dt + \big(U_n^*(t)\vee 0\big)dW_1(t), \\ dV_n^*(t) = \Big[A_2 V_n^*(t) + F_{n,2}\big(U_n^*(t)\vee 0, V_n^*(t)\vee 0\big)\Big]dt + \big(V_n^*(t)\vee 0\big)dW_2(t), \\ U_n^*(0) = U_0, V_n^*(0) = V_0. \end{cases} \tag{3.2}$$

For $i = 1, 2$, let $\lambda_i \in \rho(A_i)$, the resolvent set of A_i and $R_i(\lambda_i) := \lambda_i R_i(\lambda_i, A_i)$, with $R_i(\lambda_i, A_i)$ being the resolvent of A_i. For each small $\varepsilon > 0$, $\lambda = (\lambda_1, \lambda_2) \in \rho(A_1) \times \rho(A_2)$, by [21, Theorem 1.3.6], there exists a unique strong solution $U_{n,\lambda,\varepsilon}(t,x)$, $V_{n,\lambda,\varepsilon}(t,x)$ of the equation

$$\begin{cases} dU_{n,\lambda,\varepsilon}(t) = \Big[A_1 U_{n,\lambda,\varepsilon}(t) + R_1(\lambda_1)F_{n,1}\big(\varepsilon\Phi(\varepsilon^{-1}U_{n,\lambda,\varepsilon}(t)), \varepsilon\Phi(\varepsilon^{-1}V_{n,\lambda,\varepsilon}(t))\big)\Big]dt \\ \qquad\qquad\qquad + R_1(\lambda_1)\varepsilon\Phi(\varepsilon^{-1}U_{n,\lambda,\varepsilon}(t))dW_1(t), \\ dV_{n,\lambda,\varepsilon}(t) = \Big[A_2 V_{n,\lambda,\varepsilon}(t) + R_2(\lambda_2)F_{n,2}\big(\varepsilon\Phi(\varepsilon^{-1}U_{n,\lambda,\varepsilon}(t)), \varepsilon\Phi(\varepsilon^{-1}V_{n,\lambda,\varepsilon}(t))\big)\Big]dt \\ \qquad\qquad\qquad + R_2(\lambda_2)\varepsilon\Phi(\varepsilon^{-1}V_{n,\lambda,\varepsilon}(t))dW_2(t), \\ U_{n,\lambda,\varepsilon}(0) = R_1(\lambda_1)U_0\ ,\ V_{n,\lambda,\varepsilon}(0) = R_2(\lambda_2)V_0, \end{cases} \tag{3.3}$$

where $\Phi : \mathbb{R} \to \mathbb{R}$ is a function satisfying

$$\begin{cases} \Phi \in C^2(\mathbb{R}),\ \Phi(\xi) = 0 \text{ if } \xi < 0 \text{ and } \Phi(\xi) \geq 0 \text{ if } \xi \geq 0, \\ \varepsilon\Phi(\varepsilon^{-1}\xi) \to \xi \vee 0 \quad \text{as } \varepsilon \to 0. \end{cases}$$

For example,

$$\Phi(\xi) = \begin{cases} 0 \quad \text{if } \xi \leq 0, \\ 3\xi^5 - 8\xi^4 + 6\xi^3 \quad \text{if } 0 < \xi < 1, \\ \xi \quad \text{if } \xi \geq 1. \end{cases}$$

Combining with the convergence property in [21, Theorem 1.3.6], we have

$$(U_{n,\lambda_k,\varepsilon}(t), V_{n,\lambda_k,\varepsilon}(t)) \to (U_n^*(t), V_n^*(t)) \ \text{ in } L^p(\Omega; C([0,T], L^2(\mathcal{O}, \mathbb{R}^2)))$$

for some sequences $\{\lambda_k\} \subset \rho(A_1) \times \rho(A_2)$ and as $\varepsilon \to 0$.

Now, as in [27], let

$$\varphi(\xi) = \begin{cases} \xi^2 - \dfrac{1}{6} & \text{if } \xi \leq -1, \\ -\dfrac{\xi^4}{2} - \dfrac{4\xi^3}{3} & \text{if } -1 < \xi < 0, \\ 0 & \text{if } \xi \geq 0. \end{cases}$$

Then $\varphi''(\xi) \geq 0\ \forall\xi$ and $\varphi'(\xi)\Phi(\xi) = \varphi''(\xi)\Phi(\xi) = 0\ \forall\xi$. Because $R(\lambda_i, A_i)$ is positivity preserving, by virtue of Itô's Lemma ([9, Theorem 3.8]), we obtain

$$\begin{aligned} \int_{\mathcal{O}} \varphi(U_{n,\lambda,\varepsilon}(t,x))dx &= d_1 \int_0^t \int_{\mathcal{O}} \varphi'(U_{n,\lambda,\varepsilon}(s,x))\Delta U_{n,\lambda,\varepsilon}(s,x)dxds \\ &= -d_1 \int_0^t \int_{\mathcal{O}} \varphi''(U_{n,\lambda,\varepsilon}(s,x))\left|\nabla U_{n,\lambda,\varepsilon}(s,x)\right|^2 dxds \\ &\leq 0. \end{aligned}$$

Since $\varphi(\xi) > 0$ for all $\xi < 0$, we conclude that $\forall n \in \mathbb{N}, \varepsilon \geq 0, \lambda \in \rho(A_1) \times \rho(A_2)$ and $U_{n,\lambda,\varepsilon}(t) \geq 0$ for all $t \in [0,T]$. Similarly, we obtain the positivity of $V_{n,\lambda,\varepsilon}(t)$. Hence, $U_n^*(t), V_n^*(t) \geq 0$ for all $t \in [0,T]$ a.s. Since $(U_n^*(t), V_n^*(t))$ is the solution of

(3.2) and is positive, $U_n^*(t) = U_n(t), V_n^*(t) = V_n(t)$. As a consequence, we obtain the positivity of $U_n(t), V_n(t)$. □

We are in a position to show that the sequence $\{Z_n\}_{n=1}^\infty$ is bounded by the following lemma.

Lemma 3.2. *For all $n \in \mathbb{N}$ then*

$$\mathbb{E} \sup_{s\in[0,t]} |Z_n(s)|^p_{L^\infty(\mathcal{O},\mathbb{R}^2)} \le c_p(t)\big(1 + |Z_0|_{L^\infty(\mathcal{O},\mathbb{R}^2)}\big), \tag{3.4}$$

where $c_p(t)$ is a positive constant that may depend on p, t but is independent of n.

Proof. Without loss of the generality, we need only consider p being sufficient large such that we can choose simultaneously $\beta, \varepsilon > 0$ satisfying

$$\frac{1}{p} < \beta < \frac{1}{2} \quad \text{and} \quad \frac{l}{p} < \varepsilon < 2\big(\beta - \frac{1}{p}\big).$$

By the definition of mild solution, we have

$$U_n(t,x) = e^{tA_1}U_0(x) + \int_0^t e^{(t-s)A_1}F_{n,1}(U_n(s,x), V_n(s,x))ds + W_{U_n}(t,x),$$

where $W_{U_n}(t) = \int_0^t e^{(t-s)A_1}U_n(s)dW_1(s)$. Thus, since e^{tA_1} is positivity preserving and $U_n(t), V_n(t)$ are positive, by definition of $F_{n,1}$ and (2.2), we obtain

$$\begin{aligned}
&|U_n(t)|_{L^\infty(\mathcal{O},\mathbb{R})} = \operatorname*{ess\,sup}_{x\in\mathcal{O}} |U_n(t,x)| = \operatorname*{ess\,sup}_{x\in\mathcal{O}} U_n(t,x)\\
&= \operatorname*{ess\,sup}_{x\in\mathcal{O}} \Big[e^{tA_1}U_0(x) + \int_0^t e^{(t-s)A_1}F_{n,1}(U_n(s,x), V_n(s,x))ds + W_{U_n}(t,x)\Big]\\
&\le \operatorname*{ess\,sup}_{x\in\mathcal{O}} \Big[e^{tA_1}U_0(x) + \int_0^t e^{(t-s)A_1}U_n(s,x)a_1(x)ds + W_{U_n}(t,x)\Big]\\
&\le c(t)\Big(\big|U_0\big|_{L^\infty(\mathcal{O},\mathbb{R})} + \int_0^t \Big|U_n(s)\Big|_{L^\infty(\mathcal{O},\mathbb{R})}ds + \Big|W_{U_n}(t)\Big|_{L^\infty(\mathcal{O},\mathbb{R})}\Big),
\end{aligned} \tag{3.5}$$

where $c(t)$ is a constant depending only on t and independent of n. By using a factorization argument (see e.g., [11, Theorem 8.3]), we have

$$W_{U_n}(t) = \frac{\sin \pi\beta}{\pi} \int_0^t (t-s)^{\beta-1}e^{(t-s)A_1}Y_{U_n}(s)ds,$$

where

$$Y_{U_n}(s) = \int_0^s (s-r)^{-\beta}e^{(s-r)A_1}U_n(r)dW_1(r).$$

It is easily seen from (2.3) and Hölder's inequality that

$$\begin{aligned}
&|W_{U_n}(t)|_{\varepsilon,p}\\
&\le c_\beta \int_0^t (t-s)^{\beta-1}\big((t-s)\wedge 1\big)^{-\varepsilon/2}|Y_{U_n}(s)|_{L^p(\mathcal{O},\mathbb{R})}ds\\
&\le c_{\beta,p}(t)\Big(\int_0^t \big((t-s)\wedge 1\big)^{\frac{p}{p-1}(\beta-\varepsilon/2-1)}ds\Big)^{\frac{p-1}{p}}\Big(\int_0^t |Y_{U_n}(s)|^p_{L^p(\mathcal{O},\mathbb{R})}ds\Big)^{\frac{1}{p}}\\
&\le c_{\beta,p}(t)\Big(\int_0^t |Y_{U_n}(s)|^p_{L^p(\mathcal{O},\mathbb{R})}ds\Big)^{\frac{1}{p}} \text{ a.s. },
\end{aligned} \tag{3.6}$$

where $c_{\beta,p}(t)$ is some positive constant, independent of n. On the other hand, for all $s \in [0,t]$, almost every $x \in \mathcal{O}$, we have

$$Y_{U_n}(s,x) = \int_0^s (s-r)^{-\beta} \sum_{k=1}^{\infty} \sqrt{\lambda_{k,1}} M_1(s,r,k,x) dB_{k,1}(r),$$

where

$$M_1(s,r,k) = e^{(s-r)A_1} U_n(r) e_k.$$

Therefore, applying the Burkholder inequality, we obtain that for all $s \in [0,t]$, almost every $x \in \mathcal{O}$,

$$\mathbb{E}\,|Y_{U_n}(s,x)|^p \le c_p \mathbb{E}\Big[\int_0^s (s-r)^{-2\beta} \sum_{k=1}^{\infty} \lambda_{k,1}\, |M_1(s,r,k,x)|^2\, dr\Big]^{\frac{p}{2}}.$$

As a consequence,

$$\begin{aligned}\mathbb{E}\int_0^t |Y_{U_n}(s)|^p_{L^p(\mathcal{O},\mathbb{R})}\, ds \\ \le c_p(t) \int_0^t \mathbb{E}\Big(\int_0^s (s-r)^{-2\beta} \lambda_1 \sup_{k\in\mathbb{N}} |M_1(s,r,k)|^2_{L^\infty(\mathcal{O},\mathbb{R})}\, dr\Big)^{\frac{p}{2}} ds.\end{aligned} \tag{3.7}$$

Moreover, since the uniformly boundedness property of $\{e_k\}_{k=1}^{\infty}$ and (2.2), we have

$$\sup_{k\in\mathbb{N}} |M_1(s,r,k)|_{L^\infty(\mathcal{O},\mathbb{R})} \le c\, |U_n(r)|_{L^\infty(\mathcal{O},\mathbb{R})}\,, \tag{3.8}$$

for some constant c independent of n, s, r, u, v. Combining (3.7) and (3.8) implies that

$$\begin{aligned}\mathbb{E}\int_0^t |Y_{U_n}(s)|^p_{L^p(\mathcal{O},\mathbb{R})}\, ds &\le c_p(t) \int_0^t \mathbb{E} \sup_{r\in[0,s]} |U_n(r)|^p_{L^\infty(\mathcal{O},\mathbb{R})} \Big(\int_0^s (s-r)^{-2\beta} dr\Big)^{\frac{p}{2}} ds \\ &\le c_{\beta,p}(t) \int_0^t \mathbb{E} \sup_{r\in[0,s]} |U_n(r)|^p_{L^\infty(\mathcal{O},\mathbb{R})}\, ds.\end{aligned} \tag{3.9}$$

Since $\varepsilon > l/p$, the Sobolev inequality, (3.6), and (3.9) imply that

$$\mathbb{E} \sup_{s\in[0,t]} |W_{U_n}(s)|^p_{L^\infty(\mathcal{O},\mathbb{R})} \le c_p(t) \int_0^t \mathbb{E} \sup_{r\in[0,s]} |U_n(r)|^p_{L^\infty(\mathcal{O},\mathbb{R})}\, ds. \tag{3.10}$$

Hence, we obtain form (3.5) and (3.10) that

$$\mathbb{E} \sup_{s\in[0,t]} |U_n(s)|^p_{L^\infty(\mathcal{O},\mathbb{R})} \le c_p(t)\Big(|U_0|^p_{L^\infty(\mathcal{O},\mathbb{R})} + \int_0^t \mathbb{E} \sup_{r\in[0,s]} \Big|U_n(r)\Big|^p_{L^\infty(\mathcal{O},\mathbb{R})} ds\Big), \tag{3.11}$$

for some positive constant $c_p(t)$ that is independent of n. Therefore, we obtain from Gronwall's inequality that

$$\mathbb{E} \sup_{s\in[0,t]} |U_n(s)|^p_{L^\infty(\mathcal{O},\mathbb{R})} \le c_p(t)\big(1 + |U_0|^p_{L^\infty(\mathcal{O},\mathbb{R})}\big),$$

for some constant $c_p(t)$, is independent of n. Similarly, we have the same estimate for $V_n(t)$. Thus the Lemma is proved. □

Completion of the Proof of the Theorem. For any $n \in \mathbb{N}$, we define

$$\zeta_n := \inf\{t \ge 0 : |Z_n(t)|_{L^\infty(\mathcal{O},\mathbb{R}^2)} \ge n\}, \tag{3.12}$$

with the usual convention that $\inf\emptyset=\infty$ and define $\zeta=\sup_{n\in\mathbb{N}}\zeta_n$. Then we have

$$\mathbb{P}\{\zeta<\infty\}=\lim_{T\to\infty}\mathbb{P}\{\zeta<T\},$$

and for each $T\geq 0$

$$\mathbb{P}\{\zeta\leq T\}=\lim_{n\to\infty}\mathbb{P}\{\zeta_n\leq T\}.$$

For any fixed $n\in\mathbb{N}$ and $T\geq 0$, it follows from Lemma 3.2 that

$$\begin{aligned}\mathbb{P}\{\zeta_n\leq T\}=\mathbb{P}\Big\{\sup_{t\in[0,T]}|Z_n(t)|^p_{L^\infty(\mathcal{O},\mathbb{R}^2)}\geq n^p\Big\}&\leq\frac{1}{n^p}\mathbb{E}\sup_{t\in[0,T]}|Z_n(t)|^p_{L^\infty(\mathcal{O},\mathbb{R}^2)}\\&\leq\frac{c_p(T)\big(1+|Z_0|_{L^\infty(\mathcal{O},\mathbb{R}^2)}\big)}{n^p}.\end{aligned}$$

It leads to that $\mathbb{P}\{\zeta_n\leq T\}$ goes to zero as $n\to\infty$ and we get $\mathbb{P}\{\zeta=\infty\}=1$. Hence, for any $t\geq 0$, and $\omega\in\{\zeta=\infty\}$, there exists an $n\in\mathbb{N}$ such that $t\leq\zeta_n(\omega)$. Thus we can define

$$Z(t)(\omega):=Z_n(t)(\omega).$$

To proceed, we need to show that this definition is well-defined, i.e., for any $t\leq\zeta_n\wedge\zeta_m$ then $Z_n(t)=Z_m(t)$, $\mathbb{P}$-a.s. For $n<m$ we set $\zeta_{m,n}=\zeta_n\wedge\zeta_m$. By definition of F_n,F_m

$$\text{if}\quad|z|_{L^\infty(\mathcal{O},\mathbb{R}^2)}\leq n\quad\text{then}\quad F_n(z)(x)=F_m(z)(x)\text{ almost everywhere in }\mathcal{O}.$$

Therefore, we have

$$\begin{aligned}&Z_n(t\wedge\zeta_{m,n})-Z_m(t\wedge\zeta_{m,n})\\&=\int_0^{t\wedge\zeta_{m,n}}e^{(t-s)A}\Big(F_n(Z_n(s))-F_m(Z_m(s))\Big)ds+W_{Z_n-Z_m}(t\wedge\zeta_{m,n})\\&=\int_0^t\mathbf{1}_{\{s\leq\zeta_{m,n}\}}e^{(t-s)A}\Big(F_m(Z_n(s\wedge\zeta_{m,n}))-F_m(Z_m(s\wedge\zeta_{m,n}))\Big)ds\\&+W_{Z_n-Z_m}(t\wedge\zeta_{m,n}),\end{aligned}$$

where

$$W_{Z_n-Z_m}(t):=\Big(\int_0^t e^{(t-s)A_1}(U_n(s)-U_m(s))dW_1(s),\int_0^t e^{(t-s)A_2}(V_n(s)-V_m(s))dW_2(s)\Big).$$

Using a similar argument for getting (3.10), we obtain

$$\begin{aligned}&\mathbb{E}\sup_{s\in[0,t]}|W_{Z_n-Z_m}(s\wedge\zeta_{m,n})|^p_{L^\infty(\mathcal{O},\mathbb{R}^2)}\\&\qquad\leq c_p(t)\int_0^t\mathbb{E}\sup_{s'\in[0,s]}|Z_n(s'\wedge\zeta_{m,n})-Z_m(s'\wedge\zeta_{m,n})|^p_{L^\infty(\mathcal{O},\mathbb{R}^2)}\,ds.\end{aligned}\tag{3.13}$$

Therefore, combining with property (2.2), the Lipschitz continuity of F_m, and (3.13) yields

$$\begin{aligned}&\mathbb{E}\sup_{s\in[0,t]}|Z_n(s\wedge\zeta_{m,n})-Z_m(s\wedge\zeta_{m,n})|^p_{L^\infty(\mathcal{O},\mathbb{R}^2)}\\&\qquad\leq c_{p,m}(t)\int_0^t\mathbb{E}\sup_{s'\in[0,s]}|Z_n(s'\wedge\zeta_{m,n})-Z_m(s'\wedge\zeta_{m,n})|^p_{L^\infty(\mathcal{O},\mathbb{R}^2)}\,ds.\end{aligned}$$

Gronwall's inequality implies that $Z_n(t\wedge\zeta_{m,n})=Z_m(t\wedge\zeta_{m,n})$ $\forall t$ or

$$Z_n(t)=Z_m(t),\qquad\forall t\leq\zeta_m\wedge\zeta_n.\tag{3.14}$$

It is clear that the process $Z(t)$ defined as above is a mild solution of (2.6). Indeed, for any $t \geq 0$, $\omega \in \{\zeta = \infty\}$ then there exists $n \in \mathbb{N}$ such that $t \leq \zeta_n$ and

$$\begin{aligned} Z(t) = Z_n(t) &= e^{tA} Z_0 + \int_0^t e^{(t-s)A} F_n(Z_n(s)) ds + W_{Z_n}(t) \\ &= e^{tA} Z_0 + \int_0^t e^{(t-s)A} F(Z(s)) ds + W_Z(t). \end{aligned}$$

Next, we prove that such solution is unique. If there exists an other solution $\widehat{Z}(t)$ of (2.6). By the argument in the processing of getting (3.14), it is possible to obtain

$$Z(t \wedge \zeta_n) = \widehat{Z}(t \wedge \zeta_n), \quad \forall n \in \mathbb{N}, t \geq 0.$$

Since $\zeta_n \to \infty$ as $n \to \infty$ $\mathbb{P}$-a.s., we get $Z(t) = \widehat{Z}(t)$. Finally, we show that $Z(t) \in L^p(\Omega; C([0,T], L^2(\mathcal{O}, \mathbb{R}^2)))$. Indeed, for any $p \geq 1, T > 0$,

$$\begin{aligned} \sup_{t \in [0,T]} |Z(t)|^p_{L^2(\mathcal{O},\mathbb{R}^2)} &= \lim_{n \to \infty} \sup_{t \in [0,T]} |Z(t)|^p_{L^2(\mathcal{O},\mathbb{R}^2)} \mathbf{1}_{\{T \leq \zeta_n\}} \\ &= \lim_{n \to \infty} \sup_{t \in [0,T]} |Z_n(t)|^p_{L^2(\mathcal{O},\mathbb{R}^2)} \mathbf{1}_{\{T \leq \zeta_n\}}. \end{aligned}$$

Hence, by the boundedness of $Z_n(t)$ in Lemma 3.2, we obtain that the equation (2.5) admits a unique mild solution $Z(t) = (U(t), V(t)) \in L^p(\Omega; C([0,T], L^2(\mathcal{O}, \mathbb{R}^2)))$. The positivity of $U(t), V(t)$ follow positivity of $U_n(t), V_n(t)$.

To complete the proof, we prove that the solution depends continuously on initial data. For convenience, we use superscripts to indicate the dependence of the solution on initial values. Let $Z^{z_1}(t), Z^{z_2}(t)$ and $Z^{z_1}_n(t), Z^{z_2}_n(t)$ be the solutions of (2.6) and (3.1) with initial conditions $Z(0) = Z_n(0) = z_1$ and $Z(0) = Z_n(0) = z_2$, respectively. As in the proof of the first part, since the Lipschitz continuity of F_n, it is easy to obtain that

$$|Z^{z_1}_n - Z^{z_2}_n|^p_{L_{T,p}} \leq c_{n,p}(T)\, |z_1 - z_2|^p_{L^2(\mathcal{O},\mathbb{R}^2)}. \tag{3.15}$$

Consider the stopping times $\zeta^{z_1}_n$ and $\zeta^{z_2}_n$ as in (3.12), we have

$$\begin{aligned} |Z^{z_1} - Z^{z_2}|^p_{L_{T,p}} &= \mathbb{E} \sup_{s \in [0,T]} |Z^{z_1}(s) - Z^{z_2}(s)|^p_{L^2(\mathcal{O},\mathbb{R}^2)} \mathbf{1}_{\{\zeta^{z_1}_n \wedge \zeta^{z_2}_n > T\}} \\ &\quad + \mathbb{E} \sup_{s \in [0,T]} |Z^{z_1}(s) - Z^{z_2}(s)|^p_{L^2(\mathcal{O},\mathbb{R}^2)} \mathbf{1}_{\{\zeta^{z_1}_n \wedge \zeta^{z_2}_n \leq T\}} \\ &\leq |Z^{z_1}_n - Z^{z_2}_n|^p_{L_{T,p}} + c_p \Big(1 + |Z^{z_1}|^p_{L_{T,2p}} + |Z^{z_2}|^p_{L_{T,2p}}\Big) \Big(\mathbb{P}\{\zeta^{z_1}_n \wedge \zeta^{z_2}_n \leq T\}\Big)^{1/2}. \end{aligned} \tag{3.16}$$

Moreover, it follows from (3.4) that

$$\begin{aligned} \mathbb{P}\{\zeta^{z_1}_n \wedge \zeta^{z_2}_n \leq T\} &\leq \mathbb{P}\{\sup_{s \in [0,T]} |Z^{z_1}_n(s)|_{L^\infty(\mathcal{O},\mathbb{R}^2)} \geq n\} + \mathbb{P}\{\sup_{s \in [0,T]} |Z^{z_2}_n(s)|_{L^\infty(\mathcal{O},\mathbb{R}^2)} \geq n\} \\ &\leq \frac{c_4(T)}{n^4} \Big(1 + |z_1|^4_{L^\infty(\mathcal{O},\mathbb{R}^2)} + |z_2|^4_{L^\infty(\mathcal{O},\mathbb{R}^2)}\Big). \end{aligned}$$

Therefore, by applying (3.4) once more, we obtain from (3.16) and (3.15) that

$$|Z^{z_1} - Z^{z_2}|^p_{L_{T,p}} \leq c_{n,p}(T)\, |z_1 - z_2|^p_{L^2(\mathcal{O},\mathbb{R}^2)} + \frac{c(T)}{n^2} \Big(1 + |z_1|^{p+2}_{L^\infty(\mathcal{O},\mathbb{R}^2)} + |z_2|^{p+2}_{L^\infty(\mathcal{O},\mathbb{R}^2)}\Big). \tag{3.17}$$

Hence, for any fixed $z_1 \in L^\infty(\mathcal{O}, \mathbb{R}^2)$ and $\varepsilon > 0$, we first find $\overline{n} \in \mathbb{N}$ such that

$$\frac{c(T)}{\overline{n}^2} \Big(1 + |z_1|^{p+2}_{L^\infty(\mathcal{O},\mathbb{R}^2)} + \big(1 + |z_1|_{L^\infty(\mathcal{O},\mathbb{R}^2)}\big)^{p+2}\Big) < \frac{\varepsilon}{2}.$$

By determining $0<\delta^*<1$ such that

$$c_{\overline{n},p}(T)\,|z_1-z_2|^p_{L^2(\mathcal{O},\mathbb{R}^2)}<\frac{\varepsilon}{2}\quad\text{whenever}\quad |z_1-z_2|_{L^2(\mathcal{O},\mathbb{R}^2)}<\delta^*,$$

the continuous dependence of the solution on initial values is proved. □

Remark 2. We have following observations

(i) As the above proof, we note that the results in Theorem 3.1 still hold if we replace the space $L^p(\Omega;C([0,T],L^2(\mathcal{O},\mathbb{R}^2)))$ by $L^p(\Omega;C([0,T],L^q(\mathcal{O},\mathbb{R}^2)))$ for $q\geq 2$ (with $q=\infty$ is allowed).

(ii) From now, the solution $Z_n(t)$ of (3.1) is called as "truncated solution" of equation (2.6). By the same argument in the processing of obtaining (3.17), we conclude that

$$|Z-Z_n|_{L_{t,p}}\leq\frac{c_{p,Z_0}(t)}{n^2}\text{ for some constant } c_{p,Z_0}(t) \text{ being independent of } n.$$

As a consequence

$$\lim_{n\to\infty}|Z-Z_n|_{L_{t,p}}=0.$$

4. **Sufficient conditions for extinction and permanence.** In this section, we investigate the longtime behavior of system (2.4) by providing sufficient conditions for extinction and permanence. Because we can not apply Itô's formula to the mild solution as usual, it is very difficult to calculate and estimate. Following our idea in [25], we approximate the mild solution $(U(t),V(t))$ of (2.4) by a sequence of strong solutions (see [11] for more details about strong solutions, weak solutions, and mild solutions). Consider the following equation

$$\begin{cases}d\overline{U}_n(t,x)=\Big[d_1\Delta\overline{U}_n(t,x)+\overline{U}_n(t,x)\Big(a_1(x)-b_1(x)\overline{U}_n(t,x)\Big)\\ \qquad\qquad -\dfrac{c_1(x)\overline{U}_n(t,x)\overline{V}_n(t,x)}{m_1(x)+m_2(x)\overline{U}_n(t,x)+m_3(x)\overline{V}_n(t,x)}\Big]dt\\ \qquad\qquad +\displaystyle\sum_{k=1}^n\sqrt{\lambda_{k,1}}e_k(x)\overline{U}_n(t,x)dB_{k,1}(t)\quad\text{in}\quad\mathbb{R}^+\times\mathcal{O},\\ d\overline{V}_n(t,x)=\Big[d_2\Delta\overline{V}_n(t,x)+\overline{V}_n(t,x)\Big(-a_2(x)-b_2(x)\overline{V}_n(t,x)\Big)\\ \qquad\qquad +\dfrac{c_2(x)\overline{U}_n(t,x)\overline{V}_n(t,x)}{m_1(x)+m_2(x)\overline{U}_n(t,x)+m_3(x)\overline{V}_n(t,x)}\Big]dt\\ \qquad\qquad +\displaystyle\sum_{k=1}^n\sqrt{\lambda_{k,2}}e_k(x)\overline{V}_n(t,x)dB_{k,2}(t)\quad\text{in}\quad\mathbb{R}^+\times\mathcal{O},\\ \partial_\nu\overline{U}_n(t,x)=\partial_\nu\overline{V}_n(t,x)=0\qquad\text{on}\quad\mathbb{R}^+\times\partial\mathcal{O},\\ \overline{U}_n(x,0)=U_0(x),\overline{V}_n(x,0)=V_0(x)\qquad\text{in}\quad\mathcal{O}.\end{cases}\tag{4.1}$$

Denoted by E the Banach space $C(\overline{\mathcal{O}},\mathbb{R}^2)$ and by A_E the part of $A=(A_1,A_2)$ in E. Since we assumed the domain $\mathcal{O}$ has regular boundary in our boundary condition, $D(A_E)$ is dense in E; see [11, Appendix A.5.2]. Moreover, we use the following notation:

$$\forall u\in C(\overline{\mathcal{O}},\mathbb{R}),\quad |u|^*:=\sup_{x\in\overline{\mathcal{O}}}u(x),\quad |u|_*:=\inf_{x\in\overline{\mathcal{O}}}u(x),$$

$$\forall u\in L^p(\mathcal{O},\mathbb{R}),\quad u\geq 0,\quad |u|_p:=\Big(\int_{\mathcal{O}}u^p(x)dx\Big)^{1/p},\ p=1,2,\ldots$$

Proposition 4.1. *Assume that for each $k \in \mathbb{N}$, $e_k \in C^2(\overline{\mathcal{O}}, \mathbb{R})$. For any $0 \le (U_0, V_0) \in D(A_E)$, equation (4.1) has a unique strong solution $\overline{Z}_n(t) = (\overline{U}_n(t), \overline{V}_n(t))$. Moreover, the solution is positive, i.e., $\overline{U}_n(t), \overline{V}_n(t) \ge 0$ and for any finite $T > 0$, $(\overline{U}_n(t), \overline{V}_n(t)) \in L^p(\Omega, C([0,T], E))$.*

Proof. We apply the results in [10] or [11, Section 7.4] by verifying certain conditions. Define the following linear operators in $L^2(\mathcal{O}, \mathbb{R}^2)$

$$B_k(u,v) := \Big(\sqrt{\lambda_{k,1}} e_k u, \sqrt{\lambda_{k,2}} e_k v\Big), D(B_k) = L^2(\mathcal{O}, \mathbb{R}^2), 1 \le k \le n,$$

$$C := A - \frac{1}{2}\sum_{k=1}^n B_k^2 - 1, \quad D(C) = D(A).$$

First, the operators B_k generate mutually commuting semi-groups and all above operators and their restrictions on E generate strongly continuous and analytic semi-groups; see [11, Appendix A.5.2] or [3, Chapter 2]. As a result, the conditions $H_1, H_2(a), H_2(b')$ in [10] are satisfied. Moreover, by the arguments in [11, Example 6.31], we can conclude that the condition $H_2(c)$ in [10] is also satisfied. Second, it follows from [10, Proof of theorem 2 and Appendix A] or [1] that we can modify the condition $H_2(e)$ in [10] by an alternative one, namely, $\overline{F}_E(X^{\theta_1}) \subset X^{\theta_2}$ for some $\theta_1, \theta_2 \in (0, \frac{1}{2})$, where $X^\theta := D(-C_E)^\theta$ is the domain of the fractional power operator $(-C_E)^\theta$, $(-C_E)$ is the part of $(-C)$ in E and $\overline{F}_E$ is the part of $\overline{F}$ in E

$$\overline{F}(\overline{U},\overline{V}) = \Big[\overline{U}(1+a_1-b_1\overline{U}) - \frac{c_1\overline{UV}}{m_1+m_2\overline{U}+m_3\overline{V}}, \overline{V}(1-a_2-b_2\overline{V}) + \frac{c_2\overline{UV}}{m_1+m_2\overline{U}+m_3\overline{V}}\Big].$$

By [11, Proposition A.13], we have for all $\theta_1 > \theta_2 \in (0,1)$

$$D\big((-C_E)^{\theta_1}\big) \subset D_{C_E}(\theta_1, \infty) \subset D\big((-C_E)^{\theta_2}\big),$$

where $D_{C_E}(\theta_1, \infty)$ is defined as in [11, Appendix A] and by [11, Appendix A.5.2, p. 399]

$$D_{C_E}(\theta_1, \infty) = C^{2\theta_1}(\overline{\mathcal{O}}, \mathbb{R}^2) \text{ if } \theta_1 \in (0, \frac{1}{2}),$$

where $C^{2\theta_1}(\overline{\mathcal{O}}, \mathbb{R}^2)$ is a Hölder's space. Since the space $C^{2\theta_1}(\overline{\mathcal{O}}, \mathbb{R}^2)$ satisfies that $u, v \in C^{2\theta_1}(\overline{\mathcal{O}}, \mathbb{R}^2)$ implies $uv \in C^{2\theta_1}(\overline{\mathcal{O}}, \mathbb{R}^2)$, we obtain that $\overline{F}_E(X^{\theta_1}) \subset X^{\theta_2}$ for some $\theta_2 < \theta_1 \in (0, \frac{1}{2})$. Finally, it is needed to verify the monotonicity type hypothesis $H_2(d')$, namely, there exists $\eta \in \mathbb{R}$ such that for any $\alpha > 0$, $s \in \mathbb{R}$ and $\overline{Z} = (\overline{U}, \overline{V}) \in E$ then

$$\big|\overline{Z}\big|_E \le \big|\overline{Z} - \alpha\big(e^{-Bs}\overline{F}_E(e^{Bs}\overline{Z}) - \eta\overline{Z}\big)\big|_E, \text{where } B = \sum_{k=1}^n B_k. \tag{4.2}$$

It follows from [10, Proof of Theorem 2], this condition is needed to guarantee the strict solution of abstract problem (6) in [10] does not explode in finite time. Although reference [10] only focused on the existence and uniqueness of the strict solution of equation (6), substituting coefficients in the system we are considering into (6) of [10], a similar proof as in Lemma 3.1 leads to the positivity for the solution of (6). Hence, we need only verify the condition (4.2) for $\overline{Z} = (\overline{U}, \overline{V}) \in E$ with $\overline{U}(x), \overline{V}(x) \ge 0$ almost everywhere in $\mathcal{O}$ or $\overline{U}(x), \overline{V}(x) \ge 0$ $\forall x \in \overline{\mathcal{O}}$. As a consequence, by choosing $\eta \ge \max\left\{1 + |a_1|^*, 1 + \left|\frac{c_2}{m_2}\right|^*\right\}$, (4.2) is clearly satisfied.

Therefore, the existence and uniqueness of strong solution are obtained by applying the results in [10]. It is similar to Lemma 3.2, we have for any finite $T>0$, $p\geq 1$,

$$(\overline{U}_n(t),\overline{V}_n(t))\in L^p(\Omega, C([0,T],E)).$$

□

Proposition 4.2. *For any $t\geq 0$, $p\geq 2$ and non-negative initial data $(U_0,V_0)\in D(A_E)$, we have*

$$\lim_{n\to\infty}\mathbb{E}\left|U(t)-\overline{U}_n(t)\right|^p_{L^2(\mathcal{O},\mathbb{R})}=0, \tag{4.3}$$

and

$$\lim_{n\to\infty}\mathbb{E}\left|V(t)-\overline{V}_n(t)\right|^p_{L^2(\mathcal{O},\mathbb{R})}=0, \tag{4.4}$$

where $Z(t)=(U(t),V(t))$ is the mild solution of (2.6) and $\overline{Z}_n(t)=(\overline{U}_n(t),\overline{V}_n(t))$ is the strong solution of (4.1).

Proof. In this proof, the letter c still denotes positive constants whose values may change in different occurrences. We will write the dependence of constant on parameters explicitly if it is essential. It is similar to Lemma 3.2, we can obtain

$$\mathbb{E}\sup_{s\in[0,t]}|Z_n(s)|^p_{L^\infty(\mathcal{O},\mathbb{R}^2)}\leq c_{p,Z_0}(t) \text{ for some constant } c_{p,Z_0}(t), \text{ is independent of } n.$$

Therefore, as in part (ii) of Remark 2, we also obtain a similar convergence for the solution $Z_n(t)$ of (4.1) and their truncated solutions. Moreover, this convergence is uniform with respect to n. So, without loss of the generality, we can assume the non-linear term F is Lipschitz continuous in this proof since we can approximate solutions of (2.6) and (4.1) by their truncated solutions.

First, we still assume that each $k\in\mathbb{N}$, $e_k\in C^2(\overline{\mathcal{O}},\mathbb{R})$. Because a strong solution is also a mild one, we have

$$\overline{Z}_n(t)=e^{tA}Z_0+\int_0^t e^{(t-s)A}F(\overline{Z}_n(s))ds+W_{\overline{Z}_n}(t),\ Z_0=(U_0,V_0), \tag{4.5}$$

where $W_{\overline{Z}_n}(t)=(W_{\overline{U}_n}(t),W_{\overline{V}_n}(t))$ and

$$W_{\overline{U}_n}(t)=\sum_{k=1}^n\lambda_{k,1}\int_0^t e^{(t-s)A_1}\overline{U}_n(s)dB_{k,1}(s),$$

$$W_{\overline{V}_n}(t)=\sum_{k=1}^n\lambda_{k,2}\int_0^t e^{(t-s)A_2}\overline{V}_n(s)dB_{k,2}(s).$$

By the same argument as in the processing of getting (3.10), we obtain

$$\left|W_Z-W_{\overline{Z}_n}\right|_{L_{t,p}}\leq c_p(t)\int_0^t\left|Z-\overline{Z}_n\right|_{L_{s,p}}ds+c_p(t)\sum_{k=n}^\infty(\lambda_{k,1}+\lambda_{k,2})\left|Z\right|_{L_{t,p}}. \tag{4.6}$$

Subtracting (2.6) side-by-side from (4.5) and applying (3.4), (4.6) allows us to get

$$\left|Z-\overline{Z}_n\right|_{L_{t,p}}\leq c_{p,Z_0}(t)\Big(\sum_{k=n}^\infty(\lambda_{k,1}+\lambda_{k,2})+\int_0^t\left|Z-\overline{Z}_n\right|_{L_{s,p}}ds\Big),$$

for some constant $c_{p,Z_0}(t)$ independent of n. Hence, it follows from Gronwall's inequality that

$$\left|Z-\overline{Z}_n\right|_{L_{t,p}}\leq c_{p,Z_0}(t)\Big[\sum_{k=n}^\infty(\lambda_{k,1}+\lambda_{k,2})\Big]. \tag{4.7}$$

By (2.1), it is seen that

$$\lim_{n\to\infty}\sum_{k=n}^{\infty}(\lambda_{k,1}+\lambda_{k,2})=0. \tag{4.8}$$

Thus, we obtain from (4.7) and (4.8) that

$$\lim_{n\to\infty}\left|Z-\overline{Z}_n\right|_{L_{t,p}}=0.$$

As a consequence, for all $t\geq 0, p\geq 2$

$$\lim_{n\to\infty}\mathbb{E}\left|Z(t)-\overline{Z}_n(t)\right|^p_{L^2(\mathcal{O},\mathbb{R}^2)}=0.$$

Now, as the above proof, by the fact $C^\infty(\overline{\mathcal{O}},\mathbb{R})$ is dense in $L^2(\mathcal{O},\mathbb{R})$, we can remove the condition $e_k\in C^2(\overline{\mathcal{O}},\mathbb{R})$. To be more detailed, we will first approximate the mild solution of (2.6) by a sequence of mild solutions of (4.1) without the condition $e_k\in C^2(\overline{\mathcal{O}},\mathbb{R})$ and then these solutions are approximated by the strong solutions of (4.1) with condition $e_k\in C^2(\overline{\mathcal{O}},\mathbb{R})$ $\forall 1\leq k\leq n$. Therefore, from now, to simplify the notation, we will approximate directly the mild solution of (2.6) by the strong solutions of (4.1). Equivalently, without loss of the generality, we may assume that $e_k\in C^2(\overline{\mathcal{O}},\mathbb{R})$ $\forall k=1,2,...$ as far as the approximation is concerned. □

Remark 3. Combining Remark 2 and the above proof, the convergence in Proposition 4.2 still holds in the space $L^p(\Omega;C([0,T],L^\infty(\mathcal{O},\mathbb{R}^2)))$. In more details, by the same arguments, it is possible to obtain that for any finite $T>0$, $p\geq 1$

$$\lim_{n\to\infty}\mathbb{E}\sup_{s\in[0,T]}\left|Z(s)-\overline{Z}_n(s)\right|^p_{L^\infty(\mathcal{O},\mathbb{R}^2)}=0.$$

Now, for each $m\in\mathbb{N}, m>\left|\frac{1}{U_0}\right|_{L^\infty(\mathcal{O},\mathbb{R})},\left|\frac{1}{V_0}\right|_{L^\infty(\mathcal{O},\mathbb{R})}$ (if they are finite), let

$$\tau^n_m=\inf\Big\{t\geq 0:\text{there exists } 1\leq\overline{p}<\infty \text{ such that } \int_{\mathcal{O}}\frac{1}{\overline{U}^p_n(t,x)}dx\geq m^{p^2}\ \forall p\geq\overline{p}\Big\},$$

$$\eta^n_m=\inf\Big\{t\geq 0:\text{there exists } 1\leq\overline{p}<\infty \text{ such that } \int_{\mathcal{O}}\frac{1}{\overline{V}^p_n(t,x)}dx\geq m^{p^2}\ \forall p\geq\overline{p}\Big\}.$$

It is easy to see that for any fix $n\in\mathbb{N}$, the sequences τ^n_m and η^n_m are increasing in m. Hence, we can define

$$\tau^n_\infty:=\lim_{m\to\infty}\tau^n_m\ ,\ \eta^n_\infty:=\lim_{m\to\infty}\eta^n_m.$$

Lemma 4.1. *If $\left|\frac{1}{U_0}\right|_{L^\infty(\mathcal{O},\mathbb{R})}<\infty$ and $\left|\frac{1}{V_0}\right|_{L^\infty(\mathcal{O},\mathbb{R})}<\infty$, then for all $n\in\mathbb{N}$, $\tau^n_\infty=\eta^n_\infty=\infty$ a.s.*

Proof. First, we prove that $\forall n\in\mathbb{N}$, $\tau^n_\infty=\infty$ a.s. Indeed, if this statement is false then there exist $n_0\in\mathbb{N}$ and two constants $T_0>0$ and $\varepsilon_0\in(0,1)$ such that

$$\mathbb{P}\{\tau^{n_0}_\infty\leq T_0\}>\varepsilon_0.$$

Therefore, there is an integer m_0 such that $\mathbb{P}(\Omega^m_0)\geq\varepsilon_0$, $\forall m\geq m_0$, where

$$\Omega^m_0:=\{\tau^{n_0}_m\leq T_0\}. \tag{4.9}$$

Using Itô's Lemma ([9, Theorem 3.8]) and by direct calculations, we have

$$
\begin{aligned}
&\int_{\mathcal{O}} \frac{1}{\overline{U}_n^p(t\wedge\tau_m^n,x)}dx\\
&= \int_{\mathcal{O}} \frac{1}{U_0^p(x)}dx + \int_0^{t\wedge\tau_m^n}\int_{\mathcal{O}} \frac{-p}{\overline{U}_n^{p+1}(s,x)}\\
&\qquad \times \Big(d_1\Delta\overline{U}_n(s,x) + \overline{U}_n(s,x)\big(a_1(x)-b_1(x)\overline{U}_n(s,x)\big)\\
&\qquad - \frac{c_1(x)\overline{U}_n(s,x)\overline{V}_n(s,x)}{m_1(x)+m_2(x)\overline{U}_n(s,x)+m_3(x)\overline{V}_n(s,x)}\Big)dxds\\
&\qquad + \frac{1}{2}\int_0^{t\wedge\tau_m^n}\sum_{k=1}^n\int_{\mathcal{O}} \frac{p(p+1)\lambda_{k,1}e_k^2(x)\overline{U}_n^2(s,x)}{\overline{U}_n^{p+2}(s,x)}dxds\\
&\qquad + \sum_{k=1}^n\int_0^{t\wedge\tau_m^n}\Big[\sqrt{\lambda_{k,1}}\int_{\mathcal{O}} \frac{-pe_k(x)\overline{U}_n(s,x)}{\overline{U}_n^{p+1}(s,x)}dx\Big]dB_{k,1}(s)\\
&\le \int_{\mathcal{O}} \frac{1}{U_0^p(x)}dx + \int_0^{t\wedge\tau_m^n}\int_{\mathcal{O}} \frac{-pd_1\Delta\overline{U}_n(s,x)}{\overline{U}_n^{p+1}(s,x)}dxds\\
&\qquad + \int_0^{t\wedge\tau_m^n}\int_{\mathcal{O}} \frac{p}{\overline{U}_n^p(s,x)}\Big(K_1+pK_2+|b_1|^*\,\overline{U}_n(s,x)\Big)dxds\\
&\qquad + \sum_{k=1}^n\int_0^{t\wedge\tau_m^n}\Big[\sqrt{\lambda_{k,1}}\int_{\mathcal{O}} \frac{-pe_k(x)}{\overline{U}_n^p(s,x)}dx\Big]dB_{k,1}(s)\\
&\le \int_{\mathcal{O}} \frac{1}{U_0^p(x)}dx + \int_0^{t\wedge\tau_m^n}\Big(K_3(p) + \int_{\mathcal{O}} \frac{K_3(p)}{\overline{U}_n^p(s,x)}dx\Big)ds\\
&\qquad + \sum_{k=1}^n\int_0^{t\wedge\tau_m^n}\Big[\sqrt{\lambda_{k,1}}\int_{\mathcal{O}} \frac{-pe_k(x)}{\overline{U}_n^p(s,x)}dx\Big]dB_{k,1}(s),\ \forall p\ge 1, t\ge 0, n\in\mathbb{N},
\end{aligned}
\tag{4.10}
$$

where $K_1 = \big|\frac{c_1}{m_3}\big|^* + \frac{\lambda_1 C_0^2}{2}$, $K_2 = \frac{\lambda_1 C_0^2}{2}$ and $K_3(p) = p\big(K_1 + pK_2 + |b_1|^*\big)$. In the above, we used the following facts

$$
\int_{\mathcal{O}} \frac{-pd_1\Delta\overline{U}_n(s,x)}{\overline{U}_n^{p+1}(s,x)}dx = -p(p+1)d_1\int_{\mathcal{O}} \frac{\big|\nabla\overline{U}_n(s,x)\big|^2}{\overline{U}_n^{p+2}(s,x)}dx \le 0 \ \text{ a.s.},
$$

and

$$
\begin{aligned}
&\int_{\mathcal{O}} \frac{p}{\overline{U}_n^p(s,x)}\Big(K_1+pK_2+|b_1|^*\,\overline{U}_n(s,x)\Big)dx\\
&\quad = \int_{\mathcal{O}} \frac{p}{\overline{U}_n^p(s,x)}\Big(K_1+pK_2+|b_1|^*\,\overline{U}_n(s,x)\Big)\mathbf{1}_{\{U(s,x)\ge 1\}}dx\\
&\qquad + \int_{\mathcal{O}} \frac{p}{\overline{U}_n^p(s,x)}\Big(K_1+pK_2+|b_1|^*\,\overline{U}_n(s,x)\Big)\mathbf{1}_{\{U(s,x)\le 1\}}dx\\
&\quad \le p\big(K_1+pK_2+|b_1|^*\big) + \int_{\mathcal{O}} \frac{p\big(K_1+pK_2+|b_1|^*\big)}{\overline{U}_n^p(s,x)}dx \text{ a.s.}
\end{aligned}
$$

Hence, (4.10) leads to

$$
\mathbb{E}\int_{\mathcal{O}} \frac{1}{\overline{U}_n^p(t\wedge\tau_m^n,x)}dx \le \int_{\mathcal{O}} \frac{1}{U_0^p(x)}dx + tK_3(p) + K_3(p)\int_0^t \mathbb{E}\int_{\mathcal{O}} \frac{1}{\overline{U}_n^p(s\wedge\tau_m^n,x)}dxds.
$$

Thus, Gronwall's inequality implies that

$$\mathbb{E}\int_{\mathcal{O}} \frac{1}{\overline{U}_n^p(t\wedge\tau_m^n,x)}dx \le \Big[\int_{\mathcal{O}} \frac{1}{U_0^p(x)}dx + tK_3(p)\Big]e^{tK_3(p)}.$$

Therefore, for each fixed $t \ge 0$ and $\forall n \in \mathbb{N}$,

$$\begin{aligned}
&\sup_{p\ge 1}\Big[\mathbb{E}\int_{\mathcal{O}} \frac{1}{\overline{U}_n^p(t\wedge\tau_m^n,x)}dx\Big]^{1/p^2}\\
&\le \sup_{p\ge 1}\Big[\int_{\mathcal{O}} \frac{1}{U_0^p(x)}dx + \big(pK_1 + p\,|b_1|^* + p^2K_2\big)t\Big]^{1/p^2} e^{t(\frac{K_1+|b_1|^*}{p}+K_2)}\\
&:= M(t) < \infty.
\end{aligned}$$

In particular,

$$\sup_{p\ge 1}\Big[\mathbb{E}\int_{\mathcal{O}} \frac{1}{\overline{U}_n^p(T_0\wedge\tau_m^n,x)}dx\Big]^{1/p^2} \le M(T_0)\ \forall n\in\mathbb{N}. \tag{4.11}$$

On the other hand, for all $m \ge m_0$,

$$\begin{aligned}
\sup_{p\ge 1}\Big[\mathbb{E}\int_{\mathcal{O}} \frac{1}{\overline{U}_{n_0}^p(T_0\wedge\tau_m^{n_0},x)}dx\Big]^{1/p^2} &\ge \sup_{p\ge 1}\Big[\mathbb{E}\mathbf{1}_{\Omega_0^m}\int_{\mathcal{O}} \frac{1}{\overline{U}_{n_0}^p(T_0\wedge\tau_m^{n_0},x)}dx\Big]^{1/p^2}\\
&\ge m\varepsilon_0.
\end{aligned} \tag{4.12}$$

We deduce from (4.11) and (4.12) that

$$M(T_0) \ge m\varepsilon_0 \quad \forall m \ge m_0.$$

This is a contradiction when $m \to \infty$. Therefore

$$\tau_\infty^n = \infty \text{ a.s. } \ \forall n \in \mathbb{N}.$$

Similarly, we obtain that

$$\eta_\infty^n = \infty \text{ a.s. } \ \forall n \in \mathbb{N}.$$

□

To proceed, we introduce following numbers

$$H_0 = \Big|a_1 - \frac{c_1}{m_3}\Big|_* - \frac{3\lambda_1 C_0^2}{2} := \inf_{x\in\overline{\mathcal{O}}}\Big[a_1(x) - \frac{c_1(x)}{m_3(x)}\Big] - \frac{3\lambda_1 C_0^2}{2},$$

$$R_0 = -\,|a_2|_1 - \frac{\lambda_2}{2} + \frac{1}{\big|\frac{m_2}{c_2}\big|^* + \min\big\{\frac{1}{H_0}\,|b_1|_1\,\big|\frac{m_1}{c_2}\big|^*\,;\ \frac{1}{H_0}\,|b_1|_1^{1/2}\,(|b_1|^*)^{1/2}\big|\frac{m_1}{c_2}\big|_2\big\}}.$$

Theorem 4.1. *The following results hold.*

(i) *For any initial values $0 \le U_0, V_0 \in L^\infty(\mathcal{O},\mathbb{R})$. If*

$$\Big|a_2 - \frac{c_2}{m_2}\Big|_* := \inf_{x\in\overline{\mathcal{O}}}\Big\{a_2(x) - \frac{c_2(x)}{m_2(x)}\Big\} > 0,$$

then $V(t)$ is extinct.

(ii) *If $H_0, R_0 > 0$ and non-negative initial values $(U_0,V_0) \in E$ satisfying*

$$\Big|\frac{1}{U_0}\Big|_{L^\infty(\mathcal{O},\mathbb{R})} < \infty\ ,\ \Big|\frac{1}{V_0}\Big|_{L^\infty(\mathcal{O},\mathbb{R})} < \infty,$$

the individuals $U(t)$ and $V(t)$ are permanent.

Proof. The proof for the first part is similar to [25, Proof of Theorem 4.1]. It follows (2.5) and properties of stochastic integral ([11, Proposition 4.15] and [9, Proposition 2.9]) that $\forall t \geq 0$,

$$
\begin{aligned}
0 \leq \mathbb{E}\int_{\mathcal{O}} V(t,x)dx &\leq \int_{\mathcal{O}} V_0(x)dx + \int_0^t \mathbb{E}\int_{\mathcal{O}}\Big(-a_2(x)V(s,x) + \frac{c_2(x)V(s,x)}{m_2(x)}\Big)dxds \\
&\leq \int_{\mathcal{O}} V_0(x)dx - \Big|a_2 - \frac{c_2}{m_2}\Big|_* \int_0^t \mathbb{E}\int_{\mathcal{O}} V(s,x)dxds.
\end{aligned} \tag{4.13}
$$

As a consequence, we have

$$
\mathbb{E}\int_{\mathcal{O}} V(t,x)dx \leq -\big|a_2 - \frac{c_2}{m_2}\big|_* \int_s^t \Big(\int_{\mathcal{O}} V(s,x)dx\Big)ds, \ \forall 0 \leq s \leq t.
$$

Hence, it leads to that $\lim_{t\to\infty}\mathbb{E}\int_{\mathcal{O}} V(t,x)dx = 0$ with exponential rate or the class $V(t)$ is extinct. Now, we move to the second part. Since the density of $D(A_E)$ in E and the continuous dependence on initial values of the solution, we can assume that $(U_0, V_0) \in D(A_E)$. By Itô's formula ([9, Theorem 3.8]) and by a similar argument as in the processing of getting (4.10), we have

$$
\begin{aligned}
&e^{H_0(t\wedge\tau_m^n)}\int_{\mathcal{O}} \frac{1}{\overline{U}_n(t\wedge\tau_m^n,x)}dx \\
&\leq \int_{\mathcal{O}} \frac{1}{U_0(x)}dx + \int_0^{t\wedge\tau_m^n} e^{H_0 s}\Big(\int_{\mathcal{O}} \frac{\frac{c_1(x)}{m_3(x)} + \lambda_1 C_0^2 + H_0 - a_1(x)}{\overline{U}_n(s,x)}dx + \int_{\mathcal{O}} b_1(x)dx\Big)ds \\
&\quad + \sum_{k=1}^n \int_0^{t\wedge\tau_m^n} e^{H_0 s}\Big[\int_{\mathcal{O}} \frac{-\sqrt{\lambda_{k,1}}e_k(x)}{\overline{U}_n(s,x)}dx\Big]dB_{k,1}(s) \\
&\leq \int_{\mathcal{O}} \frac{1}{U_0(x)}dx + \int_0^{t\wedge\tau_m^n} |b_1|_1\, e^{H_0 s}ds \\
&\quad + \sum_{k=1}^n \int_0^{t\wedge\tau_m^n} e^{H_0 s}\Big[\int_{\mathcal{O}} \frac{-\sqrt{\lambda_{k,1}}e_k(x)}{\overline{U}_n(s,x)}dx\Big]dB_{k,1}(s).
\end{aligned}
$$

Therefore, taking expectations on both sides and letting $m \to \infty$, we obtain

$$
\mathbb{E}\int_{\mathcal{O}} \frac{1}{\overline{U}_n(t,x)}dx \leq e^{-H_0 t}\int_{\mathcal{O}} \frac{1}{U_0(x)}dx + |b_1|_1 \frac{e^{H_0 t}-1}{H_0 e^{H_0 t}}, \quad \forall t \geq 0, \forall n \in \mathbb{N}.
$$

As a consequence,

$$
\limsup_{t\to\infty}\limsup_{n\to\infty}\mathbb{E}\int_{\mathcal{O}} \frac{1}{\overline{U}_n(t,x)}dx \leq \frac{|b_1|_1}{H_0}. \tag{4.14}
$$

The convergence (4.3), the Hölder's inequality, and (4.14) yield

$$
\limsup_{t\to\infty}\mathbb{E}\int_{\mathcal{O}} U(t,x)dx = \limsup_{t\to\infty}\limsup_{n\to\infty}\mathbb{E}\int_{\mathcal{O}} \overline{U}_n(t,x)dx \geq \frac{H_0}{|b_1|_1}.
$$

So, the individual $U(t)$ is permanent. Similarly, we also obtain from Itô's formula ([9, Theorem 3.8]) that

$$\begin{aligned}
\mathbb{E}\int_{\mathcal{O}}\frac{1}{\overline{U}_n^2(t,x)}dx &\le e^{-2H_0t}\int_{\mathcal{O}}\frac{1}{U_0^2(x)}dx \\
&\quad + e^{-2H_0t}\mathbb{E}\int_0^t e^{2H_0s}\Big(2\int_{\mathcal{O}}\frac{\frac{c_1(x)}{m_3(x)}+\frac{3\lambda_1C_0^2}{2}+H_0-a_1(x)}{\overline{U}_n^2(s,x)}dx \\
&\quad + 2\,|b_1|^*\int_{\mathcal{O}}\frac{1}{\overline{U}_n(s,x)}dx\Big)ds \\
&\le e^{-2H_0t}\int_{\mathcal{O}}\frac{1}{U_0^2(x)}dx + 2\,|b_1|^*\,e^{-2H_0t}\int_0^t e^{2H_0s}\mathbb{E}\int_{\mathcal{O}}\frac{1}{\overline{U}_n(s,x)}dxds,
\end{aligned}$$

which implies that

$$\begin{aligned}
\limsup_{n\to\infty}\mathbb{E}\int_{\mathcal{O}}\frac{1}{\overline{U}_n^2(t,x)}dx &\le e^{-2H_0t}\int_{\mathcal{O}}\frac{1}{U_0^2(x)}dx \\
&\quad + 2\,|b_1|^*\,e^{-2H_0t}\int_0^t e^{2H_0s}\Big[\limsup_{n\to\infty}\mathbb{E}\int_{\mathcal{O}}\frac{1}{\overline{U}_n(s,x)}dx\Big]ds.
\end{aligned} \tag{4.15}$$

Applying (4.14) to (4.15), we obtain

$$\limsup_{t\to\infty}\limsup_{n\to\infty}\mathbb{E}\int_{\mathcal{O}}\frac{1}{\overline{U}_n^2(t,x)}dx \le \frac{|b_1|_1\,|b_1|^*}{H_0^2}. \tag{4.16}$$

Since the definition of R_0 and $R_0>0$, there exists a $\delta>0$ such that

$$\begin{aligned}
&-|a_2|_1-\frac{\lambda_2}{2}-\delta \\
&\quad+\frac{1}{\left|\frac{m_2}{c_2}\right|^*+\min\left\{\frac{1}{H_0}\,|b_1|_1\left|\frac{m_1}{c_2}\right|^*+\delta\ ;\ \frac{1}{H_0}\,|b_1|_1^{1/2}\,(|b_1|^*)^{1/2}\left|\frac{m_1}{c_2}\right|_2+\delta\right\}+\delta} \\
&\ge\frac{R_0}{2}.
\end{aligned} \tag{4.17}$$

Put $\widehat{\delta}=\min\left\{\frac{\delta^2}{4\,|b_2|_2^2}\ ;\ \frac{H_0^2\delta^2}{4(\left|\frac{m_3}{c_2}\right|^*)^2(|b_1|^*)^2}\right\}$. By a contradiction argument, we assume that

$$\limsup_{t\to\infty}\mathbb{E}\int_{\mathcal{O}}V^2(t,x)dx<\widehat{\delta}, \tag{4.18}$$

which means

$$\limsup_{t\to\infty}\lim_{n\to\infty}\int_{\mathcal{O}}\overline{V}_n^2(t,x)dx<\widehat{\delta}. \tag{4.19}$$

It follows from (4.19) that

$$\begin{aligned}
\limsup_{t\to\infty}\limsup_{n\to\infty}\mathbb{E}\int_{\mathcal{O}}b_2(x)\overline{V}_n(t,x)dx &\le |b_2|_2\limsup_{t\to\infty}\limsup_{n\to\infty}\Big(\mathbb{E}\int_{\mathcal{O}}\overline{V}_n^2(t,x)dx\Big)^{1/2} \\
&\le |b_2|_2\,\widehat{\delta}^{1/2}\le\frac{\delta}{2}.
\end{aligned}$$

It implies that there exists a T_1 such that

$$\limsup_{n\to\infty}\mathbb{E}\int_{\mathcal{O}}b_2(x)\overline{V}_n(t,x)dx\le\delta\quad\forall t\ge T_1. \tag{4.20}$$

On the other hand, it follows from Hölder's inequality, (4.16), and (4.19) that

$$\begin{aligned}
&\limsup_{t\to\infty}\limsup_{n\to\infty}\mathbb{E}\int_{\mathcal{O}}\frac{\overline{V}_n(t,x)}{\overline{U}_n(t,x)}dx\\
&\qquad\le \limsup_{t\to\infty}\limsup_{n\to\infty}\Big(\mathbb{E}\int_{\mathcal{O}}\overline{V}_n^2(t,x)dx\Big)^{1/2}\\
&\qquad\quad\times\limsup_{t\to\infty}\limsup_{n\to\infty}\Big(\mathbb{E}\int_{\mathcal{O}}\frac{1}{\overline{U}_n^2(t,x)}dx\Big)^{1/2}\\
&\qquad\le \big(\frac{1}{H_0^2}\widehat{\delta}\,|b_1|_1\,|b_1|^*\big)^{1/2}\le\frac{\delta}{2\left|\frac{m_3}{c_2}\right|^*}.
\end{aligned}$$

Therefore, there exists T_2 such that

$$\limsup_{n\to\infty}\left|\frac{m_3}{c_2}\right|^*\mathbb{E}\int_{\mathcal{O}}\frac{\overline{V}_n(t,x)}{\overline{U}_n(t,x)}dx\le\delta\quad\forall t\ge T_2. \tag{4.21}$$

Moreover, it is clear that

$$\begin{aligned}
\mathbb{E}\int_{\mathcal{O}}\frac{m_1(x)}{c_2(x)\overline{U}_n(t,x)}dx\le\min\Big\{\ &\left|\frac{m_1}{c_2}\right|^*\mathbb{E}\textstyle\int_{\mathcal{O}}\frac{1}{\overline{U}_n(t,x)}dx,\\
&\left|\frac{m_1}{c_2}\right|_2\Big(\mathbb{E}\textstyle\int_{\mathcal{O}}\frac{1}{\overline{U}_n^2(t,x)}dx\Big)^{1/2}\Big\}.
\end{aligned}\tag{4.22}$$

Combining (4.14),(4.16) and (4.22) obtains

$$\begin{aligned}
&\limsup_{t\to\infty}\limsup_{n\to\infty}\mathbb{E}\int_{\mathcal{O}}\frac{m_1(x)}{c_2(x)\overline{U}_n(t,x)}dx\\
&\qquad\le\min\Big\{\frac{1}{H_0}\,|b_1|_1\left|\frac{m_1}{c_2}\right|^*,\ \ \frac{1}{H_0}\,|b_1|_1^{1/2}\,(|b_1|^*)^{1/2}\left|\frac{m_1}{c_2}\right|_2\Big\}.
\end{aligned}$$

Thus, there exists T_3 such that for all $t\ge T_3$

$$\begin{aligned}
ll\ \limsup_{n\to\infty}\mathbb{E}\int_{\mathcal{O}}\frac{m_1(x)}{c_2(x)\overline{U}_n(t,x)}dx\le\min\Big\{&\frac{1}{H_0}\,|b_1|_1\left|\frac{m_1}{c_2}\right|^*+\delta,\\
\frac{1}{H_0}\,|b_1|_1^{1/2}\,(|b_1|^*)^{1/2}\left|\frac{m_1}{c_2}\right|_2+\delta\Big\}.&
\end{aligned}\tag{4.23}$$

Now, by Itô's formula ([9, Theorem 3.8]) again we have

$$\begin{aligned}
&\int_{\mathcal{O}}\ln\overline{V}_n(t\wedge\eta_m^n,x)dx\\
&\ge\int_{\mathcal{O}}\ln V_0(x)dx+\int_0^{t\wedge\eta_m^n}\int_{\mathcal{O}}\frac{d_2\left|\nabla\overline{V}_n(s,x)\right|^2}{\overline{V}_n^2(s,x)}dxds-\frac{\lambda_2}{2}(t\wedge\eta_m^n)\\
&\quad+\int_0^{t\wedge\eta_m^n}\int_{\mathcal{O}}\Big(-a_2(x)-b_2(x)\overline{V}_n(s,x)\\
&\quad+\frac{c_2(x)\overline{U}_n(s,x)}{m_1(x)+m_2(x)\overline{U}_n(s,x)+m_3(x)\overline{V}_n(s,x)}\Big)dxds\\
&\quad+\int_0^{t\wedge\eta_m^n}\sum_{k=1}^n\sqrt{\lambda_{k,2}}\Big(\int_{\mathcal{O}}e_k(x)dx\Big)dB_{k,2}(s).
\end{aligned}\tag{4.24}$$

Taking expectation on both sides and letting $m \to \infty$ imply

$$\begin{aligned}\mathbb{E}\int_{\mathcal{O}} \ln \overline{V}_n(t,x)dx \geq& \int_{\mathcal{O}} \ln V_0(x)dx - \big(|a_2|_1 + \frac{\lambda_2}{2}\big)t \\ &+ \int_0^t \mathbb{E}\int_{\mathcal{O}} \Big(- b_2(x)\overline{V}_n(s,x) \\ &+ \frac{c_2(x)\overline{U}_n(s,x)}{m_1(x)+m_2(x)\overline{U}_n(s,x)+m_3(x)\overline{V}_n(s,x)}\Big)dxds.\end{aligned} \tag{4.25}$$

By Jensen's inequality, we obtain

$$\begin{aligned}&\mathbb{E}\int_{\mathcal{O}} \frac{c_2(x)\overline{U}_n(s,x)}{m_1(x)+m_2(x)\overline{U}_n(s,x)+m_3(x)\overline{V}_n(s,x)}dx \\ &\geq \mathbb{E}\int_{\mathcal{O}} \frac{1}{\big|\frac{m_2}{c_2}\big|^* + \frac{m_1(x)}{c_2(x)\overline{U}_n(s,x)} + \big|\frac{m_3}{c_2}\big|^* \frac{\overline{V}_n(s,x)}{\overline{U}_n(s,x)}}dx \\ &\geq \frac{1}{\big|\frac{m_2}{c_2}\big|^* + \mathbb{E}\int_{\mathcal{O}} \frac{m_1(x)}{c_2(x)\overline{U}_n(s,x)}dx + \big|\frac{m_3}{c_2}\big|^* \mathbb{E}\int_{\mathcal{O}} \frac{\overline{V}_n(s,x)}{\overline{U}_n(s,x)}dx}.\end{aligned} \tag{4.26}$$

Applying (4.26) to (4.25), we have

$$\begin{aligned}&\limsup_{n\to\infty} \mathbb{E}\int_{\mathcal{O}} \ln \overline{V}_n(t,x)dx \geq \int_{\mathcal{O}} \ln V_0(x)dx - \big(|a_2|_1 + \frac{\lambda_2}{2}\big)t \\ &- \int_0^t \Big[\limsup_{n\to\infty} \mathbb{E}\int_{\mathcal{O}} b_2(x)\overline{V}_n(s,x)dx\Big]ds \\ &+ \int_0^t \frac{1}{\Big|\frac{m_2}{c_2}\Big|^* + \limsup\limits_{n\to\infty} \mathbb{E}\displaystyle\int_{\mathcal{O}} \frac{m_1(x)}{c_2(x)\overline{U}_n(s,x)}dx + \Big|\frac{m_3}{c_2}\Big|^* \limsup\limits_{n\to\infty} \mathbb{E}\displaystyle\int_{\mathcal{O}} \frac{\overline{V}_n(s,x)}{\overline{U}_n(s,x)}dx}ds.\end{aligned} \tag{4.27}$$

Let $T = \max\{T_1, T_2, T_3\}$. We obtain from (4.27),(4.20), (4.21), (4.23), and (4.17) that $\forall t \geq T$

$$\begin{aligned}&\limsup_{n\to\infty} \mathbb{E}\int_{\mathcal{O}} \ln \overline{V}_n(t,x)dx \\ &\qquad \geq \int_{\mathcal{O}} \ln V_0(x)dx - \big(|a_2|_1 + \frac{\lambda_2}{2}\big)T - \int_0^T \Big[\limsup_{n\to\infty} \mathbb{E}\int_{\mathcal{O}} b_2(x)\overline{V}_n(s,x)dx\Big]ds \\ &\qquad + \frac{R_0(t-T)}{2}.\end{aligned} \tag{4.28}$$

Since $R_0 > 0$, we have

$$\limsup_{t\to\infty} \limsup_{n\to\infty} \mathbb{E}\int_{\mathcal{O}} \ln \overline{V}_n(t,x)dx = \infty.$$

Therefore, it follows from the convergence (4.4) and Jensen's inequality that

$$\begin{aligned}\limsup_{t\to\infty} \ln \mathbb{E}\int_{\mathcal{O}} V(t,x)dx &= \limsup_{t\to\infty} \ln \big[\limsup_{n\to\infty} \mathbb{E}\int_{\mathcal{O}} \overline{V}_n(t,x)dx\big] \\ &\geq \limsup_{t\to\infty} \limsup_{n\to\infty} \mathbb{E}\int_{\mathcal{O}} \ln \overline{V}_n(t,x)dx = \infty.\end{aligned} \tag{4.29}$$

However, combining (4.18) and (4.29) leads to a contradiction. This implies

$$\limsup_{t\to\infty}\int_{\mathcal{O}} V^2(t,x)dx \ge \widehat{\delta}.$$

Thus, the individual $V(t)$ is permanent. □

5. **An example.** In this section, we consider an example when the processes driving noise processes in equation (1.1) are standard Brownian motions and the coefficients are independent of space, as following

$$\begin{cases} dU(t,x)=\Big[d_1\Delta U(t,x)+U(t,x)\big(a_1-b_1U(t,x)\big)-\dfrac{c_1U(t,x)V(t,x)}{m_1+m_2U(t,x)+m_3V(t,x)}\Big]dt \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +\sigma_1U(t,x)dB_1(t) \quad \text{in } \mathbb{R}^+\times\mathcal{O}, \\ dV(t,x)=\Big[d_2\Delta V(t,x)-V(t,x)\big(a_2+b_2V(t,x)\big)+\dfrac{c_2U(t,x)V(t,x)}{m_1+m_2U(t,x)+m_3V(t,x)}\Big]dt \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +\sigma_2V(t,x)dB_2(t) \quad \text{in } \mathbb{R}^+\times\mathcal{O}, \\ \partial_\nu U(t,x)=\partial_\nu V(t,x)=0 \qquad \text{on} \quad \mathbb{R}^+\times\partial\mathcal{O}, \\ U(x,0)=U_0(x), V(x,0)=V_0(x) \qquad \text{in} \quad \mathcal{O}, \end{cases} \tag{5.1}$$

where a_i, b_i, c_i, m_i are positive constants, and $B_1(t)$, $B_2(t)$ are independent standard Brownian motions. As we obtained above, for any initial values $0 \le U_0, V_0 \in L^\infty(\mathcal{O},\mathbb{R})$, (5.1) has unique solution $U(t,x), V(t,x) \ge 0$. Moreover, the long-time behavior of the system is shown as the following theorem.

Theorem 5.1. *Let $U(t,x), V(t,x)$ be the mild solution of equation (5.1).*

(i) *For any non-negative initial values $U_0, V_0 \in L^\infty(\mathcal{O},\mathbb{R})$, if $a_2 > \dfrac{c_2}{m_2}$, then the individuals $V(t)$ is extinct.*

(ii) *Assume that non-negative initial values $(U_0, V_0) \in C(\overline{\mathcal{O}},\mathbb{R}^2)$ satisfy*

$$\Big|\frac{1}{U_0}\Big|_{L^\infty(\mathcal{O},\mathbb{R})} < \infty \text{ and } \Big|\frac{1}{V_0}\Big|_{L^\infty(\mathcal{O},\mathbb{R})} < \infty.$$

If $d := a_1 - \dfrac{c_1}{m_3} - \dfrac{3\sigma_1^2}{2} > 0$, and $\dfrac{dc_2}{m_2d+b_1m_1} > a_2 + \dfrac{\sigma_2^2}{2}$, then the classes $U(t), V(t)$ are permanent.

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E-mail address: nhu.math.2611@gmail.com
E-mail address: gyin@wayne.edu