

Recurrence and Ergodicity for A Class of Regime-Switching Jump Diffusions

Xiaoshan Chen¹ · Zhen-Qing Chen² · Ky Tran³ · George Yin⁴

Published online: 26 December 2017

© Springer Science+Business Media, LLC, part of Springer Nature 2017

Abstract This work develops asymptotic properties of a class of switching jump diffusion processes. The processes under consideration may be viewed as a number of jump diffusion processes modulated by a random switching mechanism. The underlying processes feature in the switching process depends on the jump diffusions. In this paper, conditions for recurrence and positive recurrence are derived. Ergodicity is examined in detail. Existence of invariant probability measures is proved.

Keywords Switching jump diffusion · Regularity · Recurrence · Positive recurrence · Ergodicity

This research was partially supported by NNSF of China (No. 11601163) and NSF of Guangdong Province of China (No. 2016A030313448), NSF Grant DMS-1206276, Vietnam National Foundation for Science and Technology Development (NAFOSTED) under Grant 101.03-2017.23 and The National Science Foundation under Grant DMS-1710827.

✉ George Yin
gyin@math.wayne.edu

Xiaoshan Chen
xschen@m.scnu.edu.cn

Zhen-Qing Chen
zqchen@uw.edu

Ky Tran
quankysp@gmail.com

¹ School of Mathematical Sciences, South China Normal University, Guangdong, China

² Departments of Mathematics, University of Washington, Seattle, WA 98195, USA

³ Department of Mathematics, College of Education, Hue University, Hue, Vietnam

⁴ Department of Mathematics, Wayne State University, Detroit, MI 48202, USA

Mathematics Subject Classification 60J60 · 60J75 · 60G51

1 Introduction

This work is concerned with the asymptotic properties, in particular recurrence and ergodicity, for a class of regime-switching jump diffusion processes. The process under consideration can be thought of as a number of jump-diffusion processes modulated by a random switching device. The underlying process is $(X(t), \Lambda(t))$, a two-component process, where $X(t)$ delineates the jump diffusion behavior and $\Lambda(t)$ describes the switching involved. One of the main ingredients is that the switching process depends on the jump diffusions. If the jump process is missing, it reduces to switching diffusions that have been investigated thoroughly; see for example [11, 23] and references therein. Our primary motivation stems from the study of a family of Markov processes in which continuous dynamics, jump discontinuity, and discrete events coexist. Their interactions reflect the salient features of the underlying systems. The distinct features of the systems include the presence of non-local operators, the coupled systems of equations, and the tangled information due to the dependence of the switching process on the jump diffusions. Such systems have drawn new as well as resurgent attention because of the urgent needs of systems modeling, analysis, and optimization in a wide variety of applications. Not only do the applications arise from the traditional fields of mathematical modeling, but also they have appeared in emerging application areas such as wireless communications, networked systems, autonomous systems, multi-agent systems, flexible manufacturing systems, financial engineering, and biological and ecological systems, among others. Continuing our investigation on regime-switching systems, this paper focuses on a class of switching jump diffusion processes. The state-dependent random switching process under consideration makes the formulation more versatile and interesting with a wider range of applications. Nevertheless, it makes the analysis more difficult and challenging.

Asymptotic properties of diffusion processes and associated partial differential equations are well known in the literature. We refer to [5, 8] and references therein. Results for switching diffusion processes can be found in [7, 23, 24]. One of the important problems concerning switching jump models is their longtime behavior. In particular, similar to the case of diffusions and switching diffusion processes, several questions of crucial importance are: Under what conditions will the systems return to a prescribed compact region in finite time? Under what conditions do the systems have the desired ergodicity? In this paper, we focus on asymptotic behavior and address these issues. Despite the growing interests in treating switching jump systems, the results regarding recurrence, positive recurrence, and invariant probability measures are still scarce. One of the main difficulties is the operator being non-local. When we study switching diffusions, it has been demonstrated that although they are similar to diffusion processes, switching diffusions have some distinct features. With the non-local operator used, the distinctions are even more pronounced.

In the literature, criteria for certain types of weak stability including Harris recurrence and positive Harris recurrence for continuous time Markovian processes based on Foster–Lyapunov inequalities were developed in [13]. Using results in that paper,

some sufficient conditions for ergodicity of Lévy type operators in dimension one were presented in [19] under the assumption of Lebesgue-irreducibility. In [21, 22], sufficient conditions for stability of a class of jump diffusions and switching jump diffusions were provided by constructing suitable Lyapunov type functions. However, the class of kernels considered in aforementioned papers satisfies a different set of hypotheses than those in our current work. In [12], the existence of an invariant probability measure was established for a jump diffusion, whose drift coefficient is assumed to be only Borel measurable. Sufficient conditions for recurrence, transience and ergodicity of stable-like processes were studied in [16]. In a recent work [1], the authors treated the ergodic properties such as positive recurrence and invariant probability measures for jump processes with no diffusion part. It should be noted that for the case of switching diffusions, the state process $X(t)$ can be viewed as a diffusion process in a random environment characterized by the switching component $\Lambda(t)$. The asymptotic behavior of the diffusion process in a random environment is more complicated than that of a diffusion process in a fixed environment. In particular, there are examples (see [14, 15]) that $X(t)$ is positive recurrent in some fixed environments and is transient in some other fixed environments, we can make $X(t)$ positive recurrent or transient by choosing suitable switching rates. We refer to [17, 18] for recent works on ergodicity of regime-switching diffusion processes. We emphasize that our results in the current work are significant extensions of those in [1, 24]. In terms of the associated operators and differential equations, compared to switching diffusion processes, in lieu of an elliptic system (a system of elliptic partial differential equations), we have to deal with a system of integro-differential equations. In our recent work, maximum principle and Harnack's inequality were obtained in [6], which are used in this paper. Compared to the case of diffusion processes with jumps, even though the classical approaches such as Lyapunov function methods and Dynkin's formula are still applicable, the analysis is much more delicate because of the coupling and interactions. There is a wide range of applications. As one particular example, consider the following average cost per unit time problem for a controlled switching jump diffusion in an infinite horizon. Suppose that $(X(t), \Lambda(t))$ is given by

$$\begin{aligned} dX(t) &= b(X(t), \Lambda(t), u(t))dt \\ &\quad + \sigma(X(t), \Lambda(t))dW(t) + \int_{\mathbb{R}_0} c(X(t-), \Lambda(t-), z)\tilde{N}_0(dt, dz), \\ X(0) &= x, \quad \Lambda(0) = \lambda_0, \end{aligned}$$

where $b(\cdot)$, $\sigma(\cdot)$, and $c(\cdot)$ are suitable real-valued functions, $\tilde{N}_0(\cdot)$ is a compensated real-valued Poisson process, $W(\cdot)$ is a real-valued Brownian motion, $\Lambda(t)$ is a continuous-time Markov chain with a finite state space, $u(\cdot)$ is a control, and $\mathbb{R}_0 = \mathbb{R} - \{0\}$. [More precise notion of switching jump diffusions will be given in the next section.] Assuming that $\Lambda(\cdot)$, $W(\cdot)$, and $N_0(\cdot)$ are independent, our objective is to minimize a long-run average cost function given by

$$J(x, \lambda_0, u(\cdot)) = \lim_{T \rightarrow \infty} \mathbb{E} \frac{1}{T} \int_0^T g(X(t), \Lambda(t), u(t))dt,$$

where $g(\cdot)$ is a suitable “running cost” function. To treat such a problem, one needs to replace the instantaneous measure by an ergodic measure (if it exists). The current paper sets up the foundation for the study on such problems because it provides sufficient conditions under which the invariant measure exists. Optimal controls of controlled jump diffusions (the above process without switching) was considered in [10], in which the jump process was assumed to belong to a compact set. Here we are dealing with a more general setting.

The rest of the paper is arranged as follows. Section 2 begins with the formulation of the problem. Section 3 is devoted to recurrence. It provides definitions of recurrence, positive recurrence, and null recurrence in addition to introducing certain notation. Then Sect. 3.2 focuses on recurrence and positive recurrence. We present sufficient conditions involving Lyapunov function for recurrence and positive recurrence using Lyapunov functions. Section 4 develops ergodicity. Existence of invariant probability measures of switching jump diffusion processes is obtained. Section 5 provides some sufficient conditions for the existence of Lyapunov functions under which the theorems of this paper are applicable. Finally, Sect. 6 concludes the paper with further remarks.

2 Formulation

Throughout the paper, we use z' to denote the transpose of $z \in \mathbb{R}^{l_1 \times l_2}$ with $l_1, l_2 \geq 1$, and $\mathbb{R}^{d \times 1}$ is simply written as $\mathbb{R}^d$. If $x \in \mathbb{R}^d$, the norm of x is denoted by $|x|$. For $x \in \mathbb{R}^d$ and $r > 0$, we use $B(x, r)$ to denote the open ball in $\mathbb{R}^d$ with radius r centered at x . The term *domain* in $\mathbb{R}^d$ refers to a nonempty connected open subset of the Euclidean space $\mathbb{R}^d$. If D is a domain in $\mathbb{R}^d$, then $\bar{D}$ is the closure of D , $D^c = \mathbb{R}^d \setminus D$ is its complement. The space $C^2(D)$ refers to the class of functions whose partial derivatives up to order 2 exist and are continuous in D , and $C_b^2(D)$ is the subspace of $C^2(D)$ consisting of those functions whose partial derivatives up to order 2 are bounded. The indicator function of a set A is denoted by $\mathbf{1}_A$.

Let $(X(t), \Lambda(t))$ be a two component Markov process such that $X(\cdot)$ is an $\mathbb{R}^d$ -valued process, and $\Lambda(\cdot)$ is a switching process taking values in a finite set $\mathcal{M} = \{1, 2, \dots, m\}$. Let $b(\cdot, \cdot) : \mathbb{R}^d \times \mathcal{M} \mapsto \mathbb{R}^d$, $\sigma(\cdot, \cdot) : \mathbb{R}^d \times \mathcal{M} \mapsto \mathbb{R}^d \times \mathbb{R}^d$, and for each $x \in \mathbb{R}^d$ and $i \in \mathcal{M}$, $\pi_i(x, dz)$ is a σ -finite measure on $\mathbb{R}^d$ satisfying

$$\int_{\mathbb{R}^d} (1 \wedge |z|^2) \pi_i(x, dz) < \infty.$$

Let $Q(x) = (q_{ij}(x))$ be an $m \times m$ matrix depending on x such that

$$q_{ij}(x) \geq 0 \quad \text{for } i \neq j, \quad \sum_{j \in \mathcal{M}} q_{ij}(x) = 0.$$

Define

$$Q(x)f(x, \cdot)(i) := \sum_{j \in \mathcal{M}} q_{ij}(x)f(x, j).$$

The generator $\mathcal{G}$ of the process $(X(t), \Lambda(t))$ is given as follows. For a function $f: \mathbb{R}^d \times \mathcal{M} \mapsto \mathbb{R}$ and $f(\cdot, i) \in C^2(\mathbb{R}^d)$ for each $i \in \mathcal{M}$, define

$$\mathcal{G}\{f(x, i) = \mathcal{L}_i f(x, i) + Q(x)f(x, \cdot)(i), \quad (x, i) \in \mathbb{R}^d \times \mathcal{M}, \quad (2.1)$$

where

$$\begin{aligned} \mathcal{L}_i f(x, i) = & \frac{1}{2} \sum_{k,l=1}^d a_{kl}(x, i) \frac{\partial^2 f(x, i)}{\partial x_k \partial x_l} + \sum_{k=1}^d b_k(x, i) \frac{\partial f(x, i)}{\partial x_k} \\ & + \int_{\mathbb{R}^d} (f(x+z, i) - f(x, i) - \nabla f(x, i) \cdot z \mathbf{1}_{\{|z| \leq 1\}}) \pi_i(x, dz), \end{aligned} \quad (2.2)$$

$a(x, i) := (a_{kl}(x, i)) = \sigma(x, i)\sigma'(x, i)$ and $\nabla f(\cdot, i)$ denotes the gradient of $f(\cdot, i)$.

Let $\Omega = D([0, \infty), \mathbb{R}^d \times \mathcal{M})$ be the space of functions (mapping $[0, \infty)$ to $\mathbb{R}^d \times \mathcal{M}$) that are right continuous with left limits endowed with the Skorohod topology. Define $(X(t), \Lambda(t)) = w(t)$ for $w \in \Omega$ and let $\{\mathcal{F}_t\}$ be the right continuous filtration generated by the process $(X(t), \Lambda(t))$. A probability measure $\mathbb{P}_{x,i}$ on Ω is a solution to the martingale problem for $(\mathcal{G}, C_b^2(\mathbb{R}^d))$ started at (x, i) if

- (a) $\mathbb{P}_{x,i}(X(0) = x, \Lambda(0) = i) = 1$,
- (b) if $f(\cdot, i) \in C_b^2(\mathbb{R}^d)$ for each $i \in \mathcal{M}$, then

$$f(X(t), \Lambda(t)) - f(X(0), \Lambda(0)) - \int_0^t \mathcal{G}f(X(s), \Lambda(s))ds,$$

is a $\mathbb{P}_{x,i}$ martingale.

If for each (x, i) , there is only one such $\mathbb{P}_{x,i}$, we say that the martingale problem for $(\mathcal{G}, C_b^2(\mathbb{R}^d))$ is well-posed.

Throughout the paper, we assume conditions (A1)–(A4) hold unless otherwise noticed.

- (A1) The functions $\sigma(\cdot, i)$ and $b(\cdot, i)$ are continuous on $\mathbb{R}^d$ for each $i \in \mathcal{M}$ such that

$$\sum_{k,l=1}^d |\sigma_{kl}(x, i)| + \sum_{k=1}^d |b_k(x, i)| \leq c(1 + |x|) \quad \text{on } \mathbb{R}^d$$

for some $c > 0$, and $q_{ij}(\cdot)$ is bounded and Borel measurable on $\mathbb{R}^d$ for $i, j \in \mathcal{M}$.

- (A2) For every bounded domain $D \subset \mathbb{R}^d$, there exists a constant $\kappa_0 = \kappa_0(D) \in (0, 1]$ such that

$$\kappa_0 |\xi|^2 \leq \xi' a(x, i) \xi \leq \kappa_0^{-1} |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^d, x \in D, i \in \mathcal{M}.$$

- (A3) $\sup_{x \in \mathbb{R}^d, i \in \mathcal{M}} \pi_i(x, B(0, 1)^c) < \infty$ and $\sup_{i \in \mathcal{M}} \int_{B(0, 1)} |z|^2 \pi_i(x, dz) \leq c(1 + |x|^2)$ for some $c > 0$. Moreover, for every $n \geq 1$, there exists a σ -finite measure

$\Pi_n(dz)$ with $\int_{\mathbb{R}^d} (1 \wedge |z|^2) \Pi_n(dz) < \infty$ so that $\pi_i(x, dz) \leq \Pi_n(dz)$ for every $x \in B(0, n)$ and $i \in \mathcal{M}$.

- (A4) For any $i \in \mathcal{M}$ and $x \in \mathbb{R}^d$, $\pi_i(x, dz) = \tilde{\pi}_i(x, z)dz$. Moreover, for any $r \in (0, 1)$, any $x_0 \in \mathbb{R}^d$, any $x, y \in B(x_0, r/2)$ and $z \in B(x_0, r)^c$, we have

$$\tilde{\pi}_i(x, z - x) \leq \alpha_r \tilde{\pi}_i(y, z - y),$$

where α_r satisfies $1 < \alpha_r \leq \kappa_2 r^{-\beta}$ with κ_2 and β being positive constants.

Remark 2.1 We comment on the conditions briefly.

- (a) Under Assumptions (A1)–(A3), for each $i \in \mathcal{M}$, there is a unique solution to the martingale problem for $(\mathcal{L}_i, C_b^2(\mathbb{R}^d))$ which is conservative (see [9, Theorem 5.2]). Note that the switched Markov process $(X(t), \Lambda(t))$ can be constructed from that of $\mathcal{L}_i$ as follows. Let X^i be the Markov process associated with $\mathcal{L}_i$, that is, $\mathcal{L}_i$ is the infinitesimal generator of X^i . Suppose we start the process at (x_0, i_0) , run a subprocess $\tilde{X}^{i_0}$ of X^{i_0} that got killed with rate $-q_{i_0 i_0}(x)$; that is, via Feynman–Kac transform $\exp\left(\int_0^t q_{i_0 i_0}(X^{i_0}(s))ds\right)$. Note that this subprocess $\tilde{X}^{i_0}$ has infinitesimal generator $\mathcal{L}_{i_0} + q_{i_0 i_0}$. At the lifetime τ_1 of the killed process $\tilde{X}^{i_0}$, jump to plane $j \neq i_0$ with probability $-q_{i_0 j}(X^{i_0}(\tau_1 -))/q_{i_0 i_0}(X^{i_0}(\tau_1 -))$ and run a subprocess $\tilde{X}^j$ of X^j with killing rate $-q_{jj}(x)$ from position $X^{i_0}(\tau_1 -)$. Repeat this procedure. As pointed out in [6], since $\sum_{j \in \mathcal{M}} q_{ij}(x) = 0$ on $\mathbb{R}^d$ for every $i \in \mathcal{M}$, the resulting process $(X(t), \Lambda(t))$ is a strong Markov process with lifetime $\zeta = \infty$. It is easy to check that the law of $(X(t), \Lambda(t))$ solves the martingale problem for $(\mathcal{G}, C_b^2(\mathbb{R}^d))$ so it is the desired switched jump-diffusion. It follows from [20] that the law of $(X(t), \Lambda(t))$ is the unique solution to the martingale problem for $(\mathcal{G}, C_b^2(\mathbb{R}^d))$.
- (b) Conditions (A1) and (A2) present the local uniform ellipticity of $a(x, i)$ and the linear growth bound on $a(x, i)^{1/2}$ and $|b(x, i)|$. The measure $\pi_i(x, dz)$ can be thought of as the intensity of the number of jumps from x to $x + z$ (see [2, 3]). Condition (A4) tells us that $\pi_i(x, dy)$ is absolutely continuous with respect to the Lebesgue measure dx on $\mathbb{R}^d$, and the intensities of jumps from x and y to a point z are comparable if x, y are relatively far from z but relatively close to each other. If $\tilde{\pi}_i(x, z)$ is such that

$$\frac{c_i^{-1}}{|z|^{d+\alpha_i}} \leq \tilde{\pi}_i(x, z) \leq \frac{c_i}{|z|^{d+\alpha_i}}$$

for some $c_i \geq 1$ and $\alpha_i \in (0, 2)$, then condition (A4) is satisfied with $1 < \alpha_r < \kappa_2$ independent of $r \in (0, 1)$. Condition (A4) is needed for Harnack inequality for the switched jump-diffusion; see [6].

- (c) In our recent paper [6], various properties of switching jump diffusions including maximum principle and Harnack inequality are studied under the assumption that $a(x, i)$ is uniformly elliptic and $b(x, i)$ is bounded on $\mathbb{R}^d$ and that there is one single Π so that (A3) holds with $\Pi_n = \Pi$ for all $n \geq 1$. Using localization, one can easily see that the results of [6] hold under the current setting of this paper

when they are applied to bounded open sets. In particular, for each bounded open subset $D \subset \mathbb{R}^d$, Theorems 3.4 and 4.7 of [6] as well as Proposition 4.1 (for all $x_0 \in D$) and Proposition 4.3 hold under the setting of this paper.

Recall that a regime-switching jump diffusion $(X(t), \Lambda(t))$ can be regarded as the results of m jump diffusion processes $X^1(t), X^2(t), \dots, X^m(t)$ switching from one to another according to the dynamic movement of $\Lambda(t)$, where $X^i(t)$ is the Markov process associated with $\mathcal{L}_i$. By assumption (A4), if $\tilde{\pi}_i(x, z) = 0$ for some $(x, z) \in \mathbb{R}^d \times \mathbb{R}^d$, then $\tilde{\pi}_i(x, z) \equiv 0$ on $\mathbb{R}^d \times \mathbb{R}^d$; that is, $X^i(t)$ is a diffusion process. If $\tilde{\pi}_i(x, z) \equiv 0$ on $\mathbb{R}^d \times \mathbb{R}^d$ for all $i \in \mathcal{M}$, $(X(t), \Lambda(t))$ is just a switching diffusion process. Thus, the class of models under consideration includes switching diffusions. In this paper, sufficient conditions for recurrence, positive recurrence, and the existence of an invariant probability measure are given. We show that under the sufficient conditions derived, recurrence and positive recurrence are independent of the domain chosen. Furthermore, it is demonstrated that we can work with a fixed discrete component. Several examples are provided in which easily verifiable conditions on coefficients of the switching jump diffusions are given in lieu of the Lyapunov function type of conditions.

3 Recurrence

This section is devoted to recurrence, positive recurrence, and null recurrence. The first part gives definitions and the second part provides criteria for recurrence and positive recurrence.

3.1 Definitions: Recurrence, Positive Recurrence, and Null Recurrence

For simplicity, we introduce some notation as follows. For any $U = D \times J \subset \mathbb{R}^d \times \mathcal{M}$, where $D \subset \mathbb{R}^d$ and $J \subset \mathcal{M}$. Define

$$\begin{aligned}\tau_U &:= \inf\{t \geq 0 : (X(t), \Lambda(t)) \notin U\}, \\ \sigma_U &:= \inf\{t \geq 0 : (X(t), \Lambda(t)) \in U\}.\end{aligned}$$

In particular, if $U = D \times \mathcal{M}$ is a “cylinder”, we set

$$\begin{aligned}\tau_D &:= \inf\{t \geq 0 : X(t) \notin D\}, \\ \sigma_D &:= \inf\{t \geq 0 : X(t) \in D\}.\end{aligned}$$

Remark 3.1 Recall that the process $(X(t), \Lambda(t))$ is a strong Markov process with lifetime $\zeta = \infty$. Let $\beta_n = \inf\{t \geq 0 : |X(t)| \geq n\}$ be the first exit time of the process $(X(t), \Lambda(t))$ from the bounded set $B(0, n) \times \mathcal{M}$. Then the sequence $\{\beta_n\}$ is monotonically increasing and $\beta_n \rightarrow \infty$ almost surely as $n \rightarrow \infty$. We also refer to such property of $(X(t), \Lambda(t))$ as regularity or no finite explosion time.

Recurrence, positive recurrence, and null recurrence are defined similarly as for switching diffusions given in [23, 24].

Definition 3.2 Suppose $U = D \times J$, where $J \subset \mathcal{M}$ and $D \subset \mathbb{R}^d$ is a bounded domain. A process $(X(t), \Lambda(t))$ is said to be *recurrent* with respect to U if

$$\mathbb{P}_{x,i}(\sigma_U < \infty) = 1 \quad \text{for any } (x, i) \in U^c.$$

If $(X(t), \Lambda(t))$ is recurrent with respect to U and $\mathbb{E}_{x,i}\sigma_U < \infty$ for any $(x, i) \in U^c$, then it is said to be *positive recurrent* with respect to U ; otherwise, the process is *null recurrent* with respect to U .

3.2 Criteria for Recurrence and Positive Recurrence

We begin this section with some preparatory results, followed by sufficient conditions for recurrence and positive recurrence of the process $(X(t), \Lambda(t))$ with respect to some cylinder $U = D \times \mathcal{M}$. Under these sufficient conditions, we prove that the process $(X(t), \Lambda(t))$ is recurrent (resp., positive recurrent) with respect to some cylinder $D \times \mathcal{M}$ if and only if it is recurrent (resp., positive recurrent) with respect to $D \times \{l\}$ for every $l \in \mathcal{M}$, where $D \subset \mathbb{R}^d$ is a bounded domain. We will also prove that the properties of recurrence and positive recurrence do not depend on the choice of the domain $D \subset \mathbb{R}^d$ and $l \in \mathcal{M}$. We first prove the following theorem, which asserts that the process $(X(t), \Lambda(t))$ will exit every bounded cylinder with a finite mean exit time.

Theorem 3.3 Let $D \subset \mathbb{R}^d$ be a bounded domain. Then

$$\sup_{(x,i) \in D \times \mathcal{M}} \mathbb{E}_{x,i}\tau_D < \infty. \quad (3.1)$$

Proof By the local uniform ellipticity condition in (A2), there exists some $\kappa_0 \in (0, 1]$ such that

$$\kappa_0 \leq a_{11}(x, i) \leq \kappa_0^{-1} \quad \text{for any } (x, i) \in D \times \mathcal{M}. \quad (3.2)$$

Let $f \in C_b^2(\mathbb{R}^d)$ be such that f is nonnegative and $f(x) = (x_1 + \beta)^\gamma$ if $x \in \{y : d(y, D) < 1\}$, where the constants $\gamma > 2$ and $\beta > 0$ are to be specified, and $x_1 = e'_1 x$ is the first component of x , with $e_1 = (1, 0, \dots, 0)'$ being the standard unit vector. Define $V(x, i) = f(x)$ for $(x, i) \in \mathbb{R}^d \times \mathcal{M}$. Since D is bounded, we can choose constant $\beta > 0$ such that

$$1 \leq x_1 + \beta \quad \text{for all } (x, i) \in D \times \mathcal{M},$$

and

$$\gamma := \frac{2}{\kappa_0} \left(\sup_{(x,i) \in D \times \mathcal{M}} \left[|b_1(x, i)|(x_1 + \beta) + \kappa_1(x_1 + \beta)^2 \right] + 1 \right) + 2 < \infty,$$

where κ_1 is the constant which exists by assumption (A3). Then we have by (3.2) that

$$b_1(x, i)(x_1 + \beta) + \frac{\gamma - 1}{2} a_{11}(x, i) - \kappa_1(x_1 + \beta)^2 \geq 1 \quad \text{for all } (x, i) \in D \times \mathcal{M}. \quad (3.3)$$

Direct computation leads to

$$\begin{aligned} \mathcal{G}V(x, i) &= \gamma(x_1 + \beta)^{\gamma-2} \left[b_1(x, i)(x_1 + \beta) + \frac{\gamma - 1}{2} a_{11}(x, i) \right] \\ &\quad + \int_{|z| \leq 1} [f(x + z) - f(x) - \nabla f(x) \cdot z] \tilde{\pi}_i(x, z) (dz) \\ &\quad + \int_{|z| > 1} [f(x + z) - f(x)] \tilde{\pi}_i(x, z) (dz). \end{aligned} \quad (3.4)$$

Since $\gamma > 2$, $f(\cdot)$ is convex on $\{x \in \mathbb{R}^d : d(x, D) < 1\}$. It follows that

$$\int_{|z| \leq 1} [f(x + z) - f(x) - \nabla f(x) \cdot z] \tilde{\pi}_i(x, z) (dz) \geq 0. \quad (3.5)$$

By the nonnegativity of f , it is also clear that

$$\begin{aligned} \int_{|z| > 1} [f(x + z) - f(x)] \tilde{\pi}_i(x, z) (dz) &\geq - \int_{|z| > 1} (x_1 + \beta)^\gamma \tilde{\pi}_i(x, z) (dz) \\ &\geq -\kappa_1(x_1 + \beta)^\gamma. \end{aligned} \quad (3.6)$$

It follows from (3.4), (3.5), (3.6), and (3.3) that

$$\begin{aligned} \mathcal{G}V(x, i) &\geq \gamma(x_1 + \beta)^{\gamma-2} \left[b_1(x, i)(x_1 + \beta) + \frac{\gamma - 1}{2} a_{11}(x, i) \right] - \kappa_1(x_1 + \beta)^\gamma \\ &\geq \gamma(x_1 + \beta)^{\gamma-2} \left[b_1(x, i)(x_1 + \beta) + \frac{\gamma - 1}{2} a_{11}(x, i) - \kappa_1(x_1 + \beta)^2 \right] \\ &\geq \gamma, \end{aligned} \quad (3.7)$$

for all $(x, i) \in D \times \mathcal{M}$. Let $\tau_D(t) = \min\{t, \tau_D\}$. Then we have from Dynkin's formula and (3.7) that

$$\begin{aligned} \mathbb{E}_{x,i} V(X(\tau_D(t)), \Lambda(\tau_D(t))) - V(x, i) \\ = \mathbb{E}_{x,i} \int_0^{\tau_D(t)} \mathcal{G}V(X(s), \Lambda(s)) ds \geq \gamma \mathbb{E}_{x,i} \tau_D(t). \end{aligned}$$

Hence

$$\mathbb{E}_{x,i} \tau_D(t) \leq \frac{1}{\gamma} \sup_{(x,i) \in \mathbb{R}^d \times \mathcal{M}} V(x, i) := M. \quad (3.8)$$

Note that M is finite by our construction of functions $V(\cdot)$. Since $\mathbb{E}_{x,i} \tau_D(t) \geq t \mathbb{P}_{x,i}[\tau_D > t]$, it follows from (3.8) that

$$t \mathbb{P}_{x,i}(\tau_D > t) \leq M.$$

Letting $t \rightarrow \infty$, we obtain $\mathbb{P}_{x,i}(\tau_D = \infty) = 0$. That is, $\mathbb{P}_{x,i}(\tau_D < \infty) = 1$. This yields that $\tau_D(t) \rightarrow \tau_D$ a.s. $\mathbb{P}_{x,i}$ as $t \rightarrow \infty$. Now applying Fatou's lemma, as $t \rightarrow \infty$ in (3.8) we obtain

$$\mathbb{E}_{x,i} \tau_D \leq M < \infty. \quad (3.9)$$

This proves the theorem. $\square$

To study the recurrence and positive recurrence of the process $(X(t), \Lambda(t))$, we first present criteria based on the existence of certain Lyapunov functions. Sufficient conditions on the existence of such Lyapunov functions will be given in Sect. 5.

Theorem 3.4 *A sufficient condition for the positive recurrence of $(X(t), \Lambda(t))$ with respect to $U = D \times \mathcal{M} \subset \mathbb{R}^d \times \mathcal{M}$ is the following condition (C1) holds:*

(C1) *For each $i \in \mathcal{M}$, there exists a nonnegative function $V(\cdot, i) \in C^2(\mathbb{R}^d)$ satisfying*

$$\mathcal{G}V(x, i) \leq -1 \text{ for any } (x, i) \in D^c \times \mathcal{M}. \quad (3.10)$$

Proof Assume that there exists a nonnegative function $V(\cdot, \cdot)$ satisfying condition (C1) with respect to $U = D \times \mathcal{M}$. We show that the process $(X(t), \Lambda(t))$ is positive recurrent with respect to $U = D \times \mathcal{M}$.

Choose n_0 a positive integer sufficiently large so that $D \subset B(0, n_0)$. Fix any $(x, i) \in D^c \times \mathcal{M}$. For any $t > 0$ and $n \geq n_0$, we define

$$\sigma_D^{(n)}(t) = \min\{\sigma_D, t, \beta_n\},$$

where β_n is the first exit time from $B(0, n)$ and σ_D is the first entrance time to D . Let $f_n : \mathbb{R}^d \mapsto \mathbb{R}$ be a smooth cut-off function that takes values in $[0, 1]$ satisfying $f_n = 1$ on $B(0, n)$ and $f_n = 0$ outside of $B(0, n+1)$. Then $V_n(\cdot, j) := f_n(\cdot) V(\cdot, j) \in C_b^2(\mathbb{R}^d)$ for each $j \in \mathcal{M}$. Moreover,

$$0 \leq V_n(x, i) \leq V(x, i), \quad (x, i) \in \mathbb{R}^d \times \mathcal{M}.$$

It follows from (3.10) that

$$\mathcal{G}V_n(y, j) \leq \mathcal{G}V(y, j) \leq -1 \quad \text{for all } (y, j) \in B(0, n) - D.$$

Dynkin's formula implies that

$$\begin{aligned} & \mathbb{E}_{x,i} V_n \left(X \left(\sigma_D^{(n)}(t) \right), \Lambda \left(\sigma_D^{(n)}(t) \right) \right) - V_n(x, i) \\ &= \mathbb{E}_{x,i} \int_0^{\sigma_D^{(n)}(t)} \mathcal{G}V_n(X(s), \Lambda(s)) ds \leq -\mathbb{E}_{x,i} \sigma_D^{(n)}(t). \end{aligned}$$

Note that the function V_n is nonnegative; hence we have $\mathbb{E}_{x,i}\sigma_D^{(n)}(t) \leq V_n(x, i) = V(x, i)$. Meanwhile, since the process $(X(s), \Lambda(s))$ is regular, $\beta_n \rightarrow \infty$ a.s as $n \rightarrow \infty$. As a result, $\sigma_D^{(n)}(t) \rightarrow \sigma_D(t)$ a.s as $n \rightarrow \infty$, where $\sigma_D(t) = \min\{\sigma_D, t\}$. By virtue of Fatou's lemma, we obtain $\mathbb{E}_{x,i}\sigma_D(t) \leq V(x, i)$. Now the argument after (3.8) in the proof of Theorem 3.3 yields that

$$\mathbb{E}_{x,i}\sigma_D \leq V(x, i) < \infty. \quad (3.11)$$

Since $(x, i) \in D^c \times \mathcal{M}$ is arbitrary, we conclude that $(X(t), \Lambda(t))$ is positive recurrent with respect to U . $\square$

Theorem 3.5 *A sufficient condition for the recurrence of $(X(t), \Lambda(t))$ with respect to $U = D \times \mathcal{M} \subset \mathbb{R}^d \times \mathcal{M}$ is the following condition (C2) holds:*

(C2) *For each $i \in \mathcal{M}$, there exists a nonnegative function $V(\cdot, i) \in C^2(\mathbb{R}^d)$ satisfying*

$$\begin{aligned} \mathcal{G}V(x, i) &\leq 0, \text{ for any } (x, i) \in D^c \times \mathcal{M}, \\ \inf_{|x| \geq n, i \in \mathcal{M}} V(x, i) &\rightarrow \infty, \quad n \rightarrow \infty. \end{aligned} \quad (3.12)$$

Proof Assume that there exists a nonnegative function $V(\cdot, \cdot)$ satisfying condition (C2) with respect to U . We show that the process $(X(t), \Lambda(t))$ is recurrent with respect to U .

Choose n_0 to be a positive integer sufficiently large so that $D \subset B(0, n_0)$. Fix any $(x, i) \in D^c \times \mathcal{M}$. For any $t > 0$ and $n \geq n_0$, we define

$$\sigma_D^{(n)}(t) = \min\{\sigma_D, t, \beta_n\},$$

where β_n is the first exit time from $B(0, n)$ and σ_D is the first entrance time to D . Let $f_n : \mathbb{R}^d \mapsto \mathbb{R}$ be a smooth cut-off function that takes values in $[0, 1]$, 1 on $B(0, n)$, and 0 outside of $B(0, n+1)$. Denote $M_n := \inf_{|y| \geq n, j \in \mathcal{M}} V(y, j)$. Then

$$V_n(\cdot, j) := f_n(\cdot)V(\cdot, j) + (1 - f_n(\cdot))M_n \in C_b^2(\mathbb{R}^d), \quad \text{for each } j \in \mathcal{M}.$$

Moreover, $V_n(y, j) \leq V(y, j)$ for all (y, j) , $V_n(y, j) = V(y, j)$ for $(y, j) \in B(0, n) \times \mathcal{M}$, and $V_n(y, j) \geq M$ for $(y, j) \in B(0, n)^c \times \mathcal{M}$. Detailed computations and (3.12) yield that

$$\mathcal{G}V_n(y, j) \leq \mathcal{G}V(y, j) \leq 0 \quad \text{for all } (y, j) \in B(0, n) - D.$$

Now Dynkin's formula implies that

$$\begin{aligned} &\mathbb{E}_{x,i} V_n \left(X \left(\sigma_D^{(n)}(t) \right), \Lambda \left(\sigma_D^{(n)}(t) \right) \right) - V_n(x, i) \\ &= \mathbb{E}_{x,i} \int_0^{\sigma_D^{(n)}(t)} \mathcal{G}V_n(X(s), \Lambda(s)) ds \leq 0. \end{aligned}$$

Consequently,

$$\mathbb{E}_{x,i} V_n \left(X \left(\sigma_D^{(n)}(t) \right), \Lambda \left(\sigma_D^{(n)}(t) \right) \right) \leq V_n(x, i) = V(x, i).$$

By virtue of Fatou's lemma, we obtain

$$\mathbb{E}_{x,i} V_n (X (\sigma_D \wedge \beta_n), \Lambda ((\sigma_D \wedge \beta_n))) \leq V(x, i).$$

Then we have

$$\begin{aligned} V(x, i) &\geq \mathbb{E}_{x,i} \left[V_n (X (\beta_n), \Lambda (\beta_n)) \mathbf{1}_{\{\beta_n < \sigma_D\}} \right] \\ &\geq M_n \mathbb{P}_{x,i} (\beta_n < \sigma_D). \end{aligned}$$

It follows from (3.12) that as $n \rightarrow \infty$,

$$\mathbb{P}_{x,i} (\beta_n < \sigma_D) \leq \frac{V(x, i)}{M_n} \rightarrow 0.$$

Note that $\mathbb{P}_{x,i}(\sigma_D = \infty) \leq \mathbb{P}_{x,i}(\beta_n < \sigma_D)$. Hence $\mathbb{P}_{x,i}(\sigma_D = \infty) = 0$. Since $(x, i) \in D^c \times \mathcal{M}$ is arbitrary, we conclude that $(X(t), \Lambda(t))$ is recurrent with respect to U . $\square$

Now we recall the definition of harmonic functions with respect to process $(X(t), \Lambda(t))$, namely, $\mathcal{G}$ -harmonic functions.

Definition 3.6 Let $U = \bigcup_{i=1}^m D_i \times \{i\}$ with $D_i \subset \mathbb{R}^d$ being a bounded domain. A bounded and Borel measurable function $f : \mathbb{R}^d \times \mathcal{M} \mapsto \mathbb{R}$ is said to be $\mathcal{G}$ -harmonic in U if for any relatively compact open subset V of U ,

$$f(x, i) = \mathbb{E}_{x,i} [f(X(\tau_V), \Lambda(\tau_V))] \quad \text{for all } (x, i) \in V,$$

where $\tau_V = \inf\{t \geq 0 : (X(t), \Lambda(t)) \notin V\}$ is the first exit time of V .

Definition 3.7 The generator $\mathcal{G}$ or the matrix function $Q(\cdot)$ is said to be *strictly irreducible* on $D \subset \mathbb{R}^d$ if for any $i, j \in \mathcal{M}$ and $i \neq j$, there exists $q_{ij}^0 > 0$ such that $\inf_{x \in D} q_{ij}(x) \geq q_{ij}^0$.

In the rest of this section, we assume that the Q -matrix in the operator $\mathcal{G}$ is *strictly irreducible* on any bounded domain of $\mathbb{R}^d$. We are in a position to prove that if the process $(X(t), \Lambda(t))$ is recurrent (resp., positive recurrent) with respect to some cylinder $D \times \mathcal{M} \subset \mathbb{R}^d \times \mathcal{M}$, then it is recurrent (resp., positive recurrent) with respect to any cylinder $E \times \mathcal{M} \subset \mathbb{R}^d \times \mathcal{M}$.

Lemma 3.8 Suppose that the operator $\mathcal{G}$ is strictly irreducible on any bounded domain of $\mathbb{R}^d$. Let $D \subset \mathbb{R}^d$ be a bounded domain. Suppose that

$$\mathbb{P}_{x,i}(\sigma_D < \infty) = 1 \quad \text{for any } (x, i) \in D^c \times \mathcal{M}. \quad (3.13)$$

Then for any bounded domain $E \subset \mathbb{R}^d$, we have

$$\mathbb{P}_{x,i}(\sigma_E < \infty) = 1 \quad \text{for any } (x, i) \in E^c \times \mathcal{M}. \quad (3.14)$$

Proof Without loss of generality, we suppose that $\overline{E} \subset D$. Otherwise, let $\tilde{D}$ be a sufficiently large bounded domain containing both $\overline{D}$ and $\overline{E}$. Then (3.13) and our arguments hold for $\tilde{D}$ in place of D . It is sufficient to prove (3.14) for $(x, i) \in (D \setminus \overline{E}) \times \mathcal{M}$. Let G be a bounded domain such that $\overline{D} \subset G$. Define a sequence of stopping times by

$$\tau_0 = 0, \quad \tau_1 = \inf\{t > 0 : X(t) \in G^c\}, \quad (3.15)$$

and for $n = 1, 2, \dots$,

$$\begin{aligned} \tau_{2n} &= \inf\{t > \tau_{2n-1} : X(t) \in D\}, \\ \tau_{2n+1} &= \inf\{t > \tau_{2n} : X(t) \in G^c\}. \end{aligned} \quad (3.16)$$

It follows from (3.13), Theorem 3.3, and the strong Markov property that $\tau_n < \infty$ $\mathbb{P}_{x,i}$ a.s. for $n = 1, 2, \dots$. Define $u(y, j) = \mathbb{P}_{y,j}(\sigma_E < \tau_G)$ for $(y, j) \in \mathbb{R}^d \times \mathcal{M}$. Then $u(y, j) = \mathbb{E}_{y,j} \mathbf{1}_{\{X(\tau_{G \setminus \overline{E}}) \in E\}}$. Hence $u(y, j)$ is a $\mathcal{G}$ -harmonic function in $(G \setminus \overline{E}) \times \mathcal{M}$. By Remark 2.1(c) and [6, Theorem 3.4], $u(y, j) > 0$ for all $(y, j) \in (G \setminus \overline{E}) \times \mathcal{M}$. Let H be a domain such that $\overline{D} \subset H \subset \overline{H} \subset G$. Define

$$v(y, j) = \mathbb{E}_{y,j} u(X(\tau_H), \Lambda(\tau_H)) \quad \text{for } (y, j) \in H \times \mathcal{M}.$$

Again by Remark 2.1(c) and [6, Theorem 3.4], $v(y, j) > 0$ for $(y, j) \in H \times \mathcal{M}$. By Remark 2.1(c) and the Harnack inequality [6, Theorem 4.7], there is a constant $\delta_1 \in (0, 1)$ such that $\inf_{(y,j) \in D \times \mathcal{M}} v(y, j) \geq \delta_1$. Since $u(y, j)$ is a $\mathcal{G}$ -harmonic function in $(G \setminus \overline{E}) \times \mathcal{M}$, it is a $\mathcal{G}$ -harmonic function in $(H \setminus \overline{E}) \times \mathcal{M}$. Thus, for any $(y, j) \in D \times \mathcal{M}$,

$$\begin{aligned} u(y, j) &= \mathbb{E}_{y,j} [u(X(\tau_H), \Lambda(\tau_H)) \mathbf{1}_{\tau_H < \sigma_E}] + \mathbb{E}_{y,j} [\mathbf{1}_{\sigma_E < \tau_H}] \\ &\geq v(y, j). \end{aligned}$$

It follows that

$$\inf_{(y,j) \in D \times \mathcal{M}} \mathbb{P}_{y,j}(\sigma_E < \tau_G) \geq \delta_1. \quad (3.17)$$

Define

$$A_0 = \{X(t) \in E \text{ for some } t \in [0, \tau_1)\}, \quad (3.18)$$

and for $n = 1, 2, \dots$,

$$A_n = \{X(t) \in E \text{ for some } t \in [\tau_{2n}, \tau_{2n+1})\}. \quad (3.19)$$

Note that the event A_0^c implies $\tau_G < \sigma_E$. Hence we have from (3.17) that

$$\mathbb{P}_{x,i}(A_0^c) \leq \mathbb{P}_{x,i}(\tau_G < \sigma_E) \leq 1 - \delta_1.$$

By the strong Markov property, induction on n yields

$$\mathbb{P}_{x,i} \left(\bigcap_{k=0}^n A_k^c \right) \leq (1 - \delta_1)^{n+1}. \quad (3.20)$$

Thus, we have

$$\begin{aligned} \mathbb{P}_{x,i}(\sigma_E = \infty) &= \mathbb{P}_{x,i}(X(t) \notin E \text{ for any } t \geq 0) \\ &\leq \lim_{n \rightarrow \infty} \mathbb{P}_{x,i} \left(\bigcap_{k=0}^n A_k^c \right) \\ &\leq \lim_{n \rightarrow \infty} (1 - \delta_1)^{n+1} = 0. \end{aligned}$$

This proves the lemma. $\square$

Lemma 3.9 *Suppose that the operator $\mathcal{G}$ is strictly irreducible on any bounded domain of $\mathbb{R}^d$. Let $D \subset \mathbb{R}^d$ be a bounded domain and suppose that $\mathcal{G}$ satisfies condition (C1) with respect to $D \times \mathcal{M}$. Then for any bounded domain $E \subset \mathbb{R}^d$, we have*

$$\mathbb{E}_{x,i}\sigma_E < \infty \text{ for any } (x, i) \in E^c \times \mathcal{M}. \quad (3.21)$$

Proof Since $\mathcal{G}$ satisfies condition (C1) with respect to $D \times \mathcal{M}$,

$$\mathbb{E}_{y,j}\sigma_D < \infty \text{ for any } (y, j) \in D^c \times \mathcal{M}. \quad (3.22)$$

Using the same reasoning as in the proof for Lemma 3.8, without loss of generality, we may assume that $\overline{E} \subset D$. It suffices to prove (3.22) for $(x, i) \in (D \setminus \overline{E}) \times \mathcal{M}$. Let G be a bounded domain such that $\overline{D} \subset G$.

Define a sequence of stopping times $\{\tau_n\}$ and events $A_0, A_1, A_2, \dots$ as in (3.15), (3.16), (3.18), and (3.19) in the proof of Lemma 3.8. It follows from (3.22), Theorem 3.3, and the strong Markov property that $\tau_n < \infty$ $\mathbb{P}_{x,i}$ a.s. for $n = 1, 2, \dots$

We now prove that

$$M_2 := \sup_{(y,j) \in D \times \mathcal{M}} \mathbb{E}_{y,j}\tau_2 < \infty. \quad (3.23)$$

From Theorem 3.3, we have $M_1 := \sup_{(y,j) \in D \times \mathcal{M}} \mathbb{E}_{y,j}\tau_1 < \infty$. Meanwhile, by (3.11) in the proof of Theorem 3.4, we obtain

$$\mathbb{E}_{y,j}\sigma_D \leq V(y, j) \text{ for } (y, j) \in G^c \times \mathcal{M}.$$

Hence to prove (3.23), it suffices to show that

$$\sup_{(y,j) \in D \times \mathcal{M}} \int_{G^c} \sum_{k=1}^m V(z, k) \mathbb{P}_{\tau_1}(y, j, dz, k) < \infty,$$

where $\mathbb{P}_{\tau_1}(y, j, \cdot, \cdot)$ is the distribution of $(X^{y,j}(\tau_1), \Lambda^{y,j}(\tau_1))$. Since $V(\cdot, \cdot)$ is bounded on compact sets, it is enough if we can find an open ball $B(0, R)$ with R sufficiently large such that $\{y : d(y, G) < 2\} \subset B(0, R)$ and

$$\sup_{(y,j) \in D \times \mathcal{M}} \int_{B(0,R)^c} \sum_{k=1}^m V(z, k) \mathbb{P}_{\tau_1}(y, j, dz, k) < \infty. \quad (3.24)$$

Let a point $x^* \in \partial G$. Then for any $x \in G$ and $z \in B(0, R)^c$, there is a sequence $\{x_i : i = 0, \dots, \tilde{n}\}$ such that $x_0 = x$, $x_{\tilde{n}} = x^*$, $|x_i - x_{i-1}| < 1/2$ and $x_i \in \bar{G}$ for $i = 1, \dots, \tilde{n}$. Since G is bounded, $\tilde{n}$ can be independent of x . By assumption (A4), we have

$$\tilde{\pi}_k(x_{i-1}, z - x_{i-1}) \leq \alpha_{1/2} \tilde{\pi}_k(x_i, z - x_i), \quad i = 1, \dots, \tilde{n}, k = 1, \dots, m.$$

Thus, there is a positive constant $K = \alpha_{1/2}^{\tilde{n}}$, depending only on G such that

$$\tilde{\pi}_k(x, z - x) \leq K \tilde{\pi}_k(x^*, z - x^*) \quad \text{for } x \in G, z \in B(0, R)^c, k = 1, \dots, m.$$

Let $(y, j) \in D \times \mathcal{M}$ and $A \subset B(0, R)^c$. By virtue of Chen et al. [6, Proposition 4.3],

$$\sum_{s \leq t} \mathbf{1}_{\{X(s-) \in G, X(s) \in A, \Lambda(s) = k\}} - \int_0^t \mathbf{1}_{\{X(s) \in G, \Lambda(s) = k\}} \int_A \tilde{\pi}_k(X(s), z - X(s)) dz ds,$$

is a $\mathbb{P}_{y,j}$ -martingale. We deduce that

$$\begin{aligned} \mathbb{P}_{y,j}(X(\tau_1 \wedge t) \in A, \Lambda(\tau_1 \wedge t) = k) &= \mathbb{E}_{y,j} \left[\sum_{s \leq \tau_1 \wedge t} \mathbf{1}_{\{X(s-) \in G, X(s) \in A, \Lambda(s) = k\}} \right] \\ &= \mathbb{E}_{y,j} \left[\int_0^{\tau_1 \wedge t} \mathbf{1}_{\{X(s) \in G, \Lambda(s) = k\}} \int_A \tilde{\pi}_k(X(s), z - X(s)) dz ds \right] \\ &\leq K \mathbb{E}_{y,j} \left[\int_0^{\tau_1 \wedge t} \int_A \tilde{\pi}_k(x^*, z - x^*) dz ds \right] \\ &\leq K \mathbb{E}_{y,j}(\tau_1 \wedge t) \mu_k(A), \end{aligned} \quad (3.25)$$

where μ_k is a measure on $B(0, R)^c$ with density $\tilde{\pi}_k(x^*, z - x^*)$. Using assumption (A3) and the fact that $A \cap B(0, R) \subset B(0, R)^c \cap B(0, R) = \emptyset$, we have $\mu_k(A) < \infty$. Letting $t \rightarrow \infty$ and using Fatou's lemma on the left-hand side and the dominated convergence theorem on the right-hand side in (3.25), we have

$$\mathbb{P}_{y,j}(X(\tau_1) \in A, \Lambda(\tau_1) = k) \leq K \mathbb{E}_{y,j}(\tau_1) \mu_k(A) \leq K M_1 \mu_k(A),$$

where $M_1 = \sup_{(z,k) \in D \times \mathcal{M}} \mathbb{E}_{z,k}(\tau_1)$. Hence

$$\mathbb{P}_{\tau_1}(y, j, A, k) \leq K M_1 \mu_k(A) \quad \text{for all } k \in \mathcal{M}.$$

It follows that

$$\begin{aligned} \int_{B(0,R)^c} \sum_{k=1}^m V(z, k) \mathbb{P}_{\tau_1}(y, j, dz, k) &\leq K M_1 \sum_{k=1}^m \int_{B(0,R)^c} V(z, k) \tilde{\pi}_k(x^*, z - x^*) dz \\ &= K M_1 \sum_{k=1}^m \int_{B(-x^*, R)^c} V(z + x^*, k) \tilde{\pi}_k(x^*, z) dz. \end{aligned} \quad (3.26)$$

Choose R sufficiently large such that $B(0, R)^c - x^* \subset \{z \in \mathbb{R}^d : |z| > 1\}$. Since $\mathcal{G}$ satisfies condition (C1) with respect to $D \times \mathcal{M}$ and $x^* \notin D$, we have $\mathcal{G}V(x^*, k) < \infty$ for all $k \in \mathcal{M}$, which leads to the finiteness of the last term in (3.26). The desired inequality (3.24) then follows.

To continue, note that $\tau_{2n} \leq \sigma_E < \tau_{2n+1}$ implies $\bigcap_{k=0}^{n-1} A_k^c$. Then it follows from (3.20) that

$$\mathbb{P}_{x,i}(\tau_{2n} \leq \sigma_E < \tau_{2n+1}) \leq \mathbb{P}_{x,i}\left(\bigcap_{k=0}^{n-1} A_k^c\right) \leq (1 - \delta_1)^n.$$

Therefore we have

$$\begin{aligned} \mathbb{E}_{x,i} \sigma_E &= \sum_{n=0}^{\infty} \mathbb{E}_{x,i} \sigma_E \mathbf{1}_{[\tau_{2n} \leq \sigma_E < \tau_{2n+2}]} \\ &\leq \sum_{n=0}^{\infty} \mathbb{P}_{x,i}(\tau_{2n} \leq \sigma_E < \tau_{2n+2}) \mathbb{E}_{x,i} \tau_{2n+2} \\ &\leq \sum_{n=0}^{\infty} \mathbb{P}_{x,i}(\tau_{2n} \leq \sigma_E < \tau_{2n+1}) \sum_{k=0}^n \mathbb{E}_{x,i}(\tau_{2k+2} - \tau_{2k}) \\ &\leq \sum_{n=0}^{\infty} (1 - \delta_1)^n (n+1) M_2 < \infty. \end{aligned}$$

This completes the proof of the lemma. $\square$

In the next two lemmas, we show that, under the irreducibility assumption of the Q -matrix, if the process $(X(t), \Lambda(t))$ is recurrent (resp., positive recurrent) with respect to some cylinder $D \times \mathcal{M} \subset \mathbb{R}^d \times \mathcal{M}$, then it is recurrent (resp., positive recurrent) with respect to $D \times \{l\} \subset \mathbb{R}^d \times \mathcal{M}$ for any $l \in \mathcal{M}$, where $D \subset \mathbb{R}^d$ is any bounded domain.

Lemma 3.10 Suppose that the operator $\mathcal{G}$ is strictly irreducible on any bounded domain of $\mathbb{R}^d$. Let $D \subset \mathbb{R}^d$ be a bounded domain. Suppose that

$$\mathbb{P}_{y,j}(\sigma_D < \infty) = 1 \quad \text{for any } (y, j) \in D^c \times \mathcal{M}. \quad (3.27)$$

Then for any $(x, i) \in \mathbb{R}^d \times \mathcal{M}$ and $l \in \mathcal{M}$, we have

$$\mathbb{P}_{x,i}(\sigma_{D \times \{l\}} < \infty) = 1. \quad (3.28)$$

Proof Fix $l \in \mathcal{M}$. It suffices to prove (3.28) when $(x, i) \in D \times (\mathcal{M} \setminus \{l\})$ since the process $(X(t), \Lambda(t))$, starting from $(y, j) \in D^c \times \mathcal{M}$ will reach $D \times \mathcal{M}$ in finite time $\mathbb{P}_{y,j}$ -a.s. by (3.27).

Choose $\varepsilon > 0$ sufficiently small such that $B_0 \subset \overline{B_0} \subset B_1 \subset \overline{B_1} \subset D$, where

$$B_0 = B(x, \varepsilon) \quad \text{and} \quad B_1 = B(x, 2\varepsilon). \quad (3.29)$$

Define

$$\tau_0 = 0, \quad \tau_1 = \inf\{t > 0 : X(t) \in B_0^c\}, \quad (3.30)$$

and for $n = 1, 2, \dots$,

$$\tau_{2n} = \inf\{t \geq \tau_{2n-1} : X(t) \in B_1^c\}, \quad \tau_{2n+1} = \inf\{t \geq \tau_{2n} : X(t) \in B_0\}. \quad (3.31)$$

Note that (3.27), Theorem 3.3, and Lemma 3.8 imply that $\tau_n < \infty$ $\mathbb{P}_{x,i}$ -a.s.. Define

$$u(y, j) = \mathbb{P}_{y,j}(\sigma_{\overline{B_0} \times \{l\}} < \tau_{B_1}), \quad (y, j) \in \mathbb{R}^d \times \mathcal{M}.$$

Then $u(\cdot)$ is $\mathcal{G}$ -harmonic in $B_1 \times \mathcal{M} \setminus (\overline{B_0} \times \{l\})$, and hence in particular in $(B_1 \setminus \overline{B_0}) \times \mathcal{M}$. By Remark 2.1(c) and [6, Theorem 3.4], $u > 0$ on $(B_1 \setminus \overline{B_0}) \times \mathcal{M}$. Let H be a domain such that $\overline{B_0} \subset H \subset \overline{H} \subset B_1$. Define

$$v(y, j) = \mathbb{E}_{y,j} u(X(\tau_H), \Lambda(\tau_H)) \quad \text{for } (y, j) \in H \times \mathcal{M}.$$

Clearly, $v \geq 0$ is an $\mathcal{G}$ -harmonic function on $H \times \mathcal{M}$. By Remark 2.1(c) and the Harnack inequality [6, Theorem 4.7], there is a constant $\delta_2 \in (0, 1)$ such that $\inf_{(y,j) \in B_0 \times \mathcal{M}} v(y, j) \geq \delta_2$. For any $(y, j) \in B_0 \times \mathcal{M}$, by the definition of $u(y, j)$,

$$u(y, j) \geq \mathbb{E}_{y,j} [u(X(\tau_H), \Lambda(\tau_H)) \mathbf{1}_{\tau_H < \sigma_{B_0 \times \{l\}}}] + \mathbb{E}_{y,j} [\mathbf{1}_{\sigma_{B_0 \times \{l\}} < \tau_H}] \geq v(y, j).$$

Thus,

$$\inf_{B_0 \times \mathcal{M}} \mathbb{P}_{y,j}(\sigma_{\overline{B_0} \times \{l\}} < \tau_{B_1}) \geq \delta_2. \quad (3.32)$$

Redefine

$$A_0 = \{\Lambda(t) = l \text{ for some } t \in [\tau_0, \tau_2]\}, \quad (3.33)$$

and for $n = 1, 2, \dots$,

$$A_n = \{\Lambda(t) = l \text{ for some } t \in [\tau_{2n+1}, \tau_{2n+2}]\}. \quad (3.34)$$

The event A_0^c implies $\sigma_{\bar{B}_0 \times \{l\}} > \tau_{B_1}$. Hence we have by (3.32) that

$$\mathbb{P}_{x,i}(A_0^c) \leq \mathbb{P}_{x,i}(\sigma_{\bar{B}_0 \times \{l\}} > \tau_{B_1}) \leq 1 - \delta_2,$$

i.e., $\mathbb{P}_{x,i}(A_0^c) \leq 1 - \delta_2$. By virtue of the strong Markov property and induction on n ,

$$\mathbb{P}_{x,i}\left(\bigcap_{k=0}^n A_k^c\right) \leq (1 - \delta_2)^{n+1}. \quad (3.35)$$

Thus, we have

$$\begin{aligned} \mathbb{P}_{x,i}((X(t), \Lambda(t)) \notin D \times \{l\} \text{ for any } t \geq 0) \\ &\leq \mathbb{P}_{x,i}((X(t), \Lambda(t)) \notin B_1 \times \{l\} \text{ for any } t \geq 0) \\ &\leq \lim_{n \rightarrow \infty} \mathbb{P}_{x,i}\left(\bigcap_{k=0}^n A_k^c\right) \\ &\leq \lim_{n \rightarrow \infty} (1 - \delta_2)^{n+1} = 0. \end{aligned}$$

As a result,

$$\mathbb{P}_{x,i}(\sigma_{D \times \{l\}} = \infty) = \mathbb{P}_{x,i}((X(t), \Lambda(t)) \notin D \times \{l\} \text{ for any } t \geq 0) = 0,$$

i.e., $\mathbb{P}_{x,i}(\sigma_{D \times \{l\}} < \infty) = 1$, as desired. $\square$

Lemma 3.11 *Suppose that the operator $\mathcal{G}$ is strictly irreducible on any bounded domain of $\mathbb{R}^d$. Let $D \subset \mathbb{R}^d$ be a bounded domain and suppose that $\mathcal{G}$ satisfies condition (C1) with respect to $D \times \mathcal{M}$. Then for any $(x, i) \in \mathbb{R}^d \times \mathcal{M}$ and $l \in \mathcal{M}$, we have*

$$\mathbb{E}_{x,i} \sigma_{D \times \{l\}} < \infty. \quad (3.36)$$

Proof Fix any $l \in \mathcal{M}$. As in Lemma 3.10, it suffices to prove (3.36) when $(x, i) \in D \times (\mathcal{M} \setminus \{l\})$. Let the balls B and B_1 , stopping times $\tau_0, \tau_1, \dots$, and events $A_0, A_1, \dots$, as in (3.29)–(3.31), (3.33), and (3.34) in the proof of Lemma 3.10.

Observe that $\tau_{2n} \leq \sigma_{D,l} < \tau_{2n+2}$ implies $\bigcap_{k=0}^{n-1} A_k^c$. Hence we have by (3.32) that

$$\mathbb{P}_{x,i}(\tau_{2n} \leq \sigma_{D \times \{l\}} < \tau_{2n+2}) \leq \mathbb{P}_{x,i}\left(\bigcap_{k=0}^{n-1} A_k^c\right) \leq (1 - \delta_2)^n.$$

By virtue of Theorem 3.3, we have $M_1 := \sup_{(x,i) \in B \times \mathcal{M}} \mathbb{E}_{x,i} \tau_1 < \infty$. Similar to the proof of (3.23), we also obtain $M_2 := \sup_n \mathbb{E}_{x,i}(\tau_{2n+3} - \tau_{2n+1}) < \infty$. Therefore, we have

$$\begin{aligned} \mathbb{E}_{x,i} \sigma_{D,l} &= \sum_{n=0}^{\infty} \mathbb{E}_{x,i} \sigma_{D,l} \mathbf{1}_{[\tau_{2n} \leq \sigma_{D,l} < \tau_{2n+2}]} \\ &\leq \sum_{n=0}^{\infty} \mathbb{P}_{x,i}(\tau_{2n} \leq \sigma_{D,l} < \tau_{2n+2}) \mathbb{E}_{x,i} \tau_{2n+3} \\ &\leq \sum_{n=0}^{\infty} \mathbb{P}_{x,i}(\tau_{2n} \leq \sigma_{D,l} < \tau_{2n+2}) \left[\sum_{k=0}^n \mathbb{E}_{x,i}(\tau_{2k+3} - \tau_{2k+1}) + \tau_1 \right] \\ &\leq \sum_{n=0}^{\infty} (1 - \delta_1)^n (nM_2 + M_2 + M_1) < \infty. \end{aligned}$$

The proof of the lemma is complete. $\square$

By virtue of Lemmas 3.8 and 3.11, the process $(X(t), \Lambda(t))$ is recurrent with respect to some cylinder $D \times \mathcal{M}$ if and only if it is recurrent with respect to the product set $D \times \{l\}$ for any $l \in \mathcal{M}$. Also we have proved that the property of recurrence is independent of the choice of the bounded domain $D \subset \mathbb{R}^d$. Moreover, similar results also hold for positive recurrence. We summarize these into the following theorem.

Theorem 3.12 *Suppose that the operator $\mathcal{G}$ is strictly irreducible on any bounded domain of $\mathbb{R}^d$. Assume that $D, E \subset \mathbb{R}^d$ are bounded domains and $l \in \mathcal{M}$. The following assertions hold:*

- (i) *The process $(X(t), \Lambda(t))$ is recurrent with respect to $D \times \mathcal{M}$ if and only if it is recurrent with respect to $E \times \mathcal{M}$. The process $(X(t), \Lambda(t))$ is recurrent with respect to $E \times \mathcal{M}$, if and only if it is recurrent with respect to $E \times \{l\}$.*
- (ii) *Suppose that condition (C1) is satisfied. The process $(X(t), \Lambda(t))$ is positive recurrent with respect to $D \times \mathcal{M}$ if and only if it is positive recurrent with respect to $E \times \mathcal{M}$, and $(X(t), \Lambda(t))$ is recurrent with respect to $E \times \mathcal{M}$ if and only if it is recurrent with respect to $E \times \{l\}$.*

4 Existence of Invariant Probability Measures

In this section, we establish invariant probability measures of the process $(X(t), \Lambda(t))$ under the standing assumption (A1)–(A4) and condition (C1). That is, the process is

positive recurrent with respect to $U = E \times \{l\}$, where $E \subset \mathbb{R}^d$ is a bounded domain and $l \in \mathcal{M}$ is fixed throughout this section. By using a method introduced by Khasminskii [8], we characterize the invariant probability measure using an embedded Markov chain.

We work with a bounded domain $D \subset \mathbb{R}^d$ with $\overline{E} \subset D$. Let $\tau_0 = 0$ and redefine stopping times $\tau_1, \tau_2, \dots$, inductively as follows.

$$\begin{aligned}\tau_{2n+1} &= \inf\{t > \tau_{2n} : X(t) \notin D\}, \\ \tau_{2n+2} &= \inf\{t > \tau_{2n+1} : X(t) \in \overline{E}, \Lambda(t) = l\}.\end{aligned}\quad (4.1)$$

Now we can divide an arbitrary sample path of the process $(X(t), \Lambda(t))$ into cycles:

$$[\tau_0, \tau_2), [\tau_2, \tau_4), \dots, [\tau_{2n}, \tau_{2n+2}), \dots$$

Since the process $(X(t), \Lambda(t))$ is recurrent with respect to $E \times \{l\}$, by Theorem 3.3, all stopping times $\tau_0 < \tau_1 < \tau_2 < \tau_3 < \dots$ are finite a.s. By virtue of the recurrence of $(X(t), \Lambda(t))$, we may assume without loss of generality that $(X(0), \Lambda(0)) = (x, l) \in \overline{E} \times \{l\}$. It follows from the strong Markov property of the process $(X(t), \Lambda(t))$ that the sequence

$$(\tilde{X}_n, l) := (X(\tau_{2n}), l), n = 0, 1, \dots \text{ is a Markov chain in } \overline{E} \times \{l\}. \quad (4.2)$$

Therefore, the sequence $(\tilde{X}_n)$ is a Markov chain in $\overline{E}$.

To establish the existence of a unique stationary distribution of the Markov chain $(\tilde{X}_n)$, we first recall a result on invariant probability measures of Doob; see [4, Sect. 4.1].

Theorem 4.1 *Suppose that S is a compact metric space, that $\mathcal{B}(S)$ is the Borel σ -algebra, and that P is a linear operator from $B(S)$ into itself, where $B(S)$ is the Banach space of bounded and Borel measurable functions from S into $\mathbb{R}$, such that*

$$\begin{cases} \|P\phi\| \leq \|\phi\|, & \text{for all } \phi \in B(S), \\ P\phi = \phi & \text{if } \phi = 1, \end{cases} \quad (4.3)$$

where $\|\cdot\|$ denotes the sup norm. Assume that there exists a $\delta > 0$ such that

$$P\mathbf{1}_F(x) - P\mathbf{1}_F(y) \leq 1 - \delta \quad \text{for } x, y \in S, \quad F \in \mathcal{B}(S). \quad (4.4)$$

Then there exists a unique probability measure on $(S, \mathcal{B}(S))$ denoted by $\tilde{\mu}$ such that

$$\left| P^n \phi(x) - \int_S \phi d\tilde{\mu} \right| \leq K e^{-\rho n} \|\phi\|,$$

where $\rho = -\ln(1 - \delta)$ and $K = 2/(1 - \delta)$. The measure $\tilde{\mu}$ is the unique invariant probability measure under P on $(S, \mathcal{B}(S))$; that is, the unique probability measure on S such that

$$\int_S \phi d\tilde{\mu} = \int_S P\phi d\tilde{\mu} \quad \text{for } \phi \in B(S).$$

Lemma 4.2 Suppose that the operator $\mathcal{G}$ is strictly irreducible on D . The Markov chain $(\tilde{X}_n)$ has a unique invariant probability measure $\tilde{\mu}$ on $\bar{E}$ and for $g \in B(\bar{E})$, we have

$$\int_{\bar{E}} \mathbb{E}_{x,l} g(\tilde{X}_1) \tilde{\mu}(dx) = \int_{\bar{E}} g(x) \tilde{\mu}(dx). \quad (4.5)$$

Proof For each $\phi \in B(\bar{E})$ and $\phi \geq 0$, define

$$f(x, i) = \mathbb{E}_{x,i} \phi(X(\tau_2)), \quad (x, i) \in \mathbb{R}^d \times \mathcal{M}. \quad (4.6)$$

We claim that $f(\cdot)$ is $\mathcal{G}$ -harmonic in $D \times \mathcal{M}$. Indeed, let $\bar{f}(x, i) = \mathbb{E}_{x,i} \phi(X(\sigma_{\bar{E} \times \{l\}}))$ for $(x, i) \in D^c \times \mathcal{M}$. By the strong Markov property, we obtain

$$f(x, i) = \mathbb{E}_{x,i} \bar{f}(X(\tau_1), \Lambda(\tau_1)), \quad (x, i) \in \mathbb{R}^d \times \mathcal{M}.$$

Thus, $f(\cdot)$ is $\mathcal{G}$ -harmonic in $D \times \mathcal{M}$. Next we define

$$P\phi(x) = \mathbb{E}_{x,l} \phi(X(\tau_2)) \quad \text{for } \phi \in B(\bar{E}), x \in \bar{E}.$$

It follows from (4.6) that $P\phi(x) = f(x, l)$ for $x \in \bar{E}$. Using the Harnack inequality for $\mathcal{G}$ -harmonic functions (see Remark 2.1(c) and [6, Theorem 4.7]), there exists a positive constant κ_E such that

$$P\phi(x) \leq \kappa_E P\phi(y) \quad \text{for all } x, y \in E, \phi \in B(\bar{E}). \quad (4.7)$$

Note that κ_E is independent of $\phi \in B(\bar{E})$. It can be seen that $P : B(\bar{E}) \mapsto B(\bar{E})$ is a linear operator satisfying (4.3). To proceed, we need to verify (4.4).

In the contrary, assume that (4.4) were not true. Since $P\mathbf{1}_F(x) \in [0, 1]$ for all $x \in \bar{E}$ and $F \in \mathcal{B}(\bar{E})$, it follows from our assumption that there is a sequence $(x_k, y_k, F_k) \subset \bar{E} \times \bar{E} \times \mathcal{B}(\bar{E})$ such that

$$P\mathbf{1}_{F_k}(x_k) \rightarrow 1, \quad P\mathbf{1}_{F_k}(y_k) \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

This implies that

$$\frac{P\mathbf{1}_{F_k}(y_k)}{P\mathbf{1}_{F_k}(x_k)} \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

contradicting (4.7). Thus (4.4) holds, and the conclusion follows from Theorem 4.1. $\square$

Remark 4.3 Let τ be an $\mathcal{F}_t$ -stopping time with $\mathbb{E}_{x,i} \tau < \infty$ and let $f : \mathbb{R}^d \times \mathcal{M} \mapsto \mathbb{R}$ be a Borel measurable function. Then for any $(x, i) \in \mathbb{R}^d \times \mathcal{M}$,

$$\mathbb{E}_{x,i} \int_0^\tau f(X(t+s), \Lambda(t+s)) ds = \mathbb{E}_{x,i} \int_0^\tau \mathbb{E}_{X(s), \Lambda(s)} f(X(t), \Lambda(t)) ds. \quad (4.8)$$

Indeed, since τ is an $\mathcal{F}_t$ -stopping time, the function $\mathbf{1}_{[s < \tau]}$ is $\mathcal{F}_s$ -measurable. Therefore,

$$\begin{aligned} \mathbb{E}_{x,i} \int_0^\tau f(X(t+s), \Lambda(t+s)) ds &= \int_0^\infty \mathbb{E}_{x,i} \mathbb{E}_{x,i} (\mathbf{1}_{[s < \tau]} f(X(t+s), \Lambda(t+s)) | \mathcal{F}_s) ds \\ &= \mathbb{E}_{x,i} \int_0^\infty \mathbf{1}_{[s < \tau]} \mathbb{E}_{x,i} (f(X(t+s), \Lambda(t+s)) | \mathcal{F}_s) ds \\ &= \mathbb{E}_{x,i} \int_0^\tau \mathbb{E}_{X(s), \Lambda(s)} f(X(t), \Lambda(t)) ds, \end{aligned}$$

as desired.

Theorem 4.4 Suppose that the operator $\mathcal{G}$ is strictly irreducible on D and that condition (C1) holds. The process $(X(t), \Lambda(t))$ has an invariant probability measure $\tilde{\nu}(\cdot, \cdot)$.

Proof Let $A \in \mathcal{B}(\mathbb{R}^d)$ and $i \in \mathcal{M}$. Using the cycle times given in (4.1), we define

$$\nu(A, i) := \int_E \tilde{\mu}(dx) \mathbb{E}_{x,i} \int_0^{\tau_2} \mathbf{1}_{A \times \{i\}}(X(t), \Lambda(t)) dt, \quad (4.9)$$

where $\int_0^{\tau_2} \mathbf{1}_{A \times \{i\}}(X(t), \Lambda(t)) dt$ is the time spent by the path of $(X(t), \Lambda(t))$ in the set $(A \times \{i\})$ in the first cycle $[0, \tau_2)$. Then $\nu(\cdot, \cdot)$ is a positive measure defined on $\mathcal{B}(\mathbb{R}^d \times \mathcal{M})$.

We claim that for any bounded and Borel measurable function $g(\cdot, \cdot) : \mathbb{R}^d \times \mathcal{M} \mapsto \mathbb{R}$,

$$\sum_{j=1}^m \int_{\mathbb{R}^d} g(x, j) \nu(dx, j) = \int_E \tilde{\mu}(dx) \mathbb{E}_{x,i} \int_0^{\tau_2} g((X(s), \Lambda(s))) ds, \quad (4.10)$$

holds. Indeed, if $g(y, j) = \mathbf{1}_{A \times \{i\}}(y, j)$ for some $A \in \mathcal{B}(\mathbb{R}^d)$ and $i \in \mathcal{M}$, then (4.10) follows directly from (4.9). Similarly, we obtain that (4.10) holds for g being a simple function

$$g(y, j) = \sum_{k=1}^n c_k \mathbf{1}_{U_k}(y, j) \quad \text{where } U_k \in \mathcal{B}(\mathbb{R}^d \times \mathcal{M}), c_k \in \mathbb{R}.$$

Finally, if g is a bounded and Borel measurable function, (4.10) follows by approximating g by simple functions. Let f be a bounded and Borel measurable function on $\mathbb{R}^d \times \mathcal{M}$. It follows from (4.10) for $g(x, i) = \mathbb{E}_{x,i} f(X(t), \Lambda(t))$ and (4.8) that

$$\begin{aligned}
 & \sum_{i=1}^m \int_{\mathbb{R}^d} \mathbb{E}_{x,i} f(X(t), \Lambda(t)) v(dx, i) \\
 &= \int_{\bar{E}} \tilde{\mu}(dx) \mathbb{E}_{x,l} \int_0^{\tau_2} \mathbb{E}_{X(s), \Lambda(s)} f(X(t), \Lambda(t)) ds \\
 &= \int_{\bar{E}} \tilde{\mu}(dx) \mathbb{E}_{x,l} \int_0^{\tau_2} f(X(t+s), \Lambda(t+s)) ds \\
 &= \int_{\bar{E}} \tilde{\mu}(dx) \mathbb{E}_{x,l} \int_0^{\tau_2} f(X(u), \Lambda(u)) du \\
 &\quad + \int_{\bar{E}} \tilde{\mu}(dx) \mathbb{E}_{x,l} \int_{\tau_2}^{t+\tau_2} f(X(u), \Lambda(u)) du \\
 &\quad - \int_{\bar{E}} \tilde{\mu}(dx) \mathbb{E}_{x,l} \int_0^t f(X(u), \Lambda(u)) du. \tag{4.11}
 \end{aligned}$$

Using (4.5) with $g(x) = \mathbb{E}_{x,l} \int_{\tau_2}^{t+\tau_2} f(X(u), \Lambda(u)) du$, we obtain that

$$\begin{aligned}
 & \int_{\bar{E}} \tilde{\mu}(dx) \mathbb{E}_{x,l} \int_{\tau_2}^{t+\tau_2} f(X(u), \Lambda(u)) du \\
 &= \int_{\bar{E}} \tilde{\mu}(dx) \mathbb{E}_{x,l} \mathbb{E}_{\tilde{X}_1,l} \int_0^t f(X(u), \Lambda(u)) du \\
 &= \int_{\bar{E}} \tilde{\mu}(dx) \mathbb{E}_{x,l} \int_0^t f(X(u), \Lambda(u)) du. \tag{4.12}
 \end{aligned}$$

As noted in (4.2), $(\tilde{X}_n, l) = (X(2n), l)$, sampled from the switching jump diffusion, is a discrete time Markov chain. By (4.11), (4.12), and (4.10), we have

$$\sum_{i=1}^m \int_{\mathbb{R}^d} \mathbb{E}_{x,i} f(X(t), \Lambda(t)) v(dx, i) = \sum_{i=1}^m \int_{\mathbb{R}^d} f(x, i) v(dx, i).$$

In view of the proof of Lemma 3.11, we have

$$\sup_{x \in \bar{E}} \mathbb{E}_{x,l} \tau_2 < \infty. \tag{4.13}$$

By (4.13) and (4.9), for each $j \in \mathcal{M}$, we obtain

$$v(\mathbb{R}^d, j) \leq \int_{\bar{E}} \mathbb{E}_{x,l} [\tau_2] \tilde{\mu}(dx) < \infty.$$

Thus, the normalized measure

$$\tilde{v}(A, i) = \frac{v(A, i)}{\sum_{j=1}^m v(\mathbb{R}^d, j)}, \quad (A, i) \in \mathcal{B}(\mathbb{R}^d) \times \mathcal{M},$$

defines a desired invariant probability measure. This concludes the proof. $\square$

The following result further characterizes invariant probability measures of $(X(t), \Lambda(t))$.

Proposition 4.5 *Under conditions of Theorem 4.4, let $\tilde{\nu}$ be an invariant probability measure of $(X(t), \Lambda(t))$. Then for any bounded domain $D \subset \mathbb{R}^d$ and any $l \in \mathcal{M}$, $\tilde{\nu}(D, l) > 0$.*

Proof We argue by contradiction. Suppose that $\tilde{\nu}(D, l) = 0$. Let $x_0 \in D$ and $r \in (0, 1)$ such that $B(x_0, 2r) \in D$. By Remark 2.1(c) and [6, Proposition 4.1], there exists a constant $c_1 > 0$ such that

$$\mathbb{P}_{x,i} \left(\tau_{B(x,r)} \leq c_1 r^2 \right) \leq \frac{1}{2} \quad \text{for } (x, i) \in B(x_0, r) \times \mathcal{M}.$$

Let $t_0 = c_1 r^2$. It follows that

$$\begin{aligned} \mathbb{P}_{x,l} \left(\tau_{B(x_0, 2r)} \leq t_0 \right) &\leq \mathbb{P}_{x,l} \left(\tau_{B(x,r)} \leq t_0 \right) \\ &\leq \frac{1}{2}, \quad x \in B(x_0, r). \end{aligned} \quad (4.14)$$

Define

$$\tau^* = \inf\{t > 0 : \Lambda(t) \neq \Lambda(0)\}, \quad \tau_{B(x_0, 2r)}^* = \tau_{B(x_0, 2r)} \wedge \tau^*.$$

It can be seen that for any $x \in B(x_0, r)$, there is a number $c_2 > 0$ such that $\mathbb{P}_{x,l}(\tau^* > t_0) \geq \exp(-c_2 t_0)$. We deduce that

$$\begin{aligned} \mathbb{P}_{x,l}(\tau^* \leq t_0) &= 1 - \mathbb{P}_{x,l}(\tau^* > t_0) \\ &\leq 1 - \exp(-c_2 t_0) \\ &\leq c_2 t_0. \end{aligned}$$

Take a smaller r if needed to have $c_2 c_1 r^2 \leq 1/8$. Then we obtain $\mathbb{P}_{x,l}(\tau^* \leq t_0) \leq 1/8$ for all $x \in B(x_0, r)$. Together with (4.14), we arrive at

$$\begin{aligned} \mathbb{P}_{x,l} \left(\tau_{B(x_0, 2r)}^* \geq t_0 \right) &\geq 1 - \mathbb{P}_{x,l} \left(\tau_{B(x_0, 2r)} \leq t_0 \right) - \mathbb{P}_{x,l}(\tau^* \leq t_0) \\ &\geq \frac{1}{4}, \quad x \in B(x_0, r). \end{aligned} \quad (4.15)$$

Let $\mathcal{K} \subset \mathbb{R}^d$ be a compact set satisfying $\sum_{i=1}^m \tilde{\nu}(\mathcal{K}, i) > 1/2$. By Lemma 3.11 and its proof, there is a constant c_3 such that

$$\mathbb{E}_{x,i} \sigma_{B(x_0, r), l} \leq c_3 \quad \text{for any } (x, i) \in \mathcal{K} \times \mathcal{M}.$$

Then for $t_1 = 2c_3$,

$$\inf_{(x,i) \in \mathcal{K} \times \mathcal{M}} \mathbb{P}_{x,i}(\sigma_{B(x_0,r)}, l > t_1) \leq \frac{c_3}{t_1} = \frac{1}{2}. \quad (4.16)$$

Since $\tilde{\nu}$ is an invariant probability measure of $(X(t), \Lambda(t))$,

$$\begin{aligned} \sum_{i=1}^m \int_{\mathbb{R}^d} \mathbb{E}_{x,i} \mathbf{1}_{D \times \{l\}}(X(t), \Lambda(t)) \tilde{\nu}(dx, i) &= \sum_{i=1}^m \int_{\mathbb{R}^d} \mathbf{1}_{D \times \{l\}}(X(t), \Lambda(t)) \tilde{\nu}(dx, i) \\ &= \tilde{\nu}(D, l). \end{aligned}$$

Using (4.15) and (4.16), we obtain

$$\begin{aligned} \tilde{\nu}(D, l) &= \frac{1}{t_0 + t_1} \sum_{i=1}^m \int_0^{t_0+t_1} dt \int_{\mathbb{R}^d} \mathbb{E}_{x,i} \mathbf{1}_{D \times \{l\}}(X(t), \Lambda(t)) \tilde{\nu}(dx, i) \\ &\geq \frac{1}{t_0 + t_1} \sum_{i=1}^m \int_0^{t_0+t_1} dt \int_{\mathcal{K}} \mathbb{E}_{x,i} \mathbf{1}_{B(x_0, 2r) \times \{l\}}(X(t), \Lambda(t)) \tilde{\nu}(dx, i) \\ &= \frac{1}{t_0 + t_1} \sum_{i=1}^m \int_{\mathcal{K}} \tilde{\nu}(dx, i) \mathbb{E}_{x,i} \int_0^{t_0+t_1} \mathbf{1}_{B(x_0, 2r) \times \{l\}}(X(t), \Lambda(t)) dt \\ &\geq \frac{1}{t_0 + t_1} \sum_{i=1}^m \int_{\mathcal{K}} \tilde{\nu}(dx, i) \mathbb{E}_{x,i} \left[\mathbf{1}_{\{\sigma_{B(x_0,r)}, l \leq t_1\}} \mathbb{E}_{X(\sigma_{B(x_0,r)}, l)} \left[\mathbf{1}_{\{\tau_{B(x_0, 2r)}^* \geq t_0\}} \right] \right. \\ &\quad \left. \int_{\tau_{B(x_0,r)}^*}^{t_0+t_1} \mathbf{1}_{B(x_0, 2r) \times \{l\}}(X(t), \Lambda(t)) dt \right] \\ &\geq \frac{\sum_{i=1}^m \tilde{\nu}(\mathcal{K}, i)}{t_0 + t_1} \inf_{(x,i) \in \mathcal{K} \times \mathcal{M}} \mathbb{P}_{x,i}(\sigma_{B(x_0,r)}, l \leq t_1) \inf_{x \in B(x_0,r)} \mathbb{P}_{x,l}(\tau_{B(x_0, 2r)}^* \geq t_0) t_0 \\ &\geq \frac{t_0}{16(t_0 + t_1)}, \end{aligned}$$

contradicting to $\tilde{\nu}(D, l) = 0$. This proves the proposition. $\square$

5 Examples

This section is devoted to several examples. In the first two examples, we give sufficient conditions for the existence of Lyapunov functions needed in the theorems above and thus the main results of this paper apply. In the third example, we point out the effect of spatial jumping component to the asymptotic properties of regime-switching jump diffusions. In the last example, we mention the implication of the main results to controlled two-time-scale system.

Example 5.1 Assume that assumptions (A1)–(A4) hold and that the operator $\mathcal{G}$ is strictly irreducible. We further assume that there exists a constant $\delta \in (0, 2)$ such that for any $i \in \mathcal{M}$ and $N \in (0, \infty)$,

$$\frac{b'(x, i)x}{|x|^{2-\delta}} + N \int_{|z|>1} |z|^\delta \tilde{\pi}_i(x, z) dz \rightarrow -\infty \quad \text{as } |x| \rightarrow \infty. \quad (5.1)$$

Then condition (C1) is satisfied and the process $(X(t), \Lambda(t))$ is positive recurrent. To show this, let $f \in C^2(\mathbb{R}^d)$ be a nonnegative function such that $f(x) = |x|^\delta$ for $|x| \geq 1$. Define $V(x, i) = f(x)$ for $(x, i) \in \mathbb{R}^d \times \mathcal{M}$. Moreover, the gradient and Hessian matrix are given by

$$\nabla f(x) = \frac{\delta x}{|x|^{2-\delta}}, \quad \nabla^2 f(x) = \frac{\delta I_d}{|x|^{2-\delta}} - \frac{\delta(2-\delta)xx'}{|x|^{4-\delta}}, \quad |x| \geq 1,$$

where I_d is the $d \times d$ identity matrix. For $|x| \geq 1$, we have

$$\begin{aligned} \mathcal{G}V(x, i) &= \frac{1}{2} \text{tr}[a(x, i) \nabla^2 f(x)] + \frac{\delta b'(x, i)x}{|x|^{2-\delta}} \\ &\quad + \int_{|z| \leq 1} [f(x+z) - f(x) - \nabla f(x) \cdot z] \tilde{\pi}_i(x, z) (dz) \\ &\quad + \int_{|z| > 1} [f(x+z) - f(x)] \tilde{\pi}_i(x, z) (dz). \end{aligned} \quad (5.2)$$

Since $\nabla^2 f(\cdot)$ and $a(\cdot, i)$ are bounded, we have

$$\begin{aligned} &\frac{1}{2} \text{tr}[a(x, i) \nabla^2 f(x)] \\ &\quad + \int_{|z| \leq 1} [f(x+z) - f(x) - \nabla f(x) \cdot z] \tilde{\pi}_i(x, z) (dz) \leq c_1, \quad x \in \mathbb{R}^d, \end{aligned} \quad (5.3)$$

for some constant c_1 . Also for some constant c_2 , we have

$$\begin{aligned} &\int_{|z| > 1} [f(x+z) - f(x)] \tilde{\pi}_i(x, z) (dz) \\ &\leq c_2 \int_{|z| > 1} |z|^\delta \tilde{\pi}_i(x, z) dz \quad \text{for all } |x| \geq 1. \end{aligned} \quad (5.4)$$

It follows from (5.2), (5.3), and (5.4) that

$$\mathcal{G}V(x, i) \leq c_1 + \frac{\delta b'(x, i)x}{|x|^{2-\delta}} + c_2 \int_{|z| > 1} |z|^\delta \tilde{\pi}_i(x, z) dz.$$

Using (5.1), there exists a constant c_3 such that $\mathcal{L}V(x, i) \leq -1$ for $|x| \geq c_3$ and $i \in \mathcal{M}$. By virtue of Theorem 3.4, the process $(X(t), \Lambda(t))$ is positive recurrent.

Example 5.2 Suppose that assumptions (A1)–(A4) hold and $d = 1$. We assume that the operator $\mathcal{G}$ is strictly irreducible and $\int_{|z| \geq 1} |z| \tilde{\pi}_i(x, z) dz < \infty$ for any (x, i) .

Define

$$A(x, i) = b(x, i) + \int_{1 < |z| \leq |x|} z \tilde{\pi}_i(x, z) dz, \quad B(x, i) = \int_{|z| > |x|} |z| \tilde{\pi}_i(x, z) dz.$$

We claim that the process $(X(t), \Lambda(t))$ is positive recurrent if

$$\limsup_{|x| \rightarrow \infty} [A(x, i) \operatorname{sgn}(x) + B(x, i)] < 0. \quad (5.5)$$

Indeed, let $f \in C^2(\mathbb{R})$ be a nonnegative function such that $f(x) = |x|$ for $|x| > 1$ and $f(x) \leq |x|$ for $|x| \leq 1$. Define $V(x, i) = f(x)$ for $(x, i) \in \mathbb{R} \times \mathcal{M}$. For sufficiently large $|x|$, we claim that

$$\begin{aligned} I_1 &:= \int_{|z| \leq 1} [f(x+z) - f(x) - f'(x)z] \tilde{\pi}_i(x, z) (dz) = 0, \\ I_2 &:= \int_{1 < |z| \leq |x|} [f(x+z) - f(x)] \tilde{\pi}_i(x, z) (dz) \leq \operatorname{sgn}(x) \int_{1 < |z| \leq |x|} z \tilde{\pi}_i(x, z) dz, \\ I_3 &:= \int_{|z| > |x|} [f(x+z) - f(x)] \tilde{\pi}_i(x, z) (dz) \leq \int_{|z| > |x|} |z| \tilde{\pi}_i(x, z) dz. \end{aligned} \quad (5.6)$$

Indeed, suppose $x > 0$ and $|x|$ is large. It is clear that $I_1 = 0$. Note that for $z \in [-x, -x+1]$, $f(x+z) \leq x+z$ and for $z \in [-x+1, -1]$, $f(x+z) = x+z$. It follows that

$$\begin{aligned} I_2 &= \int_1^x [f(x+z) - f(x)] \tilde{\pi}_i(x, z) (dz) + \int_{-x}^{-1} [f(x+z) - f(x)] \tilde{\pi}_i(x, z) (dz) \\ &= \int_1^x z \tilde{\pi}_i(x, z) (dz) + \int_{-x}^{-x+1} [f(x+z) - f(x)] \tilde{\pi}_i(x, z) (dz) \\ &\quad + \int_{-x+1}^{-1} [f(x+z) - f(x)] \tilde{\pi}_i(x, z) (dz) \\ &= \int_1^x z \tilde{\pi}_i(x, z) (dz) + \int_{-x}^{-x+1} z \tilde{\pi}_i(x, z) (dz) + \int_{-x+1}^{-1} z \tilde{\pi}_i(x, z) (dz) \\ &= \int_{1 < |z| \leq |x|} z \tilde{\pi}_i(x, z) dz. \end{aligned}$$

Note also that for $z \in [-x-1, -x]$, $f(x+z) \leq -x-z$ and for $z \in [-\infty, -x-1]$, $f(x+z) = -x-z$. It follows that

$$\begin{aligned} I_3 &= \int_1^\infty [f(x+z) - f(x)] \tilde{\pi}_i(x, z) (dz) + \int_{-\infty}^{-1} [f(x+z) - f(x)] \tilde{\pi}_i(x, z) (dz) \\ &= \int_1^\infty z \tilde{\pi}_i(x, z) (dz) + \int_{-\infty}^{-x-1} [f(x+z) - f(x)] \tilde{\pi}_i(x, z) (dz) \\ &\quad + \int_{-x-1}^{-x} [f(x+z) - f(x)] \tilde{\pi}_i(x, z) (dz) \\ &= \int_1^\infty z \tilde{\pi}_i(x, z) (dz) \\ &\quad + \int_{-\infty}^{-x-1} (-z-2x) \tilde{\pi}_i(x, z) (dz) + \int_{-x-1}^{-x} (-z-2x) \tilde{\pi}_i(x, z) (dz) \\ &\leq \int_{|z|>|x|} |z| \tilde{\pi}_i(x, z) dz. \end{aligned}$$

Thus, (5.6) is proved for $x > 0$. Similar argument leads to (5.6) for $x < 0$. Detailed computations give us that $\mathcal{G}V(x, i) = A(x, i)\text{sgn}(x) + B(x, i)$ for sufficiently large $|x|$ and $i \in \mathcal{M}$. By (5.5), condition (C1) is satisfied. Thus, $(X(t), \Lambda(t))$ is positive recurrent.

Example 5.3 To illustrate the effect of the jump component on asymptotic properties of regime-switching jump diffusions, consider a Markov process $(X^0(t), \Lambda^0(t))$ associated with the operator

$$\begin{aligned} \mathcal{G}^0 f(x, i) &= \frac{1}{2} a(x, i) \frac{\partial^2 f(x, i)}{\partial x^2} + b(x, i) \frac{\partial f(x, i)}{\partial x} \\ &\quad + Q(x) f(x, \cdot)(i), \quad (x, i) \in \mathbb{R} \times \mathcal{M}, \end{aligned}$$

where $Q(\cdot)$ is strictly irreducible and assumptions (A1)–(A2) are satisfied. Then $(X^0(t), \Lambda^0(t))$ is the unique solution of the stochastic differential equations with regime-switching given by

$$\begin{aligned} dX^0(t) &= b(X^0(t), \Lambda(t)) dt + \sqrt{2a(X^0(t), \Lambda(t))} dw(t), \\ \mathbb{P}(\Lambda^0(t + \Delta t) = j | \Lambda^0(t) = i, X^0(s), \Lambda^0(s), s \leq t) &= q_{ij}(X^0(t)) \Delta t + o(\Delta t). \end{aligned}$$

Suppose $b(x, i) = 1$ for $x > 1$ and $b(x, i) = -1$ for $x < -1$. Using [23, Theorem 3.23], we can check that there exists $(y, l) \in \mathbb{R}^d \times \mathcal{M}$ and a bounded domain $D \subset \mathbb{R}^d$ such that $\mathbb{P}_{y,l}(\sigma_D = \infty) > 0$. Hence, $(X^0(t), \Lambda^0(t))$ is not recurrent (i.e., it is transient). It follows that $\mathbb{P}_{y,l}(|X^0(t)| \rightarrow \infty \text{ as } t \rightarrow \infty) = 1$ for all $(y, l) \in \mathbb{R} \times \mathcal{M}$; see [23, Chapter 3].

Next we consider a Markov process $(X(t), \Lambda(t))$ associated with the operator

$$\begin{aligned} \mathcal{G}f(x, i) &= \mathcal{G}^0 f(x, i) \\ &+ \int_{\mathbb{R}^d} (f(x+z, i) - f(x, i) - \nabla f(x, i) \cdot z \mathbf{1}_{\{|z|<1\}}) \tilde{\pi}_i(x, z) dz, \end{aligned}$$

where $\tilde{\pi}_i(x, z)$ satisfies assumptions (A3)–(A4) and (5.5). Ignoring the drift term $b(x, i)$, condition (5.5) means that for $x < 0$ and $|x|$ is sufficiently large, the intensity of large jumps to the right (i.e., $z > 0$) is much stronger than that to the left (i.e., $z < 0$); while for $x > 0$ and $|x|$ is sufficiently large, the intensity of large jumps to the left is much stronger than that to the right. By our preceding example, $(X(t), \Lambda(t))$ is positive recurrent. Thus, the jump component has certain stabilization effect on regime-switching Markov processes.

Example 5.4 In this example, we consider a controlled two-time-scale system. Because it is meant to be for demonstration purpose only, we will be very brief. Suppose that we have a fast varying switching jump diffusion $Z^\varepsilon(t) = (X^\varepsilon(t), \Lambda^\varepsilon(t))$ with the generator $\mathcal{L}^\varepsilon$ given by (2.1) (with the corresponding coefficients indexed by ε) such that $a^\varepsilon(x, i) = \hat{a}(x, i)/\varepsilon$, $b^\varepsilon(x, i) = \hat{b}(x, i)/\varepsilon$, $\pi_i^\varepsilon(x, dz) = \hat{\pi}_i(x, dz)/\varepsilon$, and $Q^\varepsilon(x) = \hat{Q}(x)/\varepsilon$, and that $\hat{b}(\cdot, \cdot)$, $\hat{a}(x, i)$, $\hat{\pi}_i(\cdot, \cdot)$, and $\hat{Q}(\cdot)$ satisfy conditions (A1)–(A4). Hence conditions (A1)–(A4) are all satisfied for this operator. Moreover, we suppose that $\hat{Q}(\cdot)$ is strictly irreducible on some bounded domain of $\mathbb{R}^d$.

Consider a second controlled process $Y^\varepsilon(\cdot)$ depending on $X^\varepsilon(\cdot)$ with operator given by

$$L_\zeta f(y) = [\nabla f(y)]' \int_U b_1(y, \zeta, c) m_t(dc) + \frac{1}{2} \text{tr}[\sigma_1(y, \zeta) \sigma_1'(y, \zeta) \nabla^2 f(y)],$$

where L_ζ denotes that the operator depends on $\zeta = (x, i)$ as a parameter, $m_t(\cdot)$ is a relaxed control representation (see [10, Chapter 3.2] for the notation), and U is a compact subset of $\mathbb{R}^{d_1}$ (for some positive integer d_1) representing the control set. Suppose that for each $i \in \mathcal{M}$ and each x and each c , $b_1(\cdot, \zeta, c)$ and $\sigma_1(\cdot, \zeta)$ are C^2 and satisfy the linear growth and Lipschitz condition. Both $X^\varepsilon(\cdot)$ and $Y^\varepsilon(\cdot)$ take values in $\mathbb{R}^d$. Note that $(X^\varepsilon(\cdot), \Lambda^\varepsilon(\cdot))$ is fast varying, whereas $Y^\varepsilon(\cdot)$ is slowly changing. Although it varies fast, $(X^\varepsilon(\cdot), \Lambda^\varepsilon(\cdot))$ does not blow up. With a time scale change, consider $X_0^\varepsilon(t) = X^\varepsilon(\varepsilon t)$ and $\Lambda_0^\varepsilon(t) = \Lambda^\varepsilon(\varepsilon t)$. As $\varepsilon \rightarrow 0$, $(X_0^\varepsilon(\cdot), \Lambda_0^\varepsilon(\cdot))$ is running on an infinite horizon $[0, \infty)$. Using the results of this paper, it can be shown that as $\varepsilon \rightarrow 0$, the fast process $(X_0^\varepsilon(\cdot), \Lambda_0^\varepsilon(\cdot))$ has an invariant measure $\hat{\nu}(\cdot, \cdot)$.

Consider a control problem in a finite horizon $[0, T_1]$ with $T_1 < \infty$. We wish to minimize an objective function

$$J_\zeta^\varepsilon(y, m^\varepsilon(\cdot)) = \mathbb{E} \int_0^{T_1} \int_U G(Y^\varepsilon(t), X^\varepsilon(t), \Lambda^\varepsilon(t), c) m_t^\varepsilon(dc),$$

where $G(\cdot)$ is a running cost function. In the above, we index the relaxed control m by ε to indicate that it is a feedback control. Using the approach in [10], we can show

that $(Y^\varepsilon(\cdot), m^\varepsilon(\cdot))$ converges weakly to $(Y(\cdot), m(\cdot))$, which is a controlled diffusion process with operator given by

$$\bar{L}f(y) = [\nabla f(y)]' \int_U \bar{b}_1(y, c) m_t(dc) + \frac{1}{2} \text{tr}[\bar{a}_1(y) \nabla^2 f(y)], \quad (5.7)$$

where

$$\begin{aligned} \bar{b}_1(y, c) &= \sum_{j \in \mathcal{M}} \int b_1(y, x, j, c) \widehat{\nu}(dx, j), \\ \bar{a}_1(y) &= \sum_{j \in \mathcal{M}} \int \sigma_1(y, x, j) \sigma_1'(y, x, j) \widehat{\nu}(dx, j). \end{aligned} \quad (5.8)$$

The limit cost function (the limit of J_ξ^ε) then becomes

$$\bar{J}(y, m(\cdot)) = \mathbb{E} \sum_{j \in \mathcal{M}} \int_0^{T_1} \int_U \int G(Y(t), x, j, c) m_t(dc) \widehat{\nu}(dx, j).$$

Then we can find the optimal control or near optimal control of the averaged system (5.7) and use it in the original system for approximation to get near optimality. Note that in [10], the controlled systems are given by using the corresponding differential equations, whereas here we present the system using the associated operators. They are in fact, equivalent. The key idea in [10] is to use controlled martingale to obtain the optimality. Here using the operators, the controlled martingales can be easily setup. In contrast to [10], the jump process in this paper can have σ -finite jump measure. The established invariant measure enables us to extend the results in [10] to jumps living in a non-compact set. Our results thus paved a way for further study on various control and optimization problems involving ergodicity.

6 Further Remarks

This paper focused on recurrence and ergodicity of a class of switching jump diffusion processes. Criteria for recurrence and positive recurrence were derived, and existence of invariant measure was obtained. The results obtained here will help future study in controlled dynamic systems, in particular, long-run average cost per unit time problems for controlled switching jump diffusions. Such study will have great impact on various applications involving optimal controls in an infinite horizon.

References

1. Arapostathis, A., Biswas, A., Caffarelli, L.: The Dirichlet problem for stable-like operators and related probabilistic representations. *Comm. Partial Differ. Equ.* **41**(9), 1472–1511 (2016)
2. Bass, R.F., Kassmann, M.: Harnack inequalities for non-local operators of variable order. *Trans. Am. Math. Soc.* **357**, 837–850 (2005)

3. Bass, R.F., Levin, D.A.: Harnack inequalities for jump processes. *Potential Anal.* **17**, 375–388 (2002)
4. Bensoussan, A.: *Perturbation Methods in Optimal Control*. Wiley, New York (1988)
5. Bhattacharya, R.N.: Criteria for recurrence and existence of invariant measures for multidimensional diffusions. *Ann. Probab.*, 541–553 (1978)
6. Chen, X., Chen, Z.-Q., Tran, K., Yin, G.: Properties of switching jump diffusions: maximum principles and Harnack inequalities, to appear in *Bernoulli* (2018)
7. Ghosh, M.K., Arapostathis, A., Marcus, S.I.: Ergodic control of switching diffusions. *SIAM J. Control Optim.* **35**(6), 1952–1988 (1997)
8. Khasminskii, R.: *Stochastic Stability of Differential Equations*, 2nd edn. Springer, Berlin (2012)
9. Komatsu, T.: Markov processes associated with certain integro-differential, *Osaka. J. Math.* **10**, 271–303 (1973)
10. Kushner, H.J.: *Weak Convergence Methods and Singularly Perturbed Stochastic Control and Filtering Problems*. Birkhäuser, Boston (1990)
11. Mao, X., Yuan, C.: *Stochastic Differential Equations with Markovian Switching*. Imperial College Press, London (2006)
12. Menaldi, J.L., Robin, M.: Invariant measure for diffusions with jumps. *Appl. Math. Optim.* **40**(1), 105–140 (1999)
13. Meyn, S.P., Tweedie, R.L.: Stability of Markovian processes III: Foster–Lyapunov criteria for continuous-time processes. *Adv. Appl. Probab.*, 518–548 (1993)
14. Pinsky, M., Pinsky, R.G.: Transience/recurrence and central limit theorem behavior for diffusions in random temporal environments. *Ann. Probab.*, 433–452 (1993)
15. Pinsky, R., Scheutzow, M.: Some remarks and examples concerning the transience and recurrence of random diffusions. *Annales de l’IHP Probabilités et statistiques* **28**(4), 519–536 (1992)
16. Sandric, N.: Long-time behavior of stable-like processes. *Stoch. Process. Appl.* **123**(4), 1276–1300 (2013)
17. Shao, J., Xi, F.: Strong ergodicity of the regime-switching diffusion processes. *Stoch. Process. Appl.* **123**(11), 3903–3918 (2013)
18. Shao, J., Xi, F.: Stability and recurrence of regime-switching diffusion processes. *SIAM J. Control Optim.* **52**(6), 3496–3516 (2014)
19. Wang, J.M.: Criteria for ergodicity of Lévy type operators in dimension one. *Stoch. Process. Appl.* **118**(10), 1909–1928 (2008)
20. Wang, J.M.: Martingale problems for switched processes. *Mathematische Nachrichten* **287**(10), 1186–1201 (2014)
21. Wee, I.S.: Stability for multidimensional jump-diffusion processes. *Stoch. Process. Appl.* **80**(2), 193–209 (1999)
22. Yin, G., Xi, F.: Stability of regime-switching jump diffusions. *SIAM J. Control Optim.* **48**(7), 4525–4549 (2010)
23. Yin, G., Zhu, C.: *Hybrid Switching Diffusions: Properties and Applications*. Springer, New York (2010)
24. Zhu, C., Yin, G.: Asymptotic properties of hybrid diffusion systems. *SIAM J. Control Optim.* **46**(4), 1155–1179 (2007)