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Explicit Milstein schemes with truncation for nonlinear stochastic differential equations: Convergence and its rate

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ABSTRACT

Although some implicit numerical procedures have been developed to treat high nonlinearity, the question whether one can use explicit schemes to achieve convergence rate similar to that of Milstein's procedure remained open. This brings us to the current work that focuses on numerical solutions of stochastic differential equations using explicit schemes. Our main goals are to obtain order one convergence in the second moment in a finite-time interval. In contrast to the implicit schemes, explicit schemes are advantageous, easily implementable, and computationally less intensive. To overcome the difficulties due to super-linear growth of the coefficients, a truncation device is used in our algorithm. In addition to reaching aforementioned goals in the analysis part, numerical examples are provided to demonstrate our results.

1. Introduction

This work aims to develop efficient numerical algorithms for highly nonlinear stochastic differential equations (SDEs) using *explicit* schemes. Our main goals are to obtain order-one convergence in the second moment in a finite-time interval. Because of the need in applications, such problems have received much attention in recent years. The main challenge that we face stems from the nonlinearity of the coefficients. The progress of numerical schemes using Milstein's method has much improved the convergence rates compared with that of the Euler–Maruyama (EM) methods. Although some implicit numerical procedures have been developed to combat the high nonlinearity, whether one can use explicit procedures to achieve higher-order convergence similar to that of Milstein's procedure remained an open question. Focusing on explicit schemes, the current paper settles these issues. It is shown that an explicit algorithm with built-in truncation device can be used to improve the rates of convergence and to achieve the same convergence rate as that of the Milstein scheme.

To begin, consider a d-dimensional stochastic differential equation of the form

$$dx(t) = f(x(t))dt + g(x(t))dB(t), \qquad t \geq 0, \qquad x(0) = x_0 \in \mathbb{R}^d, \tag{1.1}$$

where $B(t) := (B_1(t), \ldots, B_m(t))'$ is an m-dimensional standard Brownian motion, $f : \mathbb{R}^d \mapsto \mathbb{R}^d$, $g : \mathbb{R}^d \mapsto \mathbb{R}^{d\times m}$, with $f(x) = (f_1(x), \ldots, f_d(x))'$, $g(x) = (g_{ij}(x))_{d\times m}$, and z' denotes the transpose of a matrix or a vector z. To prove the convergence

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of the algorithm, we require f, g, and $L^i g_j$ (with $L^i g_j$ defined in Section 3) being locally Lipschitz. When we establish the rates of convergence, we need additional conditions, namely, these functions satisfy the polynomial growth conditions and the partial derivatives of f and g with respect to x up to the second order are continuous.

Stochastic differential equations have been used in a wide range of applications, including physics, control theory and decision making, biological and ecological models, financial market modeling and analysis, cyber–physical systems, and social networks among others; see [1] and references therein among others. More often than not, one has to deal with high nonlinearity. As a result, analytic solutions are difficult to obtain; numerical methods are often the only viable alternative. A method commonly used for solving stochastic differential equations is the Euler–Maruyama method. It is well known that EM method has order 1/2 convergence rate [2,3] (comparing also the pathwise convergence rate in [4]), whereas the Milstein scheme [5–7] has order one convergence rate. To obtain the convergence rates, linear growth was deemed crucial in the proofs. In practice, there are many cases that the linear growth condition is violated. Take for instance, the stochastic Ginzburg–Landau equation from statistical physics, which arises in the study of phase transitions [5,8],

$$dx(t) = [(\eta + \frac{1}{2}\sigma^2)x(t) - \vartheta x^3(t)]dt + \sigma x(t)dB(t), \qquad x(0) = x_0 > 0, \tag{1.2}$$

and the stochastic risk-adjusted volatility model used in finance given by [9, p. 83]

$$dv(t) = (\beta_0 - \beta_1 v(t))dt + \xi v^2(t)dB(t). \tag{1.3}$$

Neither of these equations verifies the linear growth of the coefficients. Several counterexamples for nonlinear SDEs revealed that the mean deviations of the EM method in finite time diverge to infinity [8].

Recently, by using a tamed scheme, Wang–Gan [10] modified the Milstein method and obtained order one convergence rate. The main assumptions used are the drift satisfying one-sided Lipschitz continuity and polynomial growth rate, and the Taylor expansion terms g and $L^i g_j$ being globally Lipschitz continuous. The truncated Milstein scheme was developed further for nonlinear SDEs in [11], but only convergence rate being less than order one was obtained. To design algorithms with better convergence rate, several implicit Milstein schemes were developed for SDEs with global Lipschitz coefficients; see for examples, [12–16]. Nevertheless, it is known that more computational efforts and costs are required using the implicit approximation in each iteration. Thus easily implementable explicit methods for nonlinear SDEs are more desirable. By introducing more correction terms, the tamed Milstein method with order 1.5 convergence rate was given [17]. In [18], a tamed Euler scheme was given for switching diffusions. Related work [19] considered Milstein-type schemes for switching diffusions, in which the emphasis was on treating the interactions of continuous and the switching components using certain arguments from martingale theory together with growth and smooth conditions on the coefficients.

Higher-order approximations are desirable, particularly, in the multilevel Monte-Carlo (MLMC) setting due to the need of many simulations with large discretization time step; see [20–22]. Aiming at designing an explicit numerical scheme for highly nonlinear SDEs, in this paper, we construct a class of easily implementable numerical algorithms such that the numerical solutions approximate the solutions of the highly nonlinear SDEs with better convergence rate (in the sense of strong convergence). Using a truncation mechanism (see [23]), we develop modified Milstein scheme with appropriate truncation mappings to avoid possible large excursions due to the nonlinear drift, diffusion terms, and the Brownian motions. We prove that the modified Milstein scheme preserves order one convergence rate in the sense of strong convergence for nonlinear SDEs with smooth coefficients with polynomial growth. Finally, some numerical experiments are presented to support our theoretical results. We remark that these proposed techniques can further be used to construct higher-order explicit stable schemes by adding more Taylor expansion terms.

The rest of the paper is organized as follows. Section 2 gives some notation and results on the solutions of the original stochastic differential equations. Section 3 constructs an explicit scheme and demonstrates its convergence. Section 4 provides the rate of convergence. Section 5 provides some numerical experiments to support our theoretical results. Finally, Section 6 gives further remarks to conclude the paper.

2. Preliminary

Throughout this paper, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (that is, it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $B(t) = (B_1(t), \dots, B_m(t))'$ be an m-dimensional standard Brownian motion defined on the probability space, and $|\cdot|$ denote both the Euclidean norm in $\mathbb{R}^d$ and the Frobenius norm in $\mathbb{R}^{d\times m}$. Use C to denote a generic positive constant whose value may change for different appearances. Moreover, let $C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R})$ denote the family of all nonnegative functions $V(x, t)$ on $\mathbb{R}^d \times \mathbb{R}_+$ whose partial derivatives with respect to x up to the second order and the partial derivative with respect to t are continuous. For the regularity and pth moment boundedness of the exact solution we make the following assumption.

Assumption 1. There exists a symmetric positive definite matrix $Q \in \mathbb{R}^{d\times d}$, $p > 0$, and α satisfying

$$\limsup_{|x|\to\infty} \frac{(1 + x'Qx)(2x'Qf(x) + \mathrm{trace}[g'(x)Qg(x)]) + (p-2)|x'g(x)|^2}{|x|^4} \leq \alpha. \tag{2.1}$$

Remark 2.1. For some constant $\tilde{\alpha}$, Assumption 1 is equivalent to

$$(1+x'Qx)(2x'Qf(x)+\mathrm{trace}\big[g'(x)Qg(x)\big])+(p-2)|x'Qg(x)|^2 \le \tilde{\alpha}(1+|x|^4). \tag{2.2}$$

If $Q=I_{d\times d}$ (identity matrix) (as in [23]), it becomes $(1+|x|^2)(2x'f(x)+|g(x)|^2)+(p-2)|x'g(x)|^2 \le \tilde{\alpha}(1+|x|^4)$. Furthermore, if there is a constant $p\ge 2$ such that $2x'f(x)+(p-1)|g(x)|^2 \le \tilde{\alpha}(1+|x|^2)$, which is the assumption usually used for the global existence of exact solution, for example, [3, p. 58, Theorem 3.5], then Assumption 1 holds. With the assumption above, we have the regularity and moment boundedness of the exact solution as given in the next lemma.

Lemma 2.1 *([23, pp. 851–852, Theorem 2.3, Lemma 2.5]). Let Assumption 1 hold. Then SDE (1.1) with any initial value $x_0 \in \mathbb{R}^d$ has a unique regular solution $x(t)$. For each positive integer $N>|x_0|$, define*

$$\tau_N =: \inf\{t\in[0,+\infty) : |x(t)|\ge N\}. \tag{2.3}$$

Then for any $\mathcal{F}_t$-stopping time ν satisfying $0\le\nu\le\tau_N$ a.s.,

$$\sup_{0\le t\le T}\mathbb{E}|x(t)|^p \le C, \quad \sup_{0\le t\le T}\mathbb{E}|x(t\wedge\nu)|^p\le C, \quad \forall\, T\ge 0, \tag{2.4}$$

where C is a generic positive constant dependent on T, p, and x_0, and independent of N. Moreover,

$$\mathbb{P}\{\tau_N\le T\}\le \frac{C}{N^p}. \tag{2.5}$$

The proof is an application of Dynkin's formula (see [23, pp. 851–852, Theorem 2.3, Lemma 2.5]); the details are omitted.

3. Explicit scheme and convergence in *p*th moment

In this section, our aim is to construct an easily implementable numerical algorithm and establish its strong convergence under Assumption 1. For convenience, we define

$$g_j(x) := (g_{1,j}(x),\ldots,g_{d,j}(x))', \qquad L^i g_j(x) := (\nabla_x g_j(x))g_i = \sum_{l=1}^{d} g_{l,i}\frac{\partial g_j(x)}{\partial x_l}, \qquad i,j=1,\ldots,m.$$

To construct the numerical scheme, we first estimate the growth rate of f and g. Choose a strictly increasing continuous function $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ such that $\varphi(r)\to\infty$ as $r\to\infty$ and

$$\sup_{0<|x|\vee|y|\le r}\frac{|f(x)-f(y)|}{|x-y|}\vee\frac{|g(x)-g(y)|^2}{|x-y|^2}\vee\frac{|L^i g_j(x)-L^i g_j(y)|}{|x-y|}\le\varphi(r), \qquad \forall\, r>0,\ i,j=1,\ldots,m. \tag{3.1}$$

Due to the local Lipschitz continuity of functions $f, g, L^i g_j$, the function φ is well defined. Denote by φ^{-1} the inverse function of φ. Then $\varphi^{-1}:(\varphi(0),\infty)\to\mathbb{R}_+$ is a strictly increasing continuous function. We choose a constant $K\ge |f(0)|\vee|g(0)|^2\vee|L^i g_j(0)|\vee\varphi(1)$. For a given $\triangle\in(0,1]$, let us define the truncation mapping $\pi_\triangle:\mathbb{R}^d\to\mathbb{R}^d$ by

$$\pi_\triangle(x)=\big(|x|\wedge\varphi^{-1}(K\triangle^{-\varrho})\big)\frac{x}{|x|}, \tag{3.2}$$

where $\varrho\in(0,1/2]$, and we use the convention $\frac{x}{|x|}=0$ when $x=0$. Clearly, $\forall x,y\in\mathbb{R}^d,\ i,j=1,\ldots,m$,

$$\begin{aligned} |f(\pi_\triangle(x))-f(\pi_\triangle(y))| &\le K\triangle^{-\varrho}|\pi_\triangle(x)-\pi_\triangle(y)|,\\ |g(\pi_\triangle(x))-g(\pi_\triangle(y))| &\le K^{\frac12}\triangle^{-\frac{\varrho}{2}}|\pi_\triangle(x)-\pi_\triangle(y)|,\\ |L^i g_j(\pi_\triangle(x))-L^i g_j(\pi_\triangle(y))| &\le K\triangle^{-\varrho}|\pi_\triangle(x)-\pi_\triangle(y)|. \end{aligned} \tag{3.3}$$

Remark 3.1. If (3.1) holds with $\varphi(r)\equiv C$ for any $r>0$, define $\varphi^{-1}(r)=\infty$. Then $\pi_\triangle(x)=x$, and (3.3) still holds. This is the special case that f, g, and $L^i g_j$ are all globally Lipschitz continuous.

Next we construct our numerical algorithm to approximate the exact solution of SDE (1.1). For any given step-size $\triangle\in(0,1]$, define

$$\begin{cases} \tilde{y}_0 &= x_0,\\ y_k &= \pi_\triangle(\tilde{y}_k),\\ \tilde{y}_{k+1} &= y_k+f(y_k)\triangle+\sum_{j=1}^m g_j(y_k)\triangle B_j(k)+\sum_{i,j=1}^m L^i g_j(y_k)I_{i,j}(k), \end{cases} \tag{3.4}$$

where $t_k=k\triangle$, $\triangle B(k):=B(t_{k+1})-B(t_k)$ and $I_{i,j}(k)=\int_{t_k}^{t_{k+1}}\int_{t_k}^{s}dB_i(u)dB_j(s)$. We refer to the numerical method as a *truncated Milstein scheme*.

Remark 3.2. Similar to the classical Milstein scheme for SDEs, we simply assume that the terms $I_{i,j}(k)$ can be simulated. In some applications, the considered SDEs have commutative diffusion term (that is, $L^i g_j = L^j g_i$, $i, j = 1, \ldots, m$). Thanks to the commutativity that $I_{i,j}(k) + I_{j,i}(k) = \triangle B_i(k)\triangle B_j(k)$ for $i \neq j$, the second equation of our scheme takes a simple form as

$$\tilde{y}_{k+1} = y_k + f(y_k)\triangle + \sum_{j=1}^{m} g_j(y_k)\triangle B_j(k) + \frac{1}{2}\sum_{i,j=1}^{m} L^i g_j(y_k)(\triangle B_i(k)\triangle B_j(k) - \delta_{i,j}\triangle),$$

where $\delta_{i,j} = 1$ for $i = j$, otherwise $\delta_{i,j} = 0$.

The numerical solutions y_k are obtained by truncating the intermediate terms $\tilde{y}_k$ according to the growth rate of the drift and diffusion coefficients to avoid their possible large excursions due to the nonlinearities of the coefficients and the Brownian motion increments. Consequently, from (3.3) we have the following nice linear property

$$|f(y_k)| \leq K\triangle^{-\varrho}(1 + |y_k|), \quad |g(y_k)| \leq K^{\frac{1}{2}}\triangle^{-\frac{\varrho}{2}}(1 + |y_k|), \quad |L^i g_j(y_k)| \leq K\triangle^{-\varrho}(1 + |y_k|), \tag{3.5}$$

for all $k \geq 0$, $1 \leq i, j \leq m$. Moreover, the truncated Milstein method is an explicit one, so it is easy to be used. To proceed, we define the numerical procedure by

$$y(t) := y_k, \qquad \forall t \in [t_k, t_{k+1}), \tag{3.6}$$

and

$$\bar{y}(t) = y_k + f(y_k)(t - t_k) + \sum_{j=1}^{m} g_j(y_k)(B_j(t) - B_j(t_k)) + \sum_{i,j=1}^{m} L^i g_j(y_k) I_{i,j}(t_k, t), \tag{3.7}$$

$\forall t \in [t_k, t_{k+1})$, where $I_{i,j}(s, t) := \int_s^t \int_{t_{n_s}}^r dB_i(u)dB_j(r)$ and $n_t := [t/\triangle]$.

In order to estimate the pth moment of the numerical solution $y(t)$, we cite an inequality from [24].

Lemma 3.1 (*[24, p.2929, Lemma 3.3]*)**.** *For any given positive constant p with $2i < p \leq 2(i + 1)$ (i is a nonnegative integer), the following inequality*

$$(1 + u)^{\frac{p}{2}} \leq 1 + \frac{p}{2}u + \frac{p(p-2)}{8}u^2 + u^3 P_i(u) \tag{3.8}$$

holds for any $u > -1$, where $P_i(u)$ represents a ith-order polynomial of u which coefficients depend only on p.

Lemma 3.2. *Under Assumption 1, the truncation scheme defined by (3.4) satisfies that*

$$\sup_{0<\triangle\leq 1}\sup_{0\leq t\leq T} \mathbb{E}|y(t)|^p \leq C, \qquad \forall\, T > 0, \tag{3.9}$$

where p is given by Assumption 1.

Proof. For any integer $k \geq 0$, we have

$$\begin{aligned}
&\tilde{y}'_{k+1}Q\tilde{y}_{k+1}\\
&= y'_k Q y_k + 2y'_k Q\sum_{j=1}^{m} g_j(y_k)\triangle B_j(k) + 2\triangle y'_k Q f(y_k)\\
&\;+(\sum_{j=1}^{m} g_j(y_k)\triangle B_j(k))'Q\sum_{j=1}^{m} g_j(y_k)\triangle B_j(k) + 2y'_k Q\sum_{i,j=1}^{m} L^i g_j(y_k)I_{i,j}(k)\\
&\;+2\triangle f'(y_k)Q\sum_{j=1}^{m} g_j(y_k)\triangle B_j(k) + \triangle^2 f'(y_k)Qf(y_k)\\
&\;+2\triangle f'(y_k)Q\sum_{i,j=1}^{m} L^i g_j(y_k)I_{i,j}(k) + 2(\sum_{j=1}^{m} g_j(y_k)\triangle B_j(k))'Q(\sum_{i,j=1}^{m} L^i g_j(y_k)I_{i,j}(k))\\
&\;+(\sum_{i,j=1}^{m} L^i g_j(y_k)I_{i,j}(k))'Q\sum_{i,j=1}^{m} L^i g_j(y_k)I_{i,j}(k).
\end{aligned}$$

Then

$$(1 + \tilde{y}'_{k+1}Q\tilde{y}_{k+1})^{\frac{p}{2}} = (1 + y'_k Q y_k)^{\frac{p}{2}}(1 + \sum_{i=1}^{9}\xi_{k,i})^{\frac{p}{2}}, \tag{3.10}$$

where

$$
\begin{aligned}
\xi_{k,1} &= (1+y_k'Qy_k)^{-1}2y_k'Q\sum_{j=1}^{m} g_j(y_k)\triangle B_j(k),\\
\xi_{k,2} &= (1+y_k'Qy_k)^{-1}2\triangle y_k'Qf(y_k),\\
\xi_{k,3} &= (1+y_k'Qy_k)^{-1}(\sum_{j=1}^{m} g_j(y_k)\triangle B_j(k))'Q\sum_{j=1}^{m} g_j(y_k)\triangle B_j(k),\\
\xi_{k,4} &= (1+y_k'Qy_k)^{-1}2y_k'Q\sum_{i,j=1}^{m} L^i g_j(y_k)I_{i,j}(k),\\
\xi_{k,5} &= (1+y_k'Qy_k)^{-1}2\triangle f'(y_k)Q\sum_{j=1}^{m} g_j(y_k)\triangle B_j(k),\\
\xi_{k,6} &= (1+y_k'Qy_k)^{-1}\triangle^2 f'(y_k)Qf(y_k),\\
\xi_{k,7} &= (1+y_k'Qy_k)^{-1}2\triangle f'(y_k)Q\sum_{i,j=1}^{m} L^i g_j(y_k)I_{i,j}(k),\\
\xi_{k,8} &= (1+y_k'Qy_k)^{-1}2(\sum_{j=1}^{m} g_j(y_k)\triangle B_j(k))'Q(\sum_{i,j=1}^{m} L^i g_j(y_k)I_{i,j}(k)),\\
\xi_{k,9} &= (1+y_k'Qy_k)^{-1}(\sum_{i,j=1}^{m} L^i g_j(y_k)I_{i,j}(k))'Q\sum_{i,j=1}^{m} L^i g_j(y_k)I_{i,j}(k).
\end{aligned}
$$

For the given $p > 0$, there exists a unique nonnegative integer u such that $2u < p \le 2(u+1)$. Thanks to Lemma 3.1 and (3.10), we have

$$
\begin{aligned}
&\mathbb{E}\left((1+\tilde{y}_{k+1}'Q\tilde{y}_{k+1})^{\frac{p}{2}}|\mathcal{F}_{t_k}\right)\\
&\le (1+y_k'Qy_k)^{\frac{p}{2}}\left[1+\frac{p}{2}\mathbb{E}\left(\sum_{i=1}^{9}\xi_{k,i}|\mathcal{F}_{t_k}\right)+\frac{p(p-2)}{8}\mathbb{E}\left((\sum_{i=1}^{9}\xi_{k,i})^2|\mathcal{F}_{t_k}\right)\right.\\
&\qquad\left.+\sum_{j=0}^{u} C_j\mathbb{E}\left((\sum_{i=1}^{9}\xi_{k,i})^{j+3}|\mathcal{F}_{t_k}\right)\right],
\end{aligned}
\tag{3.11}
$$

where $C_0, C_2, \dots, C_u$ are constants depending only on p. The fact that $\triangle B(k)$ is independent of $\mathcal{F}_{t_k}$ implies that

$$
\begin{aligned}
&\mathbb{E}(\triangle B_i(k)|\mathcal{F}_{t_k}) = 0,\ \mathbb{E}((\triangle B_i(k))^2|\mathcal{F}_{t_k}) = \triangle,\ \mathbb{E}(\triangle B_i(k)\triangle B_j(k)|\mathcal{F}_{t_k}) = 0,\ \forall i\ne j,\\
&\mathbb{E}(I_{i,j}(k)|\mathcal{F}_{t_k}) = 0,\ \mathbb{E}((I_{i,j}(k))^2|\mathcal{F}_{t_k}) = \triangle^2/2,\ \mathbb{E}(\triangle B_l(k)I_{i,j}(k)|\mathcal{F}_{t_k}) = 0,\\
&\qquad\qquad i,j,l = 1,\dots,m,\ \forall k\ge 0.
\end{aligned}
\tag{3.12}
$$

This implies

$$
\mathbb{E}(\xi_{k,1}|\mathcal{F}_{t_k}) = \mathbb{E}(\xi_{k,4}|\mathcal{F}_{t_k}) = \mathbb{E}(\xi_{k,5}|\mathcal{F}_{t_k}) = \mathbb{E}(\xi_{k,7}|\mathcal{F}_{t_k}) = \mathbb{E}(\xi_{k,8}|\mathcal{F}_{t_k}) = 0,
\tag{3.13}
$$

and

$$
\begin{aligned}
\mathbb{E}(\xi_{k,2}|\mathcal{F}_{t_k}) &= (1+y_k'Qy_k)^{-1}2y_k'Qf(y_k)\triangle,\\
\mathbb{E}(\xi_{k,3}|\mathcal{F}_{t_k}) &= (1+y_k'Qy_k)^{-1}\mathrm{trace}[g'(y_k)Qg(y_k)]\triangle.
\end{aligned}
\tag{3.14}
$$

Because of the symmetric positive definite matrix Q, one observes from (3.5) that

$$
\begin{aligned}
\mathbb{E}(\xi_{k,6}|\mathcal{F}_{t_k}) &\le (1+y_k'Qy_k)^{-1}\triangle^2|Q||f(y_k)|^2\\
&\le (1+y_k'Qy_k)^{-1}K^2\triangle^{2-2\varrho}|Q|(1+|y_k|)^2 \le C\triangle,
\end{aligned}
\tag{3.15}
$$

and

$$
\begin{aligned}
\mathbb{E}(\xi_{k,9}|\mathcal{F}_{t_k}) &\le \frac{\triangle^2}{2}(1+y_k'Qy_k)^{-1}|Q|\sum_{i,j=1}^{m}|L^i g_j(y_k)|^2\\
&\le \frac{\triangle^2}{2}(1+y_k'Qy_k)^{-1}|Q|\sum_{i,j=1}^{m}K^2\triangle^{-2\varrho}(1+|y_k|)^2 \le C\triangle.
\end{aligned}
\tag{3.16}
$$

Combining (3.13)–(3.16) yields

$$\mathbb{E}((\sum_{i=1}^{9}\xi_{k,i})|\mathcal{F}_{t_k}) \le (1+y_k'Qy_k)^{-1}(2y_k'Qf(y_k)+\mathrm{trace}[g'(y_k)Qg(y_k)])\triangle + C\triangle. \tag{3.17}$$

Noting that

$$\mathbb{E}((\triangle B_i(k))^{2n-1}|\mathcal{F}_{t_k}) = 0, \quad \text{and} \quad \mathbb{E}(|\triangle B_i(k)|^n|\mathcal{F}_{t_k}) \le C\triangle^{n/2}, \quad \forall n \ge 1, i = 1, \dots, m, \tag{3.18}$$

we have that

$$\mathbb{E}((\xi_{k,1})^2|\mathcal{F}_{t_k}) = 4(1+y_k'Qy_k)^{-2}|y_k'Qg(y_k)|^2\triangle. \tag{3.19}$$

One observes that

$$\begin{aligned}\mathbb{E}((\xi_{k,1}\xi_{k,2})|\mathcal{F}_{t_k}) &= \mathbb{E}((\xi_{k,1}\xi_{k,3})|\mathcal{F}_{t_k}) = \mathbb{E}((\xi_{k,1}\xi_{k,4})|\mathcal{F}_{t_k})\\ &= \mathbb{E}((\xi_{k,1}\xi_{k,6})|\mathcal{F}_{t_k}) = \mathbb{E}((\xi_{k,1}\xi_{k,7})|\mathcal{F}_{t_k}) = 0.\end{aligned}$$

Furthermore, owing to (3.5), we have

$$\begin{aligned}\mathbb{E}((\xi_{k,1}\xi_{k,5})|\mathcal{F}_{t_k}) &\le 4(1+y_k'Qy_k)^{-2}|y_k||Q|^2|f(y_k)||g(y_k)|^2\triangle^2\\ &\le 4(1+y_k'Qy_k)^{-2}K^2|Q|^2|y_k|(1+|y_k|)^3\triangle^{2-2\varrho} \le C\triangle,\end{aligned}$$

and by the Hölder inequality,

$$\begin{aligned}&\mathbb{E}((\xi_{k,1}\xi_{k,8})|\mathcal{F}_{t_k})\\ &\le 4(1+y_k'Qy_k)^{-2}|Q|^2|y_k|\mathbb{E}[(|\sum_{j=1}^{m}g_j(y_k)\triangle B_j(k)|^2\sum_{i,j=1}^{m}|L^ig_j(y_k)||I_{i,j}(k)|)|\mathcal{F}_{t_k}],\\ &\le C(1+y_k'Qy_k)^{-2}|y_k|[\mathbb{E}(|\sum_{j=1}^{m}g_j(y_k)\triangle B_j(k)|^4|\mathcal{F}_{t_k})]^{\frac{1}{2}}[\mathbb{E}((\sum_{i,j=1}^{m}|L^ig_j(y_k)||I_{i,j}(k)|)^2|\mathcal{F}_{t_k})]^{\frac{1}{2}},\\ &\le C(1+y_k'Qy_k)^{-2}|y_k||g(y_k)|^2\sum_{i,j=1}^{m}|L^ig_j(y_k)|\triangle^2,\\ &\le C(1+y_k'Qy_k)^{-2}|y_k|(1+|y_k|)^3\triangle^{2-2\varrho} \le C\triangle.\end{aligned}$$

An application of the Burkholder–Davis–Gundy inequality leads to

$$\mathbb{E}(|I_{i,j}(k)|^4) \le C\mathbb{E}|\int_{t_k}^{t_{k+1}}|B_i(s)-B_i(t_k)|^2ds|^2 \le C\triangle\mathbb{E}\int_{t_k}^{t_{k+1}}|B_i(s)-B_i(t_k)|^4ds \le C\triangle^4, \tag{3.20}$$

which together with (3.5) implies

$$\begin{aligned}&\mathbb{E}((\xi_{k,1}\xi_{k,9})|\mathcal{F}_{t_k})\\ &\le 2(1+y_k'Qy_k)^{-2}|Q|^2|y_k|\mathbb{E}[(|\sum_{j=1}^{m}g_j(y_k)\triangle B_j(k)||\sum_{i,j=1}^{m}L^ig_j(y_k)I_{i,j}(k)|^2)|\mathcal{F}_{t_k}]\\ &\le C(1+y_k'Qy_k)^{-2}|y_k|[\mathbb{E}(|\sum_{j=1}^{m}g_j(y_k)\triangle B_j(k)|^2|\mathcal{F}_{t_k})]^{\frac{1}{2}}[\mathbb{E}(|\sum_{i,j=1}^{m}L^ig_j(y_k)I_{i,j}(k)|^4|\mathcal{F}_{t_k})]^{\frac{1}{2}}\\ &\le C(1+y_k'Qy_k)^{-2}|y_k|(\sum_{j=1}^{m}|g_j(y_k)|)\triangle^{\frac{1}{2}}[\sum_{i,j=1}^{m}|L^ig_j(y_k)|^4\mathbb{E}(|I_{i,j}(k)|^4)]^{\frac{1}{2}}\\ &\le C(1+y_k'Qy_k)^{-2}|y_k|(\sum_{j=1}^{m}|g_j(y_k)|)\triangle^{\frac{1}{2}}(\sum_{i,j=1}^{m}|L^ig_j(y_k)|^2)\triangle^2\\ &\le C(1+y_k'Qy_k)^{-2}|y_k|(1+|y_k|)^3\triangle^{\frac{5}{2}(1-\varrho)} \le C\triangle^{\frac{5}{4}}.\end{aligned}$$

The above inequalities imply that

$$\mathbb{E}((\xi_{k,1}\sum_{i=2}^{9}\xi_{k,i})|\mathcal{F}_{t_k}) \le C\triangle. \tag{3.21}$$

Owing to the positive definiteness of Q and (3.5), one obtains that

$$\begin{aligned}
\mathbb{E}((\sum_{i=2}^{9}\xi_{k,i})^2|\mathcal{F}_{t_k}) &\le C\sum_{i=2}^{9}\mathbb{E}(\xi_{k,i}^2|\mathcal{F}_{t_k})\\
&\le C(1+y_k'Qy_k)^{-2}\Big(|y_k|^2|f(y_k)|^2\triangle^2+|g(y_k)|^4\triangle^2+|y_k|^2\sum_{i,j=1}^{m}\big|L^ig_j(y_k)\big|^2\triangle^2\\
&\qquad+|f(y_k)|^2|g(y_k)|^2\triangle^3+|f(y_k)|^4\triangle^4+|f(y_k)|^2\sum_{i,j=1}^{m}\big|L^ig_j(y_k)\big|^2\triangle^4\\
&\qquad+|g(y_k)|^2\sum_{i,j=1}^{m}|L^ig_j(y_k)|^2\triangle^3+\sum_{i,j=1}^{m}|L^ig_j(y_k)|^4\triangle^4\Big)\\
&\le C\Big(\triangle^{2-2\varrho}+\triangle^{3-3\varrho}+\triangle^{4-4\varrho}\Big)\le C\triangle.
\end{aligned}\tag{3.22}$$

Combining (3.19)–(3.22) yields

$$\begin{aligned}
\mathbb{E}((\sum_{i=1}^{9}\xi_{k,i})^2|\mathcal{F}_{t_k}) &= \mathbb{E}((\xi_{k,1})^2|\mathcal{F}_{t_k})+2\mathbb{E}((\xi_{k,1}\sum_{i=2}^{9}\xi_{k,i})|\mathcal{F}_{t_k})+\mathbb{E}((\sum_{i=2}^{9}\xi_{k,i})^2|\mathcal{F}_{t_k})\\
&\le 4(1+y_k'Qy_k)^{-2}\big|y_k'Qg(y_k)\big|^2\triangle+C\triangle.
\end{aligned}\tag{3.23}$$

Similar to (3.23), we have

$$\begin{aligned}
&C_0\mathbb{E}((\sum_{i=1}^{9}\xi_{k,i})^3|\mathcal{F}_{t_k})\\
&=C_0\Big[\mathbb{E}((3\xi_{k,1}^2\sum_{i=2}^{9}\xi_{k,i}+3\xi_{k,1}(\sum_{i=2}^{9}\xi_{k,i})^2+(\sum_{i=2}^{9}\xi_{k,i})^3)|\mathcal{F}_{t_k})\Big]\\
&\le C\Big(\mathbb{E}(\xi_{k,1}^2\sum_{i=2}^{9}|\xi_{k,i}||\mathcal{F}_{t_k})+\mathbb{E}(|\xi_{k,1}|\sum_{i=2}^{9}\xi_{k,i}^2|\mathcal{F}_{t_k})+\mathbb{E}(\sum_{i=2}^{9}|\xi_{k,i}|^3|\mathcal{F}_{t_k})\Big)\le C\triangle
\end{aligned}\tag{3.24}$$

due to $\mathbb{E}((\xi_{k,1})^3|\mathcal{F}_{t_k})=0$. Using the truncation convenience (3.5), we also obtain that for any integer $j\ge 1$,

$$\begin{aligned}
&C_j\mathbb{E}((\sum_{i=1}^{9}\xi_{k,i})^{j+3}|\mathcal{F}_{t_k})\le C\mathbb{E}(\sum_{i=1}^{9}|\xi_{k,i}|^{j+3}|\mathcal{F}_{t_k})\\
&\le C(1+y_k'Qy_k)^{-(j+3)}\Big(|y_k|^{j+3}|g(y_k)|^{j+3}\triangle^{\frac{j+3}{2}}+|y_k|^{j+3}|f(y_k)|^{j+3}\triangle^{j+3}\\
&\quad+|g(y_k)|^{2(j+3)}\triangle^{j+3}+|y_k|^{j+3}\sum_{i,j=1}^{m}\big|L^ig_j(y_k)\big|^{j+3}\triangle^{j+3}+|f(y_k)|^{j+3}|g(y_k)|^{j+3}\triangle^{\frac{3(j+3)}{2}}\\
&\quad+|f(y_k)|^{2(j+3)}\triangle^{2(j+3)}+|f(y_k)|^{j+3}\sum_{i,j=1}^{m}\big|L^ig_j(y_k)\big|^{j+3}\triangle^{2(j+3)}\\
&\quad+|g(y_k)|^{j+3}\sum_{i,j=1}^{m}\big|L^ig_j(y_k)\big|^{j+3}\triangle^{\frac{3(j+3)}{2}}+\sum_{i,j=1}^{m}\big|L^ig_j(y_k)\big|^{2(j+3)}\triangle^{2(j+3)}\Big)\\
&\le C\Big(\triangle^{\frac{(1-\varrho)(j+3)}{2}}+\triangle^{(1-\varrho)(j+3)}+\triangle^{\frac{3(1-\varrho)(j+3)}{2}}+\triangle^{2(1-\varrho)(j+3)}\Big)\le C\triangle.
\end{aligned}\tag{3.25}$$

Combining (3.17), (3.23)–(3.25) and using (2.2)-the equivalent form of Assumption 1, for any $k\ge 0$,

$$\begin{aligned}
&\mathbb{E}\left((1+\tilde{y}_{k+1}'Q\tilde{y}_{k+1})^{\frac{p}{2}}|\mathcal{F}_{t_k}\right)\\
&\le(1+y_k'Qy_k)^{\frac{p}{2}}\left[1+C\triangle+\frac{p}{2}\frac{(1+y_k'Qy_k)(2y_k'Qf(y_k)+\mathrm{trace}[g'(y_k)Qg(y_k)])+(p-2)|y_k'Qg(y_k)|^2}{(1+y_k'Qy_k)^2}\triangle\right]\\
&\le(1+y_k'Qy_k)^{\frac{p}{2}}(1+C\triangle).
\end{aligned}\tag{3.26}$$

Thanks to the nice property of the truncated Milstein scheme (3.4) that

$$y_k'Qy_k = \left(\frac{|\tilde{y}_k| \wedge \varphi^{-1}(K\triangle^{-\varrho})}{|\tilde{y}_k|}\right)^2 \tilde{y}_k'Q\tilde{y}_k \le \tilde{y}_k'Q\tilde{y}_k$$

for any positive integer k, we obtain

$$\begin{aligned}\mathbb{E}\left((1+y_{k+1}'Qy_{k+1})^{\frac{p}{2}}\right) &\le \mathbb{E}\left((1+\tilde{y}_{k+1}'Q\tilde{y}_{k+1})^{\frac{p}{2}}\right) = \mathbb{E}\left[\mathbb{E}\left((1+\tilde{y}_{k+1}'Q\tilde{y}_{k+1})^{\frac{p}{2}}|\mathcal{F}_{t_k}\right)\right] \\ &\le (1+C\triangle)\mathbb{E}\left((1+y_k'Qy_k)^{\frac{p}{2}}\right).\end{aligned}$$

For any $T > 0$, $0 \le k\triangle \le T$ and any $t \in [t_k, t_{k+1})$, solving the above linear first-order difference inequality, we obtain

$$\begin{aligned}\mathbb{E}\left((1+y'(t)Qy(t))^{\frac{p}{2}}\right) &\le \mathbb{E}\left((1+y_k'Qy_k)^{\frac{p}{2}}\right) \\ &\le (1+C\triangle)^k\mathbb{E}(1+y_0'Qy_0)^{\frac{p}{2}} \le e^{CT}(1+x_0'Qx_0)^{\frac{p}{2}}.\end{aligned}$$

Therefore, we get that

$$\hat{\lambda}^{\frac{p}{2}} \sup_{0<\triangle\le 1}\sup_{0\le t\le T}\mathbb{E}|y(t)|^p \le \mathbb{E}\left((1+y'(t)Qy(t))^{\frac{p}{2}}\right) \le C,$$

where $\hat{\lambda} > 0$ represents the smallest eigenvalue of Q. The desired result follows. ■

Note that $\bar{y}(t_k) = y(t_k) = y_k$, that is $\bar{y}(t)$ and $y(t)$ coincide at the grid points. For any $t \ge 0$, one observes from (3.6) and (3.7) that

$$\bar{y}(t) - y(t) = f(y(t))(t - t_{n_t}) + \sum_{j=1}^m g_j(y(t))(B_j(t) - B_j(t_{n_t})) + \sum_{i,j=1}^m L^i g_j(y(t))I_{i,j}(t_{n_t}, t). \tag{3.27}$$

Lemma 3.3. *Let Assumption 1 hold. For any $\triangle \in (0, 1]$, define*

$$\zeta_\triangle := \inf\{t \ge 0 : |\bar{y}(t)| \ge \varphi^{-1}(K\triangle^{-\varrho})\}. \tag{3.28}$$

Then for any $\mathcal{F}_t$-stopping time ν satisfying $0 \le \nu \le \zeta_\triangle$ a.s.,

$$\sup_{0<\triangle\le 1}\sup_{0\le t\le T}\mathbb{E}|\bar{y}(t\wedge\nu)|^p \le C, \quad \forall\, T > 0, \tag{3.29}$$

where C is a positive constant independent of $\triangle$ and p is given by Assumption 1. Moreover,

$$\mathbb{P}\{\zeta_\triangle \le T\} \le \frac{C}{(\varphi^{-1}(K\triangle^{-\varrho}))^p}. \tag{3.30}$$

Proof. Clearly, $\zeta_\triangle$ is a $\mathcal{F}_t$ stopping time. For any $\mathcal{F}_t$-stopping time ν satisfying $0 \le \nu \le \zeta_\triangle$ a.s. we have $|\bar{y}(t\wedge\nu)| \le \varphi^{-1}(K\triangle^{-\varrho})$ a.s. and $\bar{y}(t\wedge\nu)$, $y(t\wedge\nu)$ are $\mathcal{F}_{t\wedge\nu}$, $\mathcal{F}_{t_{n_{t\wedge\nu}}}$ measurable, respectively. Then for any $t \ge 0$, it follows from (3.27) that

$$\begin{aligned}\bar{y}(t\wedge\nu) &= y(t\wedge\nu) + f(y(t\wedge\nu))(t\wedge\nu - t_{n_{t\wedge\nu}}) + \sum_{j=1}^m g_j(y(t\wedge\nu))(B_j(t\wedge\nu) - B_j(t_{n_{t\wedge\nu}})) \\ &\quad + \sum_{i,j=1}^m L^i g_j(y(t\wedge\nu))I_{i,j}(t_{n_{t\wedge\nu}}, t\wedge\nu).\end{aligned}$$

Then

$$(1+\bar{y}'(t\wedge\nu)Q\bar{y}(t\wedge\nu))^{\frac{p}{2}} = (1+y'(t\wedge\nu)Qy(t\wedge\nu))^{\frac{p}{2}}\left(1+(1+y'(t\wedge\nu)Qy(t\wedge\nu))^{-1}\sum_{i=1}^9 \tilde{\xi}_{t\wedge\nu,i}\right)^{\frac{p}{2}}, \tag{3.31}$$

where

$$\tilde{\xi}_{t,1} = 2y'(t)Q\sum_{j=1}^m g_j(y(t))(B_j(t) - B_j(t_{n_t})),$$

$$
\begin{aligned}
\tilde{\xi}_{t,2} &= 2(t-t_{n_t})y'(t)Qf(y(t)),\\
\tilde{\xi}_{t,3} &= (\sum_{j=1}^m g_j(y(t))(B_j(t)-B_j(t_{n_t})))'Q\sum_{j=1}^m g_j(y(t))(B_j(t)-B_j(t_{n_t})),\\
\tilde{\xi}_{t,4} &= 2y'(t)Q\sum_{i,j=1}^m L^i g_j(y(t))I_{i,j}(t_{n_t},t),\\
\tilde{\xi}_{t,5} &= 2(t-t_{n_t})f'(y(t))Q\sum_{j=1}^m g_j(y(t))(B_j(t)-B_j(t_{n_t})),\\
\tilde{\xi}_{t,6} &= (t-t_{n_t})^2 f'(y(t))Qf(y(t)),\\
\tilde{\xi}_{t,7} &= 2(t-t_{n_t})f'(y(t))Q\sum_{i,j=1}^m L^i g_j(y(t))I_{i,j}(t_{n_t},t),\\
\tilde{\xi}_{t,8} &= 2(\sum_{j=1}^m g_j(y(t))(B_j(t)-B_j(t_{n_t})))'Q(\sum_{i,j=1}^m L^i g_j(y(t))I_{i,j}(t_{n_t},t)),\\
\tilde{\xi}_{t,9} &= (\sum_{i,j=1}^m L^i g_j(y(t))I_{i,j}(t_{n_t},t))'Q\sum_{i,j=1}^m L^i g_j(y(t))I_{i,j}(t_{n_t},t).
\end{aligned}
$$

As in the proof of Lemma 3.2, we prove the assertion for the given positive constant p with $2u < p \le 2(u+1)$ (u is a nonnegative integer). By virtue of (3.8), we know that

$$
\begin{aligned}
&\mathbb{E}\left((1+\bar{y}'(t\wedge\nu)Q\bar{y}(t\wedge\nu))^{\frac{p}{2}}|\mathcal{F}_{t_{n_{t\wedge\nu}}}\right)\\
&\le (1+y'(t\wedge\nu)Qy(t\wedge\nu))^{\frac{p}{2}}\Bigg[1+\frac{p}{2}(1+y'(t\wedge\nu)Qy(t\wedge\nu))^{-1}\mathbb{E}(\sum_{i=1}^9\tilde{\xi}_{t\wedge\nu,i}|\mathcal{F}_{t_{n_{t\wedge\nu}}})\\
&\qquad+\frac{p(p-2)}{8}(1+y'(t\wedge\nu)Qy(t\wedge\nu))^{-2}\mathbb{E}((\sum_{i=1}^9\tilde{\xi}_{t\wedge\nu,i})^2|\mathcal{F}_{t_{n_{t\wedge\nu}}})\\
&\qquad+\sum_{j=0}^u C_j(1+y'(t\wedge\nu)Qy(t\wedge\nu))^{-(j+3)}\mathbb{E}((\sum_{i=1}^9\tilde{\xi}_{t\wedge\nu,i})^{j+3}|\mathcal{F}_{t_{n_{t\wedge\nu}}})\Bigg].
\end{aligned}
\tag{3.32}
$$

Since $B(t)$ is a continuous martingale, by virtue of the Doob martingale stopping time theorem [3, p.11, Theorem 3.3] and [3, p.26, Theorem 5.17], one observes that for any $t \ge 0$, $\forall i,j,l = 1,\ldots,m$,

$$
\begin{aligned}
&\mathbb{E}(B_i(t\wedge\nu)-B_i(t_{n_{t\wedge\nu}})|\mathcal{F}_{t_{n_{t\wedge\nu}}})=0,\\
&\mathbb{E}(I_{i,j}(t_{n_{t\wedge\nu}},t\wedge\nu)|\mathcal{F}_{t_{n_{t\wedge\nu}}})=\mathbb{E}\Big(\int_{t_{n_{t\wedge\nu}}}^{t\wedge\nu}\int_{t_{n_{t\wedge\nu}}}^{s}dB_i(u)dB_j(s)|\mathcal{F}_{t_{n_{t\wedge\nu}}}\Big)\\
&\quad=\mathbb{E}\Big(\int_{t_{n_{t\wedge\nu}}}^{t\wedge\nu}(B_i(s)-B_i(t_{n_{t\wedge\nu}}))dB_j(s)|\mathcal{F}_{t_{n_{t\wedge\nu}}}\Big)=0,\\
&\mathbb{E}((B_i(t\wedge\nu)-B_i(t_{n_{t\wedge\nu}}))(B_j(t\wedge\nu)-B_j(t_{n_{t\wedge\nu}}))|\mathcal{F}_{t_{n_{t\wedge\nu}}})=0,\ \forall i\neq j,\\
&\mathbb{E}((B_l(t\wedge\nu)-B_l(t_{n_{t\wedge\nu}}))I_{i,j}(t_{n_{t\wedge\nu}},t\wedge\nu)|\mathcal{F}_{t_{n_{t\wedge\nu}}})\\
&\quad=\mathbb{E}\Big((B_l(t\wedge\nu)-B_l(t_{n_{t\wedge\nu}}))\int_{t_{n_{t\wedge\nu}}}^{t\wedge\nu}\int_{t_{n_{t\wedge\nu}}}^{s}dB_i(u)dB_j(s)|\mathcal{F}_{t_{n_{t\wedge\nu}}}\Big)=0,
\end{aligned}
\tag{3.33}
$$

and

$$
\mathbb{E}(|B_j(t\wedge\nu)-B_j(t_{n_{t\wedge\nu}})|^2|\mathcal{F}_{t_{n_{t\wedge\nu}}})=t\wedge\nu-t_{n_{t\wedge\nu}}.
$$

Moreover, using the Hölder inequality and the Doob martingale stopping time theorem [3, p. 11, Theorem 3.3] yields that for any constant $n \geq 2$, $\forall i, j = 1, \ldots, m$,

$$\begin{aligned}
&\mathbb{E}(|I_{i,j}(t_{n_{t\wedge\nu}}, t\wedge\nu)|^n|\mathcal{F}_{t_{n_{t\wedge\nu}}})\\
&= \mathbb{E}\Big(|\int_{t_{n_{t\wedge\nu}}}^{t\wedge\nu}\int_{t_{n_{t\wedge\nu}}}^{s} dB_i(v)dB_j(s)|^n|\mathcal{F}_{t_{n_{t\wedge\nu}}}\Big)\\
&\leq \mathbb{E}\Big(\Big(\int_{t_{n_{t\wedge\nu}}}^{t\wedge\nu}|\int_{t_{n_{t\wedge\nu}}}^{s\wedge\nu} dB_i(v)|^2 ds\Big)^{\frac{n}{2}}|\mathcal{F}_{t_{n_{t\wedge\nu}}}\Big)\\
&\leq (t\wedge\nu - t_{n_{t\wedge\nu}})^{\frac{n-2}{2}}\int_{t_{n_{t\wedge\nu}}}^{t\wedge\nu}\mathbb{E}\Big(|B_i(s\wedge\nu) - B_i(t_{n_{t\wedge\nu}})|^n|\mathcal{F}_{t_{n_{t\wedge\nu}}}\Big)ds\\
&\leq C(t\wedge\nu - t_{n_{t\wedge\nu}})^n.
\end{aligned} \tag{3.34}$$

For the case of $0 < n < 2$, the above result still holds by virtue of Jensen's inequality. By the techniques similar to Lemma 3.2, (3.33) and (3.34) together with (3.1) and (3.3) imply

$$\begin{aligned}
\mathbb{E}((\sum_{i=1}^{9}\tilde{\xi}_{t\wedge\nu,i})|\mathcal{F}_{t_{n_{t\wedge\nu}}}) &\leq (t\wedge\nu - t_{n_{t\wedge\nu}})(2y'(t\wedge\nu)Qf(y(t\wedge\nu))\\
&\quad + \mathrm{trace}[g'(y(t\wedge\nu))Qg(y(t\wedge\nu))]) + C\triangle(1 + y'(t\wedge\nu)Qy(t\wedge\nu)).
\end{aligned} \tag{3.35}$$

For any integer $n \geq 1$, $i = 1, \ldots, m$, one observes that

$$\mathbb{E}((B_i(t\wedge\nu) - B_i(t_{n_{t\wedge\nu}}))^{2n-1}|\mathcal{F}_{t_{n_{t\wedge\nu}}}) = 0, \tag{3.36}$$

and

$$\mathbb{E}(|B_i(t\wedge\nu) - B_i(t_{n_{t\wedge\nu}})|^n|\mathcal{F}_{t_{n_{t\wedge\nu}}}) \leq \triangle^{\frac{n}{2}}. \tag{3.37}$$

Using similar techniques as in Lemma 3.2, we derive from (3.34), (3.36) and (3.37) that

$$\begin{aligned}
\mathbb{E}((\sum_{i=1}^{9}\tilde{\xi}_{t,i})^2|\mathcal{F}_{t_{n_{t\wedge\nu}}}) &\leq (t\wedge\nu - t_{n_{t\wedge\nu}})|y'(t\wedge\nu)Qg(y(t\wedge\nu))|^2\\
&\quad + C\triangle(1 + y'(t\wedge\nu)Qy(t\wedge\nu))^2.
\end{aligned} \tag{3.38}$$

and

$$C_0\mathbb{E}((\sum_{i=1}^{9}\tilde{\xi}_{t,i})^3|\mathcal{F}_{t_{n_{t\wedge\nu}}}) \leq C\triangle(1 + y'(t\wedge\nu)Qy(t\wedge\nu))^3. \tag{3.39}$$

Similar to (3.25), using the truncation convenience (3.5), we can also prove that for any integer $j \geq 1$,

$$C_j\mathbb{E}((\sum_{i=1}^{9}\tilde{\xi}_{t,i})^{j+3}|\mathcal{F}_{t_{n_{t\wedge\nu}}}) \leq C\mathbb{E}(\sum_{i=1}^{9}|\tilde{\xi}_{t,i}|^{j+3}|\mathcal{F}_{t_{n_{t\wedge\nu}}}) \leq C\triangle(1 + y'(t\wedge\nu)Qy(t\wedge\nu))^{j+3}. \tag{3.40}$$

Combining (3.35), (3.38)–(3.40), by Assumption 1, for any $t \geq 0$,

$$\begin{aligned}
&\mathbb{E}\left((1 + \bar{y}'(t\wedge\nu)Q\bar{y}(t\wedge\nu))^{\frac{p}{2}}\right)\\
&= \mathbb{E}\left(\mathbb{E}\left((1 + \bar{y}'(t\wedge\nu)Q\bar{y}(t\wedge\nu))^{\frac{p}{2}}|\mathcal{F}_{t_{n_{t\wedge\nu}}}\right)\right)\\
&\leq \mathbb{E}\Big[(1 + y'(t\wedge\nu)Qy(t\wedge\nu))^{\frac{p}{2}}\Big(1 + C\triangle\\
&\quad + \frac{p}{2}\frac{2y'(t\wedge\nu)Qf(y(t\wedge\nu)) + \mathrm{trace}[g'(y(t\wedge\nu))Qg(y(t\wedge\nu))]}{1 + y'(t\wedge\nu)Qy(t\wedge\nu)}(t\wedge\nu - t_{n_{t\wedge\nu}})\\
&\quad + \frac{p(p-2)|y'(t\wedge\nu)Qg(y(t\wedge\nu))|^2}{2(1 + y'(t\wedge\nu)Qy(t\wedge\nu))^2}(t\wedge\nu - t_{n_{t\wedge\nu}})\Big)\Big]\\
&\leq \mathbb{E}(1 + y'(t\wedge\nu)Qy(t\wedge\nu))^{\frac{p}{2}}(1 + C\triangle).
\end{aligned} \tag{3.41}$$

Similar to (3.41), we arrive at that for any $0 \le t \le T$,

$$\begin{aligned}\mathbb{E}(1+y'(t\wedge\nu)Qy(t\wedge\nu))^{\frac{p}{2}} &= \mathbb{E}\left(\mathbb{E}\left((1+y'(t\wedge\nu)Qy(t\wedge\nu))^{\frac{p}{2}}|\mathcal{F}_{(t-\triangle)\wedge\nu}\right)\right)\\ &\le \mathbb{E}(1+y'((t-\triangle)\wedge\nu)Qy((t-\triangle)\wedge\nu))^{\frac{p}{2}}(1+C\triangle)\\ &\le \mathbb{E}(1+y'((t-2\triangle)\wedge\nu)Qy((t-2\triangle)\wedge\nu))^{\frac{p}{2}}(1+C\triangle)^2\\ &\le \cdots\\ &\le \mathbb{E}(1+y'((t-n_t\triangle)\wedge\nu)Qy((t-n_t\triangle)\wedge\nu))^{\frac{p}{2}}(1+C\triangle)^{n_t}\\ &= \mathbb{E}(1+y_0'Qy_0)^{\frac{p}{2}}(1+C\triangle)^{n_t}\\ &\le e^{CT}\mathbb{E}(1+x_0'Qx_0)^{\frac{p}{2}} =: C. \end{aligned} \tag{3.42}$$

Inserting (3.41) into (3.42) leads to

$$\sup_{0<\triangle\le 1}\sup_{0\le t\le T}\mathbb{E}\left((1+\bar{y}'(t\wedge\nu)Q\bar{y}(t\wedge\nu))^{\frac{p}{2}}\right)\le C.$$

Therefore the desired assertion follows from

$$(\varphi^{-1}(K\triangle^{-\varrho}))^p\mathbb{P}\{\nu\le T\}\le\mathbb{E}\left(|\bar{y}(T\wedge\nu)|^p\right)\le\hat{\lambda}^{-p/2}\mathbb{E}\left((1+\bar{y}'(T\wedge\nu)Q\bar{y}(T\wedge\nu))^{\frac{p}{2}}\right)\le C,$$

where $\hat{\lambda}>0$ represents the smallest eigenvalue of Q. The proof is complete. ■

The following theorem gives the pth-moment convergence of the truncated Milstein scheme.

Theorem 3.1. *Under Assumption 1, for any $q\in(0,p)$ and p given in Assumption 1,*

$$\lim_{\triangle\to 0}\mathbb{E}|\bar{y}(T)-x(T)|^q=\lim_{\triangle\to 0}\mathbb{E}|y(T)-x(T)|^q=0,\qquad \forall\ T\ge 0, \tag{3.43}$$

where p is given by Assumption 1.

Proof. The desired result is a slight modification of the argument in [23, Theorem 3.3]. Thus the details are omitted. ■

4. Convergence rate

In this section, our aim is to establish a better rate of convergence result than that of the classical EM method under Assumption 1 and additional conditions on f and g. We shall show that the rate is order 1 similar to the standard results for the explicit Milstein scheme with globally Lipschitz f, g and L^ig_j. To obtain the rate of convergence, we need somewhat stronger conditions compared with the convergence alone, which are stated as follows.

Assumption 2. $f,\ g\in C^2$, namely, their partial derivatives up to the second order are continuous. Moreover, there exist positive constants $p_0>1$, L, C_0, l, l_0, and symmetric positive definite matrix Q_0 such that $\forall x,y\in\mathbb{R}^d$,

$$2(x-y)'Q_0(f(x)-f(y))+p_0\text{trace}[(g(x)-g(y))'Q_0(g(x)-g(y))]\le L|x-y|^2, \tag{4.1}$$

$$|f(x)-f(y)|\vee|L^ig_j(x)-L^ig_j(y)|\le C_0(1+|x|^l+|y|^l)|x-y|, \tag{4.2}$$

$$\begin{aligned}&|(f_k)_x(x)|\vee|(f_k)_{xx}(x)|\vee|(g_{kj})_x(x)|\vee|(g_{kj})_{xx}(x)|\le C_0(1+|x|^{l_0}),\\ &\qquad k=1,\ldots,d;\ \ i,j=1,\ldots,m.\end{aligned} \tag{4.3}$$

Remark 4.1. One observes that if Assumption 2 holds, then there exists a positive constant $C_1\ge C_0$ such that

$$|g(x)-g(y)|^2\le C_1(1+|x|^l+|y|^l)|x-y|^2. \tag{4.4}$$

In addition,

$$|f(x)|\le|f(x)-f(0)|+|f(0)|\le L(1+|x|^l)|x|+|f(0)|\le C(1+|x|^{l+1}), \tag{4.5}$$

similarly,

$$|L^ig_j(x)|\le C(1+|x|^{l+1}),\qquad i,j=1,\ldots,m, \tag{4.6}$$

and by Young's inequality,

$$|g(x)|\le C[|x|^2+|x|(1+|x|^{l+1})]^{1/2}+|g(0)|\le C(1+|x|^{l/2+1}). \tag{4.7}$$

Remark 4.2. Under Assumption 2, we define φ in (3.1) by $\varphi(r)=2C_1(1+r^l)$ for any $r>0$. Then $\varphi^{-1}(r)=(r/2C_1-1)^{1/l}$ for all $r>2C_1$. Thus, $\pi_\triangle(x)=\left(|x|\wedge(K\triangle^{-\varrho}/2C_1-1)^{1/l}\right)x/|x|$ for any $x\in\mathbb{R}^d$, where $\varrho\in(0,1/2]$.

Define the stopping time $\bar\theta_\triangle := \tau_{\varphi^{-1}(K\triangle^{-\varrho})} \wedge \zeta_\triangle$, where τ_N and $\zeta_\triangle$ are defined by (2.3) and (3.28), respectively. Define another auxiliary process $\tilde x(t) := x(t_k) = x_k$ for any $t \in [t_k, t_{k+1})$. Then the moment deviations between $\bar y(t\wedge\bar\theta_\triangle)$ and $y(t\wedge\bar\theta_\triangle)$, $x(t\wedge\bar\theta_\triangle)$ and $\tilde x(t\wedge\bar\theta_\triangle)$, are estimated as follows.

Lemma 4.1. *If Assumptions 1 and 2 hold with $2p \geq l+2$, for any $q_0 \in (0, 2p/(l+2)]$,*

$$\sup_{0\le t\le T} \mathbb{E}(|\bar y(t\wedge\bar\theta_\triangle) - y(t\wedge\bar\theta_\triangle)|^{q_0}) \le C\triangle^{\frac{q_0}{2}}, \quad \sup_{0\le t\le T} \mathbb{E}(|x(t\wedge\bar\theta_\triangle) - \tilde x(t\wedge\bar\theta_\triangle)|^{q_0}) \le C\triangle^{\frac{q_0}{2}}, \tag{4.8}$$

$\forall T>0,\ \forall \triangle \in (0,1]$, *where C is a positive constant independent of $\triangle$, p and l are given by Assumptions 1 and 2, respectively.*

Proof. For any $T>0$, any $t\in[0,T]$, it follows from (3.27) that

$$\begin{aligned}
&\mathbb{E}(|\bar y(t\wedge\bar\theta_\triangle) - y(t\wedge\bar\theta_\triangle)|^{q_0})\\
&\le C\Big(\mathbb{E}(|f(y(t\wedge\bar\theta_\triangle))|^{q_0}|t\wedge\bar\theta_\triangle - t_{n_{t\wedge\bar\theta_\triangle}}|^{q_0}) + \sum_{j=1}^m \mathbb{E}(|g_j(y(t\wedge\bar\theta_\triangle))|^{q_0}|B_j(t\wedge\bar\theta_\triangle) - B_j(t_{n_{t\wedge\bar\theta_\triangle}})|^{q_0})\\
&\quad + \sum_{i,j=1}^m \mathbb{E}(|L^i g_j(y(t\wedge\bar\theta_\triangle))|^{q_0}|I_{i,j}(t_{n_{t\wedge\bar\theta_\triangle}}, t\wedge\bar\theta_\triangle)|^{q_0})\Big).
\end{aligned}$$

Since $y(t\wedge\bar\theta_\triangle)$ is $\mathcal{F}_{t_{n_{t\wedge\bar\theta_\triangle}}}$-adapted, it follows from (3.34) and (3.37) that

$$\begin{aligned}
&\mathbb{E}(|\bar y(t\wedge\bar\theta_\triangle) - y(t\wedge\bar\theta_\triangle)|^{q_0})\\
&\le C\Big(\mathbb{E}(|f(y(t\wedge\bar\theta_\triangle))|^{q_0}\triangle^{q_0}) + \sum_{j=1}^m \mathbb{E}(|g_j(y(t\wedge\bar\theta_\triangle))|^{q_0}\mathbb{E}(|B_j(t\wedge\bar\theta_\triangle) - B_j(t_{n_{t\wedge\bar\theta_\triangle}})|^{q_0}|\mathcal{F}_{t_{n_{t\wedge\bar\theta_\triangle}}}))\\
&\quad + \sum_{i,j=1}^m \mathbb{E}(|L^i g_j(y(t\wedge\bar\theta_\triangle))|^{q_0}\mathbb{E}(|I_{i,j}(t_{n_{t\wedge\bar\theta_\triangle}}, t\wedge\bar\theta_\triangle)|^{q_0}|\mathcal{F}_{t_{n_{t\wedge\bar\theta_\triangle}}}))\Big)\\
&\le C\Big(\mathbb{E}|f(y(t\wedge\bar\theta_\triangle))|^{q_0}\triangle^{q_0} + \sum_{j=1}^m \mathbb{E}|g_j(y(t\wedge\bar\theta_\triangle))|^{q_0}\triangle^{\frac{q_0}{2}} + \sum_{i,j=1}^m \mathbb{E}(|L^i g_j(y(t\wedge\bar\theta_\triangle))|^{q_0})\triangle^{q_0}\Big).
\end{aligned}$$

Due to (3.5), (4.7), and Lemma 3.3, for any $q_0 \in (0, 2p/(l+2)] \subseteq (0,p]$,

$$\begin{aligned}
&\mathbb{E}(|\bar y(t\wedge\bar\theta_\triangle) - y(t\wedge\bar\theta_\triangle)|^{q_0})\\
&\le C\mathbb{E}(1+|y(t\wedge\bar\theta_\triangle)|)^{q_0}\triangle^{q_0(1-\varrho)} + C\mathbb{E}(1+|y(t\wedge\bar\theta_\triangle)|^{\frac{l}{2}+1})^{q_0}\triangle^{\frac{q_0}{2}}\\
&\le C\triangle^{\frac{q_0}{2}} + C(\mathbb{E}|y(t\wedge\bar\theta_\triangle)|^p)^{\frac{q_0}{p}}\triangle^{\frac{q_0}{2}} + C(\mathbb{E}|y(t\wedge\bar\theta_\triangle)|^p)^{\frac{(l+2)q_0}{2p}}\triangle^{\frac{q_0}{2}}\\
&\le C\triangle^{\frac{q_0}{2}} + C(\sup_{0\le t\le T}\mathbb{E}|\bar y(t\wedge\bar\theta_\triangle)|^p)^{\frac{q_0}{p}}\triangle^{\frac{q_0}{2}} + C(\sup_{0\le t\le T}\mathbb{E}|\bar y(t\wedge\bar\theta_\triangle)|^p)^{\frac{(l+2)q_0}{2p}}\triangle^{\frac{q_0}{2}}\\
&\le C\triangle^{\frac{q_0}{2}}.
\end{aligned}$$

The first part of the desired assertion follows. Similarly, the second part of (4.8) follows by virtue of Lemma 2.1. ■

Using Assumption 2, the following results on the exact solutions follow from Jensen's inequality and Lemma 2.1 directly.

Lemma 4.2. *If Assumptions 1 and 2 hold, then for any $q_1 \in (0, p/(l_0 \vee (l+1))]$, any $T>0$, $i=1,\dots,d;\ j=1,\dots,m$, any $\triangle\in(0,1]$, we have*

$$\begin{aligned}
&\sup_{0\le t\le T}\mathbb{E}|f(x(t\wedge\bar\theta_\triangle))|^{q_1} \le C, \quad \sup_{0\le t\le T}\mathbb{E}|g_j(x(t\wedge\bar\theta_\triangle))|^{q_1} \le C,\\
&\sup_{0\le t\le T}\mathbb{E}|(f_i)_x(x(t\wedge\bar\theta_\triangle))|^{q_1} \le C, \quad \sup_{0\le t\le T}\mathbb{E}|(g_{ij})_x(x(t\wedge\bar\theta_\triangle))|^{q_1} \le C,\\
&\sup_{0\le u\le 1}\sup_{0\le t\le T}\mathbb{E}|(f_i)_{xx}(ux(t\wedge\bar\theta_\triangle) + (1-u)\tilde x(t_{n_t}\wedge\bar\theta_\triangle))|^{q_1} \le C,\\
&\sup_{0\le u\le 1}\sup_{0\le t\le T}\mathbb{E}|(g_{ij})_{xx}(ux(t\wedge\bar\theta_\triangle) + (1-u)\tilde x(t_{n_t}\wedge\bar\theta_\triangle))|^{q_1} \le C,
\end{aligned}$$

where p and l_0, l are given by Assumptions 1 and 2, respectively.

Noting that $\bar y(t)$ defined in (3.7) is related to k and t_k, for any k and any $t_k \le s \le t < t_{k+1}$,

$$\bar y(t) - \bar y(s) = f(y(s))(t-s) + \sum_{j=1}^m g_j(y(s))(B_j(t) - B_j(s)) + \sum_{i,j=1}^m L^i g_j(y(s))I_{i,j}(s,t). \tag{4.9}$$

In order to derive the convergence rate, we analyze the exact solution $x(t)$. For convenience, for any twice differential function $\psi = (\psi_1, \dots, \psi_d)' : \mathbb{R}^d \to \mathbb{R}^d$, $\forall u = (u_1, \dots, u_d)'$ and $v = (v_1, \dots, v_d)' \in \mathbb{R}^d$, define

$$\psi_x(\cdot)(u) = \langle \nabla_x \psi(\cdot), u\rangle = \sum_{i=1}^{d} \frac{\partial \psi(\cdot)}{\partial x_i} u_i, \quad |\psi_x(\cdot)| = \sqrt{\sum_{j=1}^{d} |(\psi_j)_x(\cdot)|^2},$$
$$\psi_{xx}(\cdot)(u,v) = \sum_{i,j=1}^{d} \frac{\partial^2 \psi(\cdot)}{\partial x_i \partial x_j} u_i v_j, \quad |\psi_{xx}(\cdot)| = \sqrt{\sum_{j=1}^{d} |(\psi_j)_{xx}(\cdot)|^2}.$$

Note that

$$|\psi_x(\cdot)(u)| \le |\psi_x(\cdot)||u|, \qquad |\psi_{xx}(\cdot)(u,v)| \le |\psi_{xx}(\cdot)||u||v|. \tag{4.10}$$

It follows from Assumption 2 and the Taylor formula that

$$\begin{aligned} f(x(t)) - f(x_k) &= \sum_{j=1}^{m} f_x(x_k) \int_{t_k}^{t} g_j(x(s)) dB_j(s) + f_x(x_k) \int_{t_k}^{t} f(x(s)) ds \\ &\quad + \int_0^1 (1-u) f_{xx}(ux(t) + (1-u)x_k)(x(t) - x_k, x(t) - x_k) du, \quad \forall t \ge t_k. \end{aligned}$$

Using Assumption 2 and the Taylor formula again yields that for each $1 \le j \le m$, $\forall t \ge t_k$,

$$\begin{aligned} & g_j(x(t)) - g_j(x_k) \\ &= (g_j)_x(x_k)(x(t) - x_k) + \int_0^1 (1-u)(g_j)_{xx}(ux(t) + (1-u)x_k)(x(t) - x_k, x(t) - x_k) du \\ &= \sum_{i=1}^{m} L^i g_j(x_k)(B_i(t) - B_i(t_k)) + \sum_{i=1}^{m} (g_j)_x(x_k) \int_{t_k}^{t} (g_i(x(s)) - g_i(x_k)) dB_i(s) \\ &\quad + (g_j)_x(x_k) \int_{t_k}^{t} f(x(s)) ds + \int_0^1 (1-u)(g_j)_{xx}(ux(t) + (1-u)x_k)(x(t) - x_k, x(t) - x_k) du. \end{aligned}$$

Thus, rearranging $x(t)$ and using the above equalities yields that for any $0 \le s \le t$,

$$\begin{aligned} x(t) - x(s) &= \int_s^t f(x(r)) dr + \sum_{j=1}^{m} \int_s^t g_j(x(r)) dB_j(r) \\ &= f(\tilde{x}(s))(t-s) + \sum_{j=1}^{m} g_j(\tilde{x}(s))(B_j(t) - B_j(s)) + \sum_{i,j=1}^{m} L^i g_j(\tilde{x}(s)) I_{i,j}(s,t) \\ &\quad + \sum_{i=1}^{3} \int_s^t \Big(R_i(f, s, r) dr + \sum_{j=1}^{m} R_i(g_j, s, r) dB_j(r) \Big), \end{aligned} \tag{4.11}$$

where

$$\begin{aligned} R_1(f, s, r) &= \sum_{j=1}^{m} f_x(\tilde{x}(s)) \int_{t_{n_s}}^{r} g_j(x(v)) dB_j(v), \\ R_2(f, s, r) &= f_x(\tilde{x}(s)) \int_{t_{n_s}}^{r} f(x(v)) dv, \\ R_3(f, s, r) &= \int_0^1 (1-u) f_{xx}(ux(r) + (1-u)\tilde{x}(s))(x(r) - \tilde{x}(s), x(r) - \tilde{x}(s)) du, \\ R_1(g_j, s, r) &= \sum_{i=1}^{m} (g_j)_x(\tilde{x}(s)) \int_{t_{n_s}}^{r} (g_i(x(v)) - g_i(\tilde{x}(s))) dB_i(v), \\ R_2(g_j, s, r) &= (g_j)_x(\tilde{x}(s)) \int_{t_{n_s}}^{r} f(x(v)) dv, \\ R_3(g_j, s, r) &= \int_0^1 (1-u)(g_j)_{xx}(ux(r) + (1-u)\tilde{x}(s))(x(r) - \tilde{x}(s), x(r) - \tilde{x}(s)) du. \end{aligned} \tag{4.12}$$

In order to estimate the error of estimation, we look for the moment bound on the last term of (4.11).

Lemma 4.3. *Let Assumptions 1 and 2 hold with $p/4 \geq l_0 \vee (l+2)$. Then*

$$\begin{aligned}
&\sup_{0\le k\triangle\le T} \mathbb{E}\Big|\int_{t_k\wedge\bar\theta_\triangle}^{t_{k+1}\wedge\bar\theta_\triangle} R_1(f, t_k\wedge\bar\theta_\triangle, s)ds\Big|^2 \le C\triangle^3;\\
&\sup_{0\le k\triangle\le T} \mathbb{E}\Big|\int_{t_k\wedge\bar\theta_\triangle}^{t_{k+1}\wedge\bar\theta_\triangle} R_i(f, t_k\wedge\bar\theta_\triangle, s)ds\Big|^2 \le C\triangle^4,\ \ i=2,3;\\
&\sup_{0\le k\triangle\le T} \mathbb{E}\Big|\int_{t_k\wedge\bar\theta_\triangle}^{t_{k+1}\wedge\bar\theta_\triangle} R_i(g_j, t_k\wedge\bar\theta_\triangle, s)dB_j(s)\Big|^2 \le C\triangle^3,\ \ i=1,2,3,\ j=1,\dots,m.
\end{aligned} \tag{4.13}$$

Proof. For any integer k, we estimate the remaining terms of (4.11) one by one. Using the Hölder inequality, Lemmas 2.1 and 4.2 lead to

$$\begin{aligned}
&\mathbb{E}\Big|\int_{t_k\wedge\bar\theta_\triangle}^{t_{k+1}\wedge\bar\theta_\triangle} R_1(f, t_k\wedge\bar\theta_\triangle, s)ds\Big|^2\\
&= \mathbb{E}\Big|\int_{t_k\wedge\bar\theta_\triangle}^{t_{k+1}\wedge\bar\theta_\triangle} I_{\{t_k\le\bar\theta_\triangle\}} f_x(\tilde x(t_k\wedge\bar\theta_\triangle)) \sum_{j=1}^m \int_{t_{n_{t_k\wedge\bar\theta_\triangle}}}^s g_j(x(u))dB_j(u)ds\Big|^2\\
&\le m\triangle \sum_{j=1}^m \mathbb{E}\int_{t_k}^{t_{k+1}} I_{\{t_k\le\bar\theta_\triangle\}} I_{\{t_k\wedge\bar\theta_\triangle\le s\le\bar\theta_\triangle\}} |f_x(\tilde x(t_k\wedge\bar\theta_\triangle)) \int_{t_k\wedge\bar\theta_\triangle}^{s\wedge\bar\theta_\triangle} g_j(x(u))dB_j(u)|^2 ds\\
&\le m\triangle \sum_{j=1}^m \int_{t_k}^{t_{k+1}} \left(\mathbb{E}|f_x(\tilde x(t_k\wedge\bar\theta_\triangle))|^4\right)^{\frac12} \left(\mathbb{E}\Big|\int_{t_k}^s I_{\{t_k\wedge\bar\theta_\triangle\le u\le\bar\theta_\triangle\}} g_j(x(u))dB_j(u)\Big|^4\right)^{\frac12} ds\\
&\le C\triangle \sum_{j=1}^m \int_{t_k}^{t_{k+1}} \left(\mathbb{E}\Big(\int_{t_k}^s |g_j(x(u\wedge\bar\theta_\triangle))|^2 du\Big)^2\right)^{\frac12} ds\\
&\le C\triangle^{\frac32} \sum_{j=1}^m \int_{t_k}^{t_{k+1}} \left(\int_{t_k}^s \mathbb{E}|g_j(x(u\wedge\bar\theta_\triangle))|^4 du\right)^{\frac12} ds \le C\triangle^3,
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\Big|\int_{t_k\wedge\bar\theta_\triangle}^{t_{k+1}\wedge\bar\theta_\triangle} R_2(f, t_k\wedge\bar\theta_\triangle, s)ds\Big|^2\\
&= \mathbb{E}\Big|I_{\{t_k\le\bar\theta_\triangle\}} \int_{t_k\wedge\bar\theta_\triangle}^{t_{k+1}\wedge\bar\theta_\triangle} f_x(\tilde x(t_k\wedge\bar\theta_\triangle)) \int_{t_{n_{t_k\wedge\bar\theta_\triangle}}}^s f(x(u))duds\Big|^2\\
&= \mathbb{E}\Big|\int_{t_k\wedge\bar\theta_\triangle}^{t_{k+1}\wedge\bar\theta_\triangle} I_{\{t_k\le t_k\wedge\bar\theta_\triangle\}} f_x(\tilde x(t_k\wedge\bar\theta_\triangle)) \int_{t_{n_{t_k\wedge\bar\theta_\triangle}}}^s f(x(u))duds\Big|^2\\
&\le \triangle \int_{t_k}^{t_{k+1}} \mathbb{E}(I_{\{t_k=t_k\wedge\bar\theta_\triangle\}} I_{\{t_k\wedge\bar\theta_\triangle\le s\le\bar\theta_\triangle\}} |f_x(\tilde x(t_k\wedge\bar\theta_\triangle)) \int_{t_k\wedge\bar\theta_\triangle}^{s\wedge\bar\theta_\triangle} f(x(u))du|^2)ds\\
&\le \triangle \int_{t_k}^{t_{k+1}} \left(\mathbb{E}|f_x(\tilde x(t_k\wedge\bar\theta_\triangle))|^4\right)^{\frac12} \left(\mathbb{E}\Big|\int_{t_k\wedge\bar\theta_\triangle}^{s\wedge\bar\theta_\triangle} f(x(u))du\Big|^4\right)^{\frac12} ds\\
&\le C\triangle^{\frac52} \int_{t_k}^{t_{k+1}} \left(\int_{t_k}^s \mathbb{E}|f(x(u\wedge\bar\theta_\triangle))|^4 du\right)^{\frac12} ds \le C\triangle^4.
\end{aligned}$$

Using the Hölder inequality, Lemmas 4.1 and 4.2 yields

$$\begin{aligned}
&\mathbb{E}\Big|\int_{t_k\wedge\bar\theta_\triangle}^{t_{k+1}\wedge\bar\theta_\triangle} R_3(f, t_k\wedge\bar\theta_\triangle, s)ds\Big|^2\\
&= \mathbb{E}\Big(\Big|\int_{t_k}^{t_{k+1}} I_{\{t_k\wedge\bar\theta_\triangle\le s\le\bar\theta_\triangle\}} \int_0^1 (1-u) f_{xx}(ux(s)+(1-u)\tilde x(t_k\wedge\bar\theta_\triangle))\\
&\quad (x(s)-\tilde x(t_k\wedge\bar\theta_\triangle), x(s)-\tilde x(t_k\wedge\bar\theta_\triangle))duds\Big|^2\Big)\\
&\le \triangle \int_{t_k}^{t_{k+1}} \mathbb{E}\Big|\int_0^1 (1-u) f_{xx}(ux(s\wedge\bar\theta_\triangle)+(1-u)\tilde x(t_k\wedge\bar\theta_\triangle))\\
&\quad (x(s\wedge\bar\theta_\triangle)-\tilde x(t_k\wedge\bar\theta_\triangle), x(s\wedge\bar\theta_\triangle)-\tilde x(t_k\wedge\bar\theta_\triangle))du\Big|^2 ds
\end{aligned}$$

$$\leq \triangle \int_{t_k}^{t_{k+1}} \int_0^1 \Big(\mathbb{E}|f_{xx}(ux(s\wedge\bar\theta_\triangle)+(1-u)\tilde x(t_k\wedge\bar\theta_\triangle))|^4\Big)^{\frac12} \times\Big(\mathbb{E}|x(s\wedge\bar\theta_\triangle)-\tilde x(t_k\wedge\bar\theta_\triangle)|^8\Big)^{\frac12} duds \leq C\triangle^4.$$

For each $j=1,\ldots,m$, using similar techniques yields that

$$\begin{aligned}
&\mathbb{E}|\int_{t_k\wedge\bar\theta_\triangle}^{t_{k+1}\wedge\bar\theta_\triangle} R_1(g_j, t_k\wedge\bar\theta_\triangle, s)dB_j(s)|^2\\
&=\mathbb{E}|\sum_{i=1}^m \int_{t_k\wedge\bar\theta_\triangle}^{t_{k+1}\wedge\bar\theta_\triangle} I_{\{t_k\leq\bar\theta_\triangle\}}(g_j)_x(\tilde x(t_k\wedge\bar\theta_\triangle)) \int_{t_{n_{t_k\wedge\bar\theta_\triangle}}}^{s} (g_i(x(u))-g_i(\tilde x(t_k\wedge\bar\theta_\triangle)))dB_i(u)dB_j(s)|^2\\
&\leq m\sum_{i=1}^m \mathbb{E}\int_{t_k}^{t_{k+1}} I_{\{t_k\leq\bar\theta_\triangle\}}I_{\{t_k\wedge\bar\theta_\triangle\leq s\leq\bar\theta_\triangle\}}|(g_j)_x(\tilde x(t_k\wedge\bar\theta_\triangle))\int_{t_k\wedge\bar\theta_\triangle}^{s\wedge\bar\theta_\triangle}(g_i(x(u))-g_i(\tilde x(t_k\wedge\bar\theta_\triangle)))dB_i(u)|^2 ds\\
&\leq m\sum_{i=1}^m \int_{t_k}^{t_{k+1}} \Big(\mathbb{E}|(g_j)_x(\tilde x(t_k\wedge\bar\theta_\triangle))|^4\Big)^{\frac12}\Big(\mathbb{E}|\int_{t_k\wedge\bar\theta_\triangle}^{s\wedge\bar\theta_\triangle}(g_i(x(u))-g_i(\tilde x(t_k\wedge\bar\theta_\triangle)))dB_i(u)|^4\Big)^{\frac12} ds\\
&\leq C\triangle^{\frac12}\sum_{i=1}^m \int_{t_k}^{t_{k+1}}\Big(\int_{t_k}^{s}\mathbb{E}|g_i(x(u\wedge\bar\theta_\triangle))-g_i(\tilde x(t_k\wedge\bar\theta_\triangle))|^4 du\Big)^{\frac12} ds\\
&\leq C\triangle^{\frac12}\sum_{i=1}^m \int_{t_k}^{t_{k+1}}\Big(\int_{t_k}^{s}\mathbb{E}\Big[(1+|x(u\wedge\bar\theta_\triangle)|^{2l}+|\tilde x(t_k\wedge\bar\theta_\triangle)|^{2l})|x(u\wedge\bar\theta_\triangle)-\tilde x(t_k\wedge\bar\theta_\triangle)|^4\Big]du\Big)^{\frac12} ds\\
&\leq C\triangle^{\frac12}\sum_{i=1}^m \int_{t_k}^{t_{k+1}}\Big[\int_{t_k}^{s}\Big(\mathbb{E}(1+|x(u\wedge\bar\theta_\triangle)|^{4l}+|\tilde x(t_k\wedge\bar\theta_\triangle)|^{4l})\Big)^{\frac12}\\
&\qquad\times\Big(\mathbb{E}|x(u\wedge\bar\theta_\triangle)-\tilde x(t_k\wedge\bar\theta_\triangle)|^8\Big)^{\frac12}du\Big]^{\frac12} ds\leq C\triangle^3,
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}|\int_{t_k\wedge\bar\theta_\triangle}^{t_{k+1}\wedge\bar\theta_\triangle} R_2(g_j, t_k\wedge\bar\theta_\triangle, s)dB_j(s)|^2\\
&=\mathbb{E}|\int_{t_k\wedge\bar\theta_\triangle}^{t_{k+1}\wedge\bar\theta_\triangle} I_{\{t_k\leq\bar\theta_\triangle\}}(g_j)_x(\tilde x(t_k\wedge\bar\theta_\triangle))\int_{t_{n_{t_k\wedge\bar\theta_\triangle}}}^{s} f(x(u))dudB_j(s)|^2\\
&=\mathbb{E}\int_{t_k}^{t_{k+1}} I_{\{t_k\leq\bar\theta_\triangle\}}I_{\{t_k\wedge\bar\theta_\triangle\leq s\leq\bar\theta_\triangle\}}|(g_j)_x(\tilde x(t_k\wedge\bar\theta_\triangle))\int_{t_k\wedge\bar\theta_\triangle}^{s\wedge\bar\theta_\triangle} f(x(u))du|^2 ds\\
&\leq\int_{t_k}^{t_{k+1}}\Big(\mathbb{E}|(g_j)_x(\tilde x(t_k\wedge\bar\theta_\triangle))|^4\Big)^{\frac12}\Big(\mathbb{E}|\int_{t_k\wedge\bar\theta_\triangle}^{s\wedge\bar\theta_\triangle} f(x(u))du|^4\Big)^{\frac12} ds\\
&\leq C\triangle^{\frac32}\int_{t_k}^{t_{k+1}}\Big(\int_{t_k}^{s}\mathbb{E}|f(x(u\wedge\bar\theta_\triangle))|^4du\Big)^{\frac12} ds\leq C\triangle^3.
\end{aligned}$$

Owing to Lemmas 4.1 and 4.2, one further obtains

$$\begin{aligned}
&\mathbb{E}|\int_{t_k\wedge\bar\theta_\triangle}^{t_{k+1}\wedge\bar\theta_\triangle} R_3(g_j, t_k\wedge\bar\theta_\triangle, s)ds|^2\\
&=\mathbb{E}(|\int_{t_k}^{t_{k+1}} I_{\{t_k\leq\bar\theta_\triangle\}}I_{\{t_k\wedge\bar\theta_\triangle\leq s\leq\bar\theta_\triangle\}}\int_0^1(1-u)(g_j)_{xx}(ux(s)+(1-u)\tilde x(t_k\wedge\bar\theta_\triangle))\\
&\qquad(x(s)-\tilde x(t_k\wedge\bar\theta_\triangle), x(s)-\tilde x(t_k\wedge\bar\theta_\triangle))dudB_j(s)|^2)\\
&=\int_{t_k}^{t_{k+1}}\mathbb{E}|\int_0^1(1-u)(g_j)_{xx}(ux(s\wedge\bar\theta_\triangle)+(1-u)\tilde x(t_k\wedge\bar\theta_\triangle))\\
&\qquad(x(s\wedge\bar\theta_\triangle)-\tilde x(t_k\wedge\bar\theta_\triangle), x(s\wedge\bar\theta_\triangle)-\tilde x(t_k\wedge\bar\theta_\triangle))du|^2 ds\\
&\leq\int_{t_k}^{t_{k+1}}\int_0^1\Big(\mathbb{E}|(g_j)_{xx}(ux(s\wedge\bar\theta_\triangle)+(1-u)\tilde x(t_k\wedge\bar\theta_\triangle))|^4\Big)^{\frac12}\\
&\qquad\times\Big(\mathbb{E}|x(s\wedge\bar\theta_\triangle)-\tilde x(t_k\wedge\bar\theta_\triangle)|^8\Big)^{\frac12}duds\leq C\triangle^3.
\end{aligned}$$

The proof is complete. ■

After this preparation we begin to analyze the error of strong convergence. To clarity, define $\bar{e}(t) = x(t) - \bar{y}(t)$, $e(t) := \tilde{x}(t) - y(t)$. Combining (4.9) with (4.11) yields that for any k and any $t_k \le s \le t < t_{k+1}$,

$$\begin{aligned}\bar{e}(t) - \bar{e}(s) &= \Big(f(\tilde{x}(s)) - f(y(s))\Big)(t-s) + \sum_{j=1}^{m}\Big(g_j(\tilde{x}(s)) - g_j(y(s))\Big)(B_j(t) - B_j(s)) \\ &\quad + \sum_{i,j=1}^{m}\Big(L^i g_j(\tilde{x}(s)) - L^i g_j(y(s))\Big)I_{i,j}(s,t) \\ &\quad + \sum_{i=1}^{3}\int_s^t \Big(R_i(f,s,r)dr + \sum_{j=1}^{m} R_i(g_j,s,r)dB_j(r)\Big).\end{aligned} \tag{4.14}$$

Theorem 4.1. *If Assumptions 1 and 2 hold with $p/4 \ge l_0 \vee (l+2)$, for the numerical solution defined by (3.4) and (3.7) with $\varrho \in [2l/(p-2), 1/2]$,*

$$\mathbb{E}|\bar{y}(T) - x(T)|^2 \le C\triangle^2, \quad \forall\, T > 0. \tag{4.15}$$

Proof. Since it is rather technical we divide the proof into 3 steps.

Step 1. Define $\Omega_1 := \{\omega : \ \bar{\theta}_\triangle > T\}$. Using the Young inequality, for any $\kappa > 0$, we have

$$\begin{aligned}\mathbb{E}|\bar{e}(T)|^2 &= \mathbb{E}\left(|\bar{e}(T)|^2 I_{\Omega_1}\right) + \mathbb{E}\left(|\bar{e}(T)|^2 I_{\Omega_1^c}\right) \\ &\le \mathbb{E}\left(|\bar{e}(T)|^2 I_{\Omega_1}\right) + \frac{2\triangle^\kappa}{p}\mathbb{E}\left(|\bar{e}(T)|^p\right) + \frac{p-2}{p\triangle^{2\kappa/(p-2)}}\mathbb{P}(\Omega_1^c).\end{aligned} \tag{4.16}$$

It follows from Lemmas 2.1 and 3.2 that

$$\frac{2\triangle^\kappa}{p}\mathbb{E}\left(|\bar{e}(T)|^p\right) \le C\triangle^\kappa\left[\mathbb{E}\left(|x(T)|^p\right) + \mathbb{E}\left(|\bar{y}(T)|^p\right)\right] \le C\triangle^\kappa. \tag{4.17}$$

By virtue of Lemmas 2.1 and 3.3 one observes that

$$\begin{aligned}\tfrac{p-2}{p\triangle^{2\kappa/(p-2)}}\mathbb{P}(\Omega_1^c) &\le \frac{p-2}{p\triangle^{2\kappa/(p-2)}}(\mathbb{P}\{\tau_{\varphi^{-1}(K\triangle^{-\varrho})} \le T\} + \mathbb{P}\{\zeta_\triangle \le T\}) \\ &\le \frac{p-2}{p\triangle^{2\kappa/(p-2)}}\frac{2C}{(\varphi^{-1}(K\triangle^{-\varrho}))^p} \le C\triangle^{\frac{\varrho p}{l} - \frac{2\kappa}{p-2}}.\end{aligned} \tag{4.18}$$

Step 2. We estimate the first term $\mathbb{E}\left(|\bar{e}(T)|^2 I_{\Omega_1}\right)$ on the right side of (4.16). Obviously, $\mathbb{E}\left(|\bar{e}(T)|^2 I_{\Omega_1}\right) \le \mathbb{E}\left(|\bar{e}(T \wedge \bar{\theta}_\triangle)|^2\right)$. Next, our main aim is to estimate the upper bound of $\mathbb{E}\left(|\bar{e}(T \wedge \bar{\theta}_\triangle)|^2\right)$. Since this step is rather technical and complex we divide it into 3 small parts.

Step 2. Part I Expand $\bar{e}'(t_{k+1} \wedge \bar{\theta}_\triangle)Q_0\bar{e}(t_{k+1} \wedge \bar{\theta}_\triangle)$. Note that for any $0 \le k(\omega)\triangle \le T \wedge \bar{\theta}_\triangle$, $\bar{y}(t_k) = y(t_k) = y_k = \tilde{y}_k$, then (4.14) holds with replacing $s,\ t$ by t_k, t_{k+1}, respectively. For any fixed k with $0 \le k\triangle \le T$ (if $k = n_T$, we just let $t_{k+1} = T$), it follows from (4.14) that

$$\begin{aligned}&\bar{e}(t_{k+1} \wedge \bar{\theta}_\triangle) - \bar{e}(t_k \wedge \bar{\theta}_\triangle) \\ &= \Big(f(\tilde{x}(t_k \wedge \bar{\theta}_\triangle)) - f(y(t_k \wedge \bar{\theta}_\triangle))\Big)(t_{k+1} \wedge \bar{\theta}_\triangle - t_k \wedge \bar{\theta}_\triangle) \\ &\quad + \sum_{j=1}^{m}\Big(g_j(\tilde{x}(t_k \wedge \bar{\theta}_\triangle)) - g_j(y(t_k \wedge \bar{\theta}_\triangle))\Big)(B_j(t_{k+1} \wedge \bar{\theta}_\triangle) - B_j(t_k \wedge \bar{\theta}_\triangle)) \\ &\quad + \sum_{i,j=1}^{m}\Big(L^i g_j(\tilde{x}(t_k \wedge \bar{\theta}_\triangle)) - L^i g_j(y(t_k \wedge \bar{\theta}_\triangle))\Big)I_{i,j}(t_k \wedge \bar{\theta}_\triangle, t_{k+1} \wedge \bar{\theta}_\triangle) \\ &\quad + \sum_{i=1}^{3}\int_{t_k \wedge \bar{\theta}_\triangle}^{t_{k+1} \wedge \bar{\theta}_\triangle}\Big(R_i(f, t_k \wedge \bar{\theta}_\triangle, r)dr + \sum_{j=1}^{m} R_i(g_j, t_k \wedge \bar{\theta}_\triangle, r)dB_j(r)\Big).\end{aligned} \tag{4.19}$$

Then, one observes that

$$\bar{e}'(t_{k+1} \wedge \bar{\theta}_\triangle)Q_0\bar{e}(t_{k+1} \wedge \bar{\theta}_\triangle) = \bar{e}'(t_k \wedge \bar{\theta}_\triangle)Q_0\bar{e}(t_k \wedge \bar{\theta}_\triangle) + \sum_{i=1}^{14}\eta_i(t_k \wedge \bar{\theta}_\triangle, t_{k+1} \wedge \bar{\theta}_\triangle), \tag{4.20}$$

where

$$
\begin{aligned}
\eta_1(s,t) &= 2\bar{e}'(s)Q_0\sum_{j=1}^{m}(g_j(\tilde{x}(s))-g_j(y(s)))(B_j(t)-B_j(s)),\\
\eta_2(s,t) &= 2\bar{e}'(s)Q_0(f(\tilde{x}(s))-f(y(s)))(t-s),\\
\eta_3(s,t) &= 2\bar{e}'(s)Q_0\sum_{i,j=1}^{m}(L^ig_j(\tilde{x}(s))-L^ig_j(y(s)))I_{i,j}(s,t),\\
\eta_4(s,t) &= \sum_{j=1}^{m}(B_j(t)-B_j(s))(g_j(\tilde{x}(s))-g_j(y(s)))'Q_0\sum_{j=1}^{m}(g_j(\tilde{x}(s))-g_j(y(s)))(B_j(t)-B_j(s)),\\
\eta_5(s,t) &= 2(t-s)(f(\tilde{x}(s))-f(y(s)))'Q_0\sum_{j=1}^{m}(g_j(\tilde{x}(s))-g_j(y(s)))(B_j(t)-B_j(s)),\\
\eta_6(s,t) &= 2\sum_{j=1}^{m}(B_j(t)-B_j(s))(g_j(\tilde{x}(s))-g_j(y(s)))'Q_0\sum_{i,j=1}^{m}(L^ig_j(\tilde{x}(s))-L^ig_j(y(s)))I_{i,j}(s,t),\\
\eta_7(s,t) &= 2\bar{e}'(s)Q_0\sum_{i=1}^{3}\int_s^t(R_i(f,s,r)dr+\sum_{j=1}^{m}R_i(g_j,s,r)dB_j(r)),\\
\eta_8(s,t) &= (t-s)^2(f(\tilde{x}(s))-f(y(s)))'Q_0(f(\tilde{x}(s))-f(y(s))),\\
\eta_9(s,t) &= \sum_{i,j=1}^{m}I_{i,j}(s,t)(L^ig_j(\tilde{x}(s))-L^ig_j(y(s)))'Q_0\sum_{i,j=1}^{m}(L^ig_j(\tilde{x}(s))-L^ig_j(y(s)))I_{i,j}(s,t),\\
\eta_{10}(s,t) &= 2(t-s)(f(\tilde{x}(s))-f(y(s)))'Q_0\sum_{i,j=1}^{m}(L^ig_j(\tilde{x}(s))-L^ig_j(y(s)))I_{i,j}(s,t),\\
\eta_{11}(s,t) &= 2\sum_{j=1}^{m}(B_j(t)-B_j(s))(g_j(\tilde{x}(s))-g_j(y(s)))'Q_0\\
&\quad\times\sum_{i=1}^{3}\int_s^t\Big(R_i(f,s,r)dr+\sum_{l=1}^{m}R_i(g_l,s,r)dB_l(r)\Big),\\
\eta_{12}(s,t) &= 2(t-s)(f(\tilde{x}(s))-f(y(s)))'Q_0\sum_{i=1}^{3}\int_s^t\Big(R_i(f,s,r)dr+\sum_{l=1}^{m}R_i(g_l,s,r)dB_l(r)\Big),\\
\eta_{13}(s,t) &= 2\sum_{i,j=1}^{m}I_{i,j}(s,t)(L^ig_j(\tilde{x}(s))-L^ig_j(y(s)))'Q_0\\
&\quad\times\sum_{i=1}^{3}\int_s^t\Big(R_i(f,s,r)dr+\sum_{l=1}^{m}R_i(g_l,s,r)dB_l(r)\Big),\\
\eta_{14}(s,t) &= \sum_{i=1}^{3}\int_s^t\Big(R_i(f,s,r)dr+\sum_{l=1}^{m}R_i(g_l,s,r)dB_l(r)\Big)'Q_0\\
&\quad\times\sum_{i=1}^{3}\int_s^t\Big(R_i(f,s,r)dr+\sum_{l=1}^{m}R_i(g_l,s,r)dB_l(r)\Big).
\end{aligned}
$$

Step 2. Part II. Our aim of this part is to compute the super bound of each $\mathbb{E}(\eta_i(t_k\wedge\bar{\theta}_\triangle, t_{k+1}\wedge\bar{\theta}_\triangle))$. By virtue of the Doob martingale stopping time theorem [3, p.11, Theorem 3.3], we know that $\forall i,j,l=1,\dots,m,\ k\geq 0$,

$$
\begin{aligned}
&\mathbb{E}(B_i(t_{k+1}\wedge\bar{\theta}_\triangle)-B_i(t_k\wedge\bar{\theta}_\triangle)|\mathcal{F}_{t_k\wedge\bar{\theta}_\triangle})=0,\\
&\mathbb{E}(I_{i,j}(t_k\wedge\bar{\theta}_\triangle,t_{k+1}\wedge\bar{\theta}_\triangle)|\mathcal{F}_{t_k\wedge\bar{\theta}_\triangle})=\mathbb{E}\Big(\int_{t_k\wedge\bar{\theta}_\triangle}^{t_{k+1}\wedge\bar{\theta}_\triangle}\int_{t_{n_{t_k\wedge\bar{\theta}_\triangle}}}^{s}dB_i(u)dB_j(s)|\mathcal{F}_{t_k\wedge\bar{\theta}_\triangle}\Big)\\
&=\mathbb{E}\Big(\int_{t_k\wedge\bar{\theta}_\triangle}^{t_{k+1}\wedge\bar{\theta}_\triangle}(B_i(s)-B_i(t_{n_{t_k\wedge\bar{\theta}_\triangle}}))dB_j(s)|\mathcal{F}_{t_k\wedge\bar{\theta}_\triangle}\Big)=0,\\
&\mathbb{E}((B_i(t_{k+1}\wedge\bar{\theta}_\triangle)-B_i(t_k\wedge\bar{\theta}_\triangle))(B_j(t_{k+1}\wedge\bar{\theta}_\triangle)-B_j(t_k\wedge\bar{\theta}_\triangle))|\mathcal{F}_{t_k\wedge\bar{\theta}_\triangle})=0,\ \forall i\neq j,\\
&\mathbb{E}((B_l(t_{k+1}\wedge\bar{\theta}_\triangle)-B_l(t_k\wedge\bar{\theta}_\triangle))I_{i,j}(t_k\wedge\bar{\theta}_\triangle,t_{k+1}\wedge\bar{\theta}_\triangle)|\mathcal{F}_{t_k\wedge\bar{\theta}_\triangle})
\end{aligned}
$$

$$= \mathbb{E}\Big((B_l(t_{k+1}\wedge\bar\theta_\triangle) - B_l(t_k\wedge\bar\theta_\triangle))\int_{t_k\wedge\bar\theta_\triangle}^{t_{k+1}\wedge\bar\theta_\triangle}\int_{t_{n_{t_k\wedge\bar\theta_\triangle}}}^{s} dB_i(u)dB_j(s)|\mathcal{F}_{t_k\wedge\bar\theta_\triangle}\Big) = 0,$$
$$\mathbb{E}((B_i(t_{k+1}\wedge\bar\theta_\triangle) - B_i(t_k\wedge\bar\theta_\triangle))^2|\mathcal{F}_{t_k\wedge\bar\theta_\triangle}) = \mathbb{E}(t_{k+1}\wedge\bar\theta_\triangle - t_k\wedge\bar\theta_\triangle|\mathcal{F}_{t_k\wedge\bar\theta_\triangle}) \le \triangle.$$

For convenience, define $\chi_k = t_{k+1}\wedge\bar\theta_\triangle - t_k\wedge\bar\theta_\triangle$. Then one observes that

$$\begin{aligned}&\mathbb{E}(\eta_1(t_k\wedge\bar\theta_\triangle, t_{k+1}\wedge\bar\theta_\triangle)) = 0, \quad \mathbb{E}(\eta_3(t_k\wedge\bar\theta_\triangle, t_{k+1}\wedge\bar\theta_\triangle)) = 0,\\ &\mathbb{E}(\eta_5(t_k\wedge\bar\theta_\triangle, t_{k+1}\wedge\bar\theta_\triangle)) = 0, \quad \mathbb{E}(\eta_6(t_k\wedge\bar\theta_\triangle, t_{k+1}\wedge\bar\theta_\triangle)) = 0,\\ &\mathbb{E}(\eta_{10}(t_k\wedge\bar\theta_\triangle, t_{k+1}\wedge\bar\theta_\triangle)) = 0.\end{aligned} \tag{4.21}$$

Now we begin to estimate $\mathbb{E}(\eta_2(t_k\wedge\bar\theta_\triangle, t_{k+1}\wedge\bar\theta_\triangle))$. Note that

$$\begin{aligned}&\mathbb{E}(\eta_2(t_k\wedge\bar\theta_\triangle, t_{k+1}\wedge\bar\theta_\triangle))\\ &= 2\mathbb{E}(\bar e'(t_k\wedge\bar\theta_\triangle)Q_0(f(\tilde x(t_k\wedge\bar\theta_\triangle)) - f(y(t_k\wedge\bar\theta_\triangle)))\chi_k)\\ &= 2\mathbb{E}(e'(t_k\wedge\bar\theta_\triangle)Q_0(f(\tilde x(t_k\wedge\bar\theta_\triangle)) - f(y(t_k\wedge\bar\theta_\triangle)))\chi_k)\\ &\quad + 2\mathbb{E}((x(t_k\wedge\bar\theta_\triangle) - \tilde x(t_k\wedge\bar\theta_\triangle))'Q_0(f(\tilde x(t_k\wedge\bar\theta_\triangle)) - f(y(t_k\wedge\bar\theta_\triangle)))\chi_k)\\ &\quad + 2\mathbb{E}((y(t_k\wedge\bar\theta_\triangle) - \bar y(t_k\wedge\bar\theta_\triangle))'Q_0(f(\tilde x(t_k\wedge\bar\theta_\triangle)) - f(y(t_k\wedge\bar\theta_\triangle)))\chi_k).\end{aligned} \tag{4.22}$$

By virtue of Lemma 4.2 and Hölder's inequality, one obtains from Assumption 2

$$\begin{aligned}&\mathbb{E}((x(t_k\wedge\bar\theta_\triangle) - \tilde x(t_k\wedge\bar\theta_\triangle))'Q_0(f(\tilde x(t_k\wedge\bar\theta_\triangle)) - f(y(t_k\wedge\bar\theta_\triangle)))\chi_k)\\ &= \mathbb{E}\Big(\mathbb{E}\Big((x(t_k\wedge\bar\theta_\triangle) - x(t_{n_{t_k\wedge\bar\theta_\triangle}}))'|\mathcal{F}_{t_{n_{t_k\wedge\bar\theta_\triangle}}}\Big)Q_0(f(x(t_{n_{t_k\wedge\bar\theta_\triangle}})) - f(\bar y(t_{n_{t_k\wedge\bar\theta_\triangle}})))\chi_k\Big)\\ &= \mathbb{E}\Big(\int_{t_{n_{t_k\wedge\bar\theta_\triangle}}}^{t_k\wedge\bar\theta_\triangle} f'(x(s))dsQ_0(f(x(t_{n_{t_k\wedge\bar\theta_\triangle}})) - f(\bar y(t_{n_{t_k\wedge\bar\theta_\triangle}})))\chi_k\Big)\\ &\le C\triangle\mathbb{E}\Big(|\int_{t_{n_{t_k\wedge\bar\theta_\triangle}}}^{t_k\wedge\bar\theta_\triangle} f(x(s))ds||f(x(t_{n_{t_k\wedge\bar\theta_\triangle}})) - f(\bar y(t_{n_{t_k\wedge\bar\theta_\triangle}}))|\Big)\\ &\le C\triangle\mathbb{E}\Big(\int_{t_k-\triangle}^{t_k} |f(x(s\wedge\bar\theta_\triangle))|ds(1 + |x(t_{n_{t_k\wedge\bar\theta_\triangle}})|^l + |\bar y(t_{n_{t_k\wedge\bar\theta_\triangle}})|^l)|e(t_k\wedge\bar\theta_\triangle)|\Big)\\ &\le C\triangle\mathbb{E}\Big((\int_{t_k-\triangle}^{t_k} |f(x(s\wedge\bar\theta_\triangle))|ds)^2(1 + |x(t_{n_{t_k\wedge\bar\theta_\triangle}})|^l + |\bar y(t_{n_{t_k\wedge\bar\theta_\triangle}})|^l)^2\Big)\\ &\qquad + C\triangle\mathbb{E}(|e(t_k\wedge\bar\theta_\triangle)|^2)\\ &\le C\triangle\Big(\mathbb{E}(\int_{t_k-\triangle}^{t_k} |f(x(s\wedge\bar\theta_\triangle))|ds)^4\Big)^{\frac12}\Big(\mathbb{E}(1 + |x(t_{n_{t_k\wedge\bar\theta_\triangle}})|^l + |\bar y(t_{n_{t_k\wedge\bar\theta_\triangle}})|^l)^4\Big)^{\frac12}\\ &\qquad + C\triangle\mathbb{E}(|e(t_k\wedge\bar\theta_\triangle)|^2)\\ &\le C\triangle^{\frac52}\Big(\mathbb{E}|\int_{t_k-\triangle}^{t_k} |f(x(s\wedge\bar\theta_\triangle))|^4ds|\Big)^{\frac12} + C\triangle\mathbb{E}(|e(t_k\wedge\bar\theta_\triangle)|^2)\\ &\le C\triangle^3 + C\triangle\mathbb{E}(|e(t_k\wedge\bar\theta_\triangle)|^2).\end{aligned} \tag{4.23}$$

Similarly, we obtain

$$\begin{aligned}&\mathbb{E}((y(t_k\wedge\bar\theta_\triangle) - \bar y(t_k\wedge\bar\theta_\triangle))'Q_0(f(\tilde x(t_k\wedge\bar\theta_\triangle)) - f(y(t_k\wedge\bar\theta_\triangle)))\chi_k)\\ &\qquad\qquad \le C\triangle^3 + C\triangle\mathbb{E}(|e(t_k\wedge\bar\theta_\triangle)|^2).\end{aligned} \tag{4.24}$$

Inserting (4.23) and (4.24) into (4.22) implies

$$\begin{aligned}\mathbb{E}(\eta_2(t_k\wedge\bar\theta_\triangle, t_{k+1}\wedge\bar\theta_\triangle)) &\le 2\mathbb{E}(e'(t_k\wedge\bar\theta_\triangle)Q_0(f(\tilde x(t_k\wedge\bar\theta_\triangle)) - f(y(t_k\wedge\bar\theta_\triangle)))\chi_k)\\ &\quad + C\triangle^3 + C\triangle\mathbb{E}(|e(t_k\wedge\bar\theta_\triangle)|^2).\end{aligned} \tag{4.25}$$

Note that

$$\begin{aligned}&\mathbb{E}(\eta_4(t_k\wedge\bar\theta_\triangle, t_{k+1}\wedge\bar\theta_\triangle)|\mathcal{F}_{t_k\wedge\bar\theta_\triangle})\\ &\quad = \mathrm{trace}[(g(\tilde x(t_k\wedge\bar\theta_\triangle)) - g(y(t_k\wedge\bar\theta_\triangle)))'Q_0(g(\tilde x(t_k\wedge\bar\theta_\triangle)) - g(y(t_k\wedge\bar\theta_\triangle)))]\chi_k.\end{aligned} \tag{4.26}$$

To estimate $\mathbb{E}(\eta_7(t_k \wedge \bar{\theta}_\triangle, t_{k+1} \wedge \bar{\theta}_\triangle))$, by virtue of the Doob martingale stopping time theorem [3, p.11, Theorem 3.3], we obtain

$$\begin{aligned}
&\mathbb{E}(2\bar{e}'(t_k \wedge \bar{\theta}_\triangle)Q_0 \int_{t_k\wedge\bar{\theta}_\triangle}^{t\wedge\bar{\theta}_\triangle} R_1(f, t_k \wedge \bar{\theta}_\triangle, s)ds) \\
&= \mathbb{E}(2\bar{e}'(t_k \wedge \bar{\theta}_\triangle)Q_0 \sum_{j=1}^{m} \int_{t_k\wedge\bar{\theta}_\triangle}^{t_{k+1}\wedge\bar{\theta}_\triangle} f_x(x(t_{n_{t_k\wedge\bar{\theta}_\triangle}})) \int_{t_{n_{t_k\wedge\bar{\theta}_\triangle}}}^{s} g_j(x(u))dB_j(u)ds) \\
&= \sum_{j=1}^{m} \mathbb{E}\Big(2\bar{e}'(t_k \wedge \bar{\theta}_\triangle)Q_0 f_x(\tilde{x}(t_k \wedge \bar{\theta}_\triangle))\mathbb{E}\Big(\int_{t_k\wedge\bar{\theta}_\triangle}^{t_{k+1}\wedge\bar{\theta}_\triangle} \int_{t_{n_{t_k\wedge\bar{\theta}_\triangle}}}^{s} g_j(x(u))dB_j(u)ds \mid \mathcal{F}_{t_k\wedge\bar{\theta}_\triangle}\Big)\Big) = 0.
\end{aligned}$$

The above equality implies that

$$\begin{aligned}
&\mathbb{E}(\eta_7(t_k \wedge \bar{\theta}_\triangle, t_{k+1} \wedge \bar{\theta}_\triangle)) \\
&= \sum_{i=2}^{3} \mathbb{E}\Big(2e'(t_k \wedge \bar{\theta}_\triangle)Q_0 \int_{t_k\wedge\bar{\theta}_\triangle}^{t_{k+1}\wedge\bar{\theta}_\triangle} R_i(f, t_k \wedge \bar{\theta}_\triangle, s)ds\Big) \\
&= 2\triangle\mathbb{E}(|e(t_k \wedge \bar{\theta}_\triangle)|^2) + \triangle^{-1}|Q_0|^2|\sum_{i=2}^{3} \mathbb{E}|\int_{t_k\wedge\bar{\theta}_\triangle}^{t_{k+1}\wedge\bar{\theta}_\triangle} R_i(f, t_k \wedge \bar{\theta}_\triangle, s)ds|^2.
\end{aligned}$$

Inserting the second inequality of (4.13) into the above equation arrives at

$$\mathbb{E}(\eta_7(t_k \wedge \bar{\theta}_\triangle, t_{k+1} \wedge \bar{\theta}_\triangle)) \le \triangle\mathbb{E}(|e(t_k \wedge \bar{\theta}_\triangle)|^2) + C\triangle^3. \tag{4.27}$$

It follows from the nice property (3.3) of the truncated scheme that

$$\mathbb{E}(\eta_8(t_k \wedge \bar{\theta}_\triangle, t_{k+1} \wedge \bar{\theta}_\triangle)) \le C\triangle^2\triangle^{-2\varrho}\mathbb{E}|e(t_k \wedge \bar{\theta}_\triangle)|^2 \le C\triangle\mathbb{E}|e(t_k \wedge \bar{\theta}_\triangle)|^2. \tag{4.28}$$

Similarly, using (3.3) again yields that

$$\mathbb{E}(\eta_9(t_k \wedge \bar{\theta}_\triangle, t_{k+1} \wedge \bar{\theta}_\triangle)) \le C\triangle\mathbb{E}|e(t_k \wedge \bar{\theta}_\triangle)|^2. \tag{4.29}$$

To estimate $\mathbb{E}(\eta_{11}(t_k \wedge \bar{\theta}_\triangle, t_{k+1} \wedge \bar{\theta}_\triangle))$, choose a sufficiently small constant such that $0 < \iota < p_0 - 1$. Noting

$$\begin{aligned}
\eta_{11}(s, t) &= 2\sum_{j=1}^{m}(B_j(t) - B_j(s))(g_j(\tilde{x}(s)) - g_j(y(s)))'Q_0 \sum_{i=1}^{3} \int_s^t \big[R_i(f, s, r)dr + \sum_{l=1}^{m} R_i(g_l, s, r)dB_l(r)\big] \\
&\le \iota\big[\sum_{j=1}^{m}(B_j(t) - B_j(s))(g_j(\tilde{x}(s)) - g_j(y(s)))'\big]Q_0\big[\sum_{j=1}^{m}(B_j(t) - B_j(s))(g_j(\tilde{x}(s)) - g_j(y(s)))\big] \\
&\quad + \frac{12}{\iota}|Q_0|\sum_{i=1}^{3}|\int_s^t R_i(f, s, r)dr|^2 + \frac{12m}{\iota}|Q_0|\sum_{i=1}^{3}\sum_{l=1}^{m}|\int_s^t R_i(g_l, s, r)dB_l(r)|^2,
\end{aligned}$$

by virtue of Lemma 4.3, we have

$$\begin{aligned}
&\mathbb{E}(\eta_{11}(t_k \wedge \bar{\theta}_\triangle, t_{k+1} \wedge \bar{\theta}_\triangle)) = \mathbb{E}\Big(\mathbb{E}(\eta_{11}(t_k \wedge \bar{\theta}_\triangle, t_{k+1} \wedge \bar{\theta}_\triangle))|\mathcal{F}_{t_k\wedge\bar{\theta}_\triangle}\Big) \\
&\le \iota\mathbb{E}(\text{trace}[(g(\tilde{x}(t_k \wedge \bar{\theta}_\triangle)) - g(y(t_k \wedge \bar{\theta}_\triangle)))'Q_0(g(\tilde{x}(t_k \wedge \bar{\theta}_\triangle)) - g(y(t_k \wedge \bar{\theta}_\triangle)))]\chi_k) \\
&\quad + C\sum_{i=1}^{3}\mathbb{E}|\int_{t_k\wedge\bar{\theta}_\triangle}^{t_{k+1}\wedge\bar{\theta}_\triangle} R_i(f, t_k \wedge \bar{\theta}_\triangle, s)ds|^2 + C\sum_{i=1}^{3}\sum_{l=1}^{m}\mathbb{E}|\int_{t_k\wedge\bar{\theta}_\triangle}^{t_{k+1}\wedge\bar{\theta}_\triangle} R_i(g_l, t_k \wedge \bar{\theta}_\triangle, s)dB_l(s)|^2 \\
&\le \iota\mathbb{E}(\text{trace}[(g(\tilde{x}(t_k \wedge \bar{\theta}_\triangle)) - g(y(t_k \wedge \bar{\theta}_\triangle)))'Q_0(g(\tilde{x}(t_k \wedge \bar{\theta}_\triangle)) - g(y(t_k \wedge \bar{\theta}_\triangle)))]\chi_k) \\
&\quad + C\triangle^3.
\end{aligned} \tag{4.30}$$

Using the truncation property (3.3), one derives from Lemma 4.3 that

$$\begin{aligned}
&\mathbb{E}(\eta_{12}(t_k \wedge \bar{\theta}_\triangle, t_{k+1} \wedge \bar{\theta}_\triangle)) \\
&\le 2\mathbb{E}(|t_{k+1} \wedge \bar{\theta}_\triangle - t_k \wedge \bar{\theta}_\triangle|^2|f(x(t_k \wedge \bar{\theta}_\triangle)) - f(y(t_k \wedge \bar{\theta}_\triangle))|^2) \\
&\quad + 3|Q_0|^2\sum_{i=1}^{3}\mathbb{E}|\int_{t_k\wedge\bar{\theta}_\triangle}^{t_{k+1}\wedge\bar{\theta}_\triangle} R_i(f, t_k \wedge \bar{\theta}_\triangle, s)ds|^2 \\
&\quad + 3m|Q_0|^2\sum_{i=1}^{3}\sum_{j=1}^{m}\mathbb{E}|\int_{t_k\wedge\bar{\theta}_\triangle}^{t_{k+1}\wedge\bar{\theta}_\triangle} R_i(g_j, t_k \wedge \bar{\theta}_\triangle, s)dB_j(s)|^2
\end{aligned}$$

$$\le 2K^2\triangle^{2-2\varrho}\mathbb{E}(|e(t_k\wedge\bar\theta_\triangle)|^2)+C\triangle^3+C\triangle^4$$
$$\le 2K^2\triangle\mathbb{E}(|e(t_k\wedge\bar\theta_\triangle)|^2)+C\triangle^3. \tag{4.31}$$

Furthermore, we obtain

$$\mathbb{E}(\eta_{13}(t_k\wedge\bar\theta_\triangle,t_{k+1}\wedge\bar\theta_\triangle))\le 2K^2\triangle\mathbb{E}(|e(t_k\wedge\bar\theta_\triangle)|^2)+C\triangle^3. \tag{4.32}$$

It follows from Lemma 4.3 that

$$\begin{aligned}&\mathbb{E}(\eta_{14}(t_k\wedge\bar\theta_\triangle,t_{k+1}\wedge\bar\theta_\triangle))\\ &\le 6|Q_0|\sum_{i=1}^{3}\mathbb{E}|\int_{t_k\wedge\bar\theta_\triangle}^{t_{k+1}\wedge\bar\theta_\triangle}R_i(f,t_k\wedge\bar\theta_\triangle,s)ds|^2\\ &\quad+6m|Q_0|\sum_{i=1}^{3}\sum_{j=1}^{m}\mathbb{E}|\int_{t_k\wedge\bar\theta_\triangle}^{t_{k+1}\wedge\bar\theta_\triangle}R_i(g_j,t_k\wedge\bar\theta_\triangle,s)dB_j(s)|^2\le C\triangle^3.\end{aligned} \tag{4.33}$$

Step 2. Part III. Complete the estimates of $\mathbb{E}(|\bar e(t\wedge\bar\theta_\triangle)|^2)$ and $\mathbb{E}\left(|\bar e(T)|^2 I_{\Omega_1}\right)$. Under Assumption 2, using (4.21) and (4.25)–(4.33) together with (4.20) imply

$$\begin{aligned}&\mathbb{E}(\bar e'(t_{k+1}\wedge\bar\theta_\triangle)Q_0\bar e(t_{k+1}\wedge\bar\theta_\triangle))-\mathbb{E}(\bar e'(t_k\wedge\bar\theta_\triangle)Q_0\bar e(t_k\wedge\bar\theta_\triangle))\\ &=\sum_{i=1}^{14}\mathbb{E}(\eta_i(t_k\wedge\bar\theta_\triangle,t_{k+1}\wedge\bar\theta_\triangle))\\ &\le C\triangle^3+C\triangle\mathbb{E}|e(t_k\wedge\bar\theta_\triangle)|^2+\mathbb{E}\Big(\Big(2e'(t_k\wedge\bar\theta_\triangle)Q_0(f(\tilde x(t_k\wedge\bar\theta_\triangle))-f(y(t_k\wedge\bar\theta_\triangle)))\\ &\qquad+p_0\text{trace}[(g_j(\tilde x(t_k\wedge\bar\theta_\triangle))-g_j(y(t_k\wedge\bar\theta_\triangle)))'Q_0(g_j(\tilde x(t_k\wedge\bar\theta_\triangle))-g_j(y(t_k\wedge\bar\theta_\triangle)))]\Big)\chi_t\Big)\\ &\le C\triangle^3+C\triangle\mathbb{E}(|e(t_k\wedge\bar\theta_\triangle)|^2)\\ &\le C\triangle^3+C\triangle\mathbb{E}(e'(t_k\wedge\bar\theta_\triangle)Q_0e(t_k\wedge\bar\theta_\triangle)).\end{aligned} \tag{4.34}$$

Then we arrive at

$$\mathbb{E}(\bar e'(t_{k+1}\wedge\bar\theta_\triangle)Q_0\bar e(t_{k+1}\wedge\bar\theta_\triangle))\le C\triangle^2+C\triangle\sum_{i=0}^{k}\mathbb{E}(e'(t_i\wedge\bar\theta_\triangle)Q_0e(t_i\wedge\bar\theta_\triangle)).$$

Using the same techniques as the above inequality yields that for any $0\le i\le k$ and any $t\in[t_i,t_{i+1}]$,

$$\mathbb{E}(\bar e'(t\wedge\bar\theta_\triangle)Q_0\bar e(t\wedge\bar\theta_\triangle))\le C\triangle^2+C\triangle\sum_{i=0}^{k}\mathbb{E}(e'(t_i\wedge\bar\theta_\triangle)Q_0e(t_i\wedge\bar\theta_\triangle)),$$

which implies

$$\begin{aligned}\sup_{0\le t\le T}\mathbb{E}(\bar e'(t\wedge\bar\theta_\triangle)Q_0\bar e(t\wedge\bar\theta_\triangle))&\le C\triangle^2+C\triangle\sum_{i=0}^{n_T}\mathbb{E}(e'(t_i\wedge\bar\theta_\triangle)Q_0e(t_i\wedge\bar\theta_\triangle))\\ &\le C\triangle^2+C\triangle\sum_{i=0}^{n_T}\sup_{0\le s\le t_i}\mathbb{E}(e'(s\wedge\bar\theta_\triangle)Q_0e(s\wedge\bar\theta_\triangle))\\ &\le C\triangle^2+C\triangle\sum_{i=0}^{n_T}\sup_{0\le s\le t_i}\mathbb{E}(\bar e'(s\wedge\bar\theta_\triangle)Q_0\bar e(s\wedge\bar\theta_\triangle)).\end{aligned} \tag{4.35}$$

An application of the discrete Gronwall inequality leads to

$$\sup_{0\le t\le T}\mathbb{E}(\bar e'(t\wedge\bar\theta_\triangle)Q_0\bar e(t\wedge\bar\theta_\triangle))\le C\triangle^2e^{CT}\le C\triangle^2. \tag{4.36}$$

Therefore, by the positivity of Q_0, the desired result follows, namely,

$$\mathbb{E}(|\bar e(T)|^2I_{\Omega_1})\le\sup_{0\le t\le T}\mathbb{E}(|\bar e(t\wedge\bar\theta_\triangle)|^2)\le C\triangle^2. \tag{4.37}$$

Step 3. Inserting (4.17), (4.18), and (4.37) into (4.16) yields

$$\mathbb{E}|\bar e(T)|^2\le C\triangle^2+C\triangle^\kappa+C\triangle^{\frac{\varrho p}{l}-\frac{2\kappa}{p-2}}. \tag{4.38}$$

Let $\kappa=2$ and $\frac{\varrho p}{l}-\frac{2\kappa}{p-2}\ge 2$, which implies $\varrho\ge\frac{2l}{p-2}$. Therefore, the desired assertion follows. ∎

As a consequence of Theorem 3.1, we have the following corollary.

Corollary 4.1. *Under the conditions of Theorem 4.1, for the numerical solution defined by (3.4) with $\varrho \in [2l/(p-2), 1/2]$,*

$$\sup_{0\le k\triangle\le T} \mathbb{E}|y_x - x_k|^2 \le C\triangle^2, \quad \forall\, T > 0. \tag{4.39}$$

Remark 4.3. Zong–Wu–Xu [16] and Higham–Mao–Szpruch [13] obtained the optimal rate 1 for implicit Milstein schemes of strong convergence under the conditions that the functions f and g are C^2, and there exist a constant c such that

$$|f(x)-f(y)| \vee |g(x)-g(y)| \vee |L^i g_j(x) - L^i g_j(y)| \le c|x-y|, \quad \forall x, y \in \mathbb{R}^d, i, j = 1, \ldots, m.$$

Note that a similar convergence rate result was also obtained by Wang–Gan [10] for a modified tamed Milstein scheme under the conditions that f is one-sided Lipschitz with polynomial growth, and g and $L^i g_j$ are globally Lipschitz. In contrast to the aforementioned results, we are using explicit schemes with truncations. Our conditions are much weaker. We managed to show that the order 1 convergence is preserved under weaker conditions.

5. Numerical examples

In this section, we consider an example of a nonlinear system and conduct simulations using of our numerical scheme.

Example 5.1. Let us consider the Ginzburg–Landau equation from statistical physics in the study of phase transitions. Its stochastic version with multiplicative noise is given by

$$dx(t) = [(\eta + \frac{1}{2}\sigma^2)x(t) - \vartheta x^3(t)]dt + \sigma x(t)dB(t), \quad x(0) = x_0 > 0, \tag{5.1}$$

where $\sigma,\ \vartheta > 0$; see [5,8]. It can be verified that Assumptions 1 and 2 hold with all $p,\ p_0 > 2$ and $l = l_0 = 2$. Then (5.1) has a unique regular solution.

Let $\varphi_1(r) = C_1(r^2+1)$, $\forall r > 0$, where $C_1 = |\eta| + 3\vartheta + \frac{3}{2}\sigma^2 + 1$, $\varphi_1^{-1}(r) = \sqrt{r/C_1 - 1}$, $\forall r > C_1$. Define $K = \varphi_1(1 \vee |x_0|)$, $\varrho = 0.2$. For a fixed $\triangle \in (0, 1]$, the truncated EM scheme for (5.1) is

$$\begin{cases} y_0 &= x_0, \\ y_k &= \left(\tilde{y}_k \wedge \sqrt{(1\vee x_0^2+1)\triangle^{-0.2} - 1}\right)\frac{\tilde{y}_k}{|\tilde{y}_k|}, \\ \tilde{y}_{k+1} &= y_k + (\eta + \frac{1}{2}\sigma^2)y_k\triangle - \vartheta y_k^3\triangle + \sigma y_k \triangle B(k) + \frac{1}{2}\sigma^2 y_k((\triangle B(k))^2 - \triangle). \end{cases} \tag{5.2}$$

By virtue of Theorem 4.1, the numerical solution of this scheme approximates the exact solution

$$X(t) = \frac{x_0 \exp\{\eta t + \sigma B(t)\}}{\sqrt{1 + 2x_0^2\vartheta \int_0^t \exp\{2\eta s + 2\sigma B(s)\}ds}} \tag{5.3}$$

given by [8, p. 1568] in the mean square sense with error estimate $\triangle$. To demonstrate, we carry out numerical experiments by implementing (5.2) using MATLAB. We compare the truncated EM method with the backward EM scheme and the tamed EM scheme (see, e.g., [25]) numerically. Consider (5.1) with $\eta = -1$, $\sigma = 1$, $\vartheta = 1$, $x_0 = 10$ and $T = 1$. Fig. 1 plots the root mean square approximation error $(\mathbb{E}|x(T) - X(T)|^2)^{1/2}$ between the exact solution of (5.1) and the numerical solution by the backward EM scheme, and the error $(\mathbb{E}|x(T) - y(T)|^2)^{1/2}$ between the exact solution and that of the truncated Milstein scheme, as functions of the runtime and step sizes, respectively, when $\triangle \in \{2^{-12}, 2^{-13}, 2^{-14}, 2^{-15}, 2^{-16}, 2^{-17}\}$.

6. Concluding remarks

This paper developed explicit scheme for highly nonlinear stochastic differential equations with better strong convergence rate compared to the usual EM procedure. We constructed explicit numerical schemes that allowed both drift and diffusion coefficients being not globally Lipschitzian and growth faster than linear. We obtained convergence and moment boundedness of the numerical solutions under local Lipschitz condition and structure conditions required by the analytic solutions. Under additional mild conditions, order one rate of convergence was obtained. Our results were demonstrated through numerical experiments. It is worth mentioning that, by taking higher-order Taylor expansions and truncating the local Lipschitz continuous expansion coefficients, we can obtain higher-order stable schemes for nonlinear SDEs (cf. [5, pp. 351–364, Section 10.4-10.6] although only the case of global Lipschitz and linear growth was allowed for all the functions involved in [5]). The implementation of such schemes is normally computationally more intensive, however.

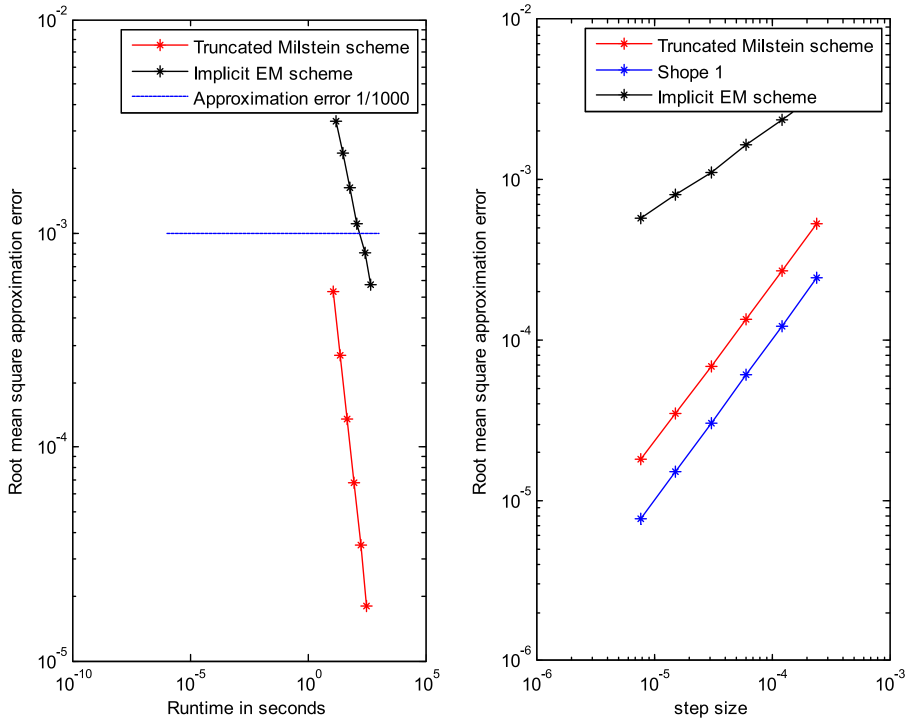

Fig. 1. The root mean square approximation errors for 1000 sample points between the exact solution $x(T)$ of SDE (5.1) and the numerical solutions: $X(T)$ by the implicit EM scheme, and $y(T)$ by the truncated Milstein scheme, as functions of the runtime and step sizes, respectively, when $\triangle \in \{2^{-12}, 2^{-13}, 2^{-14}, 2^{-15}, 2^{-16}, 2^{-17}\}$.

References

[1] G.N. Milstein, M.V. Tretyakov, Stochastic Numerics for Mathematical Physics, Scientific Computation, Springer-Verlag, Berlin, 2004.
[2] D.J. Higham, X. Mao, A.M. Stuart, Strong convergence of Euler-type methods for nonlinear stochastic differential equations, SIAM J. Numer. Anal. 40 (3) (2002) 1041–1063.
[3] X. Mao, Stochastic Differential Equations and Applications, second ed., Horwood, Chichester, UK, 2007.
[4] S.L. Nguyen, G. Yin, Pathwise convergence rate for numerical solutions of stochastic differential equations, IMA J. Numer. Anal. 32 (2012) 701–723.
[5] P.E. Kloeden, E. Platen, Numerical Solution of Stochastic Differential Equations, Springer-Verlag, Berlin, 1992.
[6] G.N. Milstein, Numerical Integration of Stochastic Differential Equations, Kluwer, Springer Science+Business Media Dordrecht, 1995.
[7] L. Yan, Asymptotic error for the milstein scheme for SDEs driven by continuous semimartingales, Ann. Appl. Probab. 15 (4) (2005) 2706–2738.
[8] M. Hutzenthaler, A. Jenzen, P.E. Kloeden, Strong and weak divergence in finite time of Euler's method for stochastic differential equations with non-globally Lipschitz continuous coefficients, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 467 (2011) 1563–1576.
[9] A.L. Lewis, Option Valuation under Stochastic Volatility, Finance Press, Newport Beach, California, 2000.
[10] X. Wang, S. Gan, The tamed milstein method for commutative stochastic differential equations with non-globally Lipschitz continuous coefficients, J. Differ. Equations Appl. 19 (3) (2013) 466–490.
[11] Q. Guo, W. Liu, X. Mao, R. Yue, The truncated Milstein method for stochastic differential equations with commutative noise, J. Comput. Appl. Math. 338 (2018) 298–310.
[12] E. Buckwar, T. Sickenberger, A comparative linear mean-square stability analysis of Maruyama-and Milstein-type methods, Math. Comput. Simulation 81 (2011) 1110–1127.
[13] D.J. Higham, X. Mao, L. Szpruch, Convergence, Non-negativity and stability of a new Milstein scheme with applications to finance, Discrete Contin. Dyn. Syst. Ser. B 18 (8) (2013) 2083–2100.
[14] M.A. Omara, A. Aboul-Hassanb, S.I. Rabiab, The composite milstein methods for the numerical solution of ito stochastic differential equations, J. Comput. Appl. Math. 235 (2011) 2277–2299.
[15] X. Wang, S. Gan, D. Wang, A family of fully implicit milstein methods for stiff stochastic differential equations with multiplicative noise, BIT 52 (2012) 741–772.
[16] X. Zong, F. Wu, G. Xu, Convergence and stability of two classes of theta-milstein schemes for stochastic differential equations, J. Comput. Appl. Math. 336 (2018) 8–29.
[17] S. Sabanis, Y. Zhang, On explicit order 1.5 approximations with varying coefficients: The case of super-linear diffusion coefficients, J. Complexity 50 (2019) 84–115.
[18] D.T. Nguyen, S.L. Nguyen, T.A. Hoang, G. Yin, Tamed-Euler method for hybrid stochastic differential equations with Markovian switching, Nonlinear Anal. Hybrid Syst. 30 (2018) 14–30.
[19] S.L. Nguyen, T.A. Hoang, D.T. Nguyen, G. Yin, Milstein-type procedures for numerical solutions of stochastic differential equations with Markovian switching, SIAM J. Numer. Anal. 55 (2017) 953–979.
[20] M. Giles, Improved multilevel Monte Carlo convergence using the milstein scheme, in: Monte Carlo and Quasi-Monte Carlo Methods, Vol. 2006, 2008, pp. 343–358.

[21] M. Giles, Multilevel Monte Carlo path simulation, Oper. Res.-Baltim. 56 (2008) 607–617.
[22] D. Higham, M. Roj, X. Mao, Q.S. Song, G. Yin, Mean exit times and the multi-level Monte Carlo method, SIAM/ASA J. Uncertain. Quantif. 1 (2013) 2–18.
[23] X. Li, X. Mao, G. Yin, Explicit numerical approximations for stochastic differential equations in finite and infinite horizons: truncation methods, convergence in pth moment and stability, IMA J. Numer. Anal. 39 (2019) 847–892.
[24] H. Yang, X. Li, Explicit approximations for nonlinear switching diffusion systems in finite and infinite horizons, J. Differential Equations 265 (2018) 2921–2967.
[25] M. Hutzenthaler, A. Jenzen, P.E. Kloeden, Strong convergence of an explicit numerical method for SDEs with nonglobally Lipschitz continuous coefficients, Ann. Appl. Probab. 22 (4) (2012) 1611–1641.