P1 FINITE ELEMENT METHODS FOR A WEIGHTED ELLIPTIC STATE-CONSTRAINED OPTIMAL CONTROL PROBLEM

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Abstract. We investigate a P_1 finite element method for a two-dimensional weighted optimal control problem arising from a three-dimensional ($3D$) axisymmetric elliptic state-constrained optimal control problem with Dirichlet boundary conditions.

1. Introduction

Let $\breve{\Omega} \subset \mathbb{R}^3$ be an axisymmetric domain, and let Ω be the restriction of $\breve{\Omega}$ onto the meridian half-plane $\mathbb{R}^2_+ = \{(r, z) \in \mathbb{R}^2 : r > 0\}$. Then,

$$\breve{\Omega} = \{(r, \theta, z) : (r, z) \in \Omega \cup \Gamma_0 \text{ and } -\pi \leq \theta < \pi\},$$

where Γ_0 is the "artificial" boundary of Ω that is on the axis of rotation (the z-axis). We will use Γ_1 to denote the subset of $\partial\Omega$ that is not on the z-axis, i.e., the rotation of Γ_1 around the z-axis will return $\partial\breve{\Omega}$.

We call a function axisymmetric if it is defined on an axisymmetric domain, and it is independent of the rotational variable θ when written in terms of cylindrical coordinates. We will write $\breve{\cdot}$ above the function to indicate that a function is axisymmetric. Similarly, we use $\breve{\cdot}$ above a function space to denote an axisymmetric function space. For example, $\breve{L}^2(\breve{\Omega})$ is the closed subspace of $L^2(\breve{\Omega})$ that consists of square-integrable functions on $\breve{\Omega}$ that are independent of the θ-variable. Similarly, $\breve{H}^k(\breve{\Omega})$ consists of square-integrable axisymmetric functions whose distributional derivatives of order k and under are also square-integrable. We use $\breve{H}^1_0(\breve{\Omega})$ to denote the closed subspace of $\breve{H}^1(\breve{\Omega})$ that consists of functions that vanish on $\partial\breve{\Omega}$.

Let β be a positive constant and $\breve{y}_d \in \breve{L}^2(\breve{\Omega})$. We are interested in efficiently solving the following axisymmetric problem after performing a dimension reduction:

$$\text{Find } (\breve{y}_\star, \breve{u}_\star) = \underset{(\breve{y},\breve{u})\in\breve{\mathbb{K}}}{\operatorname{argmin}} \left[\frac{1}{2}\|\breve{y} - \breve{y}_d\|^2_{\breve{L}^2(\breve{\Omega})} + \frac{\beta}{2}\|\breve{u}\|^2_{\breve{L}^2(\breve{\Omega})}\right], \tag{1.1}$$

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where $(\breve{y}, \breve{u}) \in \breve{H}^1_0(\breve{\Omega}) \times \breve{L}^2(\breve{\Omega})$ belongs to $\breve{\mathbb{K}}$ if and only if

$$\int_{\breve{\Omega}} \nabla\breve{y}\nabla\breve{w}dV = \int_{\breve{\Omega}} \breve{u}\breve{w}dV \qquad \text{for all } \breve{w} \in \breve{H}^1_0(\breve{\Omega}), \tag{1.2}$$

$$\breve{y} \leq \breve{\psi} \qquad \text{a.e. in } \breve{\Omega}. \tag{1.3}$$

We assume that $\breve{\psi}$ is smooth and that $\breve{\psi} > 0$ on $\partial\breve{\Omega}$. The condition $\breve{\psi} > 0$ on $\partial\breve{\Omega}$ assures that the contact/coincidence set denoted by $\breve{\mathfrak{C}}$ is a compact subset of $\breve{\Omega}$. This assumption will be important throughout the paper.

Now let us perform a dimension reduction to the problem (1.1)–(1.3). Consider the $2D$ domain $\Omega \subset \mathbb{R}^2_+$ associated with $\breve{\Omega}$ and define the following weighted Hilbert space:

$$L^2_r(\Omega) = \left\{ v : \int_\Omega v(r,z)^2 r dr dz < \infty \right\}.$$

Then, there is an isometry (up to a factor of $\sqrt{2\pi}$) between $\breve{L}^2(\breve{\Omega})$ and $L^2_r(\Omega)$, since

$$\int_{\breve{\Omega}} \breve{v}(r,\theta,z)^2 r dr d\theta dz = 2\pi \int_\Omega v(r,z)^2 r dr dz, \tag{1.4}$$

where $v(r,z) \in L^2_r(\Omega)$ is the function associated with $\breve{v}(r,\theta,z) \in \breve{L}^2(\breve{\Omega})$ that satisfies

$$v(r,z) = \breve{v}(r,\theta,z).$$

The inner-product and norm associated with $L^2_r(\Omega)$ will be denoted as follows:

$$(v_1, v_2)_r = \int_\Omega v_1 v_2 r dr dz,$$
$$\|v\|_{L^2_r(\Omega)} = \left(\int_\Omega v^2 r dr dz \right)^{1/2}.$$

Note that we are using $(\cdot,\cdot)_r$ instead of $(\cdot,\cdot)_{L^2_r(\Omega)}$ for simplicity. In general, we will use $\|\cdot\|_V$ and $|\cdot|_V$ to denote the norm and semi-norm associated with the Hilbert space V, respectively. As usual, we use $H^k_r(\Omega)$ to denote functions in $L^2_r(\Omega)$ whose distributional derivatives of order k and under are in $L^2_r(\Omega)$. Therefore,

$$H^1_r(\Omega) = \{ v \in L^2_r(\Omega) : \nabla_{rz} v \in L^2_r(\Omega) \times L^2_r(\Omega) \},$$

where

$$\nabla_{rz} v = \left(\frac{\partial v}{\partial r}, \frac{\partial v}{\partial z} \right)^T.$$

The semi-norm and norm associated with $H^1_r(\Omega)$ are

$$|v|_{H^1_r(\Omega)} = \left(\int_\Omega (\nabla_{rz} v)^2 r dr dz \right)^{1/2},$$
$$\|v\|_{H^1_r(\Omega)} = (\|v\|^2_{L^2_r(\Omega)} + |v|^2_{H^1_r(\Omega)})^{1/2}.$$

For the remainder of the paper, we will use ∂_r in place of $\dfrac{\partial}{\partial r}$, etc. The closed subspace of $H^1_r(\Omega)$ whose members have vanishing trace on Γ_1 is denoted by $H^1_{r,\diamond}(\Omega)$, i.e.,

$$H^1_{r,\diamond}(\Omega) = \{v \in H^1_r(\Omega) : v = 0 \text{ on } \Gamma_1\}.$$

It is well-known that the trace condition in this definition is well-defined. (See [3].)

Then (1.1)–(1.3) is equivalent to solving the following $2D$ weighted optimal control problem:

$$\text{Find } (y_\star, u_\star) = \underset{(y,u)\in\mathbb{K}}{\operatorname{argmin}} \left[\frac{1}{2}\|y - y_d\|^2_{L^2_r(\Omega)} + \frac{\beta}{2}\|u\|^2_{L^2_r(\Omega)}\right], \tag{1.5}$$

where $(y, u) \in H^1_{r,\diamond}(\Omega) \times L^2_r(\Omega)$ belongs to $\mathbb{K}$ if and only if

$$(\nabla_{rz} y, \nabla_{rz} w)_r = (u, w)_r \qquad \text{for all } w \in H^1_{r,\diamond}(\Omega), \tag{1.6}$$

$$y \le \psi \qquad \text{a.e. in } \Omega. \tag{1.7}$$

We assume that $y_d \in L^2_r(\Omega)$, and that ψ is smooth and $\psi > 0$ on Γ_1.

An application of optimal control problems like (1.1)–(1.3) (and therefore (1.5)–(1.7)) is in optimal stationary heating when the body to be heated is axisymmetric. Suppose $\breve{\Omega}$ is the axisymmetric body to be heated by electromagnetic induction or by microwaves. Then, $\breve{u}_\star$ can be thought as the optimal heat source (the control) in $\breve{\Omega}$, that makes the temperature distribution $\breve{y}_\star$ (the state) to be the best possible approximation of the desired stationary temperature distribution denoted by $\breve{y}_d$ while satisfying conditions like (1.1) and (1.3). Furthermore, β can be viewed as a measure of energy costs to implement the control. More details of such applications (without the axisymmetric assumptions) can be found in [27, Chapter 1].

The axisymmetric (θ-independency) assumption on the functions in (1.1)–(1.3) has two meanings. One is that this problem arises when the desired temperature distribution ($\breve{y}_d$) and the maximal temperature distribution ($\breve{\psi}$) are axisymmetric, since then the solution $(\breve{y}_\star, \breve{u}_\star)$ is also axisymmetric. The other is that the analysis of (1.5)–(1.7) is the first step of analyzing the elliptic state-constrained optimal control problem with Dirichlet boundary conditions that has the axial symmetry condition only on the domain $\breve{\Omega}$ and not necessarily on the functions. This is because, even in such a more general setting, (1.5)–(1.7) is the problem that needs to be solved to get the 0-th Fourier-mode of the optimal control and state when using Fourier finite element methods for general axisymmetric problems as in [3]. Therefore, efficient numerical methods that can approximate the solution of (1.5)–(1.7) are important for elliptic state-constrained optimal control problems whose domain is axisymmetric with axisymmetric data and also with general data.

Finite element methods (FEMs) for elliptic distributed optimal control problems with pointwise state constraints have been studied by many authors ([4, 5, 6, 7, 8, 9, 10, 12, 16, 20, 22, 24]). To our knowledge, this will be the first paper that considers the numerical solution of a $2D$ weighted optimal control problem arising from a $3D$ axisymmetric optimal control problem. In particular, we will use P_1 finite elements as in [4, 7, 12] but after making necessary changes according to the weighted norms to approximate the solution to (1.5)–(1.7). We follow the analysis done in [4] but modify it to appropriate weighted function spaces. These weighted spaces include functions with singularities at the axis of rotation, so the analysis of the weighted problem require special attention. Furthermore, to our knowledge, this paper will be the first paper that uses local weighted spaces in the analysis of axisymmetric problems.

The remainder of the paper is organized as follows. In the next section, we summarize definitions and useful properties in weighted spaces. In section 3, we analyze the solution to the continuous weighted problem of interest by using a variational inequality and a Lagrange multiplier. In the following section, we introduce the discrete weighted problem that will be used in this paper, and in section 5, we derive error estimates. Finally, section 6 provides numerical results to back up the theory.

2. Continuous Properties in Weighted Spaces

In this section, we summarize definitions and some results in weighted spaces that will be useful in this paper. Throughout this paper, we assume that $\breve{\Omega} \subset \mathbb{R}^3$ is a bounded Lipschitz domain.

Let $s \in (0, 2]$, and set

$$\begin{aligned} H^s_+(\Omega) =& H^s_r(\Omega) && \text{if } s \in (0,2), \\ H^2_+(\Omega) =& \{w \in H^2_r(\Omega) : \partial_r w \in L^2_{1/r}(\Omega)\}, \end{aligned}$$

where

$$L^2_{1/r}(\Omega) = \{w : \int_\Omega \frac{w^2}{r} drdz < \infty\}.$$

Note that $H^2_+(\Omega)$ is a Hilbert space endowed with the norm

$$\|w\|_{2,+} = (\|w\|_{H^2_r(\Omega)} + \|\partial_r w\|_{L^2_{1/r}(\Omega)})^{1/2}.$$

Furthermore, define

$$H^s_+(\Omega) = \{w \in H^s_r(\Omega) : \partial_r w = 0 \text{ on } \Gamma_0\} \qquad \text{if } s \in (2,3]$$

endowed

$$\|w\|_{H^s_+(\Omega)} = \|w\|_{H^s_r(\Omega)} \qquad \text{for } s \in (2,3].$$

The following results can be found in [3, Theorem II.2.1] and [1, subsection 3.2].

Proposition 2.1. *The trace mapping $\breve{f} \mapsto f$ is an isometry (up to a factor $\sqrt{2\pi}$) from $\breve{L}^2(\breve{\Omega})$ to $L^2_r(\Omega)$. The same holds for the reciprocal lifting, $L^2_r(\Omega) \to \breve{L}^2(\breve{\Omega})$, $f \mapsto \breve{f}$.*

Proposition 2.2. *The trace operator is an isomorphism from $\breve{H}^s(\breve{\Omega})$ to $H^s_+(\Omega)$ for all $s \in (0, 3]$.*

Let $\breve{E}_0(\Delta; \breve{L}^2(\breve{\Omega}))$ be the subspace of $\breve{H}^1_0(\breve{\Omega})$ defined by

$$\breve{E}_0(\Delta; \breve{L}^2(\breve{\Omega})) = \{\breve{v} \in \breve{H}^1_0(\breve{\Omega}) : \Delta\breve{v} \in \breve{L}^2(\breve{\Omega})\},$$

where $\Delta\breve{z}$ is understood in the sense of distributions. Due to elliptic regularity [4, 15, 19, 23], $\breve{E}_0(\Delta; \breve{L}^2(\breve{\Omega}))$ is a subspace of $\breve{H}^{1+\alpha}(\breve{\Omega})$ for some α, and we have

$$\|\breve{v}\|_{\breve{H}^{1+\alpha}(\breve{\Omega})} \le C\|\Delta\breve{v}\|_{\breve{L}^2(\breve{\Omega})} \text{ for all } \breve{v} \in \breve{E}_0(\Delta; \breve{L}^2(\breve{\Omega})). \tag{2.1}$$

In general, the index of elliptic regularity α is in $(\frac{1}{2}, 1]$.

Next, let us consider Δ in terms of cylindrical coordinates:

$$\Delta\breve{v} = \frac{1}{r}\partial_r(r\partial_r\breve{v}) + \frac{1}{r^2}\partial_{\theta\theta}\breve{v} + \partial_{zz}\breve{v}.$$

Since $\partial_\theta\breve{v} = 0$ by the definition of axisymmetric functions, the right-hand-side of the above formula reduces. With that in mind, and by using the relation $\breve{v}(r,\theta,z) = v(r,z)$, we define the following operator:

$$\Delta_{rz}v = \frac{1}{r}\partial_r(r\partial_r v) + \partial_{zz}v = \frac{1}{r}\partial_r v + \partial_{rr}v + \partial_{zz}v.$$

Then there is an isomorphism between $\breve{E}_0(\Delta; \breve{L}^2(\breve{\Omega}))$ and $E_{r,\diamond}(\Omega)$, where

$$E_{r,\diamond}(\Omega) = \{v \in H^1_{r,\diamond}(\Omega) : \Delta_{rz}v \in L^2_r(\Omega)\},$$

and by (2.1) and Proposition 2.2, we also have

$$\|v\|_{H^{1+\alpha}_+(\Omega)} \leq C\|\Delta_{rz}v\|_{L^2_r(\Omega)} \qquad \text{for all } v \in E_{r,\diamond}(\Omega). \tag{2.2}$$

Since $\breve{E}_0(\Delta; \breve{L}^2(\breve{\Omega}))$ is a subspace of $\breve{H}^{1+\alpha}(\breve{\Omega}) \cap \breve{H}^2_{loc}(\breve{\Omega}) \cap \breve{H}^1_0(\breve{\Omega})$ for some $\alpha \in (\frac{1}{2}, 1]$ ([15, 19, 23]), it follows from Proposition 2.2 that $E_{r,\diamond}(\Omega)$ is a subspace of $H^{1+\alpha}_+(\Omega) \cap H^2_{+,\widetilde{loc}}(\Omega) \cap H^1_{r,\diamond}(\Omega)$ where $H^2_{+,\widetilde{loc}}(\Omega)$ denotes the space of functions that are in $H^2_+(\omega)$ for all ω that satisfies $\breve{\omega} \subset\subset \breve{\Omega}$ (the closure of $\breve{\omega}$ is a compact subset of $\breve{\Omega}$). We use the notation $\widetilde{loc}$ instead of loc to indicate that the condition on ω is that $\breve{\omega} \subset\subset \breve{\Omega}$ instead of $\omega \subset\subset \Omega$, i.e., a side of ω may be on Γ_0 but not on Γ_1.

Moreover, we have the following integration by parts formula:

$$(\Delta_{rz}w, v)_r = -(\nabla_{rz}w, \nabla_{rz}v)_r \quad \text{for all } v \in H^1_{r,\diamond}(\Omega), w \in E_{r,\diamond}(\Omega). \tag{2.3}$$

As usual let $L^\infty(\breve{\Omega})$ denote the space of functions that satisfy

$$\text{ess} \quad \sup\{w(\boldsymbol{x}) \text{ for all } \boldsymbol{x} \in \breve{\Omega}\} < \infty.$$

Before we end this section, let us define a few more function spaces. Let $W^k_\infty(\breve{\Omega})$ denote the subspace of functions in $L^\infty(\breve{\Omega})$ whose distributional derivatives of order k and under are also in $L^\infty(\breve{\Omega})$. $L^\infty(\Omega)$ and $W^k_\infty(\Omega)$ are defined analogously without having to consider the weight r.

3. The Continuous Problem

By applying integration by parts (2.3), we can rewrite the problem (1.5)–(1.7) in the following way:

$$\begin{aligned} &\text{Find } y_\star = \operatorname*{argmin}_{y\in K} \left[\frac{1}{2}\|y - y_d\|^2_{L^2_r(\Omega)} + \frac{\beta}{2}\|\Delta_{rz}y\|^2_{L^2_r(\Omega)}\right], \\ &\text{where } K = \{y \in E_{r,\diamond}(\Omega) : y \leq \psi \text{ in } \Omega\}. \end{aligned} \tag{3.1}$$

By the classical theory of calculus of variations [17, 21], the solution $y_\star \in K$ is characterized by the variational inequality

$$(y_\star - y_d, y - y_\star)_r + \beta(\Delta_{rz}y_\star, \Delta_{rz}(y - y_\star))_r \geq 0 \qquad \text{for all } y \in K. \tag{3.2}$$

3.1. **Interior Regularity of $y_\star$.** By [11] there is a unique solution to the minimization problem (1.1)–(1.3), and

$$\breve{y}_\star \in \breve{H}^3_{loc}(\breve{\Omega}) \cap \breve{W}^{2,\infty}_{loc}(\breve{\Omega}). \tag{3.3}$$

By Proposition 2.2, this implies that

$$y_\star \in H^3_{+,\widetilde{loc}}(\Omega) \cap W^{2,\infty}_{\widetilde{loc}}(\Omega). \tag{3.4}$$

3.2. **Lagrange Multiplier.** Let $\breve{\phi} \in \breve{C}^\infty_c(\breve{\Omega})$ (the space of axisymmetric C^∞ functions with compact support in $\breve{\Omega}$) be nonnegative. Then, let ϕ be the corresponding function defined on Ω such that $\phi(r,z) = \breve{\phi}(r,\theta,z)$. Since $y = -\phi + y_\star \in K$, by (3.2), we have

$$(y_\star - y_d, \phi)_r + \beta(\Delta_{rz} y_\star, \Delta_{rz}\phi)_r \le 0.$$

It then follows from the Riesz representation theorem ([18, 25, 26]) that

$$(y_\star - y_d, v)_r + \beta(\Delta_{rz} y_\star, \Delta_{rz} v)_r = \int_\Omega v\, d\mu \quad \forall\, v \in E_{r,\diamond}(\Omega) \tag{3.5}$$

where

$$\mu \text{ is a nonpositive regular Borel measure.} \tag{3.6}$$

Let $\breve{\mathfrak{C}} = \{x \in \breve{\Omega} : \breve{y}_\star(x) = \breve{\psi}(x)\}$ be the $3D$ contact/coincidence set. Recall that we are assuming that $\breve{\psi} > 0$ on $\partial\breve{\Omega}$ so that $\breve{\mathfrak{C}}$ is a compact subset of $\breve{\Omega}$. Let $\breve{G}_{\breve{\mathfrak{C}}}$ denote an axisymmetric open neighborhood of $\breve{\mathfrak{C}}$ such that $\breve{G}_{\breve{\mathfrak{C}}} \subset\subset \breve{\Omega}$. We will use $\mathfrak{C}$ and $G_{\mathfrak{C}}$ to denote the restriction of $\breve{\mathfrak{C}}$ and $\breve{G}_{\breve{\mathfrak{C}}}$ respectively onto the meridian half plane as usual.

For any $v \in K$ whose support is disjoint from $\mathfrak{C}$, $y^\pm_\epsilon = \pm\epsilon v + y_\star$ belongs to K if ϵ is sufficiently small. Hence by (3.2) we have

$$(y_\star - y_d, \pm\epsilon v)_r + \beta(\Delta_{rz} y_\star, \Delta_{rz}(\pm\epsilon v))_r \ge 0,$$

which implies

$$(y_\star - y_d, v)_r + \beta(\Delta_{rz} y_\star, \Delta_{rz} v)_r = 0, \quad \forall\, v \in K \text{ such that } \operatorname{supp} v \cap \mathfrak{C} = \emptyset.$$

Consequently, by (3.5), μ is supported on $\mathfrak{C}$, which is equivalent to the complementarity condition

$$\int_\Omega (y_\star - \psi)\, d\mu = 0. \tag{3.7}$$

Conversely, if $y_\star \in K$ satisfies (3.5)–(3.7), then $y_\star$ is the solution to (3.2).

Another way to view μ is through the associated $3D$ axisymmetric problem, i.e.,

$$(\breve{y}_\star - \breve{y}_d, \breve{v})_{\breve{L}^2(\breve{\Omega})} + \beta(\Delta\breve{y}_\star, \Delta\breve{v})_{\breve{L}^2(\breve{\Omega})} = \int_{\breve{\Omega}} \breve{v}\, d\breve{\mu} \quad \forall\, \breve{v} \in \breve{E}_0(\Delta; \breve{L}^2(\breve{\Omega})) \tag{3.8}$$

where

$$\breve{\mu} \text{ is a nonpositive regular Borel measure.}$$

Furthermore, we have as in [8, (2.7)] and [7, (2.11)]

$$\left| \int_{\breve{\Omega}} \breve{v}\, d\breve{\mu} \right| \le C \|\breve{v}\|_{\breve{H}^1(\breve{G}_{\breve{\mathfrak{C}}})} \quad \text{for all } \breve{v} \in \breve{E}_0(\Delta; \breve{L}^2(\breve{\Omega})), \tag{3.9}$$

by (3.8), (3.3), and integration by parts. Thus, by (3.5), (3.8), and Proposition 2.2, we also have that

$$\left|\int_\Omega v\,d\mu\right| \le C\|v\|_{H^1_r(G_{\mathfrak{c}})} \quad \text{for all } v \in E_{r,\diamond}(\Omega). \tag{3.10}$$

3.3. **Global Regularity of $y_\star$ and $u_\star$.** Globally, by (2.2), we know that

$$\|y_\star\|_{H^{1+\alpha}_+(\Omega)} \le C\|\Delta_{rz}y_\star\|_{L^2_r(\Omega)}. \tag{3.11}$$

Furthermore, in view of (3.9), it is known that $\Delta\breve{y}_\star \in \breve{H}^1_0(\breve{\Omega})$. (See [10, section 2.4].) Therefore,

$$u_\star = -\Delta_{rz}y_\star \in H^1_{r,\diamond}(\Omega). \tag{3.12}$$

4. The Discrete Problem

Assume that Ω is meshed by a finite element triangulation $\mathscr{T}_h$ that satisfies the usual geometrical conformity conditions [14]. We use h_T to denote the diameter of $T \in \mathscr{T}_h$, and $h = \max_{T\in\mathscr{T}_h} h_T$ to denote the mesh parameter.

Let $V_h^0 \subset H^1_{r,\diamond}(\Omega)$ denote the P_1 finite element space with vanishing trace on Γ_1, i.e.,

$$V_h^0 = \{v \in C(\Omega) : v|_T \in V_1 \text{ for all } T \in \mathscr{T}_h \text{ and } v = 0 \text{ on } \Gamma_1\}$$

where

$$V_1 = \{ar + bz + c : a, b, c \in \mathbb{R}\}.$$

Now, let us define a discrete Laplacian $\Delta_h : H^1_{r,\diamond}(\Omega) \to V_h^0$ in the following way:

$$(\Delta_h\varphi, w_h)_r = -(\nabla_{rz}\varphi, \nabla_{rz}w_h)_r, \qquad \forall w_h \in V_h^0. \tag{4.1}$$

Integration by parts implies that

$$\Delta_h\varphi = Q_h\Delta_{rz}\varphi \text{ for all } \varphi \in E_{r,\diamond}(\Omega), \tag{4.2}$$

where $Q_h : L^2_r(\Omega) \to V_h^0$ denotes the L^2_r-orthogonal projection. The discrete problem used to approximate the solution to (3.1) is the following:

$$y_\star^h = \underset{y_h\in K_h}{\operatorname{argmin}} \left(\frac{1}{2}\|y_h - y_d\|^2_{L^2_r(\Omega)} + \frac{\beta}{2}\|\Delta_h y_h\|^2_{L^2_r(\Omega)}\right), \tag{4.3}$$

where $K_h = \{y_h \in V_h^0 : y_h \le I_h\psi \text{ in } \Omega\}$ and $I_h : H^1_{r,\diamond}(\Omega) \to V_h^0$ is the nodal interpolation operator, i.e., $I_h w$ coincides with w on all nodes in $\mathscr{T}_h$.

It follows from the classical theory that (4.3) has a unique solution $y_\star^h \in K_h$ that can be characterized by the following discrete variational inequality:

$$(y_\star^h - y_d, y_h - y_\star^h)_r + \beta(\Delta_h y_\star^h, \Delta_h(y_h - y_\star^h))_r \ge 0 \qquad \text{for all } y_h \in K_h. \tag{4.4}$$

Once the discrete state $y_\star^h \in V_h^0$ is obtained, we get the discrete control $u_\star^h \in V_h^0$ by $u_\star^h = -\Delta_h y_\star^h$.

Before we end this section we introduce an operator that we need in the convergence analysis. Let the operator $E_h : V_h^0 \to E_{r,\diamond}(\Omega)$ be defined by

$$\Delta_{rz}E_h v_h = \Delta_h v_h, \qquad \forall\, v_h \in V_h^0, \tag{4.5}$$

or equivalently

$$(\nabla_{rz} E_h v_h, \nabla_{rz} w)_r = (-\Delta_h v_h, w)_r \quad \text{for all } w \in H^1_{r,\diamond}(\Omega).$$

By definition of Δ_h, we also have that

$$(\nabla_{rz} v_h, \nabla_{rz} w_h)_r = (-\Delta_h v_h, w_h)_r \quad \text{for all } w_h \in V_h^0.$$

Recall that $E_{r,\diamond}(\Omega)$ is a subspace of $H_+^{1+\alpha}(\Omega) \cap H^2_{+,\widetilde{loc}}(\Omega) \cap H^1_{r,\diamond}(\Omega)$. Then, by the standard error estimate [13, section 5] and a duality argument, we get the following Proposition.

Proposition 4.1. *For any $v_h \in V_h^0$, $E_h v_h$ belongs to $H_+^{1+\alpha}(\Omega)$ for some $\alpha \in (\frac{1}{2}, 1]$, and*

$$\begin{aligned} |E_h v_h - v_h|_{H^1_r(\Omega)} &\leq Ch^\alpha \|\Delta_h v_h\|_{L^2_r(\Omega)}, \\ \|E_h v_h - v_h\|_{L^2_r(\Omega)} &\leq Ch^{2\alpha} \|\Delta_h v_h\|_{L^2_r(\Omega)}. \end{aligned}$$

5. Error Estimates

Let us first define a mesh-dependent norm $\|\cdot\|_h$ by

$$\|v\|_h^2 = (v, v)_r + \beta(\Delta_h v, \Delta_h v)_r. \tag{5.1}$$

The following theorem is the main result of this paper, and it provides abstract error estimates.

Theorem 5.1. *Let $y_\star$ be the unique solution to (3.1) and $y_\star^h$ be the unique solution to (4.3). Then, it follows that*

$$\|y_\star - y_\star^h\|_h \leq C \left(h + \inf_{y_h \in K_h} \left[\|y_\star - y_h\|_h + \|y_\star - y_h\|^{1/2}_{L_\infty(G_{\mathfrak{C}})} \right] \right).$$

The following Lemma will be essential in proving Theorem 5.1.

Lemma 5.1. *There exists a positive constant C independent of h such that*

$$\int_\Omega E_h(y_h - y_\star^h) d\mu \leq C \left(h \|\Delta_h (y_h - y_\star^h)\|_{L^2_r(\Omega)} + h^2 + \|y_h - I_h y_\star\|_{L^\infty(G_{\mathfrak{C}})} \right), \tag{5.2}$$

for all $y_h, y_\star^h \in K_h$.

Proof. Consider

$$\begin{aligned} &\int_\Omega E_h(y_h - y_\star^h)\, d\mu \\ &= \int_\Omega \left[E_h(y_h - y_\star^h) - (y_h - y_\star^h) \right] d\mu + \int_\Omega (I_h \psi - y_\star^h)\, d\mu + \int_\Omega I_h(y_\star - \psi)\, d\mu + \int_\Omega (y_h - I_h y_\star)\, d\mu, \\ &\leq \int_\Omega \left[E_h(y_h - y_\star^h) - (y_h - y_\star^h) \right] d\mu + \int_\Omega I_h(y_\star - \psi)\, d\mu + \int_\Omega (y_h - I_h y_\star)\, d\mu, \end{aligned} \tag{5.3}$$

where in the last inequality above, we are using (3.6) and the fact that $y_\star^h \leq I_h \psi$. We will get an upper bound for each of the terms appearing in the right-hand-side of (5.3).

First of all,

$$\begin{aligned}\int_\Omega \left[E_h(y_h - y_\star^h) - (y_h - y_\star^h)\right]\, d\mu &\le C\|E_h(y_h - y_\star^h) - (y_h - y_\star^h)\|_{H^1_r(G_{\mathfrak{C}})} && \text{by (3.10)},\\ &\le Ch\|\Delta_h(y_h - y_\star^h)\|_{L^2_r(\Omega)} && \text{by Proposition 4.1.}\end{aligned} \tag{5.4}$$

The last inequality above is true, since $E_{r,\diamond}(\Omega)$ is a subspace of $H^2_{+,\widetilde{loc}}(\Omega)$ and $\breve{G}_{\breve{\mathfrak{C}}} \subset\subset \breve{\Omega}$. Next we recall the following estimate that follows from [2, Proposition 3]:

$$\|w - I_h w\|_{L^\infty(G_{\mathfrak{C}})} \;\; \le Ch^2\|w\|_{W^2_\infty(G_{\mathfrak{C}})}, \tag{5.5}$$

for all $w \in W^2_\infty(G_{\mathfrak{C}})$. Then,

$$\begin{aligned}\int_\Omega I_h(y_\star - \psi)\, d\mu &= \int_\Omega ((\psi - y_\star) - I_h(\psi - y_\star))\, d\mu && \text{by (3.7)},\\ &\le C\|(\psi - y_\star) - I_h(\psi - y_\star)\|_{L_\infty(G_{\mathfrak{C}})} && \text{since } \mu \text{ is a finite measure},\\ &\le Ch^2\|\psi - y_\star\|_{W^2_\infty(G_{\mathfrak{C}})}.\end{aligned} \tag{5.6}$$

The last inequality follows from (3.4) and (5.5). We also have

$$\int_\Omega (y_h - I_h y_\star)\, d\mu \le C\|y_h - I_h y_\star\|_{L_\infty(G_{\mathfrak{C}})}. \tag{5.7}$$

The proof is complete by (5.3)–(5.7). □

Now we are ready to prove Theorem 5.1.

Proof. First of all, the following is straightforward:

$$\begin{aligned}&\|y_h - y_\star^h\|_h^2\\ &= (y_h - y_\star^h, y_h - y_\star^h)_r + \beta(\Delta_h(y_h - y_\star^h), \Delta_h(y_h - y_\star^h))_r,\\ &= (y_h - y_\star, y_h - y_\star^h)_r + \beta(\Delta_h(y_h - y_\star), \Delta_h(y_h - y_\star^h))_r\\ &+ (y_\star - y_d, y_h - y_\star^h)_r + \beta(\Delta_h y_\star, \Delta_h(y_h - y_\star^h))_r\\ &- (y_\star^h - y_d, y_h - y_\star^h)_r - \beta(\Delta_h y_\star^h, \Delta_h(y_h - y_\star^h))_r,\\ &\le \|y_h - y_\star\|_h\|y_h - y_\star^h\|_h + (y_\star - y_d, y_h - y_\star^h)_r + \beta(\Delta_h y_\star, \Delta_h(y_h - y_\star^h))_r \quad \text{by (4.4).}\end{aligned} \tag{5.8}$$

Furthermore,

(5.9)

$$
\begin{aligned}
&(y_\star - y_d, y_h - y_\star^h)_r + \beta(\Delta_h y_\star, \Delta_h(y_h - y_\star^h))_r \\
&= (y_\star - y_d, (y_h - y_\star^h) - E_h(y_h - y_\star^h))_r + (y_\star - y_d, E_h(y_h - y_\star^h))_r + \beta(\Delta_{rz} y_\star, \Delta_h(y_h - y_\star^h))_r \\
&\qquad \text{by (4.2)}, \\
&= (y_\star - y_d, (y_h - y_\star^h) - E_h(y_h - y_\star^h))_r + (y_\star - y_d, E_h(y_h - y_\star^h))_r + \beta(\Delta_{rz} y_\star, \Delta_{rz} E_h(y_h - y_\star^h))_r \\
&\qquad \text{by (4.5)}, \\
&= (y_\star - y_d, (y_h - y_\star^h) - E_h(y_h - y_\star^h))_r + \int_\Omega E_h(y_h - y_\star^h)\, d\mu \\
&\qquad \text{by (3.5)}, \\
&\le C\|(y_h - y_\star^h) - E_h(y_h - y_\star^h)\|_{L_r^2(\Omega)} + C\left(h\|\Delta_h(y_h - y_\star^h)\|_{L_r^2(\Omega)} + h^2 + \|y_h - I_h y_\star\|_{L_\infty(G_{\mathfrak{C}})}\right) \\
&\qquad \text{by Lemma 5.1}, \\
&\le Ch^{2\alpha}\|\Delta_h(y_h - y_\star^h)\|_{L_r^2(\Omega)} + C\left(h\|\Delta_h(y_h - y_\star^h)\|_{L_r^2(\Omega)} + h^2 + \|y_h - I_h y_\star\|_{L_\infty(G_{\mathfrak{C}})}\right) \\
&\qquad \text{by Proposition 4.1}, \\
&\le C\left(h\|y_h - y_\star^h\|_h + h^2 + \|y_h - I_h y_\star\|_{L_\infty(G_{\mathfrak{C}})}\right).
\end{aligned}
$$

Therefore, by (5.8) and (5.9), we have

$$
\begin{aligned}
\|y_h - y_\star^h\|_h^2 &\le \|y_h - y_\star\|_h \|y_h - y_\star^h\|_h + C\left(h\|y_h - y_\star^h\|_h + h^2 + \|y_h - I_h y_\star\|_{L_\infty(G_{\mathfrak{C}})}\right), \\
&\le C\left((\|y_h - y_\star\|_h + h)\|y_h - y_\star^h\|_h + h^2 + \|y_h - I_h y_\star\|_{L_\infty(G_{\mathfrak{C}})}\right),
\end{aligned}
$$

which together with the inequality of arithmetic and geometric means implies

$$
\|y_h - y_\star^h\|_h \le C\left(\|y_h - y_\star\|_h + h + \|y_h - I_h y_\star\|_{L_\infty(G_{\mathfrak{C}})}^{1/2}\right). \tag{5.10}
$$

Hence, it holds for all $y_h \in K_h$ that

$$
\begin{aligned}
\|y_\star - y_\star^h\|_h &\le \|y_\star - y_h\|_h + \|y_h - y_\star^h\|_h, \\
&\le \|y_\star - y_h\|_h + C\left(\|y_h - y_\star\|_h + h + \|y_h - I_h y_\star\|_{L_\infty(G_{\mathfrak{C}})}^{1/2}\right), \\
&\le C\left(\|y_\star - y_h\|_h + h + (\|y_h - y_\star\|_{L_\infty(G_{\mathfrak{C}})} + \|y_\star - I_h y_\star\|_{L_\infty(G_{\mathfrak{C}})})^{1/2}\right), \\
&\le C\left(\|y_\star - y_h\|_h + h + \|y_h - y_\star\|_{L_\infty(G_{\mathfrak{C}})}^{1/2} + h\right) \qquad \text{by (5.5)},
\end{aligned}
$$

and thus we have

$$
\|y_\star - y_\star^h\|_h \le C\left(h + \inf_{y_h \in K_h}\left[\|y_\star - y_h\|_h + \|y_\star - y_h\|_{L_\infty(G_{\mathfrak{C}})}^{1/2}\right]\right).
$$

□

Remark 5.1. Recall that the discrete control $u_\star^h \in V_h^0$ is obtained by $u_\star^h = -\Delta_h y_\star^h$. Therefore, by (5.1), Theorem 5.1 provides abstract error estimates for the control as well as the state.

Remark 5.2. Concrete error estimates may be obtained by constructing a $y_h \in K_h$ that satisfies

$$\|y_\star - y_h\|_h + \|y_\star - y_h\|^{1/2}_{L_\infty(G_{\mathfrak{C}})} \le C(|\ln h|^{1/2}h + h^\alpha) \tag{5.11}$$

as done in [4, Lemma 5.3]. Then it will follow by Theorem 5.1 that

$$\|y_\star - y_\star^h\|_{L^2_r(\Omega)} + |y_\star - y_\star^h|_{H^1_r(\Omega)} + \|u_\star - u_\star^h\|_{L^2_r(\Omega)} \le C(|\ln h|^{1/2}h + h^\alpha).$$

We note here that the construction of such $y_h \in K_h$ is possible under the conjecture that

$$\epsilon_h := \|y_\star - R_h y_\star\|_{L_\infty(G_{\mathfrak{C}})} \le C(|\ln h|h^2 + h^{2\alpha}),$$

where $R_h y_\star \in V_h^0$ is the finite element solution that satisfies

$$(\nabla_{rz} R_h y_\star, \nabla_{rz} v_h)_r = (\nabla_{rz} y_\star, \nabla_{rz} v_h)_r \text{ for all } v_h \in V_h^0.$$

Under this conjecture, $y_h = R_h y_\star - \epsilon_h I_h \phi$ will satisfy (5.11), where ϕ is a nonnegative function with $\phi = 1$ on $G_{\mathfrak{C}}$ and $\breve{\phi} \in \breve{C}_c^\infty(\breve{\Omega})$.

6. Numerical Results

Example 1

In this example, we modify [4, Section 7 Example 3] so that the domain and the exact solution is axisymmetric. Let the axisymmetric $3D$ domain $\breve{\Omega} \subset \mathbb{R}^3$ be the following cylinder:

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \le 4^2 \text{ and } -4 \le z \le 4\}.$$

The computational domain is then $\Omega = \{(r, z) \in \mathbb{R}^2_+ : r \le 4 \text{ and } -4 \le z \le 4\}$. We choose $\beta = 1$, $\psi(r, z) = r^2 + z^2 - 1$, and

$$y_\star(r, z) = \begin{cases} \psi(r, z) & \text{if } |x| \le 1, \\ v(|x|) + [1 - \phi(|x|)]w(r, z) & \text{if } 1 \le |x| \le 3, \\ w(r, z) & \text{if } |x| \ge 3, \end{cases}$$

where $|x|$ denotes $\sqrt{r^2 + z^2}$ and

$$v(t) = (t^2 - 1)\left(1 - \frac{t-1}{2}\right)^4 + \frac{1}{4}(t-1)^2(t-3)^4,$$

$$\phi(t) = \left[1 + 4\left(\frac{t-1}{2}\right) + 10\left(\frac{t-1}{2}\right)^2 + 20\left(\frac{t-1}{2}\right)^3\right]\left(1 - \frac{t-1}{2}\right)^4,$$

$$w(r, z) = r^4 \sin^4(\frac{\pi}{8}(r+4)) \sin(\frac{\pi}{8}(z+4)).$$

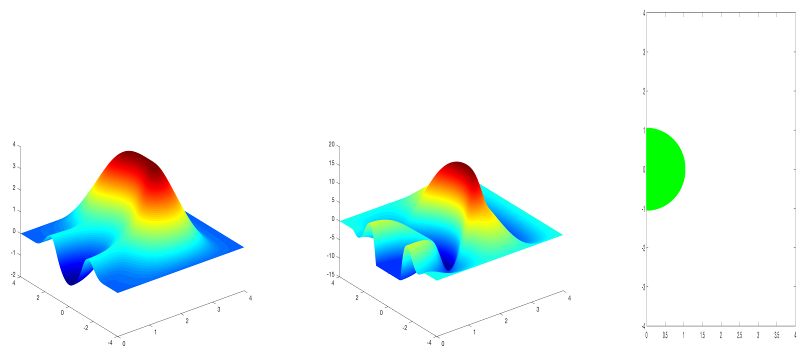
The corresponding y_d function that we need to get the constructed exact solution $y_\star$ is

$$y_d(r, z) = \begin{cases} \Delta^2_{rz} y_\star + y_\star & \text{if } |x| > 1, \\ \Delta^2_{rz} y_\star + y_\star + 2 & \text{if } |x| \le 1. \end{cases}$$

The exact control $u_\star$ is $-\Delta_{rz} y_\star$. We use Matlab “quadprog” program to solve this optimal control problem. We use y_k and u_k to denote the discrete state and control respectively

k	$\|y_\star - y_k\|_{L^2_r(\Omega)}$	rate	$\|y_\star - y_k\|_{H^1_r(\Omega)}$	rate	$\|I_k y_\star - y_k\|_{L^\infty(\Omega)}$	rate	$\|u_\star - u_k\|_{L^2_r(\Omega)}$	rate
0	5.97e+00		1.31e+01		1.41e+00		3.52e+01	
1	4.24e+00	0.49	1.07e+01	0.28	8.55e-01	0.72	2.90e+01	0.28
2	1.11e+00	1.93	5.05e+00	1.09	2.41e-01	1.83	8.71e+00	1.74
3	3.34e-01	1.73	2.56e+00	0.98	8.03e-02	1.58	2.52e+00	1.79
4	8.92e-02	1.91	1.27e+00	1.01	2.24e-02	1.85	6.73e-01	1.90
5	3.13e-02	1.51	6.36e-01	1.00	8.83e-03	1.34	1.78e-01	1.91
6	6.69e-03	2.23	3.18e-01	1.00	1.52e-03	2.53	4.82e-02	1.89
7	1.50e-03	2.16	1.59e-01	1.00	4.24e-04	1.85	1.35e-02	1.84
8	3.89e-04	1.95	7.94e-02	1.00	9.14e-05	2.21	3.96e-03	1.77

TABLE 6.1. Results on uniform meshes for Example 1



(a) State (b) Control (c) Contact Set

FIGURE 1. State, control, and contact set for Example 1

at mesh level k. In Table 6.1, we report the error in the observed norms. Figure 1 shows the graph of y_8 and u_8 and the contact set obtained in mesh level 8.

Example2

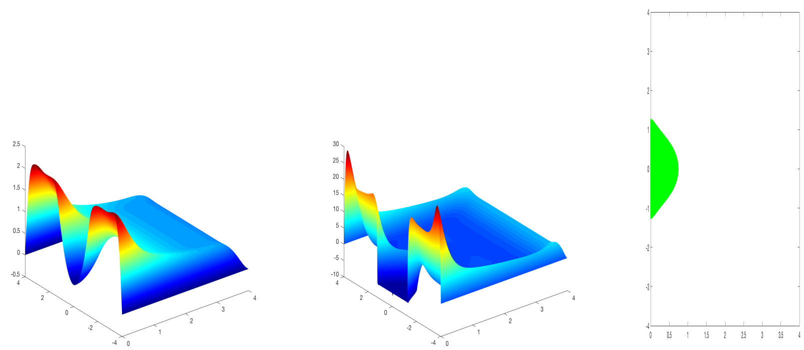
We now try an example with simpler data. We choose $y_d(r,z) = \dfrac{1}{\sqrt{r}}$, $\psi(r,z) = r^2 + z^2$, and $\beta = 0.002$. The domain is chosen to be the same as the previous example. In this case, the exact solution $(y_\star, u_\star)$ is unknown, so we calculate the error between two consecutive approximations in the observed norms in Table 6.2. Figure 2 shows the optimal state, optimal control, and contact set obtained by our method on mesh level 8.

Example 3

In this example, we let Ω to be the L-shape domain $[0,1]^2 \backslash [0.5,1]^2$. We choose $\beta = 1$, $y_d(r,z) = -2$, and $\psi(r,z) = (r-0.25)^2 + (z-0.25)^2 - 0.05$. We measure the error between two consecutive approximations.

k	$\|y_{k+1}-y_k\|_{L^2_r(\Omega)}$	rate	$\|y_{k+1}-y_k\|_{H^1_r(\Omega)}$	rate	$\|y_{k+1}-y_k\|_{L^\infty(\Omega)}$	rate	$\|u_{k+1}-u_k\|_{L^2_r(\Omega)}$	rate
1	1.81e+00		4.37e+00		1.36e+00		1.31e+01	
2	9.70e-01	0.90	3.99e+00	0.13	9.15e-01	0.57	1.59e+01	-0.28
3	3.01e-01	1.69	2.17e+00	0.88	4.96e-01	0.88	7.93e+00	1.00
4	8.50e-02	1.83	1.13e+00	0.94	1.60e-01	1.64	3.14e+00	1.34
5	2.25e-02	1.92	5.79e-01	0.97	5.14e-02	1.63	1.14e+00	1.46
6	5.75e-03	1.97	2.92e-01	0.99	1.39e-02	1.89	4.87e-01	1.23
7	1.45e-03	1.98	1.46e-01	1.00	3.53e-03	1.98	1.90e-01	1.36
8	3.67e-04	1.99	7.32e-02	1.00	8.87e-04	1.99	5.99e-02	1.67

TABLE 6.2. Results on uniform meshes for Example 2



(a) State (b) Control (c) Contact Set

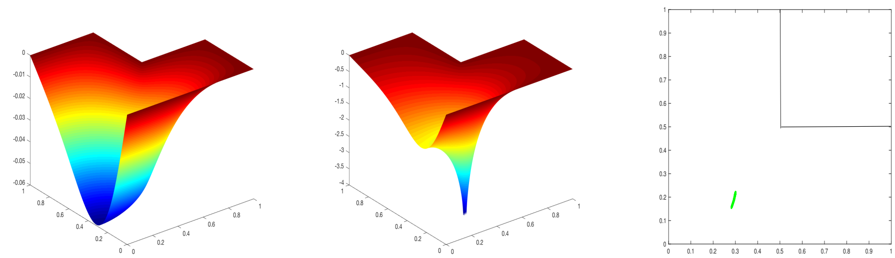
FIGURE 2. State, control, and contact set for Example 2

k	$\|y_{k+1}-y_k\|_{L^2_r(\Omega)}$	rate	$\|y_{k+1}-y_k\|_{H^1_r(\Omega)}$	rate	$\|y_{k+1}-y_k\|_{L^\infty(\Omega)}$	rate	$\|u_{k+1}-u_k\|_{L^2_r(\Omega)}$	rate
1	2.75e-03		4.37e-02		1.40e-02		1.79e-01	
2	1.52e-03	0.86	2.53e-02	0.79	7.78e-03	0.85	1.59e-01	0.17
3	5.84e-04	1.38	1.23e-02	1.04	2.30e-03	1.76	4.96e-02	1.68
4	7.19e-05	3.02	6.48e-03	0.93	1.37e-03	0.75	1.81e-02	1.45
5	2.61e-05	1.46	3.54e-03	0.87	9.02e-04	0.60	6.78e-03	1.42
6	1.35e-05	0.96	1.99e-03	0.83	5.79e-04	0.64	3.42e-03	0.99
7	4.11e-06	1.71	1.15e-03	0.79	3.67e-04	0.66	1.25e-03	1.46
8	1.46e-06	1.50	6.78e-04	0.76	2.32e-04	0.66	5.81e-04	1.10

TABLE 6.3. Results on uniform meshes for Example 3

In Table 6.3, we report the error in the observed norms. In this example, the elliptic regularity index α is $2/3$. It is clear especially from the reported error in the H^1-seminorm and the L^∞-norm that the order of convergence is dependent on the elliptic regularity index α. Figure 3 shows the graph of y_8 and u_8 and the contact set obtained in mesh level 8.

Example 4



(a) State (b) Control (c) Contact Set

FIGURE 3. State, control, and contact set for Example 3

k	$\|y_{k+1}-y_k\|_{L^2_r(\Omega)}$	rate	$\|y_{k+1}-y_k\|_{H^1_r(\Omega)}$	rate	$\|y_{k+1}-y_k\|_{L^\infty(\Omega)}$	rate	$\|u_{k+1}-u_k\|_{L^2_r(\Omega)}$	rate
1	1.41e+01		1.31e+01		1.59e+01		3.76e+00	
2	5.87e+00	1.27	9.08e+00	0.53	1.39e+01	0.19	2.78e+00	0.44
3	2.16e+00	1.44	4.36e+00	1.06	6.59e+00	1.07	8.98e-01	1.63
4	3.39e+00	-0.65	4.16e+00	0.07	6.69e+00	-0.02	9.73e-01	-0.12
5	1.72e+00	0.98	2.62e+00	0.67	5.25e+00	0.35	6.03e-01	0.69
6	1.00e+00	0.78	1.40e+00	0.90	2.84e+00	0.88	2.96e-01	1.03
7	2.53e-01	1.99	5.30e-01	1.40	1.36e+00	1.07	9.80e-02	1.59
8	1.23e-01	1.04	2.25e-01	1.24	6.10e-01	1.16	3.68e-02	1.41

TABLE 6.4. Results on uniform meshes for Example 4

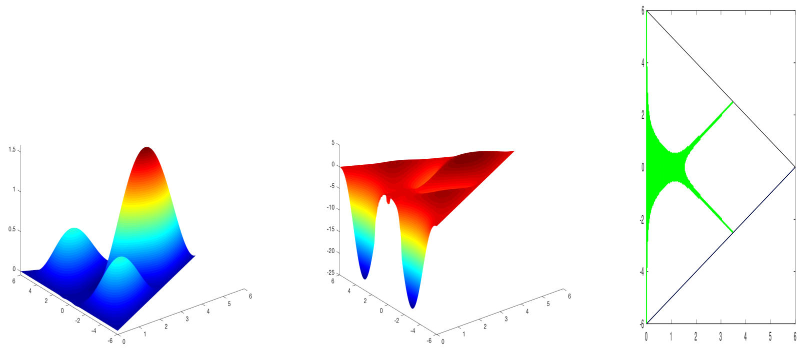
In this example, we choose Ω to be a triangular domain with vertices $(0,6), (6,0)$, and $(0,-6)$ with $\beta = 1$, $y_d(r,z) = 5$ and $\psi(r,z) = r^2(r+z-1)^2(r-z-1)^2$. Different from the previous three examples, the contact set in this example has a non-empty intersection with Γ_1, so the theory presented in this paper does not apply to this example. Nevertheless we examine the order of convergence. The error between two consecutive approximations is reported in Table 6.4 and the approximate state, control, and contact set obtained in mesh level 8 is presented in Figure 4.

Example 5

Before we end this section, we present here one more numerical example that uses an algorithm that is closely related to (4.3). Namely, as in [4], we will use a mass lumping technique to construct another P_1-FEM to approximate the solution of (3.1). First of all, let us define another inner product $(\cdot,\cdot)_{r,h}$:

$$(v_h, w_h)_{r,h} = \sum_{p\in\mathfrak{V}_h}\Big(\sum_{T\in\mathscr{T}_p}\frac{\int_T r\,dr\,dz}{3}\Big)v_h(p)w_h(p) \text{ for all } v_h, w_h \in V_h^0,$$

where $\mathfrak{V}_h$ denotes the set of nodes in $\mathscr{T}_h$, and $\mathscr{T}_p$ denotes the union of all triangles that have the node p as one of its nodes. Note that this inner-product is different from the one used in [4, 7] for mass lumping, since $\int_T r\,dr\,dz$ is used instead of $|T|$ (area of T). This is a natural modification that takes into consideration the weight r present in our problem.



(a) State (b) Control (c) Contact Set

FIGURE 4. State, control, and contact set for Example 4

k	$\|y_\star - y_k\|_{L^2_r(\Omega)}$	rate	$\|y_\star - y_k\|_{H^1_r(\Omega)}$	rate	$\|I_k y_\star - y_k\|_{L^\infty(\Omega)}$	rate	$\|u_\star - u_k\|_{L^2_r(\Omega)}$	rate
0	1.15e+01		2.21e+01		4.17e+00		3.56e+01	
1	7.72e+00	0.58	2.69e+01	-0.28	3.76e+00	0.15	3.33e+01	0.10
2	1.90e+00	2.02	1.04e+01	1.37	1.47e+00	1.35	1.28e+01	1.38
3	3.89e-01	2.29	3.19e+00	1.70	2.52e-01	2.55	3.31e+00	1.95
4	9.15e-02	2.09	1.35e+00	1.24	6.20e-02	2.02	9.60e-01	1.79
5	3.41e-02	1.43	6.46e-01	1.06	1.74e-02	1.83	3.20e-01	1.58
6	6.28e-03	2.44	3.19e-01	1.02	4.52e-03	1.94	1.15e-01	1.48
7	1.53e-03	2.03	1.59e-01	1.00	1.28e-03	1.82	4.57e-02	1.33
8	3.72e-04	2.04	7.94e-02	1.00	3.30e-04	1.95	1.95e-02	1.23

TABLE 6.5. Results on uniform meshes for Example 5

By using this inner-product, we define another discrete Laplacian $\widetilde{\Delta}_h : H^1_{r,\diamond}(\Omega) \to V_h^0$ in the following way:

$$(\widetilde{\Delta}_h \varphi, w_h)_{r,h} = -(\nabla_{rz}\varphi, \nabla_{rz} w_h)_r, \qquad \forall w_h \in V_h^0.$$

Then we solve the following discrete problem to approximate the state:

$$y_\star^h = \operatorname*{argmin}_{y_h \in K_h} \left(\frac{1}{2}\|y_h - y_d\|^2_{L^2_r(\Omega)} + \frac{\beta}{2}\|\widetilde{\Delta}_h y_h\|^2_{L^2_r(\Omega)} \right). \tag{6.1}$$

The approximate control $u_\star^h$ is then obtained by $u_\star^h = -\widetilde{\Delta}_h y_\star^h$. Since the mass matrix corresponding to $(\cdot,\cdot)_{r,h}$ is diagonal, we can use a primal dual active set method which converges superlinearly instead of "quadprog" to solve (6.1).

Table 6.5 reports the error in the observed norms when (6.1) is used instead of (4.3) with the same data provided in Example 1. These numerical results are promising, and they are similar to the ones provided in Table 6.1. The convergence analysis of (6.1) remains as future work.

7. Acknowledgement

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