# INVARIANT CONNECTIONS, LIE ALGEBRA ACTIONS, AND FOUNDATIONS OF NUMERICAL INTEGRATION ON MANIFOLDS

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Abstract. Motivated by numerical integration on manifolds, we relate the algebraic properties of invariant connections to their geometric properties. Using this perspective, we generalize some classical results of Cartan and Nomizu to invariant connections on algebroids. This has fundamental consequences for the theory of numerical integrators, giving a characterization of the spaces on which Butcher and Lie–Butcher series methods, which generalize Runge–Kutta methods, may be applied.

## 1. Introduction

1.1. **Background and motivation.** A connection on a smooth manifold $M$ can be viewed as a non-associative product on the Lie algebra of vector fields $\mathfrak{X}(M)$. The properties of the resulting algebra contain geometric information about the connection and about $M$ itself. In particular, a flat and torsion-free connection gives $\mathfrak{X}(M)$ the structure of a *pre-Lie algebra*, while a flat connection with parallel torsion gives $\mathfrak{X}(M)$ the structure of a *post-Lie algebra*. The notion of a pre-Lie algebra originates from work of Vinberg [32], Gerstenhaber [11], Agrachev and Gamkrelidze [2], while post-Lie algebras are due to Vallette [31].

Recently, Munthe-Kaas and Lundervold [26] related this algebraic perspective to certain analytical techniques for approximating flows of vector fields: Butcher series methods (Butcher [5, 6], Hairer and Wanner [16]) in the pre-Lie case and Lie–Butcher series methods (Munthe-Kaas [23, 24, 25]) in the more general post-Lie case. These techniques, which involve expressing flows as formal power series in rooted trees and forests, were originally developed for the analysis of numerical integrators.

It is natural to ask which manifolds $M$ admit such structures, and to which the techniques of Butcher and Lie–Butcher series may therefore be applied. Nomizu [29], following earlier work of E. Cartan [7], showed that $M$ admits a flat and torsion-free connection (i.e., an affine manifold structure) if and only if it is locally representable as an abelian Lie group with its canonical affine connection, while $M$ admits a flat connection with parallel torsion if and only if it is locally representable as a Lie group with its $(-)$-connection. It follows that Butcher series methods may be applied in the former case and Lie–Butcher series methods in the latter case.

However, a 1999 paper of Munthe-Kaas [25] (see also Munthe-Kaas and Wright [28]) showed that such methods may be applied, more generally, whenever a Lie algebra $\mathfrak{g}$ acts transitively on $M$. This includes not only the case where $M = G$ is the Lie group integrating $\mathfrak{g}$, but also (for example) when $M$ is a homogeneous space, or when $M$ is equipped with a frame of vector fields generating a Lie subalgebra $\mathfrak{g} \subset \mathfrak{X}(M)$. In fact, $M$ need not admit a flat connection at all, as in the example of $\mathfrak{g} = \mathfrak{so}(3)$ acting transitively on $M = \mathbb{S}^2$, the 2-sphere. The Cartan–Nomizu characterization is therefore not the end of the story.

To include these examples in the algebraic framework, Munthe-Kaas and Lundervold [26] considered connections more general than affine connections: namely, connections on a *Lie algebroid* $A \to M$, which is an anchored vector bundle with a compatible Lie bracket on the

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space of sections $\Gamma(A)$. (An affine connection is just the special case when $A = TM$ is the tangent bundle with the Jacobi–Lie bracket.) When equipped with a connection such that $\Gamma(A)$ is a pre-Lie algebra or post-Lie algebra, we say that $A$ is a *pre-Lie algebroid* or *post-Lie algebroid*. In particular, a $\mathfrak{g}$-action on $M$ has an associated *action algebroid* $\mathfrak{g} \ltimes M \to M$; as a vector bundle, this is just the trivial bundle with fiber $\mathfrak{g}$ over $M$. Munthe-Kaas and Lundervold [26] showed that the canonical flat connection makes $\mathfrak{g} \ltimes M$ into a post-Lie algebroid, and when $\mathfrak{g}$ is abelian, this is in fact a pre-Lie algebroid.

This previous work therefore gives sufficient conditions for $A \to M$ to admit a (pre-Lie) post-Lie structure: it is sufficient for $A$ to be an (abelian) action algebroid. The purpose of the present work is to prove conditions that are *both necessary and sufficient*, thereby giving a full characterization of the spaces to which Butcher and Lie–Butcher series methods may be used for numerical integration and analysis of flows on $M$.

1.2. **Overview.** The main results of this paper, Theorem 4.2 and Theorem 4.4, show that if $A \to M$ is transitive, i.e., the anchor is surjective, then it admits a (pre-Lie) post-Lie structure if and only if it is locally isomorphic to the action algebroid of some transitive (abelian) $\mathfrak{g}$-action with its canonical flat connection, and this isomorphism is global when $M$ is simply connected. (In the non-transitive case, this result holds leaf-by-leaf on the foliation induced by the anchor.) This generalizes the Cartan–Nomizu results stated above, which correspond to the special case $A = TM$.

Consequently, the *only* way to apply Butcher and Lie–Butcher series methods to manifolds, at least locally, is the one introduced twenty years ago: equip the manifold with a transitive Lie algebra action. From the perspective of applications, this means that, when faced with problems requiring numerical integration on arbitrary manifolds, one has two options. The first option is to introduce such a structure via a (generally non-canonical) choice of local coordinates or a frame of vector fields, which allows Butcher or Lie–Butcher series methods to be applied. The alternative is to use a different class of methods, such as projection methods; see Hairer et al. [15, Chapter IV] for an overview of methods on manifolds.

The paper is organized as follows:

- Section 2 begins by introducing a purely algebraic treatment of connections, relating Lie-admissible, pre-Lie, and post-Lie algebras of connections to curvature and torsion. We then bring geometry into the picture by applying this framework to algebras of affine connections on $M$, linking the pre-Lie and post-Lie conditions to the results of Cartan [7] and Nomizu [29].
- Section 3 considers connections on anchored bundles and Lie algebroids and gives necessary and sufficient conditions, in terms of curvature and torsion, for an algebroid to be Lie-admissible, pre-Lie, or post-Lie.
- Section 4 proves the main results, applying the framework of the previous sections to characterize transitive pre-Lie and post-Lie algebroids in terms of transitive $\mathfrak{g}$-actions on $M$. We also remark on the non-transitive case, in which these results hold leaf-by-leaf on the foliation of $M$ induced by the anchor, and compare our results to those of Blaom [4] and Abad and Crainic [1], who drop the transitivity assumption but require strictly stronger conditions on the connection than the pre-Lie and post-Lie conditions.

## 2. Algebras of invariant connections

In this section, we begin by considering a purely algebraic notion of a connection as a non-associative product on a Lie algebra. We then recall the definitions of Lie-admissible, pre-Lie, and post-Lie algebras, and we discuss the relationship between these algebras and the curvature and torsion of the connection corresponding to the product. Finally, we apply this framework to affine connections, obtaining necessary and sufficient conditions for $M$ to admit a connection giving $\mathfrak{X}(M)$ a Lie-admissible, pre-Lie, or post-Lie structure, and relating this to the Cartan–Nomizu classification.

2.1. **Connections on Lie algebras.** Certain properties of the connections we wish to study are purely algebraic, in the sense that they do not depend on any local or geometric arguments. Therefore, we postpone geometry to subsequent sections and begin in the following algebraic setting.

**Definition 2.1.** Let $\bigl(L, [\![\cdot,\cdot]\!]\bigr)$ be a Lie algebra over a field $\Bbbk$ of characteristic zero. A *connection* on $L$ is a $\Bbbk$-linear map $\nabla\colon L \to \operatorname{End}(L)$, $X \mapsto \nabla_X$.[1] Equivalently, a connection corresponds to a $\Bbbk$-bilinear product $\triangleright$ on $L$ defined by $X \triangleright Y \coloneqq \nabla_X Y$.

The Lie bracket on $L$ makes it possible to define algebraic notions of curvature and torsion, which are formally identical to the familiar definitions from differential geometry.

**Definition 2.2.** Given a connection $\nabla$ on $\bigl(L, [\![\cdot,\cdot]\!]\bigr)$, its *curvature* is the $\Bbbk$-bilinear map $R\colon L \times L \to \operatorname{End}(L)$ given by

$$R(X,Y) \coloneqq \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[\![X,Y]\!]}, \tag{1}$$

and its *torsion* is the $\Bbbk$-bilinear map $T\colon L \times L \to L$ given by

$$T(X,Y) \coloneqq \nabla_X Y - \nabla_Y X - [\![X,Y]\!]. \tag{2}$$

If $R = 0$, the connection is *flat*, and if $T = 0$, it is *torsion-free*.

*Remark* 2.3. A representation of a Lie algebra is precisely a flat connection.

Covariant derivatives $\nabla R$ and $\nabla T$ are defined by the usual product rules,

$$\begin{aligned}(\nabla_Z R)(X,Y)W \coloneqq{}& \nabla_Z\bigl(R(X,Y)W\bigr) \\ &- R(\nabla_Z X, Y)W - R(X, \nabla_Z Y)W - R(X,Y)\nabla_Z W,\end{aligned} \tag{3}$$

$$(\nabla_Z T)(X,Y) \coloneqq \nabla_Z\bigl(T(X,Y)\bigr) - T(\nabla_Z X, Y) - T(X, \nabla_Z Y). \tag{4}$$

The curvature (resp., torsion) is *parallel* if $\nabla R = 0$ (resp., $\nabla T = 0$), and if both the curvature and torsion are parallel, we say that $\nabla$ is an *invariant connection*.

Associated to each $\nabla$ is a *dual connection*[2] $\overline{\nabla}_X Y \coloneqq \nabla_Y X + [\![X,Y]\!]$, which is seen to satisfy $\overline{\overline{\nabla}} = \nabla$. The curvature and torsion of $\overline{\nabla}$ are denoted by $\overline{R}$ and $\overline{T}$. Observe that $T$ and $\overline{T}$ are related by

$$T(X,Y) = \nabla_X Y - \overline{\nabla}_X Y = -\overline{T}(X,Y), \tag{5}$$

so the torsion expresses the difference between the primal and dual connections, and a connection is its own dual if and only if it is torsion-free. In particular, the connection $\widetilde{\nabla} \coloneqq \frac{1}{2}(\nabla + \overline{\nabla})$ is always torsion-free.

*Example* 2.4. On any Lie algebra, we may define the trivial connection $\nabla_X Y = 0$, which has the dual connection $\overline{\nabla}_X Y = [\![X,Y]\!]$. We see that $R = 0$ trivially and $\overline{R} = 0$ by the Jacobi identity, and indeed $\nabla$ and $\overline{\nabla}$ are Lie algebra representations: the trivial representation and adjoint representation, respectively.

**Proposition 2.5.** *For a connection $\nabla$ on a Lie algebra, we have*

$$(\nabla_Z T)(X,Y) = \overline{R}(X,Y)Z + R(Y,Z)X + R(Z,X)Y. \tag{6}$$

[1] By $\operatorname{End}(L)$, we mean linear endomorphisms on $L$ as a vector space over $\Bbbk$, not necessarily Lie algebra endomorphisms.

[2] For connections on a Lie algebroid, this is the notation used by Crainic and Fernandes [9]; the name *dual connection* appears in Blaom [4], who denotes it by $\nabla^*$.

*Proof.* Consider the three terms defining $(\nabla_Z T)(X,Y)$ in (4). First,

$$\begin{aligned}\nabla_Z\bigl(T(X,Y)\bigr) &= \nabla_Z\nabla_X Y - \nabla_Z\nabla_Y X - \nabla_Z [\![X,Y]\!] \\ &= \nabla_Z\nabla_X Y - \nabla_Z\nabla_Y X - \overline{\nabla}_{[\![X,Y]\!]} Z - \bigl[\![Z,[\![X,Y]\!]\bigr]\!].\end{aligned}$$

For the second term,

$$\begin{aligned}T(\nabla_Z X, Y) &= \nabla_{\nabla_Z X} Y - \nabla_Y \nabla_Z X - [\![\nabla_Z X, Y]\!] \\ &= \overline{\nabla}_Y \nabla_Z X - \nabla_Y\nabla_Z X \\ &= \overline{\nabla}_Y\bigl(\overline{\nabla}_X Z + [\![Z,X]\!]\bigr) - \nabla_Y\nabla_Z X \\ &= \overline{\nabla}_Y\overline{\nabla}_X Z - \nabla_Y\nabla_Z X + \nabla_{[\![Z,X]\!]} Y + \bigl[\![Y,[\![Z,X]\!]\bigr]\!],\end{aligned}$$

and likewise for the third,

$$T(X,\nabla_Z Y) = \nabla_X\nabla_Z Y - \overline{\nabla}_X\overline{\nabla}_Y Z + \nabla_{[\![Y,Z]\!]} X + \bigl[\![X,[\![Y,Z]\!]\bigr]\!].$$

Combining these and applying the Jacobi identity gives (6). □

**Corollary 2.6.** *Assuming $R = 0$, we have $\overline{R} = 0$ if and only if $\nabla T = 0$.*

Cyclic sums of trilinear functions will appear repeatedly. We denote

$$\sum_{\circlearrowleft} f(X,Y,Z) \coloneqq f(X,Y,Z) + f(Y,Z,X) + f(Z,X,Y).$$

For example, the Jacobi identity may be written as $\sum_{\circlearrowleft} \bigl[\![X,[\![Y,Z]\!]\bigr]\!]$.

**Proposition 2.7.** *For a connection $\nabla$ on a Lie algebra, we have*

$$\sum_{\circlearrowleft} T\bigl(X,T(Y,Z)\bigr) = \sum_{\circlearrowleft} R(X,Y)Z + \sum_{\circlearrowleft} \overline{R}(X,Y)Z. \tag{7}$$

*Proof.* From (5), we have

$$\begin{aligned}T\bigl(Z,T(X,Y)\bigr) &= \nabla_Z\bigl(T(X,Y)\bigr) + \overline{\nabla}_Z\bigl(\overline{T}(X,Y)\bigr) \\ &= \nabla_Z\nabla_X Y - \nabla_Z\nabla_Y X - \overline{\nabla}_{[\![X,Y]\!]} Z - \bigl[\![Z,[\![X,Y]\!]\bigr]\!] \\ &\quad + \overline{\nabla}_Z\overline{\nabla}_X Y - \overline{\nabla}_Z\overline{\nabla}_Y X - \nabla_{[\![X,Y]\!]} Z - \bigl[\![Z,[\![X,Y]\!]\bigr]\!].\end{aligned}$$

The last equality comes from the expression obtained for $\nabla_Z\bigl(T(X,Y)\bigr)$ in the previous proof, together with the corresponding version for $\overline{\nabla}$. Taking the cyclic sum of both sides and applying the Jacobi identity gives (7). □

**Corollary 2.8.** *If $R = \overline{R} = 0$, then $T$ is a Lie bracket.*

*Proof.* By definition, $T$ is always bilinear and skew-symmetric. If $R = \overline{R} = 0$, the right-hand side of (7) vanishes, so $T$ also satisfies the Jacobi identity. □

**Corollary 2.9** (Bianchi's first identity)**.** *For a connection $\nabla$ on a Lie algebra, we have*

$$\sum_{\circlearrowleft} (\nabla_X T)(Y,Z) = \sum_{\circlearrowleft} R(X,Y)Z + \sum_{\circlearrowleft} T\bigl(X,T(Y,Z)\bigr) \tag{8}$$

*Proof.* Take the cyclic sum of both sides of (6) and apply (7). □

*Remark* 2.10. Bianchi's second identity,

$$\sum_{\circlearrowleft} (\nabla_X R)(Y,Z) = \sum_{\circlearrowleft} R\bigl(X,T(Y,Z)\bigr),$$

also holds in this setting. The proof is a lengthy calculation which we omit, since we will not need this identity in the sequel.

2.2. **Lie-admissible, pre-Lie, and post-Lie algebras of connections.** We now consider Lie-admissible, pre-Lie, and post-Lie algebraic structures on an algebra $(\mathcal{A}, \triangleright)$. For each of these, we show that the product $\triangleright$ may be interpreted as a connection on a Lie algebra, and we characterize these algebraic structures in terms of the curvature and torsion of this connection.

The *associator* of the product $\triangleright$ is denoted by

$$a(X, Y, Z) := X \triangleright (Y \triangleright Z) - (X \triangleright Y) \triangleright Z,$$

following the sign convention of Munthe-Kaas and Lundervold [26]. In the sequel, an important role is played by the *associator triple bracket*,

$$[X, Y, Z] := a(X, Y, Z) - a(Y, X, Z).$$

When $\triangleright$ corresponds to a connection on a Lie algebra, the following useful identity relates the associator triple bracket to curvature and torsion.

**Proposition 2.11.** *For a connection $\nabla$ on a Lie algebra, we have*

$$[X, Y, Z] = R(X, Y)Z - T(X, Y) \triangleright Z.$$

*Proof.* By definition of $R$ and $T$ and the linearity of the connection,

$$R(X, Y)Z = X \triangleright (Y \triangleright Z) - Y \triangleright (X \triangleright Z) - [\![X, Y]\!] \triangleright Z,$$
$$T(X, Y) \triangleright Z = (X \triangleright Y) \triangleright Z - (Y \triangleright X) \triangleright Z - [\![X, Y]\!] \triangleright Z,$$

so subtracting gives $[X, Y, Z]$. □

2.2.1. *Lie-admissible algebras.* The definition of a Lie-admissible algebra is due to Albert [3].

**Definition 2.12.** An algebra $(\mathcal{A}, \triangleright)$ is *Lie-admissible* if $\sum_{\circlearrowleft} [X, Y, Z] = 0$.

Such algebras are called "Lie-admissible" due to the following equivalence.

**Proposition 2.13.** $(\mathcal{A}, \triangleright)$ *is Lie-admissible if and only if the commutator bracket* $[\![X, Y]\!] := X \triangleright Y - Y \triangleright X$ *is a Lie bracket on* $\mathcal{A}$.

*Proof.* The commutator bracket is always skew-symmetric and bilinear. A short calculation shows that $\sum_{\circlearrowleft} [\![X, [\![Y, Z]\!]]\!] = \sum_{\circlearrowleft} [X, Y, Z]$, so the Jacobi identity is equivalent to the Lie-admissibility condition. □

**Proposition 2.14.** *The following are equivalent:*

*(i)* $(\mathcal{A}, \triangleright)$ *is a Lie-admissible algebra.*
*(ii)* $(\mathcal{A}, [\![\cdot, \cdot]\!])$ *is a Lie algebra with a torsion-free connection* $\nabla$.

*Proof.* $T(X, Y) = 0$ says precisely that $[\![X, Y]\!] = X \triangleright Y - Y \triangleright X$. □

2.2.2. *Pre-Lie algebras.* The notion of pre-Lie algebra appears in work of Vinberg [32] in differential geometry and Gerstenhaber [11] in algebra. They also appear in the work of Agrachev and Gamkrelidze [2] in control theory, under the name "chronological algebras."

**Definition 2.15.** An algebra $(\mathcal{A}, \triangleright)$ is *pre-Lie* if $[X, Y, Z] = 0$.

It follows immediately from this definition that every pre-Lie algebra is a Lie-admissible algebra, so $\triangleright$ corresponds to a torsion-free connection $\nabla$ on $(\mathcal{A}, [\![\cdot, \cdot]\!])$, where $[\![\cdot, \cdot]\!]$ is the commutator bracket. The next result shows that the pre-Lie condition corresponds to the case where $\nabla$ is also flat.

**Proposition 2.16.** *The following are equivalent:*

*(i)* $(\mathcal{A}, \triangleright)$ *is a pre-Lie algebra.*

*(ii)* $\big(\mathcal{A}, [\![\cdot,\cdot]\!]\big)$ *is a Lie algebra with a flat and torsion-free connection* $\nabla$.

*Proof.* If $(\mathcal{A}, \triangleright)$ is pre-Lie, then Proposition 2.14 says that $\triangleright$ corresponds to a torsion-free connection, and Proposition 2.11 with $T = 0$ gives $R(X,Y)Z = [X,Y,Z] = 0$, so the connection is also flat. The converse direction is immediate from Proposition 2.11 with $R = 0$ and $T = 0$. □

2.2.3. *Post-Lie algebras.* The notion of post-Lie algebra is due to Vallette [31].

**Definition 2.17.** A *post-Lie algebra* $\big(\mathcal{A}, [\cdot,\cdot], \triangleright\big)$ is a Lie algebra $\big(\mathcal{A}, [\cdot,\cdot]\big)$ equipped with a product $\triangleright$ satisfying the compatibility conditions

$$X \triangleright [Y,Z] = [X \triangleright Y, Z] + [X, Y \triangleright Z], \tag{9a}$$

$$[X,Y] \triangleright Z = [X,Y,Z]. \tag{9b}$$

Given a post-Lie algebra $\big(\mathcal{A}, [\cdot,\cdot], \triangleright\big)$, we immediately see from (9b) that $(\mathcal{A}, \triangleright)$ is pre-Lie if and only if $[X,Y] \triangleright Z = 0$ for all $X, Y, Z \in \mathcal{A}$. Furthermore, any pre-Lie algebra $(\mathcal{A}, \triangleright)$ admits a post-Lie structure by taking $[\cdot,\cdot]$ to be trivial.

**Proposition 2.18.** *If* $\big(\mathcal{A}, [\cdot,\cdot], \triangleright\big)$ *is a post-Lie algebra, then*

$$[\![X,Y]\!] \coloneqq X \triangleright Y - Y \triangleright X + [X,Y] \tag{10}$$

*is also a Lie bracket on* $\mathcal{A}$.

*Proof.* This bracket is always skew-symmetric and bilinear. To establish the Jacobi identity for $[\![\cdot,\cdot]\!]$, a calculation shows that

$$\sum_{\circlearrowleft} [\![X, [\![Y,Z]\!]]\!] = \sum_{\circlearrowleft} \big( X \triangleright [Y,Z] - [X \triangleright Y, Z] - [Y, X \triangleright Z] \big) + \sum_{\circlearrowleft} \big( [X,Y,Z] - [X,Y] \triangleright Z \big) + \sum_{\circlearrowleft} [X,[Y,Z]]. \tag{11}$$

On the right-hand side, the first cyclic sum vanishes by (9a), the second by (9b), and the last by the Jacobi identity for $[\cdot,\cdot]$. □

Assuming $\big(\mathcal{A}, [\cdot,\cdot], \triangleright\big)$ is post-Lie, we consider $\triangleright$ as a connection on $\big(\mathcal{A}, [\![\cdot,\cdot]\!]\big)$. It follows from (10) that $T(X,Y) = -[X,Y]$. Therefore, the post-Lie condition (9a) says that $\nabla T = 0$, while (9b) says that $R = 0$ (using Proposition 2.11 to relate the triple bracket to curvature and torsion). Furthermore, the vanishing of (11) corresponds to the first Bianchi identity (8).

Conversely, if $\big(\mathcal{A}, [\![\cdot,\cdot]\!]\big)$ is a Lie algebra with connection $\nabla$, we may define $[X,Y] = -T(X,Y)$ and ask when $\big(\mathcal{A}, [\cdot,\cdot], \triangleright\big)$ is post-Lie. The following result shows that the conditions $\nabla T = 0$ and $R = 0$ are sufficient, as well as necessary.

**Proposition 2.19.** *Let* $\triangleright$, $[\cdot,\cdot]$, *and* $[\![\cdot,\cdot]\!]$ *be related by* (10). *Then the following are equivalent:*

*(i)* $\big(\mathcal{A}, [\cdot,\cdot], \triangleright\big)$ *is a post-Lie algebra.*
*(ii)* $\big(\mathcal{A}, [\![\cdot,\cdot]\!]\big)$ *is a Lie algebra with a flat, parallel-torsion connection* $\nabla$.
*(iii)* $\big(\mathcal{A}, [\![\cdot,\cdot]\!]\big)$ *is a Lie algebra with a flat connection* $\nabla$ *and flat dual connection* $\overline{\nabla}$.

*Proof.* We have already shown, in discussion above, that (i) implies (ii), and Corollary 2.6 says that (ii) and (iii) are equivalent. To show that (ii) implies (i), observe that $\nabla T = 0$ and $R = 0$ immediately give (9a) and (9b), while Corollary 2.8 implies that $[\cdot,\cdot] = -T$ is a Lie bracket. □

2.3. **Algebras of affine connections.** We now bring geometry into the picture by considering affine connections. The main result of this section, Theorem 2.23, gives necessary and sufficient conditions for $M$ to admit a connection giving $\mathfrak{X}(M)$ a Lie-admissible, pre-Lie, or post-Lie structure. Munthe-Kaas and Lundervold [26] had previously shown sufficiency but not necessity of these conditions.

Recall that a vector field $X \in \mathfrak{X}(M)$ on a smooth manifold defines a derivation $f \mapsto X[f]$ on $C^\infty(M)$. This forms a Lie algebra $\big(\mathfrak{X}(M), [\![\cdot,\cdot]\!]_J\big)$ with respect to the *Jacobi–Lie bracket*,

$$[\![X,Y]\!]_J[f] \coloneqq X\big[Y[f]\big] - Y\big[X[f]\big].$$

An *affine connection* is not only $\mathbb{R}$-bilinear on $\mathfrak{X}(M)$, but is $C^\infty(M)$-linear in the first argument and satisfies a Leibniz rule in the second,

$$\nabla_{fX} Y = f \nabla_X Y, \qquad \nabla_X fY = X[f]Y + f\nabla_X Y,$$

for all $f \in C^\infty(M)$, $X, Y \in \mathfrak{X}(M)$. It is straightforward to show that if $\nabla$ is an affine connection, then so are $\overline{\nabla}$ and $\widetilde{\nabla}$, but we postpone the proof to the more general setting of Lie algebroids, in Section 3. The curvature and torsion of an affine connection $\nabla$ are defined with respect to $[\![\cdot,\cdot]\!]_J$, and these definitions imply that $R$ and $T$ are *tensorial*, i.e., $C^\infty(M)$-linear in all arguments.

To apply the framework developed in this section to affine connections, we first show that the brackets $[\![\cdot,\cdot]\!]$ constructed for Lie-admissible, pre-Lie, and post-Lie algebras agree with the Jacobi–Lie bracket.

**Lemma 2.20.** *If $[\![\cdot,\cdot]\!]$ is a Lie bracket on $\mathfrak{X}(M)$ satisfying the Leibniz rule*

$$[\![X, fY]\!] = X[f]Y + f[\![X,Y]\!], \tag{12}$$

*for all $f \in C^\infty(M)$, $X, Y \in \mathfrak{X}(M)$, then $[\![\cdot,\cdot]\!] = [\![\cdot,\cdot]\!]_J$.*

*Proof.* Using the Jacobi identity and Leibniz rule, a calculation gives

$$\begin{aligned} 0 &= \big[\![X, [\![Y, fZ]\!]\big]\!] + \big[\![Y, [\![fZ, X]\!]\big]\!] + \big[\![fZ, [\![X,Y]\!]\big]\!] \\ &= X\big[Y[f]\big]Z - Y\big[X[f]\big]Z - [\![X,Y]\!][f]Z \\ &= \big([\![X,Y]\!]_J - [\![X,Y]\!]\big)[f]Z, \end{aligned}$$

and the result follows since $f \in C^\infty(M)$, $X, Y, Z \in \mathfrak{X}(M)$ are arbitrary. □

**Proposition 2.21.** *Let $\nabla$ be an affine connection and $[\cdot,\cdot]$ a tensorial bracket. If $\big(\mathfrak{X}(M), [\cdot,\cdot], \triangleright\big)$ is post-Lie, then the bracket $[\![X,Y]\!] \coloneqq X \triangleright Y - Y \triangleright X + [X,Y]$ of* (10) *agrees with the Jacobi–Lie bracket.*

*Proof.* It suffices to check that the Leibniz rule (12) holds:

$$\begin{aligned} [\![X, fY]\!] &= X \triangleright (fY) - (fY) \triangleright X + [X, fY] \\ &= X[f]Y + f(X \triangleright Y) - f(Y \triangleright X) + f[X,Y] \\ &= X[f]Y + f[\![X,Y]\!]. \end{aligned}$$

The result then follows by Lemma 2.20. □

Repeating the same computation with $[\cdot,\cdot] = 0$, we find the following.

**Proposition 2.22.** *Let $\nabla$ be an affine connection. If $\big(\mathfrak{X}(M), \triangleright\big)$ is Lie-admissible, then the commutator bracket $[\![X,Y]\!] \coloneqq X \triangleright Y - Y \triangleright X$ agrees with the Jacobi–Lie bracket.*

Applying Proposition 2.14, Proposition 2.16, and Proposition 2.19 now yields our main result on algebras of affine connections.

**Theorem 2.23.** *Let $\nabla$ be an affine connection on a smooth manifold $M$.*

*(i)* $\big(\mathfrak{X}(M), \triangleright\big)$ *is a Lie-admissible algebra if and only if* $\nabla$ *is torsion-free.*
*(ii)* $\big(\mathfrak{X}(M), \triangleright\big)$ *is a pre-Lie algebra if and only if* $\nabla$ *is flat and torsion-free.*
*(iii)* $\big(\mathfrak{X}(M), [\cdot,\cdot], \triangleright\big)$ *is a post-Lie algebra, with* $[\cdot,\cdot]$ *being tensorial, if and only if* $\nabla$ *is flat with parallel torsion* $T = -[\cdot,\cdot]$.

Every smooth manifold $M$ admits an affine connection $\nabla$, and thus a torsion-free connection $\widetilde{\nabla}$, so Lie-admissibility of $\big(\mathfrak{X}(M), \triangleright\big)$ reveals nothing about $M$. By contrast, the other two algebraic structures are deeply associated with special geometries classified by Cartan [7] and Nomizu [29].

**Corollary 2.24.** *Let* $M$ *be a smooth manifold.*

- $M$ *admits an affine connection* $\nabla$ *such that* $\big(\mathfrak{X}(M), \triangleright\big)$ *is pre-Lie if and only if* $M$ *is locally representable as an abelian Lie group with its canonical affine connection.*
- $M$ *admits a connection* $\nabla$ *and a tensorial bracket* $[\cdot,\cdot]$ *such that* $\big(\mathfrak{X}(M), [\cdot,\cdot], \triangleright\big)$ *is post-Lie if and only if* $M$ *is locally representable as a Lie group with its* $(-)$*-connection.*

*Proof.* Combine Theorem 2.23 with results (a) and (b) stated in §20 of Nomizu [29]. □

## 3. The geometry and algebra of connections on Lie algebroids

In this section, we recall how connections may be generalized from the tangent bundle of $M$ (i.e., affine connections) to more general anchored bundles and Lie algebroids[3] over $M$. We then characterize connections inducing Lie-admissible, pre-Lie, and post-Lie structures in terms of their curvature and torsion, generalizing the results of Section 2.3 for affine connections.

3.1. **Lie algebroids.** Pradines [30] is credited for introducing Lie algebroids, which simultaneously generalize tangent bundles and Lie algebras (among many other things). A comprehensive treatment is given by Mackenzie [22].

**Definition 3.1.** An *anchored bundle* $(A, \rho)$ is a vector bundle $A \to M$ with a vector bundle morphism $\rho \colon A \to TM$ called the *anchor map.* A *Lie algebroid* $\big(A, \rho, [\![\cdot,\cdot]\!]\big)$ is an anchored bundle equipped with a Lie bracket $[\![\cdot,\cdot]\!]$ on the space of sections $\Gamma(A)$, satisfying the Leibniz rule

$$[\![X, fY]\!] = \rho(X)[f]Y + f[\![X,Y]\!], \tag{13}$$

for all $f \in C^\infty(M)$, $X, Y \in \Gamma(A)$. We say that an anchored bundle or Lie algebroid is *transitive* if the anchor is surjective.

*Example* 3.2. The tangent bundle $A = TM$ is a Lie algebroid over $M$, with $\rho$ the identity map on $TM$ and $[\![\cdot,\cdot]\!]$ the Jacobi–Lie bracket.

More generally, any involutive distribution $\mathcal{D} \subset TM$ is a Lie algebroid over $M$, with $\rho$ the inclusion $\mathcal{D} \hookrightarrow TM$ and $[\![\cdot,\cdot]\!]$ the restriction of the Jacobi–Lie bracket to $\mathcal{D}$. (In fact, this describes a *Lie subalgebroid* of the tangent Lie algebroid.)

*Example* 3.3. A Lie algebra is just a Lie algebroid over a single point, with trivial anchor $\rho = 0$.

More generally, a Lie algebroid with trivial anchor is called a *bundle of Lie algebras*: since $\rho = 0$, (13) implies that $[\![\cdot,\cdot]\!]$ is tensorial, so for each $x \in M$ we have a well-defined

[3] Generally, the results of this section also hold for *Lie–Rinehart algebras*, which are an algebraic abstraction of Lie algebroids (e.g., replacing smooth functions on $M$ by a commutative algebra, vector fields on $M$ by derivations on that algebra, etc.). However, since we are laying the groundwork for Section 4, which *does* use the smooth manifold structure, we have chosen to use the language of Lie algebroids throughout.

pointwise Lie bracket $[\![\cdot,\cdot]\!]_x$ on the fiber $A_x$. Note that the fibers need not be isomorphic as Lie algebras, i.e., $A$ need not be isomorphic to the trivial bundle of Lie algebras $\mathfrak{g} \times M$ for any Lie algebra $\mathfrak{g}$.

*Example* 3.4. An *action* (sometimes called an *infinitesimal action*) of a Lie algebra $\mathfrak{g}$ on $M$ is a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{X}(M)$, $\xi \mapsto \xi_M$. The *action algebroid* $A = \mathfrak{g} \ltimes M$ is the trivial vector bundle $\mathfrak{g} \times M \to M$, together with the anchor $\rho(\xi, x) \coloneqq \xi_M(x)$ induced by the action of $\mathfrak{g}$ on $M$; the bracket $[\![\cdot,\cdot]\!]$ is uniquely determined by (13) and the condition that it agrees with the bracket on $\mathfrak{g}$ for constant sections.

The action algebroid is transitive precisely when the $\mathfrak{g}$-action is transitive. In particular, if $M$ is a homogeneous space, then its transitive Lie group action has a corresponding transitive Lie algebra action.

The following is a standard (yet important) property of Lie algebroids. Some references on Lie algebroids, including Mackenzie [22], include this property as part of the definition of a Lie algebroid, but this turns out to be redundant. (An interesting account of this appears in the introduction to Grabowski [12].) The argument is essentially one due to Herz [18], later used by Kosmann-Schwarzbach and Magri [20, §6.1].

**Proposition 3.5.** *Given a Lie algebroid* $\big(A, \rho, [\![\cdot,\cdot]\!]\big)$ *over* $M$*, the anchor map induces a Lie algebra homomorphism* $\big(\Gamma(A), [\![\cdot,\cdot]\!]\big) \to \big(\mathfrak{X}(M), [\![\cdot,\cdot]\!]_J\big)$.

*Proof.* Just as in the proof of Lemma 2.20, one uses the Jacobi identity together with the Leibniz rule (13) to calculate

$$\begin{aligned} 0 &= [\![X,[\![Y,fZ]\!]]\!] + [\![Y,[\![fZ,X]\!]]\!] + [\![fZ,[\![X,Y]\!]]\!] \\ &= \rho(X)\big[\rho(Y)[f]\big]Z - \rho(Y)\big[\rho(X)[f]\big]Z - \rho\big([\![X,Y]\!]\big)[f]Z \\ &= \Big([\![\rho(X),\rho(Y)]\!]_J - \rho\big([\![X,Y]\!]\big)\Big)[f]Z, \end{aligned}$$

and the result follows since $f \in C^\infty(M)$, $X, Y, Z \in \Gamma(A)$ are arbitrary. □

*Remark* 3.6. Lemma 2.20 is actually a special case of this result. In the language just introduced, the Leibniz rule (12) implies that $\big(TM, \mathrm{id}_{TM}, [\![\cdot,\cdot]\!]\big)$ is a Lie algebroid, so $\mathrm{id}_{TM}$ induces a Lie algebra isomorphism between $\big(\mathfrak{X}(M), [\![\cdot,\cdot]\!]\big)$ and $\big(\mathfrak{X}(M), [\![\cdot,\cdot]\!]_J\big)$. That is, $[\![\cdot,\cdot]\!] = [\![\cdot,\cdot]\!]_J$.

An important consequence of this result is that the image of $\rho$ defines an involutive distribution on $M$, so there exists a (generally singular) foliation of $M$ into leaves. The restriction to each leaf $L \subset M$ defines a transitive Lie algebroid over $L$.

3.2. **Connections, curvature, and torsion.** We next discuss connections, first on anchored bundles and then on Lie algebroids, where the latter largely follows the treatment given in Fernandes [10], Crainic and Fernandes [9].

**Definition 3.7.** Given an anchored bundle $(A, \rho)$ over $M$, an *A-connection* on a vector bundle $E \to M$ is an $\mathbb{R}$-bilinear map $\Gamma(A) \times \Gamma(E) \to \Gamma(E)$, $(X, u) \mapsto \nabla_X u$, which is $C^\infty(M)$-linear in the first argument and satisfies a Leibniz rule in the second, i.e.,

$$\nabla_{fX} u = f \nabla_X u, \qquad \nabla_X f u = \rho(X)[f]u + f \nabla_X u,$$

for all $f \in C^\infty(M)$.

An affine connection is just a $TM$-connection, where, as before, the anchor is the identity map. Given a $TM$-connection on an anchored bundle $A$, the following construction gives an induced $A$-connection on $A$.

**Proposition 3.8.** *Let $(A, \rho)$ be an anchored bundle over $M$ and $\nabla$ be a $TM$-connection on $A$. Then $\nabla_X Y \coloneqq \nabla_{\rho(X)} Y$ is an $A$-connection on $A$.*

*Proof.* $\mathbb{R}$-bilinearity follows from the $\mathbb{R}$-bilinearity of the $TM$-connection, together with the fact that $\rho$ is a vector bundle morphism. For any $f \in C^\infty(M)$ and $X, Y \in \Gamma(A)$, we have

$$\nabla_{fX} Y = \nabla_{\rho(fX)} Y = \nabla_{f\rho(X)} Y = f \nabla_{\rho(X)} Y = f \nabla_X Y,$$

and

$$\nabla_X fY = \nabla_{\rho(X)} fY = \rho(X)[f]Y + f\nabla_{\rho(X)} Y = \rho(X)[f]Y + f\nabla_X Y,$$

which completes the proof. □

If $(A, \rho, [\![\cdot,\cdot]\!])$ is a Lie algebroid and $\nabla$ is an $A$-connection on $A$, then $\nabla$ is also a connection on the Lie algebra $(\Gamma(A), [\![\cdot,\cdot]\!])$, in the sense of Section 2.1. Therefore, all of the results in that section immediately hold in the Lie algebroid setting. We now show that $\overline{\nabla}_X Y \coloneqq \nabla_Y X + [\![X, Y]\!]$ and $\widetilde{\nabla} \coloneqq \frac{1}{2}(\nabla + \overline{\nabla})$ are in fact connections in the Lie algebroid sense, not just the Lie algebra sense.

**Proposition 3.9.** *If $(A, \rho, [\![\cdot,\cdot]\!])$ is a Lie algebroid, and if $\nabla$ is an $A$-connection on $A$, then so are $\overline{\nabla}$ and $\widetilde{\nabla}$.*

*Proof.* $\mathbb{R}$-bilinearity of $\overline{\nabla}$ follows from the $\mathbb{R}$-bilinearity of $\nabla$ and of $[\![\cdot,\cdot]\!]$. For any $f \in C^\infty(M)$ and $X, Y \in \Gamma(A)$,

$$\begin{aligned} \overline{\nabla}_{fX} Y &= \nabla_Y fX + [\![fX, Y]\!] \\ &= \rho(Y)[f]X + f\nabla_Y X - \rho(Y)[f]X + f[\![X, Y]\!] \\ &= f\overline{\nabla}_X Y, \end{aligned}$$

and

$$\begin{aligned} \overline{\nabla}_X fY &= \nabla_{fY} X + [\![X, fY]\!] \\ &= f\nabla_Y X + \rho(X)[f]Y + f[\![X, Y]\!] \\ &= \rho(X)[f]Y + f\overline{\nabla}_X Y, \end{aligned}$$

so $\overline{\nabla}$ is an $A$-connection. That $\widetilde{\nabla}$ is also an $A$-connection follows easily from the fact that $\nabla$ and $\overline{\nabla}$ are $A$-connections. □

*Example* 3.10. Let $\mathfrak{g}$ be a Lie algebra, considered as a Lie algebroid over a single point. The trivial connection $\nabla_\xi \eta = 0$ is a $\mathfrak{g}$-connection on $\mathfrak{g}$, and $\overline{\nabla}_\xi \eta = [\![\xi, \eta]\!] = \operatorname{ad}_\xi \eta$. We can thus identify $\nabla$ with the trivial representation and $\overline{\nabla}$ with the adjoint representation of $\mathfrak{g}$ on itself. This is readily generalized to the case where $A$ is a bundle of Lie algebras over $M$.

*Example* 3.11. Let $A = \mathfrak{g} \ltimes M$ be an action algebroid. As a vector bundle, this is just the trivial bundle $\mathfrak{g} \times M$, so we can define the obvious $TM$-connection $\nabla$ vanishing on constant sections. Identifying $\xi, \eta \in \mathfrak{g}$ with the corresponding constant sections, it follows that the corresponding $A$-connections on $A$ satisfy $\nabla_\xi \eta = 0$ and $\overline{\nabla}_\xi \eta = [\![\xi, \eta]\!]$.

In particular, if $M = G$ is the Lie group integrating $\mathfrak{g}$, then we may identify constant sections of $\mathfrak{g} \ltimes G$ with left-invariant vector fields on $G$ (and arbitrary sections with arbitrary vector fields). Under this identification, the connections $\nabla$, $\overline{\nabla}$, and $\widetilde{\nabla}$ correspond, respectively, to the affine $(-)$-, $(+)$-, and $(0)$-connections of Cartan and Schouten [8].

The curvature and torsion of an $A$-connection on a Lie algebroid $A$ are defined exactly as in (1)–(2), and all the results of Section 2.1 involving the curvature and torsion of $\nabla$, $\overline{\nabla}$, and $\widetilde{\nabla}$ immediately hold in this setting.

As with affine connections, $R$ and $T$ are *tensorial*, i.e., $C^\infty(M)$-linear in each argument, not just $\mathbb{R}$-linear, so they contain local, geometric information about the connection.[4] The proof that

$$fR(X,Y)Z = R(fX,Y)Z = R(X,fY)Z = R(X,Y)fZ$$

is by direct calculation, showing that all terms involving the anchor cancel. For $R(fX,Y)Z$, one gets the two canceling terms $\pm\rho(Y)[f]\nabla_X Z$. Similarly, for $R(X,fY)$, one gets the two canceling terms $\pm\rho(X)[f]\nabla_Y Z$. Finally, for $R(X,Y)fZ$, one gets the additional term $\big(\llbracket \rho(X),\rho(Y)\rrbracket_J - \rho(\llbracket X,Y\rrbracket)\big)[f]Z$, which vanishes by Proposition 3.5. Similarly, one gets canceling terms $\pm\rho(Y)[f]X$ when computing $T(fX,Y)$ and $\pm\rho(X)[f]Y$ when computing $T(X,fY)$, which implies the tensoriality of $T$.

3.3. **Algebras of $A$-connections.** We next relate the curvature and torsion of an $A$-connection $\nabla$ to Lie-admissible, pre-Lie, and post-Lie algebraic structures on $\Gamma(A)$ with the product $\triangleright$. This generalizes the results of Section 2.3 on affine connections, which correspond to the case $A = TM$.

3.3.1. *Lie-admissible algebroids.* We begin by introducing Lie-admissible algebroids, which are a natural generalization of Lie-admissible algebras.

**Definition 3.12.** A *Lie-admissible algebroid* $(A,\rho,\nabla)$ is an anchored bundle $(A,\rho)$, equipped with an $A$-connection $\nabla$ on $A$, such that $\big(\Gamma(A),\triangleright\big)$ is a Lie-admissible algebra.

**Proposition 3.13.** *Let $(A,\rho)$ be an anchored bundle and $\nabla$ an $A$-connection on $A$. Then $(A,\rho,\nabla)$ is a Lie-admissible algebroid if and only if $(A,\rho)$ admits a Lie algebroid structure such that $\nabla$ is torsion-free.*

*Proof.* The condition that $(A,\rho)$ admits a Lie algebroid structure such that $\nabla$ is torsion-free simply says that $\big(A,\rho,\llbracket\cdot,\cdot\rrbracket\big)$ is a Lie algebroid, where $\llbracket X,Y\rrbracket \coloneqq X\triangleright Y - Y\triangleright X$ is the commutator bracket.

First, if $\big(A,\rho,\llbracket\cdot,\cdot\rrbracket\big)$ is a Lie algebroid, then by definition, $\llbracket\cdot,\cdot\rrbracket$ is a Lie bracket on $\Gamma(A)$, so Proposition 2.13 implies Lie-admissibility.

Conversely, if $(A,\rho,\nabla)$ is Lie-admissible, then Proposition 2.13 implies that $\llbracket\cdot,\cdot\rrbracket$ is a Lie bracket, so it suffices to show that it satisfies the Leibniz rule (13). Indeed,

$$\begin{aligned}\llbracket X, fY\rrbracket &\coloneqq X\triangleright(fY) - (fY)\triangleright X\\ &= \rho(X)[f]Y + f(X\triangleright Y) - f(Y\triangleright X)\\ &= \rho(X)[f]Y + f\llbracket X,Y\rrbracket,\end{aligned}$$

which completes the proof. □

*Example* 3.14. A Lie-admissible algebra is just a Lie-admissible algebroid over a point; Proposition 3.13 gives the corresponding Lie algebra as a Lie algebroid over a point. More generally, a Lie-admissible algebroid with trivial anchor can be seen as a "bundle of Lie-admissible algebras," and Proposition 3.13 gives the corresponding bundle of Lie algebras.

The next results examine the situation where $\big(A,\rho,\llbracket\cdot,\cdot\rrbracket\big)$ is a given Lie algebroid, whose bracket is not *a priori* equal to the commutator of $\triangleright$.

**Proposition 3.15.** *Let $\big(A,\rho,\llbracket\cdot,\cdot\rrbracket\big)$ be a Lie algebroid and $\nabla$ be an $A$-connection on $A$. If $(A,\rho,\nabla)$ is Lie-admissible, then $\rho\circ T = 0$.*

[4] One also has tensorial curvature in the more general setting where $\nabla$ is an $A$-connection on a vector bundle $E\to M$. The proof is the same, just replacing $Z\in\Gamma(A)$ by $u\in\Gamma(E)$.

*Proof.* If $(A, \rho, \nabla)$ is Lie-admissible, then Proposition 3.13 implies that we have two Lie algebroid structures: one with $[\![\cdot,\cdot]\!]$, and the other with the commutator bracket. However, Proposition 3.5 implies that $\rho$ maps each of these to the Jacobi–Lie bracket on $\mathfrak{X}(M)$, so

$$\rho(X \triangleright Y - Y \triangleright X) = [\![\rho(X), \rho(Y)]\!]_J = \rho\bigl([\![X, Y]\!]\bigr).$$

Hence, $\rho\bigl(T(X,Y)\bigr) = 0$, for all $X, Y \in \Gamma(A)$. □

Unlike the situation for affine connections in Proposition 2.22, we may not necessarily conclude that $[\![\cdot,\cdot]\!]$ agrees with the commutator bracket. However, we *may* conclude this if the anchor is injective.

**Corollary 3.16.** *Let $\bigl(A, \rho, [\![\cdot,\cdot]\!]\bigr)$ be a Lie algebroid and $\nabla$ be an $A$-connection on $A$. If $\nabla$ is torsion-free, then $(A, \rho, \nabla)$ is Lie-admissible. The converse is true if $\rho$ is injective.*

*Proof.* If $T = 0$, then the commutator bracket of $\triangleright$ is precisely $[\![\cdot,\cdot]\!]$, so Proposition 3.13 implies that $(A, \rho, \nabla)$ is Lie-admissible.

Conversely, if $(A, \rho, \nabla)$ is Lie-admissible, then Proposition 3.15 says that $\rho \circ T = 0$, which implies $T = 0$ under the assumption that $\rho$ is injective. □

*Remark* 3.17. Theorem 2.23(i) is a special case of this result for $A = TM$, where the anchor $\rho = \mathrm{id}_{TM}$ is injective.

The following counterexample shows that the converse above is generally not true unless $\rho$ is injective.

*Example* 3.18. Consider a bundle of Lie algebras $\bigl(A, 0, [\![\cdot,\cdot]\!]\bigr)$. Since the anchor is trivial, we may take the trivial connection $\nabla_X Y = 0$. This is clearly Lie-admissible, but its torsion $T(X,Y) = -[\![X,Y]\!]$ generally does not vanish.

We may also obtain necessary and sufficient geometric conditions for Lie-admissibility, in cases where $\rho$ is not injective, by imposing some mild restrictions on the $A$-connection $\nabla$. In the next proposition, we assume that $\nabla_X Y = 0$ whenever $\rho(X) = 0$. This is always the case, for instance, when $\nabla$ arises from a $TM$-connection on $A$ using the construction in Proposition 3.8.

**Proposition 3.19.** *Let $\bigl(A, \rho, [\![\cdot,\cdot]\!]\bigr)$ be a Lie algebroid and $\nabla$ an $A$-connection on $A$ such that $\nabla_X Y = 0$ whenever $\rho(X) = 0$. Then $(A, \rho, \nabla)$ is Lie-admissible if and only if $\rho \circ T = 0$ and*

$$\sum_{\circlearrowleft} R(X,Y)Z = 0,$$

*for all $X, Y, Z \in \Gamma(A)$.*

*Proof.* Using Proposition 2.11, we obtain

$$R(X,Y)Z = [X,Y,Z] + T(X,Y) \triangleright Z.$$

If $\rho \circ T = 0$, then the assumption on $\nabla$ gives $T(X,Y) \triangleright Z = 0$, so

$$R(X,Y)Z = [X,Y,Z].$$

The result then follows immediately from the definition of Lie-admissibility, together with Proposition 3.15. □

Finally, note that every Lie algebroid admits an $A$-connection $\nabla$ (pick any $TM$-connection on $A$ and apply Proposition 3.8) and thus admits a torsion-free $A$-connection $\widetilde{\nabla}$. Therefore, as with the case of affine connections, Lie-admissibility does not actually reveal any information about the Lie algebroid itself.

3.3.2. *Pre-Lie algebroids.* We next introduce what we call pre-Lie algebroids, which are a natural generalization of pre-Lie algebras to the algebroid setting.[5]

**Definition 3.20.** A *pre-Lie algebroid* $(A, \rho, \nabla)$ is an anchored bundle $(A, \rho)$, with an $A$-connection $\nabla$ on $A$, such that $\big(\Gamma(A), \triangleright\big)$ is a pre-Lie algebra.

From this definition, we immediately see that every pre-Lie algebroid is Lie-admissible, so the results of the previous section apply.

**Proposition 3.21.** *Let $(A, \rho)$ be an anchored bundle and $\nabla$ an $A$-connection on $A$. Then $(A, \rho, \nabla)$ is a pre-Lie algebroid if and only if $(A, \rho)$ admits a Lie algebroid structure such that $\nabla$ is flat and torsion-free.*

*Proof.* If $(A, \rho, \nabla)$ is pre-Lie, then in particular it is Lie-admissible. Therefore, Proposition 3.13 implies that $\big(A, \rho, [\![\cdot,\cdot]\!]\big)$ is a Lie algebroid, where $[\![\cdot,\cdot]\!]$ is the commutator of $\triangleright$, with respect to which $\nabla$ is torsion-free. As in the proof of Proposition 2.16, applying Proposition 2.11 with $T = 0$ gives $R(X,Y)Z = [X,Y,Z] = 0$, so the connection is also flat, and the converse direction is immediate from Proposition 2.11 with $R = 0$ and $T = 0$. ☐

**Proposition 3.22.** *Let $\big(A, \rho, [\![\cdot,\cdot]\!]\big)$ be a Lie algebroid and $\nabla$ an $A$-connection on $A$ such that $\nabla_X Y = 0$ whenever $\rho(X) = 0$. Then $(A, \rho, \nabla)$ is pre-Lie if and only if $R = 0$ and $\rho \circ T = 0$.*

*Proof.* If $(A, \rho, \nabla)$ is pre-Lie, then in particular it is Lie-admissible, so Proposition 3.15 implies that $\rho \circ T = 0$. From the assumption on $\nabla$, we have $T(X,Y) \triangleright Z = 0$, so Proposition 2.11 implies $R(X,Y)Z = [X,Y,Z] = 0$. Conversely, if $R = 0$ and $\rho \circ T = 0$, then again the assumption on $\nabla$ gives $T(X,Y) \triangleright Z = 0$, so Proposition 2.11 implies $[X,Y,Z] = 0$. ☐

*Remark* 3.23. If $\rho$ is injective, then the condition $\rho \circ T = 0$ in Proposition 3.22 is equivalent to $T = 0$. In particular, Theorem 2.23(ii) becomes a special case of this result for $A = TM$, since the anchor $\rho = \mathrm{id}_{TM}$ is injective.

*Example* 3.24. Recall, from Example 3.18, that if $\big(A, 0, [\![\cdot,\cdot]\!]\big)$ is a bundle of Lie algebras with $\nabla$ the trivial connection, then $(A, \rho, \nabla)$ is a Lie-admissible algebroid whose torsion $T(X,Y) = -[\![X,Y]\!]$ generally does not vanish. In fact, this is also a pre-Lie algebroid, since the fact that the connection is trivial immediately gives $[X,Y,Z] = 0$. Hence, $T = 0$ is generally not a necessary condition for a pre-Lie algebroid, unless $\rho$ is injective.

3.3.3. *Post-Lie algebroids.* Unlike the definitions of Lie-admissible and pre-Lie algebroids above, which to our knowledge are new, the definition of a post-Lie algebroid appeared in Munthe-Kaas and Lundervold [26].

**Definition 3.25.** A *post-Lie algebroid* $\big(A, \rho, [\cdot,\cdot], \nabla\big)$ is an anchored bundle $(A, \rho)$ with a tensorial Lie bracket $[\cdot,\cdot]$ on $\Gamma(A)$ and an $A$-connection $\nabla$, such that $\big(\Gamma(A), [\cdot,\cdot], \triangleright\big)$ is a post-Lie algebra.

Munthe-Kaas and Lundervold [26, Proposition 2.24] showed that a Lie algebroid equipped with a flat and torsion-free connection admits a post-Lie algebroid structure. The following theorem, which is the main result of this section, strengthens this by providing both necessary and sufficient conditions for a post-Lie structure.

**Theorem 3.26.** *Let $(A, \rho)$ be an anchored bundle and $\nabla$ an $A$-connection on $A$. Then $(A, \rho)$ admits a post-Lie algebroid structure $\big(A, \rho, [\cdot,\cdot], \nabla\big)$ if and only if it admits a Lie algebroid structure $\big(A, \rho, [\![\cdot,\cdot]\!]\big)$ such that $R = \overline{R} = 0$.*

[5]We caution the reader that the term "pre-Lie algebroid" has occasionally appeared in the literature [14] to mean an almost-Lie algebroid, i.e., an algebroid where $[\![\cdot,\cdot]\!]$ is not required to satisfy the Jacobi identity [13]. This is different from our definition.

*Proof.* If $\bigl(A, \rho, [\cdot,\cdot], \nabla\bigr)$ is a post-Lie algebroid, then Proposition 2.18 implies that $[\![X, Y]\!] \coloneqq X \triangleright Y - Y \triangleright X + [X, Y]$ is a Lie bracket on $\Gamma(A)$. Moreover, for all $f \in C^\infty(M)$,

$$\begin{aligned}[\![X, fY]\!] &= X \triangleright (fY) - (fY) \triangleright X + [X, fY] \\ &= \rho(X)[f]Y + f(X \triangleright Y) - f(Y \triangleright X) + f[X, Y] \\ &= \rho(X)[f]Y + f[\![X, Y]\!],\end{aligned}$$

so the Leibniz rule (13) holds, and hence $\bigl(A, \rho, [\![\cdot,\cdot]\!]\bigr)$ is a Lie algebroid. Now, since $[X, Y] = -T(X, Y)$, Proposition 2.11 implies

$$R(X, Y)Z = [X, Y, Z] - [X, Y] \triangleright Z,$$

which vanishes by the post-Lie condition (9b). Substituting $R = 0$ into (6) and using the definition of $\nabla T$ from (4) then gives

$$\overline{R}(X, Y)Z = [Z \triangleright X, Y] + [X, Z \triangleright Y] - Z \triangleright [X, Y]$$

which vanishes by the other post-Lie condition (9a). Hence, $R = \overline{R} = 0$.

Conversely, suppose $\bigl(A, \rho, [\![\cdot,\cdot]\!]\bigr)$ is a Lie algebroid such that $R = \overline{R} = 0$, and let $[X, Y] = -T(X, Y) = \overline{T}(X, Y)$, which is tensorial. Then (7) implies that $[\cdot,\cdot]$ satisfies the Jacobi identity, so in fact this is a tensorial Lie bracket. Finally, (6) implies the post-Lie condition (9a), while $R = 0$ is equivalent to the post-Lie condition (9b). Hence, $\bigl(A, \rho, [\cdot,\cdot], \nabla\bigr)$ is a post-Lie algebroid. □

*Remark* 3.27. Theorem 2.23(iii) is a special case of this result when $A = TM$. Together with the preceding results, characterizing Lie-admissible and post-Lie algebroids in terms of curvature and torsion, we have now completed the generalization of Theorem 2.23 to the algebroid setting.

## 4. Pre-Lie, post-Lie, and action algebroids

As stated in the introduction, Munthe-Kaas [25] (see also Munthe-Kaas and Wright [28]) showed that Lie–Butcher series methods may be applied to approximate flows on a manifold $M$ equipped with a transitive $\mathfrak{g}$-action, where $\mathfrak{g}$ is a Lie algebra. This work was motivated by the question of how to construct and analyze numerical integrators on manifolds more general than Lie groups. In the language of Munthe-Kaas and Lundervold [26] and of this paper, this is due to the fact that an action algebroid $\mathfrak{g} \ltimes M$ admits a post-Lie algebroid structure, when equipped with its canonical flat connection. When $\mathfrak{g}$ is abelian, this algebroid is actually pre-Lie, and ordinary Butcher series methods, such as Runge–Kutta methods, may be used.

In this section, we prove local converses to these statements. Namely, we prove that every transitive post-Lie algebroid on $M$, whose $A$-connection arises from a $TM$-connection, is locally isomorphic to the action algebroid of a transitive $\mathfrak{g}$-action with its canonical flat connection—and in the pre-Lie case, $\mathfrak{g}$ must be abelian. These local isomorphisms are actually global when $M$ is simply connected. Essentially, this shows that there is no other way of applying Lie–Butcher series methods to $M$, other than by equipping $M$ with a $\mathfrak{g}$-action.

We note that Blaom [4] and Abad and Crainic [1] investigated the question of when a Lie algebroid is (locally) an action algebroid, dropping the assumption of transitivity but requiring assumptions on $\nabla$ that are stronger than the post-Lie condition in the non-transitive case. (This can be seen as an alternative way of generalizing the Cartan–Nomizu results to Lie algebroids.) Namely, they assume a flat $TM$-connection on $A$, which is stronger than $R = 0$ for the $A$-connection, and that it be a *Cartan connection* (in the language of Blaom [4]) or have vanishing *basic curvature* (in the language of Abad and Crainic [1]), which is stronger than $\overline{R} = 0$. These turn out to be equivalent in the transitive case but distinct in the non-transitive case, as we discuss in Section 4.2. Our proofs adapt some of the techniques developed in this previous work (especially Abad and Crainic [1, Proposition 2.12]) to the transitive pre-Lie and post-Lie cases.

In addition to these converse results, we also provide new, streamlined proofs of some of the forward results that had appeared in Munthe-Kaas and Lundervold [26], based on the characterizations developed in Section 3 and the tensoriality of the curvature and torsion.

4.1. **Main results.** We begin with the pre-Lie case, characterizing the relationship between pre-Lie algebroids and abelian action algebroids.

**Proposition 4.1.** *If an abelian Lie algebra $\mathfrak{g}$ acts on $M$, then the action algebroid $\mathfrak{g} \ltimes M$ admits a pre-Lie algebroid structure.*

*Proof.* Since $\mathfrak{g} \ltimes M$ is a trivial bundle, take $\nabla$ to be the flat $TM$-connection on $\mathfrak{g} \ltimes M$, and consider the corresponding $\mathfrak{g} \ltimes M$-connection arising from Proposition 3.8. Now, since $R$ and $T$ are tensors, we may evaluate them pointwise by extending to constant sections. However, $\nabla_\xi \eta = 0$ and $[\![\xi, \eta]\!] = 0$ for all constant sections $\xi, \eta \in \mathfrak{g}$, so $R$ and $T$ vanish. Hence, Proposition 3.21 implies that this is a pre-Lie algebroid. □

**Theorem 4.2.** *Let $(A, \rho)$ be a transitive anchored bundle over $M$ and $\nabla$ be a $TM$-connection on $A$. Then $(A, \rho, \nabla)$ is a pre-Lie algebroid if and only if $\big(A, \rho, [\![\cdot,\cdot]\!]\big)$ is locally isomorphic to the action algebroid of a transitive abelian Lie algebra action on $M$, with $\nabla$ locally the canonical flat connection. This isomorphism is global if $M$ is simply connected.*

*Proof.* The converse follows from the argument in Proposition 4.1, with the minor modification that we evaluate $R$ and $T$ by extending to *locally* constant sections. It only remains to prove the forward direction.

Suppose $(A, \rho, \nabla)$ is a pre-Lie algebroid. By Proposition 3.21, the $A$-connection $\nabla$ is flat with respect to the commutator bracket $[\![\cdot,\cdot]\!]$. Now, denoting by $R_{TM}$ the curvature of the $TM$-connection, it is straightforward to see that

$$R(X, Y)Z = R_{TM}\big(\rho(X), \rho(Y)\big)Z,$$

for all $X, Y, Z \in \Gamma(A)$. Since $\rho$ is surjective, $R = 0$ implies $R_{TM} = 0$, so $\nabla$ is a flat $TM$-connection on $A$.

Therefore, we may take a local (or global, if $M$ is simply connected) frame of $\nabla$-flat sections $e_1, \ldots, e_n$. In particular, $\nabla_{e_i} e_j = 0$ for all $i, j = 1, \ldots, n$. However, since $[\![\cdot,\cdot]\!]$ is the commutator bracket, we have

$$[\![e_i, e_j]\!] = \nabla_{e_i} e_j - \nabla_{e_j} e_i = 0.$$

Thus, $\mathfrak{g} = \operatorname{span}\{e_1, \ldots, e_n\}$ is an abelian Lie algebra, and $\rho$ is a $\mathfrak{g}$-action. □

We next consider the post-Lie case, where $\mathfrak{g}$ is generally nonabelian.

**Proposition 4.3.** *Every action algebroid admits a post-Lie algebroid structure.*

*Proof.* As in Proposition 4.1, $\mathfrak{g} \ltimes M$ is a trivial bundle, so take the flat $TM$-connection and its corresponding $\mathfrak{g} \ltimes M$-connection $\nabla$. Since $R$ and $\overline{R}$ are tensors, we may evaluate them pointwise by extending to constant sections. However, $\nabla_\xi \eta = 0$ for all constant sections $\xi, \eta \in \mathfrak{g}$, so $R = 0$ trivially and $\overline{R} = 0$ by the Jacobi identity. The result follows by Theorem 3.26. □

**Theorem 4.4.** *Let $(A, \rho)$ be a transitive anchored bundle over $M$ and $\nabla$ be a $TM$-connection on $A$. Then $\big(A, \rho, [\cdot,\cdot], \nabla\big)$ is a post-Lie algebroid if and only if $\big(A, \rho, [\![\cdot,\cdot]\!]\big)$ is locally isomorphic to the action algebroid of a transitive Lie algebra action on $M$, with $\nabla$ locally the canonical flat connection. This isomorphism is global if $M$ is simply connected.*

*Proof.* The proof is similar in spirit to that of Theorem 4.2. The converse follows from the argument in Proposition 4.3, with the minor modification that we evaluate $R$ and $T$ by extending to *locally* constant sections. It only remains to prove the forward direction.

Suppose $\bigl(A, \rho, [\cdot,\cdot], \nabla\bigr)$ is a post-Lie algebroid. By Theorem 3.26, the $A$-connection $\nabla$ satisfies $R = \overline{R} = 0$ with respect to the Lie algebroid structure defined by the bracket $[\![X, Y]\!] \coloneqq X \triangleright Y - Y \triangleright X + [X, Y]$. As in the proof of Theorem 4.2, $R = 0$ together with surjectivity of $\rho$ implies that the $TM$-connection $\nabla$ is flat.

Therefore, take a local (or global, if $M$ is simply connected) frame of $\nabla$-flat sections $e_1, \ldots, e_n$, and define the structure functions $c_{ij}^k \in C^\infty(M)$ such that $[\![e_i, e_j]\!] = \sum_{k=1}^n c_{ij}^k e_k$. Since these sections are $\nabla$-flat, for any $i, j = 1, \ldots, n$ and $X \in \Gamma(A)$, we have

$$\overline{\nabla}_{e_i} \overline{\nabla}_{e_j} X = [\![e_i, [\![e_j, X]\!]]\!], \qquad \overline{\nabla}_{e_j} \overline{\nabla}_{e_i} X = -[\![e_j, [\![X, e_i]\!]]\!],$$

$$\overline{\nabla}_{[\![e_i, e_j]\!]} X = \nabla_X [\![e_i, e_j]\!] - [\![X, [\![e_i, e_j]\!]]\!] = \sum_{k=1}^n \rho(X)[c_{ij}^k] e_k - [\![X, [\![e_i, e_j]\!]]\!].$$

These expressions, together with the Jacobi identity, immediately give

$$\overline{R}(e_i, e_j) X = \sum_{k=1}^n \rho(X)[c_{ij}^k] e_k,$$

so $\overline{R} = 0$ implies that $\rho(X)[c_{ij}^k] = 0$ for all $i, j, k = 1, \ldots, n$ and $X \in \Gamma(A)$. Since $\rho$ is surjective, this implies that the structure functions $c_{ij}^k$ are in fact constants. Therefore, $\mathfrak{g} = \operatorname{span}\{e_1, \ldots, e_n\}$ is a Lie algebra with structure constants $c_{ij}^k$, and $\rho$ is a $\mathfrak{g}$-action. □

4.2. **Remarks on the non-transitive case.** The transitivity assumption is important for numerical integration and analysis of flows on $M$ using Butcher and Lie–Butcher series methods. Transitivity allows us to locally "lift" vector fields on $M$ to sections of $A$, apply these methods using the pre-Lie or post-Lie structure of $\Gamma(A)$, and then drop back down to $M$.

However, recall that when the Lie algebroid $A \to M$ is non-transitive, the anchor induces a (generally singular) foliation of $M$ into leaves $L \subset M$, and the restriction $A_L \to L$ is a transitive Lie algebroid on each leaf. In this case, we can only "lift" vector fields on $M$ that are tangent to leaves of the foliation, so it is sufficient to restrict to each leaf and apply the results of Section 4.1.

The following example illustrates that the results of Section 4.1 generally do not hold in the non-transitive setting, although they do hold leaf-by-leaf.

*Example* 4.5. Let $A = T\mathbb{S}^2 \to \mathbb{S}^2$, but take $\rho = 0$ instead of the identity. For any affine connection $\nabla$, the induced $A$-connection is trivial, so $(A, \rho, \nabla)$ is a pre-Lie algebroid. In this case, the commutator bracket $[\![\cdot,\cdot]\!]$ is also trivial, so $\bigl(A, \rho, [\![\cdot,\cdot]\!]\bigr)$ is a bundle of abelian Lie algebras over $\mathbb{S}^2$, where each fiber is isomorphic as a Lie algebra to $\mathbb{R}^2$. (Since the fibers are all isomorphic, this is actually something stronger than a bundle of Lie algebras: it is a so-called *Lie algebra bundle*.) However, this is not isomorphic—even locally—to the trivial action algebroid $\mathbb{R}^2 \ltimes \mathbb{S}^2$ with $\nabla$ its canonical flat connection, since $\mathbb{S}^2$ does not admit a flat affine connection.

However, since the anchor is trivial, the leaves of the induced foliation are just points $x \in \mathbb{S}^2$. The transitive Lie algebroids obtained by restricting to leaves are the abelian Lie algebra fibers $A_x \to \{x\}$, and of course, each of these is isomorphic to the trivial action algebroid $\mathbb{R}^2 \ltimes \{x\}$.

Finally, we again mention that the results of Blaom [4], Abad and Crainic [1] show that $A$ is locally isomorphic to an action algebroid, even without assuming transitivity, when the connection satisfies stronger assumptions than the pre-Lie or post-Lie conditions. That this condition is *strictly* stronger is illustrated by the counterexample above: in this case, $A$ admits a pre-Lie structure but generally not a connection of the type considered by Blaom [4], Abad and Crainic [1].

## 5. Conclusion

We have characterized Lie-admissible, pre-Lie, and post-Lie algebras of connections in terms of the curvature and torsion of these connections. For affine connections on a manifold $M$, we related pre-Lie and post-Lie structures to classical results of Cartan [7] and Nomizu [29] on manifolds admitting flat affine connections with vanishing or parallel torsion. In the more general setting of connections on a transitive Lie algebroid over $M$, we showed that pre-Lie and post-Lie structures may only arise, locally (or globally, if $M$ is simply connected), from the action algebroid $\mathfrak{g} \ltimes M$ of a transitive $\mathfrak{g}$-action on $M$, equipped with its canonical flat connection. This generalizes the Cartan–Nomizu results stated above, which correspond to the special case $A = TM$. Furthermore, it implies that the approach of Munthe-Kaas [25], which equips $M$ with a transitive $\mathfrak{g}$-action and applies (Lie–)Butcher series methods, is essentially the only way to use this family of methods for numerical integration on manifolds.

Finally, we remark that Nomizu [29] also considered invariant affine connections with parallel (but not necessarily vanishing) curvature and either vanishing or parallel torsion. Manifolds admitting such connections are locally representable as symmetric homogeneous spaces (for vanishing torsion) or reductive homogeneous spaces (for parallel torsion). These do not fit into the pre-Lie or post-Lie algebraic framework. For symmetric spaces, the appropriate algebraic objects are *Lie triple systems* (Jacobson [19], Loos [21], Helgason [17]), which were used for numerical integration on symmetric spaces in Munthe-Kaas et al. [27]. Forthcoming work in progress studies algebras of connections such that the triple bracket $[\cdot,\cdot,\cdot]$ gives rise to a Lie triple system.

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