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Some rigidity results for II₁ factors arising from wreath products of property (T) groups

Ionut Chifan [a],[*],[1], Bogdan Teodor Udrea [b]

[a] *Department of Mathematics, The University of Iowa, 14 MacLean Hall, IA 52242, USA*
[b] *The Institute of Mathematics of the Romanian Academy, Calea Grivitei 21, Bucharest, Romania*

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ABSTRACT

We show that any infinite collection $(\Gamma_n)_{n \in \mathbb{N}}$ of icc, hyperbolic, property (T) groups satisfies the following von Neumann algebraic *infinite product rigidity* phenomenon. If Λ is an arbitrary group such that $L(\oplus_{n \in \mathbb{N}} \Gamma_n) \cong L(\Lambda)$ then there exists an infinite direct sum decomposition $\Lambda = (\oplus_{n \in \mathbb{N}} \Lambda_n) \oplus A$ with A icc amenable or trivial such that, for all $n \in \mathbb{N}$, up to amplifications, we have $L(\Gamma_n) \cong L(\Lambda_n)$ and $L(\oplus_{k \geq n} \Gamma_k) \cong L((\oplus_{k \geq n} \Lambda_k) \oplus A)$. The result is sharp and complements the previous finite product rigidity property found in [16]. Using this we provide an uncountable family of restricted wreath products $\Gamma \cong \Sigma \wr \Delta$ of icc, property (T) groups Σ, Δ whose wreath product structure is recognizable, up to a normal amenable subgroup, from their von Neumann algebras $L(\Gamma)$. Along the way we highlight several applications of these results to the study of rigidity in the $\mathbb{C}^*$-algebra setting.

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* Corresponding author.
 E-mail addresses: ionut-chifan@uiowa.edu (I. Chifan), bogdanudrea75@yahoo.com (B.T. Udrea).

1. Introduction

For a countable infinite group Γ we denote by $\ell^2\Gamma$ the Hilbert space of all square summable complex functions on Γ. Each element $\gamma \in \Gamma$ gives rise to a unitary operator $u_\gamma : \ell^2\Gamma \to \ell^2\Gamma$ by group translation $u_\gamma(\xi)(\lambda) = \xi(\gamma^{-1}\lambda)$, where $\lambda \in \Gamma$ and $\xi \in \ell^2\Gamma$. The bicommutant $\{u_\gamma | \gamma \in \Gamma\}''$ inside the algebra of all bounded linear operators $B(\ell^2\Gamma)$, is denoted by $L(\Gamma)$ and it is called the *group von Neumann algebra of* Γ. The algebra $L(\Gamma)$ is a II_1 factor (has trivial center) precisely when all nontrivial conjugacy classes of Γ are infinite (icc), this being the case most interesting for study [44].

Ever since their introduction, the classification of these factors is a core direction of research driven by the following fundamental question: *What aspects of the group* Γ *are remembered by* $L(\Gamma)$? This emerged as an interesting yet intriguing theme since these algebras tend to have little memory of the initial group. This is best illustrated by Connes' celebrated result asserting that all amenable icc groups give isomorphic factors, [22]. Hence very different groups like the group of all finite permutations of the positive integers, the lamplighter group, or the wreath product of the integers with itself give rise to isomorphic factors. Consequently, the von Neumann algebraic structure has no memory of the typical discrete algebraic group invariants like torsion, rank, or generators and relations. In this case the only information the von Neumann algebra retains is the amenability—an approximation property—of the group.

In the non-amenable case the situation is radically different and an unprecedented progress has been achieved through the emergence of Popa's deformation/rigidity theory [58]. Using this completely new conceptual framework it was shown that various properties of groups, such as their representation theory or their approximations, can be completely recovered from their von Neumann algebras. As a result, for large classes of group factors, many remarkable structural properties such as primeness, (strong) solidity, classification of normalizers of algebras, etc could be successfully established [54,55,37,56,59,48,49,9,10,50,61,25,32,38,28,15,64,66,17,18,33,62,30,63,34,35,4, 6,40,3,67,41,11,14,68,5,13,16,23,20]. For additional information we refer the reader to the following survey papers [58,65,34,36].

One of the most impressive milestone in this study is Ioana-Popa-Vaes' discovery of the first examples of groups that can be completely recovered from their von Neumann algebras (W^*-*superrigid*[2] groups), [38]. See also the subsequent result [3] and the more recent work [12]. These results pushed the classification problem of group factors to new boundaries and exciting possibilities. In this direction an interesting and wide open theme is to identify a comprehensive list of canonical constructions in group theory (direct sum, free product, HNN-extension, wreath product, etc.) that are recoverable from their von Neumann algebras.

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[2] Γ is W^*-superrigid if whenever Λ is an arbitrary group so that $L(\Gamma) = L(\Lambda)$ then it follows that $\Lambda = \Gamma$.

1.1. Statements of main results

Over a decade ago Ozawa and Popa discovered the first unique prime factorization results for tensor product of II_1 factors, [47]. This work has had deep consequences to the classification of II_1 factors and has generated significant subsequent developments. Some of Ozawa-Popa's results have been strengthened considerably in [16] which unveiled a large class of product groups $\Gamma_1 \times \Gamma_2$ whose product structure is a feature completely recognizable at the level of their von Neumann algebras $L(\Gamma_1 \times \Gamma_2)$. Precisely, whenever Γ_1, Γ_2 are hyperbolic icc groups (e.g. non-abelian free groups) and Λ is an *arbitrary* group such that $L(\Gamma_1 \times \Gamma_2) = L(\Lambda)$ then Λ admits a nontrivial product decomposition $\Lambda = \Lambda_1 \times \Lambda_2$ and there exists a scalar $t > 0$ such that, up to unitary conjugacy, we have $L(\Gamma_1) = L(\Lambda_1)^t$ and $L(\Gamma_2) = L(\Lambda_2)^{1/t}$. The result still holds if one assumes, more generally, that Γ_1, Γ_2 are just icc biexact groups, [45].

Isono studied unique prime factorization aspects for infinite tensor product of factors and several interesting results have emerged in [39], see also [26,27]. Motivated in part by these results it is natural to investigate whether "product rigidity" properties, similar with ones in [16], would hold in the context of infinite direct sums groups. Specifically, if one considers $\Gamma = \oplus_{n \in \mathbb{N}} \Gamma_n$ with Γ_n's icc non-amenable groups, it would be interesting to understand how much of the infinite direct sum structure of Γ is retained by its von Neumann algebra $L(\Gamma)$. Right away one may notice a sharp contrast point with the aforementioned finite product situation. Since $L(\Gamma)$ canonically decomposes as an infinite tensor product $L(\Gamma) = \bar{\otimes}_{n \in \mathbb{N}} L(\Gamma_n)$ it follows that $L(\Gamma)$ is a McDuff factor and hence $L(\Gamma) = L(\Gamma) \bar{\otimes} \mathcal{R}$, where $\mathcal{R}$ is the hyperfinite factor; consequently, we have that $L(\oplus_{n \in \mathbb{N}} \Gamma_n) = L((\oplus_{n \in \mathbb{N}} \Gamma_n) \oplus A)$, for any icc amenable group A. This observation shows that, in the best case scenario, $L(\Gamma)$ could remember the direct sum feature of the underlying group *only up to an amenable subgroup* which typically lies in the tail of the infinite tensor product. It is therefore natural to investigate under which circumstances it is possible to completely reconstruct the infinite direct sum feature only up to this obstruction. Building upon previous techniques from [38,33,16,23,20] and using the classification of normalizers from [63] we found infinitely many classes of Γ_n's for which this problem has a positive answer.

Theorem A. *Let $(\Gamma_n)_{n \in \mathbb{N}}$ an infinite collection of property (T), biexact, weakly amenable, icc groups. Assume that Λ is an* arbitrary *group satisfying $L(\oplus_{n \in \mathbb{N}} \Gamma_n) = L(\Lambda)$. Then Λ admits an infinite direct sum decomposition $\Lambda = (\oplus_{n \in \mathbb{N}} \Lambda_n) \oplus A$, where Λ_n is icc, weakly amenable, property (T) group for all n and A is a icc amenable group. Moreover, for each $k \in \mathbb{N}$ there exist scalars $t_1, t_2, ..., t_{k+1} > 0$ satisfying $t_1 t_2 ... t_{k+1} = 1$ and a unitary $u \in L(\Lambda)$ so that*

$$uL(\Gamma_n)^{t_n} u^* = L(\Lambda_n) \quad \text{for all } k \geq n \geq 1; \text{ and}$$
$$uL(\oplus_{n \geq k+1} \Gamma_n)^{t_{k+1}} u^* = L((\oplus_{n \geq k+1} \Lambda_n) \oplus A). \tag{1.1}$$

The result applies to several concrete classes of groups a such as:

(1) the uniform lattices Γ_n in $Sp(k_n, 1)$ with $k_n \geq 2$ or any icc groups in their measure equivalence class; and
(2) Gromov's random groups with density satisfying $3^{-1} < d < 2^{-1}$.

While the conclusion of the previous theorem shows a strong identification of the von Neumann algebras of the group factors Γ_n's, in general one cannot recover these groups. To see this note that Voiculescu's compression formula for free group factors gives $L(\mathbb{F}_2) = L(\mathbb{F}_5) \otimes M_2(\mathbb{C})$. This implies that $L(\oplus_{n \in \mathbb{N}} \mathbb{F}_2) = \bar{\otimes}_{n \in \mathbb{N}} L(\mathbb{F}_2) = \bar{\otimes}_{n \in \mathbb{N}} (L(\mathbb{F}_5) \otimes M_2(\mathbb{C})) = (\bar{\otimes}_{n \in \mathbb{N}} (L(\mathbb{F}_5))) \bar{\otimes} \mathcal{R} = L((\oplus_{n \in \mathbb{N}} \mathbb{F}_5) \oplus A)$, for every icc amenable group A.

Theorem A can be successfully used to shed light towards rigidity aspects in the C^*-algebraic setting. Precisely, when it is combined with [7, Theorem 1.3] one gets the following version of infinite product rigidity for reduced group C^*-algebras.

Corollary B. *Let $(\Gamma_n)_{n \in \mathbb{N}}$ an infinite collection of property (T), biexact, weakly amenable, icc groups. Assume that Λ is an* arbitrary *group satisfying $\mathbb{C}_r^*(\oplus_{n \in \mathbb{N}} \Gamma_n) = \mathbb{C}_r^*(\Lambda)$. Then Λ admits an infinite direct sum decomposition $\Lambda = \oplus_{n \in \mathbb{N}} \Lambda_n$, where the Λ_n's are icc, weakly amenable, property (T) groups.*

When compared side by side, Theorem A and Corollary B highlight again the fundamental difference between $\mathbb{C}^*$-algebras and von Neumann algebras; the absence of the infinite amenable direct summand of Λ in the conclusion of Corollary B exemplifies once more the fact that the WOT closure is considerably larger than the uniform norm closure, thus triggering significant loss of algebraic information in the von Neumann algebraic setting.

Restricted wreath product groups manifest a remarkable rigid behavior in the von Neumann algebraic setting. In fact a large portion of the groups/actions known to be recoverable from their von Neumann algebras arise from constructions involving wreath product groups or Bernoulli shifts [54,57,61,32,38,62,3]. A common feature of these examples is that the core or the wreath product groups involved are amenable and in many cases even abelian. For example, with the exception of [12], all known examples of W^*-superrigid groups are of the form $H \wr_I \Gamma$ where H is finite [38,3,2]. However significantly fewer rigidity results are known in general for wreath product factors $L(H \wr_I \Gamma)$ when H is nonamenable. In this direction we mention in passing Ioana's strong rigidity results which asserts that $L(\mathbb{F}_n \wr A) \neq L(\mathbb{F}_m \wr B)$ whenever $n, m \geq 2$ and A, B are nonisomorphic icc amenable groups, [31]. Theorem A can be successfully used to provide new insight towards this problem as well. For instance, using it in combination with various technical outgrowths of previous methods from [54,31,38,16,12] we obtain the following *wreath product rigidity* result *up to an amenable subgroup* for group factors.

Theorem C. *Let H be icc, weakly amenable, biexact property (T) group. Let $\Gamma = \Gamma_1 \times \Gamma_2$, where Γ_i are icc, biexact, property (T) group. Let $H \wr \Gamma$ be the corresponding (plain) wreath product. Let Λ be an* arbitrary *group and let $\theta : L(H \wr \Gamma) \to L(\Lambda)$ be a $*$-isomorphism. Then one can find non-amenable icc groups Σ, Ψ, an icc amenable or trivial group A, and an action $\Psi \curvearrowright^\alpha A$ such that we can decompose Λ as semidirect product $\Lambda = (\Sigma^{(\Psi)} \oplus A) \rtimes_{\beta \oplus \alpha} \Psi$, where $\Psi \curvearrowright^\beta \Sigma^{(\Psi)}$ is the Bernoulli shift. In addition, there exist a group isomorphism $\delta : \Gamma \to \Psi$, a character $\eta : \Gamma \to \mathbb{T}$, a $*$-isomorphism $\theta : L(H^{(\Gamma)}) \to L(\Sigma^{(\Psi)} \oplus A)$ and a unitary $u \in L(\Lambda)$ so that for all $x \in L(H^{(\Gamma)})$, $\gamma \in \Gamma$ we have*

$$\theta(xu_\gamma) = \eta(\gamma)u\theta(x)v_{\delta(\gamma)}u^*.$$

Here $\{u_\gamma \,|\, \gamma \in \Gamma\}$ and $\{v_\lambda \,|\, \lambda \in \Psi\}$ are the canonical group unitaries of $L(\Gamma)$ and $L(\Psi)$, respectively.

We notice the theorem still holds for slightly more general situations, e.g. generalized wreath products with finite or even possible just amenable stabilizers (see Theorem 5.4). However at this time we do not know the full extent of these cases as it seems to rely on heavy constructions on group theory (see Remarks 5.5).

As before, Theorem C in combination with [7, Theorem 1.3] lead to the following stronger version of wreath product rigidity in the context of reduced group $\mathbb{C}^*$-algebras.

Corollary D. *Let H be icc, weakly amenable, biexact property (T) group. Let $\Gamma = \Gamma_1 \times \Gamma_2$, where Γ_i are icc, biexact, property (T) group. Let $H \wr \Gamma$ be the corresponding wreath product. Let Λ be an* arbitrary *group so that $\mathbb{C}_r^*(H \wr \Gamma) = \mathbb{C}_r^*(\Lambda)$. Then $\Lambda = \Sigma \wr \Gamma$, where Σ is an icc, weakly amenable, property (T) group.*

In connection with Theorem C it is natural to investigate whether similar statements hold if one relaxes the biexactness assumption on H or the product assumption on Γ. In this situation, building upon the previous techniques from [38,43] one can show the following strong rigidity statement holds just for property (T) groups H and Γ.

Theorem E. *Let H, Γ be icc, torsion free groups. Also assume H has property (T) and Γ admits an infinite, almost normal subgroup with relative property (T). Let $\Gamma \curvearrowright I$ be a transitive action on a countable set satisfying the following conditions:*

a) There is $k \in \mathbb{N}$ such that for each $J \subseteq I$ satisfying $|J| \geq k$ we have $|\mathrm{Stab}_\Gamma(J)| < \infty$;
b) The orbit $\mathrm{Stab}_\Gamma(i) \cdot j$ is infinite for all $i \neq j$.

Denote by $G = H \wr_I \Gamma$ the corresponding generalized wreath product. Let Λ be any torsion free *group and let $\theta : L(G) \to L(\Lambda)$ be a $*$-isomorphism. Then Λ admits a generalized wreath product decomposition $\Lambda = \Sigma \wr_I \Psi$ satisfying all the properties enumerated in* a) − c). *In addition, there exist a group isomorphism $\delta : \Gamma \to \Psi$, a character $\eta : \Gamma \to \mathbb{T}$, a*

-isomorphism $\theta : L(H) \to L(\Sigma)$ and a unitary $u \in L(\Lambda)$ such that for every $x \in L(H^{(I)})$ and $\gamma \in \Gamma$ we have

$$\theta(xu_\gamma) = \eta(\gamma)u\theta^{\otimes_I}(x)v_{\delta(\gamma)}u^*.$$

Here $\{u_\gamma | \gamma \in \Gamma\}$ and $\{v_\lambda | \lambda \in \Psi\}$ are the canonical unitaries of $L(\Gamma)$ and $L(\Psi)$, respectively.

The previous theorem applies to many natural families of generalized wreath groups $H \wr_I \Gamma$, including: a) any icc torsion free prop (T) group H and any torsion free, hyperbolic, property (T) group Γ together with a maximal amenable subgroup $\Sigma < \Gamma$ and the action $\Gamma \curvearrowright I = \Gamma/\Omega$ is given by translation on right cosets Γ/Ω; b) any action of the form $\Gamma \curvearrowright \Gamma/S$ by left translations, where instances of inclusions $S < \Gamma$ can be found in [60, Examples 7.4].

Acknowledgments

2. Preliminaries

2.1. Notations

Given a von Neumann algebra M we will denote by $\mathscr{U}(M)$ its unitary group and by $\mathscr{P}(M)$ the set of all its nonzero projections. All algebras inclusions $N \subseteq M$ are assumed unital unless otherwise specified. For any von Neumann subalgebras $P, Q \subseteq M$ we denote by $P \vee Q$ the von Neumann algebra they generate in M.

All von Neumann algebras M considered in this article will be tracial, i.e., endowed with a unital, faithful, normal functional $\tau : M \to \mathbb{C}$ satisfying $\tau(xy) = \tau(yx)$ for all $x, y \in M$. This induces a norm on M by the formula $\|x\|_2 = \tau(x^*x)^{1/2}$ for all $x \in M$. The $\|\cdot\|_2$-completion of M will be denoted by $L^2(M)$.

For a countable group Γ we denote by $\{u_\gamma | \gamma \in \Gamma\} \in U(\ell^2\Gamma)$ its left regular representation given by $u_\gamma(\delta_\lambda) = \delta_{\gamma\lambda}$, where $\delta_\lambda : \Gamma \to \mathbb{C}$ is the Dirac mass at $\{\lambda\}$. The weak operator closure of the linear span of $\{u_\gamma | \gamma \in \Gamma\}$ in $B(\ell^2\Gamma)$ is the so called group von Neumann algebra and will be denoted by $L(\Gamma)$. $L(\Gamma)$ is a II$_1$ factor precisely when Γ has infinite non-trivial conjugacy classes (icc).

Given a group Γ and a subset $F \subseteq \Gamma$ we will be denoting by $\langle F \rangle$ the subgroup of Γ generated by F. Given a group action $\Gamma \curvearrowright I$ on a countable set I, for any subset $\mathcal{F} \subset I$ we denote $Stab(\mathcal{F}) = \{\gamma \in \Gamma \mid g \cdot i = i, \forall i \in \mathcal{F}\}$ and $Norm(\mathcal{F}) = \{\gamma \in \Gamma \mid \gamma \cdot \mathcal{F} = \mathcal{F}\}$.

Given a subgroup $\Lambda \leqslant \Gamma$ we will often consider the virtual centralizer of Λ in Γ, i.e. $vC_\Gamma(\Lambda) = \{\gamma \in \Gamma : |\gamma^\Lambda| < \infty\}$. Notice $vC_\Gamma(\Lambda)$ is a subgroup normalized by Λ. When $\Lambda = \Gamma$, the virtual centralizer is denoted by $vZ(\Gamma) := vC_\Gamma(\Gamma)$ and called the virtual center of Γ; this is nothing else but the FC-radical of Γ. Hence Γ is icc precisely when $vZ(\Gamma) = 1$.

2.2. Popa's intertwining techniques

Over a decade ago, Popa introduced in [54, Theorem 2.1 and Corollary 2.3] a powerful analytic criterion for identifying intertwiners between arbitrary subalgebras of tracial von Neumann algebras. This is now termed *Popa's intertwining-by-bimodules technique*.

Theorem 2.1. *[54] Let (M, τ) be a separable tracial von Neumann algebra and let $P, Q \subseteq M$ be (not necessarily unital) von Neumann subalgebras. Then the following are equivalent:*

(1) *There exist $p \in \mathcal{P}(P), q \in \mathcal{P}(Q)$, an injective $*$-homomorphism $\theta : pPp \to qQq$ and a partial isometry $0 \neq v \in qMp$ such that $\theta(x)v = vx$, for all $x \in pPp$.*
(2) *For any group $\mathcal{G} \subset \mathscr{U}(P)$ such that $\mathcal{G}'' = P$ there is no sequence $(u_n)_n \subset \mathcal{G}$ satisfying $\|E_Q(xu_ny)\|_2 \to 0$, for all $x, y \in M$.*

If one of the two equivalent conditions from Theorem 2.1 holds then we say that *a corner of P embeds into Q inside M*, and write $P \prec_M Q$. If we moreover have that $Pp' \prec_M Q$, for any projection $0 \neq p' \in P' \cap 1_P M 1_P$ (equivalently, for any projection $0 \neq p' \in \mathcal{Z}(P' \cap 1_P M 1_P)$), then we write $P \prec_M^s Q$.

2.3. Quasinormalizers of groups and algebras

Given groups $\Omega \leqslant \Gamma$, the *one-sided quasinormalizer semigroup* $QN_\Gamma^{(1)}(\Omega) \subseteq \Gamma$ is the set of all $\gamma \in \Gamma$ for which there is a finite set $F \subseteq \Gamma$ so that $\Omega\gamma \subseteq F\Omega$, [24, Section 5]; equivalently, $\gamma \in QN_\Gamma^{(1)}(\Omega)$ iff $[\Omega : \gamma\Omega\gamma^{-1} \cap \Omega] < \infty$. Thus $QN_\Gamma^{(1)}(\Omega)$ coincides with the one-sided commensurator of Ω in Γ.

Similarly, the *quasinormalizer* (also called the *commensurator*) $QN_\Gamma(\Omega)$ is the set of all $\gamma \in \Gamma$ for which there exists a finite set $F \subseteq \Gamma$ such that $\Omega\gamma \subseteq F\Omega$ and $\gamma\Omega \subseteq \Omega F$; equivalently, $\gamma \in QN_\Gamma(\Omega)$ iff $[\Omega : \gamma\Omega\gamma^{-1} \cap \Omega] < \infty$ and $[\gamma\Omega\gamma^{-1} : \gamma\Omega\gamma^{-1} \cap \Omega] < \infty$. From definitions one checks that $QN_\Gamma(\Omega) \leqslant \Gamma$ is a subgroup satisfying $\Omega \subseteq QN_\Gamma(\Omega) \subseteq QN_\Gamma^{(1)}(\Omega)$.

The von Neumann algebraic counterparts of (one-sided) quasinormalizers have played a major role in the recent classification results in this area [52–54,37]. Given an inclusion

$Q \subseteq M$, the quasinormalizer $QN_M(Q)$ is the $*$-algebra of all elements $x \in M$ such that there exist $x_1, x_2, ..., x_k \in M$ so that $Qx \subseteq \sum_i x_i Q$ and $xQ \subseteq \sum_i Qx_i$, [52]. The von Neumann algebra $QN_M(Q)''$ is called the *quasinormalizing algebra of Q inside M*. Similarly, the intertwiner space $QN_M^{(1)}(Q)$ is the set of all $x \in M$ such that there exist $x_1, x_2, ..., x_k \in M$ so that $Qx \subseteq \sum_i x_i Q$, [52,24]. The von Neumann algebra $QN_M^{(1)}(Q)''$ is called the *one-sided quasinormalizer algebra of Q inside M*.

As usual $N_M(Q) = \{u \in \mathcal{U}(M) \, | \, uQu^* = Q\}$ denotes the *normalizing group* and $N_M(Q)''$ denotes the *normalizing algebra of Q in M*. Notice that $Q \subseteq N_M(Q) \subseteq QN_M(Q) \subseteq QN_M^{(1)}(Q) \subseteq M$.

The following computations of (one-sided) quasinormalizer algebras for inclusions of group von Neumann algebras will be essential for our arguments.

Theorem 2.2 *(Corollary 5.2 [24]). If $\Omega \leqslant \Gamma$ is an inclusion of groups, the following hold:*

(1) $QN_{L(\Gamma)}(L(\Omega))'' = L(H_1)$, where $H_1 = QN_\Gamma(\Omega) = QN_\Gamma^{(1)}(\Omega) \cap QN_\Gamma^{(1)}(\Omega)^{-1}$;
(2) $QN_{L(\Gamma)}^{(1)}(L(\Omega))'' = L(H_2)$, where $H_2 = \langle QN_\Gamma^{(1)}(\Omega) \rangle \leqslant \Gamma$.

Theorem 2.3 *(Lemma 3.5 [54]; Proposition 6.2 [24]). Let $Q \subseteq M$ be an inclusion of tracial von Neumann algebras. Then for any $p \in \mathcal{P}(Q)$ we have*

(1) $QN_{pMp}(pQp)'' = pQN_M(Q)''p$, and
(2) $QN_{pMp}^{(1)}(pQp)'' = pQN_M^{(1)}(Q)''p$.

2.4. Height of elements in group von Neumann algebras

Following [32, Section 4] and [38, Section 3] the height $h_\Gamma(x)$ of an element $x \in L(\Gamma)$ is the absolute value of the largest Fourier coefficient, i.e., $h_\Gamma(x) = \sup_{\gamma \in \Gamma} |\tau(x u_\gamma)|$, where $\{u_\gamma \, | \, \gamma \in \Gamma\}$ are the canonical unitaries of M implemented by Γ. For a subset $S \subseteq L(\Gamma)$, we denote by $h_\Gamma(S) = \inf_{x \in S} h_\Gamma(x)$. In this section we prove two elementary lemmas on height that will be used in the proof Theorem C.

Lemma 2.4. *Assume that $L(\Gamma) = M$ and consider the subsets $\Sigma \subseteq \Gamma$ and $\mathcal{S} \subseteq (M)_1$. If there exist $x, y \in (M)_1$ such that $h_\Sigma(x \mathcal{S} y) > 0$ then one can find a finite subset $F \subset \Gamma$ such that $h_{F \Sigma F}(\mathcal{S}) > 0$. In particular, we have $h_\Gamma(\mathcal{S}) > 0$ iff $h_\Gamma(u \mathcal{S} u^*) > 0$ for some $u \in \mathcal{U}(M)$.*

Proof. Using Kaplansky's Theorem for every $\varepsilon > 0$ there is a finite subset $F_\varepsilon \subset \Lambda$ and $x_\varepsilon, y_\varepsilon \in (M)_1$ supported on F_ε so that $\|x - x_\varepsilon\|_2, \|y - y_\varepsilon\|_2 \leq \varepsilon$. Using these estimates together with triangle inequality, for every $s \in \mathcal{S}$ and $\gamma \in \Sigma$ we have

$$|\tau(x s y u_\gamma)| \leq 4\varepsilon + |\tau(x_\varepsilon s y_\varepsilon u_\gamma)|$$
$$\leq 4\varepsilon + \sum_{\lambda, \mu \in F_\varepsilon} |\tau(x_\varepsilon u_{\lambda^{-1}})| |\tau(y_\varepsilon u_{\mu^{-1}})| |\tau(u_\lambda s u_\mu u_\gamma)| \leq 4\varepsilon + |F_\varepsilon|^2 \max_{\nu \in F_\varepsilon \gamma F_\varepsilon} |\tau(s u_\nu)|.$$

This implies that $h_\Sigma(x\mathcal{S}y) \leq 4\varepsilon + |F_\varepsilon|^2 h_{F_\varepsilon \Sigma F_\varepsilon}(\mathcal{S})$ and hence $h_{F_\varepsilon \Sigma F_\varepsilon}(\mathcal{S}) \geq (h_\Sigma(x\mathcal{S}y) - 4\varepsilon)|F_\varepsilon|^{-2}$. Letting $\varepsilon > 0$ small enough we get the first part conclusion.

For the remaining part, the reverse implication follows from above. The direct implication follows from the reverse implication by replacing $\mathcal{S}$ with $u\mathcal{S}u^*$ and u with u^*. $\quad\square$

Lemma 2.5. *Assume that $L(\Gamma) = L(\Lambda) = M$. Consider the comultiplication along Λ i.e. the embedding $\Delta : M \to M\bar{\otimes}M$ given by $\Delta(v_\lambda) = v_\lambda \otimes v_\lambda$, where $\{v_\lambda \,|\, \lambda \in \Lambda\}$ are the canonical group unitaries of M implemented by Λ. If $\mathcal{S} \subseteq (M)_1$ and there are $x, y \in (M\bar{\otimes}M)_1$ so that $h_{\Gamma \times \Gamma}(x\Delta(\mathcal{S})y) > 0$ then $h_\Lambda(\mathcal{S}) > 0$.*

Proof. Since $h_{\Gamma \times \Gamma}(x\Delta(\mathcal{S})y) > 0$ then Lemma 2.4 implies that

$$h_{\Gamma \times \Gamma}(\Delta(\mathcal{S})) > 0. \tag{2.1}$$

Fix $s \in \mathcal{S}$. Let $s = \sum_{\lambda \in \Lambda} \tau(sv_{\lambda^{-1}})v_\lambda$ and note that $\Delta(s) = \sum_{\lambda \in \Lambda} \tau(sv_{\lambda^{-1}})v_\lambda \otimes v_\lambda$. Using these formulas and Cauchy-Schwarz inequality, for any $\gamma_1, \gamma_2 \in \Gamma$, we have

$$|\tau \otimes \tau(\Delta(s)(u_{\gamma_1} \otimes u_{\gamma_2}))| \leq \sum_{\lambda \in \Lambda} |\tau(sv_{\lambda^{-1}})||\tau(v_\lambda u_{\gamma_1})||\tau(v_\lambda u_{\gamma_2})|$$

$$\leq h_\Lambda(s) \sum_{\lambda \in \Lambda} |\tau(v_\lambda u_{\gamma_1})||\tau(v_\lambda u_{\gamma_2})| \leq h_\Lambda(s)\|u_{\gamma_1}\|_2\|u_{\gamma_2}\|_2 = h_\Lambda(s).$$

This further implies $h_{\Gamma \times \Gamma}(\Delta(\mathcal{S})) \leq h_\Lambda(\mathcal{S})$ and using (2.1) we get the conclusion. $\quad\square$

2.5. Relative amenability

A tracial von Neumann algebra (M, τ) is called amenable if there exists a state $\phi : B(L^2(M)) \to \mathbb{C}$ such that $\phi_{|M} = \tau$ and ϕ is M-central (i.e. $\phi(xT) = \phi(Tx)$ for all $x \in M, T \in B(L^2(M))$), [22]. Making use of the basic construction for inclusions of algebras [21,42] this concept was further generalized in [48] to subalgebras. Let (M, τ) be a tracial von Neumann algebra, $p \in M$ be a projection, and $P \subseteq pMp, Q \subseteq M$ be von Neumann subalgebras. Following [48, Section 2.2] we say that P *is amenable relative to Q inside M* if there exists a P-central state $\phi : p\langle M, e_Q\rangle p \to \mathbb{C}$ such that $\phi(x) = \tau(x)$, for all $x \in pMp$. Here $\langle M, e_Q\rangle$ denotes the basic construction for the inclusion $Q \subseteq M$, i.e. the commutant of the Q-right action on $B(L^2(M))$ [42].

In this section we prove a relative amenability result for subalgebras that "cluster at infinity" in an infinite tensor product of factors. The result will be essentially used to derive our infinite product rigidity result for group factors. Our proof is an adaptation of an argument due to Ioana. See also [39, Lemma 4.4] and [29, Proposition 4.2] for similar results.

Proposition 2.6. *Let $M \subseteq (\tilde{M}, \tau)$ be finite von Neumann algebras satisfying the following:*

(1) *there are subalgebras* $B, C \subseteq M$ *and* $C \subseteq \tilde{C} \subseteq \tilde{M}$ *so that* $M = B \vee C$ *and* $\tilde{M} = B \vee \tilde{C}$.

(2) *there are descending family* $B_n \subseteq M$ *of subalgebras and an ascending family* $C_n \subseteq M$ *of subalgebras such that* $\cup_n C_n = C$, $\cap_n B_n = B$ *and* $M = B_j \vee C_j$ *for all* j.

(3) *there exist* $\theta, \theta_n \in Aut(\tilde{M})$ *such that* $\theta_{n|B_n} = \mathrm{id}_{|B_n}$ *for all* n, $\theta_n \to \theta$ *pointwise.*

Let $p \in B$ *be a nonzero projection and let* $A \subseteq pMp$ *be a von Neumann subalgebra so that for each* $n \in \mathbb{N}$ *there is* $u_n \in \mathcal{U}(pMp)$ *satisfying* $u_n A u_n^* \subseteq B_n$. *Consider the* M-M *bimodule* $\mathcal{H} = L^2(\tilde{M})$ *given by the actions* $x \cdot \xi \cdot y = x \xi \theta(y)$ *for all* $x, y \in M$ *and* $\xi \in L^2(\tilde{M})$. *Then there exists a sequence of vectors* $(\xi_n)_n \subset L^2(p\tilde{M}p)$ *satisfying*

$$\lim_n \|x \cdot \xi_n - \xi_n \cdot x\|_{\mathcal{H}} = 0, \text{ for all } x \in A, \text{ and} \tag{2.2}$$

$$\lim_n \langle x \cdot \xi_n, \xi_n \rangle_{\mathcal{H}} = \tau(x), \text{ for all } x \in pMp. \tag{2.3}$$

Proof. Since $u_n A u_n^* \subseteq B_n$ and $\theta_{n|B_n} = \mathrm{id}_{|B_n}$ then for every $x \in A$ we have $\theta_n(u_n x u_n^*) = u_n x u_n^*$. This implies that $u_n^* \theta_n(u_n) \theta_n(x) = x u_n^* \theta_n(u_n)$ and letting $\xi_n = u_n^* \theta_n(u_n) \in \mathcal{U}(p\tilde{M}p)$ we conclude that for all $x \in A$ and $n \in \mathbb{N}$ we have

$$\xi_n \theta_n(x) = x \xi_n. \tag{2.4}$$

Since $u_n \in \mathcal{U}(pMp)$ and $\theta_{n|B_n} = \mathrm{id}_{|B_n}$ we have that $\|\xi_n\|_{\mathcal{H}} = \|u_n^* \theta_n(u_n)\|_2 = \|p\|_2$ for all n. Since $\|u_n\|_\infty \leq 1$ then using (2.4) and $\theta_n \to \theta$ pointwise one can check that for every $x \in A$ we have

$$\lim_n \|x \cdot \xi_n - \xi_n \cdot x\|_{\mathcal{H}} = \lim_n \|x \xi_n - \xi_n \theta(x)\|_2 = \lim_n \|\xi_n(\theta_n(x) - \theta(x))\|_2$$

$$\leq \lim_n \|\xi_n\|_\infty \|\theta_n(x) - \theta(x)\|_2 = \lim_n \|\theta_n(x) - \theta(x)\|_2 = 0.$$

Finally, since $\xi_n \in \mathcal{U}(p\tilde{M}p)$ we have $\langle x \cdot \xi_n, \xi_n \rangle_{\mathcal{H}} = \tau(\xi_n^* x \xi_n) = \tau(x \xi_n \xi_n^*) = \tau(x)$ for all $x \in pMp$. Altogether, the above relations give the desired conclusion. $\quad\square$

Proposition 2.7. *Let* $M = \bar{\otimes}_{i \in \mathbb{N}} M_i \bar{\otimes} B$. *Let* $A \subseteq M$ *be a von Neumann algebra for which there exist sequences* $(k_n)_n \subseteq \mathbb{N}$ *and* $(u_n)_n \subset \mathcal{U}(M)$ *such that* $k_n \nearrow \infty$ *and* $u_n A u_n^* \subseteq \bar{\otimes}_{i \geq k_n} M_i \bar{\otimes} B$ *for all* n. *Then* A *is amenable relative to* B *inside* M.

Proof. Denote by $\bar{\otimes}_{i \in \mathbb{N}} M_i = C$ and notice that $M = C \bar{\otimes} B$. Let $\tilde{C} = C \bar{\otimes} C$ and $\tilde{M} = \tilde{C} \bar{\otimes} B$ and notice that $M \subset \tilde{M}$. For every $n \in \mathbb{N}$ denote by $C_n = \bar{\otimes}_{i=1}^{k_n - 1} M_i$, by $D_n = \bar{\otimes}_{i \geq k_n} M_i$ and by $B_n = D_n \bar{\otimes} B$ and notice that $\cup_n C_n = C$ and $\cap_n B_n = B$. Next let $\theta_n \in Aut(\tilde{M})$ satisfying $\theta_n(x \otimes y) = y \otimes x$ for all $x, y \in C_n$ and $\theta_n = \mathrm{id}$ on $D_n \bar{\otimes} D_n \bar{\otimes} B$. Notice that $\theta_n \to \theta$ pointwise, where $\theta \in Aut(\tilde{M})$ satisfies $\theta(x \otimes y) = y \otimes x$ for all $x, y \in C$ and $\theta = \mathrm{id}$ on B. One can check all the conditions in the statement of Proposition 2.6 are satisfied. Thus if we consider the M-M bimodule $\mathcal{H} := L^2(\tilde{M}) = L^2(C) \bar{\otimes} L^2(C) \bar{\otimes} L^2(B)$ with the actions given by $x \cdot \xi \cdot y = x \xi \theta(y)$ for all $x, y \in M$ and $\xi \in \mathcal{H}$ there exists a sequence of unit vectors $(\xi_n)_n \in \mathcal{H}$ such that

$$\lim_n \|x \cdot \xi_n - \xi_n \cdot x\|_{\mathcal{H}} = 0, \text{ for all } x \in A$$
$$\lim_n \langle x \cdot \xi_n, \xi_n \rangle_{\mathcal{H}} = \lim_n \langle \xi_n \cdot x, \xi_n \rangle_{\mathcal{H}} = \tau(x), \text{ for all } x \in M. \tag{2.5}$$

Let $\langle M, e_{1 \otimes B} \rangle$ be the basic construction for $1 \bar{\otimes} B \subset M$ and let Tr be the semifinite trace on $\langle M, e_{1 \otimes B} \rangle$. Next we notice that, as M-M-bimodules, $L^2(\langle M, e_{1 \otimes B} \rangle, Tr)$ is isomorphic to $\mathcal{H}$ via the map $(x \otimes y) e_{1 \otimes B} (z \otimes 1) \to (x \otimes y) \cdot (1 \otimes 1 \otimes 1) \cdot (z \otimes 1)$, for $x, z \in C$ and $y \in B$. Indeed it is clear this is M-M-bimodular and also for all $x_i, z_i \in C$ and $y_i \in B$ we have

$$\langle (x_1 \otimes y_1) e_{1 \otimes B} (z_1 \otimes 1), (x_2 \otimes y_2) e_{1 \otimes B} (z_2 \otimes 1) \rangle_{Tr} =$$
$$= Tr((z_2 \otimes 1)^* e_{1 \otimes B} (x_2 \otimes y_2)^* (x_1 \otimes y_1) e_{1 \otimes B} (z_1 \otimes 1))$$
$$= Tr((z_1 z_2^* \otimes 1) E_{1 \otimes B} (x_2^* x_1 \otimes y_2^* y_1) e_{1 \otimes B})$$
$$= \tau_C(x_2^* x_1) \tau_{C \otimes B}(z_1 z_2^* \otimes y_2^* y_1)$$
$$= \tau_C(x_2^* x_1) \tau_C(z_2^* z_1) \tau_B(y_2^* y_1)$$
$$= \langle (x_1 \otimes 1 \otimes y_1)(1 \otimes 1 \otimes 1) \theta(z_1 \otimes 1), (x_2 \otimes 1 \otimes y_2)(1 \otimes 1 \otimes 1) \theta(z_2 \otimes 1) \rangle_{\mathcal{H}}$$
$$= \langle (x_1 \otimes y_1) \cdot (1 \otimes 1 \otimes 1) \cdot (z_1 \otimes 1), (x_2 \otimes y_2) \cdot (1 \otimes 1 \otimes 1) \cdot (z_2 \otimes 1) \rangle_{\mathcal{H}}.$$

This combined with (2.5) and [48, Theorem 2.1] show that A is amenable relative to B inside M. $\quad \square$

3. Proof of Theorem A

This section is devoted to the proof of Theorem A. In essence this result is an infinite analog of the "product rigidity" phenomenon for group factors found in [16]. In fact our methods build upon the general strategy developed in [16] and still use in a crucial way the ultrapower techniques from [33] as well as the intertwining/combinatorial aspects developed in [47,38,16,23,20] and the classification of normalizers from [63]. Since our exposition will focus primarily on the novel aspects of these techniques we recommend the reader to consult the aforementioned works as some of these results will be heavily used throughout the section.

To ease our exposition we first introduce the following notation:

Notation 3.1. Let $\{\Gamma_i\}_{i \in I}$ be a collection of icc, weakly amenable, biexact groups and denote by $\Gamma = \oplus_{i \in I} \Gamma_i$. For any subset $S \subseteq I$, we denote $\Gamma_S = \oplus_{i \in S} \Gamma_i$. Denote by $M = L(\Gamma)$, let $t > 0$ be a scalar, and assume that $M^t = L(\Lambda)$ for an arbitrary group Λ. Following [38], let $\Delta : M^t \to M^t \bar{\otimes} M^t$ be the comultiplication along Λ, i.e. $\Delta(v_\lambda) = v_\lambda \otimes v_\lambda$, where $\{v_\lambda\}_{\lambda \in \Lambda}$ are the canonical unitaries generating $L(\Lambda)$.

Proposition 3.2. *Assume Notation 3.1. Then for every $i \in I$ there exists $j \in I$ such that*
$$\Delta(L(\Gamma_{I \setminus \{i\}})^t) \prec_{M^t \bar{\otimes} M^t} M^t \bar{\otimes} L(\Gamma_{I \setminus \{j\}})^t.$$

Proof. Throughout this proof we will identify I with $\mathbb{N}$, the set of positive integers. Also, since Γ_i's are weakly amenable and biexact we show next the following

Claim 3.3. *For every $i,j \in I$ one of the following holds:*

a) $\Delta(L(\Gamma_{I \setminus \{i\}})^t) \prec_{M^t \bar{\otimes} M^t} M^t \bar{\otimes} L(\Gamma_{I \setminus \{j\}})^t$, *or*
b) $\Delta(L(\Gamma_i)) \prec_{M^t \bar{\otimes} M^t} M^t \bar{\otimes} L(\Gamma_{I \setminus \{j\}})^t$.

Proof of Claim 3.3. One has the following decomposition $M^t \bar{\otimes} M^t = M^t \bar{\otimes} L(\Gamma_{I \setminus \{j\}})^t \bar{\otimes} L(\Gamma_j)$. Fix $A \subset \Delta(L(\Gamma_i))$ a diffuse amenable subalgebra. Using [63, Theorem 1.6], we have either

c) $A \prec_{M^t \bar{\otimes} M^t} M^t \bar{\otimes} L(\Gamma_{I \setminus \{j\}})^t$, *or*
d) $\mathcal{N}_{M^t \bar{\otimes} M^t}(A)''$ is amenable relative to $M^t \bar{\otimes} L(\Gamma_{I \setminus \{j\}})^t$.

Suppose d) holds. As $\Delta(L(\Gamma_{I \setminus \{i\}})^t) \subseteq \mathcal{N}_{M^t \bar{\otimes} M^t}(A)''$, then [63, Theorem 1.6] implies either

e) $\Delta(L(\Gamma_{I \setminus \{i\}})^t) \prec M^t \bar{\otimes} L(\Gamma_{I \setminus \{j\}})^t$, *or*
f) $\mathcal{N}_{M^t \bar{\otimes} M^t} \Delta(L(\Gamma_{I \setminus \{i\}})^t)''$ is amenable relative to $M^t \bar{\otimes} L(\Gamma_{I \setminus \{j\}})^t$.

However, f) cannot hold. Indeed, since $\Delta(M^t) \subseteq \mathcal{N}_{M^t \bar{\otimes} M^t}(\Delta(L(\Gamma_{I \setminus \{i\}}))^t)''$, then [38, Theorem 7.2(2)] would imply Γ_i is finite, a contradiction. Hence, for every diffuse amenable subalgebra $A \subset L(\Gamma_i)$, either c) or e) must occur. Using [8, Corollary F.14], we get the claim. $\square$

Now assume by contradiction the conclusion does not hold. By Claim 3.3, for every $j \in I$ we have

$$\Delta(L(\Gamma_i)) \prec_{M^t \bar{\otimes} M^t} M^t \bar{\otimes} L(\Gamma_{I \setminus \{j\}})^t. \tag{3.1}$$

Next we observe that $\mathcal{Z}(\Delta(L(\Gamma_i))' \cap M^t \bar{\otimes} M^t) = \mathbb{C}1$. To see this, let $z \in \mathcal{Z}(\Delta(L(\Gamma_i))' \cap M^t \bar{\otimes} M^t)$. Since $\Delta(L(\Gamma_{I \setminus \{i\}})^t) \subset \Delta(L(\Gamma_i))' \cap M^t \bar{\otimes} M^t$, one can check that $z \in \Delta(M^t)' \cap M^t \bar{\otimes} M^t$. However, since Λ is icc we have $\Delta(M^t)' \cap M^t \bar{\otimes} M^t = \mathbb{C}1$ and our claim follows.

Thus (3.1) further implies that $\Delta(L(\Gamma_i)) \prec^s_{M^t \bar{\otimes} M^t} M^t \bar{\otimes} L(\Gamma_{I \setminus \{j\}})^t$. Hence, applying [23, Lemma 2.8 (2)], for every finite subset $F \subset I$ we have

$$\Delta(L(\Gamma_i)) \prec_{M^t \bar{\otimes} M^t} M^t \bar{\otimes} L(\Gamma_{I \setminus F})^t. \tag{3.2}$$

Next we show that (3.2) implies the following

Claim 3.4. $\Delta(L(\Gamma_i))$ *is amenable relative to* $M^t \otimes 1$.

Proof of Claim 3.4. Let $I_n = \{n, n+1, n+2, ...\}$. Since $\Delta(L(\Gamma_i))' \cap M^t \bar\otimes M^t$ is a factor, then using [47, Proposition 12], for every $n \in \mathbb{N}$ there is $t_n > 0$ and $u_n \in \mathcal{U}(M^t \bar\otimes M^t)$ so that

$$u_n \Delta(L(\Gamma_i)) u_n^* \subset (M^t \bar\otimes L(\Gamma_{I_n})^t)^{t_n}.$$

Naturally, we have the following inclusions $M^t \bar\otimes L(\Gamma_{I_n})^{tt_n} \subset M^t \bar\otimes L(\Gamma_{I_n}) \bar\otimes L(\Gamma_{n-1}) = M^t \bar\otimes L(\Gamma_{I_{n-1}})$. Thus, for every $n \in I$, there is $u_n \in \mathcal{U}(M^t \bar\otimes M^t)$ so that

$$u_n \Delta(L(\Gamma_i)) u_n^* \subset M^t \bar\otimes L(\Gamma_{I_{n-1}}). \tag{3.3}$$

Thus the claim follows from (3.3) and Proposition 2.7. □

Finally, Claim 3.4 and [38, Proposition 7.2] imply $L(\Gamma_i)$ amenable, a contradiction. □

Proposition 3.5. *Assume Notation 3.1. Then for all $i \in I$, there exists a nonamenable subgroup $\Lambda_i \leqslant \Lambda$ with nonamenable centralizer $C_\Lambda(\Lambda_i)$ such that $L(\Gamma_{I\setminus\{i\}})^t \prec_{M^t} L(\Lambda_i)$.*

Proof. This follows directly from Proposition 3.2 and [23, Theorem 4.1], (see also the proof of [16, Theorem 3.3]). □

Theorem 3.6. *Assume Notation 3.1. In addition, assume that Γ_i has property (T), for all $i \in I$. For each $i \in I$ there is a decomposition $\Lambda = \Psi_i \oplus \Theta_i$, a scalar $t_i > 0$ and $u_i \in \mathcal{U}(M)$ satisfying*

$$u_i L(\Gamma_i)^{t_i} u_i^* = L(\Psi_i) \ and \ u_i L(\Gamma_{I\setminus\{i\}})^{t/t_i} u_i^* = L(\Theta_i). \tag{3.4}$$

Proof. Fix $i \in I$ and write $M^t = L(\Gamma_I)^t = L(\Gamma_{I\setminus\{i\}})^t \bar\otimes L(\Gamma_i) = A \bar\otimes B$. By Proposition 3.5, we have $A \prec_{M^t} L(\Lambda_i)$ for some non-amenable group $\Lambda_i \leqslant \Lambda$ with non-amenable $C_\Lambda(\Lambda_i)$. By [13, Proposition 2.4], there exist nonzero projections $a \in A, q \in L(\Lambda_i)$, a partial isometry $v \in M^t$, a subalgebra $D \subseteq qL(\Lambda_i)q$, and a $*$-isomorphism $\phi : aAa \to D$ such that

$$D \vee (D' \cap qL(\Lambda_i)q) \subseteq qL(\Lambda_i)q \quad \text{has finite index, and} \tag{3.5}$$

$$\phi(x)v = vx \quad \forall x \in aAa. \tag{3.6}$$

Notice that $vv^* \in D' \cap qM^tq$ and $v^*v \in (aAa)' \cap aM^ta = a \otimes B$. Hence there is a projection $b \in B$ satisfying $v^*v = a \otimes b$. Picking $u \in \mathcal{U}(M^t)$ so that $v = u(a \otimes b)$ then (3.6) gives

$$Dvv^* = vaAav^* = u(aAa \otimes b)u^*. \tag{3.7}$$

Passing to the relative commutants, we obtain $vv^*(D' \cap qM^tq)vv^* = u(a \otimes bBb)u^*$. This further implies that there exist $s_1, s_2 > 0$

$$(D' \cap qM^t q)z = u(a \otimes bBb)^{s_1} u^* = L(\Gamma_i)^{s_2}, \tag{3.8}$$

where z is the central support projection of vv^* in $D' \cap qMq$. Now notice

$$D' \cap qM^t q \supseteq (qL(\Lambda_i)q)' \cap qM^t q = (L(\Lambda_i)' \cap M^t)q \supseteq L(C_\Lambda(\Lambda_i))q,$$

where $L(C_\Lambda(\Lambda_i))$ has no amenable direct summand since $C_\Lambda(\Lambda_i)$ is a non-amenable group. Moreover we also have $D' \cap qM^t q \supseteq D' \cap qL(\Lambda_i)q$. Thus $(L(\Lambda_i)' \cap M^t)z$ and $(D' \cap qL(\Lambda_i)q)z$ are commuting subalgebras of $(D' \cap qM^t q)z$ where $(L(\Lambda_i)' \cap M^t)z$ has no amenable direct summand. Since Γ_i was assumed to be bi-exact, then using (3.8) and [45, Theorem 1] it follows that $(D' \cap qL(\Lambda_i)q)z$ is purely atomic. Thus, cutting by a central projection $r' \in D' \cap q(\Lambda_i)q$ and using (3.5) we may assume that $D \subseteq qL(\Lambda_i)q$ is a finite index inclusion of algebras. Processing as in the second part of [16, Claim 4.4], we may assume that $D \subseteq qL(\Lambda_i)q$ is a finite index of II_1 factors. Moreover, one can check that if one replaces v by the partial isometry of the polar decomposition of $r'v \neq 0$ then all relations (3.6), (3.7) and (3.8) are still satisfied. In addition, we can assume without any loss of generality that the support projection satisfies $s(E_{L(\Lambda_i)}(vv^*) = q$. Thus, following the terminology introduced in [20, Definition 4.1] we actually have that a corner of A is spatially commensurable to a corner of $L(\Lambda_i)$, i.e.

$$A \cong_{M^t}^{com} L(\Lambda_i). \tag{3.9}$$

Performing the downward basic construction [42, Lemma 3.1.8], there exist $e \in \mathscr{P}(qL(\Lambda_i)q)$ and a II_1 subfactor $R \subseteq D \subseteq qL(\Lambda_i)q = \langle D, e \rangle$ such that $[D : R] = [qL(\Lambda_i)q : D]$ and $Re = eL(\Lambda_i)e$. Keeping with the same notation, by relation (3.6) the restriction $\phi^{-1} : R \to aAa$ is an injective $*$-homomorphism such that $T = \phi^{-1}(R) \subseteq aAa$ is a finite Jones index subfactor and

$$\phi^{-1}(y)v^* = v^* y, \text{ for all } y \in R. \tag{3.10}$$

Let $\theta' : Re \to R$ be the $*$-isomorphism given by $\theta(xe) = x$. Since e has full central support in $\langle D, e \rangle$ one can see that $ev \neq 0$. Letting w_0 be a partial isometry so that $w_0^*|v^*e| = v^*e$, then $Re = eL(\Lambda_i)e$ together with (3.10) imply that $\theta = \phi^{-1} \circ \theta' : eL(\Lambda_i)e \to aAa$ is an injective $*$-homomorphism satisfying $\theta(eL(\Lambda_i)e) = T$ and

$$\theta(y)w_0^* = w_0^* y, \text{ for all } y \in eL(\Lambda_i)e. \tag{3.11}$$

Notice that $w_0^* w_0 \in (T' \cap aAa) \bar\otimes B$ and proceeding as in the proof of [47, Proposition 12] one can further assume that $w_0^* w_0 \in \mathscr{Z}(T' \cap aAa) \bar\otimes B$. Since $[aAa : T] < \infty$ then $T' \cap aAa$ is finite dimensional and so is $\mathscr{Z}(T' \cap aAa)$. Thus, replacing the partial isometry w_0 by $w := w_0 r_0$, for some minimal projection $r_0 \in \mathscr{Z}(T' \cap aAa)$ satisfying $r_0 w_0^* |v^*e| \neq 0$, we see that all relations above still hold including relation (3.11). Moreover, we can assume

that $w^*w = z_1 \otimes z_2$, for some nonzero projections $z_1 \in \mathscr{Z}(T' \cap aAa)$ and $z_2 \in B$. Using relation (3.11) we get

$$w^*L(\Lambda_i)w = \theta(eL(\Lambda_i)e)w^*w = Tz_1 \otimes z_2. \tag{3.12}$$

Since $T \subseteq aAa$ is finite index inclusion of II_1 factors then by the local index formula [42] it follows $Tz_1 \subseteq z_1Az_1$ is a finite index inclusion of II_1 factors as well. Also, we have

$$(w^*L(\Lambda_i)w)' \cap (z_1 \otimes z_2)M^t(z_1 \otimes z_2) = ((Tz_1)' \cap z_1Az_1)\bar{\otimes}z_2Bz_2. \tag{3.13}$$

Altogether, the previous relations imply that

$$
\begin{aligned}
Tz_1\bar{\otimes}z_2Bz_2 &\subseteq Tz_1 \vee (Tz_1' \cap z_1Az_1)\bar{\otimes}z_2Bz_2 \\
&= w^*L(\Lambda_i)w \vee w^*(L(\Lambda_i)' \cap M^t)w \\
&= w^*L(\Lambda_i)w \vee \left((w^*L(\Lambda_i)w)' \cap (z_1 \otimes z_2)M^t(z_1 \otimes z_2)\right) \\
&\subseteq z_1Az_1\bar{\otimes}z_2Bz_2.
\end{aligned}
\tag{3.14}
$$

Since $Tz_1 \subseteq z_1Az_1$ if a finite index inclusion of II_1 factors then so is $Tz_1\bar{\otimes}z_2Bz_2 \subseteq z_1Az_1\bar{\otimes}z_2Bz_2$. Let $f := ww^*$ and note that $f = re$, for some projection $r \in L(\Lambda_i)' \cap M^t$. Letting $u \in \mathscr{U}(M^t)$ such that $w^* = uww^* = uf$, then relation (3.14) further implies that

$$f(L(\Lambda_i) \vee (L(\Lambda_i)' \cap M^t))f = L(\Lambda_i)f \vee f(L(\Lambda_i)' \cap M^t)f \subseteq fM^tf \tag{3.15}$$

is an inclusion of finite index II_1 factors. In addition, (3.14) gives that $\dim_{\mathbb{C}}(\mathscr{Z}(f(L(\Lambda_i) \vee (L(\Lambda_i)' \cap M^t))f)) \leq [z_1Az_1\bar{\otimes}z_2Bz_2 : Tz_1\bar{\otimes}z_2Bz_2] < \infty$. Since the central support of e in $qL(\Lambda_i)q$ equals q then (3.15) implies that

$$q(L(\Lambda_i)qr \vee r(L(\Lambda_i)' \cap M^t))rq = qr(L(\Lambda_i) \vee (L(\Lambda_i)' \cap M^t))qr \subseteq qrM^tqr, \tag{3.16}$$

is a finite index inclusion of II_1 factors. In particular, $qL(\Lambda_i)qr$ and $r(L(\Lambda_i)' \cap M^t)rq$ are commuting II_1 factors.

To this end we notice that since $0 \neq r_0w_0^*|v^*e| = w^*|v^*e|$ then $0 \neq w^*|v^*e|^{1/2}$. Thus $0 \neq w^*|v^*e|w = v^*ew$ and since v, w are partial isometries we conclude that $0 \neq vv^*eww^*$. However, since $ww^* = r_0w_0w_0^* \leq s(|v^*e|)$ then $ww^* \leq e$. Combining with the above it follows that $0 \neq vv^*ww^*$ and hence $zf = zww^* \neq 0$. Thus further implies that

$$zr \neq 0. \tag{3.17}$$

Next we show the following

Claim 3.7. $r(L(\Lambda_i)' \cap M^t)rq$ *has property (T).*

Proof of Claim 3.7. Since $D \subseteq qL(\Lambda_i)q$ is a finite index inclusion of II_1 factors then so is $Dr \vee r(L(\Lambda_i)' \cap M^t)rq \subseteq qL(\Lambda_i)qr \vee r(L(\Lambda_i)' \cap M^t)rq$. Using (3.16) it follows that $Dr \vee r(L(\Lambda_i)' \cap M^t)rq \subseteq rqM^trq$ is finite index as well. Hence $Dr \vee r(L(\Lambda_i)' \cap M^t)rq \subseteq Dr \vee r(D' \cap qM^tq)r$ is also a finite index inclusion. Since D is a factor one can check that $E_{Dr \vee r(L(\Lambda_i)' \cap M^t)rq}(x) = E_{(L(\Lambda_i)' \cap M^t)q}(x)$, for all $x \in r(D' \cap qM^tq)r$. This combined with the above entail that $r(L(\Lambda_i)' \cap M^t)rq \subseteq r(D' \cap qM^tq)r$ is finite index. By [51, Theorem 1.1.2 (ii)] $r(L(\Lambda_i)' \cap M^t)rz \subseteq r(D' \cap qM^tq)rz$ is finite index and since property (T) passes to amplifications and finite index subalgebras, then (3.8) implies that $r(L(\Lambda_i)' \cap M^t)rz$ has property (T). As $r(L(\Lambda_i)' \cap M^t)rq$ is a factor we conclude that $r(L(\Lambda_i)' \cap M^t)rq$ has property (T). □

Now consider $\Omega := vC_\Lambda(\Lambda_i) = \{\lambda \in \Lambda \,|\, |\lambda^{\Lambda_i}| < \infty\}$, the virtual centralizer of Λ_i in Λ. Using [16, Claim 4.7] we have $[\Lambda : \Lambda_i\Omega] < \infty$ and hence $\Lambda_i\Omega \leqslant \Lambda$ is an icc subgroup; in particular, $vZ(\Lambda_i\Omega) = 1$. Consider $vZ(\Omega) = \{\omega \in \Omega \,|\, |\omega^\Omega| < \infty\}$, the virtual center of Ω. Since Λ_i normalizes Ω one can check that $vZ(\Omega) \leqslant vZ(\Lambda_i\Omega)$. Since the latter is trivial we get $vZ(\Omega) = 1$ and hence Ω is icc. Let $(\mathcal{O}_n)_{n \in \mathbb{N}}$ be a countable enumeration of all the orbits under conjugation by Λ_i. Denote by $\Omega_k = \langle \mathcal{O}_1, ..., \mathcal{O}_k \rangle \leqslant \Omega$, the subgroup generated by $\mathcal{O}_n$, $n = \overline{1, k}$. Ω_k's form an ascending sequence of subgroups normalized by Λ_i such that $\Omega = \cup_{k=1}^\infty \Omega_k$. Thus $\Lambda_i\Omega_k$ is an ascending sequence satisfying $\Lambda_i\Omega = \cup_{k=1}^\infty \Lambda_i\Omega_k$. Since $r(L(\Lambda_i)' \cap M^t)rq \subset L(\Lambda_i\Omega)$ has property (T) there is $k_0 \in \mathbb{N}$ such that

$$r(L(\Lambda_i)' \cap M^t)rq \prec_{L(\Lambda_i\Omega)} L(\Lambda_i\Omega_{k_0}). \tag{3.18}$$

Next we show the following

Claim 3.8. *There exists $k \geq k_0$ such that $qL(\Lambda_i)qr \vee r(L(\Lambda_i)' \cap M^t)rq \prec_{L(\Lambda_i\Omega)} L(\Lambda_i\Omega_k)$.*

Proof of the Claim 3.8. Using Popa's intertwining techniques, (3.18) implies the existence of $x_\ell, y_\ell \in L(\Lambda_i\Omega)$, $\ell = \overline{1, j}$ and $c > 0$ satisfying

$$\sum_{\ell=1}^j \|E_{L(\Lambda_i\Omega_{k_0})}(x_\ell u y_\ell)\|_2 \geq c, \tag{3.19}$$

for all $u \in \mathcal{U}(r(L(\Lambda_i)' \cap M^t)rq)$. Since $\Lambda_i\Omega_k \nearrow \Lambda_i\Omega$ for every $\varepsilon > 0$ there is $k \geq k_0$ so that

$$\sum_{\ell=1}^j \|E_{L(\Lambda_i\Omega_k)}(x_\ell) - x_\ell\|_2 < \varepsilon, \quad \sum_{\ell=1}^j \|E_{L(\Lambda_i\Omega_k)}(y_\ell) - y_\ell\| < \varepsilon. \tag{3.20}$$

Using (3.19) together with inequalities $\|mzn\|_2 \leq \|m\|_\infty \|z\|_2 \|n\|_\infty$ for all $m, n, z \in M^t$ then for all $u \in \mathcal{U}(r(L(\Lambda_i)' \cap M^t)rq)$ we have

$$c \leq \sum_{\ell=1}^{j} \|E_{L(\Lambda_i \Omega_k)}(x_\ell u y_l)\|_2$$

$$\leq \sum_{\ell=1}^{j} \|E_{L(\Lambda_i \Omega_k)}((x_\ell - E_{L(\Lambda_i \Omega_k)}(x_\ell)) u y_\ell)\|_2$$

$$+ \sum_{\ell=1}^{j} \|E_{L(\Lambda_i \Omega_k)}(E_{L(\Lambda_i \Omega_k)}(x_\ell) u (y_\ell - E_{L(\Lambda_i \Omega_k)}(y_\ell))\|_2$$

$$+ \sum_{\ell=1}^{j} \|E_{L(\Lambda_i \Omega_k)}(E_{L(\Lambda_i \Omega_k)}(x_\ell)) u E_{L(\Lambda_i \Omega_k)}(y_\ell)\|_2$$

$$\leq \varepsilon \max_{1 \leq \ell \leq j} (\|y_\ell\|_\infty + \|x_\ell\|_\infty) + \sum_{\ell=1}^{j} \|x_\ell\|_\infty \|y_\ell\|_\infty \|E_{L(\Lambda_i \Omega_k)}(u)\|_2$$

$$\leq 2d\varepsilon + jd^2 \|E_{L(\Lambda_i \Omega_k)}(u)\|_2,$$

where $d := \max_{1 \leq \ell \leq j} \{\|x_\ell\|_\infty, \|y_\ell\|_\infty\}$. This shows there is $k \geq k_0$ so that $\|E_{L(\Lambda_i \Omega_{k_0})}(u)\|_2 \geq \frac{c - 2\varepsilon d}{jd^2}$ for all $u \in \mathcal{U}(r(L(\Lambda_i)' \cap M^t)rq)$. Letting $\varepsilon = \frac{c}{4d}$ then for all $u \in \mathcal{U}(r(L(\Lambda_i)' \cap M^t)rq)$ we have

$$\|E_{L(\Lambda_i \Omega_k)}(u)\|_2 \geq \frac{c}{2jd^2} > 0.$$

This implies for all $a \in \mathcal{U}(qL(\Lambda_i)qr)$ and $u \in \mathcal{U}(r(L(\Lambda_i)' \cap M^t)rq)$ we have

$$\|E_{L(\Lambda_i \Omega_k)}(au)\|_2 = \|a E_{L(\Lambda_i \Omega_k)}(u)\|_2 = \|E_{L(\Lambda_i \Omega_k)}(u)\|_2 \geq \frac{c}{2jd^2}. \tag{3.21}$$

As $\mathcal{U}(qL(\Lambda_i)qr)\mathcal{U}(r(L(\Lambda_i)' \cap M^t)rq)$ generates $qL(\Lambda_i)qr \vee r(L(\Lambda_i)' \cap M)rq$, (3.21) gives the claim. □

Now, since $q(L(\Lambda_i)qr \vee r(L(\Lambda_i)' \cap M)rq \subseteq rqL(\Lambda_i \Omega)rq$ is a finite index inclusion, then $rqL(\Lambda_i \Omega)rq \prec_{L(\Lambda_i \Omega)} (L(\Lambda_i \Omega_k)$ and hence $L(\Lambda_i \Omega) \prec_{L(\Lambda_i \Omega)} L(\Lambda_i \Omega_k)$. By [12, Lemma 2.2] it follows that $\Lambda_i \Omega_k \leqslant \Lambda_i \Omega$ has finite index and by increasing k we can assume that $\Lambda_i \Omega_k = \Lambda_i \Omega$. Let $\Lambda' := C_{\Lambda_i}(\Omega_k) \leqslant \Lambda_i$ and notice $[\Lambda_i : \Lambda'] < \infty$. Thus $[\Lambda_i \Omega : \Lambda' \Omega_k] < \infty$ and since $\Lambda_i \Omega$ is icc then $\Lambda' \Omega_k$ is also icc. In particular, we also have $\Lambda' \cap \Omega_k = 1$. As $\Lambda' \Omega_k \leqslant \Lambda_i \Omega$ is finite index the $\Lambda' \Omega_k \cap \Omega \leqslant \Omega$ is also finite index. In particular, since Ω is icc it follows that $\Lambda' \Omega_k \cap \Omega$ is also icc. Letting $\Lambda'' := \Lambda' \cap \Omega$ the above considerations imply that $\Lambda'' \Omega_k = \Lambda' \Omega_k \cap \Omega$. This forces Λ'' to be either trivial or icc. However, since by construction $\Lambda'' = vZ(\Lambda'')$ then $\Lambda'' = 1$. Since $\Lambda' \leqslant \Lambda_i$ finite index it follows that Λ_i is icc-by-finite and hence finite-by-icc.

This together with (3.9) and [20, Theorem 4.6] show there exists $\Sigma \leqslant C_\Lambda(\Lambda_i)$ such that $[\Lambda : \Lambda_i \Sigma] < \infty$ and $B \cong_M^{com} L(\Sigma)$. Also since Λ is icc then so are Λ_i and Σ. Finally, using [20, Theorem 4.7] there is a decomposition $\Lambda = \Psi_i \oplus \Theta_i$, $u_i \in \mathcal{U}(M^t)$ and $t > 0$ such that $u_i A^t u_i^* = u_i L(\Gamma_{I \setminus \{i\}})^t u_i^* = L(\Psi_i)$ and $u_i B^{1/t} u_i^* = u_i L(\Gamma_i)^t u_i^* = L(\Theta_i)$. □

Now we are ready to derive the proof of first of our main results.

Proof of Theorem A. Using Theorem 3.6 there exist a product decomposition $\Lambda = \Lambda_1 \oplus \Theta_1$, $v_1 \in \mathcal{U}(M)$, and $t_1 > 0$ such that $v_1 L(\Gamma_1)^{t_1} v_1^* = L(\Lambda_1)$ and $v_1 L(\Gamma_{\mathbb{N}\setminus\{1\}})^{t/t_1} v_1^* = L(\Theta_1)$. Applying Theorem 3.6 again in the last relation for the group $\Gamma_{\mathbb{N}\setminus\{1\}}$ there exist a product decomposition $\Theta_1 = \Lambda_2 \oplus \Theta_2$, $v_2 \in \mathcal{U}(v_1 L(\Gamma_{\mathbb{N}\setminus\{1\}})^{t/t_1} v_1^*)$, and $t_2 > 0$ such that $v_2 L(\Gamma_2)^{t_2} v_2^* = L(\Lambda_2)$ and $v_2 L(\Gamma_{\mathbb{N}\setminus\{1,2\}})^{t/(t_1 t_2)} v_2^* = L(\Theta_2)$. Proceeding inductively one has $\Theta_{n-1} = \Lambda_n \oplus \Theta_n$, a unitary $v_n \in \mathcal{U}(v_{n-1} L(\Gamma_{\mathbb{N}\setminus\overline{1,n-1}})^{t/(t_1 t_2 \cdots t_{n-1})} v_{n-1}^*)$ and $t_n > 0$ such that $v_n L(\Gamma_n)^{t_n} v_n^* = L(\Lambda_n)$ and $v_n L(\Gamma_{\mathbb{N}\setminus\overline{1,n}})^{t/(t_1 t_2 \cdots t_n)} v_n^* = L(\Theta_n)$. Altogether, these relations show that $\Theta_n \geqslant \Theta_{n+1}$ for all n and also $\Lambda = \oplus_{\mathbb{N}} \Lambda_n \oplus A$, where $A = \cap_n \Theta_n$. In addition, for every $k \in \mathbb{N}$ letting $u_k := v_1 v_2 \cdots v_k$ we see that

$$u_k L(\Gamma_i)^{t_i} u_k^* = L(\Lambda_i) \text{ for all } i = \overline{1,k} \text{ and}$$
$$u_k L(\Gamma_{\mathbb{N}\setminus\overline{1,k}})^{t/(t_1 t_2 \cdots t_k)} u_k^* = L(\oplus_{i \geq k+1} \Lambda_i \oplus A). \tag{3.22}$$

Since $L(\Gamma_k)$ is a II$_1$ factor the second relation in (3.22) show that for each $k \in \mathbb{N}$ one can find $u_k \in \mathcal{U}(M)$ such that $u_k^* L(A) u_k \subseteq L(\Gamma_{\mathbb{N}\setminus\overline{1,k}})^{t/(t_1 t_2 \cdots t_k)}$. Using Proposition 2.7 and the same argument as in the proof of Claim 3.4 if follows that A is icc amenable or trivial as desired. $\quad\square$

Remarks 3.9. We conjecture that Theorem A still holds true without the property (T) assumption on the Γ_i's. We point out that property (T) was used in the proof of Theorem 3.6 only to derive relation (3.18); in other words the (increasing) sequence of subgroups Ω_k becomes stationary. We believe this conclusion can still be achieved without the property (T) assumption. However at this time we are unable to prove this.

Proof of Corollary B. First we argue that the group $\Gamma = \oplus_n \Gamma_n$ has trivial amenable radical. So let $B \triangleleft \Gamma$ be a normal amenable subgroup. Thus the von Neumann subalgebra $L(B) \subseteq L(\Gamma) = L(\oplus_{n \neq k} \Gamma_k) \bar{\otimes} L(\Gamma_k)$ is regular and amenable. Applying [63, Theorem 1.4] it follows that $L(B) \prec L(\oplus_{n \neq k} \Gamma_n)$. Since B is normal Γ we further deduce from [12, Lemma 2.2] that $[B : B_k] < \infty$ where $B_k := B \cap (\oplus_{n \neq k} \Gamma_n) < \oplus_{n \neq k} \Gamma_n$. Since $B_k \triangleleft \Gamma$ is normal it follows that $B/B_k \triangleleft \Gamma/B_k$ is a finite normal subgroup. As $\Gamma/B_k = (\oplus_{n \neq k} \Gamma_n / B_k) \oplus \Gamma_k$, if $\pi_k : \Gamma/B_k \to \Gamma_k$ is the canonical projection map it follows that $\pi_k(B/B_k) \triangleleft \Gamma_k$ is a finite normal subgroup. As Γ_k is icc we have $\pi_k(B/B_k) = 1$ and hence $B/B_k < \oplus_{n \neq k} \Gamma_n / B_k$; in particular, $B = B_k < \oplus_{n \neq k} \Gamma_n$. Since this holds for every positive integer k then $B < \cap_k (\oplus_{n \neq k} \Gamma_n) = 1$, thus giving the desired claim.

[7, Theorem 1.3] implies that the reduced C^*-algebra $\mathbb{C}_r^*(\Gamma)$ has the unique trace property. Letting $\phi : \mathbb{C}_r^*(\Gamma) \to \mathbb{C}_r^*(\Lambda)$ be a $*$-isomorphism of C^*-algebras it follows that ϕ lifts to a $*$-isomorphism $\phi : L(\Gamma) \to L(\Lambda)$ of von Neumann algebras. By Theorem A we have that $\Lambda = (\oplus_n \Lambda_n) \oplus A$ with A icc amenable or trivial; moreover, the corresponding relations (1.1) also hold. Since $\mathbb{C}_r^*(\Lambda)$ has the unique trace property then [7, Theorem 1.3] implies that $A = 1$ and the first part of the conclusion is proved. The remaining part of the conclusion follows directly from relations (1.1). $\quad\square$

4. Proof of Theorem E

To ease our exposition we first introduce the following notation:

Notation 4.1. Let H_0, Γ be icc groups such that H_0 has property (T) and Γ admits an infinite, almost normal subgroup $\Gamma_0 \leqslant \Gamma$ with relative property (T). Let $\Gamma \curvearrowright I$ be an action on a countable set I satisfying the following conditions:

a) For each $i \in I$ we have $[\Gamma : \mathrm{Stab}_\Gamma(i)] = \infty$;
b) There is a $k \in \mathbb{N}$ such that for each $J \subseteq I$ with $|J| \geq k$ we have $|\mathrm{Stab}_\Gamma(J)| < \infty$.

Denote by $G = H_0 \wr_I \Gamma$ the corresponding generalized wreath product. Denote by $M = L(G)$ and assume that $M = L(\Lambda)$ for an arbitrary group Λ. Let $\Delta : M \to M \bar{\otimes} M$ be the commultiplication along Λ, i.e. $\Delta(v_\lambda) = v_\lambda \otimes v_\lambda$, where $\{v_\lambda\}_{\lambda \in \Lambda}$ are the canonical unitaries generating $L(\Lambda)$.

Proposition 4.2. *Assume Notation 4.1. Then the following hold:*

c) $\Delta(L(H_0^{(I)})) \prec^s_{M \otimes M} L(H_0^{(I)}) \bar{\otimes} L(H_0^{(I)})$;
d) *There exists* $u \in \mathcal{U}(M \bar{\otimes} M)$ *such that* $u\Delta(L(\Gamma))u^* \subseteq L(\Gamma \times \Gamma)$.

Proof. We denote $A_0 = L(H_0)$ and $A = A_0^{(I)}$. Note that $M = A \rtimes \Gamma$, the action being given by generalized Bernoulli shifts. Write $M = L(\Lambda)$ and denote by $\Delta : M \to M \bar{\otimes} M$ the associated comultiplication. Note that $M \bar{\otimes} M = (A \bar{\otimes} A) \rtimes (\Gamma \times \Gamma)$.

The inclusion $\Delta(A_0) \subset M \bar{\otimes} M = M \bar{\otimes}(A \rtimes \Gamma)$ is rigid. Denote by $P \subset M \bar{\otimes} M$ the quasinormalizer of $\Delta(A_0)$. Note that $\Delta(A) \subset P$. By applying [38, Theorem 4.2], we see that one of the following has to hold:

(1) $\Delta(A_0) \prec_{M \bar{\otimes} M} M \otimes 1$;
(2) $P \prec_{M \bar{\otimes} M} M \bar{\otimes}(A \rtimes \mathrm{Stab}_\Gamma(i))$, for some $i \in I$;
(3) $v^* P v \subset M \bar{\otimes} L(\Gamma)$ for some partial isometry $0 \neq v \in M \bar{\otimes} M$ with $vv^* \in \mathcal{Z}(P)$.

(1) is impossible since A_0 is diffuse. Suppose (3) holds. Then by the remark above we have that $v^* \Delta(A)v \subset M \bar{\otimes} L(\Gamma)$. There are two possibilities: either $\Delta(A) \prec_{M \bar{\otimes} M} M \bar{\otimes} L(\mathrm{Stab}_\Gamma(i))$ for some i, or $\Delta(A) \not\prec_{M \bar{\otimes} M} M \bar{\otimes} L(\mathrm{Stab}_\Gamma(i))$, for all $i \in I$. In the first case we again have two possibilities: either there exists a maximal finite subset $\mathcal{G} \ni i$ such that $\Delta(A) \prec_{M \bar{\otimes} M} M \bar{\otimes} L(\mathrm{Stab}_\Gamma(\mathcal{G}))$, or there is no such subset. If the first sub-case holds then [38, Lemma 4.1.3] gives that $QN_{M \bar{\otimes} M}(\Delta(A))'' \prec_{M \bar{\otimes} M} M \bar{\otimes} L(\mathrm{Norm}(\mathcal{G}))$. Since the quasinormalizer of $\Delta(A)$ contains $\Delta(M)$, this implies $\Delta(M) \prec_{M \bar{\otimes} M} M \bar{\otimes} L(\mathrm{Norm}(\mathcal{G}))$. As $\mathrm{Stab}_\Gamma(\mathcal{G})$ is a finite index subgroup of $\mathrm{Norm}(\mathcal{G})$, it follows that $\Delta(M) \prec_{M \bar{\otimes} M} M \bar{\otimes} L(\mathrm{Stab}_\Gamma(\mathcal{G}))$, and hence $\Delta(M) \prec_{M \bar{\otimes} M} M \bar{\otimes} L(\mathrm{Stab}_\Gamma(i))$ for some i, which implies by [38, Lemma 7.2.2] that $L(\mathrm{Stab}_\Gamma(i)) \subset M$ has finite index, which is a contradiction.

If the second sub-case holds, by taking $\mathcal{G}$ with $|\mathcal{G}| \geq k$ we get that $\Delta(A) \prec_{M\bar{\otimes}M} M \otimes 1$, a contradiction. Next we argue that the second possibility also does not hold. So suppose that $\Delta(A) \not\prec_{M\bar{\otimes}M} M\bar{\otimes}L(\mathrm{Stab}_\Gamma(i))$, for all $i \in I$. This further implies that $v^*\Delta(A)v \not\prec_{M\bar{\otimes}M} M\bar{\otimes}L(\mathrm{Stab}_\Gamma(i))$, for all $i \in I$ and using [38, Lemma 4.1.1] it follows that the quasinormalizer of $v^*\Delta(A)v$ is still contained in $M\bar{\otimes}L(\Gamma)$. Since A is regular in M, this further implies $v^*\Delta(M)v \subset M\bar{\otimes}L(\Gamma)$, so $\Delta(M) \prec M\bar{\otimes}L(\Gamma)$, which by [38, Lemma 7.2.2] implies that $L(\Gamma)$ has finite index in M, again a contradiction.

It follows that (2) must hold, hence $P \prec_{M\bar{\otimes}M} M\bar{\otimes}(A \rtimes \mathrm{Stab}_\Gamma(i))$, which further implies $\Delta(A) \prec_{M\bar{\otimes}M} M\bar{\otimes}(A \rtimes \mathrm{Stab}_\Gamma(i))$, for some $i \in I$. Again we have two possibilities: either there exists a finite maximal subset $\mathcal{G} \subset I$ such that $\Delta(A) \prec_{M\bar{\otimes}M} M\bar{\otimes}(A \rtimes \mathrm{Stab}_\Gamma(\mathcal{G}))$, or it doesn't. In the first sub-case we get, by [38, Lemma 4.1.3], that $QN_{M\bar{\otimes}M}(\Delta(A))'' \prec_{M\bar{\otimes}M} M\bar{\otimes}(A \rtimes \mathrm{Norm}(\mathcal{G}))$ and again as above, that $\Delta(M) \prec_{M\bar{\otimes}M} M\bar{\otimes}(A \rtimes \mathrm{Stab}_\Gamma(i))$, for some $i \in \mathcal{G}$, which by [38, Lemma 7.2.2] implies that $[M : A \rtimes \mathrm{Stab}_\Gamma(i)]$ is finite, a contradiction. In the second sub-case, by taking a $\mathcal{G}$ with $|\mathcal{G}| \geq k$, we obtain that $\Delta(A) \prec_{M\bar{\otimes}M} M\bar{\otimes}A$, which is what we wanted. The maximal projection $q \in \Delta(A)' \cap M\bar{\otimes}M$ such that $\Delta(A)q \prec^s_{M\bar{\otimes}M} M\bar{\otimes}A$ is non-zero and belongs to the center of the normalizer of $\Delta(A)$ in $M\bar{\otimes}M$. This center is contained in $\Delta(M)' \cap M\bar{\otimes}M = \mathbb{C}1$. It follows that $q = 1$, hence $\Delta(A) \prec^s_{M\bar{\otimes}M} M\bar{\otimes}A$. By symmetry we obtain that also $\Delta(A) \prec^s_{M\bar{\otimes}M} A\bar{\otimes}M$ and finally that $\Delta(A) \prec^s_{M\bar{\otimes}M} A\bar{\otimes}A$, showing part c).

Next we prove part d). First notice from the assumptions that the inclusion $\Delta(L(\Gamma_0)) \subset M\bar{\otimes}(A \rtimes \Gamma)$ is rigid. Denote by P the quasinormalizer of $\Delta(L(\Gamma_0))$ inside $M\bar{\otimes}M$. Note that P contains $\Delta(L(\Gamma))$. We apply again [38, Theorem 4.2] and we see that one of the following has to hold:

(1) $\Delta(L(\Gamma_0)) \prec_{M\bar{\otimes}M} M\bar{\otimes}1$;
(2) $P \prec_{M\bar{\otimes}M} M\bar{\otimes}(A \rtimes \mathrm{Stab}_\Gamma(i))$, for some $i \in I$;
(3) $vPv^* \subset M\bar{\otimes}L(\Gamma)$, for some $v \in \mathcal{U}(M\bar{\otimes}M)$.

Note that (1) cannot be true because $\Delta(L(\Gamma_0))$ is diffuse. Suppose (2) is true. This implies in particular that $\Delta(L(\Gamma)) \prec_{M\bar{\otimes}M} M\bar{\otimes}(A \rtimes \mathrm{Stab}_\Gamma(i))$. But since $\Delta(A) \prec^s_{M\bar{\otimes}M} A\bar{\otimes}A$, by [3, Lemma 2.3], we would get $\Delta(A \rtimes \Gamma) = \Delta(M) \prec_{M\bar{\otimes}M} M\bar{\otimes}(A \rtimes \mathrm{Stab}_\Gamma(i))$, which by [38, Lemma 7.2.2] implies that $A \rtimes \mathrm{Stab}_\Gamma(i) \subset M$ has finite index, a contradiction. So (3) must be true, hence a fortiori $v\Delta(L(\Gamma))v^* \subset M\bar{\otimes}L(\Gamma)$. Repeating the argument for the inclusion $v\Delta(L(\Gamma))v^* \subset M\bar{\otimes}L(\Gamma) = (A \rtimes \Gamma)\bar{\otimes}L(\Gamma)$, we obtain an unitary $u \in M\bar{\otimes}M$ such that $u\Delta(L(\Gamma))u^* \subset L(\Gamma)\bar{\otimes}L(\Gamma)$, as desired. □

Proposition 4.3. *Assume Notation 4.1. In addition assume that H_0, Γ and Λ are torsion free groups. Then the following hold:*

$\Delta(L(H_0^{(I)})) \prec^s_{M\otimes M} L(H_0^{(I)})\bar{\otimes}L(H_0^{(I)})$;
e) *There exists $w \in \mathcal{U}(M\bar{\otimes}M)$ such that $w\Delta(\Gamma)w^* \subseteq \mathbb{T}(\Gamma \times \Gamma)$.*

Proof. Since e) follows directly from Proposition 4.2 we will only argue for f). From Proposition 4.2 there exists $u \in \mathcal{U}(M \bar{\otimes} M)$ such that $u\Delta(L(\Gamma))u^* \subseteq L(\Gamma \times \Gamma)$.

Denote by $\mathcal{G} = \{u\Delta(u_\gamma)u^* \mid \gamma \in \Gamma\} \subset \mathcal{U}(L(\Gamma \times \Gamma))$. Since $\mathcal{G}$ normalizes $u\Delta(A)u^*$, which satisfies $u\Delta(A)u^* \prec^s A \bar{\otimes} A$, the argument in Step 5 of the proof of [38, Theorem 5.1] implies that $h_{\Gamma \times \Gamma}(\mathcal{G}) > 0$. We have that $\mathcal{G}'' = u\Delta(L(\Gamma))u^* \not\prec_{M \bar{\otimes} M} L(C_{\Gamma \times \Gamma}(\gamma_1, \gamma_2))$, for any $(\gamma_1, \gamma_2) \in \Gamma \times \Gamma - \{e\}$. Indeed, suppose this is not true, and $u\Delta(L(\Gamma))u^* \prec L(C_{\Gamma \times \Gamma}((\gamma_1, \gamma_2))) = L(C_\Gamma(\gamma_1)) \bar{\otimes} L(C_\Gamma(\gamma_2))$, with $\gamma_2 \neq e$. Again using the fact that $\Delta(A) \prec_{M \bar{\otimes} M} A \bar{\otimes} A$, we infer that $\Delta(M) = \Delta(A \rtimes \Gamma) \prec M \bar{\otimes} (A \rtimes C_\Gamma(\gamma_2))$. [38, Lemma 7.2.2] then implies that $A \rtimes C_\Gamma(\gamma_2)$ has finite index in M, which is a contradiction, since Γ is icc. Also, the representation $\{Ad(v)\}_{v \in \mathcal{G}}$ on $L^2(L(\Gamma \times \Gamma)) \ominus \mathbb{C}1$ is weakly mixing, because it is in fact weakly mixing on $L^2(M \bar{\otimes} M) \ominus \mathbb{C}1$. Indeed, let $\mathcal{H} \subset L^2(M \bar{\otimes} M)$ be a finite dimensional $\{Ad(v)\}_{v \in \mathcal{G}}$ invariant subspace. Then $\mathcal{H}_0 = u\mathcal{H}u^*$ is a finite dimensional $\{Ad\Delta(u_\gamma)\}_{\gamma \in \Gamma}$-invariant subspace of $L^2(M \bar{\otimes} M)$. Denote by $\mathcal{K}$ the closed linear span of $\mathcal{H}_0\Delta(M)$. Then $\mathcal{K}$ is a $\Delta(L\Gamma) - \Delta(M)$ bi-module, which is finitely generated as a right module. Since $L(\Gamma) \not\prec_M L(C_\Lambda(s))$, for any $s \in \Lambda - \{e\}$, [38, Proposition 7.2.3] implies that $\mathcal{K} \subset \Delta(L^2M)$, so in particular $\mathcal{H}_0 \subset \Delta(L^2M)$. Hence $\Delta^{-1}(\mathcal{H}_0) \subset L^2M$ is a finite dimensional $\{Ad(u_\gamma)\}_{\gamma \in \Gamma}$-invariant subspace. As the inclusion $\Gamma \leqslant H_0^{(I)} \rtimes \Gamma$ is icc, the representation $\{Ad(u_\gamma)\}_{\gamma \in \Gamma}$ on $L^2(M) \ominus \mathbb{C}1$ is weakly mixing, which further implies that $\mathcal{H} = \mathbb{C}1$, as claimed. Now we apply [43, Theorem 4.1] to deduce that there exists a unitary $w \in L(\Gamma \times \Gamma)$ such that $w\mathcal{G}w^* \subset \mathbb{T}(\Gamma \times \Gamma)$. By replacing w with wu, we may assume that $w\Delta(u_\gamma)w^* \in \mathbb{T}(\Gamma \times \Gamma)$ for all $\gamma \in \Gamma$. $\quad\square$

Theorem 4.4. *(Theorem E) Let H_0, Γ be icc torsion free groups such that H_0 has property (T) and Γ admits an infinite, almost normal subgroup $\Gamma_0 \leqslant \Gamma$ with relative property (T). Let $\Gamma \curvearrowright I$ be a transitive action on a countable set I satisfying the following conditions:*

a) There is a $k \in \mathbb{N}$ such that for each $J \subseteq I$ satisfying $|J| \geq k$ we have $|\mathrm{Stab}_\Gamma(J)| < \infty$;
b) For every $i \neq j$ in I we have that $|\mathrm{Stab}_\Gamma(i) \cdot j| = \infty$.

Denote by $G = H_0 \wr_I \Gamma$ the corresponding generalized wreath product. Let Λ be any torsion free group and let $\theta : L(G) \to L(\Lambda)$ be a $*$-isomorphism. Then Λ admits a wreath product decomposition $\Lambda = \Sigma_0 \wr_I \Psi$ satisfying the following properties: there exist a group isomorphism $\rho : \Gamma \to \Psi$, a character $\eta : \Gamma \to \mathbb{T}$, a $*$-isomorphism $\theta_0 : L(H_0) \to L(\Sigma_0)$ and a unitary $v \in L(\Lambda)$ such that for every $x \in L(H_0^{(I)})$ and $\gamma \in \Gamma$ we have

$$\theta(xu_\gamma) = \eta(\gamma)v^*\theta_0^{\bar{\otimes}I}(x)v_{\delta(\gamma)}v.$$

Here $\{u_\gamma \mid \gamma \in \Gamma\}$ and $\{v_\lambda \mid \lambda \in \Psi\}$ are the canonical unitaries of $L(\Gamma)$ and $L(\Psi)$, respectively.

Proof. Let $A = L(H_0^{(I)})$, and notice that $\theta(L(G)) = L(\Lambda) = M$. Using Proposition 4.3 one can find $w \in \mathcal{U}(M \bar{\otimes} M)$, group homomorphisms $\delta_i : \Gamma \to \Gamma$, and a character $\omega :$

$\Gamma \to \mathbb{T}$ such that $w\Delta(\theta(u_\gamma))w^* = \omega(\gamma)\theta(u_{\delta_1(\gamma)}) \otimes \theta(u_{\delta_2(\gamma)})$ for all $\gamma \in \Gamma$. Then applying verbatim Steps 4 and 5 in the proof of [38, Theorem 8.2] one can find an injective group homomorphism $\rho : \Gamma \to \Lambda$ and a character $\eta : \Gamma \to \mathbb{T}$ satisfying

$$\theta(u_\gamma) = \eta(\gamma)v_{\rho(\gamma)}, \text{ for all } \gamma \in \Gamma. \tag{4.1}$$

Denote by $\Psi = \rho(\Gamma)$. In addition, these proofs also show there is a $v \in \mathcal{U}(M)$ such that $w = (v^* \otimes v^*)\Delta(v)$. Henceforth the canonical unitaries $\theta(u_\gamma), \gamma \in \Gamma$ will be replaced by $v\theta(u_\gamma)v^*$ and A will be replaced by vAv^*. Under these conventions we prove that

Claim 4.5. $\Delta(\theta(A)) \subset \theta(A)\bar{\otimes}\theta(A)$.

Proof of Claim 4.5. By Proposition 4.2, $\Delta(\theta(A)) \prec^s \theta(A)\bar{\otimes}\theta(A)$. This means that for every $\epsilon > 0$, there exists a finite subset $e \in S \subset \theta(\Gamma)$ such that $\|d - P_{S\times S}(d)\|_2 \leq \epsilon$, for all $d \in \mathcal{U}(\Delta(\theta(A)))$. Here we have denoted by $P_{S\times S}$ the orthogonal projection from $L^2(M\bar{\otimes}M)$ onto the closure of the linear span of $a_1u_{s_1} \otimes a_2u_{s_2}$, where $a_1, a_2 \in A$ and $s_1, s_2 \in S$. But since, according to Proposition 4.3, $\Delta(\theta(A))$ is invariant to $Ad(\Delta(\theta(u_\gamma))) = Ad(\theta(u_\gamma)\otimes\theta(u_\gamma))$ for all $\gamma \in \Gamma$, we see that $\|d - P_{\mu S\mu^{-1}\times\mu S\mu^{-1}}(d)\|_2 \leq \epsilon$, for all $d \in \mathcal{U}(\Delta(\theta(A)))$ and $\mu \in \theta(\Gamma)$. As Γ is icc we can find $\mu \in \theta(\Gamma)$ such that $\mu S\mu^{-1} \cap S = \{e\}$ (see for instance [19, Proposition 3.4]). By the triangle inequality this further implies that

$$\|d - E_{\theta(A)\bar{\otimes}\theta(A)}(d)\|_2 = \|d - P_{(\mu S\mu^{-1}\cap S)\times(\mu S\mu^{-1}\cap S)}(d)\|_2 \leq 2\epsilon,$$

for all $d \in \mathcal{U}(\Delta(\theta(A)))$. As ϵ is arbitrary, this implies $\Delta(\theta(A)) \subset \theta(A)\bar{\otimes}\theta(A)$. $\quad\square$

From Claim 4.5 and [38, Lemma 7.1.2] it follows that $\theta(A) = L(\Sigma)$, for some $\Sigma < \Lambda$. Since the u_γ's normalize A, it follows that $\Psi \ni \rho(\gamma)$ normalizes Σ, for all γ. Consider the action of $\Psi \to Aut(\Sigma)$ given by $\Psi \ni \lambda \to Ad(\lambda) \in Aut(\Sigma)$ and observe that Λ splits as a semidirect product $\Lambda = \Sigma \rtimes \Psi$, because $L(\Lambda) = \theta(A) \rtimes \theta(\Gamma)$.

For the remaining part consider $A_0 = L(H_0)$ and denote by A_0^i the copy of A_0 in position $i \in I$. Next we show that

Claim 4.6. $\Delta(\theta(A_0^i)) \subset \theta(A_0^i)\bar{\otimes}\theta(A_0^i)$, for all $i \in I$.

Proof of Claim 4.6. Using (4.1) we note that $\Delta(\theta(A_0^i))$ is fixed by $Ad(\Delta(\theta(u_\gamma))) = Ad(\theta(u_\gamma) \otimes \theta(u_\gamma))$, for all $\gamma \in \mathrm{Stab}_\Gamma(i)$. Due to the assumption that $\mathrm{Stab}_\Gamma(i) \cdot j$ is infinite for all $i \neq j$, the representation $Ad\{\theta(u_\gamma) \otimes \theta(u_\gamma)\}_{\gamma\in\mathrm{Stab}_\Gamma(i)}$ is weakly mixing on $L^2(M\bar{\otimes}M) \ominus L^2(\theta(A_0^i)\bar{\otimes}\theta(A_0^i))$, so it follows that $\Delta(\theta(A_0^i)) \subset \theta(A_0^i)\bar{\otimes}\theta(A_0^i)$. $\quad\square$

Hence from Claim 4.6 and [38, Lemma 7.1.2] for every $i \in I$ there exists a subgroup $\Sigma_i < \Lambda$ such that $\theta(A_0^i) = L(\Sigma_i)$. Since the action $\Gamma \curvearrowright I$ is transitive, it follows that $\Sigma_i \cong \Sigma_0$ for all i, and then that $\Sigma = \bigoplus_I \Sigma_0$. Moreover, this entails that the action

$\Psi \to Aut(\Sigma) = Aut(\bigoplus_I \Sigma_i)$ is induced by the generalized Bernoulli action of $\Psi \curvearrowright I$ and hence $\Lambda = \Sigma_0 \wr_I \Gamma$. The rest of the statement follows from the previous observations. $\square$

5. Proof of Theorem C

Theorem 5.1. *Let H_0, Γ be icc, property (T) group. Also assume that $\Gamma = \Gamma_1 \times \Gamma_2$, where Γ_i are nonamenable biexact groups for all $i = 1, 2$. Let $\Gamma \curvearrowright I$ be an action on a countable set I satisfying the following conditions:*

a) The stabilizer $\mathrm{Stab}_\Gamma(i)$ is amenable for each $i \in I$;
b) There is $k \in \mathbb{N}$ such that for each $J \subseteq I$ satisfying $|J| \geq k$ we have $|\mathrm{Stab}_\Gamma(J)| < \infty$.

Denote by $G = H_0 \wr_I \Gamma$ the corresponding generalized wreath product. Let Λ be an arbitrary group and let $\theta : L(G) \to L(\Lambda)$ be a $*$-isomorphism. Then Λ admits a semidirect product decomposition $\Lambda = \Sigma \rtimes \Phi$ satisfying the following properties: there exist a group isomorphism $\delta : \Gamma \to \Phi$, a character $\zeta : \Gamma \to \mathbb{T}$, a $*$-isomorphism $\theta_0 : L(H_0^{(I)}) \to L(\Sigma)$ and a unitary $t \in L(\Lambda)$ such that for every $x \in L(H_0^{(I)})$ and $\gamma \in \Gamma$ we have

$$\theta(xu_\gamma) = \zeta(\gamma) t \theta_0(x) v_{\delta(\gamma)} t^*.$$

Here $\{u_g \, | \, \gamma \in \Gamma\}$ and $\{v_\lambda \, | \, \lambda \in \Phi\}$ are the canonical unitaries of $L(\Gamma)$ and $L(\Phi)$, respectively.

Proof. From assumptions we have that $\theta(L(G)) = L(\Lambda) = M$. Denote by $A_0 = \theta(L(H_0))$ and $A = \theta(L(H_0^I))$. Also to simplify the writing, throughout the proof we will identify Γ with $\theta(\Gamma)$, etc. Thus note that $M = A \rtimes \Gamma$, the action being given by generalized Bernoulli shifts. Consider $\Delta : M \to M \bar\otimes M$ the comultiplication along Λ. Note that $M \bar\otimes M = (A \bar\otimes A) \rtimes (\Gamma \times \Gamma)$. Proposition 4.2 implies that

(1) $\Delta(A) \prec^s_{M \bar\otimes M} A \bar\otimes A$, and
(2) there is $u \in \mathcal{U}(M \bar\otimes M)$ such that $u \Delta(L(\Gamma)) u^* \subseteq L(\Gamma \times \Gamma)$.

Next we show the following

Claim 5.2. *There exist a subgroup $\Phi < \Lambda$ with $QN_\Lambda^{(1)}(\Phi) = \Phi$, $d \in \mathcal{P}(L(\Phi))$ and $\mu \in \mathcal{U}(M)$ satisfying $h = \mu d \mu^* \in L(\Gamma)$ and $\mu d L(\Phi) d \mu^* = h L(\Gamma) h$.*

Proof of Claim 5.3. Let $\mathcal{K} := \{\Gamma \times \Gamma_1, \Gamma \times \Gamma_2, \Gamma_1 \times \Gamma, \Gamma_2 \times \Gamma\}$. Since by [8, Lemma 15.3.3] $\Gamma \times \Gamma$ is biexact relatively to $\mathcal{K}$ and $\Delta(L(\Gamma_1))$ and $\Delta(L(\Gamma_2))$ are commuting nonamenable factors then [8, Theorem 15.1.5] implies that there are $\Psi \in \mathcal{K}$ and $i = 1, 2$ so that $u \Delta(L(\Gamma_i)) u^* \prec_{L(\Gamma \times \Gamma)} L(\Psi)$. Since the flip automorphism of $M \bar\otimes M$ acts identically on $\Delta(L(\Gamma_i))$ we can assume without any loss of generality $\Psi = \Gamma \times \Gamma_1$ and $i = 1$. Hence

$$u\Delta(L(\Gamma_1))u^* \prec_{L(\Gamma \times \Gamma)} L(\Gamma \times \Gamma_1).$$

The using [23, Theorem 4.1] (see also [16, Theorem 3.3]) this further implies there exists a subgroup $\Sigma < \Lambda$ with non-amenable centralizer $\Upsilon := C_\Lambda(\Sigma)$ and $L(\Gamma_1) \prec_M L(\Sigma)$. Passing to the intertwining of the relative commutants we have that $L(\Upsilon) \subseteq L(\Sigma)' \cap M \prec_M L(\Gamma_1)' \cap M = L(\Gamma_2)$. Thus there are projections $e \in L(\Upsilon)$, $f \in L(\Gamma_2)$, a partial isometry $v \in M$, and an injective unital $*$-homomorphism $\phi : eL(\Upsilon)e \to fL(\Gamma_2)f$ such that

$$\phi(x)v = vx, \text{ for all } x \in eL(\Upsilon)e. \tag{5.1}$$

Denote by $T := \phi(eL(\Upsilon)e)$ and notice that $q' := vv^* \in T' \cap fMf$ and $p := v^*v \in eL(\Upsilon)e' \cap eMe = (L(\Upsilon)' \cap M)e$. Since T is a non-amenable factor then $a)$ implies that $T \nprec L(\mathrm{Stab}_\Gamma(i))$ for all i and using [54, Theorem 3.1] we have $QN_{fMf}(T)'' \subseteq L(\Gamma)$. In particular, $q' \in L(\Gamma)$ and by (5.1) there is $u \in \mathcal{U}(M\bar{\otimes}M)$ such that $ueL(\Upsilon)epu^* \subseteq L(\Gamma)$. Since $L(\Gamma)$ is a factor, the same argument from the proof of part (2) of [37, Theorem 5.1] shows that one can perturb u to a new unitary such that we further have

$$uL(\Upsilon)pu^* \subseteq L(\Gamma). \tag{5.2}$$

Since Υ is non-amenable then $uL(\Upsilon)pu^* \nprec L(\mathrm{Stab}_\Gamma(i))$ for all i and (5.2) combined with [54, Theorem 3.1] and the quasinormalizer formula show that $upL(QN_\Lambda(\Upsilon))''pu^* \subseteq QN_{upMpu^*}(uL(\Upsilon)pu^*)'' \subseteq L(\Gamma)$. Since $L(\Gamma)$ is a factor, the same argument as before further implies that $uL(QN_\Lambda(\Upsilon))z'u^* \subseteq L(\Gamma)$, where z' is the central support of p in $L(QN_\Lambda(\Upsilon))$. Notice $\Sigma \leqslant vC_\Lambda(\Upsilon) < QN_\Lambda(\Upsilon)$ and hence $uL(\Sigma)z'u^* \subseteq L(\Gamma)$. Letting $\Omega := vC_\Lambda(\Sigma)$, $\Theta := QN_\Lambda^{(1)}(\Sigma\Omega)$ and using the same arguments as before, we can further find $\eta \in \mathcal{U}(M\bar{\otimes}M)$ and a projection $z \in \mathcal{Z}(L(\Theta))$ such that

$$\eta L(\Theta)z\eta^* \subseteq L(\Gamma). \tag{5.3}$$

Since $\Upsilon, \Sigma < \Theta$ are commuting non-amenable groups and Γ is biexact relatively to $\{\Gamma_1, \Gamma_2\}$, [8, Theorem 15.1.5] implies that $\eta L(\Sigma)z\eta^* \prec_{L(\Gamma)} L(\Gamma_k)$, for some $k = 1, 2$. Again, without any loss of generality we can assume $k = 1$. Passing to the relative commutants intertwining we get

$$L(\Gamma_2) = L(\Gamma_1)' \cap L(\Gamma) \prec_{L(\Gamma)} (\eta L(\Sigma)z\eta^*)' \cap \eta z\eta^* L(\Gamma)\eta z\eta^* \subseteq \eta L(\Omega)z\eta^*. \tag{5.4}$$

Now let $\{\mathcal{O}_k\}_k$ be a countable enumeration of all the finite orbits under conjugation by Σ and notice that $\cup_k \mathcal{O}_k = \Omega$. Consider $\Omega_k := \langle \mathcal{O}_1, ..., \mathcal{O}_k \rangle \leqslant \Lambda$ and note that $\Omega_k \nearrow \Omega$. Since $L(\Gamma_2)$ has property (T) then (5.4) implies that

$$L(\Gamma_2) \prec_{L(\Gamma)} \eta L(\Omega_k)z\eta^* \text{ for some } k. \tag{5.5}$$

Next we continue by iterating the same argument from page 11. For convenience we include all the details. By [13, Proposition 2.4], there exist nonzero projections $a \in L(\Gamma_2), q \in L(\Omega_k)$, a partial isometry $w \in L(\Gamma)$, a subalgebra $D \subseteq \eta q L(\Omega_k) q z \eta^*$, and a *-isomorphism $\psi : aL(\Gamma_2)a \to D$ such that

$$D \vee (D' \cap \eta q L(\Omega_k) q z \eta^*) \subseteq \eta q L(\Omega_k) q z \eta^* \quad \text{has finite index, and} \qquad (5.6)$$

$$\psi(x)w = wx \quad \forall x \in aL(\Gamma_2)a. \qquad (5.7)$$

Let $r = \eta q z \eta^*$ and notice that $ww^* \in D' \cap rL(\Gamma)r$ and $w^*w \in (aL(\Gamma_2)a)' \cap aL(\Gamma)a = L(\Gamma_1)\bar{\otimes}\mathbb{C}a$. Hence there is a projection $b \in L(\Gamma_1)$ satisfying $w^*w = b \otimes a$. Picking $c \in \mathcal{U}(L(\Gamma))$ so that $w = c(b \otimes a)$ then (5.7) gives

$$Dww^* = wL(\Gamma_2)w^* = c(\mathbb{C}b \otimes aL(\Gamma_2)a)c^*. \qquad (5.8)$$

Passing to the relative commutants, we obtain $ww^*(D' \cap rL(\Gamma)r)ww^* = c(bL(\Gamma_1)b \otimes \mathbb{C}a)c^*$. Hence there exist $s_1, s_2 > 0$ satisfying

$$(D' \cap rL(\Gamma)r)y = c(bL(\Gamma_1)b \otimes \mathbb{C}a)^{s_2}c^* \cong L(\Gamma_1)^{s_1}, \qquad (5.9)$$

where y is the central support projection of ww^* in $D' \cap rL(\Gamma)r$. Notice

$$D' \cap rL(\Gamma)r \supseteq (\eta q L(\Omega_k) q z \eta^*)' \cap rL(\Gamma)r = \eta(L(\Omega_k)' \cap L(\Theta))q z \eta^* \supseteq \eta L(C_\Sigma(\Omega_k)) q z \eta^*.$$

From the definition of Ω_k it follows that $[\Sigma : C_\Sigma(\Omega_k)] < \infty$. Since Σ is non-amenable it follows that $C_\Sigma(\Omega_k)$ is also nonamenable and hence $\eta L(C_\Sigma(\Omega_k)) q z \eta^*$ has no amenable direct summand. Moreover we also have $D' \cap rL(\Gamma)r \supseteq D' \cap \eta q L(\Omega_k) q z \eta^*$. In conclusion $(\eta L(C_\Sigma(\Omega_k)) q z \eta^*)y$ and $(D' \cap \eta q L(\Omega_k) q z \eta^*)y$ are commuting subalgebras of $(D' \cap rL(\Gamma)r)y$ where $(\eta L(C_\Sigma(\Omega_k)) q z \eta^*)y$ has no amenable direct summand. Since Γ_i was assumed to be biexact, then using (5.9) and [45, Theorem 1] it follows that $(D' \cap \eta q L(\Omega_k) q z \eta^*)y$ is purely atomic. Thus, cutting by a central projection $r' \in D' \cap \eta q L(\Omega_k) q z \eta^*$ and using (5.6) we may assume that $D \subseteq \eta q L(\Omega_k) q z \eta^*$ is a finite index inclusion of algebras. Proceeding as in the second part of [16, Claim 4.4], we may assume that $D \subseteq \eta q L(\Omega_k) q z \eta^*$ is a finite index inclusion of II_1 factors. Moreover, one can check that if one replaces w by the partial isometry of the polar decomposition of $r'w \neq 0$ then all relations (5.7), (5.8) and (5.9) are still satisfied.

Using relation (5.8), the quasinormalizer compression formula, and the fact that $D \subseteq \eta q L(\Omega_k) q z \eta^*$ is a finite index inclusion of II_1 factors we can see that

$$\begin{aligned}
c(b \otimes a)L(\Gamma)(b \otimes a)c^* &= QN_{c(b \otimes a)M(b \otimes a)c^*}(c(\mathbb{C}b \otimes (aL(\Gamma_2)a))c^*)'' \\
&= QN_{ww^*Mww^*}(Dww^*)'' \\
&= ww^*QN_{rMr}(D)''ww^* \\
&= ww^*QN_{\eta q z M q z \eta^*}(\eta q L(\Omega_k) q z \eta^*)''ww^* \\
&= ww^*\eta q z QN_{L(\Lambda)}(L(\Omega_k))''q z \eta ww^*.
\end{aligned}$$

Letting $\Xi = QN_\Lambda(\Omega_k)$, then the previous relation and formula [24, Corollary 5.2] imply that $c(b \otimes a)L(\Gamma)(b \otimes a)c^* = ww^*\eta L(\Xi)\eta^*ww^*$. Since $QN_G^{(1)}(\Gamma) = \Gamma$, this formula and [24, Corollary 5.2] further imply that

$$
\begin{aligned}
ww^*\eta L(\Xi)\eta^*ww^* &= c(b \otimes a)L(\Gamma)(b \otimes a)c^* \\
&= QN_{c(b \otimes a)M(b \otimes a)c^*}^{(1)}(c(b \otimes a)L(\Gamma)(b \otimes a)c^*)'' \\
&= QN_{ww^*\eta M\eta^*ww^*}^{(1)}(ww^*\eta L(\Xi)\eta^*ww^*)'' \\
&= ww^*\eta L(\Phi)\eta^*ww^*,
\end{aligned}
\tag{5.10}
$$

where $\Phi = \langle QN_\Lambda^{(1)}(\Xi)\rangle$. Hence in particular we have $ww^*\eta L(\Xi)\eta^*ww^* = ww^*\eta L(\Phi)\eta^*ww^*$ and by [16, Proposition 2.6] it follows that $[\Phi : \Xi] < \infty$. This entails that $\Phi = QN_\Lambda^{(1)}(\Xi) = QN_\Lambda^{(1)}(\Phi)$. Note the above relations imply that $ww^* \in \eta L(\Xi)\eta^* \subseteq \eta L(\Phi)\eta^*$. Consider $d \in \mathcal{P}(L(\Phi))$ such that $ww^* = \eta d\eta^*$ and letting $\mu := c^*\eta$ and $h := b \otimes a$ then relation (5.10) gives the desired conclusion. $\square$

Claim 5.3. *There exists a unitary $w \in M$ such that $wL(\Phi)w^* = L(\Gamma)$.*

Proof of Claim 5.3. From Claim 5.2, there exists $\Phi \leqslant \Lambda$ with $\Phi = QN_\Lambda^{(1)}(\Phi), d \in \mathcal{P}(L(\Phi))$ and $\mu \in \mathcal{U}(M)$ satisfying $h = \mu d\mu^* \in L(\Gamma)$ and

$$
\mu dL(\Phi)d\mu^* = hL(\Gamma)h.
\tag{5.11}
$$

As Γ has property (T), (5.11) implies that $dL(\Phi)d$ is a property (T) von Neumann algebra. By [12, Lemma 2.13] it follows that Φ is a property (T) group. Fix $r \in \mathcal{P}((\mu L(\Phi)\mu^*)' \cap M)$ and note that $\mu L(\Phi)\mu^*r$ is a property (T) von Neumann algebra. Thus, using [31, Theorem 0.1] we have that either

(1) $\mu L(\Phi)\mu^*r \prec_M L(\Gamma)$, or
(2) $\mu L(\Phi)\mu^*r \prec_M L(H^F)$ for some finite $F \subset I$.

If (2) would hold then we would have that $L(H^{I \backslash F}) = L(H^F)' \cap M \prec_M (\mu L(\Phi)\mu^*r)' \cap rMr = r\mu(L(\Phi)' \cap M)\mu^*r$. On the other hand since $QN_\Lambda^{(1)}(\Phi) = \Phi$ we have $\mu(L(\Phi)' \cap M)\mu^* = \mathcal{Z}(\mu L(\Phi)\mu^*)$. Altogether, these would show that $H^{I \backslash F}$ is amenable and hence H is amenable, a contradiction. So (1) must hold for every $r \in \mathcal{P}((\mu L(\Phi)\mu^*)' \cap M)$. This entails that $\mu L(\Phi)\mu^* \prec_M^s L(\Gamma)$ and using (5.11) and [12, Lemma 2.6] one can find $w \in \mathcal{U}(M)$ such that $wL(\Phi)w^* = L(\Gamma)$. $\square$

Next consider the subgroup $\mathcal{G} = \{u\Delta(u_\gamma)u^* \mid \gamma \in \Gamma\} \leqslant \mathcal{U}(L(\Gamma \times \Gamma))$. Since $\mathcal{G}$ normalizes $u\Delta(A)u^*$, which by (1) satisfies $u\Delta(A)u^* \prec^s A \bar{\otimes} A$, the argument in Step 5 in the proof of [38, Theorem 5.1] implies that $h_{\Gamma \times \Gamma}(\mathcal{G}) > 0$. Then using Lemmas 2.4-2.5 we further have that $h_\Lambda(\Gamma) > 0$. Using Lemma 2.4 we get $h_\Lambda(w^*\Gamma w) > 0$ and by Claim 5.3 we

further conclude that $h_\Phi(w^*\Gamma w) > 0$. Thus by [38, Theorem 3.1] one can find $t \in \mathcal{U}(M)$, a character $\zeta : \Gamma \to \mathbb{T}$ and a group isomorphism $\delta : \Gamma \to \Phi$ such that

$$tu_\gamma t^* = \overline{\zeta(\gamma)} v_{\delta(\gamma)}, \text{ for all } \gamma \in \Gamma. \tag{5.12}$$

Letting $\Omega := (t^* \otimes t^*)\Delta(t) \in \mathcal{U}(M\bar{\otimes}M)$ we then have $\Omega\Delta(u_\gamma)\Omega^* = \zeta(\gamma)(u_\gamma \otimes u_\gamma)$ for all $\gamma \in \Gamma$. Next, we replace the canonical unitaries $u_\gamma, \gamma \in \Gamma$ by $tu_\gamma t^*$ and A by tAt^*. Then (2) combined with the argument from the proof of Claim 4.5 in Theorem 4.4 further shows that $\Delta(A) \subset A\bar{\otimes}A$. Hence using [38, Lemma 7.1.2] there exists a subgroup $\Sigma < \Lambda$ such that $A = L(\Sigma)$. Since the u_γ's normalize A, it follows that $v_{\delta(\gamma)}$ normalizes Σ, for all γ. Moreover, since $L(\Lambda) = A \rtimes \Gamma$, then Λ admits a semidirect product decomposition $\Lambda = \Sigma \rtimes \Phi$. Altogether the previous considerations give the conclusion of the theorem. $\square$

Next we prove the following result that in particular generalizes Theorem C.

Theorem 5.4. *Let H be icc, weakly amenable, biexact property (T) group. Let $\Gamma = \Gamma_1 \times \Gamma_2$, where Γ_i are icc, biexact, property (T) group. Assume that $\Gamma \curvearrowright I$ is an action on a countable infinite set I that satisfies the following properties:*

a) The stabilizer $\text{Stab}_\Gamma(i)$ is amenable for each $i \in I$;
b) There is $k \in \mathbb{N}$ such that for each $J \subseteq I$ satisfying $|J| \geq k$ we have $|\text{Stab}_\Gamma(J)| < \infty$.

Let $G = H \wr_I \Gamma$ be the corresponding generalized wreath product. Let Λ be an arbitrary group and let $\theta : L(G) \to L(\Lambda)$ be a $*$-isomorphism. Then one can find non-amenable icc groups Σ_0, Ψ, an amenable icc group A, and an action $\Psi \curvearrowright^\alpha A$ such that we can decompose Λ as semidirect product $\Lambda = (\Sigma_0^{(I)} \oplus A) \rtimes_{\beta \oplus \alpha} \Psi$, where $\Psi \curvearrowright^\beta \Sigma_0^{(I)}$ is the generalized Bernoulli action. In addition, there exist a group isomorphism $\delta : \Gamma \to \Psi$, a character $\eta : \Gamma \to \mathbb{T}$, a $*$-isomorphism $\theta_0 : L(H^{(I)}) \to L(\Sigma_0^{(I)} \oplus A)$ and $u \in \mathcal{U}(L(\Lambda))$ so that for every $x \in L(H^{(I)})$ and $\gamma \in \Gamma$ we have

$$\theta(xu_\gamma) = \eta(\gamma)u\theta_0(x)v_{\delta(\gamma)}u^*.$$

Here $\{u_g \mid \gamma \in \Gamma\}$ and $\{v_\lambda \mid \lambda \in \Psi\}$ are the canonical unitaries of $L(\Gamma)$ and $L(\Psi)$, respectively.

Proof. Let $G = H \wr_I \Gamma$ satisfy the conditions stated in the Theorem 5.1. Let Λ be an arbitrary group and assume that $\theta : L(G) \to L(\Lambda)$ is an $*$-isomorphism. Using Theorem 5.1, after composing θ with an inner automorphism of M one can find a semidirect product decomposition of $\Lambda = \Sigma \rtimes_\beta \Psi$, a group isomorphism $\delta : \Gamma \to \Psi$, and a character $\eta : \Gamma \to \mathbb{T}$ such that

$$\theta(u_\gamma) = \eta(\gamma)v_{\delta(\gamma)} \text{ for all } \gamma \in \Gamma. \tag{5.13}$$

Moreover, we have $\theta(L(\Sigma)) = L(H^{(I)})$. Since H are icc, biexact, weakly amenable, property (T) groups then Theorem A implies that one can decompose $\Sigma = \oplus_{i \in I} \Sigma_i \oplus A$, where A is trivial or amenable icc. In addition, for every finite subset $F \subset I$ there exist $u \in \mathcal{U}(L(\Lambda))$ and scalars $t_i > 0$ for $i \in F$ such that

$$uL(\Sigma_i)^{t_i} u^* = \theta(L(H_i)) \text{ for all } i \in F, \text{ and}$$
$$uL(\oplus_{i \in I \setminus F} \Sigma_i \oplus A)^{\prod_{i \in F} t_i^{-1}} u^* = \theta(L(H_{I \setminus F})).$$
(5.14)

Next we show that $\Sigma_i \cong \Sigma_0$ for all i and there is an action $\Psi \curvearrowright^\alpha A$ such that $\Lambda = (\Sigma^{(I)} \oplus A) \rtimes_{b \oplus \alpha} \Psi$, where $\Psi \curvearrowright^b \Sigma^{(I)}$ is the generalized Bernoulli action induced by $\Gamma \curvearrowright I$. Fix $i, j \in I$ and $\gamma \in \Gamma$ such that $\gamma i = j$. Let $F \subset I$ be a finite set such that $\{i, j\} \subseteq F$. Using the first relation of (5.14) for i and j in combination with (5.13) we get

$$uL(\Sigma_j)^{t_j} u^* = \theta(L(H_j)) = \theta(u_\gamma) uL(\Sigma_i)^{t_i} u^* \theta(u_\gamma^*)$$
$$= v_{\delta(\gamma)} u v_{\delta(\gamma)}^* L(\beta_{\delta(\gamma)}(\Sigma_i))^{t_i} v_{\delta(\gamma)} u^* v_{\delta(\gamma)}^*.$$

In particular this relation implies that $L(\Sigma_j) \prec_M L(\beta_{\delta(\gamma)}(\Sigma_i))$ and $L(\beta_{\delta(\gamma)}(\Sigma_i)) \prec_M L(\Sigma_j)$. Since $\Sigma_j, \beta_{\delta(\gamma)}(\Sigma_i)$ are normal subgroups of Λ these intertwinings combined with [12, Lemma 2.2] imply that Σ_j is commensurable with $\beta_{\delta(\gamma)}(\Sigma_i)$; in other words

$$[\Sigma_j : \Sigma_j \cap \beta_{\delta(\gamma)}(\Sigma_i)] < \infty \text{ and } [\beta_{\delta(\gamma)}(\Sigma_i) : \Sigma_j \cap \beta_{\delta(\gamma)}(\Sigma_i)] < \infty. \qquad (5.15)$$

Since $\beta_{\delta(\gamma)}(\Sigma_i) \leqslant \oplus_{i \in I} \Sigma_i \oplus A$ using the second relation in (5.15) there exists a finite subset $j \in J \subset I$ so that $\beta_{\delta(\gamma)}(\Sigma_i) \leqslant \Sigma^J \oplus A$. Thus we have the following normal subgroups $\Sigma_j \cap \beta_{\delta(\gamma)}(\Sigma_i) \lhd \beta_{\delta(\gamma)}(\Sigma_i) \lhd \Sigma^J \oplus A$. Taking the quotient we get a finite normal subgroup

$$\beta_{\delta(\gamma)}(\Sigma_i)/\Sigma_j \cap \beta_{\delta(\gamma)}(\Sigma_i) \lhd (\Sigma^J \oplus A)/\Sigma_j \cap \beta_{\delta(\gamma)}(\Sigma_i) = \Sigma_j/\Sigma_j \cap \beta_{\delta(\gamma)}(\Sigma_i) \oplus \Sigma_{J \setminus \{j\}} \oplus A.$$

Hence $\beta_{\delta(\gamma)}(\Sigma_i)/\Sigma_j \cap \beta_{\delta(\gamma)}(\Sigma_i) \lhd vZ(\Sigma_j/\Sigma_j \cap \beta_{\delta(\gamma)}(\Sigma_i) \oplus \Sigma_{J \setminus \{j\}} \oplus A)$. However, since $\Sigma_{J \setminus \{j\}} \oplus A)$ is icc by (5.15) we have $vZ(\Sigma_j/\Sigma_j \cap \beta_{\delta(\gamma)}(\Sigma_i) \oplus \Sigma_{J \setminus \{j\}} \oplus A) = \Sigma_j/\Sigma_j \cap \beta_{\delta(\gamma)}(\Sigma_i)$. Altogether, these relations show that $\beta_{\delta(\gamma)}(\Sigma_i)/\Sigma_j \cap \beta_{\delta(\gamma)}(\Sigma_i) \lhd \Sigma_j/\Sigma_j \cap \beta_{\delta(\gamma)}(\Sigma_i)$ and hence $\beta_{\delta(\gamma)}(\Sigma_i) \leqslant \Sigma_j$. Similarly one can show that $\beta_{\delta(\gamma)}(\Sigma_i) \geqslant \Sigma_j$ and hence $\beta_{\delta(\gamma)}(\Sigma_i) = \Sigma_j$. This shows that $\Sigma_i \cong \Sigma_0$ for all i. Moreover, there is an action $\Psi \curvearrowright I$ which induces a generalized Bernoulli action $\Psi \curvearrowright^b \oplus_{i \in I} \Sigma_i$. Also since the action of $\Psi \curvearrowright^\beta \Sigma$ leaves the subgroup $\oplus_I \Sigma_i$ invariant then Ψ will also leave invariant $C_\Sigma(\oplus_I \Sigma_i) = A$. Hence there is an action $\Psi \curvearrowright^\alpha A$ such that $\Lambda = (\Sigma_0^{(I)} \oplus A) \rtimes_{b \oplus \alpha} \Psi$. The remaining part of the statement follows directly from the above considerations. □

Proof of Corollary D. This follows proceeding in the same manner as in the proof of Corollary B and using Theorem 5.4. □

Remarks 5.5. When considering generalized Bernoulli actions it is clear the conditions presented in the statements of Theorems 5.1, 5.4 are satisfied when all the stabilizers of action $\Gamma \curvearrowright I$ are finite.

On the other hand, if one wants to tackle the infinite amenable stabilizers situation, producing examples seems far more challenging. In this direction we would like to present a possible approach for this which was suggested to us by Professor Denis Osin during the AIM workshop "Classification of group von Neumann algebras". Consider Σ_0 an icc finitely generated amenable group. By [1, Theorem 1.2] there exists an icc supragroup $\Sigma_0 < \Gamma_0$ that has property (T) and is hyperbolic relatively to Σ_0. Hence by [46] it follows that Γ_0 is biexact. Let $\Gamma = \Gamma_0 \times \Gamma_0$ and consider the diagonal subgroup $\Sigma = diag(\Sigma_0) < \Gamma$. Also let $\Gamma \curvearrowright I = \Gamma/\Sigma$ be the action by left multiplication on the right cosets Γ/Σ. Since $\Sigma_0 < \Gamma_0$ is icc and almost malnormal it follows that the one-sided quasinormalizer satisfies $QN_\Gamma^{(1)}(\Sigma) = \Sigma$. In turn this is equivalent with condition c) in Theorem 4.4. Finally, one can check that condition b) in Theorems 5.1, 5.4 is equivalent with the property that the group Σ has finite height in Γ (or it is almost malnormal). This is equivalent to the following property: there exists $k \in \mathbb{N}$ such that any subset $F < \Sigma_0$ with $|F| \geq k$ has finite centralizer $C_{\Gamma_0}(F)$. While finitely generated groups like this exist in general (e.g. monster groups) it is unclear if one can construct amenable examples.

In any case a possible positive answer to this last group theoretic question would lead to a class of generalized wreath products constructions with non-amenable core that are recognizable from the von Neumann algebraic setting. Indeed, Proposition 4.2 together with the argument from the proof of Claim 4.6 in Theorem 4.4 give the following

Corollary 5.6. *Let H_0, Γ be icc, property (T) groups. Also assume that $\Gamma = \Gamma_1 \times \Gamma_2$, where Γ_i are nonamenable biexact groups for all $i = 1, 2$. Let $\Gamma \curvearrowright I$ be an action on a countable set I satisfying the following conditions:*

a) The stabilizer $\mathrm{Stab}_\Gamma(i)$ is amenable for each $i \in I$;
b) There is $k \in \mathbb{N}$ such that for each $J \subseteq I$ satisfying $|J| \geq k$ we have $|\mathrm{Stab}_\Gamma(J)| < \infty$.
c) The orbit $\mathrm{Stab}_\Gamma(i) \cdot j$ is infinite for all $i \neq j$.

Denote by $G = H_0 \wr_I \Gamma$ the corresponding generalized wreath product. Let Λ be any torsion free group and let $\theta : L(G) \to L(\Lambda)$ be a $*$-isomorphism. Then Λ admits a wreath product decomposition $\Lambda = \Sigma_0 \wr_I \Psi$ satisfying all the properties enumerated in a)-c). In addition there exist a group isomorphism $\rho : \Gamma \to \Psi$, a character $\eta : \Gamma \to \mathbb{T}$, a $*$-isomorphism $\theta_0 : L(H_0) \to L(\Sigma_0)$ and a unitary $v \in L(\Lambda)$ such that for every $x \in L(H_0^{(I)})$ and $\gamma \in \Gamma$ we have

$$\theta(xu_\gamma) = \eta(\gamma)v^*\theta_0^{\overline{\otimes}I}(x)v_{\delta(\gamma)}v.$$

Here $\{u_\gamma \,|\, \gamma \in \Gamma\}$ and $\{v_\lambda \,|\, \lambda \in \Psi\}$ are the canonical group unitaries of $L(\Gamma)$ and $L(\Psi)$, respectively.

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