Algebra & Number Theory

Volume 14
2020
No. 4

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ALGEBRA AND NUMBER THEORY 14:4 (2020)
dx.doi.org/10.2140/ant.2020.14.961

On the Ekedahl–Oort stratification of Shimura curves

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We study the Hodge–Tate period domain associated to a quaternionic Shimura curve at a prime of bad reduction, and give an explicit description of its Ekedahl–Oort stratification.

1. Introduction

Fix a prime p, and let C be the completion of an algebraic closure of $\mathbb{Q}_p$. Denote by $\mathcal{O} \subset C$ its ring of integers, and by $k = \mathcal{O}/\mathfrak{m}$ its residue field.

1.1. ***Stratifications of p-adic periods domains.*** Let G be a p-divisible group over $\mathcal{O}$. It has a p-adic Tate module

$$T_p(G) = \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, G)$$

and a module of invariant differential forms $\Omega(G)$. These are free of finite rank over $\mathbb{Z}_p$ and $\mathcal{O}$, respectively. Using the canonical trivialization $\Omega(\mu_{p^\infty}) \cong \mathcal{O}$, we define the *Hodge–Tate morphism*

$$T_p(G) \cong \mathrm{Hom}(G^\vee, \mu_{p^\infty}) \xrightarrow{\mathrm{HT}} \mathrm{Hom}(\Omega(\mu_{p^\infty}), \Omega(G^\vee)) \cong \Omega(G^\vee), \tag{1.1.1}$$

where $G^\vee$ is the p-divisible group dual to G.

Theorem A [Scholze and Weinstein 2013]. *There is an equivalence between the category of p-divisible groups over $\mathcal{O}$ and the category of pairs (T, W) in which*

- *T is a free $\mathbb{Z}_p$-module of finite rank,*
- *$W \subset T \otimes_{\mathbb{Z}_p} C$ is a C-subspace.*

The equivalence sends G to its p-adic Tate module $T = T_p(G)$, endowed with its Hodge–Tate filtration

$$W = \ker\big(T_p(G) \otimes_{\mathbb{Z}_p} C \xrightarrow{\mathrm{HT}} \Omega(G^\vee) \otimes_{\mathcal{O}} C\big).$$

Fix a free $\mathbb{Z}_p$-module T of finite rank, and consider the $\mathbb{Q}_p$-scheme

$$X = \mathrm{Gr}_d(T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$$

parametrizing subspaces of $T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ of some fixed dimension $d \le \mathrm{rank}(T)$. By the theorem of Scholze–Weinstein, every point $W \in X(C)$ determines a p-divisible group G over $\mathcal{O}$, whose reduction to the residue field we denote by G_k. Let $G_k[p]$ be the group scheme of p-torsion points in G_k.

This research was supported in part by NSF grants DMS1501583 and DMS1801905.
MSC2010: primary 11G18; secondary 14G35.
Keywords: Shimura varieties, Ekedahl–Oort stratification.

If we declare two points $W, W' \in X(C)$ to be equivalent when the corresponding reductions G_k and G'_k are isogenous, the resulting partition is the *Newton stratification* of $X(C)$. Alternatively, if we declare $W, W' \in X(C)$ to be equivalent when the p-torsion group schemes $G_k[p]$ and $G'_k[p]$ are isomorphic, the resulting partition is the *Ekedahl–Oort stratification* of $X(C)$.

There are similar partitions when X is replaced by a more sophisticated flag variety, called the *Hodge–Tate period domain*, associated to a Shimura datum of Hodge type and a prime p. This period domain and its Newton stratification were studied by Caraiani and Scholze [2017], who proved that each Newton stratum in $X(C)$ can be realized as the C-points of a locally closed subset of the associated adic space. For the Ekedahl–Oort stratification of $X(C)$ there is nothing in the existing literature, and it is not known if it has any structure other than set-theoretic partition.

In the case of modular curves, the Hodge–Tate period domain is the projective line $\mathbb{P}^1$ over $\mathbb{Q}_p$. In this case the Newton stratification and the Ekedahl–Oort stratification agree, and there are two strata: the *ordinary locus* $\mathbb{P}^1(\mathbb{Q}_p)$, and the *supersingular locus* $\mathbb{P}^1(C) \smallsetminus \mathbb{P}^1(\mathbb{Q}_p)$.

For the compact Shimura curve determined by an indefinite quaternion algebra over $\mathbb{Q}$, and a prime p at which the quaternion algebra is ramified, the Hodge–Tate period domain X is a twisted form of $\mathbb{P}^1$. All points of $X(C)$ give rise to supersingular p-divisible groups over k, and the Newton stratification consists of a single stratum, $X(C)$ itself. In contrast, the Ekedahl–Oort stratification is nontrivial, and the goal of this paper is to make it explicit.

Although the methods used here are fairly direct, it is not clear how far they can be extended. The case of Hilbert modular surfaces may already require new ideas.

For background on the classical Ekedahl–Oort stratification of reductions of Shimura varieties (as opposed to their Hodge–Tate period domains), we refer the reader to [Oort 2001; Moonen 2004; Viehmann and Wedhorn 2013; Zhang 2018].

1.2. ***The Shimura curve period domain.*** Let $\mathbb{Q}_{p^2} \subset C$ be the unique unramified quadratic extension of $\mathbb{Q}_p$, and let $\mathbb{Z}_{p^2} \subset \mathcal{O}$ be its ring of integers. Denote by $x \mapsto \bar{x}$ the nontrivial automorphism of $\mathbb{Q}_{p^2}$. Define a noncommutative $\mathbb{Z}_p$-algebra of rank 4 by

$$\Delta = \mathbb{Z}_{p^2}[\Pi],$$

where Π is subject to the relations $\Pi^2 = p$ and $\Pi \cdot x = \bar{x} \cdot \Pi$ for all $x \in \mathbb{Z}_{p^2}$. In other words, Δ is the unique maximal order in the unique quaternion division algebra over $\mathbb{Q}_p$.

Let T be a free Δ-module of rank one, and let X be the smooth projective variety over $\mathbb{Q}_p$ with functor of points

$$X(S) = \{\mathcal{O}_S\text{-module local direct summands } W \subset T \otimes_{\mathbb{Z}_p} \mathcal{O}_S \text{ of rank 2 that are stable under } \Delta\} \quad (1.2.1)$$

for any $\mathbb{Q}_p$-scheme S. This is the Hodge–Tate period domain associated to a quaternionic Shimura curve.

As we explain in Section 4.1, our period domain becomes isomorphic to the projective line after base change to $\mathbb{Q}_{p^2}$, and any choice of Δ-module generator $\lambda \in T$ determines a bijection

$$X(C) \cong C \cup \{\infty\}. \quad (1.2.2)$$

After fixing such a choice, we normalize the valuation $\mathrm{ord} : C \to \mathbb{R} \cup \{\infty\}$ by $\mathrm{ord}(p) = 1$, extend it to $C \cup \{\infty\}$ by $\mathrm{ord}(\infty) = -\infty$, and use (1.2.2) to view ord as a function

$$\mathrm{ord} : X(C) \to \mathbb{R} \cup \{-\infty, \infty\}.$$

The theorem of Scholze–Weinstein provides a canonical bijection

$$X(C) \cong \left\{ \begin{array}{c} \text{isomorphism classes of } p\text{-divisible groups } G \text{ over } \mathcal{O} \text{ of height 4 and dimension 2,} \\ \text{endowed with an action of } \Delta \text{ and a } \Delta\text{-linear isomorphism } T_p(G) \cong T \end{array} \right\}.$$

By forgetting the level structure $T_p(G) \cong T$, reducing to the residue field, and then taking p-torsion subgroups, we obtain a function

$$X(C) \to \{\text{isomorphism classes of finite group schemes over } k\text{, endowed with an action of } \Delta/p\Delta\}$$

sending $G \mapsto G_k[p]$, whose fibers are the *Ekedahl–Oort strata* of $X(C)$.

Hypothesis. For the rest of this introduction, we assume $p > 2$. Theorems B and C below are presumably true without this hypothesis, but we are unable to provide a proof. See the remarks following Theorem 2.2.2.

It is convenient to organize the strata into two types: those on which the p-torsion group scheme $G_k[p]$ is superspecial (in the sense of Section 3.2), and those on which it is not. The two theorems that follow show that there are three superspecial strata, and two infinite families of nonsuperspecial strata. These results are proved in Section 4.2, where the reader will also find an explicit recipe for computing the Dieudonné module of the p-torsion group scheme $G_k[p]$ attached to a point of $X(C)$.

Theorem B. *The conditions*

$$\frac{1}{p+1} < \mathrm{ord}(\tau) < \frac{p}{p+1}$$

on $\tau \in X(C)$ define an Ekedahl–Oort stratum, as do each one of

$$\mathrm{ord}(\tau) < \frac{1}{p+1}, \qquad \frac{p}{p+1} < \mathrm{ord}(\tau).$$

The union of these three strata is the locus of points with superspecial reduction. In particular, the isomorphism class of the finite group scheme $G_k[p]$ is the same all on three strata, but the isomorphism class of $G_k[p]$ with its Δ-action is not.

Now consider the locus of points

$$\left\{\tau \in C : \mathrm{ord}(\tau) = \frac{1}{p+1}\right\} \cup \left\{\tau \in C : \mathrm{ord}(\tau) = \frac{p}{p+1}\right\} \subset X(C) \tag{1.2.3}$$

at which the corresponding p-divisible group does not have superspecial reduction. The isomorphism class of the p-torsion group scheme $G_k[p]$ is constant on (1.2.3), but the isomorphism class of $G_k[p]$ with its Δ-action varies. In fact, the Δ-action varies so much that (1.2.3) decomposes as an infinite disjoint union of Ekedahl–Oort strata.

Theorem C. *The fibers of the composition*

$$\left\{\tau \in C : \mathrm{ord}(\tau) = \frac{1}{p+1}\right\} \xrightarrow{\tau \mapsto p/\tau^{p+1}} \mathcal{O}^\times \to k^\times$$

are Ekedahl–Oort strata, as are the fibers of the composition

$$\left\{\tau \in C : \mathrm{ord}(\tau) = \frac{p}{p+1}\right\} \xrightarrow{\tau \mapsto \tau^{p+1}/p^p} \mathcal{O}^\times \to k^\times.$$

Both unlabeled arrows are reduction to the residue field.

Remark 1.2.1. The infinitude of Ekedahl–Oort strata is a pathology arising from the nonsmooth reduction of compact Shimura curves. Similar pathologies for the reductions of Hilbert modular varieties at ramified primes are described in the appendix to [Andreatta and Goren 2003].

1.3. ***Notation and conventions.*** Throughout the paper p is a fixed prime. We allow $p = 2$ unless otherwise stated. Let $k = \mathcal{O}/\mathfrak{m}$ as above, and denote by $\sigma : k \to k$ the absolute Frobenius $\sigma(x) = x^p$.

The rings $\mathbb{Z}_{p^2} \subset \mathcal{O}$ and $\Delta = \mathbb{Z}_{p^2}[\Pi]$ have the same meaning as above. We label the embeddings

$$j_0, j_1 : \mathbb{Z}_{p^2} \to \mathcal{O} \tag{1.3.1}$$

in such a way that j_0 is the inclusion and $j_1(x) = j_0(\bar{x})$ is its conjugate.

2. Integral p-adic Hodge theory

In this section we recall the integral p-adic Hodge theory of an arbitrary p-divisible group G over $\mathcal{O}$. The quaternion order Δ plays no role whatsoever.

Following [Fargues 2015; Lau 2018; Scholze and Weinstein 2020], we will attach to G a Breuil–Kisin–Fargues module, and explain how to extract from it invariants of G such as its Hodge–Tate morphism $T_p(G) \to \Omega(G^\vee)$, and the Dieudonné module of its reduction to k.

2.1. ***A ring of periods.*** Let $C^\flat$ be the tilt of C, with ring of integers $\mathcal{O}^\flat$. Thus

$$\mathcal{O}^\flat = \varprojlim_{x \mapsto x^p} \mathcal{O}/(p)$$

is a local domain of characteristic p, fraction field $C^\flat$, and residue field $k = \mathcal{O}^\flat/\mathfrak{m}^\flat$. An element $x \in \mathcal{O}^\flat$ is given by a sequence $(x_0, x_1, x_2, \ldots)$ of elements $x_\ell \in \mathcal{O}/(p)$ satisfying $x_{\ell+1}^p = x_\ell$. After choosing arbitrary lifts $x_\ell \in \mathcal{O}$, set

$$x^\sharp = \lim_{\ell \to \infty} x_\ell^{p^\ell}.$$

The construction $x \mapsto x^\sharp$ defines a multiplicative function $\mathcal{O}^\flat \to \mathcal{O}$, and we define $\mathrm{ord} : \mathcal{O}^\flat \to \mathbb{R} \cup \{\infty\}$ by $\mathrm{ord}(x) = \mathrm{ord}(x^\sharp)$.

Denote by $\sigma : \mathcal{O}^\flat \to \mathcal{O}^\flat$ the absolute Frobenius $x \mapsto x^p$, and in the same way the induced automorphism of the local domain

$$A_{\mathrm{inf}} = W(\mathcal{O}^\flat).$$

There is a canonical homomorphism of $\mathbb{Z}_p$-algebras

$$\Theta : A_{\mathrm{inf}} \to \mathcal{O}$$

satisfying $\Theta([x]) = x^\sharp$, where $[\,\cdot\,] : \mathcal{O}^\flat \to A_{\mathrm{inf}}$ is the Teichmüller lift.

The kernel of Θ is a principal ideal. To construct a generator, first fix a $\mathbb{Z}_p$-module generator

$$\zeta = (\zeta_p, \zeta_{p^2}, \zeta_{p^3}, \dots) \in T_p(\mu_{p^\infty})$$

and define $\epsilon = (1, \zeta_p, \zeta_{p^2}, \dots) \in \mathcal{O}^\flat$. The element

$$\xi = [1] + [\epsilon^{1/p}] + [\epsilon^{2/p}] + \cdots + [\epsilon^{(p-1)/p}] \in A_{\mathrm{inf}}$$

generates $\ker(\Theta)$. If we denote by

$$\varpi = 1 + \epsilon^{1/p} + \epsilon^{2/p} + \cdots + \epsilon^{(p-1)/p} \in \mathcal{O}^\flat$$

its image under the reduction map $A_{\mathrm{inf}} \to A_{\mathrm{inf}}/(p) = \mathcal{O}^\flat$, then $\mathrm{ord}(\varpi) = 1$, and there are canonical isomorphisms

$$\mathcal{O}/(p) \cong A_{\mathrm{inf}}/(\xi, p) \cong \mathcal{O}^\flat/(\varpi).$$

The following lemma will be needed in the proof of Proposition 3.4.4.

Lemma 2.1.1. *The reduction map $\mathcal{O}^\times \to k^\times$ sends $\varpi^\sharp/p \mapsto -1$.*

Proof. By definition, $\varpi^\sharp = \lim_{\ell\to\infty} x_\ell^{p^\ell}$, where

$$x_\ell = 1 + \zeta_{p^{\ell+1}} + \zeta_{p^{\ell+1}}^2 + \cdots + \zeta_{p^{\ell+1}}^{p-1} = \frac{\zeta_{p^\ell} - 1}{\zeta_{p^{\ell+1}} - 1} \in \mathcal{O}.$$

The binomial theorem implies that

$$\zeta_{p^\ell} = (\zeta_{p^{\ell+1}} - 1 + 1)^p = (\zeta_{p^{\ell+1}} - 1)^p + sp(\zeta_{p^{\ell+1}} - 1) + 1$$

for some $s \in \mathcal{O}$. From this we deduce first that

$$x_\ell \equiv (\zeta_{p^{\ell+1}} - 1)^{p-1} \pmod{p\mathcal{O}},$$

and then that

$$x_\ell^{p^\ell} \equiv (\zeta_{p^{\ell+1}} - 1)^{(p-1)p^\ell} \pmod{p^{\ell-1}\mathcal{O}}. \tag{2.1.1}$$

For $1 \le i \le p-1$ set

$$u_i = \frac{1-\zeta_p^i}{1-\zeta_p} = 1 + \zeta_p + \cdots + \zeta_p^{i-1} \in \mathcal{O}^\times,$$

and note that Wilson's theorem implies $u_1 \cdots u_{p-1} \equiv -1 \pmod{\mathfrak{m}}$. Taking $X = 1$ in the factorization

$$X^{p-1} + \cdots + X + 1 = (X - \zeta_p) \cdots (X - \zeta_p^{p-1})$$

shows that $p = (1-\zeta_p)^{p-1} u_1 \cdots u_{p-1}$, and hence

$$\frac{(1-\zeta_p)^{p-1}}{p} \equiv -1 \pmod{\mathfrak{m}}.$$

Combining this with (2.1.1) shows that

$$\frac{x_\ell^{p^\ell}}{p} \equiv \frac{(\zeta_{p^{\ell+1}}-1)^{(p-1)p^\ell}}{p} \equiv -\left(\frac{(\zeta_{p^{\ell+1}}-1)^{p^\ell}}{1-\zeta_p}\right)^{p-1} \pmod{\mathfrak{m}}.$$

As $\mathbb{Q}_p(\zeta_{p^{\ell+1}})$ is totally ramified over $\mathbb{Q}_p$, the reduction of

$$\frac{(\zeta_{p^{\ell+1}}-1)^{p^\ell}}{1-\zeta_p} \in \mathcal{O}^\times$$

lies in the subgroup $\mathbb{F}_p^\times \subset k^\times$. It follows that $x_\ell^{p^\ell}/p \equiv -1 \pmod{\mathfrak{m}}$, and hence $\varpi^\sharp/p \equiv -1 \pmod{\mathfrak{m}}$. □

2.2. *Breuil–Kisin–Fargues modules.* There is an equivalence of categories between p-divisible groups over $\mathcal{O}$ and Breuil–Kisin–Fargues modules, whose definition we now recall.

Definition 2.2.1. A *Breuil–Kisin–Fargues module* is a triple (M, ϕ, ψ), in which M is a free module of finite rank over A_{inf}, and

$$\phi, \psi : M \to M$$

are homomorphisms of additive groups satisfying

$$\phi(am) = \sigma(a)\phi(m), \quad \psi(\sigma(a)m) = a\psi(m),$$

for all $a \in A_{\mathrm{inf}}$ and $m \in M$, as well as $\phi \circ \psi = \xi$.

Suppose (M, ϕ, ψ) is a Breuil–Kisin–Fargues module. Denote by

$$\sigma^*M = A_{\mathrm{inf}} \otimes_{\sigma, A_{\mathrm{inf}}} M$$

the Frobenius twist of M, and by N the image of the A_{inf}-linear map

$$M \xrightarrow{x \mapsto 1\otimes\psi(x)} \sigma^*M.$$

It is easy to see that $\xi\sigma^*M \subset N \subset \sigma^*M$. We construct various realizations of M as follows:

- The *de Rham realization*

  $$M_{\mathrm{dR}} = \sigma^*M/\xi\sigma^*M,$$

  sits in the short exact sequence

  $$0 \to N/\xi\sigma^*M \to M_{\mathrm{dR}} \to \sigma^*M/N \to 0.$$

  of free $\mathcal{O}$-modules. Indeed, the freeness of M_{dR} is clear, the freeness of σ^*M/N follows from the proof of [Lau 2018, Lemma 9.5], and the freeness of $N/\xi\sigma^*M$ is a consequence of this.
- The *étale realization* is the torsion-free $\mathbb{Z}_p$-module

  $$M_{\mathrm{et}} = M^{\psi=1}.$$

Its *Hodge–Tate filtration*

$$F_{\mathrm{HT}}(M) \subset M_{\mathrm{et}} \otimes_{\mathbb{Z}_p} C$$

is the kernel of the C-linear extension of

$$M_{\mathrm{et}} \xrightarrow{x \mapsto 1 \otimes \psi(x)} N/\xi\sigma^* M.$$

- The *crystalline realization*

$$M_{\mathrm{crys}} = W(k) \otimes_{\sigma, A_{\mathrm{inf}}} M$$

is a free $W(k)$-module, endowed with operators

$$F(a \otimes m) = \sigma(a) \otimes \phi(m), \qquad V(a \otimes m) = \sigma^{-1}(a) \otimes \psi(m).$$

These give M_{crys} the structure of a Dieudonné module.

The following theorem is no doubt known to the experts, but for lack of a reference we will explain in the next subsection how to deduce it from the results of [Lau 2018].

Theorem 2.2.2 (Fargues, Scholze and Weinstein, Lau). *Assume that $p > 2$. The category of p-divisible groups over $\mathcal{O}$ is equivalent to the category of Breuil–Kisin–Fargues modules. Moreover, the Breuil–Kisin–Fargues module (M, ϕ, ψ) associated to a p-divisible group G enjoys the following properties*:

(1) *There are isomorphisms of $\mathcal{O}$-modules*

$$\Omega(G^\vee) \cong N/\xi\sigma^* M, \qquad \mathrm{Lie}(G) \cong \sigma^* M/N. \tag{2.2.1}$$

(2) *If G_k denotes the reduction of G to the residue field $k = \mathcal{O}/\mathfrak{m}$, the covariant Dieudonné module of G_k is isomorphic to M_{crys}.*

(3) *There is an isomorphism $T_p(G) \cong M_{\mathrm{et}}$ making the diagram*

$$\begin{array}{ccc} T_p(G) & = & M_{\mathrm{et}} \\ \downarrow{\scriptstyle \mathrm{HT}} & & \downarrow \\ \Omega(G^\vee) & = & N/\xi\sigma^* M \end{array} \tag{2.2.2}$$

commute, where the vertical arrow on the right is the restriction to $M_{\mathrm{et}} \subset M$ of the $\mathcal{O}$-linear map

$$M \xrightarrow{x \mapsto 1 \otimes \psi(x)} N/\xi\sigma^* M.$$

All of these isomorphisms are functorial.

Some comments on this theorem are warranted, particularly regarding the restriction to $p > 2$. A functor[1] from Breuil–Kisin–Fargues modules to p-divisible groups over $\mathcal{O}$, but not a proof that it is an equivalence of categories, first appeared in the work Fargues [2015, §4.8.1]. His construction of the

[1] Fargues only considers formal p-divisible groups, and imposes a corresponding restriction on Breuil–Kisin–Fargues modules.

functor makes essential use of the theory of *windows* introduced by Zink [2001] and extended by Lau [2010; 2018], and assumes that $p > 2$.

A proof of the equivalence of categories is found in [Scholze and Weinstein 2020, Theorem 14.1.1],[2] where the result is attributed to Fargues. The construction of the functor in [Scholze and Weinstein 2020] is very different from the construction of [Fargues 2015], and does not use of the theory of windows. Instead, what is proved in [Scholze and Weinstein 2020] is that the category of Breuil–Kisin–Fargues modules is equivalent to the category of pairs (T, W) appearing in Theorem A, and hence is equivalent to the category of p-divisible groups. This proof comes with no restriction on p.

The identification of M_{crys} with the Dieudonné module of G_k is [Scholze and Weinstein 2020, Corollary 14.4.4], and the isomorphism $T_p(G) \cong M_{\mathrm{et}}$ can be deduced by carefully tracing through the construction of the equivalence. Unfortunately, the isomorphisms of $\mathcal{O}$-modules (2.2.1) seem difficult to deduce from the description of the equivalence found in [Scholze and Weinstein 2020].

Because of this, our equivalence of categories will be the one appearing in [Lau 2018], which follows Fargues. What Lau proves is that, when $p > 2$, the categories of Breuil–Kisin–Fargues modules and p-divisible groups over $\mathcal{O}$ are both equivalent to the category of windows. The various properties of the equivalence listed in Theorem 2.2.2 can be read off from the constructions of the two functors into the category of windows, which are quite simple and direct (of course, the proof that they are equivalences is not).

The invocation of Theorem 2.2.2 in the calculations of Section 3 is the only reason why the assumption $p > 2$ is imposed in the introduction. Our approach in the sequel will be to allow arbitrary p, but to take the conclusions of Theorem 2.2.2 as hypotheses.

2.3. *Proof of Theorem 2.2.2.* As we have already indicated, Theorem 2.2.2 is proved by relating the categories of Breuil–Kisin–Fargues modules and p-divisible groups to the category of windows introduced by Zink [2001] and extended by Lau [2010; 2018].

Our windows will be modules over the ring A_{crys}, which is defined as the p-adic completion of the subring

$$A^0_{\mathrm{crys}} = A_{\mathrm{inf}}[\xi^n/n! : n = 1, 2, 3, \dots] \subset A_{\mathrm{inf}}[1/p].$$

It is an integral domain endowed with a ring homomorphism

$$\Theta_{\mathrm{crys}} : A_{\mathrm{crys}} \to \mathcal{O} \tag{2.3.1}$$

extending $\Theta : A_{\mathrm{inf}} \to \mathcal{O}$, and divided powers on the kernel $I = \ker(\Theta_{\mathrm{crys}})$.

The subring $A^0_{\mathrm{crys}} \subset A_{\mathrm{inf}}[1/p]$ is stable under σ, and there is a unique continuous extension to an injective ring homomorphism $\sigma : A_{\mathrm{crys}} \to A_{\mathrm{crys}}$ reducing to the usual p-power Frobenius on $A_{\mathrm{crys}}/pA_{\mathrm{crys}}$. Moreover, [Scholze and Weinstein 2013, Lemma 4.1.8] and the comments following [Lau 2018, (9.1)] show that

$$\sigma(I) \subset pA_{\mathrm{crys}} \quad \text{and} \quad \frac{\sigma(\xi)}{p} \in A^\times_{\mathrm{crys}}.$$

[2] Our conventions for Breuil–Kisin–Fargues modules and the equivalence of categories differ from those of [Scholze and Weinstein 2020]. The discrepancy amounts to a Tate twist.

The following definition of a window is taken from [Lau 2018, §2], where it would be called a *window over the frame*

$$\underline{A}_{\mathrm{crys}} = (A_{\mathrm{crys}}, I, \mathcal{O} = A_{\mathrm{crys}}/I, \sigma, \sigma_1),$$

with $\sigma_1 : I \to A_{\mathrm{crys}}$ defined by $\sigma_1(x) = \sigma(x)/p$.

Definition 2.3.1. A *window* is a quadruple (P, Q, Φ, Φ_1) consisting of a projective A_{crys}-module P of finite rank, a submodule $Q \subset P$, and σ-semilinear maps

$$\Phi : P \to P, \qquad \Phi_1 : Q \to P$$

satisfying the following properties:

- there exist A_{crys}-submodules $L, T \subset P$ such that

$$Q = L \oplus IT, \qquad P = L \oplus T,$$

- $a \otimes x \mapsto a\Phi_1(x)$ defines a surjection $\sigma^* Q \to P$ of A_{crys}-modules,
- $\Phi(ax) = p\Phi_1(ax)$ for all $a \in I$ and $x \in P$.

Remark 2.3.2. Taking $a = \xi$ in the final condition yields

$$\Phi(x) = \frac{p}{\sigma(\xi)} \cdot \Phi_1(\xi x)$$

for all $x \in P$. This implies $\Phi(x) = p\Phi_1(x)$ for all $x \in Q$, and shows that each one of Φ and Φ_1 determines the other.

Remark 2.3.3. Note that $IP \subset Q$, and that Q/IP and P/Q are projective (hence free) over $A_{\mathrm{crys}}/I \cong \mathcal{O}$.

Suppose G is a p-divisible group over $\mathcal{O}$. Let P be its crystalline Dieudonné module, evaluated at the divided power thickening (2.3.1). This is a projective A_{crys}-module of rank equal to the height of G, equipped with a σ-semilinear operator $\Phi : P \to P$ and a short exact sequence

$$0 \to \Omega(G^\vee) \to P/IP \to \mathrm{Lie}(G) \to 0$$

of free $\mathcal{O}$-modules. Define $Q \subset P$ as the kernel of $P \to \mathrm{Lie}(G)$. One can show that $\Phi(Q) \subset pP$, allowing us to define $\Phi_1 : Q \to P$ by

$$\Phi_1(x) = \frac{1}{p} \cdot \Phi(x).$$

The following is a special case of [Lau 2018, Proposition 9.7].

Theorem 2.3.4 (Lau). *The construction $G \mapsto (P, Q, \Phi, \Phi_1)$ just given defines a functor from the category of p-divisible groups over $\mathcal{O}$ to the category of windows. It is an equivalence of categories if $p > 2$.*

Now suppose we start with a Breuil–Kisin–Fargues module (M, ϕ, ψ). Set

$$P = A_{\mathrm{crys}} \otimes_{\sigma, A_{\mathrm{inf}}} M, \tag{2.3.2}$$

and define $\Phi : P \to P$ by $\Phi(a \otimes m) = \sigma(a) \otimes \phi(m)$ for all $a \in A_{\mathrm{crys}}$ and $m \in M$. The submodule $Q \subset P$, defined as the kernel of the composition

$$\begin{array}{ccc} A_{\mathrm{crys}} \otimes_{\sigma, A_{\mathrm{inf}}} M & \longrightarrow & A_{\mathrm{crys}}/I A_{\mathrm{crys}} \otimes_{\sigma, A_{\mathrm{inf}}} M \\ & & \downarrow \cong \\ & & A_{\mathrm{inf}}/\xi A_{\mathrm{inf}} \otimes_{\sigma, A_{\mathrm{inf}}} M \\ & & \downarrow \cong \\ & & \sigma^* M/\xi \sigma^* M \longrightarrow \sigma^* M/N \end{array}$$

is alternately characterized the A_{crys}-submodule generated by all elements of the form $1 \otimes \psi(m)$ and $a \otimes m$ with $m \in M$ and $a \in I$. There is a unique σ-semilinear map $\Phi_1 : Q \to P$ whose effect on these generators is

$$\Phi_1(1 \otimes \psi(m)) = \frac{\sigma(\xi)}{p} \otimes m, \qquad \Phi_1(a \otimes m) = \frac{\sigma(a)}{p} \otimes \phi(m).$$

The following is a special case of [Lau 2018, Theorem 1.5].

Theorem 2.3.5 (Lau). *The construction $(M, \phi, \psi) \mapsto (P, Q, \Phi, \Phi_1)$ just given defines a functor from the category of Breuil–Kisin–Fargues modules to the category of windows. It is an equivalence of categories if $p > 2$.*

Given a window (P, Q, Φ, Φ_1), define its *étale realization*

$$P_{\mathrm{et}} = \{x \in Q : \Phi_1(x) = x\}$$

as in [Lau 2019, §3]. This is a torsion-free $\mathbb{Z}_p$-module equipped with a *Hodge–Tate filtration*

$$F_{\mathrm{HT}}(P_{\mathrm{et}}) \subset P_{\mathrm{et}} \otimes_{\mathbb{Z}_p} C,$$

defined as the kernel of the C-linear extension of $P_{\mathrm{et}} \to Q/IP$.

Denote by HTpair the category of pairs (T, W) in which T is a torsion-free $\mathbb{Z}_p$-module, and $W \subset T \otimes_{\mathbb{Z}_p} C$ is a subspace. Using the obvious notation for the categories of Breuil–Kisin–Fargues modules, p-divisible groups over $\mathcal{O}$, and windows, we now have functors

$$\begin{array}{ccccc} \text{BKF-Mod} & \xrightarrow{a} & \text{Win} & \xleftarrow{b} & p\text{-DivGrp} \\ & {\scriptstyle d} \searrow & \downarrow {\scriptstyle c} & \swarrow {\scriptstyle e} & \\ & & \text{HTpair} & & \end{array} \tag{2.3.3}$$

Here a is given by Theorem 2.3.5, b is given by Theorem 2.3.4, c sends a window to its étale realization, d does the same for Breuil–Kisin–Fargues modules, and e sends a p-divisible group over $\mathcal{O}$ to its p-adic Tate module endowed with its Hodge filtration.

Remark 2.3.6. It is not obvious from the definitions that (2.3.3) commutes. When $p > 2$ the commutativity is a byproduct of the following proof.

Proof of Theorem 2.2.2. Assume that $p > 2$. In particular the functors of Theorems 2.3.4 and 2.3.5 are equivalences of categories, and their composition gives the desired equivalence of categories between p-divisible groups over $\mathcal{O}$ and Breuil–Kisin–Fargues modules.

Suppose G is a p-divisible group over $\mathcal{O}$, and let (P, Q, Φ, Φ_1) and (M, ϕ, ψ) be its corresponding window and Breuil–Kisin–Fargues module. The isomorphisms

$$\Omega(G^\vee) \cong Q/IP \cong N/\xi\sigma^* M \quad \text{and} \quad \mathrm{Lie}(G) \cong P/Q \cong \sigma^* M/N$$

are clear from the constructions of the functors of Theorems 2.3.4 and 2.3.5.

The quotient map $\mathcal{O} \to k$ induces a ring homomorphism $A_{\mathrm{inf}} \to W(k)$ sending $\xi \mapsto p$. It follows that there is a unique continuous extension to $A_{\mathrm{crys}} \to W(k)$ and, by (2.3.2), canonical isomorphisms

$$W(k) \otimes_{A_{\mathrm{crys}}} P \cong W(k) \otimes_{\sigma, A_{\mathrm{inf}}} M \cong M_{\mathrm{crys}}. \tag{2.3.4}$$

The functor of Theorem 2.3.4 is constructed in such a way that the leftmost $W(k)$-module in (2.3.4) is identified with the value of the Dieudonné crystal of G_k at the divided power thickening $W(k) \to k$, which is just the usual covariant Dieudonné module of G_k.

The window of the constant p-divisible group $\mathbb{Q}_p/\mathbb{Z}_p$ over $\mathcal{O}$ consists of

$$P^0 = A_{\mathrm{crys}} \quad \text{and} \quad Q^0 = A_{\mathrm{crys}}$$

endowed with the operators $\Phi : P^0 \to P^0$ and $\Phi_1 : Q^0 \to P^0$ defined by

$$\Phi(x) = p\sigma(x) \quad \text{and} \quad \Phi_1(x) = \sigma(x).$$

In particular there is a canonical isomorphism $Q^0/IP^0 \cong \mathcal{O}$.

The Breuil–Kisin–Fargues module of $\mathbb{Q}_p/\mathbb{Z}_p$ consists of

$$M^0 = A_{\mathrm{inf}}$$

endowed with the operators

$$\phi(x) = \xi\sigma(x) \quad \text{and} \quad \psi(x) = \sigma^{-1}(x).$$

The distinguished submodule $N^0 \subset \sigma^* M^0$ defined in Section 2.2 is all of $\sigma^* M^0 = \sigma^* A_{\mathrm{inf}}$, so is free of rank one generated by $1 \otimes 1$. Hence there is a canonical isomorphism $N^0/\xi\sigma^* M^0 \cong \mathcal{O}$.

From the equivalence of categories of Theorem 2.3.4 we obtain the commutative diagram

$$\begin{array}{ccc}
T_p(G) & \xrightarrow{\mathrm{HT}} & \Omega(G^\vee) \\
\| & & \| \\
\mathrm{Hom}_{\mathrm{p-DivGrp}}(\mathbb{Q}_p/\mathbb{Z}_p, G) & \longrightarrow & \mathrm{Hom}_{\mathcal{O}}(\Omega(\mu_{p^\infty}), \Omega(G^\vee)) \\
\| & & \| \\
\mathrm{Hom}_{\mathrm{Win}}(P^0, P) & \longrightarrow & \mathrm{Hom}_{\mathcal{O}}(Q^0/IP^0, Q/IP) \\
\| & & \| \\
P_{\mathrm{et}} & \longrightarrow & Q/IP
\end{array} \tag{2.3.5}$$

Similarly, from the equivalence of categories of Theorem 2.3.5 we obtain the commutative diagram

$$
\begin{array}{ccc}
M_{\mathrm{et}} & \longrightarrow & N/\xi\sigma^*M \\
\| & & \| \\
\mathrm{Hom}_{\mathrm{BKF}}(M^0, M) & \longrightarrow & \mathrm{Hom}_{\mathcal{O}}(N^0/\xi\sigma^*M^0, N/\xi\sigma^*M) \\
\| & & \| \\
\mathrm{Hom}_{\mathrm{Win}}(P^0, P) & \longrightarrow & \mathrm{Hom}_{\mathcal{O}}(Q^0/IP^0, Q/IP) \\
\| & & \| \\
P_{\mathrm{et}} & \longrightarrow & Q/IP
\end{array}
\tag{2.3.6}
$$

Combining these gives (2.2.2), completing the proof of Theorem 2.2.2.

As a final comment, we note that the diagrams (2.3.5) and (2.3.6) show that P_{et} and M_{et} are finitely generated $\mathbb{Z}_p$-modules, and that (2.3.3) commutes. If we denote by FinHTpair $\subset$ HTpair the full subcategory of pairs (T, W) with T of finite rank over $\mathbb{Z}_p$, we obtain a commutative diagram

$$
\begin{array}{ccccc}
\text{BKF-Mod} & \xrightarrow{\;a\;} & \text{Win} & \xleftarrow{\;b\;} & p\text{-DivGrp} \\
& {\scriptstyle d}\searrow & \downarrow{\scriptstyle c} & \swarrow{\scriptstyle e} & \\
& & \text{FinHTpair} & &
\end{array}
$$

in which the arrows a, b, and e are equivalences of categories (the last one by Theorem A). Hence all arrows are equivalences of categories. □

3. Bounding the Hodge–Tate periods

Let G be a p-divisible group of height four and dimension two over $\mathcal{O}$, endowed with an action $\Delta \to \mathrm{End}(G)$.

Throughout Section 3 we do not require $p > 2$. Instead we allow p to be arbitrary, but assume the conclusion of Theorem 2.2.2.

3.1. ***Hodge–Tate periods.*** The embeddings (1.3.1) determine a decomposition

$$\Omega(G^\vee) = \Omega_0(G^\vee) \oplus \Omega_1(G^\vee), \tag{3.1.1}$$

in which each summand is free of rank one over $\mathcal{O}$, and $\mathbb{Z}_{p^2} \subset \Delta$ acts on them through j_0 and j_1, respectively. The operator Π maps each summand injectively into the other. Applying $\otimes_{\mathcal{O}} k$ to (3.1.1) yields a decomposition

$$\Omega(G_k^\vee) = \Omega_0(G_k^\vee) \oplus \Omega_1(G_k^\vee)$$

into one-dimensional k-vector spaces.

Composing the Hodge–Tate morphism (1.1.1) with the two projections yields two *partial Hodge–Tate morphisms*

$$T_p(G) \xrightarrow{\mathrm{HT}_0} \Omega_0(G^\vee), \qquad T_p(G) \xrightarrow{\mathrm{HT}_1} \Omega_1(G^\vee).$$

By fixing isomorphisms

$$\Omega_0(G^\vee) \cong \mathcal{O}, \qquad \Omega_1(G^\vee) \cong \mathcal{O}, \tag{3.1.2}$$

we view these as $\mathcal{O}$-valued $\mathbb{Z}_p$-linear functionals on $T_p(G)$.

Lemma 3.1.1. *The* Δ*-module* $T_p(G)$ *is free of rank* 1.

Proof. As $\Delta \otimes \mathbb{Q}_p$ is a division ring, its module $T_p(G) \otimes \mathbb{Q}_p$ is necessarily free. Comparing $\mathbb{Q}_p$-dimensions shows that it is free of rank one, and hence $T_p(G)$ is isomorphic to some (left) Δ-submodule of $\Delta \otimes \mathbb{Q}_p$. As Δ admits a discrete valuation [Vignéras 1980, Lemme II.1.4] with uniformizer Π, every such submodule is principal and generated by a power of Π. ☐

Fix a Δ-module generator $\lambda \in T_p(G)$, and define

$$\tau_0 = \frac{\mathrm{HT}_0(\Pi\lambda)}{\mathrm{HT}_0(\lambda)}, \qquad \tau_1 = \frac{\mathrm{HT}_1(\Pi\lambda)}{\mathrm{HT}_1(\lambda)}.$$

These are the *Hodge–Tate periods* of G. In each fraction the numerator or denominator may vanish, but not simultaneously. Thus the Hodge–Tate periods lie in $\mathbb{P}^1(C) = C \cup \{\infty\}$. They do not depend on the choice of (3.1.2), but do depend on the choice of generator λ.

Proposition 3.1.2. *The Hodge–Tate periods satisfy* $\tau_0 \cdot \tau_1 = p$.

Proof. The action of Π on $\Omega_0(G^\vee) \oplus \Omega_1(G^\vee)$ is given by

$$(\omega_0, \omega_1) \mapsto (s_0\omega_1, s_1\omega_0)$$

for some $s_0, s_1 \in \mathcal{O}$ satisfying $s_0 s_1 = p$. From the Δ-linearity of the Hodge–Tate morphism we deduce first

$$\mathrm{HT}_0(\Pi\lambda) = s_0 \cdot \mathrm{HT}_1(\lambda), \qquad \mathrm{HT}_1(\Pi\lambda) = s_1 \cdot \mathrm{HT}_0(\lambda),$$

and then

$$\tau_0 \cdot \tau_1 = \frac{\mathrm{HT}_0(\Pi\lambda)}{\mathrm{HT}_0(\lambda)} \cdot \frac{\mathrm{HT}_1(\Pi\lambda)}{\mathrm{HT}_1(\lambda)} = s_0 \cdot s_1 = p. \qquad ☐$$

3.2. *Reduction to the residue field.* Let G_k be the reduction of G to the residue field $k = \mathcal{O}/\mathfrak{m}$, and let (D, F, V) be its covariant Dieudonné module.

Definition 3.2.1. Let H be the p-divisible group of a supersingular elliptic curve over k. In other words, H is the unique connected p-divisible group of height two and dimension one. The reduction G_k is said to be

(1) *supersingular* if it is isogenous to $H \times H$,

(2) *superspecial* if it is isomorphic to $H \times H$.

Remark 3.2.2. Our notions of supersingular and superspecial depend only on the p-divisible group G_k, and not on its Δ-action. This differs from the meaning of superspecial in some literature on Shimura curves, e.g., [Kudla and Rapoport 2000].

The following proposition, which implies that the notion of superspecial depends only on the p-torsion subgroup scheme $G_k[p] \subset G_k$, is well-known. For lack of a reference we provide the proof.

Proposition 3.2.3. *The reduction G_k is supersingular, and the following are equivalent*:

(1) *G_k is superspecial.*

(2) *There is an isomorphism of group schemes $G_k[p] \cong H[p] \times H[p]$.*

(3) *$V^2 D \subset pD$.*

(4) *$FD = VD$.*

Proof. The supersingularity of G_k follows from the Dieudonné–Manin classification of isocrystals: one can list all isogeny classes of p-divisible groups over k of height four and dimension two, and the supersingular isogeny class is the only one whose endomorphism algebra contains a quaternion division algebra.

The implication (1) $\Longrightarrow$ (2) is trivial. For the implication (2) $\Longrightarrow$ (3) it suffices to check that V^2 kills the Dieudonné module of $H[p]$, which we leave to the reader.

Next we prove (3) $\Longrightarrow$ (4). If $D' \subset D$ is any $W(k)$-lattice stable under both F and V, then its corresponding p-divisible group G'_k is isogenous to G_k. In particular it has dimension 2, and hence

$$D'/VD' \cong \mathrm{Lie}(G'_k)$$

is a 2-dimensional k vector space. Applying this with $D' = D$ and $D' = VD$ shows that D/V^2D has length 4 as a $W(k)$-module. On the other hand, D/pD also has length 4, proving the first implication in

$$V^2D \subset pD \Longrightarrow V^2D = pD \Longrightarrow VD = FD.$$

Finally, we prove (4) $\Longrightarrow$ (1). Let α_p be the finite flat group scheme whose Dieudonné module is the $W(k)$-module k, endowed with the operators $F = 0$ and $V = 0$. If $FD = VD$ then, using the self-duality of α_p, we see that

$$\begin{aligned} \mathrm{Hom}(\alpha_p, G_k^\vee) \cong \mathrm{Hom}(G_k[p], \alpha_p) &\cong \mathrm{Hom}_k(D/(FD+VD), k) \\ &\cong \mathrm{Hom}_k(D/VD, k) \cong \mathrm{Hom}_k(\mathrm{Lie}(G), k) \end{aligned}$$

is a 2-dimensional k-vector space. It follows from [Oort 1975, Theorem 2] that $G_k^\vee$ is superspecial, and hence so is G_k. □

Let (M, ϕ, ψ) be the Breuil–Kisin–Fargues module of G. The quotient

$$M^\flat = M/pM$$

is a free module over $\mathcal{O}^\flat \cong A_{\mathrm{inf}}/(p)$, endowed with operators $\phi, \psi : M^\flat \to M^\flat$ satisfying $\phi \circ \psi = \varpi$.

Denote by $N^\flat = N/pN$ the image of

$$M^\flat \xrightarrow{m \mapsto 1 \otimes \psi(m)} \sigma^* M^\flat.$$

Each of our embeddings $j_0, j_1 : \mathbb{Z}_{p^2} \to \mathcal{O}$ determines a map

$$\mathbb{Z}_{p^2} \to \mathcal{O}/p\mathcal{O} \cong \mathcal{O}^\flat/\varpi\mathcal{O}^\flat,$$

and these two maps lift uniquely to $j_0, j_1 : \mathbb{Z}_{p^2} \to \mathcal{O}^\flat$. The action of Δ on G determines an action on $M^\flat$, which induces a decomposition

$$M^\flat = M_0^\flat \oplus M_1^\flat$$

analogous to (3.1.1). It follows from the next proposition that each factor is free of rank two over $\mathcal{O}^\flat$.

Proposition 3.2.4.

(1) *D is free of rank one over $\Delta \otimes_{\mathbb{Z}_p} W(k)$.*

(2) *M is free of rank one over $\Delta \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}$.*

Proof. Reduce (1.3.1) to ring homomorphisms $j_0, j_1 : \mathbb{Z}_{p^2} \to k$, and denote again by $j_0, j_1 : \mathbb{Z}_{p^2} \to W(k)$ the unique lifts. There is a decomposition of $W(k)$-modules

$$D = D_0 \oplus D_1$$

in such a way that $\mathbb{Z}_{p^2} \subset \Delta$ acts on the two summands via j_0 and j_1, respectively. As in [Kudla and Rapoport 2000, §1], these summands are free of rank 2 over $W(k)$, and satisfy

$$pD_0 \subsetneq VD_1 \subsetneq D_0, \qquad pD_1 \subsetneq VD_0 \subsetneq D_1.$$

Moreover, either $\Pi D_0 = VD_0$ or $\Pi D_1 = VD_1$ (or both).

Without loss of generality, we may assume that $\Pi D_0 = VD_0$, and hence

$$pD_1 \subsetneq \Pi D_0 \subsetneq D_1.$$

Applying Π to these inclusions shows that

$$pD_0 \subsetneq \Pi D_1 \subsetneq D_0.$$

If we choose any $f_0 \in D_0$ and $f_1 \in D_1$ with nonzero images in $D_0/\Pi D_1$ and $D_1/\Pi D_0$, respectively, then $f_0, f_1, \Pi f_0, \Pi f_1 \in D$ reduce to a k-basis of D/pD. Using Nakayama's lemma it is easy to see that D is generated by $f_0 + f_1$ as a $\Delta \otimes W(k)$-module, and the first claim of the proposition follows.

Theorem 2.2.2 gives us an isomorphism

$$D/pD \cong \sigma^*(M/\mathfrak{m}M)$$

of $\Delta \otimes_{\mathbb{Z}_p} k$-modules, and from what was said above we deduce that $M/\mathfrak{m}M$ is free of rank one over $\Delta \otimes_{\mathbb{Z}_p} k$. The second claim of the proposition follows easily from this and Nakayama's lemma. □

3.3. ***The case*** $\mathbf{\Pi\Omega(G_k^\vee)=0}$**.** We assume throughout Section 3.3 that

$$\Pi\Omega(G_k^\vee)=0.$$

We will analyze the structure of $M^\flat$, with its operators ϕ and ψ, and use this to bound the Hodge–Tate periods of G. The first step is to choose a convenient basis.

Lemma 3.3.1. *There are $\mathcal{O}^\flat$-bases $e_0, f_0 \in M_0^\flat$ and $e_1, f_1 \in M_1^\flat$ such that the operator $\Pi \in \Delta$ satisfies*

$$\Pi e_0 = 0, \qquad \Pi e_1 = 0, \quad \Pi f_0 = e_1, \quad \Pi f_1 = e_0, \tag{3.3.1}$$

and such that ψ satisfies

$$\psi(e_0)=t_0e_1, \quad \psi(e_1)=t_1e_0, \quad \psi(f_0)=e_1+t_1f_1, \quad \psi(f_1)=e_0+t_0f_0$$

for scalars $t_0, t_1 \in \mathcal{O}^\flat$ satisfying $\mathrm{ord}(t_0)>0$, $\mathrm{ord}(t_1)>0$, *and*

$$\mathrm{ord}(t_0)+\mathrm{ord}(t_1)=1/p.$$

Proof. As $M^\flat$ is free of rank one over $\Delta\otimes_{\mathbb{Z}_p}\mathcal{O}^\flat$, we may choose a basis such that (3.3.1) holds, and the relation $\psi\circ\Pi=\Pi\circ\psi$ then implies

$$\psi(e_0)=t_0e_1, \quad \psi(e_1)=t_1e_0, \quad \psi(f_0)=u_1e_1+t_1f_1, \quad \psi(f_1)=u_0e_0+t_0f_0$$

for some $u_0, u_1, t_0, t_1 \in \mathcal{O}^\flat$. The submodule $N^\flat \subset \sigma^*M^\flat$ is generated by

$$\begin{aligned} 1\otimes\psi(e_0)&=t_0^p\otimes e_1, & 1\otimes\psi(e_1)&=t_1^p\otimes e_0,\\ 1\otimes\psi(f_0)&=u_1^p\otimes e_1+t_1^p\otimes f_1, & 1\otimes\psi(f_1)&=u_0^p\otimes e_0+t_0^p\otimes f_0. \end{aligned}$$

Recall that $\mathfrak{m}^\flat\subset\mathcal{O}^\flat$ is the maximal ideal. The first isomorphism in (2.2.1) identifies $\Omega(G_k^\vee)$ with the image of $N^\flat$ in $(\sigma^*M^\flat)/\mathfrak{m}^\flat(\sigma^*M^\flat)$, and by hypothesis this k-vector space is annihilated by Π. It is easy to see from this that $\mathrm{ord}(t_0)$ and $\mathrm{ord}(t_1)$ are positive.

Using Theorem 2.2.2, we see that

$$\sigma^*M^\flat/N^\flat \cong (\sigma^*M/N)\otimes_{\mathcal{O}}\mathcal{O}/(p)\cong \mathrm{Lie}(G)\otimes_{\mathcal{O}}\mathcal{O}/(p)$$

is free of rank two over $\mathcal{O}/(p)\cong\mathcal{O}^\flat/(\varpi)$. On the other hand, $\sigma^*M^\flat/N^\flat$ is isomorphic to the cokernel of the matrix

$$\begin{pmatrix} t_1^p & & u_0^p & \\ & t_0^p & & u_1^p \\ & & t_0^p & \\ & & & t_1^p \end{pmatrix}\in M_4(\mathcal{O}^\flat),$$

whose reduction to $M_4(k)$ must therefore have rank 2. This implies that u_0 and u_1 are units, and using elementary row and column operations one sees that

$$\sigma^*M^\flat/N^\flat\cong\mathcal{O}^\flat/(t_0t_1)^p\oplus\mathcal{O}^\flat/(t_0t_1)^p.$$

Hence $(t_0t_1)^p = (\varpi)$. Finally, having already seen that u_0 and u_1 are units, an easy calculation shows that our basis elements may be rescaled in order to make $u_0 = 1$ and $u_1 = 1$. □

Fix a basis as in Lemma 3.3.1. Theorem 2.2.2 identifies

$$T_p(G)/pT(G) = M^{\psi=1}/pM^{\psi=1} \subset (M^\flat)^{\psi=1},$$

and the image of our fixed generator $\lambda \in T_p(G)$ has the form

$$a_0e_0 + a_1e_1 + b_0 f_0 + b_1 f_1 \in M^\flat$$

for some coefficients $a_0, a_1, b_0, b_1 \in \mathcal{O}^\flat$ satisfying

$$a_0^p = a_1 t_1^p + b_1, \quad a_1^p = a_0 t_0^p + b_0, \quad b_0^p = b_1 t_0^p, \quad b_1^p = b_0 t_1^p. \tag{3.3.2}$$

The first isomorphism of (2.2.1) identifies

$$\Omega(G^\vee)/p\Omega(G^\vee) = N/(pN + \xi\sigma^* M) = N^\flat/\varpi\sigma^* M^\flat$$

with the direct summand of $\sigma^* M^\flat/\varpi\sigma^* M^\flat$ generated by the reductions of

$$1 \otimes \psi(f_0) = 1 \otimes e_1 + t_1^p \otimes f_1 \in \sigma^* M^\flat, \quad 1 \otimes \psi(f_1) = 1 \otimes e_0 + t_0^p \otimes f_0 \in \sigma^* M^\flat.$$

If we use this basis to identify

$$\Omega(G^\vee)/p\Omega(G^\vee) = N^\flat/\varpi\sigma^* M^\flat \cong \mathcal{O}^\flat/(\varpi) \oplus \mathcal{O}^\flat/(\varpi)$$

then, again using Theorem 2.2.2, the partial Hodge–Tate morphisms

$$T_p(G)/pT_p(G) \xrightarrow{\mathrm{HT}_0} \Omega_0(G^\vee)/p\Omega_0(G^\vee) \cong \mathcal{O}^\flat/(\varpi)$$
$$T_p(G)/pT_p(G) \xrightarrow{\mathrm{HT}_1} \Omega_1(G^\vee)/p\Omega_1(G^\vee) \cong \mathcal{O}^\flat/(\varpi)$$

are given by

$$\begin{aligned} \mathrm{HT}_0(\lambda) &= a_1^p, & \mathrm{HT}_0(\Pi\lambda) &= b_0^p, \\ \mathrm{HT}_1(\lambda) &= a_0^p, & \mathrm{HT}_1(\Pi\lambda) &= b_1^p. \end{aligned} \tag{3.3.3}$$

Lemma 3.3.2. *For $i \in \{0, 1\}$, we have*

$$\mathrm{ord}(b_i) = \frac{1}{p^2-1} + \frac{p \cdot \mathrm{ord}(t_i)}{p+1}.$$

Proof. As $\Pi\lambda \in T_p(G)$ has nonzero image in

$$T_p(G)/pT_p(G) \subset M^\flat,$$

we must have $b_0e_1 + b_1e_0 \neq 0$. Therefore one of b_0 and b_1 is nonzero. The relations (3.3.2) then imply first that both b_0 and b_1 are nonzero, and then that

$$b_i^{p^2-1} = (t_0t_1)^p \cdot t_i^{p(p-1)}.$$

The claim follows by applying ord to both sides of this equality. □

Lemma 3.3.3. *If we assume that*

$$\frac{1}{p^2(p-1)} < \mathrm{ord}(t_1),$$

then

$$\mathrm{ord}(a_0) = \frac{1}{p(p^2-1)} + \frac{\mathrm{ord}(t_1)}{p+1}, \qquad \mathrm{ord}(a_1) = \frac{1}{p^2-1} - \frac{\mathrm{ord}(t_1)}{p+1}.$$

Of course there is a similar result if t_1 is replaced by t_0.

Proof. Recall the equality $a_0^p = a_1 t_1^p + b_1$ from (3.3.2). The only way this can hold is if (at least) one of the three relations

- $p \cdot \mathrm{ord}(a_0) = \mathrm{ord}(b_1) \le \mathrm{ord}(t_1^p a_1)$
- $p \cdot \mathrm{ord}(a_0) = \mathrm{ord}(t_1^p a_1) \le \mathrm{ord}(b_1)$
- $\mathrm{ord}(b_1) = \mathrm{ord}(t_1^p a_1) \le p \cdot \mathrm{ord}(a_0)$

is satisfied. The second and third relations cannot be satisfied, as each implies

$$0 \le \mathrm{ord}(a_1) \le \mathrm{ord}(b_1) - p \cdot \mathrm{ord}(t_1) = \frac{1}{p^2-1} - \frac{p^2 \cdot \mathrm{ord}(t_1)}{p+1} < 0.$$

Hence the first relation holds, and Lemma 3.3.2 shows that

$$p \cdot \mathrm{ord}(a_0) = \mathrm{ord}(b_1) = \frac{1}{p^2-1} + \frac{p \cdot \mathrm{ord}(t_1)}{p+1}.$$

This proves the first equality.

For the second equality, the relations (3.3.2) imply

$$a_0^{p^2} = a_1^p \cdot (t_1^p + t_1)^p - (t_0 t_1)^p a_0,$$
$$a_1^{p^2} = a_0^p \cdot (t_0^p + t_0)^p - (t_0 t_1)^p a_1.$$

Using the second of these, along with

$$\mathrm{ord}\big(a_0^p \cdot (t_0^p + t_0)^p\big) = \mathrm{ord}(b_1) + p \cdot \mathrm{ord}(t_0) = \frac{p^2}{p^2-1} - \frac{p^2 \cdot \mathrm{ord}(t_1)}{p+1} < 1 \le \mathrm{ord}\big((t_0 t_1)^p a_1\big),$$

we find that

$$\mathrm{ord}(a_1) = \frac{\mathrm{ord}\big(a_0^p \cdot (t_0^p + t_0)^p\big)}{p^2} = \frac{1}{p^2-1} - \frac{\mathrm{ord}(t_1)}{p+1}. \qquad \square$$

Now we can prove the main result of this subsection.

Proposition 3.3.4. *If we assume, as above, that $\Pi\Omega(G_k^\vee) = 0$ then*

$$\frac{1}{p+1} < \mathrm{ord}(\tau_0) < \frac{p}{p+1} \quad \textit{and} \quad \frac{1}{p+1} < \mathrm{ord}(\tau_1) < \frac{p}{p+1}.$$

Proof. First assume that

$$\frac{1}{p^2(p-1)} < \mathrm{ord}(t_1). \tag{3.3.4}$$

The discussion leading to (3.3.3) provides us with an $\mathcal{O}$-module isomorphism

$$\Omega_0(G^\vee)/p\Omega_0(G^\vee) \cong \mathcal{O}^\flat/(\varpi) \cong \mathcal{O}/(p),$$

and we fix any lift to an isomorphism $\Omega_0(G^\vee) \cong \mathcal{O}$.

It is easy to see from Lemmas 3.3.2 and 3.3.3 that $\mathrm{ord}(a_1)$ and $\mathrm{ord}(b_0)$ lie in the open interval $(0, 1/p)$, and so a_1^p and b_0^p have nonzero images in $\mathcal{O}^\flat/(\varpi)$. By (3.3.3) these images agree with the images of $\mathrm{HT}_0(\lambda)$ and $\mathrm{HT}_0(\Pi\lambda)$ under

$$\mathcal{O} \to \mathcal{O}/(p) \cong \mathcal{O}^\flat/(\varpi).$$

Thus

$$\mathrm{ord}(\mathrm{HT}_0(\lambda)) = \mathrm{ord}(a_1^p) = \frac{p}{p^2-1} - \frac{p\cdot\mathrm{ord}(t_1)}{p+1}$$

and

$$\mathrm{ord}(\mathrm{HT}_0(\Pi\lambda)) = \mathrm{ord}(b_0^p) = \frac{p}{p^2-1} + \frac{p^2\cdot\mathrm{ord}(t_0)}{p+1}.$$

It follows that

$$\mathrm{ord}(\tau_0) = \mathrm{ord}(\mathrm{HT}_0(\Pi\lambda)) - \mathrm{ord}(\mathrm{HT}_0(\lambda)) = \frac{p}{p+1} - \frac{(p-1)}{p+1}\cdot\mathrm{ord}(t_1^p),$$

and so

$$\frac{1}{p+1} < \mathrm{ord}(\tau_0) < \frac{p}{p+1}.$$

The analogous inequalities for $\mathrm{ord}(\tau_1)$ follow from the relation $\tau_0\tau_1 = p$ of Proposition 3.1.2.

This proves Proposition 3.3.4 under the assumption (3.3.4). The proof when

$$\frac{1}{p^2(p-1)} < \mathrm{ord}(t_0) \tag{3.3.5}$$

is entirely similar.

Thus we are left to prove the claim under the assumption that both (3.3.4) and (3.3.5) fail. This assumption implies that

$$\frac{1}{p} = \mathrm{ord}(t_0) + \mathrm{ord}(t_1) \le \frac{2}{p^2(p-1)},$$

which implies that $p = 2$ and

$$\mathrm{ord}(t_0) = \tfrac{1}{4} = \mathrm{ord}(t_1).$$

In particular, Lemma 3.3.2 simplifies to

$$\mathrm{ord}(b_0) = \tfrac{1}{2} = \mathrm{ord}(b_1).$$

Consider the equality $a_0^2 = a_1 t_1^2 + b_1$ of (3.3.2). As in the proof of Lemma 3.3.3, the only way this can hold is if (at least) one of the relations

- $\mathrm{ord}(a_0) = \frac{1}{4}$
- $\mathrm{ord}(a_1) = 0$ and $\mathrm{ord}(a_0) \geq \frac{1}{4}$

holds. Similarly, the equality $a_1^2 = a_0 t_0^2 + b_0$ implies that (at least) one of the relations

- $\mathrm{ord}(a_1) = \frac{1}{4}$
- $\mathrm{ord}(a_0) = 0$ and $\mathrm{ord}(a_1) \geq \frac{1}{4}$

holds. Combining these shows that $\mathrm{ord}(a_0) = \frac{1}{4}$ and $\mathrm{ord}(a_1) = \frac{1}{4}$.

In particular, a_1^p has nonzero image in $\mathcal{O}^\flat/(\varpi)$, and

$$\mathrm{ord}(\mathrm{HT}_0(\lambda)) = \mathrm{ord}(a_1^p) = \tfrac{1}{2}.$$

On the other hand, b_0^p has trivial image in $\mathcal{O}^\flat/(\varpi)$, and so

$$\mathrm{ord}(\mathrm{HT}_0(\Pi\lambda)) \geq 1.$$

Therefore

$$\mathrm{ord}(\tau_0) = \mathrm{ord}(\mathrm{HT}_0(\Pi\lambda)) - \mathrm{ord}(\mathrm{HT}_0(\lambda)) \geq \tfrac{1}{2}.$$

The same reasoning shows that $\mathrm{ord}(\tau_1) \geq \frac{1}{2}$. As $\tau_0\tau_1 = p$ by Proposition 3.1.2, we must therefore have

$$\mathrm{ord}(\tau_0) = \tfrac{1}{2} = \mathrm{ord}(\tau_1),$$

completing the proof of Proposition 3.3.4. □

3.4. ***The case*** $\boldsymbol{\Pi\Omega_1(G_k^\vee) \neq 0}$**.** We assume throughout Section 3.4 that

$$\Pi\Omega_1(G_k^\vee) \neq 0.$$

Once again, we will analyze the structure of $M^\flat = M/pM$, and use this to bound the Hodge–Tate periods of G. As in Section 3.3, the first step is to choose a convenient basis for $M^\flat$.

Lemma 3.4.1. *There are $\mathcal{O}^\flat$-bases $e_0, f_0 \in M_0^\flat$ and $e_1, f_1 \in M_1^\flat$ such that the operator $\Pi \in \Delta$ satisfies*

$$\Pi e_0 = 0, \qquad \Pi e_1 = 0, \quad \Pi f_0 = e_1, \quad \Pi f_1 = e_0, \tag{3.4.1}$$

and such that ψ satisfies

$$\psi(e_0) = e_1, \quad \psi(e_1) = te_0, \quad \psi(f_0) = tf_1, \quad \psi(f_1) = se_0 + f_0 \tag{3.4.2}$$

for some scalars $s, t \in \mathcal{O}^\flat$ with $\mathrm{ord}(t) = 1/p$. Moreover:

(1) *For any such basis, G_k is superspecial if and only if $\mathrm{ord}(s) > 0$.*

(2) *If G_k is not superspecial such a basis can be found with $s = 1$.*

Proof. Exactly as in the proof of Lemma 3.3.1, we may choose a basis such that (3.4.1) holds, and such that

$$\psi(e_0) = t_0 e_1, \quad \psi(e_1) = t_1 e_0, \quad \psi(f_0) = u_1 e_1 + t_1 f_1, \quad \psi(f_1) = u_0 e_0 + t_0 f_0$$

for some $u_0, u_1, t_0, t_1 \in \mathcal{O}^\flat$ with $\mathrm{ord}(t_0) + \mathrm{ord}(t_1) = 1/p$.

The Δ-module $\Omega(G_k^\vee)$ is identified with the image of

$$N^\flat \to (\sigma^* M^\flat)/\mathfrak{m}^\flat(\sigma^* M^\flat),$$

and this identifies $\Omega_1(G_k^\vee)$ with the (one-dimensional) k-span of the vectors

$$1 \otimes \psi(e_1) = t_1^p \otimes e_0, \quad 1 \otimes \psi(f_1) = u_0^p \otimes e_0 + t_0^p \otimes f_0$$

in $(\sigma^* M_0^\flat)/\mathfrak{m}^\flat(\sigma^* M^\flat)$. The assumption that Π does not annihilate $\Omega_1(G_k^\vee)$ implies that $\mathrm{ord}(t_0) = 0$, which allows us to rescale our basis vectors to make $t_0 = 1$, and then add a multiple of e_0 to f_0 to make $u_1 = 0$. Setting $t = t_1$ and $s = u_0$, the relations (3.4.2) now hold.

It follows from Proposition 3.2.3 and Theorem 2.2.2 that

$$G_k \text{ is superspecial} \iff V^2(D/pD) = 0 \iff \psi^2(M^\flat/\mathfrak{m}^\flat M^\flat) = 0 \iff \mathrm{ord}(s) > 0.$$

Finally, if $\mathrm{ord}(s) = 0$ it is an easy exercise in linear algebra to see that the given basis elements can be rescaled to make $s = 1$. □

As in Section 3.3, our fixed generator $\lambda \in T_p(G)$ determines an element

$$a_0 e_0 + a_1 e_1 + b_0 f_0 + b_1 f_1 \in M^\flat,$$

where the coefficients $a_0, a_1, b_0, b_1 \in \mathcal{O}^\flat$ satisfy

$$a_0^p = a_1 t^p + b_1 s^p, \quad a_1^p = a_0, \quad b_0^p = b_1, \quad b_1^p = b_0 t^p. \tag{3.4.3}$$

As in Section 3.3, we may identify

$$\Omega(G^\vee)/p\Omega(G^\vee) = N/(pN + \xi\sigma^* M) = N^\flat/\varpi\sigma^* M^\flat$$

with the direct summand of $\sigma^* M^\flat/\varpi\sigma^* M^\flat$ generated by the reductions of

$$1 \otimes \psi(e_0) = 1 \otimes e_1 \in \sigma^* M^\flat, \quad 1 \otimes \psi(f_1) = s^p \otimes e_0 + 1 \otimes f_0 \in \sigma^* M^\flat.$$

If we use this basis to identify

$$\Omega(G^\vee)/p\Omega(G^\vee) = N^\flat/\varpi\sigma^* M^\flat \cong \mathcal{O}^\flat/(\varpi) \oplus \mathcal{O}^\flat/(\varpi)$$

then, using Theorem 2.2.2, the partial Hodge–Tate morphisms

$$T_p(G)/pT_p(G) \xrightarrow{\mathrm{HT}_0} \Omega_0(G^\vee)/p\Omega_0(G^\vee) \cong \mathcal{O}^\flat/(\varpi),$$
$$T_p(G)/pT_p(G) \xrightarrow{\mathrm{HT}_1} \Omega_1(G^\vee)/p\Omega_1(G^\vee) \cong \mathcal{O}^\flat/(\varpi)$$

satisfy

$$\begin{aligned} \mathrm{HT}_0(\lambda) &= a_0, & \mathrm{HT}_0(\Pi\lambda) &= b_1, \\ \mathrm{HT}_1(\lambda) &= b_1, & \mathrm{HT}_1(\Pi\lambda) &= 0. \end{aligned} \tag{3.4.4}$$

Lemma 3.4.2. *We have*

$$\mathrm{ord}(b_0)=\frac{1}{p^2-1},\quad \mathrm{ord}(b_1)=\frac{p}{p^2-1}.$$

Moreover,

$$\mathrm{ord}(a_0)\geq\frac{1}{p^2-1},\quad \mathrm{ord}(a_1)\geq\frac{1}{p(p^2-1)},$$

and G_k is superspecial if and only if one (equivalently, both) of these inequalities is strict.

Proof. Exactly as in the proof of Lemma 3.3.2, both b_0 and b_1 are nonzero. The relations (3.4.3) therefore imply that

$$b_0^{p^2-1}=t^p,$$

from which the stated formulas for $\mathrm{ord}(b_0)$ and $\mathrm{ord}(b_1)=\mathrm{ord}(b_0^p)$ are clear.

The relations (3.4.3) imply that a_0 is a root of $x^{p^2}-xt^{p^2}-b_1^p s^{p^2}$, and by examination of the Newton polygon we see that

$$\mathrm{ord}(a_0)\geq\frac{1}{p^2-1}$$

with strict inequality if and only if $\mathrm{ord}(s)>0$. Combining this with $a_1^p=a_0$ completes the proof. □

Lemma 3.4.3. *If G_k is not superspecial then*

$$\varpi(a_0/b_1)^{p+1}\in(\mathcal{O}^\flat)^\times\quad\textit{and}\quad\varpi s^{p+1}/t^p\in(\mathcal{O}^\flat)^\times,$$

and these units have the same reduction to $k^\times$.

Proof. We have already noted that (3.4.3) implies $t^p=b_0^{p^2-1}$, from which one easily deduces the equality

$$\left(\frac{a_1}{b_0}\right)^{p^2}=\frac{a_1}{b_0}+\frac{s^p}{b_0^{p(p-1)}}$$

in the fraction field of $\mathcal{O}^\flat$. It follows from this and Lemma 3.4.2 that

$$\varpi^{p/(p+1)}\left(\frac{a_1}{b_0}\right)^{p^2}\quad\text{and}\quad\left(\frac{\varpi^{1/(p+1)}s}{b_0^{p-1}}\right)^p$$

are units in $\mathcal{O}^\flat$ with the same reduction to $k^\times$, hence the same is true after raising both to the power $(p+1)/p$. The lemma follows easily from this and the relations (3.4.3). □

Proposition 3.4.4. *If we assume, as above, that $\Pi\Omega_1(G_k^\vee)\neq 0$ then*

$$\frac{p}{p+1}\leq\mathrm{ord}(\tau_1)\tag{3.4.5}$$

with strict inequality if and only if G_k is superspecial. Moreover, if equality holds then

$$-\frac{p}{\tau_0^{p+1}}\in\mathcal{O}^\times\quad\textit{and}\quad\frac{\varpi s^{p+1}}{t^p}\in(\mathcal{O}^\flat)^\times$$

have the same reduction to $k^\times$.

Proof. Using (3.4.4) and Lemma 3.4.2, we find that

$$\mathrm{ord}(\mathrm{HT}_0(\Pi\lambda)) = \frac{p}{p^2-1},$$

and that

$$\mathrm{ord}(\mathrm{HT}_0(\lambda)) \geq \frac{1}{p^2-1}$$

with strict inequality if and only if G_k is superspecial. This implies that

$$\mathrm{ord}(\tau_0) = \mathrm{ord}(\mathrm{HT}_0(\Pi\lambda)) - \mathrm{ord}(\mathrm{HT}_0(\lambda)) \leq \frac{1}{p+1}$$

with strict inequality if and only if G_k is superspecial. The inequality (3.4.5) follows from this and the relation $\tau_0\tau_1 = p$ of Proposition 3.1.2, with strict inequality if and only if G_k is superspecial.

Suppose that equality holds in (3.4.5), so that G_k is not superspecial. Choose an $\alpha \in \mathcal{O}^\flat$ satisfying $\alpha^{p^2-1} = \varpi$. The construction of Section 2.1 determines an element $\alpha^\sharp \in \mathcal{O}$ whose image in $\mathcal{O}/(p) \cong \mathcal{O}^\flat/(\varpi)$ agrees with α.

Combining the relations (3.4.4) with Lemma 3.4.2 shows that

$$\frac{\mathrm{HT}_0(\Pi\lambda)}{(\alpha^\sharp)^p} \in \mathcal{O}^\times \quad \text{and} \quad \frac{b_1}{\alpha^p} \in (\mathcal{O}^\flat)^\times$$

have the same reduction to $k^\times$, as do

$$\frac{\mathrm{HT}_0(\lambda)}{\alpha^\sharp} \in \mathcal{O}^\times \quad \text{and} \quad \frac{a_0}{\alpha} \in (\mathcal{O}^\flat)^\times.$$

It follows that

$$\frac{\tau_0}{(\alpha^\sharp)^{p-1}} \in \mathcal{O}^\times \quad \text{and} \quad \frac{b_1}{a_0\alpha^{p-1}} \in (\mathcal{O}^\flat)^\times$$

have the same reduction to $k^\times$. Raising both to the power $p+1$ and applying Lemma 3.4.3 proves that

$$\frac{\varpi^\sharp}{\tau_0^{p+1}} \in \mathcal{O}^\times \quad \text{and} \quad \frac{\varpi s^{p+1}}{t^p} \in (\mathcal{O}^\flat)^\times$$

have the same reduction to $k^\times$. Now apply Lemma 2.1.1. □

4. The main results

We now formulate and prove our main results on the Ekedahl–Oort stratification of the Hodge–Tate period domain X defined by (1.2.1). Throughout Section 4 we assume that the conclusions of Theorem 2.2.2 hold. For example, it is enough to assume that $p > 2$.

4.1. ***The setup.*** Let T be a free Δ-module of rank one, and fix a generator $\lambda \in T$. Use the embeddings (1.3.1) to decompose

$$T \otimes_{\mathbb{Z}_p} C = T_{C,0} \oplus T_{C,1}$$

as a direct sum of 2-dimensional C-subspaces, in such a way that the action of $\mathbb{Z}_{p^2} \subset \Delta$ on the summands is through j_0 and j_1, respectively. Using the projection maps to the two factors, we obtain injective $\mathbb{Z}_p$-linear maps

$$q_0 : T \to T_{C,0}, \qquad q_1 : T \to T_{C,1}.$$

To each $\tau \in C \cup \{\infty\}$ we associate the Δ-stable plane

$$W_\tau \subset T \otimes_{\mathbb{Z}_p} C$$

spanned by the two vectors

$$\tau q_0(\lambda) - q_0(\Pi\lambda) \in T_{C,0}, \qquad p q_1(\lambda) - \tau q_1(\Pi\lambda) \in T_{C,1}.$$

The construction $\tau \mapsto W_\tau$ establishes a bijection

$$C \cup \{\infty\} \cong X(C).$$

Remark 4.1.1. It is not hard to see that the above bijection $\mathbb{P}^1(C) \cong X(C)$ arises from an isomorphism of schemes over $\mathbb{Q}_{p^2}$. The isomorphism cannot descend to $\mathbb{Q}_p$, for the simple reason that $X(\mathbb{Q}_p) = \varnothing$.

For the rest of Section 4.1 and Section 4.2 we hold $\tau \in C \cup \{\infty\}$ fixed, and let G be the p-divisible group over $\mathcal{O}$ determined by the pair (T, W_τ). Thus G comes equipped with an action of Δ, and Δ-linear identifications

$$\begin{array}{ccc} T_p(G) & \xrightarrow{\ \mathrm{HT}\ } & \Omega(G^\vee) \otimes_{\mathcal{O}} C \\ \Vert & & \Vert \\ T & \longrightarrow & (T \otimes_{\mathbb{Z}_p} C)/W_\tau \end{array}$$

In the notation of Section 3.1, the Hodge–Tate periods of G are

$$\tau_0 = \tau \quad \text{and} \quad \tau_1 = p/\tau. \tag{4.1.1}$$

4.2. ***Computing the reduction.*** Let G_k be the reduction of G to the residue field $k = \mathcal{O}/\mathfrak{m}$, and let (D, F, V) be its covariant Dieudonné module. We will show how to compute the isomorphism class of $G_k[p]$ from the Hodge–Tate periods (4.1.1).

Let $\mathbb{D} = \Delta \otimes_{\mathbb{Z}_p} k$ with its natural action of Δ by left multiplication. The embeddings (1.3.1) induce a decomposition

$$\mathbb{D} = \mathbb{D}_0 \oplus \mathbb{D}_1$$

in which $\mathbb{Z}_{p^2} \subset \Delta$ acts on $\mathbb{D}_i$ through the composition of $j_i : \mathbb{Z}_{p^2} \to \mathcal{O}$ with the reduction map $\mathcal{O} \to k$.

Choose k-bases

$$e_0,\ f_0 \in \mathbb{D}_0, \qquad e_1,\ f_1 \in \mathbb{D}_1$$

in such a way that $\Pi \in \Delta$ acts as

$$\Pi e_0 = 0, \quad \Pi e_1 = 0, \quad \Pi f_0 = e_1, \quad \Pi f_1 = e_0. \tag{4.2.1}$$

Theorem 4.2.1. *The inequalities*

$$\frac{1}{p+1} < \mathrm{ord}(\tau) < \frac{p}{p+1} \tag{4.2.2}$$

hold if and only if $\Pi\Omega(G_k^\vee) = 0$. *When these conditions hold, there is a* Δ*-linear isomorphism* $D/pD \cong \mathbb{D}$ *under which*

$$\begin{aligned} Fe_0 &= 0, & Ff_0 &= e_1, & Fe_1 &= 0, & Ff_1 &= e_0, \\ Ve_0 &= 0, & Vf_0 &= e_1, & Ve_1 &= 0, & Vf_1 &= e_0. \end{aligned}$$

Proof. If $\Pi\Omega(G_k^\vee) \neq 0$ then either $\Pi\Omega_1(G_k^\vee) \neq 0$ or $\Pi\Omega_0(G_k^\vee) \neq 0$. In the first case Proposition 3.4.4 implies

$$\frac{p}{p+1} \leq \mathrm{ord}(\tau_1).$$

In the second case the same proof, with indices 0 and 1 interchanged throughout, shows that

$$\frac{p}{p+1} \leq \mathrm{ord}(\tau_0).$$

In either case, these bounds imply that (4.2.2) fails.

Now assume that $\Pi\Omega(G_k^\vee) = 0$. We have already proved in Proposition 3.3.4 that (4.2.2) holds, and so it only remains to prove that D/pD admits an isomorphism to $\mathbb{D}$ with the prescribed properties.

Let $e_0,\ f_0 \in M_0^\flat$ and $e_1,\ f_1 \in M_1^\flat$ be the bases of Lemma 3.3.1. Using the formula for $\psi : M^\flat \to M^\flat$ prescribed in that lemma, and the relation $\phi \circ \psi = \varpi$, one can write down an explicit formula for ϕ, and then see that the induced operators on the reduction $M^\flat/\mathfrak{m}^\flat M^\flat$ are given by

$$\begin{aligned} \phi(e_0) &= 0, & \phi(f_0) &= ue_1, & \phi(e_1) &= 0, & \phi(f_1) &= ue_0, \\ \psi(e_0) &= 0, & \psi(f_0) &= e_1, & \psi(e_1) &= 0, & \psi(f_1) &= e_0, \end{aligned}$$

where $u^{-1} \in k^\times$ is the reduction of $-t_0^p t_1^p/\varpi \in (\mathcal{O}^\flat)^\times$.

The images of $e_0,\ f_0,\ e_1,\ f_1$ under the bijection

$$M^\flat/\mathfrak{m}^\flat M^\flat \xrightarrow{x \mapsto 1 \otimes x} \sigma^*(M^\flat/\mathfrak{m}^\flat M^\flat) \cong M_{\mathrm{crys}}/pM_{\mathrm{crys}} \cong D/pD$$

provided by Theorem 2.2.2 form a k-basis of D/pD, denoted the same way, satisfying the relations (4.2.1) and

$$\begin{aligned} Fe_0 &= 0, & Ff_0 &= u^p e_1, & Fe_1 &= 0, & Ff_1 &= u^p e_0, \\ Ve_0 &= 0, & Vf_0 &= e_1, & Ve_1 &= 0, & Vf_1 &= e_0. \end{aligned}$$

It remains to prove that $u=1$. The two embeddings (1.3.1) reduce to morphisms $j_0, j_1 : \mathbb{Z}_{p^2} \to k$, which then admit unique lifts to

$$j_0, j_1 : \mathbb{Z}_{p^2} \to W(k).$$

This allows us to decompose $D = D_0 \oplus D_1$ as W-modules, where $\mathbb{Z}_{p^2} \subset \Delta$ acts on the two summands via j_0 and j_1, respectively. Choose arbitrary lifts

$$\tilde{f}_0 \in D_0, \qquad \tilde{f}_1 \in D_1$$

of f_0 and f_1, and then define

$$\tilde{e}_0 = \Pi \tilde{f}_1 \in D_0, \qquad \tilde{e}_1 = \Pi \tilde{f}_0 \in D_1.$$

Using the fact that Π and V commute, we see that

$$\begin{aligned} V\tilde{e}_0 &= pb_1\tilde{e}_1 + pa_1\tilde{f}_1, & V\tilde{f}_0 &= a_1\tilde{e}_1 + pb_1\tilde{f}_1, \\ V\tilde{e}_1 &= pb_0\tilde{e}_0 + pa_0\tilde{f}_0, & V\tilde{f}_1 &= a_0\tilde{e}_0 + pb_0\tilde{f}_0, \end{aligned}$$

for scalars

$$a_0, a_1 \in 1 + pW(k), \qquad b_0, b_1 \in W(k).$$

Denote again by $\sigma : W(k) \to W(k)$ the lift of the Frobenius on k. Applying F to the expressions for $V\tilde{e}_1$ and $V\tilde{f}_1$ results in

$$p\tilde{e}_1 = \sigma(pb_0)F\tilde{e}_0 + \sigma(pa_0)F\tilde{f}_0, \qquad p\tilde{f}_1 = \sigma(a_0)F\tilde{e}_0 + \sigma(pb_0)F\tilde{f}_0,$$

from which one deduces

$$\big(\sigma(a_0)^2 - p\sigma(b_0)^2\big) \cdot F\tilde{f}_0 = \sigma(a_0)\tilde{e}_1 - p\sigma(b_0)\tilde{f}_1.$$

Reducing this modulo p proves that $Ff_0 = e_1$, and hence $u=1$. □

Theorem 4.2.2. *The inequality*

$$\mathrm{ord}(\tau) \le \frac{1}{p+1} \tag{4.2.3}$$

holds if and only if $\Pi\Omega_1(G_k^\vee) \neq 0$. *Moreover*:

(1) *If strict inequality holds in* (4.2.3), *there is a* Δ*-linear isomorphism* $D/pD \cong \mathbb{D}$ *under which*

$$\begin{aligned} Fe_0 &= e_1, & Ff_0 &= 0, & Fe_1 &= 0, & Ff_1 &= f_0, \\ Ve_0 &= e_1, & Vf_0 &= 0, & Ve_1 &= 0, & Vf_1 &= f_0. \end{aligned}$$

(2) *If equality holds in* (4.2.3), *there is a* Δ*-linear isomorphism* $D/pD \cong \mathbb{D}$ *under which*

$$\begin{aligned} Fe_0 &= u^p e_1, & Ff_0 &= -u^p e_1, & Fe_1 &= 0, & Ff_1 &= u^p f_0, \\ Ve_0 &= e_1, & Vf_0 &= 0, & Ve_1 &= 0, & Vf_1 &= e_0 + f_0, \end{aligned} \tag{4.2.4}$$

where u *is the image of* $-p/\tau_0^{p+1} = -p/\tau^{p+1}$ *under* $\mathcal{O}^\times \to k^\times$.

Proof. If (4.2.3) holds then Theorem 4.2.1 implies that $\Pi\Omega(G_k^\vee)\neq 0$, and so either

$$\Pi\Omega_0(G_k^\vee)\neq 0 \quad \text{or} \quad \Pi\Omega_1(G_k^\vee)\neq 0.$$

The first possibility cannot occur, as then the proof of Proposition 3.4.4, with the indices 0 and 1 reversed everywhere, would give the bound

$$\frac{p}{p+1}\leq \operatorname{ord}(\tau_0),$$

contradicting (4.2.3). Conversely, if $\Pi\Omega_1(G_k^\vee)\neq 0$ then (4.2.3) holds by Proposition 3.4.4.

Assume now that (4.2.3) holds, and that $\Pi\Omega_1(G_k^\vee)\neq 0$. Let $e_0, f_0\in M_0^\flat$ and $e_1, f_1\in M_1^\flat$ be the bases of Lemma 3.4.1. As in the proof of Theorem 4.2.1, the operator ϕ on $M^\flat$ can be computed from the formula for ψ given in the lemma. The induced operators on the reduction $M^\flat/\mathfrak{m}^\flat M^\flat$ are found to be

$$\begin{aligned} &\phi(e_0)=ue_1, && \phi(f_0)=-uv^pe_1, && \phi(e_1)=0, && \phi(f_1)=uf_0,\\ &\psi(e_0)=e_1, && \psi(f_0)=0, && \psi(e_1)=0, && \psi(f_1)=ve_0+f_0, \end{aligned}$$

where $u\in k^\times$ is the reduction of $\varpi/t^p\in(\mathcal{O}^\flat)^\times$, and $v\in k$ is the reduction of $s\in\mathcal{O}^\flat$. By the final claim of Lemma 3.4.1, we may further assume that

$$v=\begin{cases}0 & \text{if } G_k \text{ is superspecial,}\\ 1 & \text{otherwise.}\end{cases}$$

Suppose that strict inequality holds in (4.2.3). Proposition 3.4.4 tells us that G_k is superspecial, and so $v=0$. The images of e_0, f_0, e_1, f_1 under the bijection

$$M^\flat/\mathfrak{m}^\flat M^\flat \xrightarrow{x\mapsto 1\otimes x} \sigma^*(M^\flat/\mathfrak{m}^\flat M^\flat)\cong M_{\mathrm{crys}}/pM_{\mathrm{crys}}\cong D/pD$$

provided by Theorem 2.2.2 form a k-basis of D/pD, denoted the same way, satisfying the relations (4.2.1) and

$$\begin{aligned} &Fe_0=u^pe_1, && Ff_0=0, && Fe_1=0, && Ff_1=u^pf_0,\\ &Ve_0=e_1, && Vf_0=0, && Ve_1=0, && Vf_1=f_0. \end{aligned}$$

One can prove that $u=1$ by lifting the basis elements to D and arguing exactly as in Theorem 4.2.1.

Suppose now that equality holds in (4.2.3). Proposition 3.4.4 implies that G_k is not superspecial, so $v=1$, and that the reduction map $\mathcal{O}^\times\to k^\times$ sends $-p/\tau_0^{p+1}\mapsto u$. Arguing as in the previous paragraph, we obtain a k-basis e_0, f_0, e_1, f_1 of D/pD satisfying (4.2.1) and (4.2.4), completing the proof. □

Theorem 4.2.3. *The inequality*

$$\frac{p}{p+1}\leq \operatorname{ord}(\tau) \tag{4.2.5}$$

holds if and only if $\Pi\Omega_0(G_k^\vee)\neq 0$. *Moreover*:

(1) *If strict inequality holds in* (4.2.5), *there is a* Δ*-linear isomorphism* $D/pD \cong \mathbb{D}$ *under which*

$$\begin{aligned} Fe_0 &= 0, & Ff_0 &= f_1, & Fe_1 &= e_0, & Ff_1 &= 0, \\ Ve_0 &= 0, & Vf_0 &= f_1, & Ve_1 &= e_0, & Vf_1 &= 0. \end{aligned}$$

(2) *If equality holds in* (4.2.5), *there is a* Δ*-linear isomorphism* $D/pD \cong \mathbb{D}$ *under which*

$$\begin{aligned} Fe_0 &= 0, & Ff_0 &= u^p f_1, & Fe_1 &= u^p e_0, & Ff_1 &= -u^p e_0, \\ Ve_0 &= 0, & Vf_0 &= e_1 + f_1, & Ve_1 &= e_0, & Vf_1 &= 0. \end{aligned} \tag{4.2.6}$$

where u *is the image of* $-p/\tau_1^{p+1} = -\tau^{p+1}/p^p$ *under* $\mathcal{O}^\times \to k^\times$.

Proof. Recalling (4.1.1), the inequality (4.2.5) is equivalent to

$$\mathrm{ord}(\tau_1) \le \frac{1}{p+1}.$$

Using this observation, the proof is identical to that of Theorem 4.2.2, but with the indices 0 and 1 reversed everywhere. ☐

Corollary 4.2.4. *The* p*-divisible group* G_k *is superspecial if and only if*

$$\mathrm{ord}(\tau) \notin \left\{ \frac{1}{p+1}, \frac{p}{p+1} \right\}.$$

Moreover, the superspecial locus of $X(C)$ *is a union of three Ekedahl–Oort strata, characterized as follows*:

(1) *The subset of* $X(C)$ *defined by*

$$\frac{1}{p+1} < \mathrm{ord}(\tau) < \frac{p}{p+1}$$

is an Ekedahl–Oort stratum. On this stratum $\Pi\Omega(G_k^\vee) = 0$.

(2) *The subset of* $X(C)$ *defined by*

$$\mathrm{ord}(\tau) < \frac{1}{p+1},$$

is an Ekedahl–Oort stratum. On this stratum $\Pi\Omega_1(G_k^\vee) \ne 0$.

(3) *The subset of* $X(C)$ *defined by*

$$\mathrm{ord}(\tau) > \frac{p}{p+1}.$$

is an Ekedahl–Oort stratum. On this stratum $\Pi\Omega_0(G_k^\vee) \ne 0$.

Proof. Recall from Proposition 3.2.3 that G_k is superspecial if and only if V^2 annihilates D/pD. Given this, all parts of the claim are clear from Theorems 4.2.1, 4.2.2, and 4.2.3. ☐

Now consider the locus of points

$$\left\{\tau \in C : \mathrm{ord}(\tau) = \frac{1}{p+1}\right\} \cup \left\{\tau \in C : \mathrm{ord}(\tau) = \frac{p}{p+1}\right\} \subset X(C)$$

at which the corresponding p-divisible group does not have superspecial reduction. This set is a union of infinitely many Ekedahl–Oort strata.

Corollary 4.2.5. *The fibers of the composition*

$$\left\{\tau \in C : \mathrm{ord}(\tau) = \frac{1}{p+1}\right\} \xrightarrow{\tau \mapsto p/\tau^{p+1}} \mathcal{O}^\times \to k^\times$$

are Ekedahl–Oort strata, *as are the fibers of the composition*

$$\left\{\tau \in C : \mathrm{ord}(\tau) = \frac{p}{p+1}\right\} \xrightarrow{\tau \mapsto \tau^{p+1}/p^p} \mathcal{O}^\times \to k^\times.$$

Proof. For each $u \in k^\times$ let F_u and V_u be the operators on $\mathbb{D}$ defined by (4.2.4). Note that V_u is actually independent of u. We claim that the existence of a Δ-linear isomorphism

$$(\mathbb{D}, F_u, V_u) \xrightarrow{\phi} (\mathbb{D}, F_{u'}, V_{u'})$$

implies $u = u'$. To see this one checks that the first relation in

$$\phi \circ V_u = V_{u'} \circ \phi, \quad \phi \circ F_u = F_{u'} \circ \phi \tag{4.2.7}$$

implies that ϕ has the form

$$\phi(e_0) = ae_0, \quad \phi(e_1) = ae_1, \quad \phi(f_0) = af_0, \quad \phi(f_1) = af_1 + be_1$$

for some $a \in \mathbb{F}_p$ and $b \in k$. Using this, one checks that ϕ commutes with both F_u and $F_{u'}$. The second relation in (4.2.7) then implies that $F_u = F_{u'}$, and hence $u = u'$.

The same is true if we replace the operators of (4.2.4) with those of (4.2.6), and so the corollary follows from Theorems 4.2.2 and 4.2.3. □

Acknowledgements

The author thanks Keerthi Madapusi Pera for helpful conversations, and the anonymous referees for numerous suggestions and corrections.

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Communicated by Michael Rapoport
Received 2019-02-28 Revised 2020-01-03 Accepted 2020-02-06

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Algebra & Number Theory

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Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

ANT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Algebra & Number Theory

Volume 14 No. 4 2020

1937-0652(2020)14:4;1-Y