Research Article

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Linear invariance of intersections on unitary Rapoport–Zink spaces

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Abstract: We prove an invariance property of intersections of Kudla–Rapoport divisors on a unitary Rapoport–Zink space.

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1 Introduction

Let p be a prime, let $\mathbf{k}$ be a quadratic extension of $\mathbb{Q}_p$, and let $\mathcal{O}_\mathbf{k} \subset \mathbf{k}$ be the ring of integers. Denote by $\breve{\mathbf{k}}$ the completion of the maximal unramified extension of $\mathbf{k}$, let $\breve{\mathcal{O}}_\mathbf{k} \subset \breve{\mathbf{k}}$ be the ring of integers, and let $\breve{\mathfrak{m}} \subset \breve{\mathcal{O}}_\mathbf{k}$ be the maximal ideal. The nontrivial automorphism of $\mathbf{k}$ is denoted by $\alpha \mapsto \overline{\alpha}$, and we denote by

$$\varphi, \overline{\varphi} : \mathcal{O}_\mathbf{k} \to \breve{\mathcal{O}}_\mathbf{k}$$

the inclusion and its conjugate $\overline{\varphi}(\alpha) = \varphi(\overline{\alpha})$, respectively.

Hypothesis A. Throughout the paper we assume that either $\mathbf{k}/\mathbb{Q}_p$ is unramified, or that $\mathbf{k}/\mathbb{Q}_p$ is ramified but $p > 2$.

In this paper, we study the intersections of special divisors on a regular n-dimensional Rapoport–Zink formal scheme

$$M = M_{(1,0)} \times_{\mathrm{Spf}(\breve{\mathcal{O}}_\mathbf{k})} M_{(n-1,1)},$$

flat over $\mathrm{Spf}(\breve{\mathcal{O}}_\mathbf{k})$. We have imposed Hypothesis A because it is assumed in [7, 10], the results of which are needed to prove the flatness and regularity of M.

The construction of M depends on the choices of supersingular p-divisible groups $\mathbf{X}_0$ and $\mathbf{X}$ of dimensions 1 and $n \geq 2$, respectively, defined over the residue field $\breve{\mathcal{O}}_\mathbf{k}/\breve{\mathfrak{m}}$ and endowed with principal polarizations and actions of $\mathcal{O}_\mathbf{k}$. The induced actions of $\mathcal{O}_\mathbf{k}$ on the Lie algebras $\mathrm{Lie}(\mathbf{X}_0)$ and $\mathrm{Lie}(\mathbf{X})$ are required to satisfy signature conditions of type $(1, 0)$ and $(n-1, 1)$, respectively.

The precise assumptions on $\mathbf{X}_0$ and $\mathbf{X}$, along with the definition of M, are explained in Section 2. We note here only that the signature condition on $\mathbf{X}$ consists of the extra data of a codimension one subspace $F_\mathbf{X} \subset \mathrm{Lie}(\mathbf{X})$ as in the work of Krämer [7]. In particular, when $\mathbf{k}/\mathbb{Q}_p$ is ramified, our formal scheme $M_{(n-1,1)}$ does not agree with the one considered in [12].

As in [8], the n-dimensional $\mathbf{k}$-vector space

$$V = \mathrm{Hom}_{\mathcal{O}_\mathbf{k}}(\mathbf{X}_0, \mathbf{X})[1/p]$$

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carries a natural hermitian form, and every nonzero vector $x \in V$ determines a *Kudla–Rapoport divisor* $Z(x) \subset M$; see Definition 2.4. Our main result concerns arbitrary intersections of Kudla–Rapoport divisors, including self-intersections.

For any nonzero $x \in V$, let $I_{Z(x)} \subset \mathcal{O}_M$ be the ideal sheaf defining $Z(x)$, and define a chain complex of locally free $\mathcal{O}_M$-modules

$$C(x) = (\cdots \to 0 \to I_{Z(x)} \to \mathcal{O}_M \to 0)$$

supported in degrees 1 and 0. We extend the definition to $x = 0$ by setting

$$C(0) = (\cdots \to 0 \to \omega \xrightarrow{0} \mathcal{O}_M \to 0),$$

where ω is the line bundle of modular forms on M of Definition 3.4. This line bundle controls the deformation theory of the Kudla–Rapoport divisors, in a sense made (somewhat) more precise in Section 4.

The following is our main result. It is stated in the text as Theorem 5.1.

Theorem B. *Fix an $r \geq 0$, and suppose $x_1, \dots, x_r \in V$ and $y_1, \dots, y_r \in V$ generate the same $\mathcal{O}_{\mathbf{k}}$-submodule. For every $i \geq 0$ there is an isomorphism of coherent $\mathcal{O}_M$-modules*

$$H_i(C(x_1) \otimes \cdots \otimes C(x_r)) \cong H_i(C(y_1) \otimes \cdots \otimes C(y_r)).$$

We can restate our main result in terms of the Grothendieck group of coherent sheaves on M. Let $K_0'(M)$ be the free abelian group generated by symbols $[F]$ as F runs over all isomorphism classes of coherent $\mathcal{O}_M$-modules, subject to the relations $[F_1] + [F_3] = [F_2]$ whenever there is a short exact sequence

$$0 \to F_1 \to F_2 \to F_3 \to 0.$$

In particular, any bounded chain complex F of coherent $\mathcal{O}_M$-modules defines a class

$$[F] = \sum_i (-1)^i \cdot [H_i(F)] \in K_0'(M),$$

allowing us to form

$$[C(x_1) \otimes \cdots \otimes C(x_r)] \in K_0'(M) \tag{1.1}$$

for any finite list of vectors $x_1, \dots, x_r \in V$. If all $x_1, \dots, x_r$ are nonzero, then

$$[C(x_1) \otimes \cdots \otimes C(x_r)] = [\mathcal{O}_{Z(x_1)} \otimes^{\mathbf{L}} \cdots \otimes^{\mathbf{L}} \mathcal{O}_{Z(x_r)}],$$

and hence one should regard (1.1) as a generalized intersection of divisors. On the right-hand side, by slight abuse of notation, we are using the pushforward via $Z(x_i) \hookrightarrow M$ to view $\mathcal{O}_{Z(x_i)}$ as a coherent sheaf on $\mathcal{O}_M$, and $\otimes^{\mathbf{L}}$ is the derived tensor product of coherent $\mathcal{O}_M$-modules.

The following is an immediate consequence of Theorem B.

Corollary C. *If $x_1, \dots, x_r \in V$ and $y_1, \dots, y_r \in V$ generate the same $\mathcal{O}_{\mathbf{k}}$-submodule, then*

$$[C(x_1) \otimes \cdots \otimes C(x_r)] = [C(y_1) \otimes \cdots \otimes C(y_r)].$$

Perhaps the most interesting aspect of Corollary C is that it encodes nontrivial information about self-intersections of Kudla–Rapoport divisors. To spell this out in the simplest case, note that Corollary C implies

$$[C(x) \otimes C(x)] = [C(x) \otimes C(0)] \tag{1.2}$$

for any nonzero $x \in V$. The right-hand side is the alternating sum in $K_0'(M)$ of the homology of the complex

$$\cdots \to 0 \to I_{Z(x)} \otimes \omega \xrightarrow{\partial_2} I_{Z(x)} \oplus \omega \xrightarrow{\partial_1} \mathcal{O}_M \to 0,$$

where $\partial_2(a \otimes b) = (0, ab)$ and $\partial_1(a, b) = a$, and so

$$[C(x) \otimes C(0)] = [\mathcal{O}_M / I_{Z(x)}] - [\omega / I_{Z(x)} \omega].$$

If we again use pushforward via $Z(x) \hookrightarrow M$ to view coherent $\mathcal{O}_{Z(x)}$-modules as coherent $\mathcal{O}_M$-modules, then (1.2) can be rewritten as a self-intersection formula

$$[\mathcal{O}_{Z(x)} \otimes^{\mathbf{L}} \mathcal{O}_{Z(x)}] = [\mathcal{O}_{Z(x)}] - [\omega|_{Z(x)}]. \tag{1.3}$$

Because of the close connection between Grothendieck groups of coherent sheaves and Chow groups, as detailed in [14, Chapter I], the global analogue of Corollary C has applications to conjectures of Kudla–Rapoport [9] on the intersection multiplicities of cycles on unitary Shimura varieties, and their connection to derivatives of Eisenstein series. This will be explored in forthcoming work of the author.

The formal $\breve{\mathcal{O}}_{\mathbf{k}}$-scheme M is locally formally of finite type, but has countably many connected components, each of which is a countable union of irreducible components. Let us fix one connected component $M^\circ \subset M$, and set $Z^\circ(x) = Z(x)|_{M^\circ}$. The following is an immediate consequence of Theorem B.

Corollary D. *Suppose $x_1, \dots, x_n \in V$ is a $\mathbf{k}$-basis. The Serre intersection multiplicity*

$$\chi(\mathcal{O}_{Z^\circ(x_1)} \otimes^{\mathbf{L}} \cdots \otimes^{\mathbf{L}} \mathcal{O}_{Z^\circ(x_n)}) \stackrel{\text{def}}{=} \sum_{i,j\geq 0} (-1)^{i+j} \operatorname{length}_{\breve{\mathcal{O}}_{\mathbf{k}}} H^j(M^\circ, H_i(\mathcal{O}_{Z^\circ(x_1)} \otimes^{\mathbf{L}} \cdots \otimes^{\mathbf{L}} \mathcal{O}_{Z^\circ(x_n)}))$$

depends only on the $\mathcal{O}_{\mathbf{k}}$-lattice spanned by $x_1, \dots, x_n$.

It is conjectured by Kudla–Rapoport that the intersection multiplicity appearing in Corollary D is related to derivatives of representation densities. When $\mathbf{k}/\mathbb{Q}_p$ is unramified, this is [8, Conjecture 1.3]. When $\mathbf{k}/\mathbb{Q}_p$ is ramified, it is perhaps not clear what the precise statement of the conjecture should be.

Relation to previous results. Weaker versions of the results stated above can be proved using a simpler argument[1] of Terstiege [15]. We clarify here what one can and cannot prove using that argument.

When $\mathbf{k}/\mathbb{Q}_p$ is unramified, Corollary D is [15, Proposition 3.2]. Terstiege's argument can also be used to prove Theorem B and Corollary C, but only under the additional assumption that the vectors $x_1, \dots, x_r$ (equivalently, $y_1, \dots, y_r$) are linearly independent. In particular, his argument does not give self-intersection formulas like (1.2) and (1.3).

The key thing that makes Terstiege's argument work is that, in the unramified case, the Kudla–Rapoport divisors $Z(x)$ and $Z(x')$ defined by linearly independent vectors $x, x' \in V$ are flat over $\breve{\mathcal{O}}_{\mathbf{k}}$, from which it follows that their intersection $Z(x) \cap Z(x')$ lies in codimension 2.

When $\mathbf{k}/\mathbb{Q}_p$ is ramified, the situation is very different: the Kudla–Rapoport divisors are usually not flat, and the intersection $Z(x) \cap Z(x')$ is often of codimension 1. In fact, it is easy to see using Proposition A.3 that one can construct a basis $x_1, \dots, x_n \in V$ and an effective Cartier divisor $D \subset M$, contained in the special fiber (in the sense that the structure sheaf $\mathcal{O}_D$ is annihilated by a uniformizer in $\mathcal{O}_{\mathbf{k}}$), such that

$$D \subset Z(x_1) \cap \cdots \cap Z(x_n).$$

Because of this, the argument used by Terstiege breaks down in a fundamental way when $\mathbf{k}/\mathbb{Q}_p$ is ramified, and seems to yield little information in the direction of Theorem B and its corollaries.

The strategy of the proof. To explain the key idea underlying the proof of Theorem B, suppose we have vectors $x_1, x_2, y_1, y_2 \in V$ related by

$$y_1 = x_1 + a x_2, \quad y_2 = x_2$$

for some $a \in \mathcal{O}_{\mathbf{k}}$. In particular, $\{x_1, x_2\}$ and $\{y_1, y_2\}$ generate the same $\mathcal{O}_{\mathbf{k}}$-submodule of V.

One should imagine that there are global sections

$$s_1, s_2, t_1, t_2 \in H^0(M, \omega^{-1}) \tag{1.4}$$

satisfying $\operatorname{div}(s_i) = Z(x_i)$ and $\operatorname{div}(t_i) = Z(y_i)$, and also satisfying

$$t_1 = s_1 + a s_2, \quad t_2 = s_2. \tag{1.5}$$

Such sections would determine complexes

$$D(x_i) = (\cdots \to 0 \to \omega \xrightarrow{s_i} \mathcal{O}_M \to 0),$$

$$D(y_i) = (\cdots \to 0 \to \omega \xrightarrow{t_i} \mathcal{O}_M \to 0),$$

1 Terstiege only considers the case $n = 3$, but his argument generalizes to all n.

along with canonical isomorphisms

$$C(x_i) \cong D(x_i), \quad C(y_i) \cong D(y_i).$$

Indeed, if $x_i \neq 0$, then

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & \omega & \xrightarrow{s_i} & \mathcal{O}_M & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle s_i} & & \| & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & I_{Z(x_i)} & \longrightarrow & \mathcal{O}_M & \longrightarrow & 0 \end{array}$$

defines an isomorphism $D(x_i) \cong C(x_i)$. If $x_i = 0$, then $s_i = 0$, and $C(x_i)$ and $D(x_i)$ are equal simply by definition. The point of replacing the complexes $C(\cdot)$ by the isomorphic complexes $D(\cdot)$ is that relations (1.5) induce relations amongst the $D(\cdot)$, which allow one to write down (see the proof of Lemma 5.3) an explicit isomorphism

$$D(x_1) \otimes D(x_2) \cong D(y_1) \otimes D(y_2).$$

In this way one would obtain from (1.4) an isomorphism of complexes

$$C(x_1) \otimes C(x_2) \cong C(y_1) \otimes C(y_2). \tag{1.6}$$

Unfortunately, sections (1.4) with the required properties need not exist globally on M, and so neither does the isomorphism (1.6). Instead, our approach is to use Grothendieck–Messing theory to construct sections s_i and t_i defined only on the first-order infinitesimal neighborhoods of $Z(x_i)$ and $Z(y_i)$ in M. Working on a sufficiently fine Zariski open cover $\mathcal{U}$ of M, we then choose local approximations of these sections, and so obtain, by the method above, an isomorphism

$$C(x_1)_U \otimes C(x_2)_U \cong C(y_1)_U \otimes C(y_2)_U \tag{1.7}$$

over each $U \in \mathcal{U}$. Because there is no canonical way to choose these local approximations, the isomorphisms (1.7) need not glue together as $U \in \mathcal{U}$ varies. However, if one imposes mild restrictions on the local approximations, the homotopy class of (1.7) is independent of the choices. The resulting isomorphisms

$$H_i(C(x_1) \otimes C(x_2))_U \cong H_i(C(y_1) \otimes C(y_2))_U$$

of $\mathcal{O}_U$-modules can therefore be glued together as $U \in \mathcal{U}$ varies.

2 The Rapoport–Zink space and its divisors

Fix a triple $(\mathbf{X}_0, \mathbf{i}_0, \boldsymbol{\lambda}_0)$ in which

- $\mathbf{X}_0$ is a supersingular p-divisible group over $\breve{\mathcal{O}}_{\mathbf{k}}/\breve{\mathfrak{m}}$ of dimension 1;
- $\mathbf{i}_0 : \mathcal{O}_{\mathbf{k}} \to \mathrm{End}(\mathbf{X}_0)$ is an action of $\mathcal{O}_{\mathbf{k}}$ on $\mathbf{X}_0$ such that the induced action on $\mathrm{Lie}(\mathbf{X}_0)$ is through the inclusion $\varphi : \mathcal{O}_{\mathbf{k}} \to \breve{\mathcal{O}}_{\mathbf{k}}$;
- $\boldsymbol{\lambda}_0 : \mathbf{X}_0 \to \mathbf{X}_0^\vee$ is a principal polarization compatible with the $\mathcal{O}_{\mathbf{k}}$-action, in the sense that the induced Rosati involution $\dagger$ satisfies $\mathbf{i}_0(\alpha)^\dagger = \mathbf{i}_0(\overline{\alpha})$ for all $\alpha \in \mathcal{O}_{\mathbf{k}}$.

From the above data one can construct a Rapoport–Zink formal scheme by specifying its functor of points. Let Nilp be the category of $\breve{\mathcal{O}}_{\mathbf{k}}$-schemes on which p is locally nilpotent. For each $S \in$ Nilp let $M_{(1,0)}(S)$ be the set of isomorphism classes of quadruples $(X_0, i_0, \lambda_0, \varrho_0)$ in which

- X_0 is a p-divisible group over S of dimension 1;
- $i_0 : \mathcal{O}_{\mathbf{k}} \to \mathrm{End}(X_0)$ is an action of $\mathcal{O}_{\mathbf{k}}$ on X_0 such that the induced action on $\mathrm{Lie}(X_0)$ is through the inclusion $\varphi : \mathcal{O}_{\mathbf{k}} \to \breve{\mathcal{O}}_{\mathbf{k}}$;
- $\lambda_0 : X_0 \to X_0^\vee$ is a principal polarization compatible with $\mathcal{O}_{\mathbf{k}}$-action in the sense above;
- and
$$\varrho_0 : X_0 \times_S \overline{S} \to \mathbf{X}_0 \times_{\mathrm{Spec}(\breve{\mathcal{O}}_{\mathbf{k}}/\breve{\mathfrak{m}})} \overline{S}$$
is an $\mathcal{O}_{\mathbf{k}}$-linear quasi-isogeny, respecting polarizations up to scaling by $\mathbb{Q}_p^\times$. Here
$$\overline{S} = S \times_{\mathrm{Spec}(\breve{\mathcal{O}}_{\mathbf{k}})} \mathrm{Spec}(\breve{\mathcal{O}}_{\mathbf{k}}/\breve{\mathfrak{m}}).$$

An isomorphism between two such tuples is an $\mathcal{O}_{\mathbf{k}}$-linear isomorphism of p-divisible groups $X_0 \cong X_0'$ identifying ϱ_0 with ϱ_0', and identifying λ_0 with λ_0' up to $\mathbb{Z}_p^\times$-scaling.

Proposition 2.1. *The functor $M_{(1,0)}$ is represented by a countable disjoint union of copies of* $\mathrm{Spf}(\breve{\mathcal{O}}_{\mathbf{k}})$.

Proof. The formal deformation space of the triple $(\mathbf{X}_0, \mathbf{i}_0, \boldsymbol{\lambda}_0)$ is $\mathrm{Spf}(\breve{\mathcal{O}}_{\mathbf{k}})$. This can be proved using Lubin–Tate theory. Alternatively, it is a special case of [3, Theorem 2.1.3], which applies to more general p-divisible groups with complex multiplication. With this fact in mind, the proof is the same as the $d = 1$ case of [13, Proposition 3.79]. □

Now fix a tuple $(\mathbf{X}, \mathbf{i}, \boldsymbol{\lambda}, F_{\mathbf{X}})$ in which

- $\mathbf{X}$ is a supersingular p-divisible group over $\breve{\mathcal{O}}_{\mathbf{k}}/\breve{\mathfrak{m}}$ of dimension n;
- $\mathbf{i} : \mathcal{O}_{\mathbf{k}} \to \mathrm{End}(\mathbf{X})$ is an action of $\mathcal{O}_{\mathbf{k}}$ on $\mathbf{X}$;
- $\boldsymbol{\lambda} : \mathbf{X} \to \mathbf{X}^\vee$ is a principal polarization compatible with the $\mathcal{O}_{\mathbf{k}}$-action in the sense above;
- $F_{\mathbf{X}} \subset \mathrm{Lie}(\mathbf{X})$ is an $\breve{\mathcal{O}}_{\mathbf{k}}/\breve{\mathfrak{m}}$-module direct summand of rank $n-1$ satisfying Krämer's [7] signature condition: the action of $\mathcal{O}_{\mathbf{k}}$ on $\mathrm{Lie}(\mathbf{X})$ induced by $\mathbf{i} : \mathcal{O}_{\mathbf{k}} \to \mathrm{End}(\mathbf{X})$ stabilizes $F_{\mathbf{X}}$, and acts on $F_{\mathbf{X}}$ and $\mathrm{Lie}(\mathbf{X})/F_{\mathbf{X}}$ through $\varphi, \overline{\varphi} : \mathcal{O}_{\mathbf{k}} \to \breve{\mathcal{O}}_{\mathbf{k}}$, respectively.

For each $S \in \mathrm{Nilp}$ let $M_{(n-1,1)}(S)$ be the set of isomorphism classes of tuples $(X, i, \lambda, F_X, \varrho)$ in which

- X is a p-divisible group over S of dimension n;
- $i : \mathcal{O}_{\mathbf{k}} \to \mathrm{End}(X)$ is an action of $\mathcal{O}_{\mathbf{k}}$ on X;
- $\lambda : X \to X^\vee$ is a principal polarization compatible with the $\mathcal{O}_{\mathbf{k}}$-action in the sense above;
- $F_X \subset \mathrm{Lie}(X)$ is a local $\mathcal{O}_S$-module local direct summand of rank $n-1$ satisfying Krämer's signature condition as above;
- and
$$\varrho : X \times_S \overline{S} \to \mathbf{X} \times_{\mathrm{Spec}(\breve{\mathcal{O}}_{\mathbf{k}}/\breve{\mathfrak{m}})} \overline{S}$$
is an $\mathcal{O}_{\mathbf{k}}$-linear quasi-isogeny respecting polarizations up to scaling by $\mathbb{Q}_p^\times$.

An isomorphism between two such tuples is an $\mathcal{O}_{\mathbf{k}}$-linear isomorphism of p-divisible groups $X \cong X'$ identifying F_X with $F_{X'}$, identifying ϱ with ϱ', and identifying λ with λ' up to $\mathbb{Z}_p^\times$-scaling.

Proposition 2.2. *The functor $M_{(n-1,1)}$ is represented by a formal $\breve{\mathcal{O}}_{\mathbf{k}}$-scheme, locally formally of finite type. Moreover:*
(i) *$M_{(n-1,1)}$ is flat over $\breve{\mathcal{O}}_{\mathbf{k}}$, and regular of dimension n.*
(ii) *If $\mathbf{k}/\mathbb{Q}_p$ is unramified, then M is formally smooth over $\breve{\mathcal{O}}_{\mathbf{k}}$.*

Proof. First suppose that $p > 2$. The representability follows from the general results of Rapoport–Zink [13, Theorem 3.25]. The remaining claims can be verified using the theory of local models, as in [10] and [13, Proposition 3.33]. In the unramified case the analysis of the local model is routine, and in the ramified case it was done by Krämer [7].

The $p = 2$ case is excluded from much of [13] by the blanket assumption imposed in [13, p. 75], and the author is unaware of a published or publicly available reference for this case.[2] However, M. Rapoport has informed the author that the necessary extensions to $p = 2$ with $\mathbf{k}/\mathbb{Q}_p$ unramified will appear in an appendix to the forthcoming work [11]. □

Following [8], we will define a family of divisors on

$$M = M_{(1,0)} \times_{\mathrm{Spf}(\breve{\mathcal{O}}_{\mathbf{k}})} M_{(n-1,1)}.$$

If $S \in \mathrm{Nilp}$, we will write S-points of M simply as $(X_0, X) \in M(S)$, rather than the cumbersome

$$(X_0, i_0, \lambda_0, \varrho_0, X, i, \lambda, F_X, \varrho).$$

2 When $p = 2$, there is a thorough study of unitary Rapoport–Zink spaces of signature $(1, 1)$ in the work of Kirch [6], even when $\mathbf{k}/\mathbb{Q}_p$ is ramified.

Lemma 2.3. *The* $\mathbf{k}$*-vector space*

$$V = \mathrm{Hom}_{\mathcal{O}_{\mathbf{k}}}(\mathbf{X}_0, \mathbf{X})[1/p]$$

has dimension n*. For any* $S \in \mathrm{Nilp}$ *and any* $(X_0, X) \in M(S)$ *there is a canonical inclusion*[3]

$$V \subset \mathrm{Hom}_{\mathcal{O}_{\mathbf{k}}}(X_0, X)[1/p]. \tag{2.1}$$

Proof. As $\mathbf{X}$ is supersingular, there is a quasi-isogeny of p-divisible groups

$$\mathbf{X} \to \mathbf{X}_0 \times \cdots \times \mathbf{X}_0.$$

The Noether–Skolem theorem implies that any two embeddings of $\mathbf{k}$ into

$$\mathrm{End}(\mathbf{X})[1/p] \cong M_n(\mathrm{End}(\mathbf{X}_0))[1/p]$$

are conjugate, and hence this quasi-isogeny can be chosen to be $\mathcal{O}_{\mathbf{k}}$-linear. It follows that

$$V \cong \mathrm{End}_{\mathcal{O}_{\mathbf{k}}}(\mathbf{X}_0)[1/p] \times \cdots \times \mathrm{End}_{\mathcal{O}_{\mathbf{k}}}(\mathbf{X}_0)[1/p].$$

Each factor on the right-hand side has dimension one, proving the first claim of the lemma.

Given $x \in V$, the quasi-isogenies ϱ_0 and ϱ allow us to identify x with

$$\varrho^{-1} \circ x \circ \varrho_0 \in \mathrm{Hom}_{\mathcal{O}_{\mathbf{k}}}(X_0 \times_S \overline{S}, X \times_S \overline{S})[1/p].$$

The reduction map

$$\mathrm{Hom}_{\mathcal{O}_{\mathbf{k}}}(X_0, X)[1/p] \to \mathrm{Hom}_{\mathcal{O}_{\mathbf{k}}}(X_0 \times_S \overline{S}, X \times_S \overline{S})[1/p]$$

is an isomorphism by [5, Lemma 1.1.3], proving the second claim of the lemma. □

The second claim of Lemma 2.3 allows us to make the following definition.

Definition 2.4. For any nonzero $x \in V$ we define the *Kudla–Rapoport divisor* to be the closed formal subscheme

$$Z(x) \subset M$$

whose functor of points assigns to any $S \in \mathrm{Nilp}$ the set of all $(X_0, X) \in M(S)$ for which $x \in \mathrm{Hom}_{\mathcal{O}_{\mathbf{k}}}(X_0, X)$ under the inclusion (2.1).

When $\mathbf{k}/\mathbb{Q}_p$ is unramified, it is proved in [8] that $Z(x) \subset M$ is defined locally by a single equation. A proof of the same claim in the ramified case can be found in [4]. We will reprove these results below in Proposition 4.3, as the arguments provide additional information that will be essential for the proof of Theorem 5.1.

3 Vector bundles

For the remainder of the paper, (X_0, X) denotes the universal object over

$$M = M_{(1,0)} \times_{\mathrm{Spf}(\breve{\mathcal{O}}_{\mathbf{k}})} M_{(n-1,1)}.$$

Let $D(X)$ be the restriction to the Zariski site of the covariant Grothendieck–Messing crystal of X. Thus $D(X)$ is a vector bundle on M of rank $2n$, sitting in a short exact sequence

$$0 \to \mathrm{Fil}(X) \to D(X) \to \mathrm{Lie}(X) \to 0.$$

3 Here one must interpret the right-hand side as global sections of the Zariski sheaf $\underline{\mathrm{Hom}}(X_0, X)[1/p]$ on S, as in [13, Definition 2.8]. If S is quasi-compact, this agrees with the naive definition. We will ignore this technical point in all that follows.

Similarly, the Grothendieck–Messing crystal of X_0 determines a short exact sequence

$$0 \to \mathrm{Fil}(X_0) \to D(X_0) \to \mathrm{Lie}(X_0) \to 0$$

of vector bundles on M.

The actions $i_0 : \mathcal{O}_{\mathbf{k}} \to \mathrm{End}(X_0)$ and $i : \mathcal{O}_{\mathbf{k}} \to \mathrm{End}(X)$ induce actions of $\mathcal{O}_{\mathbf{k}}$ on all of these vector bundles, and the above short exact sequences are $\mathcal{O}_{\mathbf{k}}$-linear. The principal polarization on X induces a perfect alternating pairing

$$\langle\,\cdot\,,\cdot\,\rangle : D(X) \times D(X) \to \mathcal{O}_M,$$

which is compatible with the action $i : \mathcal{O}_{\mathbf{k}} \to \mathrm{End}_{\mathcal{O}_M}(D(X))$, in the sense that

$$\langle i(\alpha)x, y\rangle = \langle x, i(\overline{\alpha})y\rangle \tag{3.1}$$

for all $\alpha \in \mathcal{O}_{\mathbf{k}}$ and all local sections x and y of $D(X)$. The local direct summand $\mathrm{Fil}(X) \subset D(X)$ is maximal isotropic with respect to this pairing, and hence there is an induced perfect pairing

$$\langle\,\cdot\,,\cdot\,\rangle : \mathrm{Fil}(X) \times \mathrm{Lie}(X) \to \mathcal{O}_M. \tag{3.2}$$

By virtue of the moduli problem defining $M_{(n-1,1)}$, there is a distinguished local direct summand $F_X \subset \mathrm{Lie}(X)$ of rank $n-1$, whose annihilator with respect to the pairing (3.2) is a local direct summand $F_X^{\perp} \subset \mathrm{Fil}(X)$ of rank one. Both submodules are stable under the action of $\mathcal{O}_{\mathbf{k}}$, which acts

- on F_X and $F_X^{\perp}$ via $\varphi : \mathcal{O}_{\mathbf{k}} \to \breve{\mathcal{O}}_{\mathbf{k}}$,
- on $\mathrm{Lie}(X)/F_X$ and $\mathrm{Fil}(X)/F_X^{\perp}$ via $\overline{\varphi} : \mathcal{O}_{\mathbf{k}} \to \breve{\mathcal{O}}_{\mathbf{k}}$.

There is a natural morphism of $\mathcal{O}_M$-algebras

$$\mathcal{O}_{\mathbf{k}} \otimes_{\mathbb{Z}_p} \mathcal{O}_M \xrightarrow{\alpha\otimes 1 \mapsto (\varphi(\alpha), \overline{\varphi}(\alpha))} \mathcal{O}_M \times \mathcal{O}_M.$$

If $\mathbf{k}/\mathbb{Q}_p$ is unramified, this map is an isomorphism, and we obtain a pair of orthogonal idempotents in $\mathcal{O}_{\mathbf{k}} \otimes_{\mathbb{Z}_p} \mathcal{O}_M$. Without any assumption on ramification, one can still define reasonable substitutes for these idempotents. To do so, fix a $\beta \in \mathcal{O}_{\mathbf{k}}$ satisfying $\mathcal{O}_{\mathbf{k}} = \mathbb{Z}_p + \mathbb{Z}_p\beta$, and define

$$\epsilon = \beta \otimes 1 - 1 \otimes \overline{\varphi}(\beta) \in \mathcal{O}_{\mathbf{k}} \otimes_{\mathbb{Z}_p} \mathcal{O}_M,$$
$$\overline{\epsilon} = \overline{\beta} \otimes 1 - 1 \otimes \overline{\varphi}(\beta) \in \mathcal{O}_{\mathbf{k}} \otimes_{\mathbb{Z}_p} \mathcal{O}_M.$$

The ideal sheaves in $\mathcal{O}_{\mathbf{k}} \otimes_{\mathbb{Z}_p} \mathcal{O}_M$ generated by these elements are independent of the choice of β, and there are short exact sequences of $\mathcal{O}_M$-modules

$$0 \to (\epsilon) \to \mathcal{O}_{\mathbf{k}} \otimes_{\mathbb{Z}_p} \mathcal{O}_M \xrightarrow{\alpha\otimes 1 \mapsto \overline{\varphi}(\alpha)} \mathcal{O}_M \to 0,$$
$$0 \to (\overline{\epsilon}) \to \mathcal{O}_{\mathbf{k}} \otimes_{\mathbb{Z}_p} \mathcal{O}_M \xrightarrow{\alpha\otimes 1 \mapsto \varphi(\alpha)} \mathcal{O}_M \to 0.$$

Remark 3.1. In particular, (ϵ) and $(\overline{\epsilon})$ are rank one $\mathcal{O}_M$-module local direct summands of $\mathcal{O}_{\mathbf{k}} \otimes_{\mathbb{Z}_p} \mathcal{O}_M$.

Let $\mathfrak{d} \subset \mathcal{O}_{\mathbf{k}}$ be the different of $\mathbf{k}/\mathbb{Q}_p$, and set $\breve{\mathfrak{d}} = \varphi(\mathfrak{d})\breve{\mathcal{O}}_{\mathbf{k}}$. It follows from Hypothesis A that

$$\breve{\mathfrak{d}} = \begin{cases} \breve{\mathcal{O}}_{\mathbf{k}} & \text{if } \mathbf{k}/\mathbb{Q}_p \text{ is unramified,} \\ \breve{\mathfrak{m}} & \text{if } \mathbf{k}/\mathbb{Q}_p \text{ is ramified.} \end{cases}$$

Lemma 3.2. *Suppose N is an $\mathcal{O}_M$-module endowed with an action*

$$i : \mathcal{O}_{\mathbf{k}} \to \mathrm{End}_{\mathcal{O}_M}(N).$$

If we view N as an $\mathcal{O}_{\mathbf{k}} \otimes_{\mathbb{Z}_p} \mathcal{O}_M$-module, then $N/\overline{\epsilon}N$ and $N/\epsilon N$ are the maximal quotients of N on which $\mathcal{O}_{\mathbf{k}}$ acts through φ and $\overline{\varphi}$, respectively. Moreover,

$$\epsilon N \subset \{n \in N : \text{for all } \alpha \in \mathcal{O}_{\mathbf{k}},\ i(\alpha)x = \varphi(\alpha)x\},$$
$$\overline{\epsilon} N \subset \{n \in N : \text{for all } \alpha \in \mathcal{O}_{\mathbf{k}},\ i(\alpha)x = \overline{\varphi}(\alpha)x\},$$

and both quotients are annihilated by $\breve{\mathfrak{d}}\mathcal{O}_M$.

Proof. This is an elementary exercise, left to the reader. □

Proposition 3.3. *There are inclusions of $\mathcal{O}_M$-module local direct summands $F_X^\perp \subset \epsilon D(X) \subset D(X)$. The morphism $\epsilon : D(X) \to \epsilon D(X)$ descends to a surjection*

$$\mathrm{Lie}(X) \xrightarrow{\epsilon} \epsilon D(X)/F_X^\perp, \tag{3.3}$$

whose kernel $L_X \subset \mathrm{Lie}(X)$ is an $\mathcal{O}_M$-module local direct summand of rank one. It is stable under $\mathcal{O}_{\mathbf{k}}$, which acts on $\mathrm{Lie}(X)/L_X$ and L_X via $\varphi, \overline{\varphi} : \mathcal{O}_{\mathbf{k}} \to \breve{\mathcal{O}}_{\mathbf{k}}$, respectively.

Proof. The vector bundle $D(X)$ is locally free of rank n over $\mathcal{O}_{\mathbf{k}} \otimes_{\mathbb{Z}_p} \mathcal{O}_M$, and hence $\epsilon D(X) \subset D(X)$ is a local $\mathcal{O}_M$-module direct summand by Remark 3.1. As F_X is locally free over $\mathcal{O}_M$, the perfect pairing

$$(\mathrm{Fil}(X)/F_X^\perp) \otimes F_X \to \mathcal{O}_M$$

induced by (3.2) shows that $\mathrm{Fil}(X)/F_X^\perp$ is locally free, from which it follows that $F_X^\perp$ a local direct summand of $D(X)$.

Now consider the perfect pairing

$$F_X^\perp \otimes (\mathrm{Lie}(X)/F_X) \to \mathcal{O}_M$$

induced by (3.2). As $\mathcal{O}_{\mathbf{k}}$ acts on $\mathrm{Lie}(X)/F_X$ via $\overline{\varphi}$, relation (3.1) implies that $\mathcal{O}_{\mathbf{k}}$ acts on $F_X^\perp$ via φ. Lemma 3.2 thus implies

$$\breve{\mathfrak{d}} F_X^\perp \subset \epsilon F_X^\perp \subset F_X^\perp,$$

and so $\breve{\mathfrak{d}} F_X^\perp \subset \epsilon D(X)$. The stronger inclusion $F_X^\perp \subset \epsilon D(X)$ then follows from the fact that $D(X)/\epsilon D(X)$ is $\mathcal{O}_M$-torsion free.

As $\mathcal{O}_{\mathbf{k}}$ acts on F_X through $\varphi : \mathcal{O}_{\mathbf{k}} \to \breve{\mathcal{O}}_{\mathbf{k}}$, we must have $\overline{\epsilon} F_X = 0$. Hence

$$\langle \epsilon x, y \rangle = \langle x, \overline{\epsilon} y \rangle = 0$$

for all local sections x and y of $\mathrm{Fil}(X)$ and F_X, respectively. Thus

$$\epsilon \mathrm{Fil}(X) \subset F_X^\perp,$$

and the map (3.3) is well-defined.

The kernel L_X of (3.3) is a local direct summand, as (3.3) is a surjection to a locally free $\mathcal{O}_M$-module. Moreover, Lemma 3.2 implies that $\mathcal{O}_{\mathbf{k}}$ acts on the codomain via φ, and hence acts on $\mathrm{Lie}(X)/L_X$ in the same way.

Suppose the natural map $L_X \to \mathrm{Lie}(X)/F_X$ is trivial. The inclusion $L_X \subset F_X$ then shows that $\mathcal{O}_{\mathbf{k}}$ acts on both L_X and $\mathrm{Lie}(X)/L_X$ via φ, and hence both are annihilated by $\overline{\epsilon}$. This means that $\overline{\epsilon} \cdot \overline{\epsilon}$ annihilates $\mathrm{Lie}(X)$. But $\overline{\epsilon}$ acts on $\mathrm{Lie}(X)/F_X$ via the nonzero scalar $\varphi(\beta - \overline{\beta}) \in \breve{\mathcal{O}}_{\mathbf{k}}$, a contradiction.

The map $L_X \to \mathrm{Lie}(X)/F_X$ is therefore nonzero, and hence injective as M is locally integral. As $\mathcal{O}_{\mathbf{k}}$ acts on the codomain via $\overline{\varphi}$, it acts in the same way on L_X. □

The line bundle L_X of Proposition 3.3 is, by construction, the pullback of a line bundle on $M_{(n-1,1)}$ via the projection $M \to M_{(n-1,1)}$. We will now twist it by a line bundle pulled back via $M \to M_{(1,0)}$.

Definition 3.4. The *line bundle of modular forms* ω is the invertible sheaf of $\mathcal{O}_M$-modules with inverse

$$\omega^{-1} = \underline{\mathrm{Hom}}(\mathrm{Fil}(X_0), L_X).$$

Remark 3.5. The line bundle of Definition 3.4 does not agree with the line bundle of modular forms defined in [1, 2]. In those papers the line bundle of modular forms, which we here denote by ω_{old}, is characterized by

$$\omega_{\mathrm{old}}^{-1} = \underline{\mathrm{Hom}}(\mathrm{Fil}(X_0), \mathrm{Lie}(X)/F_X).$$

The inclusion $L_X \subset \mathrm{Lie}(X)$ induces a morphism $L_X \to \mathrm{Lie}(X)/F_X$, which in turn induces $\omega_{\mathrm{old}} \to \omega$. It is not difficult to check that this latter map identifies

$$\breve{\mathfrak{d}} \cdot \omega \subset \omega_{\mathrm{old}} \subset \omega,$$

but when $\mathbf{k}/\mathbb{Q}_p$ is ramified, neither inclusion is an equality.

4 Deformation theory

Suppose $Z \subset M$ is any closed formal subscheme, and denote by $I_Z \subset \mathcal{O}_M$ its ideal sheaf. The square $I_{\widetilde{Z}} = I_Z^2$ is the ideal sheaf of a larger closed formal subscheme

$$Z \subset \widetilde{Z} \subset M$$

called the *first-order infinitesimal neighborhood* of Z in M.

Now fix a nonzero $x \in V$ and consider the first-order infinitesimal neighborhood

$$Z(x) \subset \widetilde{Z}(x) \subset M$$

of the corresponding Kudla–Rapoport divisor. By the very definition (Definition 2.4) of $Z(x)$, when we restrict the universal object (X_0, X) to $Z(x)$, we obtain a distinguished morphism of p-divisible groups

$$X_0|_{Z(x)} \xrightarrow{x} X|_{Z(x)}.$$

This induces an $\mathcal{O}_{\mathbf{k}}$-linear morphism of vector bundles

$$D(X_0)|_{Z(x)} \xrightarrow{x} D(X)|_{Z(x)} \tag{4.1}$$

on $Z(x)$, which respects the Hodge filtrations. By Grothendieck–Messing theory this morphism admits a canonical extension

$$D(X_0)|_{\widetilde{Z}(x)} \xrightarrow{\widetilde{x}} D(X)|_{\widetilde{Z}(x)}$$

to the first-order infinitesimal neighborhood, which no longer respects the Hodge filtrations. Instead, it determines a nontrivial morphism

$$\mathrm{Fil}(X_0)|_{\widetilde{Z}(x)} \xrightarrow{\widetilde{x}} \mathrm{Lie}(X)|_{\widetilde{Z}(x)}. \tag{4.2}$$

Proposition 4.1. *The morphism* (4.2) *takes values in the rank one local direct summand*

$$L_X|_{\widetilde{Z}(x)} \subset \mathrm{Lie}(X)|_{\widetilde{Z}(x)},$$

and so can be viewed as a morphism of line bundles

$$\mathrm{Fil}(X_0)|_{\widetilde{Z}(x)} \xrightarrow{\widetilde{x}} L_X|_{\widetilde{Z}(x)}. \tag{4.3}$$

The Kudla–Rapoport divisor $Z(x)$ is the largest closed formal subscheme of $\widetilde{Z}(x)$ over which (4.3) *is trivial.*

Proof. The vector bundle $D(X_0)$ is locally free of rank one over $\mathcal{O}_{\mathbf{k}} \otimes_{\mathbb{Z}_p} \mathcal{O}_M$, and its quotient

$$D(X_0)/\mathrm{Fil}(X_0) \cong \mathrm{Lie}(X_0)$$

is annihilated by $\overline{\epsilon}$. Hence $\overline{\epsilon} \cdot D(X_0) \subset \mathrm{Fil}(X_0)$, and equality holds as both are rank one local $\mathcal{O}_M$-module direct summands of $D(X_0)$; see Remark 3.1.

It follows that (4.2) takes values in the subsheaf

$$\overline{\epsilon} \cdot \mathrm{Lie}(X)|_{\widetilde{Z}(x)} \subset \mathrm{Lie}(X)|_{\widetilde{Z}(x)}.$$

On the other hand, the final claim of Proposition 3.3 implies that $\overline{\epsilon}$ annihilates $\mathrm{Lie}(X)/L_X$, and hence

$$\overline{\epsilon} \cdot \mathrm{Lie}(X)|_{\widetilde{Z}(x)} \subset L_X|_{\widetilde{Z}(x)}.$$

This proves the first claim.

For the second claim, it follows from Grothendieck–Messing theory that $Z(x)$ is the largest closed formal subscheme of $\widetilde{Z}(x)$ along which (4.2) vanishes. As $L_X \subset \mathrm{Lie}(X)$ is a local direct summand, this is equivalent to (4.3) vanishing. □

Definition 4.2. The section

$$\mathrm{obst}(x) \in H^0(\widetilde{Z}(x), \omega^{-1}|_{\widetilde{Z}(x)})$$

determined by (4.3) is called the *obstruction to deforming* x. As we have already explained, $Z(x)$ is the largest closed formal subscheme of $\widetilde{Z}(x)$ over which $\mathrm{obst}(x) = 0$.

Proposition 4.3. *For any nonzero $x \in V$, the closed formal subscheme $Z(x) \subset M$ is a Cartier divisor; that is to say, it is defined locally by a single nonzero equation.*

Proof. Let R be the local ring of M at a point $z \in Z(x)$, and let $I \supset I^2$ be the ideals of R corresponding to $Z(x) \subset \widetilde{Z}(x)$. After pulling back via $\mathrm{Spf}(R) \to M$, we may trivialize the line bundle ω, and the obstruction to deforming x becomes an R-module generator

$$\mathrm{obst}(x) \in I/I^2.$$

It follows from Nakayama's lemma that $I \subset R$ is a principal ideal, and it only remains to show that $I \neq 0$.

Suppose $I = 0$. This implies that we may find an open subset $U \subset M$ such that $Z(x)|_U = U$. As in [13, Chapter 5], M has an associated rigid analytic space M^{rig} over $\breve{\mathbf{k}}$, and $U \subset M$ determines an admissible open subset

$$U^{\mathrm{rig}} \subset M^{\mathrm{rig}}.$$

The vector bundles of Section 3 determine filtered vector bundles

$$\mathrm{Fil}(X_0)^{\mathrm{rig}} \subset D(X_0)^{\mathrm{rig}},$$
$$\mathrm{Fil}(X)^{\mathrm{rig}} \subset D(X)^{\mathrm{rig}}$$

on M^{rig}. By [13, Proposition 5.17] these admit $\mathcal{O}_{\mathbf{k}}$-linear trivializations

$$D(X_0)^{\mathrm{rig}} \cong V_0 \otimes_{\breve{\mathbf{k}}} \mathcal{O}_{M^{\mathrm{rig}}}, \tag{4.4}$$
$$D(X)^{\mathrm{rig}} \cong V \otimes_{\breve{\mathbf{k}}} \mathcal{O}_{M^{\mathrm{rig}}}, \tag{4.5}$$

where V_0 and V are vector spaces over $\breve{\mathbf{k}}$ of dimensions 2 and $2n$, respectively, endowed with actions $i_0 : \mathbf{k} \to \mathrm{End}_{\breve{\mathbf{k}}}(V_0)$ and $i : \mathbf{k} \to \mathrm{End}_{\breve{\mathbf{k}}}(V)$.

The signature $(1, 0)$ condition on X_0 implies that $\mathbf{k}$ acts on $\mathrm{Fil}(X_0)^{\mathrm{rig}}$ via $\overline{\varphi} : \mathbf{k} \to \breve{\mathbf{k}}$. From this it follows easily that (4.4) induces an identification of line bundles

$$\overline{\epsilon} \cdot \mathrm{Fil}(X_0)^{\mathrm{rig}} = (\overline{\epsilon} V_0) \otimes_{\breve{\mathbf{k}}} \mathcal{O}_{M^{\mathrm{rig}}}.$$

On the other hand, the signature $(n-1, 1)$ condition on X implies that (4.5) determines an inclusion

$$\overline{\epsilon} \cdot \mathrm{Fil}(X)^{\mathrm{rig}} \subset (\overline{\epsilon} V) \otimes_{\breve{\mathbf{k}}} \mathcal{O}_{M^{\mathrm{rig}}}$$

as a local direct summand of corank one. This inclusion determines the Grothendieck–Messing (or Gross–Hopkins) period morphism

$$\pi : M^{\mathrm{rig}} \to N^{\mathrm{rig}} \tag{4.6}$$

to the rigid analytic flag variety N^{rig} parameterizing all codimension one subspaces of $\overline{\epsilon} V$. It follows from [13, Proposition 5.17] that π is étale.

After restriction to U^{rig} the morphism (4.1) determines a morphism

$$D(X_0)^{\mathrm{rig}}|_{U^{\mathrm{rig}}} \to D(X)^{\mathrm{rig}}|_{U^{\mathrm{rig}}}$$

that respects the filtrations, and this morphism is induced by a $\mathbf{k}$-linear inclusion $V_0 \subset V$. In particular,

$$(\overline{\epsilon} V_0) \otimes_{\breve{\mathbf{k}}} \mathcal{O}_{U^{\mathrm{rig}}} \subset \overline{\epsilon} \cdot \mathrm{Fil}(X)^{\mathrm{rig}}|_{U^{\mathrm{rig}}} \subset (\overline{\epsilon} V) \otimes_{\breve{\mathbf{k}}} \mathcal{O}_{U^{\mathrm{rig}}},$$

and so the restriction of (4.6) to $U^{\mathrm{rig}} \subset M^{\mathrm{rig}}$ takes values in the closed rigid analytic subspace of N^{rig} parameterizing codimension one subspaces of $\overline{\epsilon} V$ that contain the line $\overline{\epsilon} V_0$. This contradicts (4.6) being étale. □

If $\mathbf{k}/\mathbb{Q}_p$ is unramified, it is proved in [8] that every Kudla–Rapoport divisor $Z(x)$ is flat over $\breve{\mathcal{O}}_{\mathbf{k}}$. In Appendix A we will explain why this is false when $\mathbf{k}/\mathbb{Q}_p$ is ramified.

5 Linear invariance of tensor products

Suppose $x \in V$ is nonzero. As in the introduction, let $I_{Z(x)} \subset \mathcal{O}_M$ be the ideal sheaf defining the Kudla–Rapoport divisor $Z(x) \subset M$, and define a complex of locally free $\mathcal{O}_M$-modules

$$C(x) = (\cdots \to 0 \to I_{Z(x)} \to \mathcal{O}_M \to 0)$$

supported in degrees 1 and 0. We extend the definitions to $x = 0$ by setting $Z(0) = M$ and

$$C(0) = (\cdots \to 0 \to \omega \xrightarrow{0} \mathcal{O}_M \to 0),$$

where ω is the line bundle of Definition 3.4.

Theorem 5.1. *Fix an $r \geq 0$, and suppose $x_1, \dots, x_r \in V$ and $y_1, \dots, y_r \in V$ generate the same $\mathcal{O}_{\mathbf{k}}$-submodule. For every $i \geq 0$ there is an isomorphism of coherent $\mathcal{O}_M$-modules*

$$H_i(C(x_1) \otimes \cdots \otimes C(x_r)) \cong H_i(C(y_1) \otimes \cdots \otimes C(y_r)). \tag{5.1}$$

Proof. It is an exercise in linear algebra to check that the list $x_1, \dots, x_r$ can be transformed to the list $y_1, \dots, y_r$ using a sequence of elementary operations: permute the vectors in the list, scale a vector by an element of $\mathcal{O}_{\mathbf{k}}^\times$, and add an $\mathcal{O}_{\mathbf{k}}$-multiple of one vector to another. The isomorphism class of the complex $C(x_1) \otimes \cdots \otimes C(x_r)$ is obviously invariant under the first two operations, and using this one immediately reduces to the case in which

$$\begin{aligned} y_1 &= x_1 + a x_2, \\ y_2 &= x_2, \\ &\vdots \\ y_r &= x_r \end{aligned}$$

for some $a \in \mathcal{O}_{\mathbf{k}}$.

Denote by $Z \subset M$ the closed formal subscheme

$$Z(x_1) \cap \cdots \cap Z(x_r) = Z(y_1) \cap \cdots \cap Z(y_r)$$

(here and below, we use $\cap$ as a shorthand for $\times_M$) and by $Z \subset \tilde{Z}$ its first-order infinitesimal neighborhood in M. Note that both sides of (5.1) are supported on Z in the strong sense: they are annihilated by the ideal sheaf defining Z.

For every $1 \leq i \leq r$, define sections

$$\begin{aligned} s_i &\in H^0(\tilde{Z}(x_i), \omega^{-1}|_{\tilde{Z}(x_i)}), \\ t_i &\in H^0(\tilde{Z}(y_i), \omega^{-1}|_{\tilde{Z}(y_i)}) \end{aligned}$$

by (recall Definition 4.2)

$$s_i = \begin{cases} \mathrm{obst}(x_i) & \text{if } x_i \neq 0, \\ 0 & \text{if } x_i = 0, \end{cases}$$

$$t_i = \begin{cases} \mathrm{obst}(y_i) & \text{if } y_i \neq 0, \\ 0 & \text{if } y_i = 0. \end{cases}$$

Thus the zero loci of s_i and t_i are $Z(x_i)$ and $Z(y_i)$, respectively. After restriction to

$$\tilde{Z} \subset \tilde{Z}(x_1) \cap \cdots \cap \tilde{Z}(x_r) \cap \tilde{Z}(y_1) \cap \cdots \cap \tilde{Z}(y_r)$$

these sections satisfy

$$t_1 = s_1 + a s_2,$$

and $t_i = s_i$ when $i > 1$. We will approximate s_1, s_2, and t_1, in a noncanonical way, by sections defined over open subsets of M.

Lemma 5.2. *Around every point $z \in Z$ one can find an open affine neighborhood $U = \mathrm{Spec}(R) \subset M$ over which ω_U is trivial, and sections*

$$\sigma_1, \sigma_2 \in H^0(U, \omega_U^{-1}) \quad \text{and} \quad \alpha \in H^0(U, \mathcal{O}_U) \tag{5.2}$$

such that the following assertions hold:

(i) *σ_1 has zero locus $Z(x_1)_U$ and agrees with s_1 on $\widetilde{Z}(x_1)_U$.*
(ii) *σ_2 has zero locus $Z(x_2)_U$ and agrees with s_2 on $\widetilde{Z}(x_2)_U$.*
(iii) *α restricts to the constant function a on $Z(x_2)_U$.*
(iv) *The section*

$$\tau_1 \stackrel{\mathrm{def}}{=} \sigma_1 + \alpha\sigma_2$$

has zero locus $Z(y_1)_U$ and agrees with t_1 on the closed formal subscheme, lying between $Z(y_1)_U$ and $\widetilde{Z}(y_1)_U$, defined by the ideal sheaf

$$I_{Z(y_1)_U} \cdot (I_{Z(y_1)_U} + I_{Z(x_2)_U}) \subset \mathcal{O}_U.$$

Given another collection of sections

$$\sigma_1', \sigma_2' \in H^0(U, \omega_U^{-1}) \quad \text{and} \quad \alpha' \in H^0(U, \mathcal{O}_U) \tag{5.3}$$

satisfying the same properties, there is an element $\xi \in \mathrm{Frac}(R)$ such that

$$\xi \cdot \sigma_1 \otimes \sigma_1' = \tau_1 \otimes \sigma_1' - \tau_1' \otimes \sigma_1 \tag{5.4}$$

and

$$\xi \cdot I_{Z(x_1)_U} \subset I_{Z(y_1)_U} \cdot I_{Z(x_2)_U}.$$

Proof. Start with any connected affine open neighborhood $U = \mathrm{Spf}(R)$ of $z \in U$ over which $\omega_U \cong \mathcal{O}_U$, and fix such an isomorphism. Write

$$Z(x_1)_U = \mathrm{Spf}(R/I_{x_1}),$$
$$Z(x_2)_U = \mathrm{Spf}(R/I_{x_2}),$$
$$Z(y_1)_U = \mathrm{Spf}(R/I_{y_1})$$

for ideals $I_{x_1}, I_{x_2}, I_{y_1} \subset R$, all of which are contained in the maximal ideal $\mathfrak{p} \subset R$ determined by the point $z \in U$. Identify the sections s_1, s_2, and t_1 with R-module generators

$$s_1 \in I_{x_1}/I_{x_1}^2, \quad s_2 \in I_{x_2}/I_{x_2}^2, \quad t_1 \in I_{y_1}/I_{y_1}^2.$$

Next choose, for $i \in \{1, 2\}$, an arbitrary lift $\sigma_i \in I_{x_i}$ of s_i. Nakayama's lemma implies $R_\mathfrak{p}\sigma_i = R_\mathfrak{p}I_i$, and so there is some $f \notin \mathfrak{p}$ such that $R[1/f]\sigma_i = R[1/f]I_i$. After inverting f, and hence shrinking U, we may assume that $R\sigma_i = I_i$. We now have sections σ_1 and σ_2 satisfying properties (i) and (ii).

Choose an arbitrary lift $\hat{\tau}_1 \in I_{y_1}$ of t_1. Again using Nakayama's lemma, we may shrink U in order to assume that $R\hat{\tau}_1 = I_{y_1}$. The relation $y_1 = x_1 + ax_2$ implies the equality

$$Z(y_1) \cap Z(x_2) = Z(x_1) \cap Z(x_2) \tag{5.5}$$

of closed formal subschemes of M, and hence

$$I_{y_1} + I_{x_2} = I_{x_1} + I_{x_2}. \tag{5.6}$$

Along the first-order infinitesimal neighborhood of (5.5) in M we have $t_1 = s_1 + as_2$. This implies that $\hat{\tau}_1 \equiv \sigma_1 + a\sigma_2$ modulo the square of (5.6), and so we may write

$$\hat{\tau}_1 = \sigma_1 + a\sigma_2 + (A\hat{\tau}_1^2 + B\hat{\tau}_1\sigma_2 + C\sigma_2^2)$$

for some $A, B, C \in R$. Now rewrite this as

$$\tau_1 = \sigma_1 + \alpha\sigma_2,$$

where $\tau_1 = \hat{\tau}_1 - A\hat{\tau}_1^2 - B\hat{\tau}_1\sigma_2$ and $\alpha = a + C\sigma_2$.

By construction τ_1 agrees with $\hat{\tau}_1$, hence also with t_1, in $R/I_{y_1}(I_{y_1}+I_{x_2})$. In particular, it generates $I_{y_1}/\mathfrak{p}I_{y_1}$ as an R-module, and the above argument using Nakayama's lemma allows us to shrink U in order to assume that $R\tau_1 = I_{y_1}$. The sections σ_1, σ_2, and α we have constructed satisfy properties (i), (ii), (iii), and (iv).

Now suppose we have another collection of sections (5.3) satisfying the same properties. As above, we use $\omega_U \cong \mathcal{O}_U$ to identify $\sigma_1', \sigma_2', \alpha' \in R$, so that

$$R\sigma_1 = I_{x_1} = R\sigma_1', \quad R\sigma_2 = I_{x_2} = R\sigma_2', \quad R\tau_1 = I_{y_1} = R\tau_1'.$$

In the degenerate case where $I_{x_1} = 0$ (this can only happen when $x_1 = 0$) we must have $\sigma_1 = 0 = \sigma_1'$, and any choice of $\xi \in R$ will satisfy the stated properties. Thus we may assume $I_{x_1} \neq 0$.

Define $\xi \in \mathrm{Frac}(R)$ by

$$\xi = \Big(\frac{\tau_1}{\sigma_1} - \frac{\tau_1'}{\sigma_1'}\Big) = \Big(\frac{\alpha\sigma_2}{\sigma_1} - \frac{\alpha'\sigma_2'}{\sigma_1'}\Big).$$

We need to show that $R\xi\sigma_1 \subset R\tau_1\sigma_2$. As R is regular, it is equal to the intersection of its localizations at height one primes $\mathfrak{q} \subset R$, and every such localization $R_\mathfrak{q}$ is a DVR. Thus it suffices to prove, for all such $\mathfrak{q}$,

$$\mathrm{ord}_\mathfrak{q}(\xi\sigma_1) \geq \mathrm{ord}_\mathfrak{q}(\tau_1\sigma_2). \tag{5.7}$$

The conditions imposed on our sections imply the congruences

$$\begin{aligned} \sigma_1 \equiv s_1 \equiv \sigma_1' &\pmod{I_{x_1}^2}, \\ \alpha\sigma_2 \equiv as_2 \equiv \alpha'\sigma_2' &\pmod{I_{x_2}^2}, \\ \tau_1 \equiv t_1 \equiv \tau_1' &\pmod{I_{y_1}(I_{y_1}+I_{x_2})}, \end{aligned}$$

the first and third of which imply

$$\begin{aligned} \sigma_1/\sigma_1' \equiv 1 &\pmod{R\sigma_1}, \\ \tau_1/\tau_1' \equiv 1 &\pmod{R\tau_1 + R\sigma_2}. \end{aligned} \tag{5.8}$$

First assume $\mathrm{ord}_\mathfrak{q}(\sigma_2) \geq \mathrm{ord}_\mathfrak{q}(\tau_1)$, and note that $\tau_1 = \sigma_1 + \alpha\sigma_2$ implies

$$\mathrm{ord}_\mathfrak{q}(\sigma_1) \geq \min\{\mathrm{ord}_\mathfrak{q}(\tau_1), \mathrm{ord}_\mathfrak{q}(\alpha\sigma_2)\} = \mathrm{ord}_\mathfrak{q}(\tau_1).$$

It follows from this and (5.8) that $\sigma_1/\sigma_1' \equiv 1 \pmod{R_\mathfrak{q}\tau_1}$, and hence

$$\xi\sigma_1 = \alpha\sigma_2 - \frac{\sigma_1}{\sigma_1'}\cdot\alpha'\sigma_2' \equiv \alpha\sigma_2\Big(1 - \frac{\sigma_1}{\sigma_1'}\Big) \pmod{R_\mathfrak{q}\sigma_2^2}.$$

This implies $\xi\sigma_1 \equiv 0 \pmod{R_\mathfrak{q}\tau_1\sigma_2}$, proving (5.7).

Now assume $\mathrm{ord}_\mathfrak{q}(\sigma_2) < \mathrm{ord}_\mathfrak{q}(\tau_1)$. The relation $\tau_1 = \sigma_1 + \alpha\sigma_2$ implies

$$\mathrm{ord}(\sigma_1) \geq \min\{\mathrm{ord}(\tau_1), \mathrm{ord}(\alpha\sigma_2)\} \geq \mathrm{ord}_\mathfrak{q}(\sigma_2),$$

and also $R_\mathfrak{q}\tau_1 + R_\mathfrak{q}\sigma_2 = R_\mathfrak{q}\sigma_2$. The congruences of (5.8) therefore imply

$$\begin{aligned} \sigma_1/\sigma_1' \equiv 1 &\pmod{R_\mathfrak{q}\sigma_2}, \\ \tau_1'/\tau_1 \equiv 1 &\pmod{R_\mathfrak{q}\sigma_2}, \end{aligned}$$

and hence

$$\xi\sigma_1 = \tau_1\Big(1 - \frac{\sigma_1}{\sigma_1'}\frac{\tau_1'}{\tau_1}\Big) \equiv 0 \pmod{R_\mathfrak{q}\tau_1\sigma_2}.$$

Once again, this proves (5.7). □

For a fixed $z \in Z$, choose an open neighborhood $U \subset M$ and sections (5.2) as in Lemma 5.2.

Lemma 5.3. *The choice of sections* (5.2) *determines an isomorphism*

$$f : C(x_1)_U \otimes C(x_2)_U \cong C(y_1)_U \otimes C(x_2)_U, \tag{5.9}$$

and changing the sections changes the isomorphism by a homotopy.

Proof. The choice of sections determines complexes of locally free $\mathcal{O}_U$-modules

$$D(x_1) = (\cdots \to 0 \to \omega_U \xrightarrow{\sigma_1} \mathcal{O}_U \to 0),$$
$$D(x_2) = (\cdots \to 0 \to \omega_U \xrightarrow{\sigma_2} \mathcal{O}_U \to 0),$$
$$D(y_1) = (\cdots \to 0 \to \omega_U \xrightarrow{\tau_1} \mathcal{O}_U \to 0),$$

and there are obvious isomorphisms

$$D(x_1) \cong C(x_1)_U, \quad D(x_2) \cong C(x_2)_U, \quad D(y_1) \cong C(y_1)_U.$$

Indeed, if $x_1 \neq 0$, then

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & \omega_U & \xrightarrow{\sigma_1} & \mathcal{O}_U & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \sigma_1} & & \| & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & I_{Z(x_1)_U} & \longrightarrow & \mathcal{O}_U & \longrightarrow & 0 \end{array}$$

defines an isomorphism $D(x_1) \cong C(x_1)_U$. On the other hand, if $x_1 = 0$, then $\sigma_1 = 0$, and $D(x_1) = C(x_1)_U$ by definition. The other isomorphisms are entirely similar.

To define f, it now suffices to define an isomorphism

$$g : D(x_1) \otimes D(x_2) \cong D(y_1) \otimes D(x_2).$$

The complexes in question are

$$D(x_1) \otimes D(x_2) = (\cdots \to 0 \to \omega_U \otimes \omega_U \xrightarrow{\partial_2} \omega_U \oplus \omega_U \xrightarrow{\partial_1} \mathcal{O}_U \to 0),$$
$$D(y_1) \otimes D(x_2) = (\cdots \to 0 \to \omega_U \otimes \omega_U \xrightarrow{\partial_2^*} \omega_U \oplus \omega_U \xrightarrow{\partial_1^*} \mathcal{O}_U \to 0),$$

where the boundary maps are defined by

$$\partial_1(\eta_1, \eta_2) = \sigma_1(\eta_1) + \sigma_2(\eta_2),$$
$$\partial_1^*(\eta_1, \eta_2) = \tau_1(\eta_1) + \sigma_2(\eta_2),$$
$$\partial_2(\eta_1 \otimes \eta_2) = (\sigma_2(-\eta_2)\eta_1, \sigma_1(\eta_1)\eta_2),$$
$$\partial_2^*(\eta_1 \otimes \eta_2) = (\sigma_2(-\eta_2)\eta_1, \tau_1(\eta_1)\eta_2)$$

for local sections η_1 and η_2 of ω_U. By recalling that $\tau_1 = \sigma_1 + \alpha\sigma_2$, the desired isomorphism is

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & \omega_U \otimes \omega_U & \xrightarrow{\partial_2} & \omega_U \oplus \omega_U & \xrightarrow{\partial_1} & \mathcal{O}_U & \longrightarrow & 0 \\ & & \| & & \| & & \downarrow{\scriptstyle g_1} & & \| & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & \omega_U \otimes \omega_U & \xrightarrow{\partial_2^*} & \omega_U \oplus \omega_U & \xrightarrow{\partial_1^*} & \mathcal{O}_U & \longrightarrow & 0, \end{array}$$

where $g_1(\eta_1, \eta_2) = (\eta_1, \eta_2 - \alpha\eta_1)$.

Having constructed the isomorphism (5.9), we now study its dependence on the sections (5.2). Suppose we have another collection of sections (5.3), and hence two isomorphisms

$$f, f' : C(x_1)_U \otimes C(x_2)_U \cong C(y_1)_U \otimes C(x_2)_U.$$

We must prove that f and f' are homotopic.

If $x_2 = 0$, then $y_1 = x_1$, and the conditions imposed on the sections (5.2) imply that $\sigma_1 = \tau_1$, $\sigma_2 = 0$, and $\alpha = a$. From this it is easy to see that $f = f'$, and so henceforth we assume that $x_2 \neq 0$.

If $x_1 = 0$ and $y_1 = 0$, then the conditions imposed on (5.2) imply that $\sigma_1 = 0$ and $\tau_1 = 0$. The relation $\tau_1 = \sigma_1 + \alpha\sigma_2$ and our assumption $x_2 \neq 0$ therefore imply that $\alpha = 0$. Tracing through the definitions, we again find that $f = f'$.

If $x_1 \neq 0$ and $y_1 \neq 0$, then $e = f - f'$ is given explicitly by

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_{Z(x_1)_U} \otimes I_{Z(x_2)_U} & \xrightarrow{\partial_2} & I_{Z(x_1)_U} \oplus I_{Z(x_2)_U} & \xrightarrow{\partial_1} & \mathcal{O}_U & \longrightarrow & 0 \\ & & \Big\downarrow e_2 & \overset{h_1}{\swarrow} & \Big\downarrow e_1 & \overset{h_0}{\swarrow} & \Big\downarrow 0 & & \\ 0 & \longrightarrow & I_{Z(y_1)_U} \otimes I_{Z(x_2)_U} & \xrightarrow{\partial_2^*} & I_{Z(y_1)_U} \oplus I_{Z(x_2)_U} & \xrightarrow{\partial_1^*} & \mathcal{O}_U & \longrightarrow & 0, \end{array}$$

where the boundary maps are

$$\begin{aligned} \partial_1(\eta_1, \eta_2) &= \eta_1 + \eta_2, \\ \partial_1^*(\eta_1, \eta_2) &= \eta_1 + \eta_2, \\ \partial_2(\eta_1 \otimes \eta_2) &= (-\eta_1\eta_2, \eta_1\eta_2), \\ \partial_2^*(\eta_1 \otimes \eta_2) &= (-\eta_1\eta_2, \eta_1\eta_2), \end{aligned}$$

and

$$e_1(\eta_1, \eta_2) = (\xi\eta_1, -\xi\eta_1), \quad e_2(\eta_1 \otimes \eta_2) = \xi\eta_1 \otimes \eta_2.$$

Here ξ is the rational function on U of Lemma 5.2 (ξ is uniquely determined by relation (5.4), as our assumption $x_1 \neq 0$ implies that σ_1 and σ_1' are nonzero). The dotted arrows, which exhibit the homotopy between e and 0, are defined by $h_0(\eta) = (0, 0)$ and $h_1(\eta_1, \eta_2) = -\xi\eta_1 \cdot 1 \otimes 1$. Note that the definition of h_1 only makes sense because of the inclusion

$$\xi \cdot I_{Z(x_1)_U} \subset I_{Z(y_1)_U} \cdot I_{Z(x_2)_U}$$

of Lemma 5.2.

If $x_1 = 0$ and $y_1 \neq 0$, then $e = f - f'$ is given explicitly by

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \omega_U \otimes I_{Z(x_2)_U} & \xrightarrow{\partial_2} & \omega_U \oplus I_{Z(x_2)_U} & \xrightarrow{\partial_1} & \mathcal{O}_U & \longrightarrow & 0 \\ & & \Big\downarrow e_2 & \overset{h_1}{\swarrow} & \Big\downarrow e_1 & \overset{h_0}{\swarrow} & \Big\downarrow 0 & & \\ 0 & \longrightarrow & I_{Z(y_1)_U} \otimes I_{Z(x_2)_U} & \xrightarrow{\partial_2^*} & I_{Z(y_1)_U} \oplus I_{Z(x_2)_U} & \xrightarrow{\partial_1^*} & \mathcal{O}_U & \longrightarrow & 0, \end{array}$$

where the boundary maps are

$$\begin{aligned} \partial_1(\eta_1, \eta_2) &= \eta_2, \\ \partial_1^*(\eta_1, \eta_2) &= \eta_1 + \eta_2, \\ \partial_2(\eta_1 \otimes \eta_2) &= (-\eta_2\eta_1, 0), \\ \partial_2^*(\eta_1 \otimes \eta_2) &= (-\eta_2\eta_1, \eta_2\eta_1), \end{aligned}$$

and, setting $\zeta = \tau_1 - \tau_1' \in H^0(U, \omega_U^{-1})$,

$$e_1(\eta_1, \eta_2) = (\zeta(\eta_1), -\zeta(\eta_1)), \quad e_2(\eta_1 \otimes \eta_2) = \zeta(\eta_1) \otimes \eta_2.$$

The dotted arrows, which exhibit the homotopy between e and 0, are defined by

$$h_0(\eta) = (0, 0) \quad \text{and} \quad h_1(\eta_1, \eta_2) = -\zeta(\eta_1) \cdot 1 \otimes 1.$$

To make sense of the definition of h_1, note that the relation $y_1 = ax_2$ implies $Z(x_2) \subset Z(y_1)$, and hence

$$\zeta(\eta_1) \in I_{Z(y_1)_U} \cdot (I_{Z(y_1)_U} + I_{Z(x_2)_U}) = I_{Z(y_1)_U} \cdot I_{Z(x_2)_U}.$$

The case $x_1 \neq 0$ and $y_1 = 0$ is entirely analogous to the previous case, and we leave the details to the reader. □

As $y_j = x_j$ for $j \geq 2$, the isomorphism of Lemma 5.9 determines an isomorphism

$$C(x_1)_U \otimes \cdots \otimes C(x_r)_U \cong C(y_1)_U \otimes \cdots \otimes C(y_r)_U,$$

whose homotopy class does not depend on the choices (5.2) used in its construction. Hence the induced isomorphism

$$H_i(C(x_1) \otimes \cdots \otimes C(x_r))_U \cong H_i(C(y_1) \otimes \cdots \otimes C(y_r))_U$$

of $\mathcal{O}_U$-modules also does not depend on these choices. By varying U and gluing, we obtain an isomorphism (5.1) defined over an open neighborhood of Z in M. We have already noted that both sides of (5.1) are supported on Z, and so the isomorphism extends uniquely to all of M. This completes the proof of Theorem 5.1. □

A The exceptional divisor

Throughout this appendix we assume that $\mathbf{k}/\mathbb{Q}_p$ is ramified. We want to explain why the Kudla–Rapoport divisors of Definition 2.4 are generally not flat over $\breve{\mathcal{O}}_{\mathbf{k}}$.

Denote by $\breve{\mathbb{F}} = \breve{\mathcal{O}}_{\mathbf{k}}/\breve{\mathfrak{m}}$ the residue field of $\breve{\mathcal{O}}_{\mathbf{k}}$. The two embeddings $\varphi, \overline{\varphi} : \mathcal{O}_{\mathbf{k}} \to \breve{\mathcal{O}}_{\mathbf{k}}$ necessarily reduce to the unique $\mathbb{Z}_p$-algebra morphism $\mathcal{O}_{\mathbf{k}} \to \breve{\mathbb{F}}$.

Definition A.1. The *exceptional divisor* $\mathrm{Exc} \subset M$ is the set of all points $s \in M$ at which the action

$$i : \mathcal{O}_{\mathbf{k}} \to \mathrm{End}(\mathrm{Lie}(X_s))$$

is through scalars; that is to say, the action factors through the unique morphism $\mathcal{O}_{\mathbf{k}} \to \breve{\mathbb{F}}$. This is a closed subset of the underlying topological space of M, and we endow it with its induced structure of a reduced scheme over $\breve{\mathbb{F}}$.

Proposition A.2. *The exceptional divisor* $\mathrm{Exc} \subset M$ *is a Cartier divisor, and is isomorphic to a disjoint union of copies of the projective space* $\mathbb{P}^{n-1}$ *over* $\breve{\mathbb{F}}$.

Proof. A point $s \in M(\breve{\mathbb{F}})$ corresponds to a pair (X_{0s}, X_s) over $\breve{\mathbb{F}}$, which we recall is really a tuple

$$(X_{0s}, i_0, \lambda_0, \varrho_0, X_s, i, \lambda, F_{X_s}, \varrho) \in M(\breve{\mathbb{F}}).$$

If $s \in \mathrm{Exc}(\breve{\mathbb{F}})$, then the action of $\mathcal{O}_{\mathbf{k}}$ on $\mathrm{Lie}(X)$ is through the unique $\mathbb{Z}_p$-algebra morphism $\mathcal{O}_{\mathbf{k}} \to \breve{\mathbb{F}}$. This implies that *any* codimension one subspace of $F \subset \mathrm{Lie}(X_s)$ satisfies Krämer's signature condition as in Section 2, and we obtain a closed immersion

$$\mathbb{P}(\mathrm{Lie}(X_s)^\vee) \hookrightarrow \mathrm{Exc}$$

by sending $F \mapsto (X_{0s}, i_0, \lambda_0, \varrho_0, X_s, i, \lambda, F, \varrho)$. In other words, vary the codimension one subspace in $\mathrm{Lie}(X_s)$ and leave all other data fixed.

It is clear that Exc is the disjoint union of all such closed subschemes, and that every connected component of Exc is reduced, irreducible, and of codimension one in M. The regularity of M then implies that $\mathrm{Exc} \subset M$ is defined locally by one equation. □

Proposition A.3. *Fix a nonzero* $x \in V$*, and any connected component* $D \subset \mathrm{Exc}$*. For all* $k \gg 0$ *we have* $D \subset Z(p^k x)$*. In particular,* $Z(p^k x)$ *is not flat over* $\breve{\mathcal{O}}_{\mathbf{k}}$.

Proof. If we fix one point $s \in D$, Lemma 2.3 allows us to view

$$x \in \mathrm{Hom}_{\mathcal{O}_{\mathbf{k}}}(X_{0s}, X_s)[1/p].$$

For all $k \gg 0$ we thus have $p^k x \in \mathrm{Hom}_{\mathcal{O}_{\mathbf{k}}}(X_{0s}, X_s)$. It follows from the characterization of $D \cong \mathbb{P}^{n-1}$ found in the proof of Proposition A.2 that the p-divisible groups X_{0D} and X_D are constant (that is, are pullbacks via $D \to \mathrm{Spec}(\breve{\mathbb{F}})$ of p-divisible groups over $\breve{\mathbb{F}}$), and hence the restriction map

$$\mathrm{Hom}_{\mathcal{O}_{\mathbf{k}}}(X_{0D}, X_D) \to \mathrm{Hom}_{\mathcal{O}_{\mathbf{k}}}(X_{0s}, X_s)$$

is an isomorphism. Hence $p^k x \in \mathrm{Hom}_{\mathcal{O}_{\mathbf{k}}}(X_{0D}, X_D)$ and $D \subset Z(p^k x)$. □

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