



Asymptotic behavior of acyclic and cyclic orientations of directed lattice graphs

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ABSTRACT

We calculate exponential growth constants describing the asymptotic behavior of several quantities enumerating classes of orientations of arrow variables on the bonds of several types of directed lattice strip graphs G of finite width and arbitrarily great length, in the infinite-length limit, denoted $\{G\}$. Specifically, we calculate the exponential growth constants for (i) acyclic orientations, $\alpha(\{G\})$, (ii) acyclic orientations with a single source vertex, $\alpha_0(\{G\})$, and (iii) totally cyclic orientations, $\beta(\{G\})$. We consider several lattices, including square (sq), triangular (tri), and honeycomb (hc). From our calculations, we infer lower and upper bounds on these exponential growth constants for the respective infinite lattices. To our knowledge, these are the best current bounds on these quantities. Since our lower and upper bounds are quite close to each other, we can infer very accurate approximate values for the exponential growth constants, with fractional uncertainties ranging from $O(10^{-4})$ to $O(10^{-2})$. Further, we present exact values of $\alpha(tri)$, $\alpha_0(tri)$, and $\beta(hc)$ and use them to show that our lower and upper bounds on these quantities are very close to these exact values, even for modest strip widths. Results are also given for a nonplanar lattice denoted sq_d . We show that $\alpha(\{G\})$, $\alpha_0(\{G\})$, and $\beta(\{G\})$ are monotonically increasing functions of vertex degree for these lattices. A comparison is given of these exponential growth constants with the corresponding exponential growth constant $\tau(\{G\})$ for spanning trees. Our results are in agreement with inequalities following from the Merino–Welsh and Conde–Merino conjectures.

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1. Introduction and basics

In this paper we report new results on three quantities defined on directed graphs, namely acyclic orientations, acyclic orientations with a single source vertex, and totally cyclic orientations of families of directed graphs. We present calculations of the exponential growth constants for these quantities for strip graphs of several lattices in the limit of infinite strip length. From these calculations we infer lower and upper bounds on the three types of exponential growth constants on the thermodynamic limits of the respective lattices. These are quite close to each other, even for modest maximal values of strip width used. Hence, we are able to infer the values of these growth constants to high accuracies, with fractional uncertainties ranging from the $O(10^{-4})$ to $O(10^{-2})$. We also present exact results for three exponential growth constants. From our calculations, we show that the exponential growth constants for acyclic orientations, acyclic orientations with a single source, and totally cyclic orientations are monotonically increasing functions of vertex degree.

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Comparisons are also made with the exponential growth constants for spanning trees on these lattices. The results are in agreement with inequalities implied by the Merino–Welsh and the Conde–Merino conjectures.

We begin with some basic background and definitions. Let $G = (V, E)$ be a graph defined by its vertex and edge sets V and E . Let $n(G) = |V|$, $e(G) = |E|$, and $k(G)$ denote the number of vertices, edges (= bonds), and connected components of G , respectively. We will often use the simpler symbol $n \equiv n(G)$ where no confusion will result. Without loss of generality, we restrict our analysis to connected graphs here, so $k(G) = 1$. We also restrict our analysis to graphs that do not contain loops (i.e., edges that connect a vertex to itself) or multiple edges joining a given pair of vertices. The reasons for this exclusion will be explained below. The degree $\Delta(v_i)$ of a vertex $v_i \in V$ is the number of edges that are incident on v_i . Given a graph G , one can assign an arrow to each edge of G , thereby defining a directed graph, also called a digraph, $D(G)$ [1]. We denote a directed (oriented) edge joining a vertex v_i to a vertex v_j , with the arrow pointing from v_i to v_j , as $\vec{e}_{ij}$, and the set of edges of $D(G)$ with such arrow assignments as $\vec{E}$, so that $D(G) = (V, \vec{E})$. For a given G , there are

$$N_{eo}(G) = 2^{e(G)} \quad (1.1)$$

such assignments of arrows to the edges of G and hence $2^{e(G)}$ corresponding directed graphs $D(G)$. The subscript eo on $N_{eo}(G)$ stands for “edge orientations”. Since we study edge orientations of graphs G , we assume henceforth that G has at least one edge, and hence exclude the trivial case in which G consists only of one or more disjoint vertices with no edges. Given a vertex v_i in a digraph $D(G)$, its out-degree, $\Delta^+(v_i)$, and in-degree, $\Delta^-(v_i)$, are, respectively, the number of outgoing and incoming arrows on edges incident on v_i , so that the total degree of the vertex is $\Delta(v_i) = \Delta^+(v_i) + \Delta^-(v_i)$. A graph with the property that all vertices have the same degree is denoted a Δ -regular graph. A directed cycle on a directed graph $D(G)$ is defined as a set of directed edges forming a cycle such that, as one traverses the cycle in a given direction, all of the arrows on the oriented edges point in the direction of traversal. An acyclic orientation of the edge arrows of $D(G)$ is one in which there are no directed cycles. Further background on graph is given in [Appendix](#).

An important question in the study of directed graphs concerns the enumeration of the subset of the $2^{e(G)}$ directed graphs $D(G)$ that are acyclic. Since this number depends only on the structure of G itself, it is commonly denoted $a(G)$ and is called the number of acyclic orientations of (arrows on edges of) G . In addition to its intrinsic interest in mathematical graph theory, the quantity $a(G)$ is also of interest in applications, such as manufacturing and operations research. The reason for this is that directed graphs that describe these applications are acyclic.

The number of acyclic orientations of a graph G can be calculated via an evaluation of the chromatic polynomial of G . We recall that the chromatic polynomial of a graph G , denoted $P(G, q)$, enumerates the number of assignments of q colors to the vertices of G , subject to the condition that no two adjacent vertices have the same color [1–6]. This is called a proper q -coloring of (the vertices of) G . $P(G, q)$ is a polynomial of degree n in q . For proper q colorings, q must be a positive integer, and cannot be equal to 1 if G contains at least one edge. More generally, one can consider the behavior of $P(G, q)$ at other values of q . For acyclic orientations, one has [7]

$$a(G) = (-1)^{n(G)} P(G, -1). \quad (1.2)$$

The chromatic polynomial is a special case of an important two-variable function, namely the partition function of the q -state Potts model [8], $Z(G, q, v)$ with $v = -1$ (zero-temperature Potts antiferromagnet), or equivalently, the Tutte polynomial $T(G, x, y)$ [1,9–11] with $x = 1 - q$ and $y = 0$ (see Eqs. (A.6) and (A.14) in the [Appendix](#); some recent reviews are [12–14]). Using this connection, one can equivalently express $a(G)$ as an evaluation of $T(G, x, y)$, namely

$$a(G) = T(G, 2, 0). \quad (1.3)$$

In many practical applications such as manufacturing processes and scheduling, the relevant digraph is characterized by a single source vertex, e.g., the first position of an item on an assembly line in a factory. Thus, a second quantity of interest can be defined as follows. In a graph $G = (V, E)$, let us pick a given vertex $v_i \in V$. Among the $a(G)$ acyclic orientations of the edges of G , count the number for which two conditions are satisfied: (i) v_i is a source vertex, i.e., it has only outgoing arrows on edges incident with it (and hence maximal out-degree $\Delta^+(v_i) = \Delta(v_i) \geq 1$); and (ii) v_i is the only source vertex. In order for this to be a function of G , it must be true, and it does turn out to be true, that this number is independent of which vertex v_i one selects for this enumeration [15]. This number is denoted as $a_0(G)$ and can also be calculated from a knowledge of the chromatic polynomial $P(G, q)$. This polynomial $P(G, q)$ is identical to the partition function of the zero-temperature q -state Potts antiferromagnet in statistical physics. A proper q -coloring of G is obviously not possible if $q = 0$, so $P(G, 0) = 0$, and hence, as a polynomial, $P(G, q)$ always has an overall factor of q . Hence, one can define the reduced (r) polynomial $P_r(G, q) = q^{-1}P(G, q)$, as in Eq. (A.2). Then [15,16]

$$a_0(G) = (-1)^{n(G)-1} P_r(G, 0). \quad (1.4)$$

From Eqs. (A.6) and (A.14), $a_0(G)$ can also be defined as an evaluation of the Tutte polynomial,

$$a_0(G) = T(G, 1, 0). \quad (1.5)$$

If a graph G contains a loop, this precludes the possibility of a proper q -coloring, and thus the chromatic polynomial $P(G, q)$ vanishes identically, as do both $a(G)$ and $a_0(G)$. It is to avoid these trivial zeros that we exclude graphs with loops in our analysis.

Let us illustrate these definitions in the simplest nontrivial case, namely the tree graph T_2 with two vertices and a single edge joining them. In general, for the tree graph with n vertices, $T(T_n, x, y) = x^{n-1}$. There are two edge orientations of T_2 , both of which are acyclic, so $a(T_2) = T(T_2, 2, 0) = 2$. In both of these, there is a source vertex, but the source vertex is different for the two different edge orientations. Recalling the definition of $a_0(G)$, one picks a specific vertex and then enumerates how many of the acyclic orientations have this specific vertex as a source vertex, and this number is $a_0(T_2) = T(T_2, 1, 0) = 1$. As is obvious from the fact that the reversal of all arrows is an automorphism of the set of all orientations of directed edges, one could equivalently define a_0 as the number of acyclic orientations of G such that there is a unique sink rather than a unique source.

Since acyclic orientations with a unique source are a subset of all acyclic orientations, it follows that

$$a_0(G) \leq a(G), \quad \text{i.e.,} \quad T(G, 1, 0) \leq T(G, 2, 0). \quad (1.6)$$

This inequality is evident from Eq. (A.13), since the coefficients of the nonzero terms in $T(G, x, y)$ are positive. From the relations (1.3) and (1.5), the necessary and sufficient condition for (1.6) to be an equality is clear, namely that $T(G, 1, 0) = T(G, 2, 0)$ if and only if $T(G, x, y)$ contains an overall factor of y so that $T(G, 1, 0) = T(G, 2, 0) = 0$ and thus $a_0(G) = a(G) = 0$. For the lattice graphs of interest here, (1.6) will be a strict inequality, i.e., $a_0(G) < a(G)$.

It is also of interest to enumerate, for a given graph G , the number of digraphs $D(G)$ in which every directed edge is a member of at least one directed cycle. Such digraphs are called totally cyclic, and the directed edges are called as totally cyclic orientations of $D(G)$. We denote the number of these as $b(G)$. (The number of totally cyclic orientations of G should not be confused with the number of linearly independent cycles on G , denoted $c(G)$, which is given by $c(G) = e(G) + k(G) - n(G)$.) The number $b(G)$ can be obtained as an evaluation of the Tutte polynomial, namely [17,18]

$$b(G) = T(G, 0, 2). \quad (1.7)$$

Starting with a given graph G , one can increase $b(G)$ arbitrarily by replacing each edge with multiple edges joining the same pair of vertices. In order to have a minimal measure of totally cyclic orientations, we thus exclude graphs with multiple edges in our analysis. Using the equivalence between the Tutte polynomial and the partition function of the Potts model, as given in Eq. (A.14), we can express $b(G)$ as

$$b(G) = (-1)^{k(G)} Z(G, -1, 1) = -Z(G, -1, 1). \quad (1.8)$$

where $Z(G, q, v)$ is the partition function of the Potts model on the graph G at a temperature given by the variable v defined in Eq. (A.9). Thus, $b(G)$ is obtained as an evaluation of the partition function of the Potts ferromagnet at $q = -1$ and the (finite-temperature) value $v = 1$. The quantity $b(G)$ can also be obtained as an evaluation of a one-variable polynomial, namely the flow polynomial $F(G, q)$, as

$$b(G) = (-1)^{e(G)-n(G)-1} F(G, -1). \quad (1.9)$$

The flow polynomial $F(G, q)$ enumerates the number of nowhere-zero q -flows on the graph G (with flow conservation mod q at vertices) [1]. In addition to Refs. [7–18], some related work includes Refs. [19–24]. In particular, in [21] we presented a number of results on acyclic orientations and their asymptotic behavior.

A recursive family of graphs is a family of graphs such that the $(m+1)$ 'th member, G_{m+1} , can be obtained from the m 'th member, roughly speaking, by the addition of some subgraph [25,26]. For example, a square-lattice ladder strip of length $m+1$ vertices with free boundary conditions can be obtained by adding a square to the end of the square-lattice strip of length m . For a wide variety of recursive families of graphs, these numbers $a(G)$, $a_0(G)$, and $b(G)$ grow exponentially with the number of vertices, n , for $n \gg 1$. It is thus of interest to study the associated exponential growth constants. Let us denote $\{G\}$ as the limit of the recursive family of n -vertex graphs G as $n \rightarrow \infty$. We define the three exponential growth constants for $a(G)$, $a_0(G)$, and $b(G)$ as

$$\alpha(\{G\}) = \lim_{n \rightarrow \infty} [a(G)]^{1/n} \quad (1.10)$$

$$\alpha_0(\{G\}) = \lim_{n \rightarrow \infty} [a_0(G)]^{1/n} \quad (1.11)$$

and

$$\beta(\{G\}) = \lim_{n \rightarrow \infty} [b(G)]^{1/n}. \quad (1.12)$$

For all of the lattices that we study, we find the inequality

$$\alpha_0(\{G\}) < \alpha(\{G\}). \quad (1.13)$$

Note that this inequality is not implied by the inequality (1.6), since, *a priori*, the difference, $\lim_{n(G) \rightarrow \infty} [a(G)]^{1/n(G)} - \lim_{n(G) \rightarrow \infty} [a_0(G)]^{1/n(G)}$ might vanish as $n(G) \rightarrow \infty$.

Let G be a planar graph, which we denote as G_{pl} , and denote G_{pl}^* as the planar dual graph, formed by bijectively associating the vertices (respectively faces) of G_{pl} with the faces (respectively, vertices) of G_{pl}^* , and connecting the vertices of G_{pl}^* via edges crossing the edges of G_{pl} . For such a planar graph, the Tutte polynomial satisfies the relation

$$T(G_{pl}, x, y) = T(G_{pl}^*, y, x). \quad (1.14)$$

In particular, $T(G_{pl}, 2, 0) = T(G_{pl}^*, 0, 2)$, so

$$a(G_{pl}) = b(G_{pl}^*) . \quad (1.15)$$

We denote the number of faces of a graph G as $f(G)$ and recall the Euler relation that for a planar graph G_{pl} ,

$$f(G_{pl}) - e(G_{pl}) + n(G_{pl}) = 2 . \quad (1.16)$$

From the duality relation, it follows that $n(G_{pl}^*) = f(G_{pl})$. For Δ -regular graphs G ,

$$e(G) = \frac{\Delta(G) n(G)}{2} . \quad (1.17)$$

For a Δ -regular planar graph G_{pl} we define the ratio

$$v_{\{G_{pl}\}} \equiv \lim_{n(G_{pl}) \rightarrow \infty} \frac{n(G_{pl}^*)}{n(G_{pl})} = \frac{\Delta(G_{pl})}{2} - 1 . \quad (1.18)$$

where we have used Eq. (1.17) in the last equality in (1.18). Note that

$$v(\{G_{pl}\}) = \frac{1}{v(\{G_{pl}^*\})} . \quad (1.19)$$

We record the specific values

$$v(sq) = 1 \quad (1.20)$$

and

$$v(tri) = \frac{1}{v(hc)} = 2 , \quad (1.21)$$

where the property that $v(tri) = 1/v(hc)$ follows from the fact that the triangular and honeycomb lattice are planar duals of each other. From Eq. (1.15), it follows that if a planar graph is self-dual, indicated as $G_{pl, sd.}$, then $a(G_{pl, sd.}) = b(G_{pl, sd.})$, and hence

$$\alpha(\{G_{pl, sd.}\}) = \beta(\{G_{pl, sd.}\}) . \quad (1.22)$$

In particular, since the square lattice is planar and self-dual, we have

$$\alpha(sq) = \beta(sq) , \quad (1.23)$$

so that the lower and upper bounds that we infer below for $\alpha(sq)$ also hold for $\beta(sq)$. For the triangular and honeycomb lattices, we obtain the relations

$$\alpha(hc) = [\beta(tri)]^{v(hc)} = [\beta(tri)]^{1/2} \quad (1.24)$$

and

$$\beta(hc) = [\alpha(tri)]^{v(hc)} = [\alpha(tri)]^{1/2} . \quad (1.25)$$

As was true of $a(G)$, $a_0(G)$, and $b(G)$, the corresponding exponential growth constants $\alpha(\{G\})$, $\alpha_0(\{G\})$, and $\beta(\{G\})$ have interesting connections with quantities in statistical physics. Specifically,

$$\alpha(\{G\}) = |W(\{G\}, -1)| \quad (1.26)$$

and

$$\alpha_0(\{G\}) = |W(\{G\}, 0)| , \quad (1.27)$$

where $W(\{G\}, q)$ is the ground-state (i.e., zero-temperature) degeneracy of states, normalized per vertex (site) of the q -state Potts antiferromagnet, defined in Eq. (A.11). The absolute values are used in Eqs. (1.26) and (1.27) because for (real) values of q away from the positive integers, $P(G, q)$ can be negative, so that the formal relation $W(\{G\}, q) = \lim_{n \rightarrow \infty} [P(G, q)]^{1/n}$ in Eq. (A.11) requires specification of which of the n roots of (-1) one chooses, while the value of $|W(\{G\}, q)|$ is unambiguous. (Note that for any finite $n = n(G)$, a negative sign in $P(G, q)$ is canceled by the factor of $(-1)^{n(G)}$ in Eq. (1.2) for $a(G)$ and the factor of $(-1)^{n(G)-1}$ in Eq. (1.4) for $a_0(G)$.)

Furthermore,

$$\beta(\{G\}) = e^{f(\{G\}, -1, 1)} , \quad (1.28)$$

where $f(\{G\}, q, v)$ is the dimensionless free energy per vertex of the q -state Potts model defined in Eq. (A.10). Although the Potts model partition function is naturally defined for integral $q \geq 1$ in statistical mechanics, its definition can also be extended, via Eq. (A.5) to more general values of q , and this generalization is used here. The absolute value is used in

Eq. (1.28) because $Z(G, -1, 1)$ is negative (as is clear from Eq. (1.8)), so the formal relation (A.10) requires specification of which of the n roots of (-1) one uses, whereas $|f(\{G\}, -1, 1)|$ is unambiguously determined by Eq. (A.10).

Thus, although it might initially seem that $a(G)$, $a_0(G)$, $b(G)$, and the associated exponential growth constants $\alpha(\{G\})$, $\alpha_0(\{G\})$, and $\beta(\{G\})$, are only of relevance in mathematical graph theory and applications such as operations research, Eqs. (1.2)–(1.7) and (1.26), (1.27), and (1.28), with Eqs. (A.6), (A.10), (A.11), and (A.14), show that these quantities also have interesting and fruitful connections with statistical physics. Our new results in this paper demonstrate the value of exploiting these connections.

A basic property of a digraph is that the number of edge orientations on G , $N_{eo}(G)$, grows exponentially rapidly as a function of the number of edges of G . In order to define the corresponding exponential growth constant, one first expresses $e(G)$ in terms of $n(G)$. This is done via Eq. (1.17) for Δ -regular graphs. More generally, for graphs containing vertices of different degrees, we define an effective vertex degree, as in [27] (with $n \equiv n(G)$ here and below),

$$\Delta_{eff}(G) = \frac{2e(G)}{n}. \quad (1.29)$$

In particular, in the $n \rightarrow \infty$ limit,

$$\Delta_{eff}(\{G\}) = \lim_{n(G) \rightarrow \infty} \frac{2e(G)}{n}. \quad (1.30)$$

Thus, the exponential growth constant for the total number of edge orientations $D(G)$ of a Δ -regular family of graphs G , normalized per vertex, is

$$\epsilon(\{G\}) \equiv \lim_{n \rightarrow \infty} [N_{eo}(G)]^{1/n} = \lim_{n \rightarrow \infty} (2^{e(G)})^{1/n} = 2^{\Delta(\{G\})/2}. \quad (1.31)$$

More generally, for a family of graphs G containing vertices with different degrees,

$$\epsilon(\{G\}) = 2^{\Delta_{eff}(\{G\})/2}. \quad (1.32)$$

For a given digraph, many of the edge orientations do not fall into any of the three classes (i)–(iii), i.e., they are not acyclic or totally cyclic. This is reflected in the property that for most digraphs, $a(G) + b(G) < N_{eo}(G)$. An interesting question related to this is the following: for a given graph $G = G(V, E)$, what fraction of the total number of edge orientations is comprised of those that are (i) acyclic, (ii) acyclic with a unique source vertex, and (iii) totally cyclic? Each of the corresponding numbers $a(G)$, $a_0(G)$, and $b(G)$ can be denoted generically as $N_{cond.}(G)$, where the subscript *cond.* refers to the condition that the edge orientations must satisfy to be a member of the given class. The corresponding fraction is then the ratio

$$r_{cond.}(G) \equiv \frac{N_{cond.}(G)}{N_{eo}(G)} = \frac{N_{cond.}(G)}{2^{e(G)}}. \quad (1.33)$$

For each of the specified conditions (i)–(iii), $r_{cond.}(G) \leq 1$. For certain families of graphs, $r_{cond.}(G) = 1$. Recall that a tree graph is a connected graph that does not contain any circuits. For n -vertex tree graphs, denoted T_n , all edge orientations are acyclic, so $a(T_n) = N_{eo}(T_n) = 2^{n-1}$ and thus $r_a(T_n) = 1$, while $b(T_n) = 0$, so $r_b(T_n) = 0$ (independent of n). However, as discussed above, from exact results on strip graphs of lattices with fixed width $L_y \geq 2$ and with various transverse and longitudinal boundary conditions, one finds that $r_{cond.}(G) < 1$ and, furthermore, that $N_{cond.}(G)$ grows exponentially rapidly with $n(G)$ for $n(G) \gg 1$. A relevant question is the following: for a given condition, does the ratio $r_{cond.}(G)$ approach a finite nonzero constant as $n(G) \rightarrow \infty$ or not? The answer to this question depends on the type of families graphs that one considers. If one were to consider tree graphs, for example, then the ratio $r_a(T_n)$ would be finite (and equal to its maximal value, 1) for all n and, in particular, for $n \rightarrow \infty$. However, for all of the lattice strip graphs of finite width (i.e., excluding the circuit graph) that we have studied, $r_{cond.}(G)$ vanishes as $n(G) \rightarrow \infty$. That is,

$$\lim_{n \rightarrow \infty} \frac{N_{cond.}(G)}{N_{eo}(G)} = 0, \quad \text{where } N_{cond.}(G) = a(G), \quad a_0(G), \quad \text{or } b(G). \quad (1.34)$$

Specifically, for these lattice strip graphs we find that $r_{cond.}(G)$ vanishes exponentially rapidly as $n(G) \rightarrow \infty$. Therefore, one is motivated to define a measure of this exponential decrease in the ratio $r_{cond.}(G)$. Since for most lattice strip graphs both the numerator and denominator of the ratio $r_{cond.}(G)$ increase exponentially rapidly with n , it is natural to define this measure as

$$\rho_{N_{cond.}}(\{G\}) \equiv \lim_{n \rightarrow \infty} [r_{cond.}(G)]^{1/n} = \frac{\lim_{n \rightarrow \infty} [N_{cond.}(G)]^{1/n}}{\epsilon(\{G\})}. \quad (1.35)$$

For each of the three quantities considered here corresponding to the orientations satisfying the specified conditions (i) acyclic, (ii) acyclic with a unique source vertex or sink vertex, and (iii) totally cyclic, we then have

$$\rho_\alpha(\{G\}) \equiv \lim_{n \rightarrow \infty} \left(\frac{a(G)}{N_{eo}(G)} \right)^{1/n} = \frac{\alpha(\{G\})}{\epsilon(\{G\})} \quad (1.36)$$

$$\rho_{\alpha_0}(\{G\}) \equiv \lim_{n \rightarrow \infty} \left(\frac{a_0(G)}{N_{eo}(G)} \right)^{1/n} = \frac{\alpha_0(\{G\})}{\epsilon(\{G\})} \quad (1.37)$$

and

$$\rho_\beta(\{G\}) \equiv \lim_{n \rightarrow \infty} \left(\frac{b(G)}{N_{eo}(G)} \right)^{1/n} = \frac{\beta(\{G\})}{\epsilon(\{G\})}. \quad (1.38)$$

This paper is organized as follows. In Section 2 we present a number of exact results on $a(G)$, $a_0(G)$, $b(G)$ for lattice strip graphs and show how, in the limit of infinite strip length, these yield resultant values for the corresponding exponential growth constants $\alpha(G)$, $\alpha_0(G)$, and $\beta(G)$. In Sections 3 and 4 we discuss our methods for inferring lower and upper bounds on the exponential growth constants for infinite lattices from calculations on strip graphs of varying widths in the limit of infinite width. In Sections 5 and 6 we present our numerical results on these lower and upper bounds for $\alpha(\Lambda)$, $\alpha_0(\Lambda)$, and $\beta(\Lambda)$ for various lattices Λ . In these sections, using exact values of $\alpha(tri)$, $\alpha_0(tri)$, and $\beta(hc)$, we show that our lower and upper bounds on these quantities are very close to the exact values for modest values of strip widths. We present some further discussion in Section 7, including a comparison with growth constants for spanning trees. Our conclusions are given in Section 8. Some graph theory background is included in [Appendix](#).

2. Exact results for lattice strip graphs

In this section we present some exact calculations of $a(G)$, $a_0(G)$, and $b(G)$ for lattice strip graphs of fixed transverse width L_y and arbitrarily great length L_x with certain boundary conditions, and show how one derives the corresponding exponential growth constants $\alpha(\{G\})$, $\alpha_0(\{G\})$, and $\beta(\{G\})$ from these in the limit $L_x \gg 1$. As indicated, we take the longitudinal direction to lie along the x (horizontal) axis, and the transverse direction to lie along the y (vertical) axis. We also include results on the constants $\rho_\alpha(\{G\})$, $\rho_{\alpha_0}(\{G\})$, and $\rho_\beta(\{G\})$. These examples are selected from calculations of chromatic and Tutte polynomials for a number of lattice strip graphs (e.g., [\[21,28–46\]](#)).

2.1. Cyclic square-lattice ladder graph

We first consider the square-lattice ladder strip graph L_m of width $L_y = 2$ vertices and length $L_x \equiv m$ vertices with cyclic longitudinal boundary conditions and free transverse boundary conditions. This graph has $n(L_m) = 2m$ vertices, $e(L_m) = 3m$ edges, and uniform vertex degree, $\Delta = 3$. The number of linearly independent cycles on L_m is $c(L_m) = m + 1$. We denote the infinite-length limit, $m \rightarrow \infty$, of this strip graph as $\{L\}$. The exponential growth constant for the number of edges in this limit is

$$\epsilon(\{L\}) = 2^{3/2} = 2.828427, \quad (2.1)$$

where here and below, we write non-integer numbers with the indicated number of significant figures.

Evaluating the chromatic polynomial at $q = -1$, one finds [\[21\]](#)

$$a(L_m) = 7^m - 2(2^m + 4^m) + 5. \quad (2.2)$$

Using Eq. [\(1.10\)](#), one calculates

$$\alpha(\{L\}) = \sqrt{7} = 2.645751. \quad (2.3)$$

Evaluating the expression for $a_0(G)$ in Eq. [\(1.4\)](#), we obtain

$$a_0(L_m) = (2m - 3)3^{m-1} - (m - 2). \quad (2.4)$$

Hence, in the limit $m \rightarrow \infty$,

$$\alpha_0(\{L\}) = \sqrt{3} = 1.732051. \quad (2.5)$$

The origin of the factors of m in Eq. [\(2.4\)](#) is as follows. For a cyclic strip graph G_{strip} , the chromatic polynomial $P(G_{strip}, q)$ has the form of a sum of powers of certain algebraic functions multiplied by various coefficients. These powers involve $L_x = m$, the length of the strip. Although a chromatic polynomial always has a factor of q , this factor is not explicit in the expression written as a sum of powers of these algebraic functions. Consequently, to evaluate $P_r(G_{strip}, q) = q^{-1}P(G_{strip}, q)$ at $q = 0$, one actually uses L'Hôpital's rule, calculating

$$P_r(G_{strip}, 0) = \lim_{q \rightarrow 0} P_r(G_{strip}, q) = \left. \frac{dP(G_{strip}, q)}{dq} \right|_{q=0} \quad (2.6)$$

It is this differentiation that brings down factors of m .

For the number of totally cyclic orientations of G , from the solution for $Z(L_m, q, v)$ or equivalently $T(L_m, x, y)$ in Ref. [\[39\]](#), we calculate

$$b(L_m) = 2(4^m) - 3^m - 5, \quad (2.7)$$

so that, in the limit $m \rightarrow \infty$,

$$\beta(\{L\}) = 2. \quad (2.8)$$

From these results and the expression for $\epsilon(\{L\})$, we compute

$$\rho_\alpha(\{L\}) = \frac{\alpha(\{L\})}{\epsilon(\{L\})} = \sqrt{\frac{7}{8}} = 0.935414 \quad (2.9)$$

$$\rho_{\alpha_0}(\{L\}) = \frac{\alpha_0(\{L\})}{\epsilon(\{L\})} = \sqrt{\frac{3}{8}} = 0.612372 \quad (2.10)$$

and

$$\rho_\beta(\{L\}) = \frac{\beta(\{L\})}{\epsilon(\{L\})} = \frac{1}{\sqrt{2}} = 0.707107. \quad (2.11)$$

2.2. Cyclic triangular-lattice ladder graph

We next consider a cyclic strip of the triangular lattice TL_m [where TL is an abbreviation for “triangular (lattice) ladder”] of width $L_y = 2$ vertices and length $L_x = m$ vertices. This graph can be obtained from the cyclic square-lattice strip by adding a diagonal edge to each square from, say, the lower left vertex to the upper right vertex. This graph has $n(TL_m) = 2m$ vertices, $e(TL_m) = 4m$ edges, uniform vertex degree $\Delta = 4$, and $c(TL_m) = 2m + 1$ linearly independent cycles. Evaluating the chromatic polynomial at $q = -1$, one finds [21]

$$a(TL_m) = 9^m - 2 \left[\left(\frac{7 + \sqrt{13}}{2} \right)^m + \left(\frac{7 - \sqrt{13}}{2} \right)^m \right] + 5. \quad (2.12)$$

Denoting the limit of TL_m as $m \rightarrow \infty$ as $\{TL\}$ and using Eq. (1.10), one has

$$\alpha(\{TL\}) = 3. \quad (2.13)$$

Evaluating $P_r(TL_m, q)$ at $q = 0$, we obtain

$$a_0(TL_m) = \left(\frac{2m}{3} - 1 \right) 4^m - \frac{2m}{3} + 2, \quad (2.14)$$

and hence

$$\alpha_0(\{TL\}) = 2. \quad (2.15)$$

For the number of totally cyclic orientations of G , from the solution for $Z(TL_m, q, v)$ in Ref. [40], we find

$$b(TL_m) = 2 \left[\left(\frac{11 + 3\sqrt{13}}{2} \right)^m + \left(\frac{11 - 3\sqrt{13}}{2} \right)^m \right] - 9^m - 5, \quad (2.16)$$

and hence

$$\beta(\{TL\}) = \left(\frac{11 + 3\sqrt{13}}{2} \right)^{1/2} = \frac{3 + \sqrt{13}}{2} = 3.3027756. \quad (2.17)$$

Combining these results with

$$\epsilon(\{TL\}) = 4, \quad (2.18)$$

we obtain

$$\rho_\alpha(\{TL\}) = \frac{3}{4}, \quad (2.19)$$

$$\rho_{\alpha_0}(\{TL\}) = \frac{1}{2}, \quad (2.20)$$

and

$$\rho_\beta(\{TL\}) = \frac{3 + \sqrt{13}}{8} = 0.825694. \quad (2.21)$$

2.3. Cyclic honeycomb-lattice ladder graph

We next consider a cyclic strip of the honeycomb lattice, HL_m [where HL stands for “honeycomb (lattice) ladder”] of width $L_y = 2$ vertices and length $L_x = 2m$ vertices. Here, m is the number of hexagons in a horizontal layer of the strip. This graph can be obtained from the cyclic square-lattice strip by adding a vertex on each horizontal edge (which is a homeomorphic expansion of the square-lattice ladder strip). The graph HL_m has $n(HL_m) = 4m$ vertices, $e(HL_m) = 5m$ edges,

and $c(L_m) = m + 1$ linearly independent cycles. It has two equal subsets of vertices of two different degree values, namely 2 and 3, and thus an effective vertex degree of $\Delta_{\text{eff}} = 5/2$. Evaluating the chromatic polynomial [31] at $q = -1$, we find

$$a(HL_m) = (31)^m - 2(10^m + 4^m) + 5. \quad (2.22)$$

Denoting the limit of HL_m as $m \rightarrow \infty$ as $\{HL\}$ and using Eq. (1.10), we thus obtain

$$\alpha(\{HL\}) = (31)^{1/4} = 2.359611. \quad (2.23)$$

Further, we compute

$$a_0(HL_m) = \left(\frac{6m}{5} - 1\right) 5^m - 2(m - 1), \quad (2.24)$$

and hence

$$\alpha_0(\{HL\}) = 5^{1/4} = 1.495349. \quad (2.25)$$

For the number of totally cyclic orientations of G , from the solution for $Z(HL_m, q, v)$ or equivalently $T(HL_m, x, y)$ in Ref. [42], we calculate

$$b(HL_m) = 2(4^m) - 3^m - 5, \quad (2.26)$$

and hence

$$\beta(\{HL\}) = \sqrt{2} = 1.414214. \quad (2.27)$$

Combining these results with

$$\epsilon(\{HL\}) = 2^{5/4} = 2.378414, \quad (2.28)$$

we obtain

$$\rho_\alpha(\{HL\}) = \left(\frac{31}{32}\right)^{1/4} = 0.992094 \quad (2.29)$$

$$\rho_{\alpha_0}(\{HL\}) = \left(\frac{5}{32}\right)^{1/4} = 0.628717 \quad (2.30)$$

and

$$\rho_\beta(\{HL\}) = \left(\frac{1}{8}\right)^{1/4} = 0.594604. \quad (2.31)$$

2.4. A family of self-dual planar graphs

An n -vertex wheel graph $Wh_n = K_1 + C_{n-1}$ is comprised of a circuit graph with all $n - 1$ vertices on the “rim” connected to one central “spoke” vertex. (Here, $G + H$ is the “join” of G and H). This graph has $e(Wh_n) = 2(n - 1)$ and thus $c(Wh_n) = n - 1$ linearly independent circuits. The $n - 1$ vertices on the rim have degree 3 and the central spoke vertex has degree $n - 1$, so in the limit $n \rightarrow \infty$, the effective degree is $\Delta_{\text{eff}} = 4$. The wheel graph Wh_n is a self-dual planar graph, so, as a consequence of Eq. (1.15),

$$a(Wh_n) = b(Wh_n). \quad (2.32)$$

An elementary calculation yields $P(Wh_n, q) = q[(q - 2)^{n-1} + (q - 2)(-1)^{n-1}]$, so

$$a(Wh_n) = b(Wh_n) = 3^{n-1} - 3 \quad (2.33)$$

and

$$a_0(Wh_n) = 2^{n-1} - 2. \quad (2.34)$$

Denoting $\{Wh\}$ as the $n \rightarrow \infty$ limit of the Wh_n family, we then have

$$\alpha(\{Wh\}) = \beta(\{Wh\}) = 3 \quad (2.35)$$

and

$$\alpha_0(\{Wh\}) = 2. \quad (2.36)$$

We note that Wh_n (and its $n \rightarrow \infty$ limit) share with the infinite square lattice the property of being planar and self-dual. Combining the results above with

$$\epsilon(\{Wh\}) = 4, \quad (2.37)$$

we find

$$\rho_\alpha(\{Wh\}) = \rho_\beta(\{Wh\}) = \frac{3}{4} \quad (2.38)$$

and

$$\rho_{\alpha_0}(\{Wh\}) = \frac{1}{2} . \quad (2.39)$$

2.5. The sq_d family of cyclic strip graphs

It is also of interest to investigate a family of strip graphs with a higher value of Δ . We define strip graphs of this family, denoted $(sq_d)_m$ (where the subscript d stands for “diagonals”), as follows. One starts with the cyclic square-lattice ladder graph L_m of width $L_y = 2$ and length $L_x = m$ and then adds (i) an edge connecting the upper left and lower right vertices of each square to each other, and (ii) an edge connecting the upper right and lower left vertices of each square to each other. This is a Δ -regular nonplanar lattice graph with $\Delta_{sq_d} = 5$. The cyclic $(sq_d)_m$ lattice strip has $n = 2m$ vertices and $5m$ edges. Using our calculations in [37], we obtain

$$a([sq_d]_m) = (12)^m - 2 \cdot (8)^m + 2^{m+1} \quad (2.40)$$

and

$$a_0([sq_d]_m) = \left(\frac{m}{2} - 1\right) 6^m + 3 \cdot 2^{m-1} \quad (2.41)$$

Denoting $\{sq_d\}$ as the $m \rightarrow \infty$ limit of the $[sq_d]_m$ family, we then have

$$\alpha(\{sq_d\}) = 2\sqrt{3} = 3.464102 \quad (2.42)$$

and

$$\alpha_0(\{sq_d\}) = \sqrt{6} = 2.449490 . \quad (2.43)$$

From our results in [43], we find

$$b([sq_d]_m) = 2 \left[(13 + \sqrt{181})^m + (13 - \sqrt{181})^m \right] - \left[\{2(6 + \sqrt{39})\}^m + \{2(6 - \sqrt{39})\}^m \right] - 2^{m+1} , \quad (2.44)$$

so that, in the $m \rightarrow \infty$ limit,

$$\beta(\{sq_d\}) = \sqrt{13 + \sqrt{181}} = 5.143309 . \quad (2.45)$$

In conjunction with

$$\epsilon(\{sq_d\}) = 2^{5/2} = 5.656854 , \quad (2.46)$$

these results yield

$$\rho_\alpha(\{sq_d\}) = \sqrt{\frac{3}{8}} = 0.612372 \quad (2.47)$$

$$\rho_{\alpha_0}(\{sq_d\}) = \frac{\sqrt{3}}{4} = 0.433013 \quad (2.48)$$

and

$$\rho_{\beta_0}(\{sq_d\}) = \sqrt{\frac{13 + \sqrt{181}}{32}} = 0.909217 . \quad (2.49)$$

2.6. Comparative properties

We list our results for the exponential growth constants and the corresponding $\rho_{cond.}$ constants for these infinite-length, finite-width strips in Table 1. From these results, we can observe several properties. We find that $\alpha(\{G\})$, $\alpha_0(\{G\})$, and $\beta(\{G\})$ are monotonically increasing functions of Δ (and, where applicable, of Δ_{eff}). This is also true of the ratios $\alpha_0(\{G\})/\alpha(\{G\})$ for the various strips; from the values listed in Table 1, we have

$$\frac{\alpha_0(\{HL\})}{\alpha(\{HL\})} = \left(\frac{5}{31}\right)^{1/4} = 0.633727 \quad (2.50)$$

$$\frac{\alpha_0(\{L\})}{\alpha(\{L\})} = \sqrt{\frac{3}{7}} = 0.654654 \quad (2.51)$$

Table 1

Values of ϵ , α , α_0 , β , ρ_α , ρ_{α_0} , and ρ_β for the infinite-length limits of some simple strip graphs. See text for notation. The strips are listed in order of increasing vertex degree Δ or Δ_{eff} . Short floating-point evaluations of exact expressions are included.

$\{G\}$	Δ or Δ_{eff}	ϵ	α	α_0	β	ρ_α	ρ_{α_0}	ρ_β
$\{HL\}$	5/2	$2^{5/4} = 2.378$	$(31)^{1/4} = 2.360$	$5^{1/4} = 1.495$	$\sqrt{2} = 1.414$	$\left(\frac{31}{32}\right)^{1/4} = 0.992$	$\left(\frac{5}{32}\right)^{1/4} = 0.629$	$\left(\frac{1}{8}\right)^{1/4} = 0.595$
$\{L\}$	3	$2^{3/2} = 2.828$	$\sqrt{7} = 2.646$	$\sqrt{3} = 1.732$	2	$\sqrt{\frac{7}{8}} = 0.935$	$\sqrt{\frac{3}{8}} = 0.612$	$\frac{1}{\sqrt{2}} = 0.707$
$\{Wh\}$	4	4	3	2	3	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{3}{4}$
$\{TL\}$	4	4	3	2	$\frac{3+\sqrt{13}}{2} = 3.303$	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{3+\sqrt{13}}{8} = 0.826$
$\{sq_d\}$	5	$2^{5/2} = 5.657$	$2\sqrt{3} = 3.464$	$\sqrt{6} = 2.449$	$\sqrt{13 + \sqrt{181}} = 5.143$	$\sqrt{\frac{3}{8}} = 0.612$	$\frac{\sqrt{3}}{4} = 0.433$	$\sqrt{\frac{13+\sqrt{181}}{32}} = 0.909$

$$\frac{\alpha_0(\{Wh\})}{\alpha(\{Wh\})} = \frac{\alpha_0(\{TL\})}{\alpha(\{TL\})} = \frac{2}{3} \quad (2.52)$$

and

$$\frac{\alpha_0(\{sq_d\})}{\alpha(\{sq_d\})} = \frac{1}{\sqrt{2}} = 0.707107. \quad (2.53)$$

Concerning the ratios $\rho_{\text{EGC}}(\{G\})$, we find that $\rho_\alpha(\{G\})$ and $\rho_{\alpha_0}(\{G\})$ are monotonically decreasing functions, while $\rho_\beta(\{G\})$ is a monotonically increasing function of Δ (and, where applicable, of Δ_{eff}).

2.7. Some families of graphs without exponential growth for $a(G)$, $a_0(G)$, and/or $b(G)$

For perspective, we mention some families of graphs G for which $a(G)$, $a_0(G)$, and/or $b(G)$ do not exhibit exponential growth with n . We begin with two recursive families of graphs, namely tree graphs and circuit graphs. For n -vertex tree graphs, T_n , two of the three quantities of interest here are actually constants, independent of n . These are $a_0(T_n) = 1$, and $b(T_n) = 0$. The other quantity, $a(T_n)$ does grow exponentially with n and, indeed, is maximal, namely $a(T_n) = N_{\text{eo}}(T_n) = 2^{n-1}$. Hence, denoting $\{T\}$ as the $n \rightarrow \infty$ limit of the T_n family, we have $\alpha(\{T\}) = 2$, $\alpha_0(\{T\}) = 1$, and $\beta(\{T\}) = 0$.

For a circuit graph, $a(C_n) = 2^n - 2$, $a_0(C_n) = n - 1$, and $b(C_n) = 2$, so that although $a_0(C_n)$ grows with n , it does not grow exponentially rapidly, and $b(C_n)$ is a constant, independent of n . Hence, denoting $\{C\}$ as the $n \rightarrow \infty$ limit of the C_n family, it follows that $\alpha(\{C\}) = 2$, $\alpha_0(\{C\}) = 1$, and $\beta(\{C\}) = 1$. In contrast, for strip graphs of various lattices with widths $L_y \geq 2$, we do find that $a(G)$, $a_0(G)$, and $b(G)$ grow exponentially with n .

More generally, from a physics point of view, the property that $Z(G, q, v)$ grows exponentially rapidly with $n(G)$ as $n(G) \rightarrow \infty$ is equivalent to the property that the free energy per vertex (or per d -dimensional volume, for a d -dimensional lattice) is a constant in this limit. In turn, this reflects the extensivity of the total free energy in statistical physics. However, even in this physics context, there are examples where $Z(G, q, v)$ does not grow exponentially rapidly in the large- $n(G)$ limit. We recall that the chromatic polynomial is equal to the zero-temperature partition function of the Potts antiferromagnet, Eq. (A.6). For a bipartite graph $G_{\text{bip.}}$, such as the square or honeycomb lattices, $P(G_{\text{bip.}}, 2) = 2$, independent of $n(G_{\text{bip.}})$, while for a tripartite graph $G_{\text{trip.}}$, such as the triangular lattice, $P(G_{\text{trip.}}, 3) = 3!$, independent of $n(G_{\text{trip.}})$. In both of these cases, the chromatic polynomials evaluated at these respective values of q do not exhibit exponential growth with the number of vertices and, indeed, are independent of the number of vertices.

It can also happen that for a family of graphs G , the quantities $a(G)$, $a_0(G)$, and $b(G)$ grow more rapidly than exponentially with n . An example, which is not a recursive family, is the family of complete graphs, K_n . Recall that a complete graph K_n is defined as a graph with n vertices such that each vertex is connected to every other vertex by an edge, so that there are $e(K_n) = \binom{n}{2}$ edges. The chromatic polynomial for K_n is $P(K_n, q) = \prod_{j=0}^{n-1} (q - j)$. Therefore, from Eqs. (1.2) and (1.4), one has

$$a(K_n) = n! \quad (2.54)$$

and

$$a_0(K_n) = (n - 1)! \quad (2.55)$$

Since $n!$ has the asymptotic behavior given by $n! \sim (n/e)^n (2\pi n)^{1/2}$ for $n \gg 1$ (the Stirling formula), it follows that both $a(K_n)$ and $a_0(K_n)$ grow more rapidly than exponentially as $n \rightarrow \infty$. Having mentioned these families of graphs for contrast, we return in the next section to our main subject, namely strip graphs of various lattices.

3. Calculations of $\alpha(\{G\})$ for infinite-length, finite-width lattice strips

3.1. General

In this section we present calculations of $\alpha(\{G\})$ for strip graphs of several lattices, denoted generically as Λ in the limit of infinite length, $L_x \rightarrow \infty$, for various values of the width, L_y , and various boundary conditions. The results for $\alpha(\{G\})$ and $\alpha_0(\{G\})$ for finite L_y are independent of the boundary condition (free, periodic, or twisted periodic) in the longitudinal (x) direction, denoted BC_x , but do depend on the boundary condition in the transverse (y) direction, denoted BC_y . In detail, the relevant boundary conditions (BC_y , BC_x) and their names are as follows: (i) (F,F), free, (ii) (F,P), cyclic, (iii) (P,F), cylindrical, (iv) (P,P), toroidal. In past work we have also considered (v) (F,TP), Möbius, and (vi) (P,TP), Klein-bottle, where here the symbol T stands for twisted, but since strips with twisted longitudinal boundary conditions yield the same results relevant here as the corresponding cyclic and toroidal strips, it is not necessary to consider these twisted longitudinal boundary conditions here. For our present discussion, the infinite-length strip of a lattice Λ is indicated as $\{\Lambda, (L_y)_F \times \infty\}$ and $\{\Lambda, (L_y)_P \times \infty\}$ for these two respective transverse boundary conditions (with brackets included here for clarity). For our discussion of α and α_0 on infinite-length lattice strips, we will sometimes use an equivalent notation $\{\Lambda, (L_y) \times \infty, \text{free}\}$ and $\{\Lambda, (L_y) \times \infty, \text{cyl}\}$, where the abbreviation *cyl* stands for cylindrical.

We will make use of a property of the chromatic and Tutte polynomials for these strip graphs, namely that they can be written as a sum of certain coefficients multiplied by powers of various functions, generically denoted λ , depending on Λ , L_y , and the boundary conditions, but not on L_x . The powers to which these λ functions are raised are given by the length, m , of the strip (see Appendix). As $m \rightarrow \infty$, the λ function with the largest magnitude dominates the sum. From calculations of chromatic polynomials for strip graphs of various lattices [21–47] (see also references in [13]), we know what this dominant λ function is. We remark that for the strips that we consider, the dominant λ function in $P(G, q)$ at $q = -1$ and $q = 0$ is the same as the λ function that is dominant at large q . To calculate $\alpha(\Lambda)$ and $\alpha_0(\Lambda)$, we only need the dominant λ function for the given strip with free or periodic transverse boundary conditions, which will be denoted as $\lambda_{\Lambda, L_y, \text{free}}(q)$ or $\lambda_{\Lambda, L_y, \text{cyl}}(q)$, respectively. The reason that our results for $\alpha(\{G\})$ and $\alpha_0(\{G\})$ are independent of the longitudinal boundary conditions is that, as discussed in our earlier work, the dominant λ for these is the same for free and periodic (and twisted periodic) longitudinal boundary conditions. In contrast, our results for $\beta(\{G\})$, to be discussed in Section 6, do depend on both the longitudinal and transverse boundary conditions.

The resultant exponential growth constants for acyclic orientations and acyclic orientations with a unique source on the infinite-length limits of the square and triangular lattice strips (which have $n = L_x L_y$) are

$$\alpha(\Lambda, (L_y)_{BC_y} \times \infty) = \lim_{n \rightarrow \infty} [a(\Lambda, (L_y)_{BC_y} \times L_x)]^{1/n} = [\lambda(\Lambda, L_y, BC_y)(-1)]^{1/L_y} \quad (3.1)$$

and

$$\alpha_0(\Lambda, (L_y)_{BC_y} \times \infty) = \lim_{n \rightarrow \infty} [a_0(\Lambda, (L_y)_{BC_y} \times L_x)]^{1/n} = [\lambda(\Lambda, L_y, BC_y)(0)]^{1/L_y} \quad (3.2)$$

For strips of the honeycomb lattice, $n = 2mL_y$, so one replaces the exponents $1/L_y$ by $1/(2L_y)$ in Eqs. (3.1) and (3.2).

In our previous study [21], we showed that the resultant values of $\alpha(\Lambda, (L_y)_F \times \infty)$ and $\alpha(\Lambda, (L_y)_P \times \infty)$ were monotonically increasing functions of L_y for the full range of widths L_y for which we carried out calculations with the square-lattice and triangular-lattice strips. To anticipate the new results that we present here, we continue to find this behavior both for these lattice strips and for all of the other lattice strips that we have studied. This provides strong additional support for the inference that we made in Ref. [21], that for a given infinite-length, finite-width strip of a lattice Λ with free or periodic transverse boundary conditions,

$$\alpha(\Lambda, (L_y)_{BC_y} \times \infty) \text{ is a monotonically increasing function of } L_y. \quad (3.3)$$

Furthermore, our results in [21] provide strong additional support for the inference that

$$\lim_{L_y \rightarrow \infty} \alpha(\Lambda, (L_y)_{BC_y} \times \infty) \text{ is independent of the (F or P) } BC_y. \quad (3.4)$$

With this inference, we denote the resultant common limit for either of these transverse boundary conditions as $\alpha(\Lambda)$, where Λ refers to the infinite lattice Λ .

Because there is no transverse boundary to the strip if one uses periodic transverse (cylindrical or toroidal) boundary conditions, one expects that these boundary conditions yield values of $\alpha(\Lambda, (L_y)_P \times \infty)$ that approach the $L_y = \infty$ value, $\alpha(\Lambda)$, more rapidly than if one uses free transverse boundary conditions and calculates the resultant $\alpha(\Lambda, (L_y)_F \times \infty)$ values. Our results in [21] and here are in agreement with this expectation. Provided that this monotonicity holds for all higher values of strip width L_y , it follows that the maximal value that we obtain for $\alpha(\Lambda, (L_y)_P \times \infty)$ with the largest L_y for which we have performed the calculation is a lower bound for $\alpha(\Lambda)$. As we will discuss below, a comparison of our values of $\alpha(\text{tri}, (L_y)_P \times \infty)$ with the precise value of $\alpha(\text{tri})$ that we calculate (see Eq. (5.2)) gives further strong support to this inference. In order to measure the convergence of consecutive values of $\alpha(\Lambda, (L_y)_{BC_y} \times \infty)$ to a constant limiting value, we define the ratio

$$R_{\alpha, \Lambda, (L_y+1)/L_y, BC_y} \equiv \frac{\alpha(\Lambda, (L_y+1)_{BC_y} \times \infty)}{\alpha(\Lambda, (L_y)_{BC_y} \times \infty)}. \quad (3.5)$$

As will be evident from our results for the square, triangular, and honeycomb lattices, even for modest values of the strip widths, these ratios approach very close to unity.

3.2. Strips of the square lattice

In Table 2 we list the values of $\alpha(sq, (L_y)_F \times \infty)$ and $\alpha(sq, (L_y)_P \times \infty)$ that we have calculated. The values of $\alpha(sq, (L_y)_F \times \infty)$ for $1 \leq L_y \leq 8$ and of $\alpha(sq, (L_y)_P \times \infty)$ for $1 \leq L_y \leq 12$ were given in [21], while the value of $\alpha(sq, 13_P \times \infty)$ is new here. To show the convergence quantitatively to high accuracy, we have listed the values of $\alpha(sq, (L_y)_F \times \infty)$ and $\alpha(sq, (L_y)_P \times \infty)$ to more significant figures than were given in [21], and we have also listed values of the ratio $R_{\alpha, sq, (L_y+1)/L_y, BC_y}$. Using (3.3) and (3.4), we therefore infer the lower bound

$$\alpha(sq) > 3.4932448. \quad (3.6)$$

This new lower bound may be compared with previous lower bounds on $\alpha(sq)$. (In this context, it should be mentioned that our notation is different from the notation used in Refs. [19–24]; our quantities $a(G)$, $a_0(G)$, and $b(G)$ are the same as their $\alpha(G)$, $\alpha_0(G)$, and $\beta(G)$, respectively, and our quantities $\alpha(\{G\})$, $\alpha_0(\{G\})$, and $\beta(\{G\})$ are the same as their quantities $\lim_{n(G) \rightarrow \infty} \alpha(G)^{1/n(G)}$, $\lim_{n(G) \rightarrow \infty} \alpha_0(G)^{1/n(G)}$, and $\lim_{n(G) \rightarrow \infty} \beta(G)^{1/n(G)}$, respectively. In [19], Merino and Welsh proved that

$$\frac{22}{7} \leq \alpha(sq) \leq 3.709259278 \quad (3.7)$$

and

$$\frac{7}{3} \leq \alpha_0(sq) \leq 3.21. \quad (3.8)$$

In [20], Calkin, Merino, Noble, and Noy proved more restrictive lower and upper bounds on $\alpha(sq)$, namely

$$3.41358 \leq \alpha(sq) \leq 3.55449. \quad (3.9)$$

Subsequently, in [24], Garijo et al. obtained still more restrictive lower and upper bounds on $\alpha(sq)$, namely

$$3.42351 \leq \alpha(sq) \leq 3.5477. \quad (3.10)$$

Evidently, the new lower bound (3.6) that we have inferred is consistent with, and more restrictive than these previous lower bounds. We will also infer a more restrictive upper bound on $\alpha_0(sq)$ below.

4. Method to infer lower and upper bounds on exponential growth constants

In this section we explain a method that we use to infer lower and upper bounds on the exponential growth constants $\alpha(\Lambda)$, $\alpha_0(\Lambda)$, and $\beta(\Lambda)$ for several lattices Λ . Let us discuss $\alpha(\Lambda)$ and $\alpha_0(\Lambda)$ first. We recall Eqs. (3.1) and (3.2). For definiteness, we specialize the following discussion to strip graphs of the square lattice. Corresponding results for other lattices are similar with appropriate modifications.

As discussed above, our results are consistent with the inference (3.4) so that we may equivalently use strips with free or periodic transverse boundary conditions (as well as free or periodic longitudinal boundary conditions). For strips of the square lattice with width L_y , the values of $\alpha(sq)$ and $\alpha_0(sq)$ are thus given, respectively, by the following, where, as before, BC_y denotes the transverse boundary condition, free or periodic (i.e., cylindrical)

$$\begin{aligned} \alpha(sq) &= \lim_{n \rightarrow \infty} [P([sq, (L_y)_{BC_y} \times m], -1)]^{1/n} = \lim_{L_y \rightarrow \infty} [\lambda_{sq, L_y, BC_y}(-1)]^{1/L_y} \\ &= \lim_{L_y \rightarrow \infty} \alpha(sq, (L_y)_{BC_y} \times \infty) \end{aligned} \quad (4.1)$$

and

$$\begin{aligned} \alpha_0(sq) &= \lim_{n \rightarrow \infty} [P_r([sq, (L_y)_{BC_y} \times m], 0)]^{1/n} = \lim_{L_y \rightarrow \infty} [\lambda_{sq, L_y, BC_y}(0)]^{1/L_y} \\ &= \lim_{L_y \rightarrow \infty} \alpha_0(sq, (L_y)_{BC_y} \times \infty). \end{aligned} \quad (4.2)$$

Since $a(G)$ and $a_0(G)$ can be calculated from the chromatic polynomial $P(G, q)$ without the necessity of calculating the full two-variable Tutte polynomial or equivalent Potts model partition function, the λ functions that will be used for our analysis are those that occur in $P(G, q)$ and hence depend on the single variable q . These are evaluated at $q = -1$ for $a(G)$ and at $q = 0$ for $a_0(G)$, and we indicate this in the notation. The calculation of $b(G)$ requires an evaluation of the full Tutte polynomial, $T(G, x, y)$ with $(x, y) = (0, 2)$, or equivalently, the Potts model partition $Z(G, q, v)$ with $(q, v) = (-1, 1)$, and consequently in our discussion below of $b(G)$ and the corresponding exponential growth constant, $\beta(\{G\})$, the λ functions involved will be those for the Tutte polynomial and hence will depend on two variables. As noted above, for all of these exponential growth constants, we do not actually need the full chromatic or Tutte polynomial, but only the dominant λ function.

From our explicit calculations for the full range of L_y values that we have investigated, we have observed that all the quantities $[\lambda_{sq,L_y,free}(-1)]^{1/L_y}$, $[\lambda_{sq,L_y,free}(0)]^{1/L_y}$, $[\lambda_{sq,L_y,cyl}(-1)]^{1/L_y}$, and $[\lambda_{sq,L_y,cyl}(0)]^{1/L_y}$ increase monotonically as L_y increases. Provided that this monotonic increase continues for larger L_y , our values thus yield lower bounds on the respective asymptotic values in the limit as $L_y \rightarrow \infty$, i.e., the values of $\alpha(\Lambda)$ and $\alpha_0(\Lambda)$ for the infinite lattices Λ . In [21] we had used $[\lambda_{sq,L_y,cyl}(-1)]^{1/L_y}$ with L_y up to 12 to obtain the lower bound on $\alpha(sq)$ that we gave in that paper.

As an explicit example, we consider strips of the square lattice. For $L_y = 2$, the dominant λ is

$$\lambda_{sq,2,free} = q^2 - 3q + 3. \quad (4.3)$$

Thus, $\alpha(\{L\}) = (\lambda_{sq,2,free})^{1/L_y} = \sqrt{\lambda_{sq,2,free}}$ evaluated at $q = -1$, which yields the result in Eq. (2.3) above. The corresponding evaluation at $q = 0$ yields the value in (2.5).

For $L_y = 3$, depending on the value of q , the dominant λ functions are [29,32,33]

$$\lambda_{sq,3,free,\pm} = \frac{1}{2} \left[(q-2)(q^2 - 3q + 5) \pm \sqrt{(q^2 - 5q + 7)(q^4 - 5q^3 + 11q^2 - 12q + 8)} \right]. \quad (4.4)$$

If (real) $q \geq 2$, then the function $\lambda_{sq,3,free,+}$ is dominant (i.e., has the larger magnitude) and determines the W function defined in Eq. (A.11) as $W = (\lambda_{sq,3,free,+})^{1/3}$. In contrast, for the values of q that are relevant here, namely $q = -1$ and $q = 0$, the function $\lambda_{sq,3,free,-}$ has the larger magnitude and hence is dominant. This $\lambda_{sq,3,free,-}$ function is negative for $q < 2.685$, but, as is evident in Eqs. (1.26) and (1.27), only the magnitude is relevant for the α and α_0 exponential growth constants. (For finite-length strips, the factor $(-1)^n(G)$ in Eq. (1.2) and the factor $(-1)^{n(G)-1}$ in Eq. (1.4) yield positive values for $a(G)$ and $a_0(G)$.) To avoid magnitude signs cluttering various equations, it is understood implicitly that, where necessary, we remove minus signs so that the dominant λ function is positive for $q = -1$ and $q = 0$. In the present case of the $L_y = 3$ square-lattice strips, we thus set $|\lambda_{sq,3,free,-}| \equiv \lambda_{sq,3,free}$, and similarly for other strips. Evaluating this at $q = -1$ and $q = 0$ and taking the $1/L_y = 1/3$ root, we get

$$\alpha(sq, L_y = 3, free) = \left(\frac{27 + \sqrt{481}}{2} \right)^{1/3} = 2.903043 \quad (4.5)$$

and

$$\alpha_0(sq, L_y = 3, free) = (5 + \sqrt{14})^{1/3} = 2.0599875. \quad (4.6)$$

We observe the inequalities $\alpha(sq, 3, free) > \alpha(sq, 2, free)$, where $\alpha(sq, 2, free) \equiv \alpha(\{L\})$ in Eq. (2.3), and $\alpha_0(sq, 3, free) > \alpha_0(sq, 2, free)$, where $\alpha_0(sq, 2, free) \equiv \alpha_0(\{L\})$ in Eq. (2.5). These are in accord with the monotonicity relations that were noted above.

The property that we find in our calculations, that $\alpha(\Lambda, L_y, free)$ and $\alpha_0(\Lambda, L_y, free)$ are monotonically increasing functions of strip width for a strip of the lattice Λ , is opposite to the behavior that was found for $W(\Lambda, L_y, free, q)$ for values of q used in proper q -colorings of the lattice Λ [35]. This reversal can be traced to the evaluation at different values of q , namely $q = -1$ and $q = 0$ here, as contrasted with values of q used for proper q colorings of Λ .

Related to this, we have noticed an interesting connection between our results for $\alpha(\Lambda)$ and $\alpha_0(\Lambda)$ and the analytic expressions that were proved in [27] (see also [48,49]) to be lower bounds on $W(\Lambda, q)$ for all Archimedean lattices (and dual Archimedean lattices) using a coloring-matrix method introduced in [26] to prove a lower bound on $W(sq, q)$. We find that if one evaluates these expressions at $q = -1$ and $q = 0$, then the results are consistent with being upper bounds on $\alpha(\Lambda)$ and $\alpha_0(\Lambda)$, respectively. As discussed in the Appendix, an Archimedean lattice Λ has the form $\Lambda = (\prod_i p_i^{a_i})$, where this product refers to the ordered sequence of polygons traversed in a circuit around any vertex, and the i 'th polygon has p_i sides, appearing a_i times contiguously in the sequence (it can also occur non-contiguously). The total number of occurrences of the polygon p_i in the above sequence is denoted as $a_{i,s}$. In this general notation, $(sq) = (4^4)$, $(tri) = (3^6)$, and $(hc) = (6^3)$. The number of polygons of type p_i per vertex is

$$\nu_{p_i} = \frac{a_{i,s}}{p_i}. \quad (4.7)$$

Note that ν_{p_i} coincides with ν_Λ in Eq. (1.18) and takes on the values in Eqs. (1.20)–(1.21) for the square, triangular, and honeycomb lattices. The lower bound proved in [27] for a general Archimedean lattice is

$$W(\Lambda, q) \geq W(\Lambda, q)_\ell, \quad (4.8)$$

where

$$W\left(\left(\prod_i p_i^{a_i}\right)_\ell, q\right) = \frac{\prod_i [D_{p_i}(q)]^{\nu_{p_i}}}{q-1}, \quad (4.9)$$

with

$$D_n(q) = \sum_{s=0}^{n-2} (-1)^s \binom{n-1}{s} q^{n-2-s}. \quad (4.10)$$

This lower bound applies for $q \geq \chi(G)$, where $\chi(G)$ is the chromatic number of G (i.e., the minimum value of q required for a proper q -coloring of G). In [27,48,49] this lower bound on $W(\Lambda, q)$ was shown to be very close to the actual values of $W(\Lambda, q)$ as determined from Monte-Carlo measurements, large- q series, and the result for $W(tri, q)$ from [50]. We conjecture that for each Archimedean lattice Λ , the evaluation of the right-hand side of Eq. (4.9) at the respective values $q = -1$ and $q = 0$ yields respective upper bounds on $\alpha(\Lambda)$ and $\alpha_0(\Lambda)$, which would read

$$\alpha(\Lambda) < \alpha_{u,w}(\Lambda) \quad (4.11)$$

and

$$\alpha_0(\Lambda) < \alpha_{0,u,w}(\Lambda), \quad (4.12)$$

where

$$\alpha_{u,w}(\Lambda) = \frac{\prod_i |D_{p_i}(-1)|^{v_{p_i}}}{2} \quad (4.13)$$

and

$$\alpha_{0,u,w}(\Lambda) = \prod_i |D_{p_i}(0)|^{v_{p_i}}. \quad (4.14)$$

To indicate the connection with the $W(\Lambda, q)$ bounds, we use a subscript w in these expressions. Specifically, for the three Archimedean lattices under consideration here, in order of increasing vertex degree $\Lambda(\Lambda)$, these conjectured upper bounds are

$$\alpha_{u,w}(hc) = \frac{\sqrt{31}}{2} = 2.78388218 \quad (4.15)$$

$$\alpha_{u,w}(sq) = \frac{7}{2} \quad (4.16)$$

$$\alpha_{u,w}(tri) = \frac{9}{2} \quad (4.17)$$

and

$$\alpha_{0,u,w}(hc) = \sqrt{5} = 2.236068 \quad (4.18)$$

$$\alpha_{0,u,w}(sq) = 3 \quad (4.19)$$

and

$$\alpha_{0,u,w}(tri) = 4. \quad (4.20)$$

As will be seen below, these are close to the upper bounds that we derive from our studies of strip graphs of these lattices, and to the exact values that we obtain for $\alpha(tri)$ and $\alpha_0(tri)$. (The respective values in Eqs. (4.15)–(4.20) are the $(L_y + 1)/L_y = 2/1$ entries in the corresponding tables based on strip graph calculations.) A plausible inference is that this closeness of the values (4.15)–(4.20) to the optimal inferred upper bounds is related to the fact that the lower bound (4.8)–(4.10) that was rigorously proved in [27] for all Archimedean lattices is very close to the actual values of $W(\Lambda, q)$ on these lattices.

From [39], the dominant λ function in $Z(G, q, v)$ for the relevant values $q = -1$ and $v = 1$ is

$$\lambda_{sq,2,free}(q, v) = \frac{v}{2} \left[q + v(v+4) + \sqrt{v^4 + 4v^3 + 12v^2 - 2qv^2 + 4qv + q^2} \right] \quad (4.21)$$

with the evaluation

$$\lambda_{sq,2,free} = 4 \quad \text{at } (q, v) = (-1, 1). \quad (4.22)$$

Taking the $1/(L_y) = 1/2$ root, one obtains the result for $\beta(sq, 2, free) \equiv \beta(\{L\})$ given above in Eq. (2.8).

From [44] we find that for the $L_y = 3$ strip of the square lattice with free transverse boundary conditions, the dominant λ at $(q, v) = (-1, 1)$ is a root of the sixth-degree equation given in Eqs. (A.1)–(A.7) of [44], with the value

$$\lambda_{sq,3,free} = \frac{17 + \sqrt{145}}{2} \quad \text{at } (q, v) = (-1, 1). \quad (4.23)$$

Taking the $1/3$ root, one obtains the result

$$\beta(sq, 3, free) = \left(\frac{17 + \sqrt{145}}{2} \right)^{1/3} = 2.439665. \quad (4.24)$$

We note the inequality $\beta(sq, 3, free) > \beta(sq, 2, free)$, in agreement with the general monotonicity property noted above.

We next prove a useful inequality. For this purpose, we begin by considering lattice strip graphs with width $L_y = 2^p$ for some (positive) integer power p . This inequality applies to the exponential growth constant ϕ for the Tutte polynomial of a recursive family of graphs (e.g., lattice strip graphs) for $x \geq 0$ and $y \geq 0$, where ϕ is defined as

$$\phi(\{G\}, x, y) = \lim_{n(G) \rightarrow \infty} [T(G, x, y)]^{1/n(G)}. \quad (4.25)$$

If an edge $e \in E$ is not a loop or a bridge, then the Tutte polynomial satisfies the deletion–contraction relation

$$T(G, x, y) = T(G - e, x, y) + T(G/e, x, y), \quad (4.26)$$

where $G - e$ denotes G with the edge e deleted and G/e denotes the result of deleting the edge e from G and identifying the vertices that this edge connected. (For a graph that is comprised of ℓ loops and b bridges, $T(G, x, y) = x^\ell y^b$. The proof of the inequality follows from an iterative use of the deletion–contraction relation. This leads to nested inequalities for the dominant λ function for all of the three cases $(x, y) = (2, 0)$ for $\alpha(\{G\})$, $(x, y) = (1, 0)$ for $\alpha_0(\{G\})$, and $(x, y) = (0, 2)$ for $\beta(\{G\})$. The basic observation is that if one compares the Tutte polynomial for, say, an $L_x \times 4$ strip of a lattice Λ , with the Tutte polynomial for a (disconnected) graph consisting of two copies of an $L_x \times 2$ strip, then the former has L_x more edges, the removal of which yields the latter two graphs. By iterative application of the deletion–contraction theorem, one can relate the free strip of width $L_y = 4$ to the graph consisting of two $L_y = 2$ free strips, and the inequality then follows.

From Eqs. (3.1) and (3.2), it follows that for the square and triangular lattices, as the strip width $L_y \rightarrow \infty$, $\lambda(\Lambda, (L_y)_{BC_y} \times \infty)(-1) \sim [\alpha(\Lambda)]^{L_y}$ and $\lambda(\Lambda, (L_y)_{BC_y} \times \infty)(0) \sim [\alpha_0(\Lambda)]^{L_y}$. Therefore, another measure of the asymptotic large- L_y limit is given by the ratio

$$\frac{\lambda(\Lambda, (L_y)_{BC_y} \times \infty)(q)}{\lambda(\Lambda, (L_y - 1)_{BC_y} \times \infty)(q)}, \quad (4.27)$$

where $q = -1$ for $\alpha(\Lambda)$ and where $q = 0$ for $\alpha_0(\Lambda)$. Illustrating this with our illustrative strip, we observe that the ratios of the dominant $\lambda(q)$ functions in the chromatic polynomial at $q = -1$ and $q = 0$ are

$$\frac{\lambda(sq, 3_F \times \infty)(-1)}{\lambda(sq, 2_F \times \infty)(-1)} = \frac{27 + \sqrt{481}}{14} = 3.495122 \quad (4.28)$$

and

$$\frac{\lambda(sq, 3_F \times \infty)(0)}{\lambda(sq, 2_F \times \infty)(0)} = \frac{5 + \sqrt{14}}{3} = 2.913886. \quad (4.29)$$

(The equivalent notation $\frac{\lambda(sq, 3, free)(-1)}{\lambda(sq, 2, free)(-1)}$ and $\frac{\lambda(sq, 3, free)(0)}{\lambda(sq, 2, free)(0)}$ is used in the tables.)

A corresponding discussion applies for $\beta(\Lambda)$, with the dominant $\lambda(q)$ function in the chromatic polynomial replaced with the dominant $\lambda(q, v)$ function in the Potts/Tutte polynomial, evaluated at $(x, y) = (0, 2)$ i.e., $(q, v) = (-1, 1)$. For the honeycomb lattice, one replaces these ratios by square roots. Thus, for the dominant $\lambda(q, v)$ functions in $Z(G, q, v)$ at $(q, v) = (-1, 1)$ on the $L_y = 2$ and $L_y = 3$ strips of the square lattice with free transverse boundary conditions, we have

$$\frac{\lambda(sq, 3_F \times \infty)(-1, 1)}{\lambda(sq, 2_F \times \infty)(-1, 1)} = \frac{17 + \sqrt{145}}{8} = 3.630199. \quad (4.30)$$

(The equivalent notation $\frac{\lambda(sq, 3, free)(-1, 1)}{\lambda(sq, 2, free)(-1, 1)}$ is used in the tables.)

Now consider the special case of the exponential growth constant $\alpha(\{G\})$, for which we actually only need the one-variable special case of the Tutte polynomial given by the chromatic polynomial, with the associated λ functions evaluated at $q = -1$ for the discussion of acyclic orientations, as will be indicated in the notation below. We then have the sequence of inequalities

$$\begin{aligned} \lambda_{sq, 1, free}(-1) &\leq [\lambda_{sq, 2, free}(-1)]^{1/2} \leq [\lambda_{sq, 4, free}(-1)]^{1/4} \leq [\lambda_{sq, 8, free}(-1)]^{1/8} \\ &\leq \dots \leq \lim_{L_y \rightarrow \infty} [\lambda_{sq, L_y, free}(-1)]^{1/L_y}. \end{aligned} \quad (4.31)$$

Let us focus on one of these inequalities, namely $[\lambda_{sq, 2, free}(-1)]^{1/2} \leq [\lambda_{sq, 4, free}(-1)]^{1/4}$. The others can be treated in a similar manner. Since $[\lambda_{sq, 2, free}(-1)]^{L_x}$ is the dominant λ function for the chromatic polynomial of the $2 \times L_x$ strip, and equivalently of the Tutte polynomial with $(x, y) = (2, 0)$, it determines the corresponding ϕ function in the limit $L_x \rightarrow \infty$, while $[\lambda_{sq, 4, free}(-1)]^{L_x}$ similarly gives the ϕ function for the $L_x \rightarrow \infty$ limit of the Tutte polynomial of the $4 \times L_x$ strip. Now compare two $L_y = 2$ strips with a $L_y = 4$ strip. The former has L_x fewer edges than the latter, so the Tutte polynomial of the former is smaller than that of the latter, since the coefficients of the Tutte polynomial (in terms of variables x and y) are positive. That is, $[\lambda_{sq, 2, free}(-1)]^{2L_x} \leq [\lambda_{sq, 4, free}(-1)]^{L_x}$. This completes the proof. By the same type of argument, it follows, for example, that

$$\lambda_{sq, 1, free}(-1) \leq [\lambda_{sq, 3, free}(-1)]^{1/3} \leq [\lambda_{sq, 6, free}(-1)]^{1/6} \leq [\lambda_{sq, 12, free}(-1)]^{1/12}$$

$$\leq \dots \leq \lim_{L_y \rightarrow \infty} [\lambda_{sq, L_y, free}(-1)]^{1/L_y}, \quad (4.32)$$

where here $L_y = 3 \cdot 2^s$, where s is a non-negative integer. Other corresponding inequalities with larger values of L_y follow in the same way.

It is easy to prove that

$$\lambda_{sq, L_y, free}(-1) \leq \lambda_{sq, L_y, cyl}(-1), \quad (4.33)$$

as follows. Consider an assignment of arrows on all of the edges of a free strip of the square lattice with width L_y , such that there are no cycles, i.e., an acyclic orientation of this strip. We can add L_x more edges to produce the corresponding cylindrical strip of the square lattice with the same width L_y . Now we assign an orientation for the arrow on each of these directed edges. If a choice of the direction of the arrow would result in a cycle, then we choose the opposite direction for the arrow. It is impossible that both choices will result a cycle, since that would mean that there would already have been a cycle in the original free strip, which would contradict the beginning assumption of an acyclic orientation. This statement applies to each of the additional L_x edges of the cylindrical strips, so the number of acyclic orientations on the cylindrical strip is at least the same as the number on the free strip. Therefore, for a strip graph of the lattice Λ , the quantity $[\lambda_{sq, L_y, cyl}(-1)]^{1/L_y}$ serves as a better lower bound on $\alpha(\Lambda)$ than $[\lambda_{sq, L_y, free}(-1)]^{1/L_y}$. Alternatively, one can again use the deletion-contraction relation (4.26) to prove (4.33), since the strip with cylindrical boundary conditions has L_x more edge than the strip with free boundary conditions with the same L_y . Similar discussions apply for acyclic orientations with a unique source, and for the *tri* and *sq_d* lattices. For the honeycomb lattice, it is $[\lambda_{hc, L_y, free}(-1)]^{1/(2L_y)}$ that yields the corresponding values of α and α_0 with $q = -1$ and $q = 0$, respectively. We thus infer the two inequalities

$$\alpha(\Lambda) > \alpha(\Lambda, (L_y)_p \times \infty) \quad \text{for the maximal calculated value of } L_y \quad (4.34)$$

and

$$\alpha_0(\Lambda) > \alpha_0(\Lambda, (L_y)_p \times \infty) \quad \text{for the maximal calculated value of } L_y, \quad (4.35)$$

where the right-hand sides of these inequalities are given, respectively, by $[\lambda_{\Lambda, L_y, cyl}(-1)]^{1/L_y}$ and $[\lambda_{\Lambda, L_y, cyl}(0)]^{1/L_y}$ for $\Lambda = sq, tri, sq_d$, and by the corresponding square roots of these functions for $\Lambda = hc$. The corresponding bounds also apply for free transverse boundary conditions, but, as noted, the bounds with periodic transverse boundary conditions are more restrictive.

As mentioned above, we have shown by explicit calculation that

$$\lambda_{sq, 1, free}(-1) < [\lambda_{sq, 2, free}(-1)]^{1/2} < [\lambda_{sq, 3, free}(-1)]^{1/3} < \dots < [\lambda_{sq, 8, free}(-1)]^{1/8} \quad (4.36)$$

for the square lattice. This sequence should approach $\alpha(sq)$ as the strip width $L_y \rightarrow \infty$. With the inference that

$$[\lambda_{sq, L_y, free}(-1)]^{1/L_y} < [\lambda_{sq, L_y+1, free}(-1)]^{1/(L_y+1)}, \quad (4.37)$$

this is equivalent to

$$\lambda_{sq, L_y, free}(-1) < [\lambda_{sq, L_y+1, free}(-1)]^{L_y/(L_y+1)} \quad (4.38)$$

and

$$[\lambda_{sq, L_y+1, free}(-1)]^{1/(L_y+1)} < \frac{\lambda_{sq, L_y+1, free}(-1)}{\lambda_{sq, L_y, free}(-1)}. \quad (4.39)$$

From our explicit calculation, we find that

$$\frac{\lambda_{sq, 8, free}(-1)}{\lambda_{sq, 7, free}(-1)} < \frac{\lambda_{sq, 7, free}(-1)}{\lambda_{sq, 6, free}(-1)} < \dots < \frac{\lambda_{sq, 3, free}(-1)}{\lambda_{sq, 2, free}(-1)} < \frac{\lambda_{sq, 2, free}(-1)}{\lambda_{sq, 1, free}(-1)}. \quad (4.40)$$

This leads us to infer that the ratio $\lambda_{sq, L_y+1, free}(-1)/\lambda_{sq, L_y, free}(-1)$ serves as an upper bound for $\alpha(sq)$. From Eq. (3.1) and the proof above that $[\lambda_{sq, L_y, free}(-1)]^{1/L_y}$ approaches $\alpha(sq)$ from below (and is very close to it when $L_y \gg 1$), one could infer that $\lambda_{sq, L_y+1, free}(-1)$ is close to $[\alpha(sq)]^{L_y+1}$ and $\lambda_{sq, L_y, free}(-1)$ is also close (but not as close) to $[\alpha(sq)]^{L_y}$. Therefore, $\frac{\lambda_{sq, L_y+1, free}(-1)}{\lambda_{sq, L_y, free}(-1)}$ should be slightly larger than $\alpha(sq)$, and hence should serve as an upper bound on $\alpha(sq)$. Similar discussions apply for the evaluation at $q = 0$ and for the *tri* and *sq_d* lattices. For the honeycomb lattice, one replaces this ratio by its square root to obtain the upper bound on $\alpha(hc)$. We thus infer the two inequalities

$$\alpha(\Lambda) < \frac{\lambda_{\Lambda, L_y+1, free}(-1)}{\lambda_{\Lambda, L_y, free}(-1)} \quad \text{for the maximal calculated value of } L_y \quad (4.41)$$

and

$$\alpha_0(\Lambda) < \frac{\lambda_{\Lambda, L_y+1, free}(0)}{\lambda_{\Lambda, L_y, free}(0)} \quad \text{for the maximal calculated value of } L_y, \quad (4.42)$$

for $\Lambda = sq, tri, sq_d$ lattices. We infer the corresponding inequalities for the honeycomb lattice with the ratios on the right-hand sides replaced by their square roots.

Another argument that supports this inference is the following, where we again specialize to the square-lattice strips for definiteness. Let us define the ratio of the adjacent terms in (4.36) (i.e., successive lower bounds on $\alpha(sq)$ from the infinite-length strip of width L_y and width $L_y - 1$) as

$$R_{sq, \frac{L_y}{L_y-1}, free}(-1) \equiv \frac{[\lambda_{sq, L_y, free}(-1)]^{1/L_y}}{[\lambda_{sq, L_y-1, free}(-1)]^{1/(L_y-1)}}. \quad (4.43)$$

This ratio $R_{sq, (L_y+1)/L_y, free}(-1)$ is larger than 1. We find that this ratio decreases toward 1 from above as L_y increases from 1 to 7, as listed in the next section. The same statement applies when the boundary condition is cylindrical. This is consistent with the inference that our lower bound is approaching an asymptotic constant value, namely the value for the infinite lattice.

Provided that this property continues to hold for any L_y , namely,

$$\frac{[\lambda_{sq, L_y+1, free}(-1)]^{1/(L_y+1)}}{[\lambda_{sq, L_y, free}(-1)]^{1/L_y}} < \frac{[\lambda_{sq, L_y, free}(-1)]^{1/L_y}}{[\lambda_{sq, L_y-1, free}(-1)]^{1/(L_y-1)}}, \quad (4.44)$$

then it is equivalent to

$$\begin{aligned} [\lambda_{sq, L_y+1, free}(-1)]^{1/(L_y+1)} &< \frac{[\lambda_{sq, L_y, free}(-1)]^{2/L_y}}{[\lambda_{sq, L_y-1, free}(-1)]^{1/(L_y-1)}} \\ &= \frac{\lambda_{sq, L_y, free}(-1)}{\lambda_{sq, L_y-1, free}(-1)} \times \frac{[\lambda_{sq, L_y-1, free}(-1)]^{(L_y-2)/(L_y-1)}}{[\lambda_{sq, L_y, free}(-1)]^{(L_y-2)/L_y}} \\ &< \frac{\lambda_{sq, L_y, free}(-1)}{\lambda_{sq, L_y-1, free}(-1)}. \end{aligned} \quad (4.45)$$

Let us define the ratio

$$R_{sq, \frac{L_y^2}{(L_y-1)(L_y+1)}, free}(-1) \equiv \frac{[\lambda_{sq, L_y, free}(-1)]^2}{\lambda_{sq, L_y-1, free}(-1)\lambda_{sq, L_y+1, free}(-1)}. \quad (4.46)$$

This is the ratio of adjacent upper bounds. Since the upper bounds decrease as the strip width L_y increases, the larger- L_y upper bound in the denominator is smaller than the smaller- L_y upper bound in the numerator, so this ratio is also larger than unity. We find that this ratio also decreases as L_y increases from 2 to 7, as listed in the next section. This is consistent with our upper bounds approaching a constant value as $L_y \rightarrow \infty$, namely the value of $\alpha(sq)$ for the infinite lattice.

Next, we consider the totally cyclic orientations on strip graphs and the exponential growth constant $\beta(\Lambda)$ of the lattice Λ . As stated above in Eqs. (1.7) and (1.8), for a finite graph, G , the number of totally cyclic orientations, $b(G)$, is given by the evaluations $b(G) = T(G, 0, 2)$, or equivalently, by $b(G) = -Z(G, -1, 1)$. From our earlier calculations of Potts model partition functions for strip graphs of various lattices with cyclic and toroidal boundary conditions, we showed that the dominant λ function in the Potts partition function evaluated at $(q, v) = (-1, 1)$ has the coefficient $q - 1$. Related to this, the result for β in the limit of infinite length depends on both the longitudinal and transverse boundary conditions of the strip. To minimize finite-size effects, we therefore restrict to strips with periodic longitudinal boundary conditions in our analysis of the β exponential growth constant. The dominant λ functions of this type will be denoted as $\lambda_{sq, L_y, cyc}(q = -1, v = 1) \equiv \lambda_{sq, L_y, cyc}(-1, 1)$ and $\lambda_{sq, L_y, tor}(q = -1, v = 1) \equiv \lambda_{sq, L_y, tor}(-1, 1)$, respectively. We have obtained inequalities similar to those discussed above in this case also. That is, $[\lambda_{sq, L_y, cyc}(-1, 1)]^{1/L_y}$ and $[\lambda_{sq, L_y, tor}(-1, 1)]^{1/L_y}$ increase monotonically as L_y increases, and

$$\lambda_{sq, L_y, cyc}(-1, 1) \leq \lambda_{sq, L_y, tor}(-1, 1). \quad (4.47)$$

We find similar results for other lattices. This leads us to infer that $[\lambda_{\Lambda, L_y, tor}(-1, 1)]^{1/L_y}$ is a lower bound for $\beta(\Lambda)$ for $\Lambda = sq, tri, sq_d$:

$$\beta(\Lambda) > [\lambda_{\Lambda, L_y, tor}(-1, 1)]^{1/L_y} \quad \text{for the maximal calculated value of } L_y. \quad (4.48)$$

for these lattices. For the honeycomb lattice, we infer that the corresponding lower bound holds with the right-hand side replaced by its square root, i.e., with the power $1/(2L_y)$ rather than $1/L_y$. We define the ratio

$$R_{sq, \frac{L_y}{L_y-1}, BC_y}(-1, 1) \equiv \frac{[\lambda_{sq, L_y, BC_y}(-1, 1)]^{1/L_y}}{[\lambda_{sq, L_y-1, BC_y}(-1, 1)]^{1/(L_y-1)}} \quad (4.49)$$

for the adjacent lower bounds, where BC_y can be either cyclic or toroidal boundary conditions. For the honeycomb lattice, L_y can only be an even number for the strips with cylindrical or toroidal boundary conditions, and the ratio analogous to (4.49) is defined by the results for strips with width L_y and $L_y - 2$ rather than L_y and $L_y - 1$.

Table 2

Values of $\alpha(\{G\})$ for the infinite-length limits of strip graphs of the square lattice with width L_y vertices and free (F) or periodic (P) transverse boundary conditions, BC_y . Here, as discussed in the text, $\alpha(sq, (L_y)_{BC_y} \times \infty) = [\lambda_{sq, (L_y)_{BC_y}}(-1)]^{1/L_y}$, and these values are inferred to be lower bounds on $\alpha(sq)$, with the values for periodic BC_y and the maximal L_y being the most restrictive. Here and in subsequent tables, a blank entry means that the evaluation is not applicable.

BC_y	L_y	$\alpha(sq, (L_y)_{BC_y} \times \infty)$	$R_{sq, BC_y, \frac{L_y}{L_y-1}}$
F	1	2	
F	2	$\sqrt{7} = 2.64575131$	1.32287566
F	3	$\left(\frac{27+\sqrt{481}}{2}\right)^{1/3} = 2.90304302$	1.09724713
F	4	3.04073149	1.04742901
F	5	3.12642125	1.02818064
F	6	3.18487566	1.01869691
F	7	3.22729404	1.01331934
F	8	3.25947731	1.00997215
P	3	$(34)^{1/3} = 3.2396118$	1.22445817
P	4	$\left(\frac{139+\sqrt{16009}}{2}\right)^{1/4} = 3.39445098$	1.04779560
P	5	$\left(\frac{527+\sqrt{200585}}{2}\right)^{1/5} = 3.44812570$	1.01581249
P	6	3.47054571	1.00650209
P	7	3.48113984	1.00305258
P	8	3.48658682	1.00156471
P	9	3.48956089	1.00085301
P	10	3.49125850	1.00048648
P	11	3.49226085	1.00028710
P	12	3.49286857	1.00017402
P	13	3.493244875	1.000107736

Table 3

Upper bounds and their ratios for $\alpha(sq)$ as functions of strip width L_y .

$\frac{L_y+1}{L_y}$	$\frac{\lambda_{sq, L_y+1, free}(-1)}{\lambda_{sq, L_y, free}(-1)}$	$R_{sq, \frac{L_y^2}{(L_y-1)(L_y+1)}, free}(-1)$
2/1	3.5	
3/2	$\frac{27+\sqrt{481}}{14} = 3.49512230$	1.00139557
4/3	3.49423306	1.00025449
5/4	3.49401836	1.00006145
6/5	3.49395589	1.00001788
7/6	3.49393533	1.00000588
8/7	3.493927961	1.000002100

We also find that the ratio

$$R_{sq, \frac{L_y^2}{(L_y-1)(L_y+1)}, cyc}(-1, 1) \equiv \frac{[\lambda_{sq, L_y, cyc}(-1, 1)]^2}{\lambda_{sq, L_y-1, cyc}(-1, 1)\lambda_{sq, L_y+1, cyc}(-1, 1)} \quad (4.50)$$

decreases when L_y increases and therefore infer that $\lambda_{sq, L_y+1, cyc}(-1, 1)/\lambda_{sq, L_y, cyc}(-1, 1)$ provides an upper bound on $\beta(sq)$. We find similar behavior for other lattices and thus infer the upper bound

$$\beta(\Lambda) < \frac{\lambda_{\Lambda, L_y+1, cyc}(-1, 1)}{\lambda_{\Lambda, L_y, cyc}(-1, 1)} \quad \text{for the maximal calculated value of } L_y. \quad (4.51)$$

for $\Lambda = sq, tri, sq_d$. For the honeycomb lattice, $\Lambda = hc$, we infer the corresponding inequality with the ratio on the right-hand side replaced by its square root.

5. Numerical values of lower and upper bounds for $\alpha(\Lambda)$ and $\alpha_0(\Lambda)$

In this section we present our results for numerical values of lower and upper bounds for $\alpha(\Lambda)$ and $\alpha_0(\Lambda)$ on various two-dimensional lattices Λ . For a given lattice Λ , we denote our lower (ℓ) and upper (u) bounds with respective subscripts ℓ and u as $\alpha_\ell(\Lambda)$, $\alpha_u(\Lambda)$, $\alpha_{0,\ell}(\Lambda)$, and $\alpha_{0,u}(\Lambda)$. Since we use the entries with the highest values of strip width L_y for our lower and upper bounds, we quote these to slightly higher precision than the smaller- L_y entries. As noted above, we obtain our best lower and upper bounds from the strips with periodic transverse boundary conditions. To begin, we present these results for the square lattice in [Tables 2–5](#).

Table 4

Lower bounds and their ratios for $\alpha_0(sq)$ as functions of strip width L_y . In this table and the others, the abbreviation cyl stands for "cylindrical".

BC	L_y	$[\lambda_{sq,L_y,free/cyl}(0)]^{1/L_y}$	$R_{sq, \frac{L_y}{L_y-1}, free/cyl}(0)$
free	1	1	
free	2	$\sqrt{3} = 1.73205081$	1.73205081
free	3	$(5 + \sqrt{14})^{1/3} = 2.05998754$	1.18933436
free	4	2.24131157	1.08802190
free	5	2.35572295	1.05104662
free	6	2.43432494	1.03336640
free	7	2.49159809	1.02352733
free	8	2.53516365	1.01748499
cyl	3	$(13)^{1/3} = 2.35133469$	1.35754371
cyl	4	$(23 + 2\sqrt{111})^{1/4} = 2.57655243$	1.09578294
cyl	5	$(74 + 11\sqrt{34})^{1/5} = 2.67956432$	1.03998052
cyl	6	2.73462860	1.02054971
cyl	7	2.76735961	1.01196909
cyl	8	2.78834612	1.00758359
cyl	9	2.80258484	1.00510651
cyl	10	2.81267772	1.00360127
cyl	11	2.82008605	1.00263390
cyl	12	2.82568101	1.00198397
cyl	13	2.830007783	1.001531233

Table 5

Upper bounds and their ratios for $\alpha_0(sq)$ as functions of strip width L_y .

$\frac{L_y+1}{L_y}$	$\frac{\lambda_{sq,L_y+1,free}(0)}{\lambda_{sq,L_y,free}(0)}$	$R_{sq, \frac{L_y}{L_y-1}, free}(0)$
2/1	3	
3/2	$\frac{5+\sqrt{14}}{3} = 2.91388580$	1.02955305
4/3	2.88678970	1.00938624
5/4	2.87482980	1.00416021
6/5	2.86846939	1.00221735
7/6	2.86467029	1.00132619
8/7	2.862213752	1.000858265

Table 6

Lower bounds on $\alpha(tri)$ and their ratios relative to the exact value (5.2), as functions of strip width L_y .

BC	L_y	$[\lambda_{tri,L_y,free/cyl}(-1)]^{1/L_y}$	$\frac{[\lambda_{tri,L_y,free/cyl}(-1)]^{1/L_y}}{\alpha(tri)}$	$R_{tri, \frac{L_y}{L_y-1}, free/cyl}(-1)$
free	2	3	0.67044390	
free	3	$\left(\frac{43+\sqrt{1417}}{2}\right)^{1/3} = 3.429090932116...$	0.766337701111...	1.143030310705...
free	4	3.66535037	0.81913727	1.06889856
free	5	3.81466660	0.85250665	1.04073723
free	6	3.91752078	0.87549264	1.02696282
free	7	3.99266294	0.89228551	1.01918105
free	8	4.04995674	0.90508960	1.01434977
free	9	4.09508340	0.91517457	1.01114251
cyl	2	$2\sqrt{3} = 3.46410162$	0.77416193	
cyl	3	$(71)^{1/3} = 4.14081775$	0.92539534	1.19535112
cyl	4	$(2^5 \times 11)^{1/4} = 4.33147354$	0.96800334	1.04604303
cyl	5	$[3(299 + 113\sqrt{5})]^{1/5} = 4.40312504$	0.98401611	1.01654206
cyl	6	4.43528747	0.99120381	1.00730445
cyl	7	4.45150713	0.99482860	1.00365696
cyl	8	4.46037926	0.99681136	1.00199306
cyl	9	4.46552972	0.99796239	1.00115471
cyl	10	4.46865768	0.99866143	1.00070047
cyl	11	4.47062537	0.99910117	1.00044033
cyl	12	4.471898356	0.999385660	1.0002847452

Next, for the triangular lattice, we can use Eqs. (1.26) and (1.27) together with exact expressions for the W function on the triangular lattice from [50] to obtain precise values of $\alpha(tri)$ and $\alpha_0(tri)$. As discussed below, by duality, one thus obtains a precise value of $\beta(hc)$. These results provide a quantitative measure of how close our lower and upper bounds

Table 7

Upper bounds on $\alpha(tri)$, their ratios relative to the exact value (5.2), and ratios of adjacent upper bounds, as functions of strip width L_y .

$\frac{L_y+1}{L_y}$	$\frac{\lambda_{tri,L_y+1,free}(-1)}{\lambda_{tri,L_y,free}(-1)}$	$\frac{\lambda_{tri,L_y+1,free}(-1)/\lambda_{tri,L_y,free}(-1)}{\alpha(tri)}$	$R_{tri, \frac{L_y^2}{(L_y-1)(L_y+1)}, free}(-1)$
2/1	4.5	1.00566585	
3/2	$\frac{43+\sqrt{1417}}{18} = 4.48017002$	1.00123422	1.00442617
4/3	4.47635966	1.00038268	1.00085122
5/4	4.47528766	1.00014311	1.00023954
6/5	4.47491635	1.000060125	1.00008298
7/6	4.47476968	1.00002735	1.00003278
8/7	4.47470626	1.000013175	1.00001417
9/8	4.474676977	1.000006630	1.0000065451

Table 8

Lower bounds on $\alpha_0(tri)$ and their ratios relative to the exact value (5.3), and ratios of adjacent bounds, as functions of strip width L_y .

BC	L_y	$[\lambda_{tri,L_y,free/cyl}(0)]^{1/L_y}$	$\frac{[\lambda_{tri,L_y,free/cyl}(0)]^{1/L_y}}{\alpha_0(tri)}$	$R_{tri, \frac{L_y}{L_y-1}, free/cyl}(0)$
free	2	2	0.53037459	
free	3	$\left(\frac{17+\sqrt{193}}{2}\right)^{1/3} = 2.49042857$	0.66043002	1.24521429
free	4	2.77154840	0.73497943	1.11288010
free	5	2.95242249	0.78294494	1.06526102
free	6	3.07822224	0.81630543	1.04260899
free	7	3.17066700	0.84082061	1.03003187
free	8	3.24142502	0.85958474	1.02231645
free	9	3.29730594	0.87440365	1.01723962
cyl	2	$\sqrt{6} = 2.44948974$	0.64957356	
cyl	3	$2^{5/3} = 3.17480210$	0.84191719	1.29610753
cyl	4	$[6(12 + \sqrt{129})]^{1/4} = 3.440692605$	0.91242797	1.08375026
cyl	5	$(307 + 29\sqrt{85})^{1/5} = 3.56318084$	0.94491029	1.03559988
cyl	6	3.628852235	0.96232551	1.01843055
cyl	7	3.667909685	0.97268305	1.01076303
cyl	8	3.69293928	0.97932058	1.00682394
cyl	9	3.70990510	0.98381970	1.00459412
cyl	10	3.72191820	0.98700543	1.003238115
cyl	11	3.73072654	0.98934128	1.00236661
cyl	12	3.737371971	0.991103569	1.001781271

Table 9

Upper bounds on $\alpha_0(tri)$, their ratios relative to the exact value (5.3), and ratios of adjacent bounds, as functions of strip width L_y .

$\frac{L_y+1}{L_y}$	$\frac{\lambda_{tri,L_y+1,free}(0)}{\lambda_{tri,L_y,free}(0)}$	$\frac{\lambda_{tri,L_y+1,free}(0)/\lambda_{tri,L_y,free}(0)}{\alpha_0(tri)}$	$R_{tri, \frac{L_y^2}{(L_y-1)(L_y+1)}, free}(0)$
2/1	4	1.06074919	
3/2	$\frac{17+\sqrt{193}}{8} = 3.86155550$	1.02403546	1.03585200
4/3	3.82003723	1.01302535	1.01086855
5/4	3.80191720	1.00822014	1.00476603
6/5	3.79234033	1.00568048	1.00252532
7/6	3.78664508	1.00417017	1.00150404
8/7	3.78297452	1.003196785	1.00097028
9/8	3.780466270	1.002531630	1.000663476

are to the exact values and show the very high degree of precision that we achieve with these bounds for the square, triangular, and honeycomb lattices, even with modest values of strip width L_y . For the relevant range of real $q \leq 0$, with $q = 2 - \xi - \xi^{-1}$, an infinite-product expression for $W(tri, q)$, applicable for $0 \leq \xi \leq 1$, was given in [50], from which one has

$$|W(tri, q)| = \frac{1}{\xi} \prod_{j=1}^{\infty} \frac{(1 - \xi^{6j-3})(1 - \xi^{6j-2})^2(1 - \xi^{6j-1})}{(1 - \xi^{6j-5})(1 - \xi^{6j-4})(1 - \xi^{6j})(1 - \xi^{6j+1})}. \quad (5.1)$$

Table 10Lower bounds on $\alpha(hc)$ and their ratios, as functions of strip width L_y .

BC	L_y	$[\lambda_{hc,L_y,free/cyl}(-1)]^{1/(2L_y)}$	$R_{hc, \frac{L_y}{L_y-1} / \frac{L_y}{L_y-2}, free/cyl}(-1)$
free	2	$(31)^{1/4} = 2.35961106$	
free	3	2.49321528	1.05662129
free	4	2.56281578	1.02791596
free	5	2.60550411	1.01665681
free	6	2.63435715	1.01107388
cyl	2	$\sqrt{7} = 2.64575131$	
cyl	4	2.77349764	1.04828357
cyl	6	2.782197008	1.003136606

Table 11Upper bounds on $\alpha(hc)$ and their ratios, as functions of strip width L_y .

$(L_y + 1)/L_y$	$\sqrt{\lambda_{hc,L_y+1,free}(-1)/\lambda_{hc,L_y,free}(-1)}$	$R_{hc, \frac{L_y^2}{(L_y-1)(L_y+1)}, free}(-1)$
2/1	$\frac{\sqrt{31}}{2} = 2.78388218$	
3/2	2.78354659	1.00012056
4/3	2.78349352	1.00001907
5/4	2.78348737	1.00000221
6/5	2.783486470	1.000000323

Table 12Lower bounds on $\alpha_0(hc)$ and their ratios, as functions of strip width L_y .

BC	L_y	$[\lambda_{hc,L_y,free/cyl}(0)]^{1/(2L_y)}$	$R_{hc, \frac{L_y}{L_y-1} / \frac{L_y}{L_y-2}, free/cyl}(0)$
free	2	$5^{1/4} = 1.49534878$	
free	3	1.69793365	1.13547667
free	4	1.80571700	1.06347913
free	5	1.87241553	1.03693742
free	6	1.91770572	1.02418811
cyl	2	$\sqrt{3} = 1.73205081$	
cyl	4	2.04591494	1.18120954
cyl	6	2.106218408	1.029475062

Table 13Upper bounds on $\alpha_0(hc)$ and their ratios, as functions of strip width L_y .

$(L_y + 1)/L_y$	$\sqrt{\lambda_{hc,L_y+1,free}(0)/\lambda_{hc,L_y,free}(0)}$	$R_{hc, \frac{L_y^2}{(L_y-1)(L_y+1)}, free}(0)$
2/1	$\sqrt{5} = 2.23606798$	
3/2	2.18915819	1.02142823
4/3	2.17188387	1.00795361
5/4	2.16477332	1.00328466
6/5	2.161128567	1.001686502

We have evaluated this infinite product numerically for $q = -1$, i.e., $\xi = (3 - \sqrt{5})/2$, using Maple and Mathematica. We obtain

$$\alpha(tri) = |W(tri, -1)| = 4.47464730907. \quad (5.2)$$

The values that we obtain from Maple and Mathematica serve to check each other and agree with each other; their precision extends well beyond the twelve significant figures listed in Eq. (5.2), but the numerical result in Eq. (5.2) will be sufficient for our present purposes.

We combine an analytic evaluation of $W(tri, q)$ at $q = 0$ from [50] with the relation (1.27) to obtain an exact analytic result for $\alpha_0(tri)$, namely

$$\alpha_0(tri) = |W(tri, 0)| = \frac{3^{3/2} [\Gamma(\frac{1}{3})]^9}{(2\pi)^5} = \frac{(2\pi)^4}{3^3 [\Gamma(\frac{2}{3})]^9} = 3.77091969752, \quad (5.3)$$

where $\Gamma(z)$ is the Euler gamma function. The equality of the two analytic expressions on the right-hand side of Eq. (5.3) follows from the reflection formula $\Gamma(z)\Gamma(1-z) = \pi/[\sin(\pi z)]$ with $z = 1/3$.

Table 14Lower bounds on $\alpha(sq_d)$ and their ratios, as functions of strip width L_y .

BC	L_y	$[\lambda_{sq_d, L_y, free/cyl}(-1)]^{1/L_y}$	$R_{sq_d, \frac{L_y}{L_y-1}, free/cyl}(-1)$
free	2	$2\sqrt{3} = 3.464101615$	
free	3	$[4(9 + \sqrt{69})]^{1/3} = 4.10604888$	1.18531421
free	4	4.46677215	1.08785167
cyl	3	$2 \times (15)^{1/3} = 4.93242415$	
cyl	4	5.354782509	1.085628962

Table 15Lower bounds on $\alpha_0(sq_d)$ and their ratios, as functions of strip width L_y .

BC	L_y	$[\lambda_{sq_d, L_y, free/cyl}(0)]^{1/L_y}$	$R_{sq_d, \frac{L_y}{L_y-1}, free/cyl}(0)$
free	2	$\sqrt{6} = 2.44948974$	
free	3	$[\frac{3(11+\sqrt{97})}{2}]^{1/3} = 3.15058481$	1.28622086
free	4	3.55858048	1.12949839
cyl	3	$(60)^{1/3} = 3.91486764$	
cyl	4	4.417285760	1.128335915

We list the ratios of lower and upper bounds to these exact values in [Tables 6–9](#). For the triangular lattice, $\epsilon(tri) = 8$, so we also obtain the exact results

$$\rho_\alpha(tri) = 0.55933091363 \quad (5.4)$$

and

$$\rho_{\alpha_0}(tri) = \frac{2\pi^4}{3^3[\Gamma(\frac{2}{3})]^9} = 0.47136496219 \quad (5.5)$$

We present our results on lower and upper bounds on $\alpha(hc)$ and $\alpha_0(hc)$ in [Tables 10–13](#).

Summarizing the lower and upper bounds for the these lattices Δ , listed in order of increasing (uniform) vertex degree, $\Delta(\Delta)$, we have, for $\alpha(\Delta)$, the bounds, given to the indicated number of significant figures (and, where available, the exact results):

$$2.782197008 < \alpha(hc) < 2.783486470 \quad (5.6)$$

$$3.493244874 < \alpha(sq) < 3.493927960 \quad (5.7)$$

and

$$4.471898355 < \alpha(tri) < 4.474676977 \quad (5.8)$$

(with the precise value $\alpha(tri) = 4.47464731$ in Eq. (5.2) from the exact result). For $\alpha_0(\Delta)$ we have

$$2.106218408 < \alpha_0(hc) < 2.161128567 \quad (5.9)$$

$$2.830007782 < \alpha_0(sq) < 2.862213752 \quad (5.10)$$

and

$$3.737371971 < \alpha_0(tri) < 3.780466270 \quad (5.11)$$

(with the precise value $\alpha_0(tri) = 3.7709196975$ from the exact result in Eq. (5.3)). Aside from $\alpha(tri)$ and $\alpha_0(tri)$, for which we have given exact results, these lower and upper bounds are, to our knowledge, the best current bounds on these exponential growth constants.

The exact values of $\alpha(tri)$ and $\alpha_0(tri)$, and our upper bounds on these other exponential growth constants, are close to the conjectured upper bounds in Eq. (4.11)–(4.20), especially for $\alpha_u(\Delta)$, as is evident from the following ratios (note that we use the exact values of $\alpha(tri)$ and $\alpha_0(tri)$ in the numerators of Eqs. (5.14) and (5.17)):

$$\frac{\alpha_u(hc)}{\alpha_{u,w}(hc)} = 0.999858 \quad (5.12)$$

$$\frac{\alpha_u(sq)}{\alpha_{u,w}(sq)} = 0.998265 \quad (5.13)$$

$$\frac{\alpha(tri)}{\alpha_{u,w}(tri)} = 0.994366 \quad (5.14)$$

Table 16

Lower bounds and their ratios for $\beta(sq)$ as functions of strip width L_y . The abbreviations cycl and tor stand for “cyclic” and “toroidal”, respectively.

BC	L_y	$[\lambda_{sq, L_y, \text{cyc/tor}}(-1, 1)]^{1/L_y}$	$R_{sq, \frac{L_y}{L_y-1}, \text{cyc/tor}}(-1, 1)$
cycl	1	1	
cycl	2	2	2
cycl	3	$\left(\frac{17+\sqrt{145}}{2}\right)^{1/3} = 2.43966477$	1.21983238
cycl	4	2.68228611	1.09944864
cycl	5	2.83465313	1.05680491
tor	2	$\sqrt{10} = 3.16227766$	
tor	3	$\left(\frac{41+\sqrt{1345}}{2}\right)^{1/3} = 3.38648385$	1.07090022
tor	4	3.449673447	1.018659353

$$\frac{\alpha_{0,u}(hc)}{\alpha_{0,u,w}(hc)} = 0.966486 \quad (5.15)$$

$$\frac{\alpha_{0,u}(sq)}{\alpha_{0,u,w}(sq)} = 0.954071 \quad (5.16)$$

and

$$\frac{\alpha_0(tri)}{\alpha_{0,u}(tri)} = 0.942730. \quad (5.17)$$

A very important property of our lower and upper bounds on these exponential growth constants is that they are quite close to each other. To show this quantitatively, we first calculate the average of these values for each exponential growth constant (EGF),

$$EGC_{ave}(\Lambda) = \frac{(EGC)_\ell(\Lambda) + (EGF)_u(\Lambda)}{2}, \quad (5.18)$$

and then calculate the fractional difference between the upper and lower bounds, i.e., the difference divided by the average of these bounds,

$$\frac{EGC_u(\Lambda) - EGC_\ell(\Lambda)}{EGC_{ave}(\Lambda)}, \quad (5.19)$$

where $EGC = \alpha, \alpha_0$ (and β , as discussed below).

In order of increasing vertex degree, we obtain, for $\alpha(\Lambda)$,

$$\frac{\alpha_u(hc) - \alpha_\ell(hc)}{\alpha_{ave}(hc)} = 0.463 \times 10^{-3} \quad (5.20)$$

$$\frac{\alpha_u(sq) - \alpha_\ell(sq)}{\alpha_{ave}(sq)} = 1.96 \times 10^{-4} \quad (5.21)$$

$$\frac{\alpha_u(tri) - \alpha_\ell(tri)}{\alpha_{ave}(tri)} = 0.621 \times 10^{-3} \quad (5.22)$$

and for $\alpha_0(\Lambda)$,

$$\frac{\alpha_{0,u}(hc) - \alpha_{0,\ell}(hc)}{\alpha_{0,ave}(hc)} = 2.57 \times 10^{-2} \quad (5.23)$$

$$\frac{\alpha_{0,u}(sq) - \alpha_{0,\ell}(sq)}{\alpha_{0,ave}(sq)} = 1.13 \times 10^{-2} \quad (5.24)$$

$$\frac{\alpha_{0,u}(tri) - \alpha_{0,\ell}(tri)}{\alpha_{0,ave}(tri)} = 1.15 \times 10^{-2} \quad (5.25)$$

Since our lower and upper bounds are so close to each other, we can use them to obtain an approximate (ap) value of the given exponential growth constant. One way to get this is simply to use the average of the lower and upper bounds for each exponential growth constant,

$$EGC_{ap}(\Lambda) = EGC_{ave}(\Lambda) \pm \delta_{EGC(\Lambda)} \quad (5.26)$$

where the uncertainty $\delta_{EGC(\Lambda)}$ is defined as

$$\delta_{EGC(\Lambda)} = (EGF)_u(\Lambda) - EGC_{ave}(\Lambda) = EGC_{ave}(\Lambda) - (EGF)_\ell(\Lambda) \quad (5.27)$$

Carrying out this procedure, we obtain the following approximate values, again listed in order of increasing vertex degree:

$$\alpha_{ap}(hc) = 2.78284 \pm 0.00064 \quad (5.28)$$

$$\alpha_{ap}(sq) = 3.49359 \pm 0.00034 \quad (5.29)$$

$$\alpha_{ap}(tri) = 4.4733 \pm 0.0014 \quad (5.30)$$

$$\alpha_{0,ap}(hc) = 2.134 \pm 0.027 \quad (5.31)$$

$$\alpha_{0,ap}(sq) = 2.846 \pm 0.016 \quad (5.32)$$

$$\alpha_{0,ap}(tri) = 3.7589 \pm 0.0215 . \quad (5.33)$$

These values are listed in Table 22. As is evident from these results, we achieve high accuracies in the determinations of these exponential growth constants, with fractional uncertainties ranging from $O(10^{-4})$ to $O(10^{-2})$.

For each exponential growth constant $EGC(\Lambda)$ that is not known exactly, we define the estimated ratio from Eq. (1.35) as

$$\rho_{EGC,ap}(\Lambda) \equiv \frac{EGC_{ave}(\Lambda)}{\epsilon(\Lambda)} . \quad (5.34)$$

Regarding the EGCs $\alpha(tri)$, $\alpha_0(tri)$, and $\beta(hc)$, for which we have presented exact values, we define $\rho_{EGC}(\Lambda)$ as these exact EGCs divided by $\epsilon(\Lambda)$ for the given lattice, i.e., $\rho_{\alpha}(tri) = \alpha(tri)/\epsilon(tri)$, etc. We list these in Table 23.

Combining our calculations of lower and upper bounds for the triangular lattice with our exact results for this lattice yields another demonstration of the very high precision of our bounds. The fractional difference between our upper bounds and the exact values of $\alpha(tri)$ and $\alpha_0(tri)$ are extremely small:

$$\frac{\alpha_u(tri) - \alpha(tri)}{\alpha(tri)} = 0.663 \times 10^{-5} \quad (5.35)$$

and

$$\frac{\alpha_{0,u}(tri) - \alpha_0(tri)}{\alpha_0(tri)} = 2.53 \times 10^{-3} . \quad (5.36)$$

The corresponding fractional differences relative to our lower bounds are

$$\frac{\alpha(tri) - \alpha_{\ell}(tri)}{\alpha(tri)} = 0.614 \times 10^{-3} \quad (5.37)$$

and

$$\frac{\alpha_0(tri) - \alpha_{0,\ell}(tri)}{\alpha_0(tri)} = 0.890 \times 10^{-2} . \quad (5.38)$$

Thus, our upper bounds for $\alpha(tri)$ and $\alpha_0(tri)$ are closer to the respective exact results than are the lower bounds. Consequently, the average quantities $\alpha_{ave}(tri)$ and quantities $\alpha_{0,ave}(tri)$ lie slightly below the respective exact values:

$$\frac{\alpha(tri) - \alpha_{ave}(tri)}{\alpha(tri)} = 3.04 \times 10^{-4} \quad (5.39)$$

and

$$\frac{\alpha_0(tri) - \alpha_{0,ave}(tri)}{\alpha_0(tri)} = 3.18 \times 10^{-3} . \quad (5.40)$$

Provided that this pattern also holds for the square and honeycomb lattices, then the average quantities $\alpha_{ave}(\Lambda)$ and $\alpha_{0,ave}(\Lambda)$ and thus the central values of $\alpha_{ap}(\Lambda)$ and $\alpha_{0,ap}(\Lambda)$ would also lie slightly below the respective exact values $\alpha(\Lambda)$ and $\alpha_0(\Lambda)$ for $\Lambda = sq, hc$.

We have also carried out corresponding calculations of lower and upper bounds for a nonplanar lattice denoted sq_d . We described above how one constructs a strip of this lattice. The construction here is analogous. One starts with the square lattice and then adds (i) an edge connecting the upper left and lower right vertices of each square to each other, and (ii) an edge connecting the upper right and lower left vertices of each square to each other. A finite section of this lattice with doubly periodic boundary conditions is a Δ -regular lattice graph with $\Delta_{sq_d} = 8$, and this also describes the infinite planar lattice. We present the resultant lower bounds and their ratios for $\alpha(sq_d)$ and for $\alpha_0(sq_d)$ in Tables 14–15. For upper bounds on this lattice, our relevant results are, first, that the ratio $\lambda_{sq_d, Ly+1, free}(-1)/\lambda_{sq_d, Ly, free}(-1)$ takes

the respective values $6, 3 + (1/3)\sqrt{69} = 5.76887462$, and 5.75046353 for $L_y = 1, 2, 3$, respectively. For the quantity $R_{sq_d, \frac{L_y^2}{(L_y-1)(L_y+1)}, free}(-1)$, the indicated pairs yield the respective values 1.04006421 and 1.00320167 . Second, we find that the ratio $\lambda_{sq_d, L_y+1, free}(0)/\lambda_{sq_d, L_y, free}(0)$ has the values $6, (11 + \sqrt{97})/4 = 5.21221445$, and 5.12783026 for $L_y = 1, 2, 3$. For the quantity $R_{sq_d, \frac{L_y^2}{(L_y-1)(L_y+1)}, free}(0)$, the indicated pairs yield the respective values 1.15114220 and 1.01645612 . Our results for the sq_d lattice are

$$5.354782509 < \alpha(sq_d) < 5.750463529 \quad (5.41)$$

and

$$4.417285760 < \alpha_0(sq_d) < 5.127830256. \quad (5.42)$$

These bounds for the sq_d lattice (with $\Delta(sq_d) = 8$) are included mainly for the general insight that they yield concerning the dependence of the exponential growth constants on vertex degree. This goal is already achieved with the widths that we have included, showing that for the set of honeycomb, square, triangular, and sq_d lattices Λ , the quantities $\alpha(\Lambda)$ and $\alpha_0(\Lambda)$ are monotonically increasing functions of the vertex degree, $\Delta(\Lambda)$. Accordingly, we have not attempted to carry out calculations on wider strips of the sq_d lattice to obtain the same precision in the lower and upper bounds that we did for the planar lattices considered here, and the bounds (5.41) and (5.42) are not as restrictive as the corresponding bounds for the other lattices considered here.

Given the rapid convergence of our results for these exponential growth constants on these lattice strips, even for modest strip widths, one could use extrapolation techniques to infer the actual respective values for $L_y \rightarrow \infty$ in the cases where exact results are not known. However, this extrapolation analysis is beyond the scope of our present paper, since the estimation of the uncertainty in the inferred value of $\alpha(\Lambda)$ and $\alpha_0(\Lambda)$ would depend on the extrapolation method used. These comments also apply to our bounds on $\beta(\Lambda)$ to be presented below. It is straightforward to calculate corresponding lower and upper bounds for the quantities $\rho_\alpha(\Lambda)$ and $\rho_{\alpha_0}(\Lambda)$ for these lattices; we do not list these explicitly.

6. Lower and upper bounds for $\beta(\Lambda)$

Using results on calculations of Potts/Tutte polynomials for a variety of families of lattice strip graphs, we have also obtained lower and upper bounds on the exponential growth constant β for totally cyclic orientations on these lattices. As discussed above, these involve dominant λ functions evaluated at $(q, v) = (-1, 1)$ or equivalently, $(x, y) = (0, 2)$. We recall the lower and upper bounds that we have inferred in Eqs. (4.48) and (4.51).

We list our numerical values of lower and upper bounds for $\beta(\Lambda)$ on various two-dimensional lattices in Tables 16–21. The format of these tables is analogous to the format in the corresponding tables presented above for α and α_0 .

The relation (1.25) enables us to obtain a precise value of $\beta(hc)$ from our evaluation of the exact expression for $\alpha(tri)$ in [50]. We find

$$\beta(hc) = \sqrt{\alpha(tri)} = 2.11533621655. \quad (6.1)$$

Since $\epsilon(hc) = 2^{3/2}$, it follows that

$$\rho_\beta(hc) = 0.7478842916. \quad (6.2)$$

Summarizing the lower and upper bounds for these lattices Λ from the above calculations, listed in order of increasing (uniform) vertex degree, $\Delta(\Lambda)$, we have

$$(*) \quad 2.09444676 < \beta(hc) < 2.12591038 \quad (6.3)$$

$$(*) \quad 3.449673447 < \beta(sq) < 3.535730951 \quad (6.4)$$

and

$$(*) \quad 7.696127303 < \beta(tri) < 7.832553170. \quad (6.5)$$

where $(*)$ means that by using duality and our previous calculations of lower and upper bounds on $\alpha(\Lambda)$ for $\Lambda = sq, tri, hc$, we can improve upon these bounds. Thus, first, using Eq. (1.25), we improve upon the bounds (6.3):

$$2.114686349 < \beta(hc) < 2.115343229. \quad (6.6)$$

Evidently, these lower and upper bounds on $\beta(hc)$ are very close to the precise value, $\beta(hc) = 2.11533621655$ in (6.1). Second, using Eq. (1.23) in conjunction with our bounds (5.7), we improve upon the bounds (6.4):

$$3.493244874 < \beta(sq) < 3.493927960. \quad (6.7)$$

Third, using Eq. (1.24), we improve upon the bounds (6.5):

$$7.740620193 < \beta(tri) < 7.747796928. \quad (6.8)$$

Aside from $\beta(hc)$, for which we have given an exact value, these lower and upper bounds on $\beta(\Lambda)$, (6.7) and (6.8) are, to our knowledge, the best current bounds on these quantities.

As was the case with our lower and upper bounds on $\alpha(\Lambda)$ and $\alpha_0(\Lambda)$, our lower and upper bounds on $\beta(\Lambda)$ are very close to each other. To show this, we exhibit the fractional differences of the upper and lower bounds on these lattices, in order of increasing vertex degree:

$$\frac{\beta_u(hc) - \beta_\ell(hc)}{\beta_{ave}(hc)} = 3.11 \times 10^{-4} \quad (6.9)$$

$$\frac{\beta_u(sq) - \beta_\ell(sq)}{\beta_{ave}(sq)} = 1.96 \times 10^{-4} \quad (6.10)$$

(which is the same as Eq. (5.21) by duality) and

$$\frac{\beta_u(tri) - \beta_\ell(tri)}{\beta_{ave}(tri)} = 0.927 \times 10^{-3} . \quad (6.11)$$

Hence, as before, since our lower and upper bounds are quite close to each other, we can use them to obtain the approximate value of $\beta(\Lambda)$ on the various lattices Λ . Using the same procedure as discussed above in Eqs. (5.18), (5.26), and (5.27), we calculate the approximate values

$$\beta_{ap}(hc) = 2.11501 \pm 0.00033 \quad (6.12)$$

$$\beta_{ap}(sq) = 3.49359 \pm 0.00034 \quad (6.13)$$

(which is the same as Eq. (5.29) by duality) and

$$\beta_{ap}(tri) = 7.7442 \pm 0.0036 . \quad (6.14)$$

These values of $\beta_{ap}(sq)$ and $\beta_{ap}(tri)$ are listed in Table 22, which also includes our exact value for $\beta(hc)$. As is evident, we achieve very high accuracy with our determination of these approximate values, with a fractional uncertainty of less than 10^{-4} for $\beta(sq)$ and 5×10^{-4} for $\beta(tri)$.

In Table 23 we list the values of the ratios $\rho_\alpha(\Lambda)$, $\rho_{\alpha_0}(\Lambda)$, and $\rho_\beta(\Lambda)$ obtained from our calculations. For $\rho_\alpha(tri)$, $\rho_{\alpha_0}(tri)$, and $\rho_\beta(hc)$, we list the exact values, and for the others we list the ratios calculated using $EGC_{ave}(\Lambda)$, where $EGC = \alpha$, α_0 , β .

We can use our exact value of $\beta(hc)$ to obtain a further measure of the accuracy of our bounds. The fractional differences between our upper and lower bounds on $\beta(hc)$ and this exact value are

$$\frac{\beta_u(hc) - \beta(hc)}{\beta(hc)} = 3.31 \times 10^{-6} \quad (6.15)$$

and

$$\frac{\beta(hc) - \beta_\ell(hc)}{\beta(hc)} = 3.07 \times 10^{-4} . \quad (6.16)$$

Thus, as was true of our lower and upper bounds on $\alpha(tri)$ and $\alpha_0(tri)$, here we observe that the upper bound in (6.6) is closer to the exact value, $\beta(hc)$ than is the lower bound in (6.6). Hence, the average, $\beta_{ave}(hc)$ is slightly below the exact value:

$$\frac{\beta(hc) - \beta_{ave}(hc)}{\beta(hc)} = 1.52 \times 10^{-4} . \quad (6.17)$$

As before, if this pattern also holds for the square and triangular lattices, then the average quantities $\beta_{ave}(\Lambda)$ and thus the central values of $\beta_{ap}(\Lambda)$ would also be slightly smaller than the exact values $\beta(\Lambda)$ for $\Lambda = sq, tri$.

As discussed above, we have included results for the sq_d lattice here for the information that they give on the dependence of the exponential growth constants on vertex degree. For the sq_d lattice strip with cyclic BCs and $L_y = 2$, we calculate

$$[\lambda_{sq_d, L_y, cyc}(-1, 1)]^{1/2} = \sqrt{13 + \sqrt{181}} = 5.14330867 \quad (6.18)$$

and for toroidal BCs and $L_y = 3$, we obtain the numerical value

$$[\lambda_{sq_d, L_y, tor}(-1, 1)]^{1/2} = 15.85636130 . \quad (6.19)$$

These yield lower bounds on $\beta(sq_d)$. In addition, we calculate the ratio

$$\frac{\lambda_{sq_d, 2, cyc}(-1, 1)}{\lambda_{sq_d, 1, cyc}(-1, 1)} = 13 + \sqrt{181} = 26.45362405 , \quad (6.20)$$

which yields an upper bound on $\beta(sq_d)$. Thus, we have the loose bounds

$$15.8563613 < \beta(sq_d) < 26.4536240. \quad (6.21)$$

These bounds are much less stringent than our bounds on $\beta(\Lambda)$ for the other lattices, but they are sufficient to show the monotonic increase of $\beta(\Lambda)$ with vertex degree $\Delta(\Lambda)$ among these lattices. For this reason, we have not tried to include results from larger-width strips for this sq_d lattice.

7. Comparative analysis

7.1. General

From our calculations, we observe that for these lattices Λ , the values of $\alpha(\Lambda)$, $\alpha_0(\Lambda)$, and $\beta(\Lambda)$ that are consistent with our lower and upper bounds (and the exact values where we have calculated them) are monotonically increasing functions of $\Delta(\Lambda)$. In particular, this is true of the quantities $\alpha_{ave}(\Lambda)$, $\alpha_{0,ave}(\Lambda)$, and $\beta_{ave}(\Lambda)$. This is the opposite of the behavior of the ground-state degeneracy of the q -state Potts antiferromagnet, $W(\Lambda, q)$, which, for values of q used in proper q -colorings of Λ , is a monotonically decreasing function of $\Delta(\Lambda)$. This dependence of $W(\Lambda, q)$ on $\Delta(\Lambda)$ was also shown in lower and upper bounds on $W(\Lambda, q)$ [27,28,28–46,48]. The fact that, for a given value of q used for a proper q -coloring of the lattice Λ , $W(\Lambda, q)$ is a monotonically decreasing function of $\Delta(\Lambda)$, was shown to be a consequence of the fact that increasing $\Delta(\Lambda)$ places more constraints on this proper q -coloring [27,28,35]. The reversal in the dependence of $W(\Lambda, q)$ on $\Delta(\Lambda)$ going from (positive) values of q used in proper q -colorings of Λ to $q \leq 0$ was evident in (Fig. 5 of) Ref. [28]. Our present results extend these earlier ones with quite restrictive upper and lower bounds and high-accuracy approximate values for $\alpha(\Lambda) = |W(\Lambda, -1)|$ and $\alpha_0(\Lambda) = |W(\Lambda, 0)|$. A property that is pertinent here is the fact that the signs of successive terms in the chromatic polynomial alternate, starting with a positive sign (and, indeed, a coefficient of unity) for the highest-degree term, $q^{n(G)}$, then a negative sign for the $q^{n(G)-1}$ term, and so forth for lower-power terms. Hence, if q is positive, as in the evaluation of $W(G, q)$ for the ground state degeneracy of the q -state Potts antiferromagnet, then alternate terms contribute with opposite sign, whereas if q is negative, as in the evaluations at $q = -1$ for $\alpha(G)$ and $\alpha_0(G)$, then all of the terms in $P(G, q)$ and $P_r(G, q) = q^{-1}P(G, q)$ contribute with the same sign.

As regards the relative sizes of $\alpha(\Lambda)$ and $\alpha_0(\Lambda)$, on the one hand, and $\beta(\Lambda)$ on the other, we find that $\alpha(\Lambda)$ and $\alpha_0(\Lambda)$ may be larger or smaller than $\beta(\Lambda)$, while the duality of the square lattice implies the equality of $\alpha(sq)$ and $\beta(sq)$. We also observe that the property that two families of recursive lattice graphs have the same value of Δ (or Δ_{eff}) does not imply that they have the same values of $\alpha(\{G\})$, $\alpha_0(\{G\})$, or $\beta(\{G\})$. For example, the cyclic square-lattice strip graph L_m has the same value of (uniform) vertex degree $\Delta = 3$ as the honeycomb lattice, but the values of $\alpha(\{L\})$, $\alpha_0(\{L\})$, and $\beta(\{L\})$ are different from the respective values of $\alpha(hc)$, $\alpha_0(hc)$, and $\beta(hc)$.

We recall the inequality (1.13). From our bounds and exact results, we compute the ratios $\alpha_0(\Lambda)/\alpha(\Lambda)$ for various lattices Λ , using $\alpha_{ave}(\Lambda)$ and $\alpha_{0,ave}(\Lambda)$ for $\Lambda = hc$, sq and our exact values $\alpha(tri)$ and $\alpha_0(tri)$. In order of increasing vertex degree, we have

$$\frac{\alpha_{0,ave}(hc)}{\alpha_{ave}(hc)} = 0.767 \quad (7.1)$$

$$\frac{\alpha_{0,ave}(sq)}{\alpha_{ave}(sq)} = 0.815 \quad (7.2)$$

and

$$\frac{\alpha_0(tri)}{\alpha(tri)} = 0.8427300. \quad (7.3)$$

We note that for these lattices, this ratio is a monotonically increasing function of $\Delta(\Lambda)$. This is the same dependence that we showed for the infinite-length, finite-width strips discussed in Section 2. We have also obtained results on these exponential growth constants for a number of heteropolygonal Archimedean lattices (i.e., Archimedean lattices comprised of more than one type of regular polygon) [51].

Concerning the ratios $\rho_{EGC}(\Lambda)$, we find that $\rho_\alpha(\Lambda)$ and $\rho_{\alpha_0}(\Lambda)$ are monotonically decreasing functions, while $\rho_\beta(\Lambda)$ is a monotonically increasing function of $\Delta(\Lambda)$. Again, this is the same dependence that we found for infinite-length, finite-width strips as a function of $\Delta(\{G\})$ (or, where appropriate, $\Delta_{eff}(\{G\})$).

Using similar methods, we have also obtained results on exponential growth constants for spanning forests and connected spanning subgraphs on a variety of lattices. A spanning forest in a graph G is a spanning subgraph of G that does not contain any circuits. Denote $N_{SF}(G)$ as the number of spanning forests of a graph G and $\phi(\{G\}) \equiv \lim_{n(G) \rightarrow \infty} [N_{SF}(G)]^{1/n(G)}$. For example, for the square lattice, we have found $3.675183 \leq \phi(sq) < 3.699659$, improving on the bounds $3.32 \leq \phi(sq) \leq 3.8416195$ in [19], the bounds $3.64497 \leq \phi(sq) \leq 3.74101$ in [20], the bounds $3.65166 \leq \phi(sq) \leq 3.73635$ in [24], and the upper bound $\phi(sq) \leq 3.705603$ in [23]. For the triangular and honeycomb lattices we obtain $5.393333 \leq \phi(tri) \leq 5.494840$ and $2.803787 \leq \phi(hc) \leq 2.804781$.

7.2. Comparison with spanning trees

The quantities $a(G)$, $a_0(G)$, and $b(G)$ enumerate classes of orientations of arrows defined on the edges of the directed graph $D(G)$, but depend only on G itself, and similarly with the resultant exponential growth constants in the $n(G) \rightarrow \infty$ limit. Because of this, it is appropriate to compare them with the number of spanning trees on G and the associated exponential growth constant. We do this in the present section. Recall that a tree graph is defined as a connected graph that does not contain any circuits, and a spanning tree of a graph G is a subgraph of G that is a tree and that contains all of the vertices of G (and a subset of the edges of G).

From the definition (A.12), it is evident that the number of spanning trees in a graph G is

$$N_{ST}(G) = T(G, 1, 1). \quad (7.4)$$

Since the nonzero coefficients of each term in Eq. (A.13) are positive, and since $a_0(G) = T(G, 1, 0)$ (recall Eq. (1.5)), a basic inequality is [19]

$$a_0(G) \leq N_{ST}(G), \quad \text{i.e.,} \quad T(G, 1, 0) \leq T(G, 1, 1). \quad (7.5)$$

The necessary and sufficient condition for this to be an equality is clear from Eqs. (1.5), (7.4), and (A.13); thus, $T(G, 1, 0) = T(G, 1, 1)$ if and only if $T(G, x, y)$ does not contain any nonzero terms of the form $t_{ij}x^i y^j$ with $j \geq 1$. From the definition (A.12), this condition is equivalent to the condition that G does not contain any cycles.

For the families of graphs under consideration here, $N_{ST}(G)$ grows exponentially rapidly with the number of vertices, $n(G)$. This motivates one to analyze the associated exponential growth constant,

$$\tau(\{G\}) \equiv \lim_{n(G) \rightarrow \infty} [N_{ST}(G)]^{1/n(G)}. \quad (7.6)$$

An equivalent quantity is $z(\{G\})$, defined as

$$z(\{G\}) \equiv \ln[\tau(\{G\})]. \quad (7.7)$$

The exponential growth constants $\tau(\Lambda)$ have been calculated exactly for the square, triangular, and honeycomb lattices under consideration here [52], and, indeed, for all Archimedean lattices [53–55] (as well as some higher-dimensional lattices). The relevant exponential growth constants for the planar lattices studied here are, in order of increasing vertex degree, [52]

$$\tau(hc) = \exp\left[\frac{\ln(3)}{4} + \frac{3}{\pi} \text{Ti}_2\left(\frac{1}{\sqrt{3}}\right)\right] = 2.24266494889 \quad (7.8)$$

$$\tau(sq) = \exp\left(\frac{4C}{\pi}\right) = 3.20991230073 \quad (7.9)$$

and

$$\tau(tri) = \exp\left[\frac{\ln(3)}{2} + \frac{6}{\pi} \text{Ti}_2\left(\frac{1}{\sqrt{3}}\right)\right] = 5.02954607297. \quad (7.10)$$

(to the indicated precision). In Eqs. (7.8) and (7.10), $\text{Ti}_2(x)$ is the tangent inverse integral,

$$\text{Ti}_2(x) = \int_0^x \frac{\arctan(y)}{y} dy = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)^2}, \quad (7.11)$$

and in Eq. (7.9), C is the Catalan constant,

$$C = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = \text{Ti}_2(1) = 0.915965594177 \quad (7.12)$$

Owing to the fact that the triangular lattice is the (planar) dual of the honeycomb lattice and using Eq. (1.21), we have

$$\tau(hc) = [\tau(tri)]^{1/2}, \quad (7.13)$$

which is evident in Eqs. (7.8) and (7.10). For the sq_d lattice, we have [55]

$$\tau(sq_d) = \exp\left[\frac{4C}{\pi} + \ln(2 - \sqrt{3}) - \frac{4}{3} \text{arctanh}\left(\frac{1}{\sqrt{3}}\right) + \frac{4}{\pi} \text{Ti}_2(2 + \sqrt{3})\right] = 6.984820959. \quad (7.14)$$

For the strip graphs of the honeycomb, square, and triangular lattices discussed in Section 2, the exponential growth constants τ are, again in order of increasing Δ or Δ_{eff} [39,40,42,43,46],

$$\tau(\{HL\}) \equiv \tau(hc, 2_F \times \infty) = (3 + 2\sqrt{2})^{1/4} = \sqrt{1 + \sqrt{2}} = 1.55377397 \quad (7.15)$$

$$\tau(\{L\}) \equiv \tau(sq, 2_F \times \infty) = \sqrt{2 + \sqrt{3}} = 1.93185165 \quad (7.16)$$

$$\tau(\{Wh\}) = \frac{3 + \sqrt{5}}{2} = 2.61803399 \quad (7.17)$$

$$\tau(\{TL\}) \equiv \tau(tri, 2_F \times \infty) = \sqrt{\frac{7 + 3\sqrt{5}}{2}} = \frac{3 + \sqrt{5}}{2} = 2.61803399 \quad (7.18)$$

and

$$\tau(\{sq_d\}) \equiv \tau((sq_d)_{2,F} \times \infty) = 2\sqrt{3} = 3.4641016 . \quad (7.19)$$

(Note that $\alpha(\{sq_d\}) = \tau(\{sq_d\})$ in the case of the sq_d strip.) For both the infinite planar lattices and for these infinite-length, finite-width lattice strip graphs, the exponential growth constant $\tau(\{G\})$ is a monotonically increasing function of $\Delta(\Lambda)$ (and, where appropriate, $\Delta_{eff}(\{G\})$).

We discuss some inequalities. A theorem of Thomassen [56] states that if G is a Δ -regular graph of degree $\Delta(G) \leq 3$ which has no loops (which may have bridges and multiple edges), then $N_{ST}(G) \leq a(G)$. Considering a family of graphs of this type and taking the limit $n(G) \rightarrow \infty$, this implies that in this limit, $\tau(\{G\}) \leq \alpha(\{G\})$. It is readily checked that our results for Δ -regular families of graphs of degree $\Delta(G) \leq 3$ satisfy this theorem. For example, for the cyclic square-ladder strip L_m , which has $\Delta(L_m) = 3$, the value of $\tau(\{L\})$ given in Eq. (7.16), namely $\tau(\{L\}) = 1.932$, is less than $\alpha(\{L\}) = \sqrt{7} = 2.646$ given in Eq. (2.3), and for the infinite honeycomb lattice, with $\Delta(hc) = 3$, the exactly known value of $\tau(hc)$ given in Eq. (7.8) from [52], is less than our value for $\alpha(hc)$ given in Eq. (5.28), namely $\alpha_{ap}(hc) = 2.78284 \pm 0.00064$. We calculate the ratio

$$\frac{\tau(hc)}{\alpha_{ap}(hc)} = 0.80589 \pm 0.00019 . \quad (7.20)$$

Another theorem of Thomassen [56] states that if G is a graph with no loops or bridges (but which may have multiple edges) with $e(G) \geq 4[n(G) - 1]$, then $N_{ST}(G) \leq b(G)$. For a family of graphs satisfying this condition, in the limit $n(G) \rightarrow \infty$, this theorem implies that $\tau(\{G\}) \leq \beta(\{G\})$. Among the graphs that we consider, sections of the sq_d lattice with doubly periodic boundary conditions, and also the infinite sq_d lattice have $\Delta(sq_d) = 8$ and hence $e(G) = 4n(G)$. Again, it is readily checked that, with the value of $\tau(sq_d) = 6.985$ in Eq. (7.14), it follows that any value of $\beta(sq_d)$ in the range allowed by our inferred lower and upper bounds in (6.21) satisfies this theorem.

For all of the lattice graphs that we consider,

$$\alpha_0(\{G\}) < \tau(\{G\}) . \quad (7.21)$$

Note that this inequality is not implied by the inequality (7.5), since, *a priori*, the difference, $\lim_{n(G) \rightarrow \infty} [N_{ST}]^{1/n(G)} - \lim_{n(G) \rightarrow \infty} [a_0(G)]^{1/n(G)}$ might vanish as $n(G) \rightarrow \infty$.

Furthermore, given the lower and upper bounds (5.7) and the duality relation that $\alpha(sq) = \beta(sq)$, it is evident that

$$\alpha(sq) > \tau(sq) . \quad (7.22)$$

Numerically, using our determination of the approximate value $\alpha_{ap}(sq)$ in Eq. (5.29), we have

$$\frac{\tau(sq)}{\alpha_{ap}(sq)} = \frac{\tau(sq)}{\beta_{ap}(sq)} = 0.9188005 \pm 0.0000894 . \quad (7.23)$$

From the exact value of $\alpha(tri)$ in Eq. (5.2), we have

$$\alpha(tri) < \tau(tri) . \quad (7.24)$$

Numerically,

$$\frac{\alpha(tri)}{\tau(tri)} = 0.8896722 . \quad (7.25)$$

Given our bounds on $\beta(tri)$, (6.8), we have

$$\beta(tri) > \tau(tri) . \quad (7.26)$$

Using our determination of $\beta_{ap}(tri)$ in Eq. (6.14), we compute

$$\frac{\tau(tri)}{\beta_{ap}(tri)} = 0.64946 \pm 0.00030 . \quad (7.27)$$

Finally, from the exact value (6.1), we have the inequality

$$\beta(hc) < \tau(hc) \quad (7.28)$$

Table 17Upper bounds and their ratios for $\beta(sq)$ as functions of strip width L_y .

$(L_y + 1)/L_y$	$\lambda_{sq, L_y+1, cyc}(-1, 1)/\lambda_{sq, L_y, cyc}(-1, 1)$	$R_{sq, \frac{L_y^2}{(L_y-1)(L_y+1)}, cyc}(-1, 1)$
2/1	4	
3/2	$\frac{17+\sqrt{145}}{8} = 3.63019932$	1.10186787
4/3	3.56475709	1.01835812
5/4	3.535730951	1.0082093749

Table 18Lower bounds and their ratios for $\beta(tri)$ as functions of strip width L_y .

BC	L_y	$[\lambda_{tri, L_y, cyc/tor}(-1, 1)]^{1/L_y}$	$R_{tri, \frac{L_y}{L_y-1}, cyc/tor}(-1, 1)$
cycl.	2	$\frac{3+\sqrt{13}}{2} = 3.302775637731...$	
cycl.	3	4.48070229	1.35664749
cycl.	4	5.16971535	1.15377345
cycl.	5	5.61764092	1.08664414
tor.	2	$\sqrt{2(14 + \sqrt{202})} = 7.51168029$	
tor.	3	7.696127303	1.024554694

Table 19Upper bounds and their ratios for $\beta(tri)$ as functions of strip width L_y .

$(L_y + 1)/L_y$	$\lambda_{tri, L_y+1, cyc}(-1, 1)/\lambda_{tri, L_y, cyc}(-1, 1)$	$R_{tri, \frac{L_y^2}{(L_y-1)(L_y+1)}, cyc}(-1, 1)$
2/1	$\frac{11+3\sqrt{13}}{2} = 10.90832691$	
3/2	8.24669860	1.32275077
4/3	7.94014180	1.03860848
5/4	7.832553170	1.013736086

Table 20Lower bounds, their ratios with respect to the exact value of $\beta(hc)$, and ratios of adjacent bounds, as functions of strip width L_y .

BC	L_y	$[\lambda_{hc, L_y, cyc/tor}(-1, 1)]^{1/(2L_y)}$	$\frac{[\lambda_{hc, L_y, cyc/tor}(-1, 1)]^{1/(2L_y)}}{\beta(hc)}$	$R_{hc, \frac{L_y}{L_y-1} / \frac{L_y}{L_y-2}, cyc/tor}(-1, 1)$
cycl.	2	$\sqrt{2} = 1.41421356$	0.66855262	
cycl.	3	1.62353902	0.76750873	1.14801545
cycl.	4	1.73776398	0.82150722	1.07035554
cycl.	5	1.80926267	0.85530738	1.04114408
tor.	2	2	0.94547618	
tor.	4	2.094446760	0.990124758	1.047223380

and numerically,

$$\frac{\beta(hc)}{\tau(hc)} = 0.943224. \quad (7.29)$$

It is of interest to discuss how our results relate to the Merino–Welsh conjecture (MWC) [19] and a later conjecture by Conde and Merino (CMC) [57]. The Merino–Welsh conjecture is as follows: Let G be a connected graph without loops or bridges (which may have multiple edges). Then the Merino–Welsh conjecture is the inequality [19]

$$N_{ST}(G) \leq \max[a(G), b(G)], \quad \text{i.e., } T(G, 1, 1) \leq \max[T(G, 2, 0), T(G, 0, 2)] \quad (\text{MWC}). \quad (7.30)$$

Subsequently, in [57], Conde and Merino conjectured the stronger inequality that if G is a connected graph without loops or bridges (which may have multiple edges), then

$$[N_{ST}(G)]^2 \leq a(G)b(G), \quad \text{i.e., } [T(G, 1, 1)]^2 \leq T(G, 2, 0)T(G, 0, 2) \quad (\text{CMC}). \quad (7.31)$$

As observed in [57], the inequality (7.31) implies the inequality (7.30). Some works related to these conjectures include [56,58–61]. In particular, the Merino–Welsh conjecture has been proved for wheel graphs Wh_n , complete graphs K_n , and complete bipartite graphs $K_{r,s}$ with $r \geq s \geq 2$ in [58], and for series–parallel graphs in [60].

We first note that the Merino–Welsh and Conde–Merino conjectures imply the following inequalities on exponential growth constants:

$$\tau(\{G\}) \leq \max[\alpha(\{G\}), \beta(\{G\})] \quad \text{from MWC}. \quad (7.32)$$

Table 21

Upper bounds, their ratios relative to the exact $\beta(hc)$, and ratios of adjacent bounds, as functions of strip width L_y .

$(L_y + 1)/L_y$	$\sqrt{\frac{\lambda_{hc,L_y+1,cyc}(-1,1)}{\lambda_{hc,L_y,cyc}(-1,1)}}$	$\frac{\sqrt{\lambda_{hc,L_y+1,cyc}(-1,1)/\lambda_{hc,L_y,cyc}(-1,1)}}{\beta(hc)}$	$R_{hc, \frac{i_y^2}{(L_y-1)(L_y+1)}, cyc}(-1, 1)$
3/2	2.13972616	1.01153005	
4/3	2.13095839	1.0073820	1.00411447
5/4	2.125910381	1.004998810	1.002374517

Table 22

Values of $EGC_{ap}(\Lambda)$ defined in Eq. (5.26) with Eqs. (5.18) and (5.27) for honeycomb, square, and triangular lattices, where EGC denotes exponential growth constant. In the cases where we have obtained exact values, namely $\alpha(tri)$, $\alpha_0(tri)$, and $\beta(hc)$, these are listed instead of the $EGC_{ap}(\Lambda)$ quantity. For reference, we also list the (exactly known) values of $\tau(\Lambda)$ for these lattices. See text for further discussion.

Λ	$\Delta(\Lambda)$	$\alpha(\Lambda)$	$\alpha_0(\Lambda)$	$\beta(\Lambda)$	$\tau(\Lambda)$
hc	3	2.78284 ± 0.00064	2.134 ± 0.027	2.11533621655	2.24266494889
sq	4	3.49359 ± 0.00034	2.846 ± 0.016	3.49359 ± 0.00034	3.20991230073
tri	6	4.47464730907	3.77091969752	7.7442 ± 0.0036	5.02954607297

and

$$[\tau(\{G\})]^2 \leq \alpha(\{G\})\beta(\{G\}) \quad \text{from CMC.} \quad (7.33)$$

We discuss these in turn. As is evident in Table 22, our results are in agreement with the inequality (7.32). This is also true for all of the infinite-length, finite-width strip graphs discussed in Section 2, for which the $\tau(\{G\})$ values were given in Eqs. (7.15)–(7.19). Recall that the values of $\tau(\Lambda)$ are exactly known for all of the (infinite limits of) lattice graphs that we consider here. For the honeycomb lattice and the cyclic square-lattice strip graph, the validity of the inequality (7.32) is guaranteed by the first theorem from Thomassen [56] mentioned above, namely that because $\Delta(hc) = \Delta(L_m) = 3$, it follows that $\tau(hc) \leq \alpha(hc)$ and $\tau(\{L\}) \leq \alpha(\{L\})$. In either of the hypothetical cases in which $\alpha(hc) \geq \beta(hc)$ or $\alpha(hc) \leq \beta(hc)$, this implies that $\tau(hc) \leq \max[\alpha(hc), \beta(hc)]$ and similarly with $\{L\}$. In fact, we find that $\alpha(hc) > \beta(hc)$ and $\alpha(\{L\}) > \beta(\{L\})$. For the square lattice, we find

$$\tau(sq) < \max[\alpha(sq), \beta(sq)] \quad \text{where } \alpha(sq) = \beta(sq), \quad (7.34)$$

with the approximate value of $\tau(sq)/\alpha(sq) = \tau(sq)/\beta(sq)$ given by Eq. (7.23). For the triangular lattice, we have

$$\tau(tri) < \max[\alpha(tri), \beta(tri)] = \beta(tri), \quad (7.35)$$

with the approximate value of $\tau(tri)/\beta(tri)$ given by Eq. (7.27).

Our results are also in agreement with the inequality on exponential growth constants implied by the Conde–Merino conjecture (7.33). This is clear for our results on the infinite-length finite-width strip graphs discussed in Section 2 in conjunction with Eqs. (7.15)–(7.19), all of which are exact. Further, for the infinite planar lattices we have

$$[\tau(hc)]^2 = 5.029546 < \alpha_{ap}(hc)\beta(hc) = 5.8866 \quad (7.36)$$

$$[\tau(sq)]^2 = 10.30354 < \alpha_{ap}(sq)\beta_{ap}(sq) = [\alpha_{ap}(sq)]^2 = 12.205 \quad (7.37)$$

and

$$[\tau(tri)]^2 = 25.29633 < \alpha(tri)\beta_{ap}(tri) = 34.653, \quad (7.38)$$

where we have used the approximate (ap) values that we have determined for $\alpha_{ap}(hc)$, $\alpha_{ap}(sq) = \beta_{ap}(sq)$, and $\beta_{ap}(tri)$ in Eqs. (5.28), (5.29), and (6.14), respectively, together with our exact values for $\alpha(tri)$ and $\beta(hc)$ in Eqs. (5.2) and (6.1), in computing the right-hand sides of (7.36)–(7.38). As is clear from these results, the accuracy with which we have determined the approximate values $\alpha_{ap}(hc)$, $\alpha_{ap}(sq) = \beta_{ap}(sq)$, and $\beta_{ap}(tri)$ is more than adequate to establish the validity of the inequalities (7.36)–(7.38) (see Table 24).

8. Conclusions

In this paper we have calculated the exponential growth constants α , α_0 , and β describing the asymptotic behavior of, respectively, acyclic orientations, acyclic orientations with a single source vertex, and totally cyclic orientations of directed lattice strip graphs. We have considered several different types of lattices, including square, triangular, and honeycomb. From our calculations, we have inferred new lower and upper bounds on these exponential growth constants for the respective infinite lattices. Our bounds from calculations on infinite-length, finite-width lattice strips converge rapidly

Table 23

Values of $\epsilon(\Lambda)$ and $\rho_{EGC}(\Lambda)$, defined in (1.31) and (1.36)–(1.38), for honeycomb, square, and triangular lattices Λ , with the exponential growth constants (EGCs) $\alpha(\Lambda)$, $\alpha_0(\Lambda)$, $\beta(\Lambda)$, and $\tau(\Lambda)$. For the EGCs, we use the exact values of $\rho_\alpha(tri)$, $\rho_{\alpha_0}(tri)$, and $\rho_\beta(hc)$ that we have presented here; for the other EGCs, we use $EGC_{ave}(\Lambda)$ from Eq. (5.18) as in Eq. (5.34). See text for further discussion.

Λ	$\Delta(\Lambda)$	$\epsilon(\Lambda)$	$\rho_\alpha(\Lambda)$	$\rho_{\alpha_0}(\Lambda)$	$\rho_\beta(\Lambda)$
hc	3	$2\sqrt{2}$	0.984	0.7545	0.7478842912
sq	4	4	0.873	0.7115	0.873
tri	6	8	0.5593309136	0.4713649622	0.968

Table 24

Graph-theoretic numbers $a(G)$, $a_0(G)$, and $b(G)$ and their expressions as valuations of the chromatic polynomial $P(G, q)$, the reduced chromatic polynomial $P_r(G, q) = q^{-1} P(G, q)$, the Tutte polynomial $T(G, x, y)$, and/or the Potts polynomial $Z(G, q, v)$.

Quantity	x	y	q	v	Expression
$a(G)$	2	0	−1	−1	$a(G) = T(G, 2, 0) = (-1)^n P(G, -1)$
$a_0(G)$	1	0	0	−1	$a_0(G) = T(G, 1, 0) = (-1)^{n-1} P_r(G, 0)$
$b(G)$	0	2	−1	1	$b(G) = T(G, 0, 2) = -Z(G, -1, 1)$

even for modest values of strip widths. Using exact results for $\alpha(tri)$, $\alpha_0(tri)$, and $\beta(hc)$, we have shown that our lower and upper bounds are very close to the exact values of these quantities. In addition to the above-mentioned exact results, our bounds are, to our knowledge, the best current bounds on these exponential growth constants. Since our lower and upper bounds are quite close to each other, we infer quite accurate approximate values for the exponential growth constants that are not exactly known. These values have fractional uncertainties ranging from $O(10^{-4})$ to $O(10^{-2})$. Comparisons of these values with the growth constants for spanning trees on these lattices are given. Our results are in agreement with the Merino–Welsh and Conde–Merino conjectures. We have also presented corresponding bounds for a nonplanar lattice denoted sq_d with a higher vertex degree, $\Delta = 8$. Our results show that $\alpha(\Lambda)$, $\alpha_0(\Lambda)$, and $\beta(\Lambda)$ are monotonically increasing functions of vertex degree $\Delta(\Lambda)$ for these lattices. We have conjectured that the analytic expression that was proved to be a lower bound on $W(\Lambda, q)$ for values of q used in proper q -colorings of Λ is an upper bound on $\alpha(\Lambda)$ and $\alpha_0(\Lambda)$ when evaluated at $q = -1$ and $q = 0$, respectively.

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Appendix. Some graph theory background

In this appendix we include some graph theory background relevant for our analysis in the paper (for further details, see, e.g., [1]). As in the text, let $G = (V, E)$ be a graph defined by its vertex and edge sets V and E . Let $n = n(G) = |V|$, $e(G) = |E|$, and $k(G)$ denote the number of vertices, edges, and connected components of G , respectively. Denote $\Delta(v_i)$ as the degree of the vertex $v_i \in V$. A loop is defined as an edge that connects a vertex to itself, and a bridge (co-loop) is defined as an edge that has the property that if it is deleted, then this increases the number of components in the resultant graph, relative to the number of components in the initial graph that contained the bridge. If a graph has no bridges, then it is said to be 2-connected. Two adjacent vertices may have more than one edge joining them; if so, one says that the graph has multiple edges (and sometimes calls it a multigraph). However, as explained in the text, in order to have minimal measures of acyclic orientations, acyclic orientations with a unique source, and totally cyclic orientations, we exclude graphs with loops or multiple edges from our analysis. A cycle on G is defined as a set of edges that form a closed circuit (cycle). Let $c(G)$ denote the number of linearly independent cycles in G . This satisfies $c(G) = e(G) + k(G) - n(G)$. The “join” of two graphs G and H is denoted $G + H$ and is defined as the graph constructed by adding edges to each of the vertices of G connecting to each of the vertices of H .

The chromatic polynomial of G , denoted $P(G, q)$, counts the number of ways of assigning q colors to the vertices of G subject to the condition that no two adjacent vertices have the same color [1–6]. Such a color assignment is called a proper q -coloring of (the vertices of) G . The chromatic number $\chi(G)$ is defined as the minimum value of q required for a proper q -coloring of G . A spanning subgraph of G , denoted G' , is a graph with the same vertex set V and a subset of the edge set E , i.e., $G' = G'(V, E')$ with $E' \subseteq E$. $P(G, q)$ is given by

$$P(G, q) = \sum_{G' \subseteq G} (-1)^{e(G')} q^{k(G')}. \quad (\text{A.1})$$

As is clear from Eq. (A.1), $P(G, q)$ is a polynomial of degree $n = n(G)$ in q . Since one obviously cannot perform a proper q -coloring of a graph G if the number of colors is zero, i.e., $q = 0$, it follows that $P(G, q)$ always contains a factor q . This property is also clear from Eq. (A.1). Consequently, one may define a reduced polynomial

$$P_r(G, q) \equiv \frac{P(G, q)}{q}. \quad (\text{A.2})$$

One can write the chromatic polynomial as

$$P(G, q) = \sum_{j=1}^{n(G)} \kappa_j(G) q^j, \quad (\text{A.3})$$

with $\kappa_n(G) = 1$, etc. A general property is that the signs of the $\kappa_j(G)$ alternate as j decreases from n to 1. From Eq. (1.4) and (A.3), it follows that

$$a_0(G) = (-1)^{n(G)-1} \kappa_1(G). \quad (\text{A.4})$$

The chromatic polynomial is a special case of the partition function of the q -state Potts model, $Z(G, q, v)$. A convenient expression for this partition function is as a sum of contributions from spanning subgraphs [62],

$$Z(G, q, v) = \sum_{G' \subseteq G} v^{e(G')} q^{k(G')}, \quad (\text{A.5})$$

where, in the physics context, v is a temperature-dependent variable given by Eq. (A.9) below. As is obvious from Eq. (A.5), $Z(G, q, v)$ is a polynomial in q and v with the property that the nonzero coefficients are positive integers. The expression for $Z(G, q, v)$ allows one to define the Potts partition function for values of q that are more general than just the positive integers. From (A.1) and (A.5), it is evident that

$$P(G, q) = Z(G, q, -1). \quad (\text{A.6})$$

We recall the definition of $Z(G, q, v)$ in terms of a spin-type Hamiltonian [8]. Consider a graph G with a set of classical spins σ_i taking values in the set of positive integers $\{1, \dots, q\}$ at each site (vertex) v_i of G , whose interactions with spins on adjacent sites are described by the Hamiltonian

$$\mathcal{H} = -J \sum_{e_{ij}} \delta_{\sigma_i, \sigma_j}, \quad (\text{A.7})$$

where J is the spin-spin interaction constant. Let us define $\beta = 1/(k_B T)$ (not to be confused with $\beta(G)$), where k_B is the Boltzmann constant, and denote $K = \beta J$. Then the partition function of this model on a graph G is given by

$$Z(G, q, v) = \sum_{\{\sigma\}} e^{-\beta \mathcal{H}} = \sum_{\{\sigma\}} \left[\prod_{e_{ij}} e^{K \delta_{\sigma_i, \sigma_j}} \right] = \sum_{\{\sigma\}} \prod_{e_{ij}} \left[(1 + v \delta_{\sigma_i, \sigma_j}) \right], \quad (\text{A.8})$$

where here $\{\sigma\}$ denotes the set of all values of the σ variables on the vertices of G and

$$v \equiv e^K - 1. \quad (\text{A.9})$$

The intervals $v \geq 0$ and $-1 \leq v \leq 0$ correspond, respectively to the ferromagnetic and antiferromagnetic Potts models. The value $v = -1$, i.e., $K = -\infty$, corresponds to the zero-temperature Potts antiferromagnet. This provides an understanding of the relation (A.6); in the limit $K \rightarrow -\infty$, the only spin configurations that contribute to the Potts model partition function are those for which $\sigma_i \neq \sigma_j$ on adjacent vertices v_i and v_j of G , and, with the isomorphism between the values of these σ_i and σ_j variables and assignments of colors to the vertices of G , these spin configurations are precisely isomorphic to a proper q -coloring of the vertices of G .

The dimensionless free energy, per vertex, of the Potts model on a graph G (usually a regular lattice graph) in the limit $n(G) \rightarrow \infty$, is

$$f(\{G\}, q, v) = \lim_{n(G) \rightarrow \infty} \frac{1}{n(G)} \ln[Z(G, q, v)]. \quad (\text{A.10})$$

The ground-state degeneracy, per vertex, of the q -state Potts antiferromagnet on a graph G in the limit $n(G) \rightarrow \infty$ is

$$W(\{G\}, q) = \lim_{n(G) \rightarrow \infty} [P(G, q)]^{1/n(G)}. \quad (\text{A.11})$$

As noted above, in statistical physics, one is commonly interested in integral values of $q \geq \chi(G)$, is usually a positive integer, but Eq. (A.5) enables one to generalize q to other values. Since $P(G, q)$ and/or $Z(G, q, v)$ may be negative for (real) values of q away from the positive integers, the evaluation of the relations (A.10) and (A.11) requires specification of which of the n roots of (-1) one uses [28,39]. However, the magnitudes $|f(\{G\}, q, v)|$ and $|W(\{G\}, q)|$ are unambiguously defined by Eqs. (A.10) and (A.11). The values $q = -1$ and $q = 0$ are relevant for the evaluation of the exponential growth

constants of acyclic orientations of directed edges of G and acyclic orientations of directed edges of G that have a unique source vertex, as specified in Eqs. (1.26) and (1.27).

The Tutte polynomial of a graph G is defined as

$$T(G, x, y) = \sum_{G' \subseteq G} (x-1)^{k(G')-k(G)} (y-1)^{c(G')} . \quad (\text{A.12})$$

Two basic properties that are relevant here are that (i) if G contains a loop, then it has no acyclic orientations, so $a(G) = T(G, 2, 0) = 0$ and $a_0(G) = T(G, 1, 0) = 0$; and (ii) if G contains a bridge, then it has no totally cyclic orientations, so $b(G) = T(G, 0, 2) = 0$. One can write the Tutte polynomial of a graph G as

$$T(G, x, y) = \sum_{i,j} t_{ij} x^i y^j , \quad (\text{A.13})$$

where the t_{ij} can be determined from the definition (A.12). A basic property of $T(G, x, y)$ is that the nonzero t_{ij} are positive (integers) [1,10].

The Tutte polynomial and Potts model partition functions are equivalent, and are related according to

$$Z(G, q, v) = (x-1)^{k(G)} (y-1)^{n(G)} T(G, x, y) , \quad (\text{A.14})$$

with the definitions

$$x = 1 + \frac{q}{v} , \quad (\text{A.15})$$

and

$$y = v + 1 \quad (\text{A.16})$$

so that

$$q = (x-1)(y-1) . \quad (\text{A.17})$$

Hence,

$$P(G, q) = q^{k(G)} (-1)^{n(G)-k(G)} T(G, 1-q, 0) . \quad (\text{A.18})$$

Without loss of generality, we restrict ourselves to connected graphs here, i.e., $k(G) = 1$, so Eq. (A.18) reduces to

$$P(G, q) = q(-1)^{n(G)-1} T(G, 1-q, 0) . \quad (\text{A.19})$$

From the representation of the Potts model partition function $Z(G, q, v)$ as a sum of contributions from spanning subgraphs, Eq. (A.5), it follows that $Z(G, q, v)$ also has an overall factor of q , so that one can define a reduced Potts model partition function that is a polynomial in q and v ,

$$Z_r(G, q, v) \equiv \frac{Z(G, q, v)}{q} \quad (\text{A.20})$$

Thus,

$$a_0(G) = (-1)^{n(G)-1} P_r(G, 0) = (-1)^{n(G)-1} Z_r(G, 0, -1) = T(G, 1, 0) \quad (\text{A.21})$$

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