A positive fraction mutually avoiding sets theorem

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Abstract

Two sets A and B of points in the plane are *mutually avoiding* if no line generated by any two points in A intersects the convex hull of B, and vice versa. In 1994, Aronov, Erdős, Goddard, Kleitman, Klugerman, Pach, and Schulman showed that every set of n points in the plane in general position contains a pair of mutually avoiding sets each of size at least $\sqrt{n/12}$. As a corollary, their result implies that, for every set of n points in the plane in general position, one can find at least $\sqrt{n/12}$ segments, each joining two of the points, such that these segments are pairwise crossing.

In this note, we prove a fractional version of their theorem: for every $k > 0$ there is a constant $\varepsilon_k > 0$ such that any sufficiently large point set P in the plane contains $2k$ subsets $A_1, \ldots, A_k, B_1, \ldots, B_k$, each of size at least $\varepsilon_k|P|$, such that every pair of sets $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_k\}$, with $a_i \in A_i$ and $b_i \in B_i$, are mutually avoiding. Moreover, we show that $\varepsilon_k = \Omega(1/k^4)$. Similar results are obtained in higher dimensions.

1 Introduction

Let P be an n-element point set in the plane in general position, that is, no three members are collinear. For $k > 0$, we say that P contains a *crossing family* of size k if there are k segments whose endpoints are in P that are pairwise crossing. Crossing families were introduced in 1994 by Aronov, Erdős, Goddard, Kleitman, Kluggerman, Pach, and Schulman [1], who showed that for any given set of n points in the plane in general position, there exists a crossing family of size at least $\sqrt{n/12}$. They raised the following problem (see also Chapter 9 in [3]).

Problem 1.1 ([1]). *Does there exist a constant $c > 0$ such that every set of n points in the plane in general position contains a crossing family of size at least cn?*

There have been several results on crossing families over the past several decades [7, 11, 13]. Very recently, Pach, Rudin, and Tardos showed that any set of n points in general position in the plane determines $n^{1-o(1)}$ pairwise crossing segments. More precisely, they proved the following theorem.

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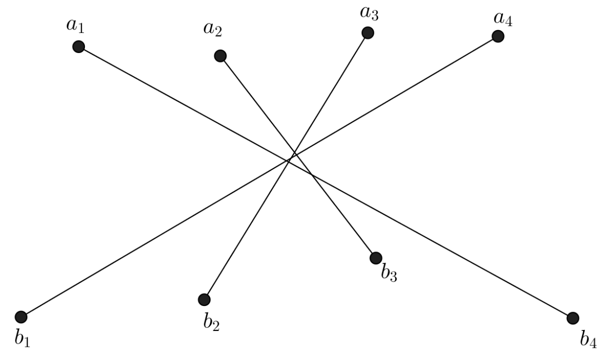

Figure 1: Two mutually avoiding sets $A = \{a_1, a_2, a_3, a_4\}$ and $B = \{b_1, b_2, b_3, b_4\}$ yield a crossing family of size four.

Theorem 1.2 ([10]). *Any set P of n points in general position in the plane determines at least $n/2^{O(\sqrt{\log n})}$ pairwise crossing segments.*

The result of Aronov et al. on crossing families was actually obtained by finding point sets that are *mutually avoiding*. Let A and B be two disjoint point sets in the plane. We say that A *avoids* B if no line subtended by a pair of points in A intersects the convex hull of B. The sets A and B are *mutually avoiding* if A avoids B and B avoids A. In other words, A and B are mutually avoiding if and only if each point in A "sees" every point in B in the same clockwise order, and vice versa. Hence two mutually avoiding sets A and B, where $|A| = |B| = k$, would yield a crossing family of size k. See Figure 1.

Theorem 1.3 ([1]). *Any set of n points in the plane in general position contains a pair of mutually avoiding sets, each of size at least $\sqrt{n/12}$.*

It was shown by Valtr [14] that this bound is best possible up to a constant factor. In this note, we give a fractional version of Theorem 1.3.

Theorem 1.4. *For every $k > 0$ there is a constant $\varepsilon_k > 0$ such that every sufficiently large point set P in the plane in general position contains $2k$ disjoint subsets $A_1, \ldots, A_k, B_1, \ldots, B_k$, each of size at least $\varepsilon_k|P|$, such that every pair of sets $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_k\}$, with $a_i \in A_i$ and $b_i \in B_i$, are mutually avoiding. Moreover, $\varepsilon_k = \Omega(1/k^4)$.*

As an immediate corollary, we establish the following fractional version of the crossing families theorem.

Theorem 1.5. *For every $k > 0$ there is a constant $\varepsilon_k > 0$ such that every sufficiently large point set P in the plane in general position contains $2k$ subsets $A_1, \ldots, A_k, B_1, \ldots, B_k$, each of size at least $\varepsilon_k|P|$, such that every segment that joins a point from A_i and B_{k+1-i} crosses every segment that joins a point from A_{k+1-i} and B_i, for $1 \leq i \leq k$. Moreover, $\varepsilon_k = \Omega(1/k^4)$.*

Let us remark that if we are not interested in optimizing ε_k in the theorems above, one can combine the well-known same-type lemma due to Barany and Valtr [2] (see section 3.1) with Theorem 1.3 to establish Theorems 1.4 and 1.5 with $\varepsilon_k = 2^{-O(k^4)}$. Hence, the main advantage in the theorems above is that ε_k decays only polynomially in k. We will however, use this approach in higher dimensions with a more refined same-type lemma.

Higher dimensions. Mutually avoiding sets in $\mathbb{R}^d$ are defined similarly. A point set P in $\mathbb{R}^d$ is in *general position* if no $d+1$ members of P lie on a common hyperplane. Given two point sets A and B in $\mathbb{R}^d$, we say that A *avoids* B if no hyperplane generated by a d-tuple in A intersects the convex hull of B. The sets A and B are *mutually avoiding* if A avoids B and B avoids A. Aronov et al. proved the following.

Theorem 1.6 ([1]). *For fixed $d \geq 3$, any set of n points in $\mathbb{R}^d$ in general position contains a pair of mutually avoiding subsets each of size $\Omega_d(n^{1/(d^2-d+1)})$.*

In the other direction, Valtr showed in [14] that by taking a $k \times \cdots \times k$ grid, where $k = \lfloor n^{1/d} \rfloor$, and slightly perturbing the n points so that the resulting set is in general position, one obtains a point set that does not contain mutually avoiding sets of size $cn^{1-1/d}$, where $c = c(d)$.

Our next result is a fractional version of Theorem 1.6.

Theorem 1.7. *For $d \geq 3$ and $k \geq 2$, there is a constant $\varepsilon_{d,k}$, such that every sufficiently large point set P in $\mathbb{R}^d$ in general position contains $2k$ subsets $A_1, \ldots, A_k, B_1, \ldots, B_k$, each of size at least $\varepsilon_k |P|$, such that every pair of sets $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_k\}$, with $a_i \in A_i$ and $b_i \in B_i$, are mutually avoiding. Moreover, $\varepsilon_{d,k} = 1/k^{c_d}$ where $c_d > 0$ depends only on d.*

Similar to Theorem 1.4, $\varepsilon_{d,k}$ in Theorem 1.7 also decays only polynomially in k for fixed $d \geq 3$. However, c_d does have a rather bad dependency on d, $c_d \approx 2^{O(d)}$.

Finally, we establish a result on crossing families in higher dimensions which was also observed by Aronov et al. in [1].

2 Proof of Theorem 1.4

Proof. In this section we give the proof of Theorem 1.4 which closely follows an argument of Pór and Valtr in [12]. Let $k > 2$ and let P be a set of n points in the plane in general position where $n > (1500k)^4$. It follows from Theorem 1.2 that among any $12(40k+1)^2$ points P, it is always possible to find two mutually avoiding sets $A \subseteq P$ and $B \subseteq P$ each of size at least $40k+1$. It follows that P contains at least

$$\frac{\binom{n}{12(40k+1)^2}}{\binom{n-(80k+2)}{12(40k+1)^2-(80k+2)}} = \frac{\binom{n}{80k+2}}{\binom{12(40k+1)^2}{80k+2}} \tag{2.1}$$

pairs of mutually avoiding sets, each set of size $40k+1$. Note that (2.1) follows from the equality

$$\frac{\binom{m}{a}}{\binom{m-b}{a-b}} = \frac{\binom{m}{b}}{\binom{a}{b}},$$

for positive integers m, a, b where $1 \leq b \leq a \leq m$.

Let A and B be a pair of mutually avoiding sets each of size $40k+1$. For $b \in B$, label the points in A with $a_1, \ldots, a_{40k+1}$ in radial clockwise order with respect to b. Likewise, for $a \in A$, label the points in B with $b_1, \ldots, b_{40k+1}$ in radial counterclockwise order with respect to a. We say that the pair (A', B') *supports* the pair (A, B) if $A' = \{a_i \in A; i \equiv 1 \bmod 4\}$ and $B' = \{b_i \in B; i \equiv 1 \bmod 4\}$. Clearly, $|A'| = |B'| = 10k+1$.

Since P has at most $\binom{n}{10k+1}^2$ pairs of disjoint subsets with size $10k+1$ each, there is a pair of subsets (A', B') such that $A', B' \subset P, |A'| = |B'| = 10k+1$, and (A', B') supports at least

$$\frac{\binom{n}{80k+2}}{\binom{12(40k+1)^2}{80k+2}\binom{n}{10k+1}^2} > \frac{\left(\frac{n}{80k+2}\right)^{80k+2}}{\left(\frac{12(40k+1)^2 e}{80k+2}\right)^{80k+2}\left(\frac{ne}{10k+1}\right)^{20k+2}}$$
$$> \frac{n^{60k}}{e^{100k+4}12^{80k+2}(50k)^{141k}}$$
$$> \frac{n^{60k}}{(430k)^{141k}}$$

mutually avoiding pairs (A, B) in P, where $|A| = |B| = 40k+1$. Notice that for the first inequality, we use the inequality $\left(\frac{m}{r}\right)^r < \binom{m}{r} < \left(\frac{me}{r}\right)^r$, where $1 < r < m$. To see why the second inequality holds, we claim that

$$\frac{(10k+1)^{20k+2}}{(40k+1)^{160k+4}} > \frac{1}{(50k)^{141k}} \quad \text{as long as } k > 2.$$

To prove the claim, we need to show that

$$(50k)^{141k} > (40k+1)^{140k+2}\left(\frac{40k+1}{10k+1}\right)^{20k+2}.$$

Since $k > 2$, $(40k+1)^{140k+2}\left(\frac{40k+1}{10k+1}\right)^{20k+2} < (40k+1)^{141k}\left(\frac{40k+1}{10k+1}\right)^{21k}$. Therefore, it is enough to show

$$(50k)^{141}(10k+1)^{21} > (40k+1)^{162}.$$

It is easy to check that $50^{141}10^{21} > (40.5)^{162}$ (since $k > 2,\ 40k+1 < 40.5k$) and this completes the proof of the claim. For the last inequality, it is easy to observe that $e^{100k+4}12^{80k+2}(50)^{141k} < (430)^{141k}$, for $k > 2$. Note that

$$e^{100k} < \left(\frac{43}{5}\right)^{46.5k} \quad \text{and} \quad 12^{80k} < \left(\frac{43}{5}\right)^{92.5k}.$$

Therefore,

$$e^{100k}12^{80k}12^2e^4 < \left(\frac{43}{5}\right)^{46.5k}\left(\frac{43}{5}\right)^{92.5k}\left(\frac{43}{5}\right)^{5} < \left(\frac{43}{5}\right)^{141k}.$$

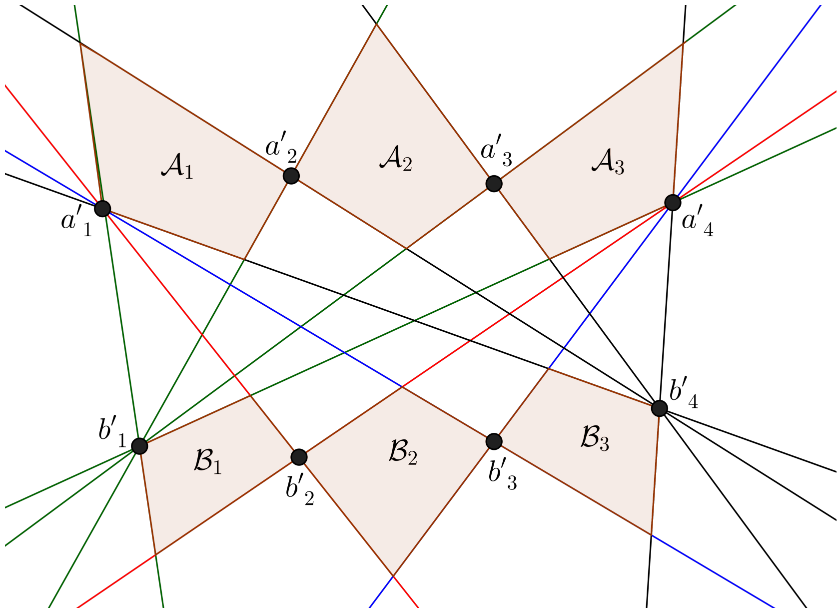

Figure 2: The regions $\mathcal{A}_i$ and $\mathcal{B}_i$ defined by support $A' = \{a'_1, a'_2, a'_3, a'_4\}$ and $B' = \{b'_1, b'_2, b'_3, b'_4\}$. Let us remark that $4 \neq 10k + 1$ for $k \in \mathbb{Z}$. The purpose of this figure is to give some intuition on how the regions A_i and B_i are formed.

Set $A' = \{a'_1, \dots, a'_{10k+1}\}$ and $B' = \{b'_1, \dots, b'_{10k+1}\}$. For any two consecutive points $a'_i, a'_{i+1} \in A', 1 \leq i \leq 10k$, consider the region $\mathcal{A}_i$ produced by the intersection of regions bounded by the lines $b'_1a'_i, b'_1a'_{i+1}$ and $b'_{10k}a'_i, b'_{10k}a'_{i+1}$. Similarly, we define the region $\mathcal{B}_i$ produced by the intersection of regions bounded by the lines $a'_1b'_i, a'_1b'_{i+1}$ and $a'_{10k}b'_i, a'_{10k}b'_{i+1}$ for $1 \leq i \leq 10k$. Therefore, we have $20k$ regions $\mathcal{A}_1, \dots, \mathcal{A}_{10k}, \mathcal{B}_1, \dots, \mathcal{B}_{10k}$. See Figure 2.

Observation 2.1. Let A and B be a pair of mutually avoiding sets each of size $40k + 1$. If (A', B') supports (A, B), where $A' = \{a'_1, \dots, a'_{10k+1}\}$ and $B' = \{b'_1, \dots, b'_{10k+1}\}$, then $A = A' \cup A_1 \cup \dots \cup A_{10k}$ and $B = B' \cup B_1 \cup \dots \cup B_{10k}$, where $|A_i| = |B_i| = 3$ for all $1 \leq i \leq 10k$, and A_i lies in region $\mathcal{A}_i$ and B_i lies in region $\mathcal{B}_i$.

For $i = 1, \dots, 10k$, let α_i, respectively β_i, denote the number of points of P lying in the interior of $\mathcal{A}_i$, respectively $\mathcal{B}_i$. It follows from Observation 2.1 that (A', B') supports at most $\prod_{i=1}^{10k} \binom{\alpha_i}{3} \prod_{i=1}^{10k} \binom{\beta_i}{3}$ pairs of mutually avoiding sets (A, B), each of size $40k + 1$. Therefore,

$$\frac{n^{60k}}{(430k)^{141k}} \leq \prod_{i=1}^{10k} \binom{\alpha_i}{3} \prod_{i=1}^{10k} \binom{\beta_i}{3} \leq \prod_{i=1}^{10k} (\alpha_i \beta_i)^3.$$

Without loss of generality, let us relabel the regions $\mathcal{A}_1, \dots, \mathcal{A}_{10k}, \mathcal{B}_1, \dots, \mathcal{B}_{10k}$ so that $\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_{10k}$ and $\beta_1 \leq \beta_2 \leq \dots \leq \beta_{10k}$.

Claim 2.2. *There exists an i such that $1 \leq i \leq 9k$, and $\alpha_i, \beta_i \geq \frac{n}{(1320k)^4}$.*

Proof. For the sake of contradiction, suppose for each i, $1 \leq i \leq 9k$, we have $\alpha_i < \frac{n}{(1320k)^4}$. Therefore,

$$\begin{aligned}
\frac{n^{20k}}{(430k)^{47k}} \leq \prod_{i=1}^{10k} \alpha_i \beta_i = \prod_{i=1}^{9k} \alpha_i \left(\prod_{i=9k+1}^{10k} \alpha_i \prod_{i=1}^{10k} \beta_i \right) \\
&\leq \left(\frac{n}{(1320k)^4} \right)^{9k} \left(\frac{\Sigma_{i=9k+1}^{10k} \alpha_i + \Sigma_{i=1}^{10k} \beta_i}{11k} \right)^{11k} \\
&< \left(\frac{n}{(1320k)^4} \right)^{9k} \left(\frac{n}{11k} \right)^{11k} \\
&= \frac{n^{20k}}{(1320k)^{36k}(11k)^{11k}}.
\end{aligned}$$

Hence, we have

$$\frac{n^{20k}}{(430k)^{47k}} < \frac{n^{20k}}{(1320k)^{36k}(11k)^{11k}}. \tag{2.2}$$

After simplifying (2.2), we get $\frac{1320^{36} 11^{11}}{430^{47}} < 1$ which is a contradiction as $\frac{1320^{36} 11^{11}}{430^{47}} \approx 1.054$. Thus, there exists an i, $1 \leq i \leq 9k$, with $\alpha_i \geq \frac{n}{(1320k)^4}$. With a similar calculation, there exists an i, $1 \leq i \leq 9k$ with $\beta_i \geq \frac{n}{(1320k)^4}$. □

By setting $A_i^* = P \cap \mathcal{A}_{9k+i}$ and $B_i^* = P \cap \mathcal{B}_{9k+i}$, for $1 \leq i \leq k$, we have $2k$ subsets $A_1^*, \ldots, A_k^*, B_1^*, \ldots, B_k^*$, each of size at least $\frac{n}{(1320k)^4}$, such that every pair of subsets $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_k\}$, where $a_i \in A_i^*$ and $b_i \in B_i^*$, is mutually avoiding. □

3 Mutually avoiding sets in higher dimensions

In this section we will prove Theorem 1.7. Let $P = (p_1, \ldots, p_n)$ be an n-element point sequence in $\mathbb{R}^d$ in general position. The *order type* of P is the mapping $\chi : \binom{P}{d+1} \to \{+1, -1\}$ (positive orientation, negative orientation), assigning each $(d+1)$-tuple of P its orientation. More precisely, by setting $p_i = (a_{i,1}, a_{i,2}, \ldots, a_{i,d}) \in \mathbb{R}^d$,

$$\chi(\{p_{i_1}, p_{i_2}, \ldots, p_{i_{d+1}}\}) = sgn\ det \begin{bmatrix} 1 & 1 & \ldots & 1 \\ a_{i_1,1} & a_{i_2,1} & \ldots & a_{i_{d+1},1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_1,d} & a_{i_2,d} & \ldots & a_{i_{d+1},d} \end{bmatrix},$$

where $i_1 < i_2 < \cdots < i_{d+1}$.
Hence two point sequences $P = (p_1, \ldots, p_n)$ and $Q = (q_1, \ldots, q_n)$ have the same order-type if and only if they are "combinatorially equivalent." See [6] and [9] for more background on order-types.

Given k disjoint subsets $P_1, \ldots, P_k \subset P$, a *transversal* of $(P_1, \ldots, P_k)$ is any k-element sequence $(p_1, \ldots, p_k)$ such that $p_i \in P_i$ for all i. We say that the k-tuple $(P_1, \ldots, P_k)$ has *same-type transversals* if all of its transversals have the same order-type. In 1998, Bárány and Valtr proved the following same-type lemma.

Lemma 3.1 ([2])**.** *Let $P = (p_1, \ldots, p_n)$ be an n-element point sequence in $\mathbb{R}^d$ in general position. Then for $k > 0$, there is an $\varepsilon = \varepsilon(d, k)$, such that one can find disjoint subsets $P_1, \ldots, P_k \subset P$ such that $(P_1, \ldots, P_k)$ has same-type transversals and $|P_i| \geq \varepsilon n$.*

Their proof shows that $\varepsilon = 2^{-O(k^{d-1})}$. This was later improved by Fox, Pach, and Suk [5] who showed that Lemma 3.1 holds with $\varepsilon = 2^{-O(d^3 k \log k)}$. We will use the following result, which was communicated to us by Jacob Fox, which shows that Lemma 3.1 holds with ε decaying only polynomially in k for fixed $d \geq 3$.

Lemma 3.2. *Lemma 3.1 holds for $\varepsilon = k^{-c_d}$, where c_d depends only on d.*

The proof of Lemma 3.2 is a simple application of the following regularity lemma due to Fox, Pach, and Suk. A partition on a finite set P is called *equitable* if any two parts differ in size by at most one.

Lemma 3.3 (Theorem 1.3 in [5])**.** *For $d > 0$, there is a constant $c = c(d)$ such that the following holds. For any $\varepsilon > 0$ and for any n-element point sequence $P = (p_1, \ldots, p_n)$ in $\mathbb{R}^d$, there is an equitable partition $P = P_1 \cup \cdots \cup P_K$, with $1/\varepsilon < K < (1/\varepsilon)^c$, such that all but at most $\varepsilon\binom{K}{d+1}$ $(d+1)$-tuples of parts $(P_{i_1}, \ldots, P_{i_{d+1}})$ have same-type transversals.*

Let us note that $K > 1/\varepsilon$ follows by first arbitrarily partitioning P into $\lceil 1/\varepsilon \rceil$ parts, such that any two parts differ in size by at most one, and then following the proof of Theorem 1.3 in [5].

The next lemma we will use is Turán's Theorem for hypergraphs. Given an r-uniform hypergraph $\mathcal{H}$, let $ex(n, \mathcal{H})$ denote the maximum number of edges in any $\mathcal{H}$-free r-uniform hypergraph on n vertices.

Lemma 3.4 (de Caen [4])**.** *Let K_k^r denote the complete r-uniform hypergraph on k vertices. Then*

$$ex(n, K_k^r) \leq \left(1 - \frac{1}{\binom{k-1}{r-1}} + o(1)\right)\binom{n}{r}.$$

Proof of Lemma 3.2. Let $P = (p_1, \ldots, p_n)$ be an n-element point sequence in $\mathbb{R}^d$ in general position. Set $\varepsilon = 1/(2k)^d$, and apply Lemma 3.3 to P with parameter ε to obtain the equitable partition $P = P_1 \cup \cdots \cup P_K$ with the desired properties. Hence $|P_i| \geq n/(2k)^{d \cdot c}$, where c is defined in Lemma 3.3. Since all but at most $\varepsilon\binom{K}{d+1}$ $(d+1)$-tuples of parts $(P_{i_1}, \ldots, P_{i_{d+1}})$ have same-type transversals, we can apply Lemma 3.4 to obtain k parts $P'_1, \ldots, P'_k \in \{P_1, \ldots, P_K\}$ such that all $(d+1)$-tuples $(P'_{i_1}, \ldots, P'_{i_{d+1}})$ in $\{P'_1, \ldots, P'_k\}$ have same-type transversals. $\square$

Proof of Theorem 1.7. Let $k > 0$ and let P be an n-element point set in $\mathbb{R}^d$ in general position. We will order the elements of $P = \{p_1, \ldots, p_n\}$ by increasing first coordinate, breaking ties arbitrarily. Let $c' = c'(d)$ be a sufficiently large constant that will be determined later. We apply Lemma 3.2 to P with parameter $k' = \lceil k^{c'} \rceil$ to obtain subsets $P_1, \ldots, P_{k'} \subset P$ such that $|P_i| \geq k^{-c_d c'} n$, where c_d is defined in Lemma 3.2, such that all $(d+1)$-tuples $(P_{i_1}, \ldots, P_{i_{d+1}})$ have same-type transversals. Let P' be a k'-element subset obtained by selecting one point from each subset P_i. By applying Theorem 1.6 to P', we obtain subsets $A, B \subset P'$ such that A and B are mutually avoiding, and $|A|, |B| \geq \Omega((k')^{1/(d^2-d+1)})$. By choosing $c' = c'(d)$ sufficiently large, we have $|A|, |B| \geq k$. Let $\{a_1, \ldots, a_k\} \subset A$ and $\{b_1, \ldots, b_k\} \subset B$. Then the subsets $A_1, \ldots, A_k, B_1, \ldots, B_k \in \{P_1, \ldots, P_{k'}\}$, where $a_i \in A_i$ and $b_i \in B_i$, are as required in the theorem. $\square$

3.1 Crossing Families in Higher Dimensions

Let P be an n-element point set in $\mathbb{R}^d$ in general position. A $(d-1)$-*simplex* in P is a $(d-1)$-dimensional simplex generated by taking the convex hull of d points in P. We say that two $(d-1)$-simplices *strongly cross* in P if their interiors intersect and they do not share a common vertex. A *crossing family* of size k in P is a set of k pairwise strongly crossing $(d-1)$-simplices in P.

In [1], Aronov et al. stated that Theorem 1.6 implies that every point set P in $\mathbb{R}^d$ in general position contains a polynomial-sized crossing family, that is, a collection of $(d-1)$-simplices in P such that any two strongly cross. Since they omitted the details, below we provide the construction of a crossing family using mutually avoiding sets in $\mathbb{R}^d$.

Corollary 3.5. *Let $d \geq 2$ and let P be a set of n points in $\mathbb{R}^d$ in general position. Then P contains a crossing family of size $\Omega(\sqrt{n})$ for $d = 2$, and of size $\Omega_d(n^{\frac{1}{2\prod_{i=3}^{d}(i^2-i+1)}})$ for $d \geq 3$.*

Proof. We proceed by induction on d. The base case $d = 2$ follows from Theorem 1.3: a pair of mutually avoiding sets A and B in the plane, each of size $\Omega(\sqrt{n})$, gives rise to a crossing family of size $\Omega(\sqrt{n})$. For the inductive step, assume the statement holds for all $d' < d$.

Let P be a set of n points in $\mathbb{R}^d$ in general position. By Theorem 1.6, there is a pair of mutually avoiding sets A and B such that $|A| = |B| = k = \Omega_d(n^{\frac{1}{d^2-d+1}})$. Let $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_k\}$. Since $conv(A) \cap conv(B) = \emptyset$, by the separation theorem (see Theorem 1.2.4 in [9]), there is a hyperplane $\mathcal{H}$ such that A lies in one of the closed half-spaces determined by $\mathcal{H}$, and B lies in the opposite closed half-space.

For each $a_i \in A$, let $a_i b$ be the line generated by points a_i and $b \in B$. Then set $B_i = \{a_i b \cap \mathcal{H} : b \in B\}$. Since P is in general position, B_i is also in general position in $\mathcal{H}$ for each i. Moreover, since A and B are mutually avoiding, B_i has the same order-type as B_j for every $i \neq j$. Indeed, for any d-tuple $b_{i_1}, b_{i_2}, \ldots, b_{i_d} \in B$, every point in A lies on the same side of the hyperplane generated by $b_{i_1}, b_{i_2}, \ldots, b_{i_d}$. Hence the orientation of the corresponding d-tuple in $B_i \subset \mathcal{H}$ will be the same as the orientation of the corresponding d-tuple in $B_j \subset \mathcal{H}$ for $i \neq j$. Therefore, let us just consider $B_1 \subset \mathcal{H}$. By the induction hypothesis, there exists a crossing family of $(d-2)$-simplices of size

$$k' = \Omega_d\left(k^{\frac{1}{2\prod_{i=3}^{d-1}(i^2-i+1)}}\right) = \Omega_d\left(n^{\frac{1}{2\prod_{i=3}^{d}(i^2-i+1)}}\right),$$

in $B_1 \subset \mathcal{H}$. Let $\mathcal{S} = \{\mathcal{S}_1, \ldots, \mathcal{S}_{k'}\}$ be our set of pairwise crossing $(d-2)$-simplices in $B_1 \subset \mathcal{H}$ and let $\mathcal{S}' = \{\mathcal{S}'_1, \ldots, \mathcal{S}'_{k'}\}$ be the corresponding $(d-2)$-simplices in B (which may or may not intersect).

Set $\mathcal{S}_i^* = \text{conv}(a_i \cup \mathcal{S}_i')$. Then $\mathcal{S}_1^*, \ldots, \mathcal{S}_{k'}^*$ is a set of k' pairwise crossing $(d-1)$-simplices in $\mathbb{R}^d$. Indeed, consider $\mathcal{S}_i^*$ and $\mathcal{S}_j^*$. If $\mathcal{S}_i' \cap \mathcal{S}_j' \neq \emptyset$, then we are done. Otherwise, we would have $\mathcal{S}_j' \cap \mathcal{S}_i^* \neq \emptyset$ or $\mathcal{S}_i' \cap \mathcal{S}_j^* \neq \emptyset$ since B_i and B_j have the same order type and $\mathcal{S}_i \cap \mathcal{S}_j \neq \emptyset$. More precisely, let r_i be a ray from a_i through an intersection point of $\mathcal{S}_i$ and $\mathcal{S}_j$. The ray r_i intersects both $\mathcal{S}_i'$ and $\mathcal{S}_j'$ by the definition of $\mathcal{S}_i$ and $\mathcal{S}_j$. Without loss of generality assume r_i intersects $\mathcal{S}_i$ first. It follows that $\mathcal{S}_i' \cap \mathcal{S}_j^* \neq \emptyset$. □

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