Calc. Var. (2019) 58:182
https://doi.org/10.1007/s00526-019-1635-8

Calculus of Variations

General least gradient problems with obstacle

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Received: 29 April 2019 / Accepted: 31 August 2019 / Published online: 3 October 2019

Abstract
We study existence, structure, uniqueness and regularity of solutions of the obstacle problem

$$\inf_{u\in BV_f(\Omega)} \int_\Omega \phi(x, Du),$$

where $BV_f(\Omega) = \{u \in BV(\mathbb{R}^n) : u \geq \psi \text{ in } \Omega \text{ and } u|_{\partial\Omega} = f|_{\partial\Omega}\}$, $f \in W_0^{1,1}(\mathbb{R}^n)$, ψ is the obstacle, and $\phi(x, \xi)$ is a convex, continuous and homogeneous function of degree one with respect to the ξ variable. We show that every minimizer of this problem is also a minimizer of the least gradient problem

$$\inf_{u\in \mathcal{A}_f(\Omega)} \int_{\mathbb{R}^n} \phi(x, Du),$$

where $\mathcal{A}_f(\Omega) = \{u \in BV(\Omega) : u \geq \psi, \text{ and } u = f \text{ in } \Omega^c\}$. Moreover, there exists a vector field T with $\nabla \cdot T \leq 0$ in Ω which determines the structure of all minimizers of these two problems, and T is divergence free on $\{x \in \Omega : u(x) > \psi(x)\}$ for any minimizer u. We also present uniqueness and regularity results that are based on maximum principles for minimal surfaces. Since minimizers of the least gradient problems with obstacle do not hit small enough obstacles, the results presented in this paper extend several results in the literature about least gradient problems without obstacle.

Mathematics Subject Classification 35B65 · 35R35 · 49N60

Communicated by P. Rabinowitz.

Amir Moradifam is supported by NSF Grant DMS-1715850.

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1 Introduction

Least gradient problems are closely related to the study of minimal surfaces and appeared in the pioneering work of Bombieri et al. [4] (see also [11] for stability results on functions of least gradient). They also arise as the limiting case of p-harmonic functions as $p \to 1$ [8], and due to their important applications in conductivity imaging, such problems have received an extensive attention in the past decade (see [6,7,10,12–18,24]). In [21], the authors studied exietence, uniqueness, and regularity of functions of least gradient, and the results were later extended to least gradient problems with obstacle in [23]. Least gradient problems with constraint have been carefully studied in [9,22]. In this paper we investigate existence, structure, uniqueness, and regularity of minimizers of general least gradient problems with obstacle (see the problem (3) below).

Let Ω be a bounded open set in $\mathbb{R}^n$ with Lipschitz boundary and $\phi : \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuous function satisfying the following conditions:

(C1) There exists $\alpha > 0$ such that $\alpha|\xi| \leq \phi(x, \xi) \leq \alpha^{-1}|\xi|$ for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$.
(C2) $\xi \mapsto \phi(x, \xi)$ is a norm for every x.

For our results concerning the regularity of solutions we will also assume the following three additional assumptions

(C3) $\phi \in W^{2,\infty}_{\text{loc}}$ away from $\{\xi = 0\}$, and there exists $C > 0$ such that

$$\phi_{\xi_i\xi_j}(x, \xi)p^i p^j \geq C|p'|^2,$$

for all $\xi \in S^{n-1}$ and $p \in \mathbb{R}^n$, where $p' := p - (p \cdot \xi)\xi$.

(C4) ϕ and $D_\xi\phi$ are $W^{2,\infty}$ away from $\{\xi = 0\}$, and there are positive constants ρ and λ such that

$$\begin{aligned}\phi(x, \xi) + |D_\xi\phi(x, \xi)| + |D^2_\xi\phi(x, \xi)| + |D^3_\xi\phi(x, \xi)| + \rho|D_x D_\xi\phi(x, \xi)| \\ + \rho|D_x D^2_\xi\phi(x, \xi)| + \rho^2|D^2_x D_\xi\phi(x, \xi)| \leq \lambda, \qquad \text{for all } x \in \Omega, \xi \in S^{n-1}.\end{aligned}$$

(C5) For the result of regularity we need to assume that the integrand $\phi(x, \xi) = \phi(\xi)$ is independent of x.

It is elementary to verify that if $\phi : \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfies (C1)–(C4), then for every $p, q \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ we have

$$\phi_\xi(x, \lambda p) = \phi_\xi(x, p), \quad \text{and} \quad p \cdot \phi_\xi(x, p) = \phi(x, p). \tag{1}$$

For $u \in BV_{loc}(\mathbb{R}^n)$, let $\phi(x, Du)$ denote the measure defined by

$$\int_A \phi(x, Du) = \int_A \phi(x, \nu^u(x))|Du| \quad \text{for any bounded Borel set } A,$$

where $|Du|$ is the total variation measure associated to the vector-valued measure Du, and ν^u is the Radon–Nikodym derivative $\nu^u(x) = \frac{d\,Du}{d\,|Du|}$. Basic facts about BV functions imply that if U is an open set, then

$$\int_U \phi(x, Du) = \sup\left\{\int_U u\nabla \cdot Y dx : Y \in C^\infty_c(U; \mathbb{R}^n),\ \sup \phi^0(x, Y(x)) \leq 1\right\}, \tag{2}$$

where $\phi^0(x, \cdot)$ denotes the norm on $\mathbb{R}^n$ dual to $\phi(x, \cdot)$, defined by

$$\phi^0(x, \xi) := \sup\{\xi \cdot p : \phi(x, p) \leq 1\},$$

(see [2,7]). For $u \in BV(\Omega)$, $\int_\Omega \phi(x, Du)$ is called the ϕ-total variation of u in Ω. Also, if A, E are subsets of $\mathbb{R}^n$, with A Borel and E having finite perimeter, then we shall write $P_\phi(E; A)$ to denote the ϕ-perimeter of E in A, defined by

$$P_\phi(E; A) := \int_A \phi(x, D\chi_E),$$

where χ_E is the characteristic function of E. We will also write $P_\phi(E)$ to denote $P_\phi(E; \mathbb{R}^n)$. We shall need the following lemma.

Lemma 1.1 (Lemma 2.2 in [7]) *Let $A \subset \mathbb{R}^n$ be a Borel set and E_1, $E_2 \subset \mathbb{R}^n$ be of locally finite perimeter with respect ϕ. Then*

$$P_\phi(E_1 \cup E_2; A) + P_\phi(E_1 \cap E_2; A) \leq P_\phi(E_1; A) + P_\phi(E_2; A).$$

Definition 1.2 We say that a function $u \in BV(\mathbb{R}^n)$ is a ϕ-total variation minimizing in a set $\Omega \subset \mathbb{R}^n$ if

$$\int_{\mathbb{R}^n} \phi(x, Du) \leq \int_{\mathbb{R}^n} \phi(x, Dv) \text{ for all } v \in BV(\mathbb{R}^n) \text{ such that } u = v \text{ a.e. in } \Omega^c.$$

Similarly, we say that $E \subset \mathbb{R}^n$ of finite perimeter is ϕ-area minimizing in Ω if

$$P_\phi(E) \leq P_\phi(F) \text{ for all } F \subset \mathbb{R}^n \text{ such that } F \cap \Omega^c = E \cap \Omega^c \text{ a.e..}$$

Moreover, $E \subset \mathbb{R}^n$ is called ϕ-super (sub) area minimizing in Ω, if

$$P_\phi(E) \leq P_\phi(E \cup F) \text{ (respectively } P_\phi(E) \leq P_\phi(E \cap F))$$

for all $F \subset \mathbb{R}^n$ such that $F \cap \Omega^c = E \cap \Omega^c$ almost everywhere, i.e. $(F \cap \Omega^c)\Delta(E \cap \Omega^c)$ has zero Lebesgue measure.

Let $f \in BV(\mathbb{R}^n)$ and $\psi \in W^{1,1}(\Omega)$, and consider the obstacle least gradient problem

$$\inf_{u \in BV_f} \int_\Omega \phi(x, Du), \tag{3}$$

where

$$BV_f(\Omega) := \{u \in BV(\Omega) : u \,|_{\partial\Omega} = f \text{ and } u(x) \geq \psi(x) \text{ for a.e. } x \in \Omega\}.$$

In general the problem (3) may not have a minimizer (see [7,10,21]). However the relaxed problem

$$\min_{u \in \mathcal{A}_f} \left(\int_\Omega \phi(x, Du) + \int_{\partial\Omega} \varphi(x, \nu_\Omega)|u - f| \right), \tag{4}$$

always has a solution, where $\mathcal{A}_f = \{u \in BV(\mathbb{R}^n) : u \geq \psi, \text{ and } u = f \text{ in } \Omega^c\}$, and ν_Ω is the outer pointing unit normal vector on $\partial\Omega$. Indeed let $\{v_n\}_{n=1}^\infty$ be a minimizing sequence for

$$F(v) := \int_{\mathbb{R}^n} \varphi(x, Dv). \tag{5}$$

Since $BV(\mathbb{R}^n) \hookrightarrow \mathcal{L}^1_{loc}$, F is coercive in $BV(\mathbb{R}^n)$ (a consequence of $(C1)$) and weakly lower semicontinuous (see [7] for more details), it follows from standard arguments that $\{v_n\}_{n=1}^\infty$ has a subsequence converging strongly in $\mathcal{L}^1_{loc}$ to a function $u \in \mathcal{A}_f$ with

$$\int_{\mathbb{R}^n} \varphi(x, Du) \leq \inf_{v \in BV_f(\Omega)} \int_{\mathbb{R}^n} \varphi(x, Dv),$$

and hence v is also a minimizer of (4). However, in general, the trace $u|_{\partial\Omega}$ on $\partial\Omega$ may not be equal to f, leading to possible nonexistence for the problem (3). In addition, we shall prove the following result.

Remark 1.3 Since $u|_{\partial\Omega} = f$ for every $u \in BV_f(\Omega)$, the compatibility condition $f \geq \psi$ on $\partial\Omega$ must be satisfied. Every $f \in \mathcal{L}^1(\partial\Omega)$ can be extended to a function in $BV(\mathbb{R}^n)$ (denoted by f again) with $f \geq \psi$ in Ω, and throughout the paper we shall naturally assume that $f \geq \psi$ in $\bar{\Omega}$.

Proposition 1.4 *Let Ω be a bounded open set in $\mathbb{R}^n$ with Lipschitz boundary, $f \in BV(\mathbb{R}^n)$ and $\psi \in W^{1,1}(\Omega)$ with $f \geq \psi$ in Ω, and $\phi : \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuous function satisfying (C1)–(C2). Then (4) has a solution and*

$$\inf_{u\in BV_f} \int_\Omega \phi(x, Du) = \min_{u\in \mathcal{A}_f} \left(\int_\Omega \phi(x, Du) + \int_{\partial\Omega} \varphi(x, \nu_\Omega)|u - f| \right).$$

In particular, every minimizer of (3) is also a minimizer of (4).

Indeed in order to prove existence of solutions to (3) we need a condition on Ω which is defined as follows. Remind that for a measurable subset E of $\mathbb{R}^n$, we define

$$E^{(1)} := \left\{ x \in \mathbb{R}^n : \lim_{r\to 0} \frac{\mathcal{H}^n(B(r, x) \cap E)}{\mathcal{H}^n(B(r))} = 1 \right\}. \tag{6}$$

Definition 1.5 Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $\phi : \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is continuous function that satisfies (C1)–(C2). We say that Ω satisfies the barrier condition if for $x_0 \in \partial\Omega$ and $\epsilon > 0$ sufficiently small, if V minimizes $P_\phi(\cdot\,; \mathbb{R}^n)$ in

$$\{W \subset \Omega : W \setminus B(\epsilon, x_0) = \Omega \setminus B(\epsilon, x_0)\},$$

then

$$\partial V^{(1)} \cap \partial\Omega \cap B(\epsilon, x_0) = \emptyset,$$

where $V^{(1)}$ is defined as in (6).

Remark 1.6 Intuitively, if Ω satisfies the barrier condition, then at every point on $\partial\Omega$ one can decrease the perimeter of $\partial\Omega$ by pushing the boundary inwards. In [7], a convenient interpretation of the barrier condition, when $\partial\Omega$ is sufficiently smooth, is provided:

$$-\sum_{i=1}^{n} \partial_{x_i} \phi_{\xi_i}(x, Dd(x)) > 0, \quad \text{on a dense subset of } \partial\Omega, \tag{7}$$

where $d(\cdot)$ is the signed distance to $\partial\Omega$ by

$$d(x) := \begin{cases} dist(x, \partial\Omega), & \text{if } x \in \Omega, \\ -dist(x, \partial\Omega), & \text{if not.} \end{cases}$$

We will show that if Ω satisfies the barrier condition, then every solution of (4) is also a solution of (3).

Theorem 1.7 *Suppose that $\phi : \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is a continuous function that satisfies (C1)–(C2) in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $f \in C(\partial\Omega)$ with $f \geq \psi$, and $\psi \in W^{1,1}(\Omega)$. If Ω satisfies the barrier condition with respect to ϕ, then every solution of (4) is also a solution of (3). In particular, (3) has a solution.*

We only require the barrier condition for our existence result, Theorem 1.7. Indeed our regularity and uniqueness do not assume this condition which is contrast with the results in [21].

We shall also prove that there exists a fixed vector field T that determines the structure of level sets of the minimizers of (3) and (4).

Theorem 1.8 *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $\phi : \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is continuous function that satisfies (C1)–(C2), and $f \in W_0^{1,1}(\mathbb{R}^n)$. Then there exists a vector field $T \in (\mathcal{L}^\infty(\Omega))^n$ with $\phi^0(x,T) \leq 1$ a.e. in Ω, and $\nabla \cdot T \leq 0$ such that*

$$\phi\left(x, \frac{Dw}{|Dw|}\right) = T \cdot \frac{Dw}{|Dw|}, \quad |Dw| - a.e. \text{ in } \Omega, \tag{8}$$

$$\phi(x, \nu_\Omega)|f - w| = [T, (f-w)\nu_\Omega], \quad \mathcal{H}^{n-1} - a.e. \text{ in } \partial\Omega \cap \{w > \psi\}, \tag{9}$$

for every minimizer w of (3) or (4). Moreover T is divergence-free in $\{x \in \Omega : w(x) > \psi(x)\}$.

The above result generalizes Theorem 1.2 in [12] and simplifies to the following result in the special case $\varphi(x,\xi) = a(x)|\xi|$.

Corollary 1.9 *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and assume that $a \in C(\bar{\Omega})$ is a non-negative function, and $f \in W_0^{1,1}(\mathbb{R}^n)$. Then there exists a vector field $T \in (\mathcal{L}^\infty(\Omega))^n$ with $|T| \leq a$ a.e. in Ω, and $\nabla \cdot T \leq 0$ such that every minimizer $w \in \mathcal{A}_f$ of the least gradient problem*

$$\inf_{v \in \mathcal{A}_f} \int_\Omega a|Dv|, \tag{10}$$

satisfies

$$T \cdot \frac{Dw}{|Dw|} = |T| = a, \quad |Dw| - a.e. \text{ in } \Omega,$$

$$a|f - w| = [T, (f-w)\nu_\Omega], \quad \mathcal{H}^{n-1} - a.e. \text{ in } \partial\Omega \cap \{w > \psi\}.$$

Moreover T is divergence-free in $\{x \in \Omega : w(x) > \psi(x)\}$.

The above corollary asserts that there exists a vector field T such that for every minimizer w of (10) the vector field $\frac{Dw}{|Dw|}$ is parallel to T, $|Dw|$-a.e. in Ω. Moreover, if T is regular enough so that the trace of T can be represented by a function $T_{tr} \in (\mathcal{L}^\infty(\partial\Omega))^n$, then up to a set with $\mathcal{H}^{n-1}$-measure zero

$$\{x \in \partial\Omega \cap \{w > \psi\} : w|_{\partial\Omega} > f\} \subseteq \{x \in \partial\Omega : T_{tr} \cdot \nu_\Omega = |T_{tr}|\},$$

and similarly

$$\{x \in \partial\Omega \cap \{w > \psi\} : w|_{\partial\Omega} < f\} \subseteq \{x \in \partial\Omega : T_{tr} \cdot \nu_\Omega = -|T_{tr}|\}.$$

In other words $w|_{\partial\Omega} = f$, $\mathcal{H}^{n-1}$-a.e. in

$$\{x \in \partial\Omega \cap \{w > \psi\} : |T_{tr} \cdot \nu_\Omega| < |T_{tr}|\},$$

for every minimizer w of (10). These results extend the second authors results about structure of minimizers of least gradient problems [12] for least gradient problems with obstacle.

We will also prove the following results about the uniqueness and regularity of minimizers of the obstacle least gradient problem (3).

Theorem 1.10 (Comparison principle) *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with connected boundary, and assume $\phi : \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfies (C1)–(C5). Suppose that u_1 and u_2 are solutions of (3) for boundary conditions $f_1, f_2 \in C(\partial\Omega)$ respectively and the obstacle $\psi \in C(\bar{\Omega})$. Then*

$$|u_1 - u_2| \leq \sup_{\partial\Omega} |f_1 - f_2| \qquad \text{a.e. in } \Omega.$$

Moreover,

$$u_2 \geq u_1 \text{ a.e. in } \Omega, \text{ if } f_2 \geq f_1 \text{ on } \partial\Omega. \tag{11}$$

In particular, for every $f \in C(\partial\Omega)$, there is at most one solution for (3).

Theorem 1.11 (Holder regularity) *Suppose that $\phi : \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfies (C1)–(C5), and let Ω be a bounded, open subset of $\mathbb{R}^n$ with C^2 boundary such that the signed distance $d(\cdot)$ to $\partial\Omega$ satisfies (7). Assume $f \in C^{0,\alpha}(\partial\Omega)$, and $\psi \in C^{0,\alpha/2}(\bar{\Omega})$ for some $0 < \alpha \leq 1$. If $u \in BV(\Omega)$ is a solution of (3), then $u \in C^{0,\alpha/2}(\overline{\Omega})$.*

Theorem 1.12 (Lipschitz regularity) *Suppose that $\phi : \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfies (C1)–(C5) and let Ω be a bounded, open subset of $\mathbb{R}^n$ with C^2 boundary which the signed distance $d(\cdot)$ to $\partial\Omega$ satisfies the relation (7). Assume $f \in C^{1,\alpha}(\partial\Omega)$, and $\psi \in C^{0,\frac{1+\alpha}{2}}(\bar{\Omega})$ for some $0 < \alpha \leq 1$. If $u \in BV(\Omega)$ is a solution of (3), then $u \in C^{0,\frac{1+\alpha}{2}}(\overline{\Omega})$.*

Since minimizers of the least gradient problems with obstacle do not hit small enough obstacles, the results in this paper extend and unify several results in the literature about least gradient problems without obstacle. Indeed if

$$\sup_{x\in\Omega} \psi \leq \inf_{x\in\partial\Omega} f,$$

then u is a minimizer of (3) if and only if it is a minimizer of

$$\inf_{\{u\in BV(\Omega):u|_{\partial\Omega}=f\}} \int_\Omega \phi(x, Du).$$

Our results in Sect. 3 are inspired by the second author's work in [7] and use several ideas from this paper. However, analysis of the obstacle problem requires a careful refinement of those ideas. We shall refer to the results in [7] without repeating the details of the arguments.

2 Structure of minimizers

In this section we study the relationship between minimizers of the least gradient problems (3) and (4), and prove several results about existence and structure of minimizers of these problems.

Let ν_Ω denote the outer unit normal vector to $\partial\Omega$. Then for every $V \in (\mathcal{L}^\infty(\Omega))^n$ with $\nabla \cdot V \in \mathcal{L}^n(\Omega)$ there exists a unique function $[V, \nu_\Omega] \in \mathcal{L}^\infty_{\mathcal{H}^{n-1}}(\partial\Omega)$ such that

$$\int_{\partial\Omega} [V, \nu_\Omega] u \, d\mathcal{H}^{n-1} = \int_\Omega u \nabla \cdot V dx + \int_\Omega V \cdot Du dx, \quad u \in C^1(\bar{\Omega}). \tag{12}$$

Moreover, for $u \in BV(\Omega)$ and $V \in (\mathcal{L}^\infty(\Omega))^n$ with $\nabla \cdot V \in \mathcal{L}^n(\Omega)$, the linear functional $u \mapsto (V \cdot Du)$ gives rise to a Radon measure on Ω, and (12) is valid for every $u \in BV(\Omega)$ (see [1,3] for a proof).

We first show that there exists a vector field T that determines the structure of all minimizers of (3) and (4). Next we define the dual of the least gradient problem (3). Let $E : (\mathcal{L}^1(\Omega))^n \to \mathbb{R}$ and $G : W_0^{1,1}(\Omega) \to \mathbb{R}$ be defined as follows

$$E(P) := \int_\Omega \phi(x, P + \nabla f)dx, \quad G(u) = \begin{cases} 0 & u \in \mathcal{K} \\ +\infty & u \notin \mathcal{K}, \end{cases} \tag{13}$$

where

$$\mathcal{K} := \{u \in W_0^{1,1}(\Omega) : u \geq \psi - f\}.$$

Then the problem (3) can be written as

$$(P) \qquad \inf_{u \in W_0^{1,1}(\Omega)} E(Du) + G(u).$$

By Fenchel duality (see Chapter III in [5]) the dual problem is given by

$$(P^*) \qquad \sup_{V \in (\mathcal{L}^\infty(\Omega))^n} \{-E^*(V) - G^*(\nabla \cdot V)\},$$

where E^* and G^* are the Legendre–Fenchel transform of F and G. By Lemma 2.1 in [12] we have

$$E^*(V) = \begin{cases} -\langle Df, V\rangle & \text{if } \phi^0(x, V(x)) \leq 1 \text{ in } \Omega \\ +\infty, & \text{otherwise.} \end{cases}$$

One can also compute $G^* : W^{-1,\infty}(\Omega) \to \mathbb{R}$ as follows.

Lemma 2.1 *Suppose $v = \nabla \cdot V$ for some $V \in (\mathcal{L}^\infty(\Omega))^n$. Then*

$$G^*(v) = \begin{cases} < \infty, & v \in \mathcal{C}^*, \\ +\infty, & v \notin \mathcal{C}^*, \end{cases}$$

where

$$\mathcal{C}^* := \{v \in W^{-1,\infty}(\Omega) : \langle v, u\rangle \leq 0, \textit{for all } 0 \leq u \in W_0^{1,1}(\Omega)\}.$$

Moreover there exists a real valued function $C(V)$ which only depends on V near $\partial\Omega$, i.e.

$$C(V_1) = C(V_2) \ \textit{ if } \ V_1 - V_2 \in (\mathcal{L}_c^\infty(\Omega))^n,$$

and for $v \in C^*$, we have

$$G^*(v) = -\int_\Omega V \cdot D(\psi - f) + C(V). \tag{14}$$

Proof First note that

$$G^*(v) = \sup_{u \in W_0^{1,1}(\Omega)} \big(\langle v, u\rangle - G(u)\big) = \sup_{u \in \mathcal{K}} \langle v, u\rangle.$$

Then if $v \notin \mathcal{C}^*$, there exists $0 \leq u_0 \in W_0^{1,1}(\Omega)$ such that $\langle v, u_0\rangle > 0$. Hence for any $u \in \mathcal{K}$ and $\lambda > 0$, we have $u + \lambda u_0 \in \mathcal{K}$ and $\langle v, u + \lambda u_0\rangle \to \infty$ when $\lambda \to \infty$.

For $v \in \mathcal{C}^*$, consider the decomposition $u = u_+ - u_-$ where $u_\pm = \max\{\pm u, 0\}$. Then $\langle v, u\rangle \leq \langle v, -u_-\rangle$, and hence

$$G^*(v) = \sup_{0 \geq u \in \mathcal{K}} \langle v, u\rangle.$$

Now consider the Lipschitz function $\eta_\epsilon \in C_0^{0,1}(\Omega)$ with value in $[0, 1]$ such that $\eta_\epsilon \equiv 1$ in $\Omega_\epsilon = \{x \in \Omega : dist(x, \partial\Omega) \geq \epsilon\}$ and $\nabla\eta_\epsilon = -\frac{1}{\epsilon}\nu_\Omega$ a.e. in $\Omega \setminus \Omega_\epsilon$, in which ν_Ω is a Lipschitz extension of the boundary normal vector of $\partial\Omega$ to its neighborhood. If $\psi - f \leq u \leq 0$ in Ω, we have $\eta_\epsilon(\psi - f) \in \mathcal{K}$, and

$$\eta_{\epsilon_2}(\psi - f) \geq \eta_{\epsilon_1}(\psi - f) \text{ if } 0 < \epsilon_1 \leq \epsilon_2.$$

Since $v \in C^*$, $\langle v, \eta_\epsilon(\psi - f)\rangle$ is monotone in ϵ and the limit

$$\lim_{\epsilon\to 0}\langle v, \eta_\epsilon(\psi - f)\rangle \tag{15}$$

exists. Thus we have

$$\begin{aligned} G^*(v) &\geq \lim_{\epsilon\to 0}\langle v, \eta_\epsilon(\psi - f)\rangle \\ &= \lim_{\epsilon\to 0}\left(\int_{\Omega\setminus\Omega_\epsilon}\frac{1}{\epsilon}(\psi - f)V\cdot\nu_\Omega - \eta_\epsilon V\cdot D(\psi - f)\,dx - \int_{\Omega_\epsilon} V\cdot D(\psi - f)\,dx\right) \\ &= \lim_{\epsilon\to 0}\left(\int_{\Omega\setminus\Omega_\epsilon}\frac{1}{\epsilon}(\psi - f)V\cdot\nu_\Omega\right) - \int_\Omega V\cdot D(\psi - f)\,dx \\ &= C(V) - \int_\Omega V\cdot D(\psi - f)\,dx, \end{aligned}$$

where

$$C(V) := \lim_{\epsilon\to 0}\left(\int_{\Omega\setminus\Omega_\epsilon}\frac{1}{\epsilon}(\psi - f)V\cdot\nu_\Omega\right).$$

Note that, in view of (15), the above limit exists and only depends on V near $\partial\Omega$.

On the other hand, for every $\psi - f \leq u \leq 0$, we have $0 \leq \eta_\epsilon(u - (\psi - f)) \in W_0^{1,1}(\Omega)$, so $0 \geq \langle v, \eta_\epsilon(u - (\psi - f))\rangle$. Thus $\langle v, \eta_\epsilon u\rangle \leq \langle v, \eta_\epsilon(\psi - f)\rangle$. Since $\lim_{\epsilon\to 0}\|\eta_\epsilon u - u\|_{W_0^{1,1}(\Omega)} = 0$, we have $\lim_{\epsilon\to 0}\langle v, \eta_\epsilon u\rangle = \langle v, u\rangle$. Hence

$$\begin{aligned} \langle v, u\rangle &\leq C(V) - \int_\Omega V\cdot D(\psi - f)\,dx, \\ G^*(v) &\leq C(V) - \int_\Omega V\cdot D(\psi - f)\,dx. \end{aligned}$$

The proof is now complete. □

Proof of Theorem 1.8 The dual problem (P^*) has a solution. This follows from Theorem III.4.1 in [5]. Indeed it easily follows from (2) that $I(v) = \int_\Omega \varphi(x, Dv)$ is convex, and $J : \mathcal{L}^1(\Omega) \to \mathbb{R}$ with $J(p) = \int_\Omega \varphi(x, p)dx$ is continuous at $p = 0$ (a consequence of $(C2)$. Therefore the condition (4.8) in the statement of Theorem III.4.1 in [5] is satisfied, duality gap is zero, and the dual problem (P^*) has a solution. Let T be a solution of the dual problem (P^*), then it must satisfy $\nabla\cdot T \in \mathcal{C}^*$ (i.e. $\nabla\cdot T \leq 0$ in the sense of distributions) as well as $\phi^0(x, T(x)) \leq 1$. The later relation yields that

$$p\cdot T(x) \leq \phi(x, p), \quad \text{for all vectors } p. \tag{16}$$

Moreover, we have

$$\sup(P^*) = \langle T, Df\rangle + \langle T, D(\psi - f)\rangle - C(T) = \langle T, D\psi\rangle - C(T).$$

Let $w \in \mathcal{A}_f$ be a minimizer of (4), and $\epsilon > 0$. Then by (16), we have

$$\begin{aligned}
\int_\Omega \phi(x, Dw) &= \int_\Omega \phi(x, \frac{Dw}{|Dw|})|Dw| \geq \int_\Omega T \cdot \frac{Dw}{|Dw|}|Dw| \qquad (17)\\
&= \int_\Omega T \cdot Dw\\
&= \sup(P^*) + \int_\Omega T \cdot D(w-\psi) + C(T)\\
&= \sup(P^*) - \langle T, D(\psi - f)\rangle + C(T) + \int_\Omega T \cdot D(w-f) \qquad (18)\\
&= \sup(P^*) + G^*(\nabla \cdot T) + \int_\Omega T \cdot D(w-f)\\
&= \sup(P^*) + G^*(\nabla \cdot T) + \int_\Omega T \cdot D(\eta_\epsilon(w-f))\\
&\quad + \int_\Omega T \cdot D\left[(1-\eta_\epsilon)(w-f)\right]\\
&\geq \sup(P^*) + G^*(\nabla \cdot T) + \inf_{\psi - f \leq u \in BV_0(\Omega)} \int_\Omega T \cdot Du \qquad (19)\\
&\quad + \int_\Omega T \cdot D\left[(1-\eta_\epsilon)(w-f)\right]\\
&= \sup(P^*) + \sup_{u\in\mathcal{K}} \langle \nabla \cdot T, u\rangle + \inf_{u\in\mathcal{K}} \int_\Omega T \cdot Du + \int_\Omega T \cdot D\left[(1-\eta_\epsilon)(w-f)\right]\\
&= \sup(P^*) + \sup_{u\in\mathcal{K}} \langle \nabla \cdot T, u\rangle - \sup_{u\in\mathcal{K}} \langle \nabla \cdot T, u\rangle + \int_\Omega T \cdot D\left[(1-\eta_\epsilon)(w-f)\right]\\
&= \sup(P^*) + \int_\Omega T \cdot D\left[(1-\eta_\epsilon)(w-f)\right]\\
&= \sup(P^*) + \int_\Omega T \cdot \left[(w-f)D(1-\eta_\epsilon) + (1-\eta_\epsilon)D(w-f)\right]\\
&\geq \sup(P^*) - \int_{\Omega\setminus\Omega_\epsilon} \phi(x, \frac{\nu_\Omega}{\epsilon}(w-f)) + \int_\Omega (1-\eta_\epsilon) T \cdot D(w-f)\\
&= \sup(P^*) - \int_{\Omega\setminus\Omega_\epsilon} \phi(x, \frac{\nu_\Omega}{\epsilon})|w-f| - \|T\|_{(\mathcal{L}^\infty(\Omega))^n} \int_{\Omega\setminus\Omega_\epsilon} \left|D(w-f)\right|, \qquad (20)
\end{aligned}$$

where we have used (16) to obtain the inequality (20). Letting $\epsilon \to 0$, we have $\int_{\Omega\setminus\Omega_\epsilon} \left|D(w-f)\right| \to 0$ and get

$$\int_\Omega \phi(x, Dw) + \int_{\partial\Omega} \phi(x, \nu_\Omega)|w-f| \geq \sup(P^*) = \inf(P).$$

On the other hand since $BV_f(\Omega) \subset \mathcal{A}_f$, the above inequality also holds in the opposite direction. Thus

$$\inf_{w\in\mathcal{A}_f} \left(\int_\Omega \phi(x, Dw) + \int_{\partial\Omega} \phi(x, \nu_\Omega)|w-f| \right) = \inf_{w\in BV_f(\Omega)} \int_\Omega \phi(x, Dw). \qquad (21)$$

Note also that if $w \in \mathcal{A}_f$ is a minimizer of (4), then all the above inequalities are equalities. In particular (8) and (9) hold because of (17) and (20), and we can deduce by (19) that

$$\inf_{\psi - f \leq u \in BV_0(\Omega)} \int_\Omega T \cdot Du = \lim_{\epsilon \to 0} \int_\Omega T \cdot D(\eta_\epsilon(w - f)). \tag{22}$$

Now let $\omega \Subset \Omega$ and suppose $w > \psi$ on ω. Then for $\varphi \in C_c^\infty(\omega)$ and $|t|$ small, we have $w + t\varphi > \psi$ in ω. Hence for ϵ small enough

$$w + t\varphi - f = \eta_\epsilon(w + t\varphi - f) \text{ in } \omega,$$

and $\psi - f \leq \eta_\epsilon(w + t\varphi - f) \in BV_0(\Omega)$. Thus it follows from (22) that

$$\lim_{\epsilon \to 0} \int_\Omega T \cdot D(\eta_\epsilon(w - f)) \leq \int_\Omega T \cdot D(\eta_\epsilon(w + t\varphi - f)),$$

and hence

$$\begin{aligned} 0 \leq \lim_{\epsilon \to 0} \int_\Omega T \cdot D(t\eta_\epsilon\varphi) &= \lim_{\epsilon \to 0} \int_\omega T \cdot D(t\eta_\epsilon\varphi) \\ &= \lim_{\epsilon \to 0} \int_\omega T \cdot D(t\varphi) \\ &= \int_\omega T \cdot D(t\varphi). \end{aligned}$$

Therefore

$$\langle \nabla \cdot T, \varphi \rangle = 0, \quad \forall \varphi \in C_0^\infty(\omega),$$

and consequently $T \in (\mathcal{L}^\infty(\Omega))^n$ is divergence-free on $\{w > \psi\}$. □

Proof of Proposition 1.4 The proof follows from (21) in the proof of Theorem 1.8, and the argument right before the statement of Proposition 1.4. □

3 Existence

In this section we study the existence of the obstacle least gradient problem (3), and prove Theorem 1.7. Consider an arbitrary function $u \in \mathcal{A}_f$ and let

$$\begin{aligned} E_t &:= \{x \in \mathbb{R}^n : u(x) > t\}, \\ L_t &:= \{x \in \mathbb{R}^n : f(x) > t\}, \\ O_t &:= \{x \in \mathbb{R}^n : \psi(x) > t\}. \end{aligned}$$

The following theorem shows that the level sets of the solutions of (4) solve a ϕ-area minimizing problem with obstacle.

Theorem 3.1 *Let Ω be a bounded Lipschitz domain and u be a solution of (4), then E_t is a solution of the following variational problem,*

$$\min\{P_\phi(E; \Omega) : E \cap \Omega^c = L \cap \Omega^c \text{ and } E \supset O \cap \Omega\}, \tag{23}$$

in which $O = O_t$ and $L = L_t$.

Remark 3.2 It is not difficult to see that $\partial E_t \setminus \bar{O}_t$ is locally ϕ-minimizing in Ω as well as $\partial E_t \cap \bar{O}_t$ is locally ϕ-super minimizing in Ω.

In order to prove Theorem 3.1, we need the following lemma. It will also help us to study the relation between the minimizers of (3) and (4). Therein v^+ and v^- stand for the outer and inner trace of $v \in BV(\mathbb{R}^n)$ on $\partial\Omega$.

Lemma 3.3 *Assume u_k is a solution of (4) for the obstacle ψ_k such that $\psi_k \nearrow \psi$ and*

$$u_k \longrightarrow u \text{ in } \mathcal{L}^1(\Omega) \quad \text{and} \quad u_k^{\pm} \longrightarrow u^{\pm} \text{ in } \mathcal{L}^1(\partial\Omega).$$

Then u is a solution of (4) for the obstacle ψ.

Proof The proof is similar to the proof of Lemma 2.7 in [7] and we present it here for the sake of completeness. Given $g \in \mathcal{L}^1(\partial\Omega; \mathcal{H}^{n-1})$, define

$$I_\phi(v; \Omega, g) := \int_{\partial\Omega} \phi(x, \nu_\Omega)|g - v^-|\, d\mathcal{H}^{n-1} + \int_\Omega \phi(x, Dv),$$

where ν_Ω denotes the outer unit normal to Ω. Since (5) is lower semicontinuous,

$$\int_\Omega \phi(x, Du) \le \liminf_k \int_\Omega \phi(x, Du_k),$$

and the $\mathcal{L}^1$ convergence of the trace, implies that

$$I_\phi(u; \Omega, u^+) \le \liminf_k I_\phi(u_k; \Omega, u_k^+). \tag{24}$$

Now for any $v \in BV(\mathbb{R}^n)$ such that $v \ge \psi$, then $v \ge \psi_k$ and we have

$$\begin{aligned} I_\phi(u_k; \Omega, u_k^+) &\le I_\phi(v; \Omega, u_k^+) \\ &\le I_\phi(v; \Omega, u^+) + \int_{\partial\Omega} \phi(x, \nu_\Omega)|u^+ - u_k^+|\, d\mathcal{H}^{n-1} \\ &\le I_\phi(v; \Omega, u^+) + \alpha^{-1} \int_{\partial\Omega} |u^+ - u_k^+|\, d\mathcal{H}^{n-1}. \end{aligned}$$

It follows from this and (24) that $I_\phi(u; \Omega, u^+) \le I_\phi(v; \Omega, u^+)$. □

Proof of Theorem 3.1 For $t \in \mathbb{R}$, let $u_1 := \max(u, t)$, $u_2 := u - u_1$, $\psi_1 := \max(\psi, t)$. Consider $v \in BV(\mathbb{R}^n)$ such that $v = u_1$ a.e. in Ω^c and $\psi_1 \le v$, then $\psi \le \psi_1 + u_2 \le v + u_2$ and $v + u_2 = u$ a.e. in Ω^c, where we have used the assumption $\psi \le u$. Since u is a solution of (4), we can write

$$\begin{aligned} \int_\Omega \phi(x, Du_1) + \int_\Omega \phi(x, Du_2) &= \int_\Omega \phi(x, Du) \\ &\le \int_\Omega \phi(x, D(v + u_2)) \\ &\le \int_\Omega \phi(x, Dv) + \int_\Omega \phi(x, Du_2). \end{aligned}$$

The last inequality, the triangle inequality, is an immediate consequence of (2). Hence u_1 is also a solution of (4) for the obstacle ψ_1 and the boundary condition $f_1 := \max(f, t)$. Repeating the same argument, one verifies that

$$\chi_{\epsilon,t} := \min(1, \frac{1}{\epsilon} u_1) = \begin{cases} 0 & \text{if } u \le t, \\ \epsilon^{-1}(u - t) & \text{if } t \le u \le t + \epsilon, \\ 1 & \text{if } t + \epsilon \le u, \end{cases}$$

is also a solution of (4) for the obstacle $\psi_{\epsilon,t} := \min(1, \frac{1}{\epsilon}\psi_1)$, and boundary condition $f_{\epsilon,t} := \min(1, \frac{1}{\epsilon} f_1)$.

It is straightforward to check that

$$\chi_{\epsilon,t} \to \chi_t := \chi_{E_t} \text{ in } \mathcal{L}^1_{\text{loc}}(\mathbb{R}^n), \qquad \chi^{\pm}_{\epsilon,t} \to \chi^{\pm} \text{ in } \mathcal{L}^1(\partial\Omega; \mathcal{H}^{n-1}).$$

Notice that $\psi_{\epsilon,t} \nearrow \chi_{O_t}$. Thus Lemma 3.3 implies that χ_{E_t} is a solution of (4) for the obstacle χ_{O_t} and the boundary condition χ_{L_t}. □

Next we use the barrier condition to prove the following lemma. The proof is similar to the proof of Lemma 3.4 in [7] and we omit it.

Lemma 3.4 *Let Ω be a bounded Lipschitz domain satisfying the barrier condition with respect to ϕ, and assume that E is a solution of (23). Then*

$$\{x \in \partial\Omega \cap \partial E^{(1)} : B(\epsilon, x) \cap \partial E^{(1)} \subset \bar{\Omega} \text{ for some } \epsilon > 0\} \subset \bar{O}.$$

Proof of Theorem 1.7 The proof follows from Proposition 1.4, Lemma 3.4, Theorem 3.1, and an argument similar to that of Theorem 1.1 in [7]. □

4 Maximum and comparison principles

This section is devoted maximum and comparison principles which will be our main tools in proving uniqueness and regularity results. At the first, we review some well-known definition and results about the regularity theory for minimal surfaces.

Definition 4.1 Let $E \subset \mathbb{R}^n$. A point $x \in \partial E$ is called a regular point if there exists $\rho > 0$ such that $\partial E \cap B(x, \rho)$ is a C^2 hypersurface. We denote the set of all regular points of ∂E by $reg(\partial E)$. We say that x is a singular point if $x \in sing(\partial E) = \partial E \setminus reg(\partial E)$.

The following estimate on the size of singular sets of ϕ-area minimizing sets has been proved in [19] (see also Remarks 2.7 and 2.8 in [7]).

Theorem 4.2 *Let $\Omega \subset \mathbb{R}^n$, and assume $\phi : \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfies (C1)–(C4). If E is ϕ-area minimizing in Ω, then*

$$\begin{cases} \mathcal{H}^{n-3}(\text{sing}(\partial E^{(1)}) \cap \Omega) < \infty, & \text{if } n \geq 4, \\ \text{sing}(\partial E^{(1)}) \cap \Omega = \emptyset, & \text{if } n \leq 3. \end{cases}$$

We shall also need the following proposition which states that every connected components of regular points of a ϕ-area minimizing set E in Ω must reach the boundary $\partial\Omega$.

Proposition 4.3 *Let Ω be a bounded Lipschitz domain with connected boundary and assume that $E \subset \mathbb{R}^n$ is a solution of (23) for some sets (L, O). If R is a nonempty connected component of* $\text{reg}(\partial E^{(1)}) \cap \Omega$, *then $\bar{R} \cap \partial\Omega \neq \emptyset$ or $\bar{R} \cap \bar{O} \neq \emptyset$.*

Proof The proof follows directly from Lemma 4.2 in [7]. In fact, if $\bar{R} \cap \bar{O} = \emptyset$, it will be a ϕ-area minimizer and we can apply that lemma. □

In order to prove the strict maximum principle, we first prove a couple of intermediate results.

Lemma 4.4 *Assume that ϕ satisfies conditions (C1)–(C2). Let E be a ϕ-sub (or ϕ-super) area minimizing in Ω. There exists a ϕ-area minimizing G such that $G \cap \Omega^c = E \cap \Omega^c$ as well as $G \supseteq E$ (or $G \subseteq E$).*

Proof First note that there is a ϕ-area minimizing set G in Ω such that $G \cap \Omega^c = E \cap \Omega^c$. Since E is ϕ-sub area minimizing,

$$P_\phi(E) \leq P_\phi(E \cap G).$$

Thus it follows from Lemma 1.1 that

$$P_\phi(E \cup G) \leq P_\phi(G).$$

Hence $\tilde{G} = E \cup G$ is also ϕ-area minimizing and $E \subseteq \tilde{G}$. One can similarly show that every ϕ-super area minimizing set contains a ϕ-area minimizing set G with the stated properties. □

We will deduce the uniqueness of the solution and the comparison principle (Theorem 1.10) from the following theorem.

Theorem 4.5 *Assume that ϕ satisfies conditions (C1)–(C5). Suppose that E_1 and E_2 are solutions of (23) respectively for pairs of sets (L_1, O_1) and (L_2, O_2). Also, we have*

$$L_1 \Subset L_2 \text{ and } O_1 \Subset O_2.$$

Suppose Ω satisfies the barrier condition, or

$$\partial E_1^{(1)} \setminus E_2^{(1)} \subset \Omega \text{ and } \partial E_2^{(1)} \cap \overline{E}_1^{(1)} \subset \Omega, \tag{25}$$

then $E_1^{(1)} \Subset E_2^{(1)}$.

Proof In view of Theorem 4.2, $int(E_i^{(1)})$ differs from E_i in a set of measure zero and we replace E_i by $int(E_i^{(1)})$. We prove the result in a series of steps.

Step 1 We will show that $G = E_1 \cap E_2$ and $F = E_1 \cup E_2$ are solutions of (23) for the pairs of sets (L_1, O_1) and (L_2, O_2), respectively. Since E_1 and E_2 are solutions of (23),

$$P_\phi(E_1) \leq P_\phi(G), \quad \text{and} \quad P_\phi(E_2) \leq P_\phi(F).$$

By Lemma 1.1, we have

$$P_\phi(G) + P_\phi(F) \leq P_\phi(E_1) + P_\phi(E_2),$$

and hence $P_\phi(G) = P_\phi(E_1)$ and $P_\phi(F) = P_\phi(E_2)$. Thus G and F are also solutions of the problem (23).

Step 2 If $x_0 \in \partial E_1 \cap \partial F$, then there is a neighborhood of x_0 in which E_1 is a ϕ-area minimizing and F is ϕ-super area minimizing. This immediately follows from the observation that $x_0 \notin \bar{O}_1 \cup \partial\Omega$. Otherwise, $x_0 \in \bar{O}_1 \Subset O_2$ or $x_0 \in \partial\Omega$. In the first case, $x_0 \in O_2$ is an interior point of F which is not possible. In the second case, $x_0 \in \partial\Omega \cap \partial E_1 \cap \partial F$ violates the barrier condition.

Step 3 In this step we show that if $\partial E_1 \cap \partial F \neq \emptyset$, then $\mathcal{H}^{n-2}(\partial E_\nu \cap \partial F) > 0$, where $E_\nu = E_1 + \nu$ for some small vector $\nu \in \mathbb{R}^n$. In order to see this, define

$$\Omega_\delta = \{x \in \Omega : dist(x, \partial\Omega) > \delta\},$$

and choose $\delta > 0$ such that

$$dist(\partial E_1 \cap \Omega_\delta^c, \partial F \cap \Omega_\delta^c) > \delta, \quad dist(\bar{O}_1, O_2^c) > \delta.$$

Let $x_0 \in \partial E_1 \cap \partial F$ and choose $y \in B(x_0, \delta) \cap F^c$. Set $\nu := y - x_0$ and $E_\nu = E_1 + \nu$. By (C5), $\phi(x, \xi) = \phi(\xi)$ and hence E_ν is also a solution of (23) for the pair of sets $(L_1 + \nu, O_1 + \nu)$ in Ω_δ. Then it follows from an argument similar to the one used in the proof of Theorem 4.6 in [7] that

$$\mathcal{H}^{n-2}(\partial E_\nu \cap \partial F) > 0. \tag{26}$$

As in step 1, replace F by $F \cup E_\nu$.

Step 4 In view of Theorem 4.2 and (26), there exists a regular point x_1 of ∂E_ν such that $x_1 \in \partial E_\nu \cap \partial F$ and x_1 is a Lebesgue point of $\partial E_\nu \cap \partial F$ with respect to the measure $\mathcal{H}^{n-2}$. In this step, we will show that there is a neighborhood of x_1 in ∂E_ν that is a subset of $\partial E_\nu \cap \partial F$. Consider a ball $B = B_r(x_1)$ such that $E_\nu \cap B$ is a C^2 hypersurface, and towards a contradiction assume that $E_\nu \cap \partial B \neq F \cap \partial B$. According to Lemma 4.4, there is a ϕ-area minimizing G, such that $G \subseteq F$ and $G \cap B^c = F \cap B^c$. Notice that $\mathcal{H}^{n-2}(\partial E_\nu \cap \partial G \cap B) > 0$, since either ∂G intersects ∂E_ν transversally or contains $\partial E_\nu \cap \partial F$.

Now repeat Step 1 to find two ϕ-area minimizing $E_\nu \cup G$ and $E_\nu \cap G$, which intersects in a set with positive $\mathcal{H}^{n-2}$-measure. Then by Theorem 4.2 there is a point x_* such that $E_\nu \cup G$ and $E_\nu \cap G$ are regular at that. By Lemma 4.4 in [7] we conclude that $\partial(E_\nu \cup G) = \partial(E_\nu \cap G)$ in a neighborhood of x_*. This yields that $E_\nu = G$ in an open subset of $\partial E_\nu \cap B$. The boundary of this set has positive $\mathcal{H}^{n-2}$-measure, and we can repeat the above argument to prove that $E_\nu \cap B = G \cap B$ (see the proof of Theorem 4.6 in [7] for more details). Therefore, $E_\nu \cap \partial B = G \cap \partial B = F \cap \partial B$. This is a contradiction, and hence ∂E_ν is a subset of $\partial E_\nu \cap \partial F$ in a neighborhood of x_1.

Step 5: In this step we show that $E_1 \Subset (E_1 \cup E_2)^{(1)}$. Towards a contradiction suppose this is not the case. Then by steps 3 and 4, we know that each connected component of $\partial E_\nu \cap \partial F$ is an open subset of ∂E_ν for some $\nu \in \mathbb{R}^n$. It follows from Proposition 4.3 that $\partial E_\nu \cap \partial F$ intersects the boundary $\partial\Omega$ or the obstacle $O_1 + \nu$, which contradicts the assumptions of the theorem, and hence $E_1 \Subset (E_1 \cup E_2)^{(1)}$.

Step 6: Finally we prove that $E_1 \Subset E_2$. First we will show that $E_1 \subset E_2$, toward a contradiction assume that $E_1 \setminus E_2$ has non-empty interior. Since $E_1 \Subset F = (E_1 \cup E_2)^{(1)}$, then we have $\partial F \subseteq \partial E_2$. On the other hand, from topological point of view

$$\partial E_2 \subseteq \partial F \cup \partial(E_1 \setminus E_2). \tag{27}$$

If there exists some point $x_0 \in \partial(E_1 \setminus E_2) \setminus \partial E_2$, then we must have

$$x_0 \in int(E_2^c) \cap \partial E_1 \subset int(E_2^c) \cap F \subseteq E_1,$$

which contradicts $x_0 \in \partial E_1$ (E_1 is open). It yields that $\partial(E_1 \setminus E_2) \subset \partial E_2$. Therefore, $\partial E_2 = \partial F \cup \partial(E_1 \setminus E_2)$ by (27), which means that the perimeter of F is less than the perimeter of E_2 unless $\mathcal{H}^{n-1}(\partial(E_1 \setminus E_2)) = 0$. This contradicts the assumption $int(E_1 \setminus E_2) \neq \emptyset$. Hence $E_1 \cup E_2$, and consequently $E_1 \Subset F = E_2$ by the conclusion in Step 5. □

Remark 4.6 When $n = 2$ or 3, the statement in Theorem 4.5 holds without condition (C5). Because all ϕ-area minimizing sets are regular even ϕ depends on variable x (Theorem 4.2). Hence we does not need steps 3, and in step 4 we can choose $\nu = 0$. A similar argument implies $E_1 \Subset E_2$.

Proof of Theorem 1.10 The proof is inspired by Theorem 1.4 from [7]. Suppose that (11) is not true. Since

$$\{x \in \Omega : u_1(x) > u_2(x)\} = \bigcup_{(\lambda_1,\lambda_2)\in\mathbb{Q}\times\mathbb{Q}} \{x \in \Omega : u_1(x) > \lambda_1 > \lambda_2 \geq u_2(x)\},$$

there must be some rational numbers $\lambda_1 > \lambda_2$ such that

$$\mathcal{H}^n(\{x \in \Omega : u_1(x) > \lambda_1 > \lambda_2 \geq u_2(x)\}) > 0.$$

Now define

$$E_i := \{x \in \mathbb{R}^n : u_i(x) > \lambda_i\},$$

then we have $\mathcal{H}^n(E_1 \setminus E_2) > 0$. On the other hand, we can easily verify that the conditions of Theorem 4.5 are satisfies, and hence $E_1^{(1)} \Subset E_2^{(1)}$. □

The idea in the proof of Theorem 4.5 allow us to prove a strict maximum principle for ϕ-sub and super area minimizing sets. This result generalizes the result in [20,24].

Theorem 4.7 (Strict maximum principle) *Assume that ϕ satisfies the conditions (C1)–(C5). Let $E \subset \mathbb{R}^n$ be ϕ-sub area minimizing and $F \subset \mathbb{R}^n$ be ϕ-super area minimizing relative to an open set Ω, and*

$$E \setminus \Omega \Subset F \setminus \Omega.$$

Suppose Ω satisfies the barrier condition, then

$$E^{(1)} \Subset F^{(1)}.$$

Proof By Lemma 4.4, there exists ϕ-area minimizing sets $\tilde{E}$ and $\tilde{F}$ such that $\tilde{E} \supseteq E$ and $\tilde{F} \subseteq F$. Since Ω satisfies the barrier condition,

$$\partial\tilde{E}^{(1)} \setminus \tilde{F}^{(1)} \subset \Omega \text{ and } \partial\tilde{F}^{(1)} \cap \overline{\tilde{E}}^{(1)} \subset \Omega.$$

By Theorem 4.6 in [7] we have $\tilde{E}^{(1)} \subset \tilde{F}^{(1)}$. Moreover $\tilde{E}^{(1)} \Subset \tilde{F}^{(1)}$ if $n \leq 3$. In order to prove the theorem for $n \geq 4$, note that since $E \cap \Omega^c \Subset F \cap \Omega^c$, there is a $\delta > 0$ such that

$$dist(\partial E \cap \Omega_\delta^c, \partial F \cap \Omega_\delta^c) > \delta.$$

Let $x_0 \in \partial\tilde{E}^{(1)} \cap \partial\tilde{F}^{(1)}$ and choose $y \in B(x_0, \delta) \cap \tilde{F}^c$. Set $\nu := y - x_0$ and $E_\nu = \tilde{E} + \nu$. Since we have assumed (C5), $\phi(x, \xi) = \phi(\xi)$, and hence E_ν is also a ϕ-area minimizer in Ω_δ. Observe that

$$\partial E_\nu^{(1)} \setminus \tilde{F}^{(1)} \subset \Omega_\delta \text{ and } \partial\tilde{F}^{(1)} \cap \overline{E}_\nu^{(1)} \subset \Omega_\delta.$$

It again follows from Theorem 4.6 in [7] that $E_\nu^{(1)} \subset \tilde{F}^{(1)}$ which is a contradiction. Thus $\partial\tilde{E}^{(1)} \cap \partial\tilde{F}^{(1)} = \emptyset$, and the proof is complete. □

We shall need the following proposition to prove regularity results for solutions of (3).

Proposition 4.8 *Under the assumption of Theorem 4.5, if $d = dist(\partial E_1 \cap \bar{\Omega}, \partial E_2 \cap \bar{\Omega})$ and this distance is taken in points $|x - y| = d$, such that $x \in \partial E_1 \cap \bar{\Omega}$ and $y \in \partial E_2 \cap \bar{\Omega}$, then either $x \in \bar{O}_1 \cup \partial\Omega$ or $y \in \partial\Omega$.*

Proof Consider the points x and y such that violate the statement. Let $v = y - x$, the translation $\tilde{E}_1 = v + E_1$ remains a solution of (23) for the pair of sets $(v + L_1, v + O_1) =: (\tilde{L}_1, \tilde{O}_1)$ in $\tilde{\Omega} := v + \Omega$. According to our assumption $\tilde{L}_1 \Subset L_2$ and $\tilde{O}_1 \cap \partial E_2 = \emptyset$. Choose $\epsilon > 0$ such that $\tilde{O}_1 + B_\epsilon \Subset E_2$, and define $\tilde{O}_2 := O_2 \cup (\tilde{O}_1 + B_\epsilon)$ which satisfies $\tilde{O}_2 \Supset \tilde{O}_1$. Then E_2 is also a solution for $(L_2, \tilde{O}_2)$. On the other hand, $y \in \partial \tilde{E}_1 \cap \partial E_2$ and this contradicts Theorem 4.5, for $\tilde{E}_1$ and E_2 in the domain $\Omega \cap \tilde{\Omega}$. □

5 Regularity of solutions

First of all we shall notice that the continuity of the solution of (3) is a straightforward result of the geometric comparison principle, Theorem 4.5. The proof is similar to Theorem 1.3 in [7], then we just give the statement without proof in the following proposition.

Proposition 5.1 (Continuity) *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with connected boundary, and assume $\phi : \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfies (C1)–(C5). If u is a solution of (3) for the boundary condition $f \in C(\partial\Omega)$ and $\psi \in C(\bar{\Omega})$, then u is continuous.*

In order to study the Holder regularity, we need the following property for the norm $\phi(x, \xi)$.

Lemma 5.2 *If $\phi : \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfies (C1)–(C4), then for every p and q we have*

$$p \cdot \phi_\xi(x, q) \leq \phi(x, p).$$

Proof By the norm property (1), we can assume $\phi(x, p) = \phi(x, q) = 1$. Let $f(t) := \phi(x, tp + (1-t)q)$, we have $f(0) = 1$ and for $0 < t < 1$

$$f(t) \leq t\phi(x, p) + (1-t)\phi(x, q) = 1.$$

Alos, for $t < 0$ we have

$$f(t) \geq \phi(x, (1-t)q) - \phi(x, tp) = (1-t) - |t| = 1.$$

Thus $f'(0) \leq 0$ which yields

$$\phi_\xi(x, q) \cdot (p - q) \leq 0.$$

Using the norm property (1), $q \cdot \phi_\xi(x, q) = \phi(x, q) = 1$ to deduce the lemma. □

Now we are going to construct barriers and prove a comparison principle for such barriers. The results and the proofs in this section are inspired by [23].

Lemma 5.3 *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Suppose $u \in C^0(\bar{\Omega}) \cap BV(\Omega)$ is a solution of (3) and $v \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies*

(i) $|\nabla v| > 0$ *in* Ω,
(ii) $u \geq v$ *on* $\partial\Omega$,
(iii) $\mathcal{L}v > 0$ *in* Ω,

where $\mathcal{L}v = \sum_{i=1}^n \partial_{x_i} \phi_{\xi_i}(x, Dv(x))$. Then $u \geq v$ in Ω. Similarly, if inequalities (ii) and (iii) are reserved, then $u \leq v$ in Ω.

Proof Let $E = \{x \in \Omega : v(x) > u(x) + \epsilon\}$ for some $\epsilon > 0$, and $w = \max(u, v - \epsilon)$. Notice that $w \in BV(\Omega) \cap C^0(\bar{\Omega})$, $w = u$ on $\partial\Omega$ and $w \geq \psi$. Now let $\eta \in C_0^\infty(\Omega)$ satisfy $\eta = 1$ on E and $0 \leq \eta \leq 1$ in Ω. Set

$$g = \eta\phi_\xi(x, Dv),$$

so that $g \in [C_0^1(\Omega)]^n$. Note that $u = w$ in $\Omega \setminus E$, then by Theorem 2.1 in [2], we can write

$$\begin{aligned}\int_E (u-w)\nabla \cdot g\, dx &= \int_\Omega (u-w)\nabla \cdot g\, dx = -\int_\Omega g \cdot D(u-w)\, dx \\ &= -\int_E g \cdot D(u-w) = -\int_E g \cdot Du + \int_E g \cdot Dv \\ &= -\int_E \phi_\xi(x, Dv) \cdot \frac{Du}{|Du|}|Du| + \int_E Dv \cdot \phi_\xi(x, Dv)\, dx \\ &\geq -\int_E \phi(x, \frac{Du}{|Du|})|Du| + \int_E \phi(x, Dv),\end{aligned}$$

where in the last line we use the norm properties in Lemma 5.2 and relation (1). Since $u - w < 0$ and $\nabla \cdot g > 0$ in E (condition (iii)), we have

$$\int_E \phi(x, Du) = -\int_E \phi(x, \frac{Du}{|Du|})|Du| > \int_E \phi(x, Dw),$$

which violates the fact that u is a minimal solution of (3). □

Here, we first prove the regularity of the solutions near the boundary.

Lemma 5.4 *Suppose Ω is a bounded, open subset of $\mathbb{R}^n$ with C^2 boundary which the signed distance $d(\cdot)$ to $\partial\Omega$ satisfies the relation (7). Assume $f \in C^{0,\alpha}(\partial\Omega)$, and $\psi \in C^{0,\alpha/2}$ for some $0 < \alpha \leq 1$. If $u \in C^0(\overline{\Omega}) \cap BV(\Omega)$ is a solution of (3), then there exists positive constants δ and C depending only on $\|f\|_{C^{0,\alpha}(\partial\Omega)}$, $\|\psi\|_{C^{0,\alpha/2}}$ and $\|u\|_{C^0(\bar{\Omega})}$ such that*

$$|u(x) - u(x_0)| \leq C|x - x_0|^{\alpha/2},$$

whenever $x_0 \in \partial\Omega$ and $x \in \bar{\Omega}$ with $|x - x_0| < \delta$.

Proof For each $x_0 \in \partial\Omega$ we will construct functions $w^+, w^- \in C^2(U) \cap C^0(\bar{U})$ where $U = B(x_0, \delta) \cap \Omega$ for some $\delta > 0$ is to be determined later, such that

(i) $w^+(x_0) = w^-(x_0) = f(x_0)$,
(ii) $|w^+(x) - f(x_0)| \leq C|x - x_0|^{\alpha/2}$ and $|w^-(x) - f(x_0)| \leq C|x - x_0|^{\alpha/2}$ for every $x \in U$.
(iii) $|\nabla w^+| > 0$ and $|\nabla w^-| > 0$ in U.
(iv) $w^- \leq u \leq w^+$ on ∂U.
(v) $\mathcal{L}w^+ < 0 < \mathcal{L}w^-$ in U.

By applying Lemma 5.3 to w^+ and w^-, we obtain the inequality $w^- \leq u \leq w^+$ in U. This accomplishes the proof by the property (ii).

In order to construct the function w^+, notice that $d \in C^2(\{x : 0 \leq d(x) < \delta_0\})$ for some $\delta_0 > 0$, because $\partial\Omega \in C^2$. We choose $\delta < \delta_0$ and let

$$\begin{aligned}v(x) &= |x - x_0|^2 + \lambda d(x), \\ w^+(x) &= Kv^{\alpha/2}(x) + f(x_0),\end{aligned}$$

where K and λ are to be determined. Obviously (i) and (ii) are valid. To establish (iii), observe that

$$\begin{aligned}|\nabla w^+| &= K\frac{\alpha}{2}v^{\frac{\alpha}{2}-1}|\nabla v|,\\ |\nabla v| &= |2(x-x_0)+\lambda\nabla d| \geq \lambda|\nabla d| - 2|x-x_0|\\ &\geq \lambda - 2|x-x_0| > 0,\end{aligned}$$

provided $\lambda > 2\delta$. We also have used the fact that $|\nabla d| = 1$ in the last relation.

For (iv) on $\partial B(x_0,\delta)\cap\Omega$, we have $w^+(x) \geq K\delta^\alpha \geq \|u\|_{C^0(\bar{\Omega})}$ if K is chosen large enough. On $\partial\Omega\cap B(x_0,\delta)$ we have

$$u(x) = f(x) \leq f(x_0) + \|f\|_{C^{0,\alpha}}|x-x_0|^\alpha \leq w^+(x),$$

provided $K \geq \|f\|_{C^{0,\alpha}(\partial\Omega)}$.

To establish (v), we note that $\phi_\xi(x,tp) = \phi_\xi(x,p)$ and $Dw^+ = K\frac{\alpha}{2}v^{\frac{\alpha}{2}-1}Dv$, then

$$\begin{aligned}\mathcal{L}w^+ &= div_x(\phi_\xi(x,Dv)) = div_x(\phi_\xi(x,2(x-x_0)+\lambda Dd(x)))\\ &= div_x(\phi_\xi(x,\frac{2}{\lambda}(x-x_0)+Dd(x))).\end{aligned}$$

Since d is C^2 near the boundary $\partial\Omega$ and satisfies the relation (7), then for a large value of λ, we will have uniformly $\mathcal{L}w^+ < 0$ in the δ-neighborhood of the boundary.

A similar construction provides function $w^-(x) = -Kv^{\alpha/2}(x) + f(x_0)$ for a suitable positive constant K. □

Proof of Theorem 1.11 For $s < t$, consider the supersets E_s, E_t of u and assume that $dist(\partial E_s, \partial E_t) = |x-y|$ where $x \in E_t$ and $y \in E_s$. It is sufficient to show that $|u(x)-u(y)| = |t-s| \leq C|x-y|^{\alpha/2}$ whenever $|x-y| < \delta$, where δ is given by Lemma 5.4. Observe that $O_t \subset E_t \Subset E_s$. By Proposition 4.8, we just have two following cases:

(i) If either x or y belongs to $\partial\Omega$, then our result follows from Lemma 5.4.
(ii) $x \in \partial E_t \cap \bar{O}_t$, then $u(x) = \psi(x)$ and $u(y) \geq \psi(y)$, so

$$0 < t-s = u(x)-u(y) \leq \psi(x)-\psi(y) \leq [\psi]_{0,\alpha/2}|x-y|^{\alpha/2}.$$

□

Proof of Theorem 1.12 The proof is similar to that of Lemma 5.4. Indeed it is enough to construct functions w^+ and w^- satisfying conditions (i)–(v) while the condition (ii) is replaced by

$$|w^\pm(x) - f(x_0)| \leq C|x-x_0|^{\frac{1+\alpha}{2}}.$$

For this, put

$$w^+(x) := Kv^{\frac{1+\alpha}{2}}(x) + \nabla f(x_0)\cdot(x-x_0) + f(x_0), \tag{28}$$

and notice that on $\partial\Omega\cap B(x_0,\delta)$, by the $C^{1,\alpha}$ regularity of f there is a positive constant C_1 such that the following inequality is established

$$u(x) = f(x) \leq f(x_0) + \nabla f(x_0)\cdot(x-x_0) + C_1|x-x_0|^{\frac{1+\alpha}{2}}.$$

Therefore, the relation (iv), $u \leq w^+$, will be obtained provided $K \geq C_1$. The rest of the proof is exactly the same. □

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