Proceedings of the Edinburgh Mathematical Society (2020) **63**, 398–412
doi:10.1017/S0013091519000439

MAXIMAL FUNCTIONS ASSOCIATED WITH FAMILIES OF HOMOGENEOUS CURVES: L^P BOUNDS FOR $P \leq 2$

SHAOMING GUO[1], JORIS ROOS[1], ANDREAS SEEGER[1] AND PO-LAM YUNG[2]*

[1] *Department of Mathematics, University of Wisconsin-Madison, 480 Lincoln Dr., Madison, WI 53706, USA* (shaomingguo@math.wisc.edu; jroos@math.wisc.edu; seeger@math.wisc.edu)

[2] *Department of Mathematics, The Chinese University of Hong Kong, Ma Liu Shui, Shatin, Hong Kong* (plyung@math.cuhk.edu.hk)

(Received 13 June 2019; first published online 3 February 2020)

Abstract Let $M^{(u)}$, $H^{(u)}$ be the maximal operator and Hilbert transform along the parabola (t, ut^2). For $U \subset (0,\infty)$ we consider L^p estimates for the maximal functions $\sup_{u\in U} |M^{(u)} f|$ and $\sup_{u\in U} |H^{(u)} f|$, when $1 < p \leq 2$. The parabolas can be replaced by more general non-flat homogeneous curves.

Keywords: maximal functions; Hilbert transforms along curves; maximal and singular Radon transforms

2010 *Mathematics subject classification:* Primary 42B15; 42B20; 42B25; 44A12

1. Introduction and statement of results

Let $b > 1$, $u > 0$, and $\gamma_b : \mathbb{R} \to \mathbb{R}$ homogeneous of degree b, that is, $\gamma_b(st) = s^b \gamma_b(t)$ for $s > 0$. Also suppose $\gamma_b(\pm 1) \neq 0$. For a Schwartz function f on $\mathbb{R}^2$ we let

$$M^{(u)} f(x) = \sup_{R>0} \frac{1}{R} \int_0^R |f(x - (t, u\gamma_b(t)))| \,\mathrm{d}t,$$

$$H^{(u)} f(x) = p.v. \int_{\mathbb{R}} f(x - (t, u\gamma_b(t))) \frac{\mathrm{d}t}{t},$$

denote the maximal function and Hilbert transform of f along the curve $(t, u\gamma_b(t))$. For an arbitrary non-empty $U \subset (0,\infty)$ we consider the maximal functions

$$\mathcal{M}^U f(x) = \sup_{u\in U} M^{(u)} f(x), \quad \mathcal{H}^U f(x) = \sup_{u\in U} |H^{(u)} f(x)|. \tag{1.1}$$

For $2 < p < \infty$ the operators $\mathcal{M}^U$ are bounded on $L^p(\mathbb{R}^2)$ for all U; this was shown by Marletta and Ricci [**8**]. For the operators $\mathcal{H}^U$ a corresponding satisfactory theorem was

*Current address: Mathematical Sciences Institute, Australian National University, Canberra, ACT 2600, Australia; polam.yung@anu.edu.au.

proved in a previous paper [6] of the authors. To describe the result let

$$\mathfrak{N}(U) = 1 + \#\{n \in \mathbb{Z} : [2^n, 2^{n+1}] \cap U \neq \emptyset\}.$$

Then, for $2 < p < \infty$, $\mathcal{H}^U$ is bounded on $L^p(\mathbb{R}^2)$ if and only if $\mathfrak{N}(U)$ is finite, and we have the equivalence

$$c_p \leq \frac{\|\mathcal{H}^U\|_{L^p\to L^p}}{(\log \mathfrak{N}(U))^{1/2}} \leq C_p, \quad 2 < p < \infty,$$

with non-zero constants c_p, C_p. Moreover, for all $p > 1$ we have the lower bound $\|\mathcal{H}^U\|_{L^p\to L^p} \gtrsim \sqrt{\log \mathfrak{N}(U)}$. The consideration of such results in Guo *et al.* [6] and in this paper has multiple motivations. First, there is an analogy (although not a close relation) with similar results on maximal operators and Hilbert transforms for families of straight lines; here we mention the lower bounds by Karagulyan [7], and the currently best upper bounds for $p > 2$ by Demeter and Di Plinio [3]. The second motivation comes from the above-mentioned work by Marletta and Ricci [8] on the maximal function for $p > 2$, and the third motivation comes from a *curved version* of the Stein–Zygmund vector-field problem concerning the L^p boundedness of $M^{(u(\cdot))}$ and $H^{(u(\cdot))}$ where $x \mapsto u(x)$ is a Lipschitz function. In this case the L^p boundedness of $M^{(u(\cdot))}$ for the full range $1 < p < \infty$ was proved by Guo *et al.* [5], and the analogous result for $H^{(u(\cdot))}$ by Di Plinio *et al.* [4]. We refer to the bibliography of Guo *et al.* [6] for a list of related works.

Regarding the operators $\mathcal{M}^U$, $\mathcal{H}^U$, most satisfactory results (except for certain lacunary sequences) have so far been obtained in the range $p > 2$. In this paper we seek to find efficient upper bounds for the L^p operator norms of $\mathcal{M}^U$ and $\mathcal{H}^U$ in the case $1 < p \leq 2$. It turns out that there is a striking dichotomy between the cases $2 < p < \infty$ and $1 < p \leq 2$. In the latter case, the operator norms of $\mathcal{M}^U$ and $\mathcal{H}^U$ depend on an additional quantity that involves the local behaviour of the set U on each dyadic interval. The formulation of the results, using some variant of Minkowski dimension, is in part motivated by considerations for spherical maximal functions in the work of Seeger *et al.* [11] (see also [10, 12]).

As pointed out in Guo *et al.* [6], with reference to Seeger *et al.* [10], L^p boundedness for $p \leq 2$ fails, for both $\mathcal{M}^U$ and $\mathcal{H}^U$, when $U = [1, 2]$; therefore some additional sparseness condition needs to be imposed. To formulate such results let, for each $r > 0$,

$$U^r = r^{-1}U \cap [1, 2] = \{\rho \in [1, 2] : r\rho \in U\}.$$

For $0 < \delta < 1$ we let $N(U^r, \delta)$ be the δ-covering number of U^r, that is, the minimal number of intervals of length δ needed to cover U^r. It is obvious that $\sup_{r>0} N(U^r, \delta) \lesssim \delta^{-1}$. Define

$$\mathcal{K}_p(U, \delta) = \delta^{1-1/p} \sup_{r>0} N(U^r, \delta)^{1/p}. \tag{1.2}$$

Define

$$p_{\mathrm{cr}}(U) = 1 + \limsup_{\delta\to0+} \frac{\sup_{r>0} \log N(U^r, \delta)}{\log(\delta^{-1})}. \tag{1.3}$$

Notice that $1 \leq p_{\mathrm{cr}}(U) \leq 2$ always. If $p_{\mathrm{cr}}(U) < p < 2$ there exists an $\varepsilon = \varepsilon(p, U) > 0$ such that $\sup_{0<\delta<1} \delta^{-\varepsilon}\mathcal{K}_p(U, \delta) < \infty$. If $1 < p < p_{\mathrm{cr}}(U)$ then there is $\varepsilon' = \varepsilon'(p, U) > 0$ and a sequence $\delta_n \to 0$ such that $\limsup_n \delta_n^{\varepsilon'}\mathcal{K}_p(U, \delta_n) > 0$.

Theorem 1.1. *Let $1 < p \le 2$.*

(i) *If $p_{\mathrm{cr}}(U) < p \le 2$ then $\mathcal{M}^U$ is bounded on $L^p(\mathbb{R}^2)$.*

(ii) *If $1 < p < p_{\mathrm{cr}}(U)$ then $\mathcal{M}^U$ is not bounded on $L^p(\mathbb{R}^2)$.*

(iii) *For every $\varepsilon > 0$ we have*

$$c_p \sup_{\delta>0} \mathcal{K}_p(U,\delta) \le \|\mathcal{M}^U\|_{L^p\to L^p} \le C_{\varepsilon,p} \sup_{\delta>0} \delta^{-\varepsilon}\mathcal{K}_p(U,\delta).$$

Here $c_p, C_{p,\varepsilon}$ are constants only depending on p or p, ε, respectively.

Theorem 1.2. *Let $1 < p \le 2$ and $p_{\mathrm{cr}}(U)$ as in (1.3).*

(i) *If $p_{\mathrm{cr}}(U) < p \le 2$ then $\mathcal{H}^U$ is bounded on $L^p(\mathbb{R}^2)$ if and only if $\mathfrak{N}(U) < \infty$.*

(ii) *If $1 < p < p_{\mathrm{cr}}(U)$ then $\mathcal{H}^U$ is not bounded on $L^p(\mathbb{R}^2)$.*

(iii) *For every $\varepsilon > 0$ we have*

$$\|\mathcal{H}^U\|_{L^p\to L^p} \le C_p\sqrt{\log(\mathfrak{N}(U))} + C_{\varepsilon,p} \sup_{\delta>0} \delta^{-\varepsilon}\mathcal{K}_p(U,\delta)$$

and

$$c_p\left(\sqrt{\log(\mathfrak{N}(U))} + \sup_{\delta>0} \mathcal{K}_p(U,\delta)\right) \le \|\mathcal{H}^U\|_{L^p\to L^p}.$$

Here $c_p, C_p, C_{p,\varepsilon}$ are constants only depending on p or p, ε, respectively.

We note that parts (i) and (ii) of each theorem follow immediately from part (iii) of the same theorem.

We discuss some examples. We have $p_{\mathrm{cr}}(U) = 1$ for lacunary U and we have $p_{\mathrm{cr}}(U) = 2$ if U contains any intervals. There are many interesting intermediate examples with $1 < p_{\mathrm{cr}}(U) < 2$; see Seeger *et al.* [**11**]. One may take for U a self-similar Cantor set $\mathcal{C}_\beta$ of Minkowski dimension β, contained in $[1,2]$; then $p_{\mathrm{cr}}(\mathcal{C}_\beta) = 1+\beta$. This remains true if for U we take $\bigcup_{k\in\mathbb{Z}} 2^k\mathcal{C}_\beta$ in Theorem 1.1, or with finite $F \subset \mathbb{Z}$, we take $U = \bigcup_{k\in F} 2^k\mathcal{C}_\beta$ in Theorem 1.2.

Another set of examples comes from considering convex sequences. One may take $S_a = \{1 + n^{-a} : n \in \mathbb{N}\}$, then $p_{cr}(S_a) = (2+a)/(1+a)$. Again we may also take suitable unions of dilates of S_a; that is, for U we can take $\bigcup_{k\in\mathbb{Z}} 2^k S_a$ in Theorem 1.1, or $U = \bigcup_{k\in F} 2^k S_a$ in Theorem 1.2, provided that $F \subset \mathbb{Z}$ is finite.

We shall in fact prove sharper but more technical versions of Theorems 1.1 and 1.2. The term $C_{\varepsilon,p}\delta^{-\varepsilon}\mathcal{K}_p(U,\delta)$ can be replaced with one with logarithmic dependence, namely

$$C_p[\log(2/\delta)]^A\mathcal{K}_p(U,\delta)$$

for $A > 14/p - 6$. More precisely, we have the following theorem.

Theorem 1.3. *Let $1 < p \le 2$. Then there is C independent of p and U so that*

$$\|\mathcal{M}^U\|_{L^p\to L^p} \le C\sum_{\ell\ge 1}\vartheta_{p,\ell}\mathcal{K}_p(U,2^{-\ell}), \tag{1.4}$$

where $\vartheta_{p,\ell} = (p-1)^{3-10/p}$ *if* $\ell \le (p-1)^{-1}$ *and* $\vartheta_{p,\ell} = \ell^{7(2/p-1)}$ *if* $\ell > (p-1)^{-1}$. *Moreover,*

$$\|\mathcal{H}^U\|_{L^p\to L^p} \le C(p-1)^{-7}\sqrt{\log(\mathfrak{N}(U))} + C(p-1)^{-2}\sum_{\ell\ge 1}\vartheta_{p,\ell}\mathcal{K}_p(U,2^{-\ell}). \tag{1.5}$$

Structure of the paper. In § 2 we decompose the operators $\mathcal{M}^U$, $\mathcal{H}^U$ in the spirit of Guo *et al.* [6] in order to prepare for the proof of Theorem 1.3. The proof of Theorem 1.3 is then completed in § 3 and § 4. Finally, the lower bounds claimed in Theorems 1.1 and 1.2 are addressed in § 5.

2. Basic reductions

We recall some notation and basic reductions from Guo *et al.* [6]. By the assumption of homogeneity and $\gamma_b(\pm 1) \ne 0$ there are $c_\pm \ne 0$ such that $\gamma_b(t) = c_+ t^b$ for $t > 0$, and $\gamma_b(t) = c_-(-t)^b$ for $t < 0$, and finally $\gamma_b(0) = 0$. We note that by scaling we may always assume that $c_- = 1$. Let $\chi_+ \in C_c^\infty$ be supported in $(1/2, 2)$ such that

$$\sum_{j\in\mathbb{Z}}\chi_+(2^j t) = 1 \quad \text{for } t > 0.$$

Let $\chi_-(t) = \chi_+(-t)$ and $\chi = \chi_+ + \chi_-$. We define measures τ_0, σ_0, $\sigma_\pm$ by

$$\begin{aligned}\langle\tau_0, f\rangle &= \int f(t,\gamma_b(t))\chi_+(t)\,\mathrm{d}t,\\ \langle\sigma_\pm, f\rangle &= \int f(t,\gamma_b(t))\chi_\pm(t)\frac{\mathrm{d}t}{t},\\ \sigma_0 &= \sigma_+ + \sigma_-.\end{aligned}$$

For $j \in \mathbb{Z}$, let the measures τ_j^u, σ_j^u be defined by

$$\begin{aligned}\langle\tau_j^u, f\rangle &= \int f(t,u\gamma_b(t))2^j\chi_+(2^j t)\,\mathrm{d}t,\\ \langle\sigma_j^u, f\rangle &= \int f(t,u\gamma_b(t))\chi(2^j t)\frac{\mathrm{d}t}{t}.\end{aligned}$$

By homogeneity of γ_b we have $\tau_j^u = 2^{j(1+b)}\tau_0^u(\delta_{2^j}^b\cdot)$ with $\delta_t^b x = (tx_1, t^b x_2)$, as well as the analogous relation between σ_j^u and σ_0^u. We note that the τ_j^u are positive measures and the σ_j^u have cancellation.

For Schwartz functions f the Hilbert transform along Γ_b^u can be written as

$$H^{(u)}f = \sum_{j\in\mathbb{Z}} \sigma_j^u * f.$$

For the maximal function it is easy to see that there is the pointwise estimate

$$M^{(u)}f(x) \le C \sup_{j\in\mathbb{Z}} \tau_j^u * |f|. \tag{2.1}$$

Following Guo *et al.* [6, § 2], we further decompose σ_0 and τ_0. Choose Schwartz function η_0, supported in $\{|\xi| \le 100\}$ and equal with $\eta_0(\xi) = 1$ for $|\xi| \le 50$. Let $\varsigma_+ \in C_c^\infty(\mathbb{R})$ be supported in $\big(b(1/4)^{b-1}, b4^{b-1}\big)$ and equal to 1 on $\big[b(2/7)^{b-1}, b(7/2)^{b-1}\big]$. Let $\varsigma_- \in C_c^\infty(\mathbb{R})$ be supported on $\big(-b4^{b-1}, -b(1/4)^{b-1}\big)$ and equal to 1 on $\big[-b(7/2)^{b-1}, -b(2/7)^{b-1}\big]$.

One then decomposes

$$\begin{aligned}\sigma_0 &= \phi_0 + \mu_{0,+} + \mu_{0,-},\\ \tau_0 &= \varphi_0 + \rho_0,\end{aligned}$$

where ϕ_0, φ_0 are given by

$$\begin{aligned}\widehat{\phi_0}(\xi) = \eta_0(\xi)\widehat{\sigma}_0(\xi) &+ (1-\eta_0(\xi))\left(1 - \varsigma_-\left(\frac{\xi_1}{c_+\xi_2}\right)\right)\widehat{\sigma}_+(\xi)\\ &+ (1-\eta_0(\xi))\left(1 - \varsigma_+\left(\frac{\xi_1}{c_-\xi_2}\right)\right)\widehat{\sigma}_-(\xi)\end{aligned}$$

and

$$\widehat{\varphi_0}(\xi) = \eta_0(\xi)\widehat{\tau}_0(\xi) + (1-\eta_0(\xi))\left(1 - \varsigma_-\left(\frac{\xi_1}{c_+\xi_2}\right)\right)\widehat{\tau}(\xi).$$

The measures and $\mu_{0,\pm}$ and ρ_0 are given via the Fourier transform by

$$\begin{aligned}\widehat{\mu}_{0,+}(\xi) &= (1-\eta_0(\xi))\varsigma_-\left(\frac{\xi_1}{c_+\xi_2}\right)\widehat{\sigma}_+(\xi),\\ \widehat{\mu}_{0,-}(\xi) &= (1-\eta_0(\xi))\varsigma_+\left(\frac{\xi_1}{c_-\xi_2}\right)\widehat{\sigma}_-(\xi)\end{aligned}$$

and

$$\widehat{\rho}_0(\xi) = (1-\eta_0(\xi))\varsigma_-\left(\frac{\xi_1}{c_+\xi_2}\right)\widehat{\tau}_0(\xi). \tag{2.2}$$

As in Lemma 2.1 of Guo *et al.* [6], the functions φ_0, ϕ_0 are Schwartz functions. In addition, we have $\widehat{\phi}_0(0) = 0$.

For $j \in \mathbb{Z}$, define φ_j and ϕ_j by scaling via $\widehat{\varphi}_j(\xi) = \widehat{\varphi}_0(2^{-j}\xi_1, 2^{-jb}\xi_2)\widehat{f}(\xi)$ and $\widehat{\phi}_j(\xi) = \widehat{\phi}_0(2^{-j}\xi_1, 2^{-jb}\xi_2)\widehat{f}(\xi)$. Define $A^u_{j,0}f$ by

$$\widehat{A^u_{j,0}f}(\xi) = \widehat{\varphi}_j(\xi_1, u\xi_2)\widehat{f}(\xi)$$

and let $\mathcal{M}_0 f(x) = \sup_{j\in\mathbb{Z}} \sup_{u\in\mathbb{R}} |A^u_{j,0}f(x)|$. Let

$$\widehat{S^{(u)}f}(\xi) = \sum_{j\in\mathbb{Z}} \widehat{\phi}_j(\xi_1, u\xi_2)\widehat{f}(\xi).$$

Let $M^{\rm str} f$ denote the strong maximal function of f. For $p \in (1, 2)$ we have

$$\|M^{\rm str}\|_{L^p\to L^p} \le C(p-1)^{-2}. \tag{2.3}$$

This follows from the pointwise bound $M^{\rm str} \le M^{(1)} \circ M^{(2)}$, where $M^{(k)}$ denotes the Hardy–Littlewood maximal operator taken in the kth variable. Indeed, $M^{(k)}$ is of weak type $(1, 1)$ so Marcinkiewicz interpolation gives $\|M^{(k)}\|_{L^p\to L^p} \le C(p-1)^{-1}$ for some constant $C > 0$ and all $p \in (1, 2]$, which implies (2.3).

Lemma 2.1. *There exists a constant C such that, for all $p \in (1, 2)$,*

(i)

$$\|\mathcal{M}_0 f\|_p \le C(p-1)^{-2}\|f\|_p,$$

(ii)

$$\|\sup_{u\in U} |S^{(u)}f|\|_p \le C(p-1)^{-7}\sqrt{\log \mathfrak{N}(U)}\|f\|_p.$$

Proof. Part (i) follows from the estimate

$$|A^u_{j,0}f(x)| \le CM^{\rm str}f(x). \tag{2.4}$$

Part (ii) is more substantial and relies on the Chang–Wilson–Wolff bounds for martingales [**2**]. This is the subject of Theorem 2.2 in Guo *et al.* [**6**]. The dependence on p was not specified there, but can be obtained by a literal reading of the proof provided in Guo *et al.* [**6**, § 4]. We remark that the exponent 7 can probably be improved, but it is satisfactory for our purposes here. □

We also decompose $\widehat{\rho_0}$ and $\widehat{\mu_{0,\pm}}$ further by making an isotropic decomposition for large frequencies. Let $\zeta_0 \in C^\infty_c(\mathbb{R}^2)$ supported in $\{\xi : |\xi| < 2\}$ and such that $\zeta_0(\xi) = 1$ for $|\xi| \le 5/4$. For $\ell = 1, 2, 3, \ldots$, let

$$\zeta_\ell(\xi) = \zeta_0(2^{-\ell}\xi) - \zeta_0(2^{1-\ell}\xi).$$

Then for $\ell > 0$, ζ_ℓ is supported in the annulus $\{\xi : 2^{\ell-1} < |\xi| < 2^{\ell+1}\}$ and we have $1 = \sum_{\ell>0} \zeta_\ell(\xi)$ for ξ in the support of $\widehat{\rho_0}, \widehat{\mu_{0,\pm}}$.

Define operators $A^u_{j,\ell}$ and $T^u_{j,\ell,\pm}$ by

$$\widehat{A^u_{j,\ell}f}(\xi) = \zeta_\ell(2^{-j}\xi_1, 2^{-jb}u\xi_2)\widehat{\rho_0}(2^{-j}\xi_1, 2^{-jb}u\xi_2)\widehat{f}(\xi), \tag{2.5}$$

$$\widehat{T^u_{j,\ell,\pm}f}(\xi) = \zeta_\ell(2^{-j}\xi_1, 2^{-jb}u\xi_2)\widehat{\mu_{0,\pm}}(2^{-j}\xi_1, 2^{-jb}u\xi_2)\widehat{f}(\xi). \tag{2.6}$$

We shall show the following proposition.

Proposition 2.2. *There is $C > 0$ such that for each $\ell > 0$, $p \in (1, 2]$, we have*

$$\| \sup_{u\in U} \sup_{j\in\mathbb{Z}} |A^u_{j,\ell}f| \|_p \leq C\vartheta_{p,\ell}\mathcal{K}_p(U, 2^{-\ell})\|f\|_p, \tag{2.7}$$

where $\vartheta_{p,\ell} = (p-1)^{3-10/p}\mathbb{1}_{\ell\leq(p-1)^{-1}} + \ell^{7(2/p-1)}\mathbb{1}_{\ell>(p-1)^{-1}}$ and

$$\left\| \sup_{u\in U} \left| \sum_{j\in\mathbb{Z}} T^u_{j,\ell,\pm}f \right| \right\|_p \leq C(p-1)^{-2}\vartheta_{p,\ell}\mathcal{K}_p(U, 2^{-\ell})\|f\|_p. \tag{2.8}$$

We claim that Proposition 2.2 implies Theorem 1.3. Indeed, we have for non-negative f,

$$\mathcal{M}^U f \lesssim \mathcal{M}_0 f + \sum_{\ell>0} \sup_{u\in U} \sup_{j\in\mathbb{Z}} |A^u_{j,\ell}f|$$

and thus (1.4) follows from part (i) of Lemma 2.1 and (2.7). It remains to show (1.5). But in view of the decomposition

$$H^{(u)} = S^{(u)} + \sum_{\pm}\sum_{\ell>0}\sum_{j\in\mathbb{Z}} T^u_{j,\ell,\pm},$$

this follows from part (ii) of Lemma 2.1 and (2.8). This finishes the proof of Theorem 1.3.

We conclude this section with some estimates that will be used in the proof of Proposition 2.2. We will harvest the required decay in ℓ from the following simple estimate. For $p \in [1, 2]$, $\ell > 0$, $j \in \mathbb{Z}$, $u \in (0, \infty)$, we have

$$\|A^u_{j,\ell}f\|_p \leq C2^{-\ell(1-1/p)}\|f\|_p. \tag{2.9}$$

Indeed, the endpoint $p = 2$ is a consequence of Plancherel's theorem and van der Corput's lemma, while $p = 1$ follows because the convolution kernel of $A^u_{j,\ell}f$ is L^1-normalized. Another key ingredient will be the following pointwise estimate. From the definition of $A^u_{j,\ell}$ in (2.5) we have, for $\ell > 0$, $j \in \mathbb{Z}$, $u \in (0, \infty)$, that

$$|A^u_{j,\ell}f| \leq CM^{\mathrm{str}}(\tau^u_j * |f|). \tag{2.10}$$

This follows because we have

$$A^u_{j,\ell}f = (f * \tau^u_j) * \kappa^u_{j,\ell},$$

with $\kappa^u_{j,\ell}$ certain Schwartz functions that can be read off from definitions (2.2) and (2.5) and satisfy $|f * \kappa^u_{j,\ell}| \leq CM^{\mathrm{str}}f$ with $C > 0$ not depending on j, ℓ, u.

We also need to introduce appropriate Littlewood–Paley decompositions. Let $\chi^{(1)}$ be an even C^∞ function supported on

$$\{\xi_1 : |c_+|b2^{-3b-1} \le |\xi_1| \le |c_+|b2^{3b+1}\}$$

and equal to 1 for $|c_+|b2^{-3b} \le |\xi_1| \le |c_+|b2^{3b}$. Let $\chi^{(2)}$ be an even C^∞ function supported on

$$\{\xi_2 : 2^{-2b-1} \le |\xi_2| \le 2^{2b+1}\}$$

and equal to 1 for $2^{-2b} \le |\xi_2| \le 2^{2b}$. Define $P^{(1)}_{k_1,\ell}$, $P^{(2)}_{k_2,\ell,b}$ by

$$\widehat{P^{(1)}_{k_1,\ell}f}(\xi) = \chi^{(1)}(2^{-k_1-\ell}\xi_1)\widehat{f}(\xi),$$
$$\widehat{P^{(2)}_{k_2,\ell,b}f}(\xi) = \chi^{(2)}(2^{-k_2 b-\ell}\xi_2)\widehat{f}(\xi).$$

Then for $s \in [1, 2^b]$,

$$A^{2^{bn}s}_{j,\ell} = A^{2^{bn}s}_{j,\ell} P^{(2)}_{j-n,\ell,b} P^{(1)}_{j,\ell} = P^{(1)}_{j,\ell} P^{(2)}_{j-n,\ell,b} A^{2^{bn}s}_{j,\ell}. \tag{2.11}$$

For $p \in (1, 2)$ we have the Littlewood–Paley inequalities

$$\left\| \left(\sum_{k_1\in\mathbb{Z}} \sum_{k_2\in\mathbb{Z}} \left| P^{(1)}_{k_1,\ell} P^{(2)}_{k_2,\ell,b} f \right|^2 \right)^{1/2} \right\|_p \le C(p-1)^{-2}\|f\|_p \tag{2.12}$$

and

$$\left\| \sum_{k_1\in\mathbb{Z}} \sum_{k_2\in\mathbb{Z}} P^{(1)}_{k_1,\ell} P^{(2)}_{k_2,\ell,b} f_{k_1,k_2} \right\|_p \le C(p-1)^{-2} \left\| \left(\sum_{k_1\in\mathbb{Z}} \sum_{k_2\in\mathbb{Z}} |f_{k_1,k_2}|^2 \right)^{1/2} \right\|_p, \tag{2.13}$$

which also hold for Hilbert-space-valued functions. Similarly to (2.3), both of these inequalities follows from two applications of appropriate one-dimensional Littlewood–Paley inequalities and the fact that these come with a constant of $(p-1)^{-1}$ each, owing to Marcinkiewicz interpolation with the weak $(1, 1)$ endpoint.

3. A positive bilinear operator

In this section we are given for every $n \in \mathbb{Z}$ an at most countable set

$$\mathfrak{S}(n) = \{s_n(i) : \ i = 1, 2, \ldots\} \subset [1, 2^b].$$

Proposition 3.1. *There is a constant C, independent of the choice of the sets $\mathfrak{S}(n) = \{s_n(i)\}$, $n \in \mathbb{N}$, such that, for $1 < p \le 2$ and $\ell > 0$,*

$$\Big\| \Big(\sum_{j,n\in\mathbb{Z}} \sum_{i\in\mathbb{N}} \Big| w_n(i)\, \mathcal{A}_{j,\ell}^{2^{bn}s_n(i)} f \Big|^2 \Big)^{1/2} \Big\|_p \le C(p-1)^{3-10/p} 2^{-\ell(p-1)/2} \sup_{n\in\mathbb{Z}} \|w_n\|_{\ell^p} \|f\|_p$$

for all functions f and $w_n : \mathbb{N} \to \mathbb{C}$. This holds for $\mathcal{A}_{j,\ell}^{2^{bn}s_n(i)}$ being any one of the following:

$$A_{j,\ell}^{2^{bn}s_n(i)}, \quad 2^{-\ell} \frac{\mathrm{d}}{\mathrm{d}s} A_{j,\ell}^{2^{bn}s} \Big|_{s=s_n(i)}, \quad T_{j,\ell,\pm}^{2^{bn}s_n(i)}, \quad 2^{-\ell} \frac{\mathrm{d}}{\mathrm{d}s} T_{j,\ell,\pm}^{2^{bn}s} \Big|_{s=s_n(i)}.$$

We will only detail the proof in the case $\mathcal{A}_{j,\ell}^{2^{bn}s_n(i)} = A_{j,\ell}^{2^{bn}s_n(i)}$. The other cases follow *mutatis mutandis*. To this end note that the corresponding variants of the main ingredients (2.9)–(2.11) also hold for each of the other cases, the underlying reasoning being identical in each case.

In the proof of the proposition we use a bootstrapping argument by Nagel *et al.* [**9**] in a simplified and improved form given in unpublished work by Christ (see Carbery [**1**] for an exposition).

We first introduce an auxiliary maximal operator. For $R \in \mathbb{N}$, let

$$\mathfrak{M}_R[f, w](x) = \sup_{-R\le j,n\le R} \sup_{i\in\mathbb{N}} |w_n(i)\, \tau_j^{2^{bn}s_n(i)} * f(x)|.$$

We let $B_p(R)$ be the best constant C in the inequality

$$\|\mathfrak{M}_R[f, w]\|_p \le C \sup_{n\in\mathbb{Z}} \|w_n\|_{\ell^p} \|f\|_p,$$

that is,

$$B_p(R) = \sup \Big\{ \|\mathfrak{M}_R[f, w]\|_p : \|f\|_p \le 1, \sup_{n\in\mathbb{Z}} \|w_n\|_{\ell^p} \le 1 \Big\}. \tag{3.1}$$

The positive number $B_p(R)$ is finite, as from the uniform L^p-boundedness of the operator $f \mapsto \tau_j^u * f$ we have $B_p(R) \le C(2R+1)^{2/p}$. It is our objective to show that $B_p(R)$ is independent of R. More precisely, we claim that there is a constant C independent of the choice of the sets $\mathfrak{S}(n)$, such that for $1 < p \le 2$,

$$B_p(R) \le C(p-1)^{2-10/p}. \tag{3.2}$$

We begin with an estimate for a vector-valued operator.

Lemma 3.2. *Let* $1 < p \le 2$, $p \le q \le \infty$. *Then*

$$\Big\| \Big(\sum_{-R\le j,n\le R} \sum_{i\in\mathbb{N}} |w_n(i)\, A_{j,\ell}^{2^{bn}s_n(i)} g_{j,n}|^q \Big)^{1/q} \Big\|_p$$
$$\le C(p-1)^{-2(1-p/q)} B_p(R)^{1-p/q} 2^{-\ell(1-1/p)pq} \sup_{n\in\mathbb{Z}} \|w_n\|_{\ell^p} \Big\| \Big(\sum_{j,n\in\mathbb{Z}} |g_{j,n}|^q \Big)^{1/q} \Big\|_p. \quad (3.3)$$

Proof. The case $q = p$ of (3.3) follows from (2.9). For $q = \infty$ we use (2.10) to estimate

$$\Big\| \sup_{-R\le j,n\le R} \sup_{i\in\mathbb{N}} |w_n(i)\, A_{j,\ell}^{2^{bn}s_n(i)} g_{j,n}| \Big\|_p$$
$$\le C \Big\| \sup_{-R\le j,n\le R} \sup_{i\in\mathbb{N}} |w_n(i)|\, M^{\mathrm{str}}[\tau_j^{2^{bn}s_n(i)} * |g_{j,n}|] \Big\|_p$$
$$\le C \Big\| M^{\mathrm{str}} \Big[\sup_{-R\le j,n\le R} \sup_{i\in\mathbb{N}} |w_n(i)|\, \tau_j^{2^{bn}s_n(i)} * \Big(\sup_{j',n'\in\mathbb{Z}} |g_{j',n'}| \Big) \Big] \Big\|_p,$$

where we have used the positivity of the operators $f \mapsto \tau_j^u * f$. By (2.3) we can dominate the last displayed expression by

$$C'(p-1)^{-2} \Big\| \sup_{-R\le j,n\le R} \sup_{i\in\mathbb{N}} |w_n(i)|\, \tau_j^{2^{bn}s_n(i)} * \Big[\sup_{j',n'\in\mathbb{Z}} |g_{j',n'}| \Big] \Big\|_p$$
$$\lesssim (p-1)^{-2} B_p(R) \sup_{n\in\mathbb{Z}} \|w_n\|_{\ell^p} \Big\| \sup_{j',n'\in\mathbb{Z}} |g_{j',n'}| \Big\|_p$$

which establishes the case $q = \infty$. The case $p < q < \infty$ follows by interpolation. □

Proof of Proposition 3.1. We use the decomposition $\tau_j^u * f = \sum_{\ell=0}^{\infty} A_{j,\ell}^u f$. By (2.4) we get

$$\Big\| \sup_{j,n\in\mathbb{Z}} \sup_{i\in\mathbb{N}} |w_n(i)\, A_{j,0}^{2^{bn}s_n(i)} f| \Big\|_p \lesssim (p-1)^{-2} \sup_{n\in\mathbb{Z}} \|w_n\|_{\ell^\infty} \|f\|_p.$$

For $\ell > 0$, we have

$$\Big\| \sup_{-R\le j,n\le R} \sup_{i\in\mathbb{N}} |w_n(i)\, A_{j,\ell}^{2^{bn}s_n(i)} f| \Big\|_p \le \Big\| \Big(\sum_{-R\le j,n\le R} \sum_{i\in\mathbb{N}} |w_n(i)\, A_{j,\ell}^{2^{bn}s_n(i)} f|^2 \Big)^{1/2} \Big\|_p$$

and, by (2.11) and Lemma 3.2 for $q = 2$, and (2.12),

$$\Big\| \Big(\sum_{-R\le j,n\le R} \sum_{i\in\mathbb{N}} |w_n(i)\, A_{j,\ell}^{2^{bn}s_n(i)} f|^2 \Big)^{1/2} \Big\|_p$$
$$\lesssim (p-1)^{-2(1-p/2)} B_p(R)^{1-p/2} 2^{-\ell(1-1/p)p/2} \sup_{n\in\mathbb{Z}} \|w_n\|_{\ell^p} \Big\| \Big(\sum_{j,n\in\mathbb{Z}} |P_{j-n,\ell,b}^{(2)} P_{j,\ell}^{(1)} f|^2 \Big)^{1/2} \Big\|_p$$
$$\lesssim (p-1)^{p-4} 2^{-\ell(p-1)/2} B_p(R)^{1-p/2} \sup_{n\in\mathbb{Z}} \|w_n\|_{\ell^p} \|f\|_p. \quad (3.4)$$

This implies, for $1 < p \le 2$,

$$B_p(R) \lesssim \left[(p-1)^{-2} + \sum_{\ell>0}(p-1)^{p-4}2^{-\ell(p-1)/2}B_p(R)^{1-p/2}\right]$$
$$\lesssim (p-1)^{-2} + (p-1)^{p-5}B_p(R)^{1-p/2}$$

which leads to

$$B_p(R) \lesssim (p-1)^{2-10/p}.$$

If we use this inequality in (3.4) and observe

$$p-4+(2-10/p)(1-p/2) = 3-10/p,$$

then the claimed inequality in Proposition 3.1 follows by the monotone convergence theorem. □

4. Proof of Proposition 2.2

For $n \in \mathbb{Z}$, let $U_n \subset [1, 2^b]$ be defined by

$$U_n = \{2^{-bn}u : u \in [2^{bn}, 2^{b(n+1)}] \cap U\}$$

and let

$$\mathcal{N}_{n,\ell}(U) = \#\{k : [2^{-\ell}k, 2^{-\ell}(k+1)) \cap U_n \neq \emptyset\}.$$

Then we have

$$2^{-\ell(1-1/p)} \sup_{n\in\mathbb{Z}} \mathcal{N}_{n,\ell}(U) \approx \mathcal{K}_p(U, 2^{-\ell}).$$

We cover each set U_n with dyadic intervals of the form

$$I_{k,\ell} = [k2^{-\ell}, (k+1)2^{-\ell}),$$

where $k \in \mathbb{N}$. Denote by $\mathfrak{S}_{n,\ell}$ the left endpoints of these intervals and note that $\mathcal{N}_{n,\ell}(U) = \#\mathfrak{S}_{n,\ell}$. We label the set of points in $\mathfrak{S}_{n,\ell}$ by $\{s_{n,\ell}(i)\}_{i=1}^{\mathcal{N}_{n,\ell}(U)}$ and write

$$\sup_{j\in\mathbb{Z}}\sup_{u\in U}|A^u_{j,\ell}f(x)| = \sup_{j\in\mathbb{Z}}\sup_{n\in\mathbb{Z}}\sup_{s\in U_n}|A^{2^{nb}s}_{j,\ell}f(x)|$$
$$\le \sup_{j,n\in\mathbb{Z}}\ \sup_{i=1,\dots,\mathcal{N}_{n,\ell}(U)}|A^{2^{nb}s_{n,\ell}(i)}_{j,\ell}f(x)|$$
$$+ \sup_{j,n\in\mathbb{Z}}\ \sup_{i=1,\dots,\mathcal{N}_{n,\ell}(U)}\int_0^{2^{-\ell}}\left|\frac{\mathrm{d}}{\mathrm{d}\alpha}A^{2^{nb}(s_{n,\ell}(i)+\alpha)}_{j,\ell}f(x)\right|\mathrm{d}\alpha.$$

Hence

$$\Big\| \sup_{j\in\mathbb{Z}} \sup_{u\in U} |A^u_{j,\ell} f| \Big\|_p \le \Big\| \Big(\sum_{j,n\in\mathbb{Z}} \sum_{i=1}^{\mathcal{N}_{n,\ell}(U)} |A^{2^{nb} s_{n,\ell}(i)}_{j,\ell} f|^2 \Big)^{1/2} \Big\|_p$$

$$+ \int_0^{2^{-\ell}} \Big\| \Big(\sum_{j,n\in\mathbb{Z}} \sum_{i=1}^{\mathcal{N}_{n,\ell}(U)} \Big| \frac{\mathrm{d}}{\mathrm{d}\alpha} A^{2^{nb}(s_{n,\ell}(i)+\alpha)}_{j,\ell} f \Big|^2 \Big)^{1/2} \Big\|_p \,\mathrm{d}\alpha,$$

and by part (ii) of Proposition 3.1 both expressions on the right-hand side can be estimated by

$$C(p-1)^{3-10/p} 2^{-\ell(p-1)/2} \sup_{n\in\mathbb{Z}} \mathcal{N}_{n,\ell}(U)^{1/p} \|f\|_p. \tag{4.1}$$

This estimate is efficient for $1 < p < 1 + \ell^{-1}$. Note that in this range $2^{-C\ell(1-1/p)} \approx 1$ and $\mathcal{N}_{n,\ell}(U)^{1/p} \approx \mathcal{K}_p(U, 2^{-\ell})$. For $p = 2$ we have the inequality

$$\Big\| \Big(\sum_{j,n\in\mathbb{Z}} \sum_{i=1}^{\mathcal{N}_{n,\ell}(U)} |A^{2^{nb} s_{n,\ell}(i)}_{j,\ell} f|^2 \Big)^{1/2} \Big\|_2$$

$$+ \int_0^{2^{-\ell}} \Big\| \Big(\sum_{j,n\in\mathbb{Z}} \sum_{i=1}^{\mathcal{N}_{n,\ell}(U)} \Big| \frac{\mathrm{d}}{\mathrm{d}\alpha} A^{2^{nb}(s_{n,\ell}(i)+\alpha)}_{j,\ell} f \Big|^2 \Big)^{1/2} \Big\|_2 \,\mathrm{d}\alpha$$

$$\lesssim 2^{-\ell/2} \sup_{n\in\mathbb{Z}} \mathcal{N}_{n,\ell}(U)^{1/2} \|f\|_2. \tag{4.2}$$

For $p_\ell := 1 + \ell^{-1} < p < 2$ we use the Riesz–Thorin interpolation theorem (together with the fact that $(p_\ell - 1)^{C/\ell} \approx_C 1$ and $(p_\ell - 1)^{-A} = \ell^A$). We then obtain, for $p_\ell < p < 2$,

$$\Big\| \Big(\sum_{j,n\in\mathbb{Z}} \sum_{i=1}^{\mathcal{N}_{n,\ell}(U)} |A^{2^{nb} s_{n,\ell}(i)}_{j,\ell} f|^2 \Big)^{1/2} \Big\|_p$$

$$+ \int_0^{2^{-\ell}} \Big\| \Big(\sum_{j,n\in\mathbb{Z}} \sum_{i=1}^{\mathcal{N}_{n,\ell}(U)} \Big| \frac{\mathrm{d}}{\mathrm{d}\alpha} A^{2^{nb}(s_{n,\ell}(i)+\alpha)}_{j,\ell} f \Big|^2 \Big)^{1/2} \Big\|_p \,\mathrm{d}\alpha$$

$$\lesssim 2^{-\ell(1-1/p)} \sup_{n\in\mathbb{Z}} \mathcal{N}_{n,\ell}(U)^{1/p} \ell^{7(2/p-1)} \|f\|_p. \tag{4.3}$$

Thus we have established (2.7). The proof of (2.8) is similar but the reduction to a square-function estimate requires one more use of a Littlewood–Paley estimate. We have, using the analogue of (2.11) for $T^{2^{bn}s}_{j,\ell,+}$,

$$\Big\| \sup_{n\in\mathbb{Z}} \sup_{u\in U\cap[2^{nb},2^{(n+1)b}]} \Big| \sum_{j\in\mathbb{Z}} T^u_{j,\ell,+} f \Big| \Big\|_p$$

$$\leq \Big\| \Big(\sum_{n\in\mathbb{Z}} \sum_{i=1}^{\mathcal{N}_{n,\ell}} \Big| \sum_{j\in\mathbb{Z}} P^{(1)}_{j,\ell} P^{(2)}_{j-n,\ell,b} T^{2^{nb}s_{n,\ell}(i)}_{j,\ell,+} f \Big|^2 \Big)^{1/2} \Big\|_p$$

$$+ \int_0^{2^{-\ell}} \Big\| \Big(\sum_{n\in\mathbb{Z}} \sum_{i=1}^{\mathcal{N}_{n,\ell}(U)} \Big| \sum_{j\in\mathbb{Z}} P^{(1)}_{j,\ell} P^{(2)}_{j-n,\ell,b} \frac{\mathrm{d}}{\mathrm{d}\alpha} T^{2^{nb}(s_{n,\ell}(i)+\alpha)}_{j,\ell,+} f \Big|^2 \Big)^{1/2} \Big\|_p \,\mathrm{d}\alpha,$$

which by (2.13) is bounded by

$$C(p-1)^{-2} \Big[\Big\| \Big(\sum_{n\in\mathbb{Z}} \sum_{i=1}^{\mathcal{N}_{n,\ell}(U)} \sum_{j\in\mathbb{Z}} |T^{2^{nb}s_{n,\ell}(i)}_{j,\ell,+} f|^2 \Big)^{1/2} \Big\|_p$$

$$+ \int_0^{2^{-\ell}} \Big\| \Big(\sum_{n\in\mathbb{Z}} \sum_{i=1}^{\mathcal{N}_{n,\ell}(U)} \sum_{j\in\mathbb{Z}} \Big| \frac{\mathrm{d}}{\mathrm{d}\alpha} T^{2^{nb}(s_{n,\ell}(i)+\alpha)}_{j,\ell,+} f \Big|^2 \Big)^{1/2} \Big\|_p \,\mathrm{d}\alpha \Big].$$

From here on the estimation is exactly analogous to the previous square function: just replace $A^u_{j,\ell}$ with $T^u_{j,\ell,+}$. The arguments for the corresponding terms with $T^u_{j,\ell,-}$ are similar (or could be reduced to the previous case by a change of variable and curve). This concludes the proof of Theorem 2.2.

5. Lower bounds for $p \leq 2$

As mentioned before, the lower bound $(\log \mathfrak{N}(U))^{1/2}$ for $\|\mathcal{H}^U\|_{L^p\to L^p}$, based on ideas of Karagulyan [7], was established in Guo *et al.* [6]. We now show the easier lower bound in terms of the quantity $\sup_{\delta>0} \mathcal{K}_p(U,\delta)$ (where we only have to consider the cases $\delta < 1$). The same calculation gives the same type of lower bound for $\|\mathcal{M}^U\|_{L^p\to L^p}$.

By rescaling in the second variable and reflection we may assume that $c_+ = 1$. For $u \in U$ and $\delta \in (0,1)$ we define

$$V_\delta(u) = \{(x_1,x_2) : 1 \leq x_1 \leq 2, |x_2 - ux_1^b| \leq \delta/4\},$$

and let f_δ be the characteristic function of the ball of radius δ centred at the origin. Observe that for $1 \leq x_1, u \leq 2$, $\varepsilon < 1$ and $x_1 \leq t \leq x_1 + \varepsilon\delta$, we have $u(t^b - x_1^b) \leq 2b \cdot 3^{b-1}\varepsilon\delta$. Thus for $\varepsilon_b = (8b \cdot 3^{b-1})^{-1}$ we get $f_\delta(x_1 - t, x_2 - ut^b) = 1$ and thus

$$H^{(u)} f_\delta(x) \geq \frac{1}{3} \int_{x_1}^{x_1+\varepsilon_b\delta} f_\delta(x_1 - t, x_2 - ut^b)\,\mathrm{d}t \geq \frac{\varepsilon_b}{3}\delta, \quad x \in V_\delta(u).$$

By rescaling in the second variable we have, for every $r > 0$, that

$$\|\mathcal{H}^U\|_{L^p\to L^p} \geq \|\mathcal{H}^{U^r}\|_{L^p\to L^p},$$

where $U^r = r^{-1}U \cap [1,2]$. Let $U^r(\delta)$ be a maximal $2^b\delta$-separated subset of U^r; then $\#U^r(\delta) \gtrsim N(U^r,\delta)$. This implies

$$\mathcal{H}^{U^r(\delta)} f_\delta(x) \gtrsim \delta \quad \text{for } x \in V_{r,\delta} := \bigcup_{u\in U^r(\delta)} V_\delta(u).$$

For different $u_1, u_2 \in U^r(\delta)$ the sets $V_\delta(u_1)$ and $V_\delta(u_2)$ are disjoint and therefore we have $\text{meas}(V_{r,\delta}) \gtrsim \delta\#(U_r(\delta))$. Hence we get

$$\|\mathcal{H}^{U^r(\delta)} f_\delta\|_p \geq c\delta^{1+1/p}\#(U_r(\delta))^{1/p}.$$

Since also $\|f_\delta\|_p \lesssim \delta^{2/p}$, we obtain

$$\|\mathcal{H}^U\|_{L^p\to L^p} \geq \|\mathcal{H}^{U^r(\delta)}\|_{L^p\to L^p} \gtrsim \delta^{1-1/p}\#(U^r(\delta))^{1/p} \gtrsim \delta^{1-1/p}N(U^r,\delta)^{1/p},$$

which gives the uniform lower bound

$$\|\mathcal{H}^U\|_{L^p\to L^p} \gtrsim \mathcal{K}_p(U,\delta) \tag{5.1}$$

for sufficiently small δ.

Acknowledgements. S.G. was supported in part by NSF grant 1800274. A.S. was supported in part by NSF grant 1764295. P.Y. was partially supported by a General Research Fund CUHK14303817 from the Hong Kong Research Grant Council, and a direct grant for research from the Chinese University of Hong Kong (4053341).

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