

Bidding Strategies with Gender Nondiscrimination Constraints for Online Ad Auctions

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ABSTRACT

Interactions between bids to show ads online can lead to an advertiser's ad being shown to more men than women even when the advertiser does not target towards men. We design bidding strategies that advertisers can use to avoid such emergent discrimination without having to modify the auction mechanism. We mathematically analyze the strategies to determine the additional cost to the advertiser for avoiding discrimination, proving our strategies to be optimal in some settings. We use simulations to understand other settings.

CCS CONCEPTS

• Information systems → Online advertising.

KEYWORDS

targeted advertising, online auctions, fairness constraints, MDPs

ACM Reference Format:

Milad Nasr and Michael Carl Tschantz. 2020. Bidding Strategies with Gender Nondiscrimination Constraints for Online Ad Auctions. In *Conference on Fairness, Accountability, and Transparency (FAT* '20)*, January 27–30, 2020, Barcelona, Spain. ACM, New York, NY, USA, 15 pages. <https://doi.org/10.1145/3351095.3375783>

1 INTRODUCTION

Prior work found Google showing an ad for the Barrett Group, a career coaching service promoting the seeking of high paying jobs, more often to simulated men than to simulated women [7]. Later work enumerates possible causes of this disparity [6].

One possibility, raised by Google itself [25], is that the Barrett Group targeted both men and women equally, but other advertisers, on average, focused more on women, which would be in line with subsequent findings [18]. In this possibility, the Barrett Group found itself outbid for just women by the other advertisers who were willing to pay more than it was for reaching women but not for men. These other advertisers might be promoting products that many find acceptable to target toward women, such as dresses. Thus, it's possible that each advertiser's targeting appears reasonable in isolation but interacts to bring about emergent discrimination for a job-related ad.

For conscientious advertisers of products that should be broadcasted to women and men at equal rates, such an outcome is unacceptable but currently difficult to avoid. While Google offers the ability to skew ads toward men or toward women, it provides no way to ensure that both men and women see the ad an equal number of times. As discussed above, simply not targeting by gender is not enough to guarantee parity. Even running two ad campaigns of equal size is insufficient since the size is determined by budget and not the number of ads shown, which means that parity would only be achieved if women and men are equally expensive to reach.

We consider how advertisers can ensure approximate demographic parity for its ads without changing Google's ad auction mechanism, which is based on a second-price auction [10]. Given that an advertiser wishes to maximize its utility by reaching the people most likely to respond to its ads, we model the advertiser's utility function along with the parity goal as a constrained bidding problem. We consider both a strict absolute parity constraint and a more relaxed relative constraint inspired by the US EEOC's four-fifths rule on disparate impact [9]. Although using a second-price auction suggests that the advertisers should bid their true value of showing an ad, a parity constraint and multiple rounds of the auction interact to make deviations from this truthful strategy optimal. Intuitively, as in multi-round second-price auctions with budget constraints [12], it is sometimes better to bid less to preserve the ability to participate in later auctions that might have a lower cost of winning. More interestingly, unlike with just budget constraints, it is also sometimes better to bid more to ensure an acceptable degree of parity, enabling participation in other auctions later.

Given these complexities, finding an optimal bidding strategy for such a constrained bidding problem is non-trivial. We do so by modeling them as Markov Decision Problems (MDPs). Solving these MDPs using traditional methods, such as value iteration, is made difficult by the continuous space of possible bid values over which to optimize. To avoid this issue, we find recursive formulae for each type of constraint providing the optimal bid value and solve for their values instead. This approach allows us to solve the MDPs without needing to explicitly maximize over the possible actions as in value iteration.

We compare this optimal constrained bidding strategy to the optimal unconstrained strategy for both real and simulated data sets. The cost to the advertiser for ensuring parity varies by setting, but is manageable under the more realistic settings explored. In all cases, the revenue of the simulated ad auctioneer (Google's role) remains roughly the same or goes up.

By not modifying the core auction algorithm used by Google and instead suggesting bidding strategies that could be deployed by the advertisers, we believe this work provides a practical path towards nondiscriminatory advertising.

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FAT* '20, January 27–30, 2020, Barcelona, Spain

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ACM ISBN 978-1-4503-6936-7/20/02...\$15.00

<https://doi.org/10.1145/3351095.3375783>

2 RELATED WORK

The most closely related work, recently looked at enforcing parity constraints with auction mechanisms, whereas we do so with bidding strategies [4, 5]. While both approaches have their use cases, we believe ours is easier to deploy since just the advertisers wanting the feature need to make changes to implement it. We further discuss tradeoffs between deployment approaches in Section 8. Our approach also differs by using strict constraints whereas theirs uses probabilistic constraints. Probabilistic constraints allow more utility but may be insufficient in cases where approximate parity is required, as when disparate impact is prohibited. At an algorithmic level, they differ by using gradient decent.

A similar alternative approach could use auction mechanisms with Guaranteed ad Delivery (GD) [22, 26]. An advertiser can act as two parties to the auction, one for each gender, and use GD to ensure an equal number of wins for each party. Unlike our bidding strategy, which an advertiser can unilaterally employ, this approach requires the ad exchange to change its auction mechanisms.

Prior works have looked at how to enforce (proportional) parity constraints on the classifications produced by ML algorithms [2, 3, 16, 28]. We instead look at auctions.

Prior works have used MDPs to model ad slot auctions. Li et al. [19] and Iyer et al. [13–15] have used them to find optimal bidding strategies when advertisers do not know the exact values of each type of ad slot and learn values by winning them. They showed that advertisers should overbid to learn more information. Gummadi et al. [12] described the optimal bidding strategy for the second-price auction in which each advertiser has a limited budget, which leads to underbidding. Zhang et al. [29] derived optimal real-time bidding strategies when each ad slot have different properties.

3 ONLINE AD AUCTIONS

When a person visits a webpage, the webpage will often contain dynamically loaded ads at fixed locations on the page. These ads each occupy an ad *slot*, a location at a time (or page load) on the webpage. In some cases, the website selects which ads to show in which slots itself, such as with Facebook. In other cases, the website contracts with a third-party, to fill and charge for the slots in exchange for payments to the website. In either case, we call the entity choosing how to fill the slots an *ad exchange*. For example, Google runs an ad exchange, Google Ad Manager, which includes slots put up for sale by websites with its AdSense tool.

Typically, an ad exchange auctions off the slots it controls to advertisers. It can use *real-time bidding* to auction off the slots as the webpage loads. The website and the ad exchange can offer advertisers various amounts of information about the slot, such as the webpage it is on and demographics about who is loading the page. Advertisers performing *programmatic advertising* use a dynamic bidding strategy that adjusts their bids according to how well they expect their ads to perform in the offered slot. To avoid having to create programs for executing such strategies on their own, advertisers often use a *demand-side platform* (DSP). Figure 1 demonstrates a sketch of the interactions.

An ad exchange may accept bids that are more complex than just a single price, such as including an offer to pay a bonus if the website visitor clicks the ad [10]. Exchanges wishing to maximize

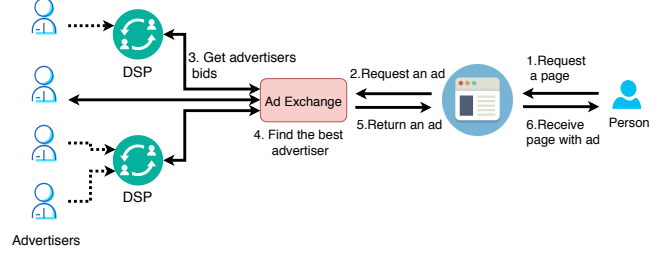


Figure 1: Online advertisement interactions

the amount of bonuses it receives, or to avoid annoying visitors, might consider the quality of the ad and its fit for the slot. For simplicity, we will defer further consideration of these complications to Section 9 and presently presume that all bids are simply offers to pay for showing the ad.

Second price auctions is a common mechanism for resolving such auctions, with Google using a variation of one [10], and we will presume the ad exchange uses one. In this auction mechanism, the exchange selects the highest bidder as the winner but only charges the bidder the price offered by the second highest bidder. Under certain circumstances, this mechanism ensures that each bidder's optimal strategy is to bid the actual amount it values the slot at, making the mechanism *truthful*. Since ad exchanges sometimes sell more than one slot at a time, such as for a webpage with multiple slots, they often use generalized second-price auctions, known as position auctions [8, 27].

We model the above economy as a sequential game of incomplete information, where in each round of the game a set of self-interested rational advertisers bid to win an ad slot through a second-price auction. We allow bids to vary over auctions and assume that each advertiser has a geometric lifespan. For simplicity, we make the total number of advertisers α equal in all auctions by assuming that every time an advertiser dies a new advertiser joins.

At time t , each advertiser i submits a bid b_i^t . Let b_{-i}^t be the bids of other the advertisers. The ad exchange platform runs a second-price auction where advertiser i wins the ad slot if its bid is higher than all other bids: $b_i^t > \max b_{-i}^t$. For simplicity, we assume no ties, ensuring that such a winner exists. Let a_i^t be 1 if the advertiser i wins at round t and be 0 otherwise. If the advertiser i wins it will pay the second highest bid $d_i^t = \max b_{-i}^t$. The cost of the auction t is $c_i^t = a_i^t * d_i^t$ since the advertiser i only pays if it wins.

The ad slot auctioned at t has a value v_i^t for the advertiser i . When an advertiser i wins auction t , it gets an immediate reward, which is the value v_i^t less its price d_i^t . Thus, the utility of advertiser i gained at each round is $u_i^t = a_i^t * v_i^t - c_i^t = a_i^t (v_i^t - d_i^t)$. Let the geometric parameter for the lifespan distribution for advertiser i be δ_i . The total utility for each advertiser is $U_i = \sum_{t=0}^{\infty} \delta_i^t * a_i^t (v_i^t - d_i^t)$ where δ_i^t is δ_i raised to the t th power, not indexing by t .

The advertiser i should select its bids b_i^t to maximize the expected value of U_i , where the expectation is over its value v_i^t and the bids of other advertisers b_{-i}^t . The advertiser can use market research, its prior experiences, and any information provided by the ad exchange to estimate these uncertain values. In the case of a pure second-price auction, the values of the other bids b_{-i}^t are irrelevant and the

optimal strategy is to always set its bid b_i^t equal to its estimation of its value v_i^t .

However, this result does not carry over to all second-price auctions with constraints, including the parity constraints we consider. In this case, the behavior the other advertisers matters, but estimating it for individual ad slots is difficult. Furthermore, the advertiser is unlikely to estimate the value of every ad slot individually even for a pure second-price auction. Rather, the advertiser will likely model ad slots as each having a *type* belonging to a set Θ of reasonable size. The types will represent the most important information to the advertiser about the slot. For simplicity, we will typically assume that Θ has just two types, one for each gender. For each type θ , the advertiser will estimate the expected value v_i^θ of a slot of type θ .

For estimating the other bids, prior research [13, 14] has shown it reasonable to model them as coming from a stationary fixed distribution, due to the large number of other advertisers. To simplify our analysis, we denote the CDF of other bids for a slot of type θ by g_i^θ . Finally, let p_i^θ be the probability that advertiser i assigns to the type θ .

With these estimations, we compute estimations of other key quantities. The probability of winning on auction t for a slot of type θ with a bid of x is $q(x; g_i^\theta) = \Pr(\max b_{-i}^t \leq x) = g_i^\theta(x)^{(\alpha-1)}$ where α is the number of advertisers at each ad slot auction. The expected value of the utility for the advertiser i for a single auction given the distribution of the other advertisers' bids g_i is

$$\mathbb{E}[u_i^t] = \sum_{\theta} p_i^\theta * q(v_i^t; g_i^\theta) * (v_i^t - d_i^t) \quad (1)$$

The expected value of the total utility for each advertiser is

$$\mathbb{E}[U_i] = \sum_{t=1}^{\infty} \delta_i^{t-1} \sum_{\theta} p_i^\theta q(v_i^t; g_i^\theta) (v_i^t - d_i^t) \quad (2)$$

4 PARITY CONSTRAINTS

Advertisers may have concerns in addition to attempting to maximize the utility U_i , such as complying with laws and social norms. In some cases, this will include ensuring that its ads reach various protected groups to the same degree. For example, an employer may desire that a job ad be shown to an equal number of women and men to comply with laws prohibiting gender discrimination in hiring [6]. Such advertisers would like to place their bids in a manner to ensure such demographic parity.

However, the above auction mechanism, as well Google's actual mechanism as far as we can tell, does not offer any way of ensuring that a job ad is shown to an approximately equal number of women and men, as required by laws prohibiting gender discrimination in hiring [6]. Furthermore, ad exchanges may be unwilling to support such constraints given that only some advertisers have such concerns. Thus, our goal is to provide advertisers with a bidding strategy that dynamically adjusts bids to preserve the gender parity of the viewers, which advertisers can unilaterally use without needing changes to the auction mechanism of the ad exchange.

As an additional benefit of not modifying the ad auction mechanism, our bidding strategy can be used for any type of auction. However, we design and analyze them with only with second-price auctions in mind.

We will focus on an advertiser that wants to show an ad to equal numbers of men and women. A particularly careful advertiser may desire that this parity constraint holds not only at the end of ad campaign but throughout. Such continuous parity ensures that the advertiser would pass an audit checking for this property at any point in time. It also ensures meeting the parity goal if the the ad campaign must be cut short or if a sudden influx of competing advertisers prevents winning addition slots.

Meeting this strict goal is impossible since the first ad must go to either a man or woman, and not both. To account for this, we relax this goal to allow for approximate parity. We distinguish between absolute (additive) and relative (ratio) approximate parity and first consider an absolute constraint. An advertiser has *K-parity* if after each auction, the maximum difference between the number of auctions that it wins for each gender is not more than K .

Our goal is to find the optimal bidding strategy for advertisers obeying such a constraint. This task is difficult since a constrained advertiser must consider not just the immediate reward of winning a slot, but also how it may close or open the possibility of winning additional slots later. To see this, we will consider three examples involving a simplified setting in which an advertiser i is subject to 1-parity and knows exactly how long it will live. In each example, it values men and women both at 20 (with no variance), but other advertisers value women at an expected value of 21 and men at an expected value of 5. This setting reflects that advertisers are willing to pay more, on average, for women than men [18].

In the first example we consider, the advertiser knows that it will live for exactly one ad auction. In this case, the advertiser i will bid the value of the immediate reward 20 that it receives for winning an auction regardless of whether it is subject to a 1-parity constraint, since winning the auction has no effect other than that immediate reward. It will win an auction for a man and lose an auction for a woman.

Next, consider the advertiser's behavior for a series of two auctions. The interesting case is two men in a row. In this case, advertiser i can only win one of the slots since it is subject to a 1-parity constraint. Thus, the utility of the advertiser will be smaller from having 1-parity, but it need not be half that of when it is unrestricted. If the number of women is small enough ($p \ll 0.5$), the advertiser can assume it will get two men in a row and can lower the value of its bid on the first man in hopes of winning at a discount, given the fluctuations in the other advertisers' bids. We call this *underbidding*, although we emphasize that it is underbidding with respect to its immediate reward, not with respect to what is overall rational. Underbidding effectively allows the advertiser i to skip the first auction if the variance in the other advertisers' bids produces a high competing bid. This is similar to how underbidding is optimal in some repeated second-price auctions with a constrained budget [12]. The degree of underbidding must balance the chance of getting a male slot at a discount with the risk of either losing both auctions or getting a female slot for the second auction.

The opposite, *overbidding*, can also occur. To see this, consider a series of three auctions with a woman followed by two men. In this case, the advertiser i can win both men, despite the 1-parity constraint, provided that it first wins the woman. Thus, winning the woman produces not just an immediate reward, but also a future reward by unlocking the ability to win more men. If we presume

negligible variance in the other bids, the advertiser i will have to bid 22 to win the woman and pay the second price of 21, yielding an immediate reward of -1 by bidding 1 over the advertiser's inherent value of 20 for the female ad slot. Since the immediate reward of a male slot is $20 - 5 = 15$, being able to win the second man means a net positive gain of $15 - 1 = 14$. (We ignore the effects of underbidding since we are now considering negligible variance in the other advertiser's bids, which makes the effect go away.)

We find this distinction between the immediate reward and the future rewards coming from future flexibility useful for determining the optimal bidding strategy. However, doing so requires not only making the above intuitions quantitative, but also dealing with additional probabilistic factors, such as the genders of ad slots not being known in advance and the uncertain duration of the auction sequence. To overcome these difficulties, we switch to a more systematic model for each type of constraint.

5 ABSOLUTE PARITY CONSTRAINTS

To make absolute constraints precise, let Γ denote a set of groups. We are typically interested in the case where $\Gamma = \{m, w\}$ with m denoting men and w women. We use p to denote the probability of a male ad slot (i.e., p_i^m). We use n_i^Y to denote the number of ad slots for people in group Y won by the constrained advertiser i .

DEFINITION 1 (K -PARITY). *An advertiser i obeys a K -strict absolute parity constraint, or K -parity for short, for a set of groups Γ iff, after each auction, for all groups Y and Y' in Γ , the number of auctions that it wins satisfies $n_i^Y - n_i^{Y'} \leq K$.*

The constraint is *strict* by requiring that approximate parity holds at all times with certainty and not merely with high probability.

We study approximating the optimal bidding strategy that an advertiser desiring to meet a K -parity constraint can use to do so. In our analysis, we assume all of the advertisers have an unlimited budget. Thus, they can bid on all auctions in its lifespan, unless maintaining the K -parity constraint precludes it.

5.1 Modeling

To find the optimal bidding strategy for an advertiser subject to K -parity, we model the problem as a Markov Decision Problem (MDP). The obvious state space for such an MDP would have states of the form $\langle n_i^m, n_i^w, \theta \rangle$, where n_i^m and n_i^w is the current number male and female slots won, respectively, and θ is the type of the ad slot currently being auctioned off, which we presume corresponds to a gender. (θ could be generalized to allow targeting toward certain men and women.) Observing that only $n_i^m - n_i^w$ matters, we instead use a smaller space of $(2K+1) * |\Theta|$ states. We denote each state by a tuple $\langle k, \theta \rangle$, where k is the difference between male and female slots won. When the advertiser wins an ad slot for a male viewer, the advertiser goes from state k to $k+1$; for a female, it goes from k to $k-1$. The value of the θ is decided by a random process depending upon the value of p , where p is probability of the viewer being male.

To find the optimal solution, we write the Bellman equation for the MDP in the steady state. Since we consider the steady state regime, we also replace the value of each ad slot by its expected value (i.e., v_i^θ). The value function for each state except for the two

states $\langle K, m \rangle$ and $\langle -K, w \rangle$ has two parts: a reward function R that indicates the immediate reward of taking action b_i and N that is the future value the advertiser gets by doing that action. We write the value functions as follows:

$$V(k, \theta; g_i) = \max_{b_i} \left\{ R^\theta(b_i; g_i^\theta) + \delta N^\theta(b_i, k; g_i) \right\} \quad (3)$$

$R^w(b_i; g_i^w)$ and $R^m(b_i; g_i^m)$ show the reward value that advertiser i will receive if it wins an ad slot auction viewed by female or male:

$$R^\theta(b_i; g_i^\theta) = q(b_i; g_i^\theta)(v_i^\theta - d_i^\theta) \quad (4)$$

$N^\theta(b_i, k; g_i)$, the future value that advertiser i gets by bidding b_i at state $\langle k, \theta \rangle$, consists of two parts. The first part N_{win}^θ is the value that the advertiser gets if it wins while the second part N_{lose}^θ is the value when it loses. We treat g_i as providing both g_i^m and g_i^w .

$$N^\theta(b_i, k; g_i) = q(b_i; g_i^\theta) * N_{\text{win}}^\theta(k; g_i) + (1 - q(b_i; g_i^\theta)) * N_{\text{lose}}^\theta(k; g_i)$$

with

$$N_{\text{win}}^m(k; g_i) = pV(k+1, m; g_i) + (1-p)V(k+1, w; g_i)$$

$$N_{\text{win}}^w(k; g_i) = pV(k-1, m; g_i) + (1-p)V(k-1, w; g_i)$$

$$N_{\text{lose}}^\theta(k; g_i) = pV(k, m; g_i) + (1-p)V(k, w; g_i)$$

As for the two edge cases, their values are solely determined by the values of their successor states since the advertiser cannot win the current auction:

$$V(K, m; g_i) = \delta * (pV(K, m; g_i) + (1-p)V(K, w; g_i))$$

$$V(-K, w; g_i) = \delta * (pV(-K, m; g_i) + (1-p)V(-K, w; g_i))$$

5.2 Computing Optimal Bidding Strategies

Computing V with an MDP solver, such as value iteration, is complicated by the bid space being continuous. Computing V for a discretization of this space will require a fine discretization to avoid rounding errors, which will mean slow convergence. Using numerical optimization methods is complicated by V not being a linear function in b_i . To avoid these complexities, we instead rewrite V in a form that can be solved without any optimization.

To identify the optimal bidding strategy, we observe that the two edge cases do not involve a decision and the strategy of bidding 0 is forced for them. We also observe that for the remaining states the valuation function (3) includes many terms that do not change under various bidding strategies. We collect these constants into a term Λ_i , which we can ignore while optimizing the strategy. We replace $q(b_i; g_i^\theta)d_i^\theta$ by $c(b_i; g_i^\theta)$, the estimated cost of each ad slot. The remainder of the valuation function provides the *conjoint valuation* function Φ_i^θ , a decomposition previously used by Iyer et al. [13, 14]. Our Lemma 1 in Appendix A [21] shows that

$$\begin{aligned} V(k, \theta; g_i) &= \max_{b_i} \left\{ q(b_i; g_i^\theta) \Phi_i^\theta(k; g_i) - c(b_i; g_i^\theta) + \Lambda_i(k; g_i) \right\} \\ &= \max_{b_i} \left\{ q(b_i; g_i^\theta) \Phi_i^\theta(k; g_i) - c(b_i; g_i^\theta) \right\} + \Lambda_i(k; g_i) \end{aligned} \quad (5)$$

where

$$\Lambda_i(k; g_i^m, g_i^w) = \delta(pV(k, m; g_i) + (1-p)V(k, w; g_i))$$

The conjoint valuation Φ represents the reward for winning, both immediate and long-term, which is why it is multiplied by the

ALGORITHM 1: Iterative approach to find V

Input: $K, g_i, \alpha, v^m, v^w, \epsilon$
Initialize $V[-K : K, m] \leftarrow \frac{v^m + v^w}{2}; V[-K : K, w] \leftarrow \frac{v^m + v^w}{2}$
repeat
 $\Delta \leftarrow 0$
 for k in $\{-K, \dots, K\}$ **do**
 for θ in $\{m, w\}$ **do**
 $V'[k, \theta] \leftarrow R^\theta(\Phi^\theta(k); g_i^\theta) + \delta N^\theta(\Phi^\theta(k), k; g_i)$
 $\Delta \leftarrow \max(\Delta, |V'[k, \theta] - V[k, \theta]|)$
 end
 end
 $V \leftarrow V'$
until $\Delta < \epsilon$

probability of winning $q(b_i; g_i^\theta)$. The expected cost of winning $c(b_i; g_i^\theta)$ is subtracted from this product. Φ breaks down along the lines of winning and losing cases, as N did:

$$\Phi^\theta(k; g_i) = v_i^\theta + \delta(\Phi_{\text{win}}^\theta(k; g_i) - \Phi_{\text{lose}}^\theta(k; g_i)) \quad (6)$$

where

$$\Phi_{\text{win}}^m(k; g_i) = pV(k+1, m; g_i) + (1-p)V(k+1, w; g_i)$$

$$\Phi_{\text{win}}^w(k; g_i) = pV(k-1, m; g_i) + (1-p)V(k-1, w; g_i)$$

$$\Phi_{\text{lose}}^\theta(k; g_i) = pV(k, m; g_i) + (1-p)V(k, w; g_i)$$

The term v_i^θ represents the immediate value of winning the ad slot. The reminder considers the gain that the advertiser gets from the future by winning (moving to a new state) or losing (staying put). The difference between future rewards for winning and those for losing corresponds to the amount of overbidding (or underbidding) called for, which explains the subtraction in (6).

The following theorem shows the use of this decomposition.

THEOREM 1. *For all K, i, g_i , and states $\langle k, \theta \rangle$ other than the edge cases $\langle K, m \rangle$ and $\langle -K, w \rangle$, the optimal bid for i subject to K -parity under the distribution g_i at the state $\langle k, \theta \rangle$ is $\Phi_i^\theta(k; g_i)$.*

The proof is in Appendix B [21].

This theorem means that we do not need to search the space of possible bid values to find the optimal bid. Rather, we can just compute the optimal bid using Φ . While Φ depends upon the value function V , we can recursively make use of this fact to compute V without such a search either. In particular, the theorem implies that

$$V(k, \theta; g_i) = R^\theta(\Phi_i^\theta(k; g_i); g_i^\theta) + \delta N^\theta(\Phi_i^\theta(k; g_i), k; g_i)$$

However, this equation is still not a closed-form solution. Thus, Algorithm 1 does this calculation iteratively to converge to the states' values. Although, showing the convergence in general is an open problem, as discussed in Section 7, our experiments find convergence within a reasonable tolerance within a feasible number of iterations.

To use our approach, an advertiser (or DSP) runs Algorithm 1 to compute the value function V and stores it as a look-up table. Then, for each new ad auction, the advertiser first checks if it winning the auction would violate the parity constraint. If so, it will not participate in the auction (i.e., bids zero). Otherwise, The advertiser bids the value of $\Phi_i^\theta(k; g_i)$, which can be easily computed from value functions.

6 RATIO CONSTRAINTS

While constraints on the difference between the number of ads shown to each gender are intuitive, the EEOC's four-fifths rule found in US regulations against disparate impact in employment instead focuses on a ratio [9]. The ratio considered is not simply between the number of ads shown to each gender. Rather, it acknowledges that parity can be unrealistic due to having differing numbers of male and female applicants. It adjusts for that factor by comparing the fraction of female applicants receiving a job offer to the fraction male applicants receiving a job offer. It requires that this ratio of ratios be between 5/4 and 4/5. We provide a similar constraint that compares two ratios, checking whether the fraction of female ad slots won is within a factor of r to the fraction of male ad slots won.

Strictly enforcing such a check creates problems when the number of slots seen so far is small since the fractions won may be very far apart even when the number of ads shown to each gender only differs by 1. To avoid this issue, we also allow an additive difference in the number of ads show to each gender. The resulting rule may be viewed as a hybrid between a pure ratio constraint and the absolute constraint we have already presented.

We use similar notation as in Section 5.1 to express this constraint in a manner that avoids division by zero.

DEFINITION 2 ((r, K)-RATIO). *An advertiser obeys a (r, K)-strict relative constraint, or (r, K)-ratio constraint for short, for groups Γ iff, after each auction, for all groups γ and γ' in Γ , the number of auctions that it wins satisfies $rp_\gamma n_{\gamma'} \leq p_{\gamma'} n_\gamma + K$ where p_γ and $p_{\gamma'}$ are the probability of seeing slots for groups γ and γ' , respectively.*

6.1 Modeling

Similar to the K -parity constraint, we limit ourselves to the case where Γ and Θ only contain two types, which we treat as male and female. We use p as the probability of a male. We denote each state by a triplet $\langle n_i^m, n_i^w, \theta \rangle$, where n_i^m and n_i^w is the current number male and female slots won, respectively.

While we reuse the immediate reward function R^θ from (4), we rewrite the value function V and future value function N . When winning the slot would not violate the constraint,

$$V(n_i^m, n_i^w, \theta; g_i) = \max_{b_i} \left\{ R^\theta(b_i; g_i^\theta) + \delta N^\theta(b_i, n_i^m, n_i^w; g_i) \right\}$$

When offered a male that may not be won because $r(1-p)(n_i^m+1) > pn_i^w + K$ where n_i^m is the current number of males won,

$$V(n_i^m, n_i^w, m; g_i) = \delta (pV(n_i^m, n_i^w, m; g_i) + (1-p)V(n_i^m, n_i^w, w; g_i))$$

When a female may not be won since $rp(n_i^w+1) > (1-p)n_i^m + K$,

$$V(n_i^m, n_i^w, w; g_i) = \delta (pV(n_i^m, n_i^w, m; g_i) + (1-p)V(n_i^m, n_i^w, w; g_i))$$

We call these two cases *edge cases*.

We set the future value $N^\theta(b_i, n_i^m, n_i^w; g_i)$ at

$$q(b_i; g_i^\theta) * N_{\text{win}}^\theta(n_i^m, n_i^w; g_i) + (1 - q(b_i; g_i^\theta)) * N_{\text{lose}}^\theta(n_i^m, n_i^w; g_i)$$

with

$$N_{\text{win}}^m(n_i^m, n_i^w; g_i) = pV(n_i^m+1, n_i^w, m; g_i) + (1-p)V(n_i^m+1, n_i^w, w; g_i)$$

$$N_{\text{win}}^w(n_i^m, n_i^w; g_i) = pV(n_i^m, n_i^w+1, m; g_i) + (1-p)V(n_i^m, n_i^w+1, w; g_i)$$

$$N_{\text{lose}}^\theta(n_i^m, n_i^w; g_i) = pV(n_i^m, n_i^w, m; g_i) + (1-p)V(n_i^m, n_i^w, w; g_i)$$

ALGORITHM 2: Iterative approach to find V

Input: $r, K, p, g_i, \alpha, v^m, v^w, \epsilon, M$
Initialize $V[0 : M, 0 : r \frac{(1-p)}{p} M + K, m] \leftarrow \frac{v^m + v^w}{2};$
 $V[0 : M, 0 : r \frac{(1-p)}{p} M + K, w] \leftarrow \frac{v^m + v^w}{2}$
repeat
 $\Delta \leftarrow 0$
 for n_i^m in $\{0, \dots, M\}$ **do**
 for n_i^w in $\{0, \dots, r \frac{(1-p)}{p} M + K\}$ **do**
 for θ in $\{m, w\}$ **do**
 $V'[n_i^m, n_i^w, \theta] \leftarrow R^\theta(\Phi^\theta(n_i^m, n_i^w); g_i^\theta) +$
 $\delta N^\theta(\Phi^\theta(n_i^m, n_i^w), n_i^m, n_i^w; g_i)$
 $\Delta \leftarrow \max(\Delta, |V'[n_i^m, n_i^w, \theta] - V[n_i^m, n_i^w, \theta]|)$
 end
 end
 end
 $V \leftarrow V'$
until $\Delta < \epsilon$

6.2 Computing Optimal Bidding Strategies

We use a similar approach as in Section 5.2 to find optimal strategies. As before, we force the strategy to bid zero when winning would violate the constraint and do not include these cases in the optimization. As justified in Appendix C [21], we rewrite the value $V(n_i^m, n_i^w, \theta; g_i)$ as

$$\max_{b_i} \left\{ q(b_i; g_i^\theta) \Phi_i^\theta(n_i^m, n_i^w; g_i) - c(b_i; g_i^\theta) \right\} + \Lambda_i(n_i^m, n_i^w; g_i) \quad (7)$$

where

$$\Lambda_i(n_i^m, n_i^w; g_i^\theta) = \delta(pV(n_i^m, n_i^w, m; g_i) + (1-p)V(n_i^m, n_i^w, w; g_i))$$

and

$$\Phi^\theta(n_i^m, n_i^w; g_i) = v_i^\theta + \delta(\Phi_{\text{win}}^\theta(n_i^m, n_i^w; g_i) - \Phi_{\text{lose}}^\theta(n_i^m, n_i^w; g_i))$$

where

$$\Phi_{\text{win}}^m(n_i^m, n_i^w; g_i) = pV(n_i^m+1, n_i^w, m; g_i) + (1-p)V(n_i^m+1, n_i^w, w; g_i)$$

$$\Phi_{\text{win}}^w(n_i^m, n_i^w; g_i) = pV(n_i^m, n_i^w+1, m; g_i) + (1-p)V(n_i^m, n_i^w+1, w; g_i)$$

$$\Phi_{\text{lose}}^\theta(n_i^m, n_i^w; g_i) = pV(n_i^m, n_i^w, m; g_i) + (1-p)V(n_i^m, n_i^w, w; g_i)$$

THEOREM 2. For all r, K, i, g_i , and states $\langle n_i^m, n_i^w, \theta \rangle$ other than the edge cases, the optimal bid for i subject to (r, K) -ratio parity under the distribution g_i at the state $\langle n_i^m, n_i^w, \theta \rangle$ is $\Phi_i^\theta(n_i^m, n_i^w; g_i)$.

The proof is similar to that of Theorem 1.

This theorem eliminates the need for searching the space of possible bids at each state to find the optimal one. Whereas we could bound the state space for K -parity by tracking the difference k instead of the actual numbers of male and female ad slots won, we cannot similarly bound the state space for the (r, K) -ratio constraint. In practice, however, each advertiser either has a limited budget or is advertising for a limited time allowing us to estimate a finite set of reachable states. We use M to indicate estimated the maximum number of male ad slots won. Algorithm 2 computes the value of each state reachable assuming that M is not surpassed.

An advertiser using our approach, does so in the same manner as with our approach to parity constraints. That is, it first runs Algorithm 2 and stores V as a look-up table. It skips auctions when

winning would violate the constraint and otherwise bids $\Phi_i^\theta(k; g_i)$, computed from V .

We can extend this approach to recover if the advertiser underestimates M . In this case, the advertiser can use a linear approximation to estimate the optimal bid. To do so, let $\rho = \frac{n_i^w}{n_i^m}(M-1)$. If ρ is an integer value, then the advertiser bids $\Phi^\theta(\rho, M-1)$. Otherwise, the advertiser bids $\Phi^\theta(\lfloor \rho \rfloor, M-1) + (\rho - \lfloor \rho \rfloor) * (\Phi^\theta(\lceil \rho \rceil, n_i^m) - \Phi^\theta(\lfloor \rho \rfloor, n_i^m))$.

7 EXPERIMENTS

We simulate various scenarios to show the feasibility of our method and to measure the impact of our fairness constraints on utility. To do so, we implemented a second-price auction simulator in Python, where each advertiser gets the gender of the website viewer before selecting its bid and participating in the ad slot auction. To simulate the viewer, we draw their genders independent and identically from a binomial distribution with probability p where p is the probability of the viewer be male.

We focus on one advertiser i and measure how its utility changes when it has either a fairness constraint or not. When having a fairness constraint, it uses our bidding strategy, with δ set to 0.999 (unless otherwise noted) and ϵ set to 0.001. When not, it bids its immediate value v_i^t for the ad slot t , as is rational for an unconstrained second-price auction. We assume that the other advertisers are unconstrained and they always bid their immediate values.

To obtain distributions over ad values, we used both a real dataset, the Yahoo! A1 Search Marketing Advertiser Bidding Dataset, and a simulated one. Since the Yahoo! A1 data does not have exact timestamps, we could not use it to estimate the number of advertisers (i.e., α) for each ad auction. To estimate α , we visited top websites, according to <https://www.alexa.com/topsites>, that have ads using header bidding [23]. For one month (June 2019), we collected how many advertisers bid on a specific ad slot. In our observations of these sites, we never saw more than 10 advertisers bid on an ad slot. In line with this observation, we assume that there are $\alpha = 10$ advertisers bidding for each ad slot.

7.1 Real Dataset

The Yahoo! A1 Search Marketing Advertiser Bidding Dataset contains anonymized bids of advertisers participating in Yahoo! Search Marketing auctions for the top 1000 search queries from June 15, 2002, to June 14, 2003. The dataset includes 18 millions bids from more than 10,000 advertisers, but without the exact timestamps or information about the ad viewer. Each record in this dataset indicates a course timestamp with 15 minutes precision, the advertiser, the keyword, and the bid.

In our analysis, we assumed bids have a stationary distribution. We evaluate this assumption on our dataset. We use a specific keyword (keyword number 2) and we gathered all of bids from different advertisers in four days period (starting 2/15/2003). Then, we compute the empirical distribution of the bids of the first two days and the second two days. Figure 2 presents the distribution of the bids for these periods, showing that the distributions are very similar in both periods, supporting our stationarity assumption.

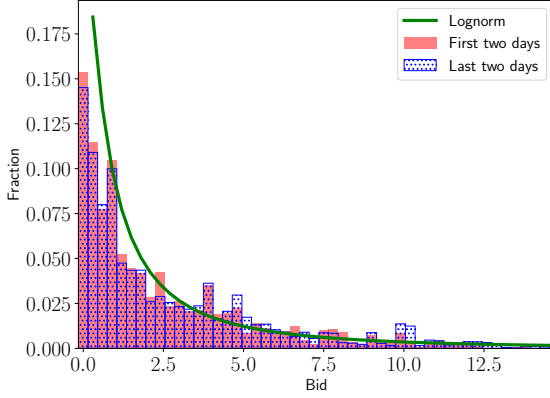


Figure 2: Distribution of the bids for two different periods. The distributions are similar in both periods which supports the assumption that bids distribution are stationary.

The figure also shows that the bids follow a log-normal distribution, in line with the findings of Balseiro et al. [1].

We enforce a separate parity constraint on each keyword in the dataset to ensure that they each get roughly equal numbers of male and female viewers. Each keyword has a different bid value distribution and we have our simulated constrained advertiser model each keyword with a separate MDP to compute the optimal bidding strategy for each. We assume that the constrained advertiser updates its model parameters every two days.

As mentioned, the Yahoo A1 dataset does not contain the exact timestamps. Therefore, we cannot exactly determine which advertisers participated in any single ad auction. We randomly select a set of advertisers' bids from each 15-minute interval for each of our ad auctions. Since the dataset does not include information about the viewers, we sample the bids for both female and male viewers from the same set of bid values, making their values equal.

Figures 3(a), 3(b), and 3(c) show the total utility ratio of the K -parity and (r, K) -ratio versions to the unconstrained version of the advertiser i for various values of K , r , and p on Yahoo A1 bid dataset. Here, and in the other simulations, we compute this ratio by simulating constrained and unconstrained versions of the advertiser i , using the same draw of values across the two versions. We do this 100 times, computing the average of total utilities U_i for each version. We then plot the ratio of these two averages. Since the value of ad slots for both female and male viewers are equal, the total utility of an unconstrained advertiser will not change for different values of p . On the other hand, a constrained advertiser will get different utilities based on the distribution of the men and women viewers. K -parity and (r, K) -ratio constraints are harder to achieve for extreme values of p . Turning to the effects of K , the results show that when K is large, the K -parity advertiser can reach the utility of the unconstrained advertiser. Also by relaxing r , (r, K) -ratio advertiser achieves higher utility.

To show the benefit of our approach compared to simply bidding immediate values, we compare the utility ratio across both

Table 1: Parameters for the log-normal distribution used in the modeling the bids in the ad slot auctions. σ^2 is always 0.7.

Scenario name	Others		Advertiser i	
	μ_{-i}^m	μ_{-i}^w	μ_i^m	μ_i^w
Equal price - Female valuable	-2.8	-2.8	-3.5	-2.4
Expensive female - Equal value	-3.5	-2.4	-2.8	-2.8

approaches. Figures 3(d) and 3(e) show that our bidding strategy allows the advertiser to achieve a higher utility ratio. We note that the difference in these utility ratios is muted by assuming that each gender is equally valuable to the advertiser. This assumption comes from the real dataset not showing which ad slots are for men and which are for women. We avoid this limitation when we use our synthetic data.

7.2 Synthetic Data

We use a synthetic dataset to explore how changing their relative values of men and women affects the advertiser's utility. We generate two synthetic datasets using a log-normal distribution to sample the advertisers bids. Table 1 shows the model parameter settings used for the two scenarios.

To show the effect of assigning different values to men and women, consider an advertiser that gives more value to female slots than to male ones, as shown in the *Equal price - Female valuable* scenario. Figure 4(a) shows the utility ratio for the K -parity and unconstrained versions of the advertiser in this scenario. The K -parity version has its maximum utility ratio when there are more male than female slots. This may seem counterintuitive since the advertiser values females more, but the measured ratio reflects that an abundance of males means that the K -parity version will not have to operate much differently from the unconstrained one. This is due to their abundance making overbidding less needed, decreasing the K -parity version's costs. Figures 4(b) and 4(c) show similar results for the (r, K) -ratio constraint.

Lambrech et al. [18] empirically showed that ad slots for young women are more expensive. To simulate this setting, we considered a scenario in which the other advertisers prefer females (i.e., $g_j^w(x) \leq g_j^m(x)$ for all $j \neq i$ and x). We used the *Expensive female - Equal value* parameter settings for this scenario. We have advertiser i value both types equally, at the average of the two different values used by the other advertisers. Figure 4(d) plots the total utility ratio as before (solid line). Note that as women become rare, the K -parity version struggles relative to the unconstrained one since the other advertisers snap up the few women leaving the constrained version unable to bid for men. The figure also shows the total utility ratio for a constrained version of the advertiser i that uses the same simple bidding strategy as the unconstrained advertisers (dashed line). Note that ratio is lower than with our optimal bidding strategy, showing its value. This difference comes from our optimal bidding strategy overbidding for the female viewers, delaying the aforementioned effect. Figure 4(e) shows that under the ratio constraint, which is less strict, the optimal strategy can do even better.

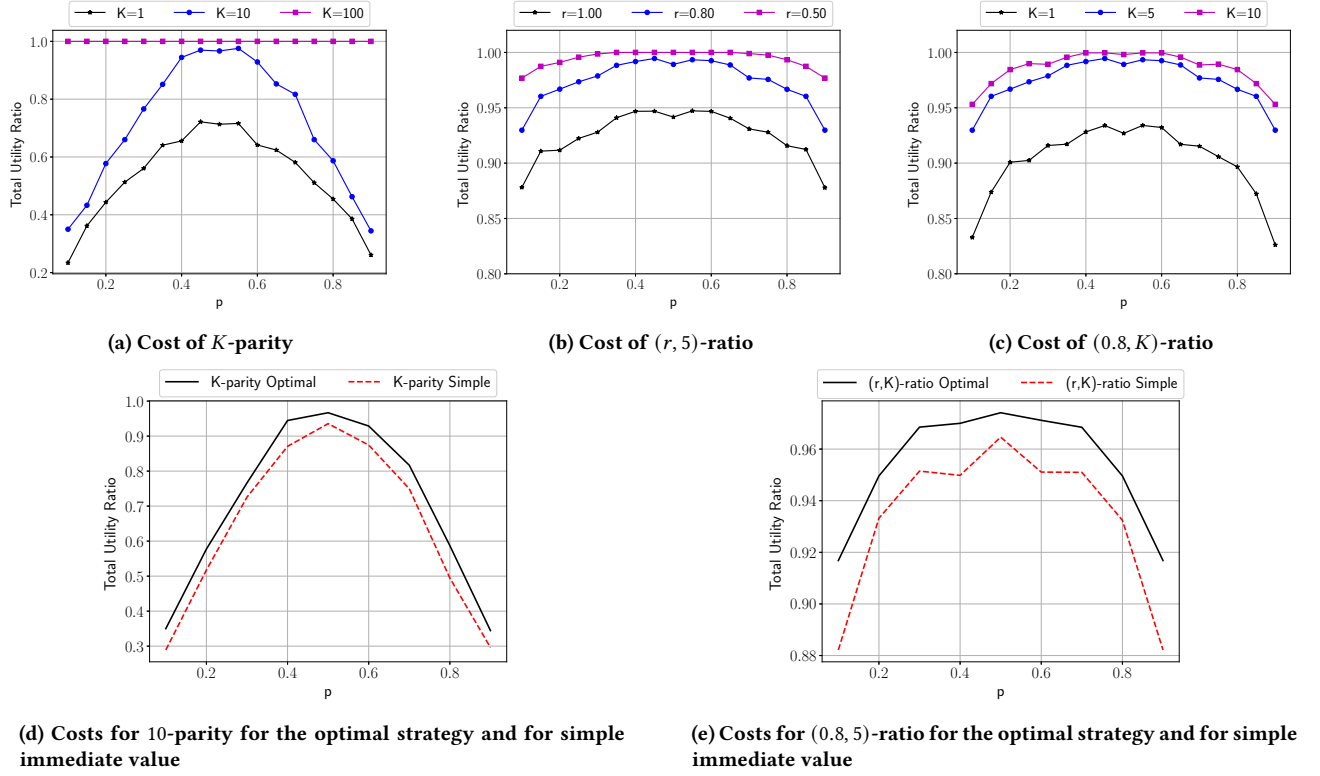


Figure 3: Experimental results for Yahoo A1 bidding dataset

Figures 4(f) and 4(g) further explore overbidding using a variation on the *Expensive female - Equal value* scenario. Rather than keep the value that the advertiser i assigns to males fixed at $\mu_i^m = -2.5$, we vary it to see the effect on overbidding. Rather than plot μ_i^m itself, we plot the ratio of μ_i^m to the value assigned to males by the other advertisers. Figure 4(f) shows this value ratio by using various lines. Overbidding is not needed when the ratio is 1 since the advertiser i can outbid the other advertisers without surpassing its immediate reward. For ratios above 1, as the rate p of male viewers increases, the optimal K -parity advertiser will increase its overbidding on the female viewers since they become more scarce. Figure 4(g) shows that as μ_i^m (and, thus, the male value ratio of advertiser i to the other advertisers) increases, the overbidding for females increases.

Figure 4(h) plots the utility ratio as the value of the rate δ at which the advertiser i will leave the ad network changes. Rather than plot δ directly, it plots the expected lifespan of the advertiser computed from δ . It shows that for short lived advertisers, K -parity has no effect since the advertiser is unlikely to reach K wins for either gender. However, the constraint rapidly has an effect as the advertiser lives long enough to win this number of slots.

Ad Exchange Revenue. Also important is how our strategy impacts the revenue of the ad exchange. We explored the ratio of the ad exchange’s revenue when there is one constrained advertisers for each ad slot auction to the case where all advertisers are unconstrained for all of our scenarios (for both the real and synthetic datasets). In most cases, the ad exchange revenue did not decrease

at all. The worst case happens for a $(1.0, 1)$ -ratio constrained advertiser on the Yahoo! A1 data, where the ratio of revenues is 0.993. The ad exchange can avoid this worst case by having lower bounds on K and r . Therefore, implementing this feature will not significantly reduce the ad exchange’s revenue.

In fact, supporting constraints may raise the ad exchange’s revenues. Our observations show constrained advertisers often overbid, which increases the exchange’s revenue. Figure 5 compares the revenue of the exchange for various numbers of constrained advertisers (p) using the Yahoo! A1 dataset. As expected, by increasing the number of constrained advertisers, the the revenue of ad exchange’s increases above a ratio of 1 with the revenue from having only unconstrained advertisers.

Performance. Algorithms 1 and 2, each of which only has to run once for each parameter setting, completed in under 2 minutes and under 10 minutes, respectively. Calculating bids during auctions, each took the 2 microseconds. We used a 2013 MacBook Pro with a 2.3 GHz Intel Core i7 and 16 GB of 1600 MHz DDR3 memory.

8 DEPLOYMENT

We envision two ways in which advertisers could use our bidding strategy. First, ad exchanges can implement it for them as a feature in the ad buying interface. Such exchanges could use the data it has to determine the demographics of individuals viewing ad slots and adjust bids accordingly. While this would require a change to

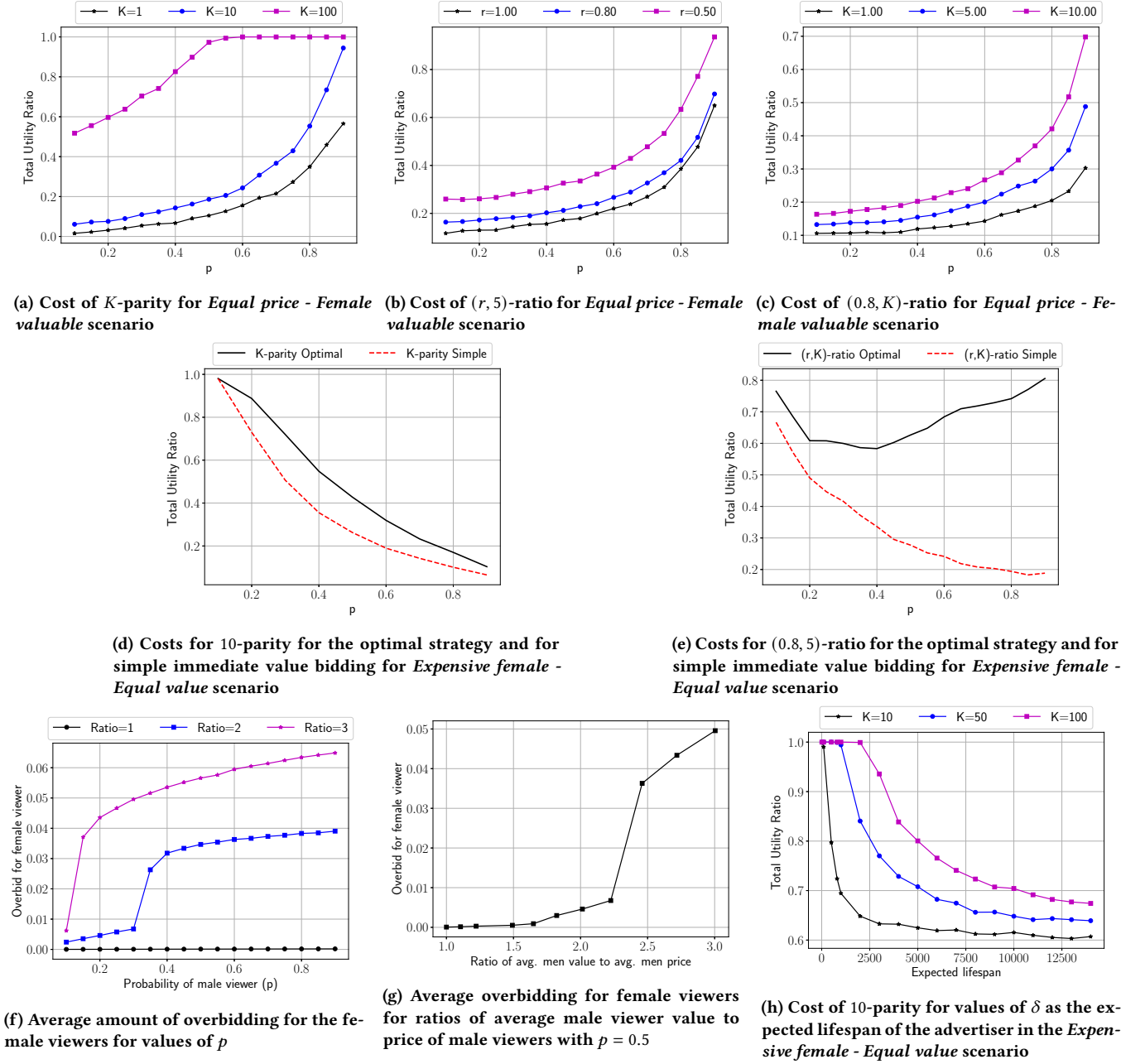


Figure 4: Experimental results for synthetic datasets

the ad exchange, it would not require modifying the core auction mechanism, making it a more straightforward feature to add.

Second, the strategy could be used either directly by the advertiser or offered to them by demand-side platforms as a feature. This approach does not require any changes to the ad exchange. It has the disadvantage of only working for ad exchanges that support real-time bidding and programmatic advertising with rich enough data to infer the group membership of the people viewing ad slots. Additionally, such rich data can pose privacy concerns.

We believe that either of these approaches to deployment would be more straightforward than any way of deploying an auction mechanism that enforces parity constraints [5] or Guaranteed ad Delivery (GD) [22, 26]. Only the ad exchange would be able to implement such functionality. Presumably, ad exchanges have already selected the auction mechanism that they believe would be best for their business and would be reluctant to change it in a way that could have wide-ranging effects. Given that Google uses a generalization of second-price auctions [10], it may believe that the

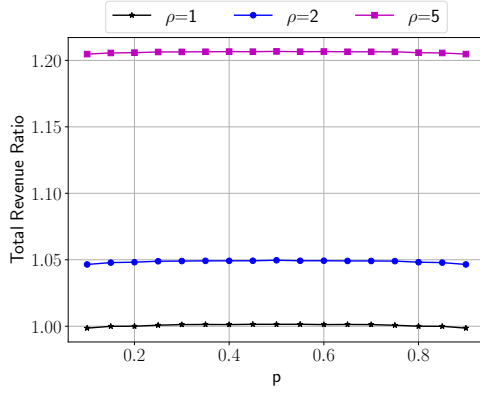


Figure 5: Ratio of the ad revenue with constrained advertisers and without for various numbers of advertisers subject to a (0.8, 5)-ratio constraint, computed with the Yahoo A1 dataset.

theoretical result that second-price auctions are optimal in certain settings has some bearing on its setting. Thus, it may believe that any change to its auction mechanism is likely to reduce its profits, a strong disincentive. We believe that ad exchanges would be more willing to implement a change that instead only alters the bids of advertisers who opt in since it would be equivalent to one that advertisers could already implement unilaterally by altering their bids. Furthermore, since our approach changes just opted-in advertisers' bids, there is a sense in which they pay for it.

We believe that our approach is computationally feasible. In some cases, an advertiser may have to solve more than one MDP. For example, the advertiser may wish to enforce a separate parity constraint for two different webpages whose ad slots differ in characteristics included in the MDP. The advertiser will have to solve an MDP for each ad campaign targeting a different type of ad auction, with its own expected values, bid distributions, and parity constraints. Given modern approaches to solving large MDPs [24] and the ability to rent cloud computing resources, we believe that advertisers sophisticated enough to model a large number of different auction types will be able to solve them as well. Once solved, the advertiser can reuse the solution for every instance of the auction type. Thus, addressing each incoming bid request from the ad exchange takes little more than a table lookup.

9 EXTENSIONS AND FUTURE WORK

Our model of ad exchanges does not include that advertisers often pay exchanges more when the viewer clicks on an ad. Thus, the expected value v_i and cost c_i of a slot depends upon not just the bid prices but also the fit of the ad for the slot, which can be estimated with online tracking and machine learning. It may seem that these changes will have to be linked since a clicked ad both increases the value and cost, but by the linearity of expectations, we can consider the average of each. This changes the reward on the transitions in the MDP, but not structure of the MDP (the state space and the transition relation). Therefore, an advertiser can still use our approach to find optimal bids.

We used a simple model in which the expected value of each female slot is equal to the others, and the expected value of each male slot is equal to the others. Advertisers can use online tracking, machine learning, and other techniques to compute more fine-grained estimations of slot values. We can accommodate such fine-grained modeling by increasing the size of the set Θ of types, which will increase the size of state space.

Future work could accommodate constraints for non-binary sensitive attributes, such as age and location (a proxy for race, which is apparently not explicitly tracked by any ad exchange), leading to a larger set Γ of protected groups. Multiple simultaneous constraints can lead to having multiple Γ . Although our MDPs can be extended to such cases using a cross-product-like construction, the MDP size will be exponential in the number of constraints and their values, motivating more significant theoretical future work. Nevertheless, we suspect that the limited number of protected attributes tracked by ad exchanges, fast MDP solvers [24], and cloud computing will make our approach fast enough in practice.

The constraints we explore are very strict in that they must hold at all times, as opposed to holding with high probability or asymptotically, which might be acceptable in some settings. In related problems, parity may only be required at the end of certain checkpoints, such as at the end of a hiring season. Exploring such relaxations can be future work.

Our work is an example of the tension between not using protected attributes and avoiding disparity in outcomes. Our ratio constraint avoids disparity in outcomes and is based on the four-fifths rule found in U.S. antidiscrimination rules around disparate impact [9]. However, that rule is with respect to employment practices that are, on their face, neutral toward protected groups. Our constraints explicitly use protected attributes to enforce strict quotas. In some cases, "Quotas are expressly forbidden" by U.S. regulations [11]. Resolving this tension between competing antidiscrimination goals is tricky (see, e.g., [20]). An analysis of when our approach would be legal would be context and country specific and beyond the scope of this work. However, we will mention that, in some cases, U.S. antidiscrimination laws would not apply, such as to private clubs [17]. In such cases, the advertiser may be guided by its ethics to use our approach to avoid disparity in outcomes.

10 CONCLUSION

Adding parity constraints results in a surprisingly complex bidding problem, exhibiting both over- and underbidding relative to the advertiser's immediate value of an ad slot. Despite this complexity, we show a practical way of computing optimal bids, to within a small approximation factor ϵ . This enables us to characterize how the cost of parity depends upon not just its level of strictness K or r , but also the base rate p of types, their relative values to both the governed advertiser i and to other advertisers, and the lifespan (or discounting factor) δ , in sometimes counterintuitive ways.

ACKNOWLEDGMENTS

Milad Nasr is supported by a Google PhD Fellowship. We gratefully acknowledge funding support from the National Science Foundation (Grants 1237265 and 1704985). The opinions in this paper are

those of the authors and do not necessarily reflect the opinions of any funding sponsor or the United States Government.

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A CONJOINT VALUATION FOR ABSOLUTE PARITY CONSTRAINTS

The following lemma justifies Equation (5).

LEMMA 1. *For all K and g_i , and for all states $\langle k, \theta \rangle$ other than the edge cases $\langle K, m \rangle$ and $\langle -K, w \rangle$,*

$$V(k, \theta; g_i) = \max_{b_i} \left\{ q(b_i; g_i^\theta) \Phi_i^\theta(k; g_i) - c(b_i; g_i^\theta) \right\} + \Lambda_i(k; g_i)$$

PROOF. First note that

$$N_{\text{win}}^\theta(k; g_i) = \Phi_{\text{win}}^\theta(k; g_i) \quad (8)$$

since

$$N_{\text{win}}^m(k; g_i) = pV(k+1, m; g_i) + (1-p)V(k+1, w; g_i) = \Phi_{\text{win}}^m(k; g_i)$$

and

$$N_{\text{win}}^w(k; g_i) = pV(k-1, m; g_i) + (1-p)V(k-1, w; g_i) = \Phi_{\text{win}}^w(k; g_i)$$

Thus,

$$N^\theta(b_i, k; g_i) = q(b_i; g_i^\theta) N_{\text{win}}^\theta(k; g_i) + (1 - q(b_i; g_i^\theta)) * N_{\text{lose}}^\theta(k; g_i) \quad (9)$$

$$= q(b_i; g_i^\theta) \Phi_{\text{win}}^\theta(k; g_i) + (1 - q(b_i; g_i^\theta)) [pV(k, m; g_i) + (1-p)V(k, w; g_i)] \quad (10)$$

$$= q(b_i; g_i^\theta) \Phi_{\text{win}}^\theta(k; g_i) - q(b_i; g_i^\theta) [pV(k, m; g_i) + (1-p)V(k, w; g_i)] + [pV(k, m; g_i) + (1-p)V(k, w; g_i)] \quad (11)$$

$$= q(b_i; g_i^\theta) \left(\Phi_{\text{win}}^\theta(k; g_i) - [pV(k, m; g_i) + (1-p)V(k, w; g_i)] \right) + [pV(k, m; g_i) + (1-p)V(k, w; g_i)] \quad (12)$$

$$= q(b_i; g_i^\theta) \left(\Phi_{\text{win}}^\theta(k; g_i) - \Phi_{\text{lose}}^\theta(k; g_i) \right) + [pV(k, m; g_i) + (1-p)V(k, w; g_i)] \quad (13)$$

where (10) follows from (8). Thus,

$$V(k, \theta; g_i) = \max_{b_i} \left\{ R^\theta(b_i; g_i^\theta) + \delta N^\theta(b_i, k; g_i) \right\} \quad (14)$$

$$= \max_{b_i} \left\{ q(b_i; g_i^\theta) * (v_i^\theta - d_i^\theta) + \delta \left[q(b_i; g_i^\theta) \left(\Phi_{\text{win}}^\theta(k; g_i) - \Phi_{\text{lose}}^\theta(k; g_i) \right) + [pV(k, m; g_i) + (1-p)V(k, w; g_i)] \right] \right\} \quad (15)$$

$$= \max_{b_i} \left\{ q(b_i; g_i^\theta) v_i^\theta - q(b_i; g_i^\theta) d_i^\theta + \delta q(b_i; g_i^\theta) \left(\Phi_{\text{win}}^\theta(k; g_i) - \Phi_{\text{lose}}^\theta(k; g_i) \right) + \delta [pV(k, m; g_i) + (1-p)V(k, w; g_i)] \right\} \quad (16)$$

$$= \max_{b_i} \left\{ q(b_i; g_i^\theta) v_i^\theta - c(b_i; g_i^\theta) + \delta q(b_i; g_i^\theta) \left(\Phi_{\text{win}}^\theta(k; g_i) - \Phi_{\text{lose}}^\theta(k; g_i) \right) + \delta [pV(k, m; g_i) + (1-p)V(k, w; g_i)] \right\} \quad (17)$$

$$= \max_{b_i} \left\{ q(b_i; g_i^\theta) v_i^\theta + \delta q(b_i; g_i^\theta) \left(\Phi_{\text{win}}^\theta(k; g_i) - \Phi_{\text{lose}}^\theta(k; g_i) \right) - c(b_i; g_i^\theta) + \Lambda_i(k; g_i) \right\} \quad (18)$$

$$= \max_{b_i} \left\{ q(b_i; g_i^\theta) \left[v_i^\theta + \delta \left(\Phi_{\text{win}}^\theta(k; g_i) - \Phi_{\text{lose}}^\theta(k; g_i) \right) \right] - c(b_i; g_i^\theta) + \Lambda_i(k; g_i) \right\} \quad (19)$$

$$= \max_{b_i} \left\{ q(b_i; g_i^\theta) \Phi_i^\theta(k; g_i) - c(b_i; g_i^\theta) + \Lambda_i(k; g_i) \right\} \quad (20)$$

$$= \max_{b_i} \left\{ q(b_i; g_i^\theta) \Phi_i^\theta(k; g_i) - c(b_i; g_i^\theta) \right\} + \Lambda_i(k; g_i) \quad (21)$$

where (15) follows from (4) and (13). \square

B PROOF OF THEOREM 1

The following lemma is based on Lemma C.1 of Iyer et al. [14, p. 28].

LEMMA 2. *For all ϕ and all functions $h: \mathbb{R} \rightarrow \mathbb{R}$ that is continuous and non-decreasing, the function $f: \mathbb{R} \rightarrow \mathbb{R}$ such that $f_\phi(x) = h(x) * (\phi - x) + \int_0^x h(u) du$ is maximized at $x = \phi$.*

PROOF. Consider any ϕ . Since, for all x ,

$$\int_x^\phi h(x) du = [h(x) * u + C]_{u=x}^\phi \quad \text{For any constant } C \quad (22)$$

$$= (h(x) * \phi + C) - (h(x) * x + C) \quad \text{For any constant } C \quad (23)$$

$$= h(x) * (\phi - x) \quad (24)$$

the following holds for all x

$$f_\phi(x) - f_\phi(\phi) = \left[h(x) * (\phi - x) + \int_0^x h(u) du \right] - \left[h(\phi) * (\phi - \phi) + \int_0^\phi h(u) du \right] \quad (25)$$

$$= h(x) * (\phi - x) + \int_0^x h(u) du - \int_0^\phi h(u) du \quad (26)$$

$$= \int_x^\phi h(x) du + \int_0^x h(u) du - \int_0^\phi h(u) du \quad (27)$$

For all x such that $\phi < x$,

$$\int_0^x h(u) du - \int_0^\phi h(u) du = \int_\phi^x h(u) du \quad (28)$$

and, thus,

$$f_\phi(x) - f_\phi(\phi) = \int_x^\phi h(x) du + \int_0^x h(u) du - \int_0^\phi h(u) du \quad (29)$$

$$= \int_x^\phi h(x) du + \int_\phi^x h(u) du \quad (30)$$

$$= - \int_\phi^x h(x) du + \int_\phi^x h(u) du \quad (31)$$

$$= \int_\phi^x h(u) - h(x) du \quad (32)$$

$$(33)$$

For all u such that $\phi \leq u \leq x$, $h(u) \leq h(x)$ since h is a non-decreasing function. Thus, $h(u) - h(x) \leq 0$ and

$$f_\phi(x) - f_\phi(\phi) = \int_\phi^x h(u) - h(x) du \leq 0 \quad (34)$$

For all x such that $x < \phi$,

$$\int_0^\phi h(u) du - \int_0^x h(u) du = \int_x^\phi h(u) du \quad (35)$$

$$(36)$$

and, thus,

$$f_\phi(x) - f_\phi(\phi) = \int_x^\phi h(x) du + \int_0^x h(u) du - \int_0^\phi h(u) du \quad (37)$$

$$= \int_x^\phi h(x) du - \left(\int_0^\phi h(u) du - \int_0^x h(u) du \right) \quad (38)$$

$$= \int_x^\phi h(x) du - \int_x^\phi h(u) du \quad (39)$$

$$= \int_x^\phi h(x) - h(u) du \quad (40)$$

For all u such that $x \leq u \leq \phi$, $h(x) \leq h(u)$ since h is a non-decreasing function. Thus, $h(x) - h(u) \leq 0$ and

$$f_\phi(x) - f_\phi(\phi) = \int_x^\phi h(x) - h(u) du \leq 0 \quad (41)$$

Thus, for any x , $f_\phi(x) - f_\phi(\phi) \leq 0$, which implies that $f_\phi(\phi) \geq f_\phi(x)$ for all x . Therefore, for all ϕ , f_ϕ is maximized at ϕ . \square

THEOREM 1. For all K, i, g_i , and states $\langle k, \theta \rangle$ other than the edge cases $\langle K, m \rangle$ and $\langle -K, w \rangle$, the optimal bid for i subject to K -parity under the distribution g_i at the state $\langle k, \theta \rangle$ is $\Phi_i^\theta(k; g_i)$.

PROOF. Without loss of generality we assume all of the bids are between 0 and 1. By Lemma 1, the optimal bidding strategy maximizes term given by (5):

$$\max_{b_i} \left\{ q(b_i; g_i^\theta) * \Phi_i^\theta(k; g_i) - c(b_i; g_i^\theta) \right\} + \Lambda_i(k; g_i)$$

To maximize this term, we can omit the function Λ_i since it is constant for each b_i . Similar to line (4) of Iyer et al. [14, p. 9], we rewrite the cost function $c(b_i; g_i^\theta)$ as

$$c(b_i; g_i^\theta) = q(b_i; g_i^\theta) * d_i^\theta \quad (42)$$

$$= q(b_i; g_i^\theta) * \max b_{-i}^\theta \quad (43)$$

$$= \mathbb{E}[\chi[\max b_{-i}^\theta \leq b_i] * \max b_{-i}^\theta] \quad (44)$$

$$= \int_0^1 q'(u; g_i^\theta) * \chi[u \leq b_i] * u \, du \quad (45)$$

$$= \int_0^{b_i} u * q'(u; g_i^\theta) \, du \quad (46)$$

$$= q(b_i; g_i^\theta) * b_i - q(0; g_i^\theta) * 0 - \int_0^{b_i} b'_i * q(b_i; g_i^\theta) \, du \quad (47)$$

$$= q(b_i; g_i^\theta) * b_i - \int_0^{b_i} q(u; g_i^\theta) \, du \quad (48)$$

where χ is the Iverson indicator function, (45) follows from $q(b; g_i^\theta) = \Pr(\max b_{-i} \leq b) = \text{cdf}[\max b_{-i}](b) = \int_0^b \text{pdf}[\max b_{-i}](u) \, du$ and (47) uses integration by parts.

Next, we rewrite the decision problem of the advertiser i as

$$\begin{aligned} \max_{b_i} \left\{ q(b_i; g_i^\theta) * \Phi_i^\theta(k; g_i) - c(b_i; g_i^\theta) \right\} &= \max_{b_i} \left\{ q(b_i; g_i^\theta) * \Phi_i^\theta(k; g_i) - \left(q(b_i; g_i^\theta) b_i - \int_0^{b_i} q(u; g_i^\theta) \, du \right) \right\} \\ &= \max_{b_i} \left\{ q(b_i; g_i^\theta) * (\Phi_i^\theta(k; g_i) - b_i) + \int_0^{b_i} q(u; g_i^\theta) \, du \right\} \end{aligned}$$

We know that $q(b_i; g_i^\theta)$ is a continuous non-decreasing function in b_i for all g_i^θ . That is, $h(b_i) = q(b_i; g_i^m)$ is a continuous non-decreasing function in b_i for any fixed distribution g_i^m and $h(b_i) = q(b_i; g_i^w)$ is a continuous non-decreasing function in b_i for any fixed distribution g_i^w . Therefore, we can use Lemma 2 with b_i playing the role of x and with $h(x) = q(x; g_i^\theta)$ and $\phi = \Phi_i^\theta(k; g_i)$. This allows us to conclude that equation (5) is at its maximum when the bid is $\Phi_i^\theta(k; g_i)$ for θ equal to either m or w . \square

C CONJOINT VALUATION FOR RATIO CONSTRAINTS

The following lemma justifies Equation (7). The proof follows the same outline as that of Lemma 1.

LEMMA 3. For all r, K , and g_i , for all states $\langle n_i^m, n_i^w, \theta \rangle$ other than the edge cases when either $\theta = m$ and $r(1-p)(n_i^m + 1) > pn_i^w + K$ or $\theta = w$ and $rp(n_i^w + 1) > (1-p)n_i^m + K$,

$$V(n_i^m, n_i^w, \theta; g_i) = \max_{b_i} \left\{ q(b_i; g_i^\theta) \Phi_i^\theta(n_i^m, n_i^w; g_i) - c(b_i; g_i^\theta) \right\} + \Lambda_i(n_i^m, n_i^w; g_i)$$

PROOF. First note that

$$N_{\text{win}}^\theta(n_i^m, n_i^w; g_i) = \Phi_{\text{win}}^\theta(n_i^m, n_i^w; g_i) \quad (49)$$

since

$$N_{\text{win}}^m(n_i^m, n_i^w; g_i) = pV(n_i^m + 1, n_i^w, m; g_i) + (1-p)V(n_i^m + 1, n_i^w, w; g_i) = \Phi_{\text{win}}^m(n_i^m, n_i^w; g_i)$$

and

$$N_{\text{win}}^w(n_i^m, n_i^w; g_i) = pV(n_i^m, n_i^w + 1, m; g_i) + (1-p)V(n_i^m, n_i^w + 1, w; g_i) = \Phi_{\text{win}}^w(n_i^m, n_i^w; g_i)$$

Thus,

$$N^\theta(b_i, n_i^m, n_i^w; g_i) \quad (50)$$

$$= q(b_i; g_i^\theta) * N_{\text{win}}^\theta(n_i^m, n_i^w; g_i) + (1 - q(b_i; g_i^\theta)) * N_{\text{lose}}^\theta(n_i^m, n_i^w; g_i) \quad (51)$$

$$= q(b_i; g_i^\theta) * \Phi_{\text{win}}^\theta(n_i^m, n_i^w; g_i) + (1 - q(b_i; g_i^\theta)) * [pV(n_i^m, n_i^w, m; g_i) + (1-p)V(n_i^m, n_i^w, w; g_i)] \quad (52)$$

$$= q(b_i; g_i^\theta) * \Phi_{\text{win}}^\theta(n_i^m, n_i^w; g_i) - q(b_i; g_i^\theta) [pV(n_i^m, n_i^w, m; g_i) + (1-p)V(n_i^m, n_i^w, w; g_i)] + [pV(n_i^m, n_i^w, m; g_i) + (1-p)V(n_i^m, n_i^w, w; g_i)] \quad (53)$$

$$= q(b_i; g_i^\theta) \left(\Phi_{\text{win}}^\theta(n_i^m, n_i^w; g_i) - [pV(n_i^m, n_i^w, m; g_i) + (1-p)V(n_i^m, n_i^w, w; g_i)] \right) + [pV(n_i^m, n_i^w, m; g_i) + (1-p)V(n_i^m, n_i^w, w; g_i)] \quad (54)$$

$$= q(b_i; g_i^\theta) \left(\Phi_{\text{win}}^\theta(n_i^m, n_i^w; g_i) - \Phi_{\text{lose}}^\theta(n_i^m, n_i^w; g_i) \right) + [pV(n_i^m, n_i^w, m; g_i) + (1-p)V(n_i^m, n_i^w, w; g_i)] \quad (55)$$

where (52) follows from (49). Thus,

$$V(n_i^m, n_i^w, \theta; g_i) \quad (56)$$

$$= \max_{b_i} \left\{ R^\theta(b_i; g_i^\theta) + \delta N^\theta(b_i, n_i^m, n_i^w; g_i) \right\} \quad (57)$$

$$= \max_{b_i} \left\{ q(b_i; g_i^\theta) * (v_i^\theta - d_i^\theta) + \delta \left[q(b_i; g_i^\theta) \left(\Phi_{\text{win}}^\theta(n_i^m, n_i^w; g_i) - \Phi_{\text{lose}}^\theta(n_i^m, n_i^w; g_i) \right) + [pV(n_i^m, n_i^w, m; g_i) + (1-p)V(n_i^m, n_i^w, w; g_i)] \right] \right\} \quad (58)$$

$$= \max_{b_i} \left\{ q(b_i; g_i^\theta) v_i^\theta - q(b_i; g_i^\theta) d_i^\theta + \delta q(b_i; g_i^\theta) \left(\Phi_{\text{win}}^\theta(n_i^m, n_i^w; g_i) - \Phi_{\text{lose}}^\theta(n_i^m, n_i^w; g_i) \right) + \delta [pV(n_i^m, n_i^w, m; g_i) + (1-p)V(n_i^m, n_i^w, w; g_i)] \right\} \quad (59)$$

$$= \max_{b_i} \left\{ q(b_i; g_i^\theta) v_i^\theta - c(b_i; g_i^\theta) + \delta q(b_i; g_i^\theta) \left(\Phi_{\text{win}}^\theta(n_i^m, n_i^w; g_i) - \Phi_{\text{lose}}^\theta(n_i^m, n_i^w; g_i) \right) + \Lambda_i(n_i^m, n_i^w; g_i) \right\} \quad (60)$$

$$= \max_{b_i} \left\{ q(b_i; g_i^\theta) v_i^\theta + \delta q(b_i; g_i^\theta) \left(\Phi_{\text{win}}^\theta(n_i^m, n_i^w; g_i) - \Phi_{\text{lose}}^\theta(n_i^m, n_i^w; g_i) \right) - c(b_i; g_i^\theta) + \Lambda_i(n_i^m, n_i^w; g_i) \right\} \quad (61)$$

$$= \max_{b_i} \left\{ q(b_i; g_i^\theta) \left[v_i^\theta + \delta \left(\Phi_{\text{win}}^\theta(n_i^m, n_i^w; g_i) - \Phi_{\text{lose}}^\theta(n_i^m, n_i^w; g_i) \right) \right] - c(b_i; g_i^\theta) + \Lambda_i(n_i^m, n_i^w; g_i) \right\} \quad (62)$$

$$= \max_{b_i} \left\{ q(b_i; g_i^\theta) * \Phi^\theta(n_i^m, n_i^w; g_i) - c(b_i; g_i^\theta) + \Lambda_i(n_i^m, n_i^w; g_i) \right\} \quad (63)$$

$$= \max_{b_i} \left\{ q(b_i; g_i^\theta) * \Phi^\theta(n_i^m, n_i^w; g_i) - c(b_i; g_i^\theta) \right\} + \Lambda_i(n_i^m, n_i^w; g_i) \quad (64)$$

where (58) follows from (4) and (55). \square