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Spectral analysis of non-Hermitian matrices and directed graphs

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We generalize classical results in spectral graph theory and linear algebra more broadly, from the case where the underlying matrix is Hermitian to the case where it is non-Hermitian. New admissibility conditions are introduced to replace the Hermiticity condition. We prove new variational estimates of the Rayleigh quotient for non-Hermitian matrices. As an application, a new Delsarte-Hoffman-type bound on the size of the largest independent pair in a directed graph is developed. Our techniques consist in quantifying the impact of breaking the Hermitian symmetry of a matrix and are broadly applicable.

1. Introduction

The eigendecomposition is among the most powerful tools for analyzing Hermitian matrices $\mathbf{B} \in \mathbb{C}^{n\times n}$, i.e. matrices satisfying $\mathbf{B}^* = \mathbf{B}$, where $\mathbf{B}^*[j,\ell] = \overline{\mathbf{B}[\ell,j]}$. Several classical results in linear algebra can be derived from decomposing a Hermitian matrix

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$\mathbf{B} \in \mathbb{C}^{n\times n}$ as $\mathbf{B} = \mathbf{U}\,\mathrm{diag}(\boldsymbol{\lambda}(\mathbf{B}))\mathbf{U}^*$, where $\mathbf{U}$ is unitary and $\boldsymbol{\lambda}(\mathbf{B}) = (\lambda_\ell(\mathbf{B}))_{0\le\ell<n} \subset \mathbb{R}$. In particular, under the assumption that $\mathbf{B}$ is Hermitian, variational estimates on the *Rayleigh quotient* [19] can be stated in terms of $\boldsymbol{\lambda}(\mathbf{B})$:

$$\forall \mathbf{f} \in \mathbb{C}^{n\times 1}, \mathbf{f} \neq 0,\ \min\{\boldsymbol{\lambda}(\mathbf{B})\} \le \frac{\mathbf{f}^*\mathbf{B}\mathbf{f}}{\|\mathbf{f}\|_2^2} \le \max\{\boldsymbol{\lambda}(\mathbf{B})\}. \tag{1}$$

In case that $\mathbf{B}$ is non-Hermitian, the eigenvalues of $\mathbf{B}$ may be complex, or even worse the eigendecomposition may not exist at all.

When $\mathbf{B}$ is non-Hermitian but *diagonalizable*, its eigendecomposition is of the form $\mathbf{B} = \mathbf{P}\,\mathrm{diag}(\boldsymbol{\lambda}(\mathbf{B}))\mathbf{P}^{-1}$ for some invertible matrix $\mathbf{P}$ and scalars $\boldsymbol{\lambda}(\mathbf{B})$. Compared to the Hermitian case, the columns of $\mathbf{P}$ need not form an orthonormal basis. A different decomposition that is available to all matrices is the *singular value decomposition (SVD)*: $\mathbf{B} = \mathbf{U}\,\mathrm{diag}(\boldsymbol{\sigma}(\mathbf{B}))\mathbf{V}^*$, where the singular values $\boldsymbol{\sigma}(\mathbf{B}) = (\sigma_\ell(\mathbf{B}))_{0\le\ell<n}$ are non-negative and $\mathbf{U}, \mathbf{V}$ are unitary matrices. However, it needs not be the case that $\mathbf{U}\mathbf{V}^* = \mathbf{U}^*\mathbf{V} = \mathbf{I}$, which is the primary contrast with the eigendecomposition.

Powerful tools of linear algebra can be applied to the study of graphs via spectral graph theory [7,3,18]. Indeed, let $\mathcal{G} = (V, \mathbf{B})$ be a graph, where V is the set of vertices and $\mathbf{B} \in \{0,1\}^{|V|\times|V|}$ an adjacency matrix such that $\mathbf{B}[j,\ell] = 1$ if there is an edge between the jth and ℓth nodes. By analyzing the spectral properties of $\mathbf{B}$, a variety of mathematical ideas may be adapted to $\mathcal{G}$, including notions of geometry [25,26], Fourier and wavelet analysis [11,21,30], random diffusion processes [10,9], and clusters [28,27]. While these tools have contributed to a renaissance in the analysis of data, spectral graph methods almost uniformly require the underlying graph $\mathcal{G}$ to be *undirected*, or in linear algebra terms, $\mathbf{B}$ must be Hermitian. This is a severe limitation in practice, as a variety of real data does not lend itself to representation as an undirected graph, for example social networks [22], models for the spread of contagious disease in a heterogenous population [24], and predator-prey relationships [32].

1.1. Summary of contributions

This article develops new approaches for the analysis of non-Hermitian matrices. The primary contributions are twofold. First, we prove *a generalized version of the classical variational estimates on the Rayleigh quotient.* New *admissibility conditions* are introduced to replace the Hermiticity condition. Second, the *Delsarte-Hoffman bound on the size of independent sets in undirected graphs is generalized to the directed setting.* Our major tool consists in quantifying the discrepancy between Hermitian and non-Hermitian matrices, and proposing additional hypotheses in the theorems to address this discrepancy. The gap between Hermiticity and non-Hermiticity is made precise, and moreover in the case of the Delsarte-Hoffman bound, the gaps between $\mathbf{B}$ being Hermitian, diagonalizable (with potentially complex eigenvalues) and $\mathbf{B}$ arbitrary is considered by analyzing the SVD. Our proof methods are flexible, and may be applicable to settings not considered in the present article.

1.2. Related work

Spectral graph theory has been attempted for operators defined on directed graphs in a variety of contexts, including for the graph Laplacian [1,8,6,2,5,33,14,12] and for non-reversible Markov chains [15]. Combinatorial results for directed graphs have also been studied [31,4,23,17]. These results do not, however, develop precise characterizations of the ways in which classical results can be modified in the non-Hermitian setting. In particular, the admissibility conditions proposed in this article explicitly illustrate what is lost when a matrix is perturbed to deviate from Hermiticity, and suggest how to compensate for the loss of Hermiticity. Moreover, the proposed generalized Delsarte-Hoffman bound makes no assumptions of normality of the adjacency matrix of the underlying graph.

1.3. Notation

Throughout, bold typography is used to denote matrices and vectors. Let $\mathbf{I}$ denote the identity matrix with size clear from context. For a collection of points $\{\alpha_\ell\}_{0\leq\ell<n} \subset \mathbb{C}$, let $\operatorname{diag}\left\{(\alpha_\ell)_{0\leq\ell<n}\right\}$ denote the $n \times n$ diagonal matrix with ℓth diagonal entry α_ℓ. Let $\mathbf{1}_{m\times n}$ and $\mathbf{0}_{m\times n}$ respectively denote the $m \times n$ matrix of all 1s and all 0s. Let $\mathbf{B}[j,:]$ and $\mathbf{B}[:,\ell]$ denote the jth row and ℓth column of the matrix $\mathbf{B}$, respectively. For matrices $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{m\times n}$, we denote by $\mathbf{A} \circ \mathbf{B} \in \mathbb{C}^{m\times n}$ the Hadamard product $(\mathbf{A} \circ \mathbf{B})[j,\ell] = \mathbf{A}[j,\ell]\,\mathbf{B}[j,\ell]$. For matrices $\mathbf{A} \in \mathbb{C}^{m\times n}, \mathbf{B} \in \mathbb{C}^{p\times q}$, we denote by $\mathbf{A} \otimes \mathbf{B} \in \mathbb{C}^{mp\times nq}$ the Kronecker product $(\mathbf{A} \otimes \mathbf{B})_{p(k-1)+v,q(\ell-1)+w} = \mathbf{A}_{k\ell}\mathbf{B}_{vw}$.

2. Generalized Rayleigh quotient estimation

2.1. Rayleigh quotient for complex diagonalizable matrices

Let $\mathbf{B} \in \mathbb{C}^{n\times n}$. There corresponds to $\mathbf{B}$ a (in general non-symmetric) bilinear form $\langle \mathbf{f}, \mathbf{g}\rangle_{\mathbf{B}} \mapsto \mathbf{f}^*\mathbf{B}\mathbf{g}$, defined for $\mathbf{f}, \mathbf{g} \in \mathbb{C}^{n\times 1}$. The behavior of this bilinear form can be analyzed in a scale-invariant manner through the Rayleigh quotient. When $\mathbf{B}$ is Hermitian, (1) states that the Rayleigh quotient $\langle \mathbf{f}, \mathbf{f}\rangle_{\mathbf{B}}/\|\mathbf{f}\|_2^2$ is controlled for $\mathbf{f} \neq \mathbf{0}_{n\times 1}$ by the largest and smallest eigenvalues of $\mathbf{B}$. We extend this result to the case when $\mathbf{B}$ has complex eigenvalues.

Theorem 2.1. *Let $\mathbf{B} \in \mathbb{C}^{n\times n}$ be a diagonalizable matrix decomposed as $\mathbf{B} = \mathbf{U}\operatorname{diag}\{\boldsymbol{\lambda}(\mathbf{B})\}\mathbf{V}^*$ where $\mathbf{U}^{-1} = \mathbf{V}^*$ and $\boldsymbol{\lambda}(\mathbf{B}) \subset \mathbb{C}$. Write $\lambda_\ell(\mathbf{B}) = |\lambda_\ell(\mathbf{B})|\, e^{i\theta_\ell}$ for $\theta_\ell \in [0, 2\pi)$. Let $\mathbf{f}, \mathbf{g} \in \mathbb{C}^{n\times 1}$ be such that there exist $\mathbf{F}, \mathbf{G} \in \mathbb{C}^{n\times 1}$ satisfying $\mathbf{F}\circ\mathbf{G} \neq \mathbf{0}_{n\times 1}$ and*

$$\mathbf{f} = \mathbf{V}\overline{\mathbf{F}},\ \ \mathbf{g} = \mathbf{U}\mathbf{G}\ \ s.t.\ \forall 0 \leq \ell < n,\ \left(\mathbf{F}[\ell]\,\mathbf{G}[\ell]\,e^{i\theta_\ell}\right) \in \mathbb{R}. \tag{2}$$

Then

$$\min_{0\le\ell<n}\left\{|\lambda_\ell(\mathbf{B})|\operatorname{sgn}(\mathbf{F}[\ell]\,\mathbf{G}[\ell]\,e^{i\theta_\ell})\right\}\le\frac{\mathbf{f}^*\mathbf{B}\mathbf{g}}{\sum_{0\le\ell<n}|\mathbf{F}[\ell]\,\mathbf{G}[\ell]|}$$
$$\le\max_{0\le\ell<n}\left\{|\lambda_\ell(\mathbf{B})|\operatorname{sgn}(\mathbf{F}[\ell]\,\mathbf{G}[\ell]\,e^{i\theta_\ell})\right\}.$$

Proof. Using the eigendecomposition of $\mathbf{B}$ we have

$$\begin{aligned}\mathbf{f}^*\mathbf{B}\mathbf{g}&=\mathbf{f}^*\mathbf{U}\operatorname{diag}\{\boldsymbol{\lambda}(\mathbf{B})\}\mathbf{V}^*\mathbf{g}\\&=\left(\mathbf{V}\bar{\mathbf{F}}\right)^*\mathbf{U}\operatorname{diag}\{\boldsymbol{\lambda}(\mathbf{B})\}\mathbf{V}^*(\mathbf{U}\mathbf{G})\\&=\mathbf{F}^\top\operatorname{diag}\{\boldsymbol{\lambda}(\mathbf{B})\}\mathbf{G}\\&=\sum_{0\le\ell<n}\mathbf{F}[\ell]\,\lambda_\ell(\mathbf{B})\,\mathbf{G}[\ell].\end{aligned}$$

Writing each eigenvalue in polar form as $\lambda_\ell(\mathbf{B})=|\lambda_\ell(\mathbf{B})|\,e^{i\theta_\ell}$ and applying the admissibility condition (2) yields

$$\begin{aligned}&\min_{0\le\ell<n}\left\{|\lambda_\ell(\mathbf{B})|\operatorname{sgn}(\mathbf{F}[\ell]\,\mathbf{G}[\ell]\,e^{i\theta_\ell})\right\}\sum_{0\le\ell<n}|\mathbf{F}[\ell]\,\mathbf{G}[\ell]|\\\le&\sum_{0\le\ell<n}\mathbf{F}[\ell]\,\lambda_\ell(\mathbf{B})\,\mathbf{G}[\ell]\\\le&\max_{0\le\ell<n}\left\{|\lambda_\ell(\mathbf{B})|\operatorname{sgn}(\mathbf{F}[\ell]\,\mathbf{G}[\ell]\,e^{i\theta_\ell})\right\}\sum_{0\le\ell<n}|\mathbf{F}[\ell]\,\mathbf{G}[\ell]|.\end{aligned}$$

The result follows by algebraic manipulation. □

The condition (2) is an *admissibility condition* on the vectors $\mathbf{f},\mathbf{g}$. In the case where $\mathbf{B}$ is Hermitian, the eigenvalues of $\mathbf{B}$ are real and $\mathbf{V}=\mathbf{U}$ is unitary. If $\mathbf{F}[\ell]\,\mathbf{G}[\ell]\ge 0$ for all ℓ, then

$$\sum_{0\le\ell<n}|\mathbf{F}[\ell]\,\mathbf{G}[\ell]|=\sum_{0\le\ell<n}\mathbf{F}[\ell]\,\mathbf{G}[\ell]=\langle\mathbf{f},\mathbf{g}\rangle$$

since $\mathbf{U}=\mathbf{V}$ is unitary. If $\mathbf{B}$ is Hermitian and moreover $\mathbf{f}=\mathbf{g}$, which implies that $\overline{\mathbf{F}}=\mathbf{G}$, we have

$$\begin{aligned}&\sum_{0\le\ell<n}|\mathbf{F}[\ell]\,\mathbf{F}[\ell]|=\|\mathbf{f}\|_2^2,\\&\min_{0\le\ell<n}|\lambda_\ell(\mathbf{B})|\operatorname{sgn}(\mathbf{F}[\ell]\,\mathbf{F}[\ell]\,e^{i\theta_\ell})=\min\{\boldsymbol{\lambda}(\mathbf{B})\},\\&\max_{0\le\ell<n}|\lambda_\ell(\mathbf{B})|\operatorname{sgn}(\mathbf{F}[\ell]\,\mathbf{F}[\ell]\,e^{i\theta_\ell})=\max\{\boldsymbol{\lambda}(\mathbf{B})\},\end{aligned}$$

which recovers (1). In the general case, $\mathbf{f}, \mathbf{g}$ must interact in a particular way for Theorem 2.1 to hold, as quantified by the admissibility condition (2).

A slightly more general result holds, using a decomposition that bears resemblance to the singular value decomposition, but with negative singular values permitted.

Theorem 2.2. *Let $\mathbf{B} \in \mathbb{C}^{n\times n}$ be decomposed as $\mathbf{B} = \mathbf{U}\operatorname{diag}\{\boldsymbol{\sigma}(\mathbf{B})\}\mathbf{V}^*$ where $\mathbf{UU}^* = \mathbf{I} = \mathbf{VV}^*$ and $\boldsymbol{\sigma}(\mathbf{B}) \subset \mathbb{C}$. Write $\sigma_\ell(\mathbf{B}) = |\sigma_\ell(\mathbf{B})|\, e^{i\theta_\ell}$ for $\theta_\ell \in \{0, \pi\}$. Let $\mathbf{f}, \mathbf{g} \in \mathbb{C}^{n\times 1}$ be such that there exist $\mathbf{F}, \mathbf{G} \in \mathbb{C}^{n\times 1}$ satisfying $\mathbf{F} \circ \mathbf{G} \neq \mathbf{0}_{n\times 1}$ and*

$$\mathbf{f} = \mathbf{U}\overline{\mathbf{F}},\ \ \mathbf{g} = \mathbf{VG}\ \ s.t.\ \forall 0 \le \ell < n,\ \left(\mathbf{F}[\ell]\,\mathbf{G}[\ell]\,e^{i\theta_\ell}\right) \in \mathbb{R}. \tag{3}$$

Then

$$\min_{0\le\ell<n}\left\{|\sigma_\ell(\mathbf{B})|\operatorname{sgn}(\mathbf{F}[\ell]\,\mathbf{G}[\ell]\,e^{i\theta_\ell})\right\} \le \frac{\mathbf{f}^*\mathbf{B}\mathbf{g}}{\sum_{0\le\ell<n}|\mathbf{F}[\ell]\,\mathbf{G}[\ell]|}$$
$$\le \max_{0\le\ell<n}\left\{|\sigma_\ell(\mathbf{B})|\operatorname{sgn}(\mathbf{F}[\ell]\,\mathbf{G}[\ell]\,e^{i\theta_\ell})\right\}.$$

Proof. Using the decomposition of $\mathbf{B}$, we have

$$\begin{aligned}\mathbf{f}^*\mathbf{B}\mathbf{g} &= \mathbf{f}^*\mathbf{U}\operatorname{diag}\{\boldsymbol{\sigma}(\mathbf{B})\}\mathbf{V}^*\mathbf{g}\\ &= \left(\mathbf{U}\bar{\mathbf{F}}\right)^*\mathbf{U}\operatorname{diag}\{\boldsymbol{\sigma}(\mathbf{B})\}\mathbf{V}^*(\mathbf{VG})\\ &= \mathbf{F}^\top\operatorname{diag}\{\boldsymbol{\sigma}(\mathbf{B})\}\mathbf{G}\\ &= \sum_{0\le\ell<n}\mathbf{F}[\ell]\,\sigma_\ell(\mathbf{B})\,\mathbf{G}[\ell].\end{aligned}$$

Writing each element of $\boldsymbol{\sigma}(\mathbf{B})$ in polar form as $\sigma_\ell(\mathbf{B}) = |\sigma_\ell(\mathbf{B})|\, e^{i\theta_\ell}$ and applying the admissibility condition (3) yields

$$\begin{aligned}&\min_{0\le\ell<n}\left\{|\sigma_\ell(\mathbf{B})|\operatorname{sgn}(\mathbf{F}[\ell]\,\mathbf{G}[\ell]\,e^{i\theta_\ell})\right\}\sum_{0\le\ell<n}|\mathbf{F}[\ell]\,\mathbf{G}[\ell]|\\ &\le \sum_{0\le\ell<n}\mathbf{F}[\ell]\,\sigma_\ell(\mathbf{B})\,\mathbf{G}[\ell]\\ &\le \max_{0\le\ell<n}\left\{|\sigma_\ell(\mathbf{B})|\operatorname{sgn}(\mathbf{F}[\ell]\,\mathbf{G}[\ell]\,e^{i\theta_\ell})\right\}\sum_{0\le\ell<n}|\mathbf{F}[\ell]\,\mathbf{G}[\ell]|.\end{aligned}$$

The result follows by algebraic manipulation. □

We remark that the decomposition in Theorem 2.2 may be thought of as a singular value decomposition in which the singular values are permitted to be negative. This type of decomposition will be developed further in Section 3.2.

2.2. Illustration of admissible vectors via index rotations

Admissible vectors may be constructed as follows. Consider the matrix transformation prescribed by performing a rotation to the entry indices of a matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$:

$$\left(\mathbf{A}^{\mathrm{R}_\theta}\right)[j,\ell] = \mathbf{A}\left[\left(j-\frac{n-1}{2}\right)\cos\theta + \left(\frac{n-1}{2}-\ell\right)\sin\theta + \frac{n-1}{2}, \left(j-\frac{n-1}{2}\right)\sin\theta - \left(\frac{n-1}{2}-\ell\right)\cos\theta + \frac{n-1}{2}\right]$$

where rotation angles are restricted to $\theta \in \left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}$. Together with the transpose, these transformations generate the dihedral group of order 8. Given a 3×3 matrix $\mathbf{A}$, we have

$$\mathbf{A}^{\mathrm{R}_0} = \mathbf{A} = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix},$$

$$\mathbf{A}^{\mathrm{R}_{\frac{\pi}{2}}} = \begin{pmatrix} a_{20} & a_{10} & a_{00} \\ a_{21} & a_{11} & a_{01} \\ a_{22} & a_{12} & a_{02} \end{pmatrix}, \mathbf{A}^{\mathrm{R}_\pi} = \begin{pmatrix} a_{22} & a_{21} & a_{20} \\ a_{12} & a_{11} & a_{10} \\ a_{02} & a_{01} & a_{00} \end{pmatrix}, \mathbf{A}^{\mathrm{R}_{\frac{3\pi}{2}}} = \begin{pmatrix} a_{02} & a_{12} & a_{22} \\ a_{01} & a_{11} & a_{21} \\ a_{00} & a_{10} & a_{20} \end{pmatrix}.$$

Note that these matrix transformations may be written in terms of the $n \times n$ auxiliary matrix

$$\mathbf{Q} = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Indeed, a straightforward calculation gives that for any $\mathbf{A} \in \mathbb{C}^{n\times n}$, $\mathbf{A}^{R_{\frac{\pi}{2}}} = \mathbf{A}^\top\mathbf{Q}$, $\mathbf{A}^{R_\pi} = \mathbf{QAQ}$ and $\mathbf{A}^{R_{\frac{3\pi}{2}}} = \mathbf{QA}^\top$.

These rotation transformations preserve unitarity:

Lemma 2.3. *Suppose* $\mathbf{A}$ *is unitary. Then* $\forall\, \theta \in \left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}$, $\mathbf{A}^{R_\theta}$ *is also unitary.*

Proof. It suffices to prove that $\left(\mathbf{A}^{\mathrm{R}_{\frac{\pi}{2}}}\right)^* \left(\mathbf{A}^{\mathrm{R}_{\frac{\pi}{2}}}\right) = \mathbf{I}$; the other angles follow similarly. Noting that $\mathbf{A}^\top$ is unitary because $\mathbf{A}$ is,

$$\begin{aligned}\left(\mathbf{A}^{\mathrm{R}_{\frac{\pi}{2}}}\right)^{*}\left(\mathbf{A}^{\mathrm{R}_{\frac{\pi}{2}}}\right) &= \left(\mathbf{A}^{\top}\mathbf{Q}\right)^{*}\mathbf{A}^{\top}\mathbf{Q}^{\top} \\ &= \mathbf{Q}^{*}\left(\mathbf{A}^{\top}\right)^{*}\mathbf{A}^{\top}\mathbf{Q} \\ &= \mathbf{Q}^{*}\mathbf{Q} \\ &= \mathbf{I}. \quad \square\end{aligned}$$

Moreover, index rotations obey a convenient multiplicative identity:

Lemma 2.4. *Let* $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{n\times n}$*. Then* $(\mathbf{AB})^{R_{\frac{\pi}{2}}} = \mathbf{B}^{\top}\left(\left(\mathbf{A}^{\top}\right)^{R_{\frac{3\pi}{2}}}\right)^{\top}$*.*

Proof. Note that

$$\begin{aligned}(\mathbf{AB})^{\mathrm{R}_{\frac{\pi}{2}}} &= (\mathbf{AB})^{\top}\mathbf{Q} \\ &= \mathbf{B}^{\top}\mathbf{A}^{\top}\mathbf{Q} \\ &= \mathbf{B}^{\top}(\mathbf{QA})^{\top} \\ &= \mathbf{B}^{\top}\left(\left(\mathbf{A}^{\top}\right)^{\mathrm{R}_{\frac{3\pi}{2}}}\right)^{\top} \quad \square\end{aligned}$$

We now establish the existence of a class of admissible vectors.

Theorem 2.5. *Let* $\mathbf{M} \in \mathbb{R}^{n\times n}$ *be symmetric such that* $\mathbf{M}^{R_{\frac{\pi}{2}}}$ *is diagonalizable with*

$$\mathbf{M}^{R_{\frac{\pi}{2}}} = \mathbf{U}\operatorname{diag}\left\{\boldsymbol{\lambda}\left(\mathbf{M}^{R_{\frac{\pi}{2}}}\right)\right\}\mathbf{V}^{*}, \ \mathbf{U}^{-1} = \mathbf{V}^{*}.$$

Write the eigenvalues of $\mathbf{M}^{R_{\frac{\pi}{2}}}$ *in the polar form* $\lambda_{\ell}\left(\mathbf{M}^{R_{\frac{\pi}{2}}}\right) = \left|\lambda_{\ell}\left(\mathbf{M}^{R_{\frac{\pi}{2}}}\right)\right| e^{i\theta_{\ell}}$, $\ell = 0, \ldots, n-1$*. Then there exist admissible vector pairs* $\mathbf{f}, \mathbf{g} \in \mathbb{C}^{n\times 1}$ *for which*

$$\mathbf{f} = \mathbf{V}\bar{\mathbf{F}}, \ \mathbf{g} = \mathbf{U}\mathbf{G}, \ \mathbf{F}\circ\mathbf{G} \neq \mathbf{0}_{n\times 1} \ \textit{s.t.}\ \forall\, 0 \leq \ell < n, \ \left(\mathbf{F}[\ell]\,\mathbf{G}[\ell]\,e^{i\theta_{\ell}}\right) \in \mathbb{R}.$$

Proof. For an arbitrary matrix $\mathbf{B} \in \mathbb{C}^{n\times n}$ the following identity follows from Lemma 2.3

$$\mathbf{f}^{*}\mathbf{B}\mathbf{g} = \mathbf{f}^{*}\left(\mathbf{B}^{\top}\right)^{\mathrm{R}_{\frac{\pi}{2}}}\mathbf{Q}\mathbf{g},$$

since for any matrix $\mathbf{B}$ we have $\left(\mathbf{B}^{\top}\right)^{\mathrm{R}_{\frac{\pi}{2}}}\mathbf{Q} = \mathbf{B}$. In particular, for a real Hermitian $\mathbf{M}$,

$$\mathbf{f}^{*}\mathbf{M}\mathbf{f} = \mathbf{f}^{*}\mathbf{M}^{\mathrm{R}_{\frac{\pi}{2}}}\mathbf{Q}\mathbf{f}.$$

For some fixed j let

$$\mathbf{f} = \mathbf{QU}[:, j], \ \mathbf{g} = \mathbf{U}[:, j].$$

Note that there exist a unique pair of vectors $\bar{\mathbf{F}}, \mathbf{G} \in \mathbb{C}^{n\times 1}$ such that $\mathbf{f} = \mathbf{V}\bar{\mathbf{F}}$ and $\mathbf{g} = \mathbf{U}\mathbf{G}$. In particular, $\mathbf{G} = (0, 0, \ldots, 0, 1, 0, \ldots, 0)^\top$, with the 1 in the jth coordinate. So, to show admissibility of the vector pair $\mathbf{f}, \mathbf{g} \in \mathbb{C}^{n\times 1}$, it thus suffices to show $\mathbf{F}[j]\mathbf{G}[j]e^{i\theta_j} \in \mathbb{R}$. This follows from the fact that

$$\mathbf{f}^*\mathbf{M}^{\mathrm{R}\frac{\pi}{2}}\mathbf{g} = \mathbf{f}^*\mathbf{M}^{\mathrm{R}\frac{\pi}{2}}\mathbf{Q}\mathbf{f} = (\mathbf{f}^*\mathbf{M}\mathbf{f}) \in \mathbb{R}$$

and also

$$\mathbf{f}^*\mathbf{M}^{\mathrm{R}\frac{\pi}{2}}\mathbf{g} = \mathbf{F}^\top \operatorname{diag}(\{\boldsymbol{\lambda}(\mathbf{M}^{\mathrm{R}\frac{\pi}{2}})\})\mathbf{G} = \mathbf{F}[j]\lambda_j\left(\mathbf{M}^{\mathrm{R}\frac{\pi}{2}}\right)\mathbf{G}[j]. \quad \square$$

We remark that the matrix $\mathbf{M}^{\mathrm{R}\frac{\pi}{2}}$ in Theorem 2.5 is a *persymmetric matrix* [16], that is, $\mathbf{M}^{R\frac{\pi}{2}} = \mathbf{Q}\left(\mathbf{M}^{R\frac{\pi}{2}}\right)^\top \mathbf{Q}$.

Note that the rotation operator preserves the magnitude of the eigenvalues of a Hermitian matrix, though their phases may change in general, and the associated eigenvectors may also change.

Corollary 2.6. *Let* $\mathbf{M}$ *be a Hermitian matrix with spectral decomposition* $\mathbf{M} = \mathbf{V}^* \operatorname{diag}\{\boldsymbol{\lambda}(\mathbf{M})\}\mathbf{V}$, $\mathbf{V}^*\mathbf{V} = \mathbf{I}$. *Then* $\mathbf{M}^{R\frac{\pi}{2}} = \mathbf{V}^\top \operatorname{diag}\{\boldsymbol{\lambda}(\mathbf{M})\}\left(\left(\mathbf{V}^{*\top}\right)^{R\frac{3\pi}{2}}\right)^\top$.

Proof. Applying Lemma 2.4 to the factorization $\mathbf{V}^*(\operatorname{diag}\{\boldsymbol{\lambda}(\mathbf{M})\}\mathbf{V})$ yields the desired result. $\square$

3. Estimating independent set cardinalities in directed graphs

As an application of our method to graph theory, we develop estimates on the size of the largest independent set in certain directed graphs. We will consider the inner product on $n \times n$ matrices $\langle \mathbf{B}, \mathbf{M}\rangle = \operatorname{tr}(\mathbf{B}^*\mathbf{M})$, which has associated norm

$$\|\mathbf{B}\| = \|\mathbf{B}\|_{\mathrm{Fro}} = \sqrt{\sum_{0\le j,\ell<n} |\mathbf{B}[j,\ell]|^2}.$$

Definition 3.1. Let $\mathcal{G}$ be a directed, unweighted graph on n vertices. The graph $\mathcal{G}$ has *adjacency matrix* $\mathbf{B} \in \{0,1\}^{n\times n}$ where $\mathbf{B}[i,j] = 1$ if and only if there is a directed edge from the ith node to the jth node in $\mathcal{G}$. The graph $\mathcal{G}$ is *d-regular* if all row sums and column sums are equal to d. The graph $\mathcal{G}$ is *undirected* if it has symmetric adjacency matrix.

The *Delsarte-Hoffman bound* [13,20] is a classical estimate on the cardinality of the largest independent set of an undirected, unweighted, d-regular graph in terms of the spectrum of its adjacency matrix. The proof of the Theorem 3.2 is well-known [29]. For completeness, it is given in the appendix.

Theorem 3.2 *(Undirected Delsarte-Hoffman bound). Let* $\mathbf{B} \in \{0,1\}^{n\times n}$ *be the adjacency matrix of an undirected d-regular graph* $\mathcal{G}$ *with spectral decomposition*

$$\mathbf{B} = \mathbf{U}\,\mathrm{diag}(\boldsymbol{\lambda}(\mathbf{B}))\mathbf{U}^*,\ \ \mathbf{U}\mathbf{U}^* = \mathbf{I}$$

where $\boldsymbol{\lambda}(\mathbf{B}) \subset \mathbb{R}$*. Let* I *be the indices of an independent set in* $\mathcal{G}$*. Then*

$$\frac{|I|}{n} \leq \frac{-\min\left\{\boldsymbol{\lambda}\left(\mathbf{B}\right)\right\}}{d-\min\left\{\boldsymbol{\lambda}\left(\mathbf{B}\right)\right\}}. \tag{4}$$

The condition on the maximum size on an independent set may be characterized as the maximum value $\|\mathbf{1}_I\|_2^2$, where $\mathbf{1}_I^*\mathbf{B}\mathbf{1}_I = 0$ for some $I \subset V$. The spectral decomposition of $\mathbf{B}$ decouples the rank-one matrix $\mathbf{1}_{n\times n}$ associated with the eigenvalue $\lambda_0(\mathbf{B}) = d$, from whence the analysis flows.

Note that the Delsarte-Hoffman bound is sharp. Indeed, let n be a positive integer and consider a complete bipartite graph with $2n$ nodes in which each partition has n nodes. For such a graph,

$$\max_{I \text{ independent}} |I| = n,\ \min\{\boldsymbol{\lambda}(\mathbf{B})\} = -n,\ d = n.$$

3.1. Directed Delsarte-Hoffman bound

We now consider independent sets in *directed* regular graphs and broaden the scope to adjacency matrices whose entries are not necessarily binary.

Definition 3.3. Let $\mathcal{G}$ be a directed graph with n nodes. A matrix $\mathbf{B} \in \mathbb{C}^{n\times n}$ is a *pseudo-adjacency matrix* for $\mathcal{G}$ if $\mathbf{B}_{ij} = 0$ if and only if there is no directed edge from the ith node to the jth node in $\mathcal{G}$. For a real number $\delta > 0$, the pseudo-adjacency matrix $\mathbf{B}$ is said to be δ*-regular* if all row sums and column sums of $\mathbf{B}$ are equal to δ.

We develop Delsarte-Hoffman-type bounds based on the spectral decomposition and the singular value decomposition of the (non-Hermitian) pseudo-adjacency matrix.

Definition 3.4. Let $\mathbf{B} \in \mathbb{C}^{n\times n}$ be a pseudo-adjacency matrix for a directed graph. Let $S, T \subset \{0, 1, 2, \ldots, n-1\}$ be subsets of nodes in the graph. The pair (S, T) is an *independent pair for* $\mathbf{B}$ if $\mathbf{1}_S^*\mathbf{B}\mathbf{1}_T = 0$, where $\mathbf{1}_S, \mathbf{1}_T \in \{0,1\}^{n\times 1}$ are indicator sets for S, T respectively. We say $\mathbf{1}_S, \mathbf{1}_T$ are *indicator vectors for the independent pair* (S, T).

The Delsarte-Hoffman bound we prove estimates the maximal size of $|S||T|$, where (S, T) is an independent pair in a directed graph with (non-symmetric) adjacency matrix $\mathbf{B}$.

Theorem 3.5. *Let* $\mathbf{B} \in \mathbb{C}^{n\times n}$ *denote a* δ*-regular pseudo-adjacency matrix for a directed graph* $\mathcal{G}$*. Let* $\mathbf{B}$ *be diagonalizable and decomposed as* $\mathbf{B} = \mathbf{U}\,\mathrm{diag}\left\{\boldsymbol{\lambda}\left(\mathbf{B}\right)\right\}\mathbf{V}^*$ *where*

$\mathbf{U}^{-1} = \mathbf{V}^*$ *and the first columns of* $\mathbf{U}, \mathbf{V}$ *are constant. Let* $\lambda_\ell(\mathbf{B}) = \alpha_\ell e^{i\theta_\ell}$, $\alpha_\ell \in \mathbb{R}, \theta_\ell \in [0, 2\pi)$ *be a polar form of the* ℓ*-th eigenvalue of* $\mathbf{B}$*. Let* $\boldsymbol{\alpha}(\mathbf{B}) = (\alpha_0, \ldots, \alpha_{n-1})$*. Let* $\mathbf{f}, \mathbf{g} \in \{0,1\}^{n\times 1}$ *be such that there exist* $\mathbf{F}, \mathbf{G} \in \mathbb{C}^{n\times 1}$ *satisfying*

$$\mathbf{f} = \mathbf{V}\overline{\mathbf{F}},\ \mathbf{g} = \mathbf{U}\mathbf{G} \text{ such that } \forall 0 \le \ell < n,\ \left(\mathbf{F}[\ell]\,\mathbf{G}[\ell]\,e^{i\theta_\ell}\right) \ge 0. \tag{5}$$

Then if $\mathbf{f}$ *and* $\mathbf{g}$ *denote respectively indicator vectors for an independent pair in* $\mathbf{B}$*,*

$$\frac{\|\mathbf{f}\|_2^2\|\mathbf{g}\|_2^2}{n} \le \frac{-\min\{\boldsymbol{\alpha}(\mathbf{B})\} \sum\limits_{0\le\ell<n} \mathbf{F}[\ell]\,\mathbf{G}[\ell]\,e^{i\theta_\ell}}{\delta - \min\{\boldsymbol{\alpha}(\mathbf{B})\}}.$$

Proof. Note that by δ-regularity, $\lambda_0(\mathbf{B}) = \delta$. We also have that $\mathbf{U}[:,0] = \mathbf{V}[:,0] = \frac{1}{\sqrt{n}}(1,1,\ldots,1)^\top$. Hence, $\mathbf{B} = \frac{\delta}{n}\mathbf{1}_{n\times n} + \left(\mathbf{I} - \frac{\mathbf{1}_{n\times n}}{n}\right)\mathbf{B}$. Thus, for all $\mathbf{f}, \mathbf{g} \in \mathbb{C}^{n\times 1}$ subject to (5),

$$\langle \mathbf{f}\,\mathbf{g}^*, \mathbf{B}\rangle = \delta\left\langle \mathbf{f}\,\mathbf{g}^*, \frac{\mathbf{1}_{n\times n}}{n}\right\rangle + \left\langle \mathbf{f}\,\mathbf{g}^*, \left(\mathbf{I} - \frac{\mathbf{1}_{n\times n}}{n}\right)\mathbf{B}\right\rangle.$$

We analyze the second term of the right hand side as follows:

$$\begin{aligned}
\left\langle \mathbf{f}\,\mathbf{g}^*, \left(\mathbf{I} - \frac{\mathbf{1}_{n\times n}}{n}\right)\mathbf{B}\right\rangle &= \mathbf{f}^*\left(\mathbf{I} - \frac{\mathbf{1}_{n\times n}}{n}\right)\mathbf{B}\mathbf{g} \\
&= \mathbf{f}^*\left(\mathbf{I} - \frac{\mathbf{1}_{n\times n}}{n}\right)\mathbf{U}\,\mathrm{diag}(\boldsymbol{\lambda}(\mathbf{B}))\mathbf{V}^*\mathbf{g} \\
&= \mathbf{f}^*\left(\mathbf{U}\,\mathrm{diag}(\boldsymbol{\lambda}(\mathbf{B}))\mathbf{V}^* - \frac{1}{n}\sum_{0\le\ell<n}\lambda_\ell(\mathbf{B})\mathbf{1}_{n\times n}\mathbf{U}[:,\ell]\mathbf{V}^*[\ell,:]\right)\mathbf{g} \\
&= \mathbf{f}^*\left(\sum_{0\le\ell<n}\lambda_\ell(\mathbf{B})\mathbf{U}[:,\ell]\mathbf{V}^*[\ell,:] - \lambda_0(\mathbf{B})\mathbf{U}[:,0]\mathbf{V}^*[0,:]\right)\mathbf{g} \\
&= \mathbf{f}^*\left(\sum_{1\le\ell<n}\lambda_\ell(\mathbf{B})\mathbf{U}[:,\ell]\mathbf{V}^*[\ell,:]\right)\mathbf{g} \\
&= \mathbf{F}^\top\mathbf{V}^*\left(\sum_{1\le\ell<n}\lambda_\ell(\mathbf{B})\mathbf{U}[:,\ell]\mathbf{V}^*[\ell,:]\right)\mathbf{U}\mathbf{G} \\
&= \sum_{1\le\ell<n}\mathbf{F}[\ell]\,\lambda_\ell(\mathbf{B})\mathbf{G}[\ell] \\
&= \sum_{1\le\ell<n}\alpha_\ell\mathbf{F}[\ell]\,\mathbf{G}[\ell]\,e^{i\theta_\ell}
\end{aligned}$$

$$\geq \min\{\boldsymbol{\alpha}(\mathbf{B})\} \sum_{1\leq \ell<n} \mathbf{F}\left[\ell\right] \mathbf{G}\left[\ell\right] e^{i\theta_\ell}$$

$$= \min\{\boldsymbol{\alpha}(\mathbf{B})\} \left(-\mathbf{F}\left[0\right] \mathbf{G}\left[0\right] + \sum_{0\leq \ell<n} \mathbf{F}\left[\ell\right] \mathbf{G}\left[\ell\right] e^{i\theta_\ell} \right)$$

$$= \min\{\boldsymbol{\alpha}(\mathbf{B})\} \left(-\frac{1}{n}\|\mathbf{f}\|_2^2\|\mathbf{g}\|_2^2 + \sum_{0\leq \ell<n} \mathbf{F}\left[\ell\right] \mathbf{G}\left[\ell\right] e^{i\theta_\ell} \right).$$

Note that $\mathbf{F}\left[0\right] = \frac{1}{\sqrt{n}}\|\mathbf{f}\|_1 = \frac{1}{\sqrt{n}}\|\mathbf{f}\|_2^2$ follows from the observation that $\mathbf{f}$ takes values in $\{0,1\}, \mathbf{U}^*\mathbf{f} = \mathbf{U}^*\mathbf{V}\overline{\mathbf{F}} = \overline{\mathbf{F}}$, and $\mathbf{U}[:,0] = \frac{1}{\sqrt{n}}(1,\ldots,1)^\top$. That $\mathbf{G}\left[0\right] = \frac{1}{\sqrt{n}}\|\mathbf{g}\|_2^2$ follows similarly. Hence,

$$\langle \mathbf{f}\,\mathbf{g}^*, \mathbf{B}\rangle \geq \delta \left\langle \mathbf{f}\,\mathbf{g}^*, \frac{\mathbf{1}_{n\times n}}{n} \right\rangle + \min\{\boldsymbol{\alpha}(\mathbf{B})\} \left(-\frac{1}{n}\|\mathbf{f}\|_2^2\|\mathbf{g}\|_2^2 + \sum_{0\leq \ell<n} \mathbf{F}\left[\ell\right] \mathbf{G}\left[\ell\right] e^{i\theta_\ell} \right)$$

$$= \frac{\delta}{n}\|\mathbf{f}\|_2^2\|\mathbf{g}\|_2^2 + \min\{\boldsymbol{\alpha}(\mathbf{B})\} \left(-\frac{1}{n}\|\mathbf{f}\|_2^2\|\mathbf{g}\|_2^2 + \sum_{0\leq \ell<n} \mathbf{F}\left[\ell\right] \mathbf{G}\left[\ell\right] e^{i\theta_\ell} \right).$$

Since $\mathbf{f}$ and $\mathbf{g}$ are indicator vectors for an independent pair in $\mathbf{B}$, $\mathbf{f}^*\mathbf{B}\mathbf{g} = 0$. It follows that

$$0 \geq \frac{\delta}{n}\|\mathbf{f}\|_2^2\|\mathbf{g}\|_2^2 + \min\{\boldsymbol{\alpha}(\mathbf{B})\} \left(-\frac{1}{n}\|\mathbf{f}\|_2^2\|\mathbf{g}\|_2^2 + \sum_{0\leq \ell<n} \mathbf{F}\left[\ell\right] \mathbf{G}\left[\ell\right] e^{i\theta_\ell} \right),$$

from whence the result follows by algebraic manipulation. □

We note that the condition that the first columns of $\mathbf{U}$, $\mathbf{V}$ are constant is unnecessary in the case that the eigenvalue δ has multiplicity 1. If it has multiplicity greater than 1, then this condition is simply to specify that the constant vector of ℓ^2 norm 1 is a generator for the subspace spanned by the eigenvectors with eigenvalue δ.

In the case that $\mathbf{B}$ is the adjacency matrix of a d-regular graph, the result may be interpreted as a generalization of Theorem 3.2.

Corollary 3.6 *(Directed Delsarte-Hoffman bound). Let $\mathbf{B} \in \{0,1\}^{n\times n}$ denote a d-regular adjacency matrix for a directed graph $\mathcal{G}$ on n vertices. Let $\mathbf{B}$ be decomposed as $\mathbf{B} = \mathbf{U}\,\mathrm{diag}\,\{\boldsymbol{\lambda}(\mathbf{B})\}\,\mathbf{V}^*$ where $\mathbf{U}^{-1} = \mathbf{V}^*$ and the first columns of $\mathbf{U}, \mathbf{V}$ are constant. Let $\lambda_\ell(\mathbf{B}) = \alpha_\ell e^{i\theta_\ell}$, $\alpha_\ell \in \mathbb{R}, \theta_\ell \in [0, 2\pi)$ be a polar decomposition of $\boldsymbol{\lambda}(\mathbf{B})$. Let $\boldsymbol{\alpha}(\mathbf{B}) = (\alpha_1, \ldots, \alpha_n)$. Let $\mathbf{f}, \mathbf{g} \in \{0,1\}^{n\times 1}$ be such that there exist $\mathbf{F}, \mathbf{G} \in \mathbb{C}^{n\times 1}$ satisfying*

$$\mathbf{f} = \mathbf{V}\overline{\mathbf{F}},\ \mathbf{g} = \mathbf{U}\mathbf{G} \textit{ such that } \forall 0 \leq \ell < n,\ \left(\mathbf{F}\left[\ell\right] \mathbf{G}\left[\ell\right] e^{i\theta_\ell}\right) \geq 0. \tag{6}$$

Then if $\mathbf{f}$ *and* $\mathbf{g}$ *are indicator vectors for an independent pair in* $\mathbf{B}$*,*

$$\frac{\|\mathbf{f}\|_2^2\|\mathbf{g}\|_2^2}{n} \leq \frac{-\min\{\boldsymbol{\alpha}(\mathbf{B})\} \sum_{0\leq\ell<n} \mathbf{F}[\ell]\,\mathbf{G}[\ell]\,e^{i\theta_\ell}}{d-\min\{\boldsymbol{\alpha}(\mathbf{B})\}}.$$

Note that the quantity bounded in Corollary 3.6 may be interpreted as the size of an independent pair (S,T). Indeed, if $\mathbf{f} = \mathbf{1}_S, \mathbf{g} = \mathbf{1}_T$ are the indicator vectors for the independent pair, then $\|\mathbf{f}\|_2^2\|\mathbf{g}\|_2^2 = |S|\,|T|$.

If $\mathbf{B}$ is Hermitian and $\mathbf{f} = \mathbf{g}$, the admissibility condition (6) is always satisfied as $\theta_\ell = 0, \ell = 0,\dots,n-1$. Indeed, in this case,

$$\sum_{0\leq\ell<n} \mathbf{F}[\ell]\,\mathbf{G}[\ell]\,e^{i\theta_\ell} = \|\mathbf{f}\|_2^2,$$

so that the conclusion of Theorem 3.2 holds. Hence, Corollary 3.6 is a strict generalization of the classical Delsarte-Hoffman inequality.

3.1.1. Tightness of directed Delsarte-Hoffman bound

When n is a multiple of 4, adjacency matrices of the form

$$\mathbf{B}_n = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \otimes \mathbf{1}_{\frac{n}{4}\times\frac{n}{4}}$$

show Corollary 3.6 is tight. In the case $n = 4$, $\mathbf{B}_4$ is the adjacency matrix for a $\frac{n}{4} = 1$-regular directed graph, which may be decomposed as:

$$\begin{pmatrix} 1/2 & -i/2 & i/2 & 1/2 \\ 1/2 & i/2 & -i/2 & 1/2 \\ -1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & -i/2 & i/2 & 1/2 \\ 1/2 & i/2 & -i/2 & 1/2 \\ -1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & -1/2 & 1/2 \end{pmatrix}^* .$$

In this case, the largest independent pair has size 4, corresponding to the zero block on the upper left. Note that the zero block on the lower right also corresponds to an independent pair of size 4. Let $\mathbf{f}_4 = \mathbf{g}_4 = (1,1,0,0)^\top$, so that for coefficients $\mathbf{F}_4, \mathbf{G}_4$ as in (6), $\mathbf{F}_4 = \mathbf{G}_4 = (1,0,0,1)^\top$. Decomposing the first and fourth eigenvalues as $\alpha_0 = -1, \alpha_3 = 1, \theta_0 = \theta_3 = 0$, it is seen that the admissibility condition is satisfied, and that $\min\{\boldsymbol{\alpha}\} = -1$. Moreover,

$$\sum_{0\leq\ell<4} \mathbf{F}_4[\ell]\,\mathbf{G}_4[\ell]\,e^{i\theta_\ell} = 2$$

so that the estimate of Corollary 3.6 is

$$\frac{\|\mathbf{f}_4\|_2^2\|\mathbf{g}_4\|_2^2}{4} \leq 1,$$

which is tight since $\|\mathbf{f}_4\|_2^2\|\mathbf{g}_4\|_2^2/4 = 1$. This shows the maximal size of an independent pair in $\mathbf{B}_4$ is tightly estimated by Corollary 3.6. Note that a similar argument holds for the block corresponding to indicator vectors $\mathbf{f}_4 = \mathbf{g}_4 = (0, 0, 1, 1)^\top$.

In the general case when $n > 4$ is a multiple of 4, the above argument generalizes to show the size of the largest independent pair in $\mathbf{B}_n$ is tightly estimated by Corollary 3.6. Let $\mathbf{f}_n = \mathbf{f}_4 \otimes \mathbf{1}_{\frac{n}{4}\times 1}$, $\mathbf{g}_n = \mathbf{g}_4 \otimes \mathbf{1}_{\frac{n}{4}\times 1}$. Then $\mathbf{f}_n, \mathbf{g}_n$ are indicator functions for an independent pair in $\mathbf{B}_n$ of largest size, and $\|\mathbf{f}_n\|_2^2\|\mathbf{g}_n\|_2^2/n = \frac{n}{4}$. Note that since $\mathbf{1}_{\frac{n}{4}\times\frac{n}{4}}$ has eigenvalue $\frac{n}{4}$ with multiplicity 1 and eigenvalue 0 with multiplicity $(\frac{n}{4}-1)$, $\mathbf{B}_n$ has eigenvalues $-\frac{n}{4}, \frac{ni}{4}, -\frac{ni}{4}, \frac{n}{4}$, each of multiplicity 1, and eigenvalue 0 with multiplicity $(n-4)$. In particular, the eigenvectors corresponding to the non-zero eigenvalues are just the eigenvectors of $\mathbf{B}_4$, but appropriately inflated and normalized, namely $\frac{2}{\sqrt{n}}(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})^\top \otimes \mathbf{1}_{\frac{n}{4}\times 1}$, $\frac{2}{\sqrt{n}}(-\frac{i}{2}, \frac{i}{2}, \frac{1}{2}, -\frac{1}{2})^\top \otimes \mathbf{1}_{\frac{n}{4}\times 1}$, $\frac{2}{\sqrt{n}}(\frac{i}{2}, -\frac{i}{2}, \frac{1}{2}, -\frac{1}{2})^\top \otimes \mathbf{1}_{\frac{n}{4}\times 1}$, $\frac{2}{\sqrt{n}}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})^\top \otimes \mathbf{1}_{\frac{n}{4}\times 1}$. It follows that for coefficients $\mathbf{F}_n, \mathbf{G}_n$ as in (6), $\mathbf{F}_n = \mathbf{G}_n = (\sqrt{\frac{n}{4}}, 0, 0, \sqrt{\frac{n}{4}}, 0, 0, \dots, 0)^\top$. Decomposing the first and fourth eigenvalues as $\alpha_0 = -\frac{n}{4}, \alpha_3 = \frac{n}{4}, \theta_0 = \theta_3 = 0$, it is seen that the admissibility condition is satisfied, and that $\min\{\boldsymbol{\alpha}\} = -\frac{n}{4}$. Moreover,

$$\sum_{0\leq\ell<n} \mathbf{F}_n[\ell]\,\mathbf{G}_n[\ell]\,e^{i\theta_\ell} = \frac{n}{2}.$$

Noting that $\mathbf{B}_n$ is $\frac{n}{4}$-regular, we see that Corollary 3.6 estimates $\|\mathbf{f}_n\|_2^2\|\mathbf{g}\|_2^2/n$ as $\frac{n}{4}$, which verifies the tightness of this estimate.

3.2. A Delsarte-Hoffman bound using the singular value decomposition

Consider the singular value decomposition of $\mathbf{B} \in \mathbb{C}^{n\times n}$ expressed by

$$\mathbf{B} = \mathbf{U}\,\text{diag}(\boldsymbol{\sigma}(\mathbf{B}))\mathbf{V}^* \;\; \text{s.t. } \mathbf{U}\mathbf{U}^* = \mathbf{I} = \mathbf{V}\mathbf{V}^*,$$

where each element of $\boldsymbol{\sigma}(\mathbf{B})$ is positive. Theorem 3.7 provides a Delsarte-Hoffman estimate on the maximal size of an independent set using a decomposition similar to the SVD, which holds for *all matrices*, not just diagonalizable ones.

Theorem 3.7. *Let $\mathbf{B} \in \mathbb{C}^{n\times n}$ be a δ-regular pseudo-adjacency matrix of a directed graph. Suppose $\mathbf{B}$ has a decomposition $\mathbf{B} = \mathbf{U}\,\text{diag}(\boldsymbol{\sigma})\,\mathbf{V}^*$ such that $\mathbf{U}\mathbf{U}^* = \mathbf{I} = \mathbf{V}\mathbf{V}^*$, the first columns of $\mathbf{U}, \mathbf{V}$ are constant and $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$. Let $\sigma_{min} = \min_\ell \sigma_\ell$. Suppose that $\mathbf{f}, \mathbf{g} \in \{0,1\}^{n\times 1}$ are indicator vectors for an independent pair in $\mathbf{B}$, and that there exist $\mathbf{F}, \mathbf{G} \in \mathbb{C}^{n\times 1}$ such that*

$$\mathbf{f} = \mathbf{U}\overline{\mathbf{F}},\;\; \mathbf{g} = \mathbf{V}\mathbf{G},\;\; \textit{and}\; \forall 0 \leq \ell < n,\; \mathbf{F}[\ell]\,\mathbf{G}[\ell] \geq 0. \tag{7}$$

Then

$$\frac{\|\mathbf{f}\|_2^2\|\mathbf{g}\|_2^2}{n} \leq \frac{-\sigma_{min} \sum_{0\leq\ell<n} \mathbf{F}[\ell]\,\mathbf{G}[\ell]}{\delta - \sigma_{min}}.$$

Proof. By the SVD and by δ-regularity,

$$\mathbf{B} = \frac{\delta}{n}\mathbf{1}_{n\times n} + \left(\mathbf{I} - \frac{\mathbf{1}_{n\times n}}{n}\right)\mathbf{B}.$$

Analyzing the second term for all $\mathbf{f},\mathbf{g} \in \mathbb{C}^{n\times 1}$, subject to (7),

$$\begin{aligned}
\left\langle \mathbf{f}\,\mathbf{g}^*, \left(\mathbf{I} - \frac{\mathbf{1}_{n\times n}}{n}\right)\mathbf{B}\right\rangle =& \mathbf{f}^*\left(\mathbf{I} - \frac{\mathbf{1}_{n\times n}}{n}\right)\left(\sum_{0\leq\ell<n}\sigma_\ell \mathbf{U}[:,\ell]\mathbf{V}^*[\ell,:]\right)\mathbf{g} \\
=& \mathbf{f}^*\left(\sum_{1\leq\ell<n}\sigma_\ell \mathbf{U}[:,\ell]\mathbf{V}^*[\ell,:]\right)\mathbf{g} \\
=& \mathbf{F}^\top\mathbf{U}^*\left(\sum_{1\leq\ell<n}\sigma_\ell \mathbf{U}[:,\ell]\mathbf{V}^*[\ell,:]\right)\mathbf{V}\mathbf{G} \\
=& \sum_{1\leq\ell<n}\sigma_\ell \mathbf{F}[\ell]\,\mathbf{G}[\ell] \\
\geq& \sigma_{\min}\sum_{1\leq\ell<n}\mathbf{F}[\ell]\,\mathbf{G}[\ell] \\
=& \sigma_{\min}\left(-\mathbf{F}[0]\,\mathbf{G}[0] + \sum_{0\leq\ell<n}\mathbf{F}[\ell]\,\mathbf{G}[\ell]\right) \\
=& \sigma_{\min}\left(-\frac{1}{n}\|\mathbf{f}\|_2^2\|\mathbf{g}\|_2^2 + \sum_{0\leq\ell<n}\mathbf{F}[\ell]\,\mathbf{G}[\ell]\right).
\end{aligned}$$

Note that $\mathbf{F}[0] = \frac{1}{\sqrt{n}}\|\mathbf{f}\|_1 = \frac{1}{\sqrt{n}}\|\mathbf{f}\|_2^2$ follows from $\mathbf{U}^*\mathbf{f} = \mathbf{U}^*\mathbf{U}\overline{\mathbf{F}} = \mathbf{F}$ and that fact that $\mathbf{f} \in \{0,1\}^{n\times 1}$, $\mathbf{U}[:,0] = \frac{1}{\sqrt{n}}(1,\ldots,1)^\top$; $\mathbf{G}[0] = \frac{1}{\sqrt{n}}\|\mathbf{g}\|_2^2$ follows similarly. Thus,

$$\begin{aligned}
\langle \mathbf{f}\,\mathbf{g}^*, \mathbf{B}\rangle =& \delta\left\langle \mathbf{f}\,\mathbf{g}^*, \frac{\mathbf{1}_{n\times n}}{n}\right\rangle + \left\langle \mathbf{f}\,\mathbf{g}^*, \left(\mathbf{I} - \frac{\mathbf{1}_{n\times n}}{n}\right)\mathbf{B}\right\rangle \\
\geq& \delta\left\langle \mathbf{f}\,\mathbf{g}^*, \frac{\mathbf{1}_{n\times n}}{n}\right\rangle + \sigma_{\min}\left(-\frac{1}{n}\|\mathbf{f}\|_2^2\|\mathbf{g}\|_2^2 + \sum_{0\leq\ell<n}\mathbf{F}[\ell]\,\mathbf{G}[\ell]\right) \\
=& \frac{\delta}{n}\|\mathbf{f}\|_2^2\|\mathbf{g}\|_2^2 + \sigma_{\min}\left(-\frac{1}{n}\|\mathbf{f}\|_2^2\|\mathbf{g}\|_2^2 + \sum_{0\leq\ell<n}\mathbf{F}[\ell]\,\mathbf{G}[\ell]\right).
\end{aligned}$$

Since $\mathbf{f}, \mathbf{g}$ are indicator vectors for an independent pair in $\mathbf{B}$, $\mathbf{f}^*\mathbf{B}\mathbf{g} = 0$. For such an $\mathbf{f}$, $\mathbf{g}$ pair also subject to the admissibility condition (7) we have

$$0 \geq \delta \left(\frac{\|\mathbf{f}\|_2^2 \|\mathbf{g}\|_2^2}{n} \right) - \sigma_{\min} \left(\frac{\|\mathbf{f}\|_2^2 \|\mathbf{g}\|_2^2}{n} \right) + \sigma_{\min} \sum_{0 \leq \ell < n} \mathbf{F}[\ell] \mathbf{G}[\ell],$$

from whence the result follows by algebraic manipulation. □

Theorem 3.7 requires a decomposition which bears resemblance to the SVD in the fact that $\mathbf{B} = \mathbf{U}\,\mathrm{diag}(\boldsymbol{\sigma})\mathbf{V}^*$, where $\mathbf{U}\mathbf{U}^* = \mathbf{I} = \mathbf{V}\mathbf{V}^*$, but without the condition that $\sigma_\ell \geq 0$ for all ℓ. Note that if $\sigma_\ell \mapsto -\sigma_\ell$, and $\mathbf{U}[:, \ell] \mapsto -\mathbf{U}[:, \ell]$ or $\mathbf{V}^*[\ell, :] \mapsto -\mathbf{V}^*[\ell, :]$, this still expresses such a decomposition for $\mathbf{B}$. In this sense, there are 2^n decompositions to consider in Theorem 3.7, corresponding to the 2^n possible sign assignments. Thus, one can think of the decomposition in Theorem 3.7 as a (non-unique) signed SVD, and the condition (7) as an admissibility condition with respect to this decomposition.

If in particular $\mathbf{B}$ is the adjacency matrix of a d-regular directed graph, the following result holds.

Corollary 3.8. *Let* $\mathbf{B} \in \{0,1\}^{n\times n}$ *be a d-regular adjacency matrix of a directed graph. Suppose* $\mathbf{B}$ *has a decomposition* $\mathbf{B} = \mathbf{U}\,\mathrm{diag}(\boldsymbol{\sigma})\,\mathbf{V}^*$ *such that* $\mathbf{U}\mathbf{U}^* = \mathbf{I} = \mathbf{V}\mathbf{V}^*$*, the first columns of* $\mathbf{U}, \mathbf{V}$ *are constant, and* $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$*. Let* $\sigma_{min} = \min_\ell \sigma_\ell$*. Suppose that* $\mathbf{f} \in \{0,1\}^{n\times 1}, \mathbf{g} \in \{0,1\}^{n\times 1}$ *are indicator vectors for an independent pair in* $\mathbf{B}$*, and that there exist* $\mathbf{F}, \mathbf{G} \in \mathbb{C}^{n\times 1}$ *such that*

$$\mathbf{f} = \mathbf{U}\overline{\mathbf{F}}, \;\; \mathbf{g} = \mathbf{V}\mathbf{G}, \;\; \textit{and}\; \forall 0 \leq \ell < n,\; \mathbf{F}[\ell]\,\mathbf{G}[\ell] \geq 0.$$

Then

$$\frac{\|\mathbf{f}\|_2^2 \|\mathbf{g}\|_2^2}{n} \leq \frac{-\sigma_{min} \sum\limits_{0 \leq \ell < n} \mathbf{F}[\ell]\,\mathbf{G}[\ell]}{d - \sigma_{min}}.$$

3.2.1. Tightness of directed Delsarte-Hoffman SVD bound

We note that Theorem 3.7 is tight. Indeed, consider the 2-regular directed graph with adjacency matrix

$$\mathbf{B} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The adjacency matrix $\mathbf{B}$ is not symmetric and is not even diagonalizable. However, $\mathbf{B}$ admits the SVD decomposition $\mathbf{B} = \mathbf{U}\,\mathrm{diag}(\boldsymbol{\sigma})\mathbf{V}^*$ where $\mathbf{U}, \mathbf{V}$ are orthogonal and have the numerical expression

$$\mathbf{U} = \begin{pmatrix}
\frac{\sqrt{7}}{7} & -\frac{1}{5}\sqrt{\frac{10}{7}} & 0 & \sqrt{\frac{2}{5}} & 0 & -\sqrt{\frac{2}{5}} & 0 \\
\frac{\sqrt{7}}{7} & -\frac{1}{5}\sqrt{\frac{10}{7}} & \frac{1}{\sqrt{5-\sqrt{5}}} & \frac{\sqrt{\frac{5}{2}}(\sqrt{5}+1)}{5(\sqrt{5}+3)} & \frac{\sqrt{5}-1}{2\sqrt{5-\sqrt{5}}} & \frac{2\sqrt{\frac{5}{2}}(\sqrt{5}+2)}{5(\sqrt{5}+3)} & 0 \\
\frac{\sqrt{7}}{7} & -\frac{1}{5}\sqrt{\frac{10}{7}} & -\frac{1}{\sqrt{5-\sqrt{5}}} & \frac{\sqrt{\frac{5}{2}}(\sqrt{5}+1)}{5(\sqrt{5}+3)} & -\frac{\sqrt{5}-1}{2\sqrt{5-\sqrt{5}}} & \frac{2\sqrt{\frac{5}{2}}(\sqrt{5}+2)}{5(\sqrt{5}+3)} & 0 \\
\frac{\sqrt{7}}{7} & -\frac{1}{5}\sqrt{\frac{10}{7}} & \frac{\sqrt{5}-1}{2\sqrt{5-\sqrt{5}}} & -\frac{2\sqrt{\frac{5}{2}}(\sqrt{5}+2)}{5(\sqrt{5}+3)} & -\frac{1}{\sqrt{5-\sqrt{5}}} & -\frac{\sqrt{\frac{5}{2}}(\sqrt{5}+1)}{5(\sqrt{5}+3)} & 0 \\
\frac{\sqrt{7}}{7} & -\frac{1}{5}\sqrt{\frac{10}{7}} & -\frac{\sqrt{5}-1}{2\sqrt{5-\sqrt{5}}} & -\frac{2\sqrt{\frac{5}{2}}(\sqrt{5}+2)}{5(\sqrt{5}+3)} & \frac{1}{\sqrt{5-\sqrt{5}}} & -\frac{\sqrt{\frac{5}{2}}(\sqrt{5}+1)}{5(\sqrt{5}+3)} & 0 \\
\frac{\sqrt{7}}{7} & \frac{1}{2}\sqrt{\frac{10}{7}} & 0 & 0 & 0 & 0 & -\frac{\sqrt{2}}{2} \\
\frac{\sqrt{7}}{7} & \frac{1}{2}\sqrt{\frac{10}{7}} & 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2}
\end{pmatrix}$$

$$\mathbf{V} = \begin{pmatrix}
\frac{\sqrt{7}}{7} & \frac{1}{2}\sqrt{\frac{10}{7}} & 0 & 0 & 0 & 0 & -\frac{\sqrt{2}}{2} \\
\frac{\sqrt{7}}{7} & \frac{1}{2}\sqrt{\frac{10}{7}} & 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} \\
\frac{\sqrt{7}}{7} & -\frac{1}{5}\sqrt{\frac{10}{7}} & \frac{\sqrt{5}-1}{2\sqrt{5-\sqrt{5}}} & \frac{2\sqrt{\frac{5}{2}}(\sqrt{5}+2)}{5(\sqrt{5}+3)} & \frac{1}{\sqrt{5-\sqrt{5}}} & -\frac{\sqrt{\frac{5}{2}}(\sqrt{5}+1)}{5(\sqrt{5}+3)} & 0 \\
\frac{\sqrt{7}}{7} & -\frac{1}{5}\sqrt{\frac{10}{7}} & -\frac{\sqrt{5}-1}{2\sqrt{5-\sqrt{5}}} & \frac{2\sqrt{\frac{5}{2}}(\sqrt{5}+2)}{5(\sqrt{5}+3)} & -\frac{1}{\sqrt{5-\sqrt{5}}} & -\frac{\sqrt{\frac{5}{2}}(\sqrt{5}+1)}{5(\sqrt{5}+3)} & 0 \\
\frac{\sqrt{7}}{7} & -\frac{1}{5}\sqrt{\frac{10}{7}} & \frac{1}{\sqrt{5-\sqrt{5}}} & -\frac{\sqrt{\frac{5}{2}}(\sqrt{5}+1)}{5(\sqrt{5}+3)} & -\frac{\sqrt{5}-1}{2\sqrt{5-\sqrt{5}}} & \frac{2\sqrt{\frac{5}{2}}(\sqrt{5}+2)}{5(\sqrt{5}+3)} & 0 \\
\frac{\sqrt{7}}{7} & -\frac{1}{5}\sqrt{\frac{10}{7}} & -\frac{1}{\sqrt{5-\sqrt{5}}} & -\frac{\sqrt{\frac{5}{2}}(\sqrt{5}+1)}{5(\sqrt{5}+3)} & \frac{\sqrt{5}-1}{2\sqrt{5-\sqrt{5}}} & \frac{2\sqrt{\frac{5}{2}}(\sqrt{5}+2)}{5(\sqrt{5}+3)} & 0 \\
\frac{\sqrt{7}}{7} & -\frac{1}{5}\sqrt{\frac{10}{7}} & 0 & -\sqrt{\frac{2}{5}} & 0 & -\sqrt{\frac{2}{5}} & 0
\end{pmatrix}$$

and $\boldsymbol{\sigma} = (2, 2, \frac{\sqrt{5}}{2} + \frac{1}{2}, \frac{\sqrt{5}}{2} + \frac{1}{2}, \frac{\sqrt{5}}{2} - \frac{1}{2}, \frac{\sqrt{5}}{2} - \frac{1}{2}, 0)$. Let $\tilde{\mathbf{V}} = \mathbf{V}$, and $\tilde{\mathbf{U}}$ and $\boldsymbol{\sigma}$ be the same as $\mathbf{U}$ and $\boldsymbol{\sigma}$ but with the second of column of $\mathbf{U}$ and the second singular value multiplied by -1, respectively. Then $\mathbf{B} = \tilde{\mathbf{U}}\,\mathrm{diag}(\tilde{\boldsymbol{\sigma}})\tilde{\mathbf{V}}^*$. This corresponds to a particular signing of the SVD, as discussed immediately after the proof of Theorem 3.7.

The largest independent pair in $\mathbf{B}$ has size 10, corresponding to $\mathbf{f} = (1, 1, 1, 1, 1, 0, 0)^\top$ and $\mathbf{g} = (1, 1, 0, 0, 0, 0, 0)^\top$. For the decomposition $\mathbf{B} = \tilde{\mathbf{U}}\,\mathrm{diag}(\tilde{\boldsymbol{\sigma}})\tilde{\mathbf{V}}^*$, the coefficients for $\mathbf{f}, \mathbf{g}$ are respectively

$$\begin{aligned}
\mathbf{F} &= (5/\sqrt{7}, \sqrt{10}/\sqrt{7}, 0, 0, 0, 0, 0)^\top \\
\mathbf{G} &= (2/\sqrt{7}, \sqrt{10}/\sqrt{7}, 0, 0, 0, 0, 0)^\top.
\end{aligned}$$

Thus, the admissibility condition of Theorem 3.7 is satisfied for the decomposition $\mathbf{B} = \tilde{\mathbf{U}}\,\mathrm{diag}(\tilde{\boldsymbol{\sigma}})\tilde{\mathbf{V}}^*$. Noting that $\tilde{\boldsymbol{\sigma}}_{\min} = -2$, a short calculation shows the resulting estimate on $(\|\mathbf{f}\|_2^2\|\mathbf{g}\|_2^2)/n$ is tight.

4. Discussion and future research

This article proposes generalizations of classical linear algebraic and spectral graph theoretic results to the case in which the underlying matrix $\mathbf{B}$ is non-Hermitian. This is done by constraining certain vectors to satisfy admissibility conditions. When $\mathbf{B}$ is Hermitian, these admissibility conditions hold and the classical results are recovered. The admissibility condition take slightly different forms, depending on which decomposition is used in place of the spectral decomposition into an orthonormal eigenbasis.

In Theorems 2.1, 3.5, $\mathbf{B}$ is assumed diagonalizable as $\mathbf{B} = \mathbf{U}\,\mathrm{diag}(\boldsymbol{\lambda}(\mathbf{B}))\mathbf{V}^*$ where $\boldsymbol{\lambda}(\mathbf{B})$ may be complex and $\mathbf{U}, \mathbf{V}$ need not be unitary, merely inverses: $\mathbf{U}\mathbf{V}^* = \mathbf{V}^*\mathbf{U} = \mathbf{I}$. The analysis of $\mathbf{f}^*\mathbf{B}\mathbf{g}$ proceeds by assuming $\mathbf{f}$ admits an expansion in terms of the rows of $\mathbf{V}$ and $\mathbf{g}$ an expansion in terms of the rows of $\mathbf{U}$. Of course, when $\mathbf{U} = \mathbf{V}$ these conditions are the same, and when $\mathbf{f} = \mathbf{g}$, this condition always holds. On the other hand, Theorem 3.7 takes advantage of the singular value decomposition $\mathbf{B} = \mathbf{U}\,\mathrm{diag}(\boldsymbol{\sigma}(\mathbf{B}))\mathbf{V}^*$ where $\mathbf{U}$ and $\mathbf{V}$ are unitary but $\mathbf{U} \neq \mathbf{V}$. The analysis of $\mathbf{B}$ in this situation requires a different condition on $\mathbf{f}, \mathbf{g}$, namely that $\mathbf{f}$ has an admissible decomposition with respect to the rows of $\mathbf{U}$, and $\mathbf{g}$ with respect to the rows of $\mathbf{V}$. We remark that in all of these cases, the crucial property is that for d-regular unweighted graphs (or δ-regular weighted graphs), the first eigenvector or singular vector (both left and right) is the vector $\frac{1}{\sqrt{n}}(1, 1, \ldots, 1)^\top \in \mathbb{R}^{n\times 1}$ with corresponding eigenvalue or singular value d. All subsequent analysis is downstream from this observation.

Intuitively, as $\mathbf{B}$ deviates from being Hermitian, the admissibility conditions will still hold for a large class of vectors $\mathbf{f}, \mathbf{g}$. A topic of future research is to develop a rigorous perturbation theory of Hermitian matrices that quantifies how likely the admissibility conditions are to hold in a probabilistic sense. That is, if $\mathbf{B}$ is Hermitian, then the admissibility condition holds automatically for all $\mathbf{f} = \mathbf{g}$. As $\mathbf{f}$ deviates from $\mathbf{g}$ and $\mathbf{B}$ deviates from Hermiticity, it is of interest to determine which vectors (or, what proportion of them in a probabilistic sense) satisfy the admissibility condition.

Declaration of competing interest

None.

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Appendix

Proof of Theorem 3.2

Proof. Note that by d-regularity, $\mathbf{B}$ has an eigenvalue of d; without loss of generality, let $\lambda_0(\mathbf{B}) = d$. Then $\mathbf{B} = \frac{d}{n}\mathbf{1}_{n\times n} + \left(\mathbf{I} - \frac{\mathbf{1}_{n\times n}}{n}\right)\mathbf{B}$. Thus, for all $\mathbf{f} \in \mathbb{C}^{n\times 1}$,

$$\langle \mathbf{f}\,\mathbf{f}^*, \mathbf{B}\rangle = d\left\langle \mathbf{f}\,\mathbf{f}^*, \frac{\mathbf{1}_{n\times n}}{n}\right\rangle + \left\langle \mathbf{f}\,\mathbf{f}^*, \left(\mathbf{I} - \frac{\mathbf{1}_{n\times n}}{n}\right)\mathbf{B}\right\rangle.$$

Let $\mathbf{B} = \mathbf{U}\,\mathrm{diag}\{\lambda(\mathbf{B})\}\mathbf{U}^*$ where $\mathbf{U}\mathbf{U}^* = \mathbf{I}$. We analyze the second summand on the right-hand side as follows:

$$\begin{aligned}
\left\langle \mathbf{f}\,\mathbf{f}^*, \left(\mathbf{I} - \frac{\mathbf{1}_{n\times n}}{n}\right)\mathbf{B}\right\rangle =& \mathbf{f}^*\left(\mathbf{I} - \frac{\mathbf{1}_{n\times n}}{n}\right)\mathbf{B}\mathbf{f} \\
=& \mathbf{f}^*\left(\sum_{0\le \ell<n} \lambda_\ell(\mathbf{B})\mathbf{U}[:,\ell]\mathbf{U}^*[\ell,:] - \lambda_0(\mathbf{B})\mathbf{U}[:,0]\mathbf{U}^*[0,:]\right)\mathbf{f} \\
=& \mathbf{f}^*\left(\sum_{1\le \ell<n} \lambda_\ell(\mathbf{B})\mathbf{U}[:,\ell]\mathbf{U}^*[\ell,:]\right)\mathbf{f} \\
=& \sum_{1\le \ell<n} \lambda_\ell(\mathbf{B})|\mathbf{f}^*\mathbf{U}[:,\ell]|^2 \\
\ge& \min\{\boldsymbol{\lambda}(\mathbf{B})\}\sum_{1\le \ell<n} |\mathbf{f}^*\mathbf{U}[:,\ell]|^2 \\
=& \min\{\boldsymbol{\lambda}(\mathbf{B})\}\sum_{1\le \ell<n} \mathbf{f}^*\mathbf{U}[:,\ell]\mathbf{U}^*[\ell,:]\mathbf{f} \\
=& \min\{\boldsymbol{\lambda}(\mathbf{B})\}\left\langle \mathbf{f}\,\mathbf{f}^*, \left(\mathbf{I} - \frac{\mathbf{1}_{n\times n}}{n}\right)\right\rangle.
\end{aligned}$$

Hence,

$$\begin{aligned}
\langle \mathbf{f}\,\mathbf{f}^*, \mathbf{B}\rangle \ge& d\left\langle \mathbf{f}\,\mathbf{f}^*, \frac{\mathbf{1}_{n\times n}}{n}\right\rangle + \min\{\boldsymbol{\lambda}(\mathbf{B})\}\left\langle \mathbf{f}\,\mathbf{f}^*, \left(\mathbf{I} - \frac{\mathbf{1}_{n\times n}}{n}\right)\right\rangle \\
=& \frac{d}{n}\|\mathbf{f}\|_2^4 + \min\{\boldsymbol{\lambda}(\mathbf{B})\}\left(\|\mathbf{f}\|_2^2 - \frac{1}{n}\|\mathbf{f}\|_2^4\right).
\end{aligned}$$

To every independent set I, there is a corresponding indicator vector $\mathbf{f} = \mathbf{1}_I$ for which by definition $\langle \mathbf{B}, \mathbf{f}\,\mathbf{f}^* \rangle = \mathbf{f}^*\mathbf{B}\mathbf{f} = 0$. For such an indicator vector $\mathbf{f}$, it follows that

$$0 \geq \frac{d}{n}\|\mathbf{f}\|_2^4 - \frac{\min\{\boldsymbol{\lambda}(\mathbf{B})\}}{n}\|\mathbf{f}\|_2^4 + \min\{\boldsymbol{\lambda}(\mathbf{B})\}\|\mathbf{f}\|_2^2.$$

Noting that $|I| = \|\mathbf{f}\|_2^2$, we get

$$0 \geq \frac{d}{n}|I|^2 - \frac{\min\{\boldsymbol{\lambda}(\mathbf{B})\}}{n}|I|^2 + \min\{\boldsymbol{\lambda}(\mathbf{B})\}|I|.$$
$$\implies \frac{|I|}{n} \leq \frac{-\min\{\boldsymbol{\lambda}(\mathbf{B})\}}{d - \min\{\boldsymbol{\lambda}(\mathbf{B})\}},$$

thus completing the proof. □

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