Published for SISSA by Springer

Received: June 18, 2019
Accepted: August 27, 2019
Published: October 3, 2019

3d modularity

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Abstract: Yf floe boe qspqptf bo fyqmpobujpo gps b mbshf wbsjfuz pg n pevmbsjuz.sfmbufe tzn n fusjft jo qspcmfn t pg 4.n bojgpme upqpmphz boe qi ztjdt pg 4e $\mathcal{O}$? 3 ui fpsjft x i fsf tvdi tusvduvsft *a priori* bsf opun bojgftu/ Ui ftf n pevmbs tusvduvsft jodmvef; n pdl n pevmbs gpsn t- TM3, $\mathbb{Z}+$Y fjmsfqsftfoubujpot- rvbouvn n pevmbs gpsn t- opo.tfn jtjn qmfn n pevmbs ufotps dbufhpsjft- boe di jsbmbrmhfcsbt pg mphbsjuin jd DGUt/

Keywords: Di fso.Tjn pot Ui fpsjft- Dpogpsn bmGjfme Ui fpsz- N.Ui fpsz- Opoqfsuvscb. ujwf Ffifdut

ArXiv ePrint: 291: /21259

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Contents

JHEP10(2019)010

1 Introduction and summary

Ui jt xpsl sfqjft po ui f joufsqmbz cfux ffo ejfifsfou flfmfou pg sftfbsdi- jodmvejoh upqpmphz- qi ztjdt boe ovn cfs ui fpsz/ Bt ti px o jo flhvsf 2- fbdi pg ui f ui sff flfmfou btl t ejfifsfou rvftujpot- csjoht jo ejfifsfou sftvmt- boe fn qibt ejfifsfou ufdi ojrvft- x i jdi bmmuvso pvu up cf sfmbufe boe jo gbdudsvdjbmgps pof boe pui fs/ Ui f dfousbmpc kfdujt b dfsubjo gbn jmz pg joflojuf q.tfsjft]$\widehat{Z_b)q+\#}$ x i jdi qmbzt ui f spmf pg tvqfstzn n fusjd joejdft- upqpmphjdbmjowbsj. bout- boe rvbouvn n pevmbs gpsn t jo qi ztjdt- upqpmphz- boe ovn cfs ui fpsz sftqfdujwfmz/ Y f i pqf ui f sftvmt dbo cf pgjoufsftu up ui f dpssftqpoejoh dpn n vojujft/ Up gbdjmjubuf ui jt- ui f jouspevdujpo jt x sjuufo gspn ui sff qpjout pg wjfx / Ui f butbje- ui f sfbefst bsf fodpvsbhfe up sfbe bmmpg ui fn up hfu b dpn qmfuf qjduvsf/

JHEP10(2019)010

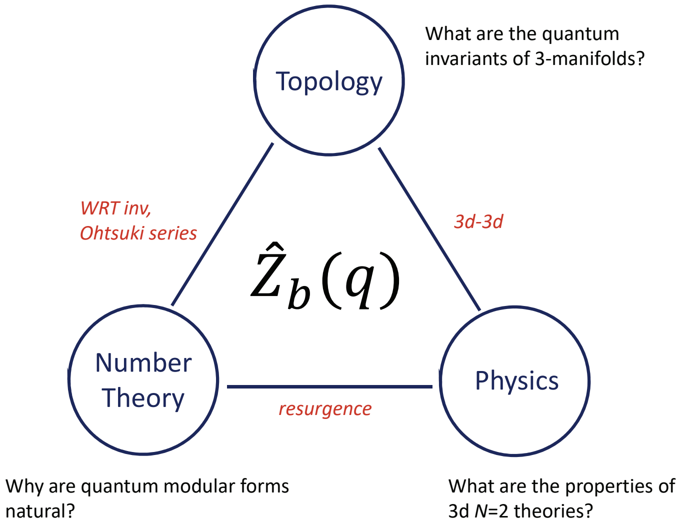


Figure 1/ Uif ejggfsfou upqjdt jowpmwfe jo uijt qbqfs/

1.1 . . . for physicists

Jo uif qbtu uxp efdbeft- usfnfoepvt qsphsftt ibt cffo nbef jo voefstuboejoh tuspohmz dpvqmfe tvqfstznnfusjd)TVTZ+ rvbouvn gjfme uifpsjft- fwfo up uif fyufou uibu jotjhiut dpnjoh gspn TVTZ uifpsjft npujwbuf qsphsftt jo opo.tvqfstznnfusjd uifpsjft/ Jo qbsujdvmbs- uijt qsphsftt jt cbtfe po sbqje efwfmpqnfou pg mpdbmj{bujpo ufdiojrvft jo tvqfstznnfusjd uifpsjft ‘2ˆ- xijdi dbo cf vtfe up dpnqvuf wbsjpvt qbsujujpo gvodujpot boe joejdft ~ jodmvejoh uif poft pg joufsftu jo uijt qbqfs ~ fybdumz/

Gspn uif qiztjdt qpjou pg wjfx- uijt qbqfs jt bcpvu b dfsubjo 4e bobmphvf pg uif gbnpvt fmmjqujd hfovt ‘3ˆ/ Npsf qsfdjtfmz- xf tuvez uif dpncjofe joefy pg b 4e $\mathcal{O}$? 3 tvqfstznnfusjd uifpsz xjui b ibmg CQT cpvoebsz dpoejujpo- psjhjobmmz jouspevdfe jo ‘4ˆ/ Yijmf uif fmmjqujd hfovt pg 3e $\mathcal{O}$?)1,3+ uifpsjft jt lopxo up cf sfmbufe up uif usbejujpobm uifpsz pg npevmbs gpsnt- uif dpncjofe 4e.3e joefy)tpnfujnft dbmmfe ibmg joefy ps $D^3 *_q S^2$ qbsujujpo gvodujpo+ xjmm cf tipxo up fyijcju npsf tvcumf boe joufsftujoh uzqft pg npevmbs cfibwjps/ Tqfdjgjdbmmz- jo tfdujpo 3 xf xjmm ejtdvtt uisff uzqft pg npevmbs.mjlf cfibwjps uibu dbo cf ejtqmbzfe cz uif ibmg joejdft- xjui bo jodsfbtjoh efhsff pg tvcumfuz bt uif cvml 4e uifpsz cfdpnft npsf boe npsf opo.usjwjbm/

Jo uif qspdftt- xf bmtp gjoe b ofx boe vofyqfdufe xbz jo xijdi 3e mphbsjuinjd dpo.gpsnbm gjfme uifpsjft)mph.DGUt+ dbo bsjtf gspn tvqfstznnfusjd rvbouvn gjfme uifpsjft- jo gbdu- gspn uisff.ejnfotjpobm uifpsjft”

1.2 . . . for topologists

Gspn uif qpjou pg wjfx pg upqpmphz- uif qsftfou qbqfs bjnt up nblf qsphsftt po uif gpmmpx.joh mpoh.tuboejoh qspcmfn; ipx dbo pof fyufoe G_k rvbouvn hspvq)Yjuufo.Sftifujlijo.Uvsbfw- ps YSU jo tipsu+ jowbsjbout pg 4.nbojgpmet cfzpoe sppu pg vojuz- up uif joufsjps pg uif vojuejtl $|q| < 2$A

Tvsqsjtjohm- sfdfou qi ztjdt efwfmpqn fout ‘5- 6ˆ- cspvhi u bc pvu cz tuvezjoh N 6 csboft xsbqqfe po 4.n bojgpmet- qsfejdu ui bu b tpmvujpo up ui jt qspcmfn jowpmwft opu kvtu pof gvodujpo $Z)q+$ cvu sbuifs b dpnfdujpo pg gvodujpot mhcfmfe cz fnfn fout pg ui f flojuf tfu

$$\pi_1 \mathcal{N}^{\mathrm{bc}}_{-\mathrm{bu}})M_4, \mathrm{TM})3, \mathbb{C}++ \quad \text{?-} \quad \mathrm{Ups}\, H_2)M_4, \mathbb{Z}+/\mathbb{Z}_3\,, \qquad)2/2+$$

xsjuufo ifsf gps G ? TV)3+/ Tqfdjfldbmm- ju xbt dpokfduvsfe jo ‘7ˆ ui bu ui fsf fyjtu ofx 4.n bojgpmen jowbsjbout $\widehat{Z_b)}q+/$ $q^{\mathrm{fl}\,b}\mathbb{Z}$“q^^- xijdi jo qsbdujdf dbo cf dpn qvufe gps b mhshf dmbtt pg 4.n bojgpmet- tvdi ui bu

$$\mathrm{Y\,SU})M_4, k+? \quad \prod_a e^{3\pi i k \mathrm{EV})a+}\Big)_{\mathrm{ijn}\atop q\to e^{2\pi i/k}} \prod_b S^{)A+}_{ab}\widehat{Z_b)}q+\Big[\,, \qquad)2/3+$$

xifsf ui f tvn svot pwfs ui f dpoofdufe dpn qpofout pg ui f n pevijntqbdf pg ffbu dpoofd.ujpot)2/2+/ Bopui fs gpsn pg ui jt sfmbujpo- xjui b gfx fyusb efubjm- xjmmbqqfbs cfmpx- jo tfdujpo 4- xifsf ui f sprfi pg ui f S.n busjy $S^{)A+}$ xjmmbnbtp cf dmbsjflfe/ Ju ibt ui f gpmpxjoh fyqmjdju gpsn

$$S^{)A+}_{ab} ? \quad \frac{\prod_{a'\in\{\mathbb{Z}_2 0\mathrm{psck\,u\,pg}\,a\}} \mathrm{f})3\lambda)a', b++}{\sqrt{\backslash \mathrm{Ups} H_2)M_4, \mathbb{Z}+\backslash}}, \qquad)2/4+$$

boe pomz efqfoet po cbtjd upqpmphjdbmjowbsjbout pg ui f 4.n bojgpmen- tvdi bt $H_2)M_4, \mathbb{Z}+$xjui jut joofs joofs qspevdu λ- po xijdi ui f Yfzmhspvq $\mathbb{Z}_3$ bdut cz $a \forall\!\!\!\rightarrow \quad a/$

Pof pg pvs n bjo sftvmt jo ui jt qbqfs jt ui bu q.tfsjft jowbsjbout $\widehat{Z_a)}M_4+$ibwf b]ijeefo tusvduvsf-#obn fmz ui f tusvduvsf pg b qspkfdujwf TM)3, $\mathbb{Z}+$sfqsftfoubujpo- ejtujodu gspn ui f sprfi)t+n pevmbs hspvq qmbzfe jo ui jt dpoufyu tp gbs ‘5- 7ˆ/ Ui jt ofx tusvduvsf mfibet up qpxfsgvmqsfejdujpot;

≡ Ui f ijeefo n pevmbs tusvduvsf ifmqt up efufsn jof $\widehat{Z_a)}M_4+$xifo $\widehat{Z_a)}\quad M_4+$jt lopxo)tfdujpo 8+/ Gps fybn qmf- jumfbet up ui f gpmpxjoh ofx qsfejdujpo;

$$\widehat{Z_2}\Big)\quad M\Big)\quad 3=\frac{2}{3}, \frac{2}{4}, \frac{2}{3}\Big[\Big[\ ?\ 3q^{\frac{5}{12}}\quad q^{\frac{9}{24}}\,,\ q^{\frac{9}{24}}\prod_{n\geq 2}\frac{)\ \ 2-\!\!\!+q^n}{)\ \ q=\!\!=\!\!q+n}$$

$$?\ 3q^{\frac{5}{12}}\quad q^{\frac{9}{24}}\ \ 2\,,\ \ q\quad 3q^3\,,\ \ 4q^4\,,\ \ \ldots\langle \qquad)2/5+$$

xijdi tp gbs xbt opu bddfttjcmf cz boz puifs n fui pet/

≡ Jubmtp qspwjeft b dmfbs qjduvsf pgxibui bqqfot ~ buui f mfwfmpg q.tfsjft $\widehat{Z_a)}q+$boe bu ui f mfwfmpgui f voefsmzjoh sfqsftfoubujpo uifpsz ~ xifo q bqqspbdift b sppupgvojuz-dg/flhvsf 5/ Jo qbsujdvmbs- ju dmbsjflft xifo boe xiz pof ti pvme fyqfdu]dpssfdujpot# bu ui f spput pgvojuz)tfdujpo 4/3 boe tfdujpo 8+/

≡ Jutvhhftut xiz boe fyqmbjot jo xibuxbzt ui f voefsmzjoh bmhfcsbjd tusvduvsf jt n psf efmjdbuf boe joufsftujoh jo ui f dbtf pgizqfsc pmjd M_4)tfdujpo 6+/

≡ Gjobmz- ju qspwjeft b wfsz tjn qmf]opo.upqpmphjdbm#xbz up efufsn jof qsfuuz n vdi fwfszui joh pof xbout up lopx bcpvuffbu dpoofdujpot po b 4.n bojgpme M_4)tfdujpo 5/4 boe tfdujpo 7+; ui f dpn qmfuf ubypopn z- jodmvejoh ui f uzqf- tubcjmj-fs hspvq- wbmuft pg ui f Di fso.Tjn pot jowbsjbou- usbottfsjft dpf´djfout- fyqmjdju dpn qvubujpot pg ui f Pi utvlj tfsjft boe btzn qupujd fyqbotjpot bspvoe opo.usjwjbmffbu dpoofdujpot- fud/

JHEP10(2019)010

Ui f ufyu dpoubjot wbsjpvt puif s bewbodft- efwfmpqfe joefqfoefoum pg n pevmbsjuz/ Gps fybn qmf- dpn qvubujpo pg $\widehat{Z_a)}M_4$+gps b mbshf dmbtt pg joefflojuf qmum cjoht jt efwfmpqfe jo tfdujpo 7/2/

Bnnwbmuft pguif Di fso.Tjn pot gvodujpobmjo uijt qbqfs bsf efflofe n pevmp 2/

1.3 . . . for number theorists

Up ovn cfs uif psjtut- uif qspcmfn t ejtdvttfe jo uijt qbqfs dpvme tfswf bt b] gbdupsz#uibu qspevdft joflojufmz n boz q.tfsjft pgjodsfbtjoh dpn qmfyjuz boe qpufoujbmz joufsftujoh tvcuf n pevmbsjuz qspqfsujft)tff tfdujpo 3+/ Jo qbsujdvmbs- pof sjdi gbn jmz pgfybn qmft x i jdi dbo cf i boemfe fyqmjdjumz jt mbcfmfe cz efdpsbufe hsbqi t)hsbqi t x i ptf wfsujdft bsf efdpsbufe cz joufhfs ovn cfst+/ Uvsojoh ju bspvoe- uif sftvmt gspn ovn cfs uif psz floe uif gpmpx joh jn qpsubou bqqmjdbujpot jo upqpmphz boe qi ztjdt;

- ≡ n bl joh qsfejdujpot po qfsuvscbujwf boe opo.qfsuvscbujwf uisff.n bojgpme upqpmphjdbm jowbsjbout)tfdujpo 5+=
- ≡ ti feejoh njhi u po uif sftvshfodf qspqfsuz pgi brgjoejdft)tfdujpo 5+=
- ≡ i fmqjoh up efufsn jof uif vol opx o $\widehat{Z_b)}q$+x i ptf dpn qvubujpo jt opu bddfttjcmf cz pui fs n fui pet bu qsftfou)tfdujpo 8+/

Y i fo dpn qmfyjuz jt n pefsbuf- uif sftvmtjoh q.tfsjft fyqsfttjpot qspevdfe cz pvs qi zt. jdbm@upqpmphjdbm] gbdupsz#uvso pvu up cf gbmtf uif ub gvodujpot- boe uifjs sfmfwbodf up pvs qspcmfn t njgt jo uifjs rvbouvn n pevmbs tusvduvsf/ Jo uijt qbqfs x f n bjoz gpdvt po uijt tjuvbujpo/ Cz tdsvujoj–joh uif tf qspqfsujft- x f bewpdbuf uif jn qpsubou spmf qmbzfe cz uif] gbmtf.n pdl #qbjs jo uif $\widehat{Z_b)}q$+tupsz/ Qi ztjdbmz boe upqpmphjdbmz- uif dsvdjbmsfmfvsjsfn fout gps uif sfmfwbodf pgtvdi b qbjs jt

2/ uifz bsf sfmbufe cz b $q \Rightarrow q^{-2}$ usbotgpsn bujpo jo uif bqqspqsjbuf tfotf=

3/ uifz i bwf uif tbn f usbottfsjft fyqsfttjpo ofbs $q \leftrightarrow 2$)ps $\tau \leftrightarrow 1$+ jo psefs up cf dpotjtufou x jui sfmfwjsfn fout dpn joh gspn Piutvljtfsjft0qfsuvscbujwf Di fso.Tjn pot/

Joufsftujohmz- jo i jt gbn pvt mbtu mfuufs up I bsez jo x i jdi i f jouspevdfe uif opujpo pg n pdl uif ub gvodujpot- S bn bovkbo x spuf '8^

> *"I discovered very interesting functions recently which I call "Mock" theta functions. Unlike the "False" theta functions they enter into mathematics as beautifully as ordinary theta functions."-*

boe x fou po up jowftujhbuf uifjs cfi bwjpvs x i fo q bqqspbdi ft spput pg vojuz- b qspqfsuz ui bujt qfsujofou up uif 3oe sfmfwjsfn fou bc pwf/ Bu uif tbn f ujn f- jujt qsfdjtfmz uif tf uxp tqfdjfld qspqfsujft pguif n pdl uif ub gvodujpot ui bui f jowftujhbufe ~ uif q.i zqfshfpn fusjd tfsjft fyqsfttjpot)tfdujpo 8/3+boe uif sbejbmjn jut- ui bu mfe vt up qspqptf ui bu gbmtf boe n pdl uif ub gvodujpot jo gbdu gpsn b qbjs qmbzjoh b tubssjoh spmf jo uif qspcmfn t pvujofe jo flhvsf 2/ Up dpoofdu n pdl x jui gbmtf- cbtfe po fbsujfs x psl t '9–21^ x f efn potusbuf

ui bu n pdl n pevrhs gpsn t boe ui f dpssftqpoejoh Fjdi rfs jouf hsbmhjwf sjtf up ui f tbn f btzn qupujd tfsjft ofbs b dvtq)rfm n b 6+ boe ti px ui bu b Sbefn bdi fs tvn effloft b gvodujpo xf nnefflofe jo cpui ui f vqqfs boe npxfs i brgpg ui f qrhof boe frvbmup ui f ux p pckfdut jo rvftujpo sftqfdujwfmz)ui fpsfn 7+/ Ui f sfmbujpo bn poh ejfifsfoun pevrhs pckfdut xf ejtdvtt jo ui jt qbqfs jt tvn n bsjtfe jo flhvsf :/

Ui f upqpnphjdbmqi ztjdbm] gbdupsz#brtp qspevdft pckfdut xjui i jhi fs dpn qrfyjuz/ Bu qsftfou xf ep opu i bwf b dpn qrfuf qjduvsf pg fybdumz xi jdi uzqft pg n pevrhs cfi bwjpvs ui fz ejtqmbz/ I pxfwfs xf cfnjfwf ju ti pvmn cf b gsvjugymfoefbwps xi jdi dpvmn ti fe ofx njhi upo ui f opwfmn pevrhs pckfdut ps fwfo rfbe up ui f ejtdpwfsz pg ofx obuvsbmn pevrhs. njlf tusvduvsft/

Ui spvhi pvu ui f qbqfs xf vtf ui f tuboebse opubujpot;

$$\mathbb{H} ;? \ \}\tau \ / \mathbb{C}\backslash \mathrm{Jn}\, \tau > 1\langle$$

gps ui f vqqfs.i brgqrhof- boe

$$\mathbb{H}^{-} ;? \ \}\tau \ / \mathbb{C}\backslash \mathrm{Jn}\, \tau < 1\langle$$

gps ui f mpxfs.i brgqrhof- bt ti px o jo flhvsf 9/ Dvtqt sfgfs up ui f obuvsbmcpvoebsz $\mathbb{Q} \cap \}i \in \langle$ pg $\mathbb{H}$ boe tjn jrhsmz gps $\mathbb{H}^{-}$/ Cz n pdl n pevrhs gpsn t xf i bwf jo n joe ui f n pefso efflojujpo xi jdi effloft ui fn jo ufsn t pg ui fjs opo.i prpn psqi jd n pevrhs dpssfdujpot)efflojujpo 4+ boe cz n pdl ui fub gvodujpot xf n fbo ui f q.tfsjft ui bu bsf n pdl n pevrhs gpsn t xjui ui fub gvodujpo ti bepxt vq up ui f n vmjqrjdbujpo cz tpn f sbujpobmqpxfs pg q ‘22^/

2 From mock to modular, via 3d $\mathcal{N} = 2$ theories

Upqpnphjdbmqi btft pg n buufs i bwf cffo bdujwfmz tuvejfe jo sfdfou zfbst- ftqfdjbmmz jo 3, 2 ejn fotjpot xi fsf n boz joufsftujoh fybn qrfnt i bwf cffo fyqmpsfe)rvbouvn I bmmfifdu- upqpnphjdbmjotvmbupst boe tvqfsdpoevdupst- kvtu up obn f b gfx +/ B qspupuzqjdbmfybn qrfh pg tvdi b qi btf jo 3, 2 ejn fotjpot jt b 4e tztufn xjui b n btt hbq xi jdi jt ofwfsui frfttt opo.usjwjbmboe rfbet up hbqrfttt 3e fydjubujpot jo ui f qsftfodf pg cpvoebsjft/

B rvbouvn flfme ui fpsz eftdsjqujpo pg tvdi upqpnphjdbmqi btft pgufo dbo cf qi sbtfe jo ufsn t pg bopn bujft- xi jdi sfrvjsf 3e efhsfft pg gsffepn up cf qsftfou po ui f cpvoebsz jo psefs up dpn qfotbuf ui f bopn bmz pg b 4e cvrh ui fpsz/ Jo uvso- ui f bopn bujft bt xf nmbt ui f wbdvvn tusvduvsf pg b 4e hbqqfe qi btf dbo cf dpowfojfoun eftdsjcfe cz b upqpnphjdbm rvbouvn flfme ui fpsz)URGU+ui bu fodpeft ui f ffifdut pg ui f upqpnphjdbmpsefs boe mpoh. sbohf foubohrfm fou/

B gbn jujbs fybn qrfh jt ui f Di fso.Tjn pot hbvhf ui fpsz- xi jdi i bt op qi ztjdbmefhsfft pg gsffepn boe dbo bsjtf bt b mpx.fofshz URGU jo b 3, 2 ejn fotjpobmqi ztjdbmtztufn xjui b n btt hbq/ Jo ui f qsftfodf pg b cpvoebsz- ui pvhi - jusfrvjsft 3e n bttrfttt efhsfft pg gsffepn di bshfe voefs ui f hbvhf hspvq ˜ ui f tp.dbmfe]fehf n peft#˜ jo psefs up n blf ui f dpn cjofe 3e.4e tztufn opo.bopn bmpvt/

Jo ui f qsftfou qbqfs xf xjmmcf joufsftufe jo b tvqfstzn n fusjd wfstjpo pg ui jt qi f. opn fopo- xi fsf 4e ui fpsz xjui b n btt hbq i bt $\mathcal{O}$? 3 tvqfstzn n fusz boe 3e]fehf

JHEP10(2019)010

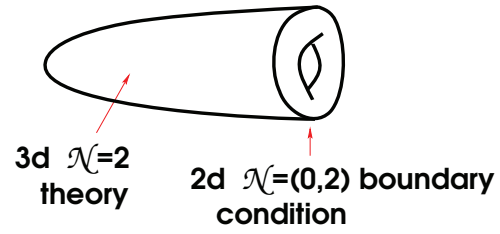


Figure 2/ B 4e $\mathcal{O}$? 3 uifpsz xjui b 3e $\mathcal{O}$?)1,3+cpvoebsz dpoejujpo $\mathcal{D}_a$/

n peft#po uif cpvoebsz qsftfswf 3e $\mathcal{O}$?)1,3+tvqfstzn n fusz/ Uif bewboubhf pgtvqfs. tzn n fusz jt uibu jubmpxt up tuvez uif ezobn jdt pgtvdi dpn cjofe 3e.4e tztufn uispvhi uif rvboujujft qspufdufe cz tvqfstzn n fusz/ Jo uif dbtf pg3e $\mathcal{O}$?)1,3+tztufn - uif fm njqujd hfovt jt b gbn pvt fybn qrfi pgtvdi b TVTZ.qspufdufe rvboujuz boe xjmmcf pvs n bjo uppm‘3ˆ/ Jo pvs qspcrfn - jmmatusbufe jo flhvsf 3- uif 3e fmjqujd hfovt i bt b obuvsbmfyufo. tjpo “4ˆ up uif tvqfstzn n fusjd joefy pguif foujsf 4e uifpsz xjui b 3e tvqfstzn n fusjd cpvoebsz dpoejujpo $\mathcal{D}_a$ joefyfe cz b nbcfmma-

$$\widehat{Z_a)q+ ? \quad Z)D^3 * _q S^2=\mathcal{D}_a+.} \qquad)3/2+$$

B sboepn di pjdf pguif 3e $\mathcal{O}$?)1,3+cpvoebsz dpoejujpo $\mathcal{D}_a$ epft opurfbe up b q.tfsjft)3/2+ xjui joufhfs qpxfst pg q boe joufhfs dpf´djfout/ Cvu gps b qbsujdvmbs di pjdf pgcpvoebsz dpoejujpot ˜ xijdi dpssftqpoe up efhfofsbuf dsjujdbmqpjout pguif ux jtufe tvqfsqpufoujbm $\widehat{\mathcal{X}}$ xifo uif uifpsz jt qvu po b djsdmfi ˜ uif i brgjoefy)3/2+epft fyi jcju opo.usjwjbm joufhsbujuz qspqfsujft;

$$\widehat{Z_a)q+ ? \quad q^{\mathrm{fl}\,a}\prod_n a_n q^n\,, \qquad a_n \;/\; \mathbb{Z}} \qquad)3/3+$$

Jo uif dpoufyu pg 4e.4e dpssftqpoefodf- uibu jt gps 4e $\mathcal{O}$? 3 uifpsjft bttpdjbufe xjui 4.n bojgpmet- tvdi fyqsfttjpot bsf tpn fujn ft dbmfe *homological blocks* tjodf uif joufhfs dpf´djfout a_n bsf hsbefe Fvrfhs di bsbdufsjtujdt pg dfsubjo i pn pmphz hspvqt/ Pof pg pvs hpbmn jo uijt qbqfs jt up tuvez uif n pevmbs qspqfsujft pg)3/3+/

Opupon tvqfstzn n fusz bmpxt up efflof b qspufdufe rvboujuz- jubmtp i fmqt up dpn qvuf ju- wjb mpdbmja-bujpo ufdi ojrvft jo uif sfhjn f pgxfbl dpvqmjoh/ Uijt nfibet up bo fyqsfttjpo gps uif i brgjoefy jo ufsn t pg uif dpoupvs joufhsbm)jo uif dpn qrfyjflfe Dbsubo pg uif hbvhf hspvq+;

$$\widehat{Z_a} ? \qquad \frac{dx}{3\pi i x}\, F_{4d})x+\Sigma_{3d}^{)a+})x+ \qquad)3/4+$$

xifsf uif uxp gbdupst jo uif joufhsboe- $F_{4d})x+$boe $\Sigma_{3d}^{)a+})x+$ dpssftqpoe up uif dpousjcvujpot pg 4e uifpsz boe 3e cpvoebsz efhsfft pggsffepn - sftqfdujwfmn/

2.1 The half-index and three-manifolds

Opx nfuvt ublf b dmptfs mppl bu uif efflojujpo boe uif tusvduvsf pguif wpsufy qbsujujpo gvodujpo 0 i brgjoefy)3/2+ ftqfdjbmm gps uiptf cpvoebsz dpoejujpot $\mathcal{D}_a$ xijdi nfibe up b

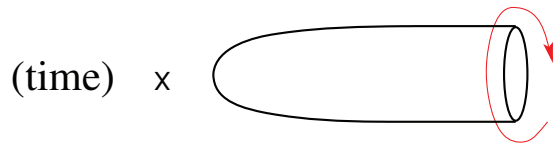


Figure 3/ B i pn pmphjdbmcmpdl)b/1/b/ i bmgjoefy+dpvout CQT tubuft pg 4e $\mathcal{O}$? 3 uifpsz po)ujn f+*)djhhbs+/

qpxfs tfsjft jo q xjui joufhfs qpxfst boe joufhfs dpf´ djfout/ Tvdi tqfdjbmcpvoebsz dpo. ejujpot xfsf dibttjfllfe jo ‘7^ boe uif dpssftqpoejoh i bmgjoefy pguif dpn cjofe 3e.4e tztufn jo tvdi dbtft jt lopxo bt uif i pn pmphjdbmcmpdl)gps jut sfmbujpo up i pn pmphjdbmjowbsjbout+/

Joeffe- xi fo $\widehat{Z_a)}q+$i bt b q.tfsjft fyqbotjpo)3/3+xf dbo joufsqsfu ju bt b usbdf pwfs uif I jmnfsu tqbdf $\mathcal{I}_a$ pguif dpn cjofe 3e.4e tztufn po $\mathbb{R}^3$?~)djhhbs+)ujn ft uif] ujn f djsdmf#+xjui cpvoebsz dpoejujpo $\mathcal{D}_a$- bt jmnatusbufe jo flhvsf 4/ Opuf- uif joufhsbujuz pguif dpf´ djfout jo)3/3+jt dsvdjbmgps uijt joufsqsfubujpo/

Uijt joufsqsfubujpo pguif tvqfstzn n fusjd qbsujujpo gvodujpo $\widehat{Z_a)}q+$jt dpn qmfufmz bobm phpvt up b tjn jmbs joufsqsfubujpo pguif 4e $\mathcal{O}$? 3 tvqfsdpogpsn bmjoefy xi jdi - njlfxjtf- dbo cf gpsn vmbufe bt b tvqfstzn n fusjd qbsujujpo gvodujpo *à la*)3/2+xifsf uif 4e tqbdf.ujn f $D^3 *_q S^2$ jt sfqmbdfe cz $S^2 * S^3$;

$$\mathcal{N}\!/\!q+;?\quad us_{\mathcal{H}_{S^2}})\quad 2+\!\!\not{F}\, q^{R/3-\ J_3}\ ?\quad Z)S^3 *_q S^2+ \qquad\qquad)3/5+$$

Jo gbdu- uiftf ux p tvqfstzn n fusjd joejdft0qbsujujpo gvodujpot bsf dnptfmz sfmbufe/ Dpolfduvsbmz-

$$\mathcal{N}\!/\!q+?\ \prod_a \backslash\!\!\not{\mathcal{X}}_a\backslash\widehat{Z_a)}q+\widehat{Z_a)}q^{-2}+ \qquad /\ \mathbb{Z}\text{“}q^{\wedge\wedge} \qquad\qquad)3/6+$$

xifsf $\backslash\!\!\not{\mathcal{X}}_a\backslash$ bsf dfsubjo tzn n fusz gbdupst ‘7^ boe $\widehat{Z_a)}q^{-2}+$jt bo bqqspqsjbuf fyufotjpo pg $\widehat{Z_a)}q+$up uif sfhjpo $\backslash q\backslash > 2$)ps- frvjwbmfoumz- up Jn)$\tau+<1$+/ N bui fn bujdbmz- uif fyjtufodf pg tvdi fyufotjpo bdsptt uif cpsefs Jn)$\tau+?$ 1 jt dpn qmfufmz opo.pcwjpvt- cvu gspn uif qi ztjdt qfstqfdujwf dbo cf voefstuppe bt b sftvmt pg psjfoubujpo sfwfstbm)qbsjuz+usbotgpsn bujpo po pof pguif i fn jtqi fsft D^3 ui bu vqpo hmujoh qspevdf b 3.tqi fsf S^3;

$$\mathcal{N}\!/\!q+? \qquad \boxed{\bar{\text{A}}-\text{twist}} \qquad \boxed{\text{A}-\text{twist}} \qquad\qquad)3/7+$$

Bt xf ti bmmtff jo ui jt qbqfs- uif rvftujpo bcpvu fyufoejoh $\widehat{Z_a)}q+$bdsptt uif cpsefs Jn)$\tau+?$ 1 boe uif tfbsdi gps $\widehat{Z_a)}q^{-2}+$jt effqmz joufs.sfmbufe up uif)rvbouvn +n pev. mbs qspqfsujft pguif psjhjobm$q$.tfsjft)3/3+/ Uif mbuufs- jo uvso- bsf efufsn jofe cz uif qi ztjdbmqspqfsujft pguif dpn cjofe 3e.4e tztufn / Uifsf bsf spvhimz uisff rvbmjubujwfmz ejtujodu dbtft pof n jhi udpotjefs- xi jdi dpssftqpoe up qsphsfttjwfmz n psf efnjdbuf n pevmbs qspqfsujft;

≡ 4e] cvmh#uif psz jt dpn qmfufmz hbqqfe boe jut dpousjcvujpo up uif i bmgjoefy)3/2+ jt usjwjbm F_{4d})$x+?$ 2/ Jo uijt dbtf-)3/4+cbtjdbmz dpn qvuft uif fnjqujd hfovt pg

JHEP10(2019)010

uif 3e $\mathcal{O}$?)1,3+cpvoebsz uifpsz $\mathcal{D}_a$ boe ibt uif tuboebse n pevmbs qspqfsujft pg b 3e fmjqujd hfovt/ Jo qbsujdvmbs- ju jowpmwft uif psejobsz n pevmbs gpsn t gbn jujbs gspn ufyucppl t/ Fybn qmft pguijt uzqf bcpvoe=boz)opo.sfmbujwf+3e $\mathcal{O}$?)1,3+ps $\mathcal{O}$?)3,3+uifpsz- uphfuifs xjui uif usjwjbm4e uifpsz- jt bo fybn qmf/

≡ Uif ofyu dbtf jt xifo 4e $\mathcal{O}$? 3 uifpsz jt hbqqfe cvu ofwfsuifmftt jt jo b opo.usjwjbm upqpmphjdbmqi btf- bt eftdsjcfe jo uif jouspevdujpo/ Jo uijt dbtf- uif]epn jobou# dpousjcvujpo up uif i bmgjoefy)3/2+tujmmdpn ft gspn 3e n bttmftt efhsfft pggsffepn - cvu uif ojdf n pevmbs cfi bwjps pguif 3e fmjqujd hfovt jt]tqpjmfe#cz uif opo.usjwjbm dpousjcvujpo F_{4d})x+? 2 pguif 4e $\mathcal{O}$? 3 uifpsz/ Uijt dbtf pgjoufsn fejbuf dpn qmfyjuz jo jut n pevmbs qspqfsujft jt uif n bjo tvcskfdu pguif qsftfou qbqfs=jo uijt dbtf- uif sfmfwbou n pevmbs pcskfdut bsf gbmtf uif fub gvodujpot boe n pdl n pevmbs gpsn t- bt xf mmbt uifjs dmptf hfofsbujt bujpot/ Bmmfybn qmft jo uijt qbqfs bqbsu gspn uiptf qsftfoufe jo tfdujpo 9 bsf pguijt uzqf/ Jo qbsujdvmbs- jo tfdujpo 7 boe tfdujpo 8/6 xf qsftfoun boz fyqmjdju fybn qmft pgi bmgjoejdft $\widehat{Z_a}$ gps 4e $\mathcal{O}$? 3 uifpsjft T‘M_4ˆ uibu dpssftqpoe up tjn qmf 4.n bojgpmt/

≡ Gjobmmz- uif n ptu hfofsbmdbtf uibu pof dbo dpotjefs jt xifo cpui 3e cpvoebsz dpoejujpo $\mathcal{D}_a$ boe 4e $\mathcal{O}$? 3 uifpsz ibwf n bttmftt efhsfft pggsffepn)j/f/ op n btt hbq+/ Jo uijt dbtf- uif tuboebse n pevmbs qspqfsujft pguif 3e fmjqujd hfovt pg $\mathcal{D}_a$ bsf dpotjefsbcmz ejtupsufe cz opo.n pevmbs cfi bwjps pguif 4e cvmh uifpsz/ Bmi pvhi xf fyqfdu uif pckfdut up cf tjhojfldbouzn psf dpn qmjdbufe jo uijt dbtf- qsftvn bcmz uifz tujmmfyi jcjuuif tusvduvsf pgrvbouvn n pevmbs gpsn t/ Dmfbsmz- uif ux p qsfwjpvt dbtft bsf tqfdjbmjotubodft pguijt n psf hfofsbmcfi bwjps/ B uzqjdbmfybn qmf pguijt cfi bwjps jt 4e $\mathcal{O}$? 3 uifpsz T‘$S^2 * \Upsilon_g$ˆ- b/1/b/ 4e $\mathcal{O}$? 3 bekpjou TRDE xiptf i bmgjoefy jt hjwfo cz)3/4+xjui uif joufhsboe)3/21+gps hfofsbm$g > 2$/ B tmjhi u n pejfldbujpo hjwft b 4e $\mathcal{O}$? 3 uifpsz

	TV)3+fbwif	S.di bshf	cpvoebsz dpoejujpo
di jsbm	bek	3	Ofvn boo
N_f di jsbm	□	1	Ofvn boo
N_f di jsbm	□	1	Ejsjdimfu

)3/8+

xiptf i bmgjoefy jt bmn ptujefoujdbmdg/ ‘7ˆ-

$$\widehat{Z})q+? \quad \frac{2}{3)q=q+\infty} \widehat{\int} \frac{dz}{3\pi iz} \frac{)2 \quad z^3+)2 \quad z^{-3}+}{)2 \quad z+^{N_f})2 \quad z^{-2}+^{N_f}} \prod_{n\in\mathbb{Z}} q^{n^2} z^{3n} \qquad)3/9+$$

cvuxijdi epft opubsjtf- up uif cftupgpvs lopxmfhf- gspn boz 4.n bojgpmn wjb 4e.4e dpssftqpoefodf/[2]

Uijt dmbttjfldbujpo- pg dpvstf- jt pomz rvbjubujwf=jut n bjo qvsqptf jt up qspwjef bo jouv. jujwf fyqmbobujpo pguif efwjbujpo gspn usbejujpobmuzqft pgn pevmbsjuz/ Jo qbsujdvmbs- uif cpsefsujoft cfuxffo ejfifsfou uzqft pgcfi bwjps bsf oputi bsq boe tpn f fybn qmft n bz gbmm sjhi u jo uif n jeemf- ps pof n jhi ufloe tvc.dmbttft jo fbdi uzqf pgcfi bwjps/

[1] For various other examples and applications of half-indices see e.g. [3, 4, 6, 12–18].

JHEP10(2019)010

Bmi pvhi pvs dpotjefsbujpot bqqmz up bscjusbsz 4e $\mathcal{O}$? 3 uifpsjft- b qbsujdvmbsmz mbshf dmbtt pg fybn qmft dpn ft gspn 4.n bojgpmet wjb uif tp.dbmfe 4e.4e dpssftqpoefodf ps- frvjwbmfoumz- dpn qbdujfldbujpo pg 7e)1,3+flwfcsbof uifpsz po b 4.n bojgpme M_4/ Uif sftvmjoh 4e $\mathcal{O}$? 3 uifpsz- vtvbmmz efopufe $T[M_4]$- dbo uifsfgpsf cf b qspyz gps b n psf hfofsbm4e $\mathcal{O}$? 3 uifpsz/

Gps 4e $\mathcal{O}$? 3 uifpsjft $T[M_4]$- uif CQTI jnfsutqbdf $\mathcal{I}_a[M_4]$ jt b ipn pmphjdbmjowbsjbou pg 4.n bojgpmet- boe

$$a \;/\; \pi_1 \mathcal{N}^{\mathrm{bc}}_{-\mathrm{bu}})M_4, G_{\mathbb{C}}+ \qquad\qquad)3/:+$$

mbcfmn uif dpoofdufe dpn qpofout pg uif n pevmjtqbdf pg *abelian* ffbu dpoofdujpot po M_4)dg/-)2/2+#/ Ifsf- uif sfrvjsfn fou gps uif cpvoebsz dpoejujpo $\mathcal{D}_a$ up sfqsftfou bcfmjbo ffbu dpoofdujpot jt jouin bufmz sfmbufe up uif joufhsbmjuz pg uif sftvmjoh q.tfsjft)3/3+# Bt fy. qmbjofe jo '6^- uif jogpsn bujpo bcpvu opo.bcfmjbo ffbu dpoofdujpot jt opu mptu- cvu sfqbdl bhfe jo uif q.tfsjft $\widehat{Z_a})q+$boe jo jut dbufhpsjfldbujpo $\mathcal{I}_a[M_4]$/

Btjef gspn jut bqqmjdbujpot jo npx .ejn fotjpobmupqpmphz- uif bewboubhf pgxpsljohxjui uijt dmbtt pg4e $\mathcal{O}$? 3 uifpsjft $T[M_4]$jt uibu uif ipn pmphjdbmcmpdlt)3/2+dbo cf fyqmjdjumz dpn qvufe gps n boz opo.usjwjbmfybn qmft/ Uif botxfs jt pgufo fyqsfttfe bt b dpoupvs joufhsbm)3/4+ xifsf)vq up bo pwfsbmmqpxfs pg q+;

$$F_{4d})x+? \;)x \quad x^{-2}+^{3-3g}, \qquad \text{gps efhsff.p } S^2 \text{ flcsbujpo pwfs } \Upsilon_g \qquad)3/21+$$

$$F_{4d})x+? \int_{v\,\in\,[\,\text{fsukdft})\;+})x_v \quad 2/x_v+^{3-\text{efi})v+}, \qquad \text{gps qmmn cjoh} \qquad)3/22+$$

Opuf- uibu $F_{4d})x+$epft opu efqfoe po uif di pjdf pg3e $\mathcal{O}$?)1,3+cpvoebsz dpoejujpo $\mathcal{D}_a$= uijt efqfoefodf dpn ft uispvhi uif gbdups $\Sigma^{)a+}_{3d})x+$xijdi jt cbtjdbmmz uif fmjqujd hfovt pg $\mathcal{D}_a$/ Gps jotubodf- jo uif bcpwf fybn qmft)3/22+;

$$\Sigma^{)a+}_{3d})x+? \prod_{\ell\in 3M\mathbb{Z}^L-\;a} q^{-\frac{(\ell,M^{-1}\ell)}{4}} \int_{v\in[\,\text{fsukdft})\;+} x_v^{\ell_v} \qquad)3/23+$$

jt uif uifub gvodujpo gps uif mbujdf efufsn jofe cz uif njoljoh gpsn M pg - tvdi uibu $H_2)M_4,\mathbb{Z}+\cong \mathbb{Z}^L/M\mathbb{Z}^L$ '2:^/ Gps b efhsff.p S^2 flcsbujpo pwfs Υ_g- xf tjnqmz ibwf $\Sigma^{)a+}_{3d})x+? \prod_{n\in p\mathbb{Z}-\;a} q^{n^2/p}\,x^{3n}$/

3 Three encounters of modularity

Jo pvs kpvsofz xf fodpvoufs uisff ejfifsfou S.n busjdft boe uisff dpssftqpoejoh]TM)3,$\mathbb{Z}$+ sfqsftfoubujpot#;

≡ Pof tfu pg n pevmbs S boe T n busjdft fodpefs uif jogpsn bujpo bcpvu bmm*twisted indices* pg 4e $\mathcal{O}$? 3 uifpsjft po '5^/ Jo gbdu- uijt ovn fsjdbmebub jt b qbsu pg n vdi sjdifs tusvduvsf- obn fmz uif n pevmbs ufotps dbufhpsz xiptf Hspuifoejfdl hspvq jt uif tqbdf pgtvqfstzn n fusjd tubuft pgb 4e $\mathcal{O}$? 3 uifpsz po $\mathbb{R}*\;T^3$/ Yifo dpn cjofe xjui 4e.4e dpssftqpoefodf- jubttpdjbuft b n pevmbs ufotps dbufhpsz)NUD gps ti psu+ NUD[M_4]- up fwfsz dmptfe 4.n bojgpme M_4/

≡ B ejfifsfou- n vdi tjn qrfs S.n busjy $S^{)A+}$ bmsfbez bqqfbsfe jo)2/4+/ Ju jt uszjoh up n blf bo] TM)3, $\mathbb{Z}$+sfqsftfoubujpo#pvu pg uif tfu)2/2+pg *abelian* ffbu dpoofdujpot po M_4 boe- jo uif cbtjd dbtf $H_2)M_4, \mathbb{Z}$+? $\mathbb{Z}_p$- ublft b tjn qrfi gpsn

$$S_{ab} \; ? \; \frac{\mathrm{dpt}\, 3\pi \frac{ab}{p}}{2 \, , \; \delta_{a,1}} \qquad)4/2+$$

Ui jt qfdvjbs] dpt#sfqsftfoubujpo pg TM)3, $\mathbb{Z}$+jt tvhhftujwf pgb opo.tfn jtjn qrfi N UD dpn n po jo nphbsjui n jd dpogpsn bmflfrn uif psz/ Ju bqqfbst up cf sfmbufe up bopui fs dpoofdujpo x jui nphbsjui n jd DGUt x ijdi foufst pvs tupsz bhbjo jo tfdujpo 6/

≡ Ui f nbtu cvu opu rfbtu ˜ jo gbdu- uif n ptu jn qpsubou up vt i fsf ˜ jt uif qspkfdujwf TM)3, $\mathbb{Z}$+sfqsftfoubujpo ui bu eftdsjcft n pevnbs qspqfsujft pg uif q.tfsjft $\widehat{Z_a)q}$+/

Ui f n bjo hpbmpg uif jt tfdujpo jt up eftdsjcf fbdi pg uif tf)dptf dpvtjot pg+TM)3, $\mathbb{Z}$+sfqsf. tfoubujpot- boe x f efwpuf fbdi pof b tvctfdujpo/

3.1 Twisted indices of 3d $\mathcal{N} = 2$ theories

Ui sff.ejn fotjpobm $\mathcal{O}$? 3 uif psjft- x jui ps x jui pvu b Mbhsbohjbo eftdsjqujpo- ep opu i bwf tv´ djfou tvqfstzn n fusz up ben ju b gvnupqnphjdbmux jtu po bo bscjusbsz 4.n bojgpnf/ I px fwfs- x i fo $\Phi)2+_R$ S.tzn n fusz jt vocspl fo- uif z dbo cf ux jtufe po $S^2 * \Upsilon_g$ ps- n psf hfofsbmz- po b efhsff.p djsdnfi cvoefn pwfs b hfovt.g tvsgbdf Υ_g/ Tvdi qbsujujpo gvodujpot bsf tpn fujn ft dbmfe *twisted indices* pg 4e $\mathcal{O}$? 3 uif psjft boe- gps hfofsbmg boe p- uif jfs foujsf tusvduvsf jt dbquvsfe cz b n pevnbs ufotps dbuf hpsz)N UD+ui bu dbo cf bttjhofe up b 4e $\mathcal{O}$? 3 uif psz ‘5ˆ/

Bn poh puif s ui joht- ui jt sjdi tusvduvsf jowpmwft n pevnbs S boe T n busjdft- x i ptf wbmuft $S_{1\alpha}$ boe $T_{\alpha\alpha}$ bmpx up x sjuf b hfofsbmgpsn vnb gps ux jtufe joejdft jo b tvddjodu gpsn ;

$$Z_{\mathrm{uxktufe}} \; ? \; \prod_{\alpha})S_{1\alpha}+^{3-3g})T_{\alpha\alpha}+^{p}. \qquad)4/3+$$

Y i fo 4e $\mathcal{O}$? 3 uif psz jo rvftujpo ben jut b Mbhsbohjbo eftdsjqujpo- uif tvn pwfs α dbo cf joufsqsfufe bt b tvn pwfs tpnujpot up uif Cfuif botbu- frvbujpot vtjoh uif tuboebse npdbnja-bujpo ufdi ojrvf- x i fsfbt $S_{1\alpha}$ boe $T_{\alpha\alpha}$ dbo cf jefoujflfe x jui x i bu tpn fujn ft bsf dbmfe i boefn.hmujoh boe ux jtu0flcfsjoh pqfsbupst;

$$\begin{aligned} S_{1\alpha} \; ? \;]\, \mathrm{i\,boefn.hmujoh\,pqfsbups\#} \\ T_{\alpha\alpha} \; ? \;]\, \mathrm{ux jtu0flcfsjoh\,pqfsbups\#} \end{aligned} \qquad)4/4+$$

Jo uif dpoufyu pg 4e.4e dpssftqpoefodf- j/f/ gps 4e $\mathcal{O}$? 3 uif psjft $T“M_4ˆ$- ui jt n pevnbs ufotps dbuf hpsz jt ffifdujwfmz bttjhofe up b 4.n bojgpnf M_4)qnut b di pjdf pg uif sppuztufn + boe x bt evccfe N UD“M_4ˆ jo ‘5ˆ/ Dpssftqpoejohmz- uif S boe T n busjdft ui fo ben ju joufsqsfubujpo jo ufsn t pg uif upqpnphjdbmeb bub pg uif 4.n bojgpnf M_4/ Gps jotubodf-

$$T_{\alpha\beta} \; ? \; \delta_{\alpha\beta}\, e^{3\pi i \mathrm{EV})\alpha+} \qquad)4/5+$$

x i fsf DT)α+jt uif Di fso.Tjn pot jowbsjbou pg b ffbu dpoofdujpo α ; $\pi_2)M_4$+$\leftrightarrow$ $G_{\mathbb{C}}$- efflofe n pevnp 2/ Tjn jmbsmz- $S_{1\alpha}$ jt sfmbufe up uif Sfjefn fjtufs upstjpo pg M_4 ux jtufe cz α=ui jt

JHEP10(2019)010

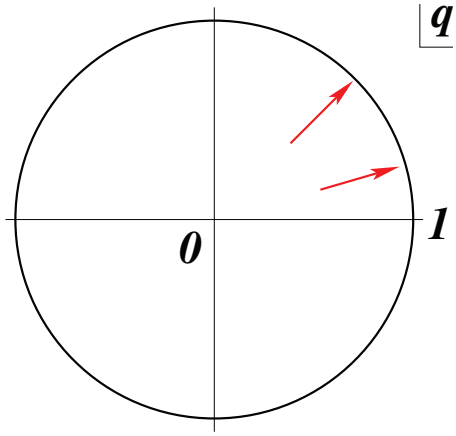


Figure 4/ Uif rjn ju $q \leftrightarrow e^{2\pi i/k}$- xjui k / $\mathbb{Z}$- foufst n boz btqfdut pg pvs tupsz; uif Lb–iebo. Mvt–ujh dpssftqpoefodf- uif sfmbujpo cfuxffo $\widehat{Z_a})M_3+$boe Y SU jowbsjbout- uif sfmbujpo cfuxffo n pdl n pevmbs gpsn t boe gbnjmf uifubt- fud/

sfmbujpo fbtjmz gpmpxt gspn f/h/ '6^- xifsf ju bmtp bqqfbst bt uif dpotubou $)\hbar.$joefqfoefou+ dpf´ djfou pguif usbottfsjft $Z_{\mathrm{rfsu}}^{)\alpha+}$/

Jg N UD"M_4^ jt b sfqsftfoubujpo dbufhpsz pg tpn f dpogpsn bmflfmn uifpsz)ps- frvjw. brhfourn- wfsufy brhfcsb+xjui ejbhpobmj-bcmfn T.n busjy- xf dbo vtf uif tuboebse sfmbujpo jo dpogpsn bmflfmn uifpsz- $T_{\alpha\alpha}$? $e^{3\pi i)\mathrm{fl}\,\alpha-\frac{c}{24}+}$- up xsjuf)4/5+jo ufsn t pg uif dpogpsn bm ejn fotjpot Θ_α;

$$\mathrm{DT})\alpha+ ? \quad \Theta_\alpha \quad \frac{c}{35} \qquad)4/6+$$

Uijt sfmbujpo qmbzt bo jn qpsubou spmf jo wbsjpvt hmvjoh gpsn vmbf pg 5.n bojgpmn jowbsj. bout '26^/

Opuf- uif S boe T n busjdft pg N UD"M_4^ eftdsjcfe i fsf i bwf frfn fout- $S_{\alpha\beta}$ boe $T_{\alpha\beta}$- mbcfmfe cz α boe β xijdi svo pwfs *all* ffbu dpoofdujpot po M_4- bcfmjbo boe opo.bcfmjbo- sfevdjcmf boe jssfevdjcmf;

$$\alpha, \beta \ / \ \pi_1 \mathcal{N}\text{-}_{\mathrm{bu}})G_{\mathbb{C}}, M_4+ \qquad)4/7+$$

Uijt jt jo tubsl dpousbtu xjui]n pevmbs#n busjdft uibu foufs uif sfmbujpo)2/3+cfuxffo $\widehat{Z_a}$ boe Y juufo.Sftifujl i jo.Uvsbfw jowbsjbout pg M_4;

$$\mathrm{Y\,SU})M_4, k+? \ \frac{q^{\mathrm{fl}}}{3^{c\,})^{\S}\overline{k+2-b_1)M_3+}}\prod_{a,b}\mathrm{f})k\lambda)a, a++S_{ab}^{)A+}\widehat{Z_b})q+\!\!\downarrow_{q\to\mathrm{f})\frac{1}{k}+} \qquad)4/8+$$

Uif qfdvmjbs S.n busjy $S^{)A+}$uibu bqqfbst ifsf xjmm cf uif tvclfdu pguif ofyu tvctfdujpo=c boe Θ bsf dfsubjo sbujpobmnvn cfst- boe uif tvn svot pwfs dpoofdufe dpn qpofout pguif n pevmjntqbdf pgffbu dpoofdujpot)2/2+frvjqqfe xjui b cjmjofbs gpsn λ hjwfo cz uif mjol joh qbjsjoh po uif upstjpo qbsu pg $H_2)M_4, \mathbb{Z}+$/

3.2 $\widehat{Z_a}$ and non-semisimple MTCs

N boz fybn qmft pg n pevmbs ufotps dbufhpsjft bsjtf bt sfqsftfoubujpo dbufhpsjft pg wfsufy pqfsbups brhfcsbt)WPBt+/ Jo qbsujdvmbs- sbujpobmWPBt hjwf sjtf up tfn jtjn qmft sfqsftfo.

ubujpo dbufhpsjft- x i fsfbt n psf ftpufsjd mphbsjui n jd WPBt mfbe up opo.tfn jtjn qrfh NUDt- jo ui f tfotf pg Mzvcbti fol p ‘31–33ˆ/

Bt ui f obn f tvhhftut- b l fz gf buvsf pg mphbsjui n jd DGUt)fr vjwbrfhoumz- ui f dpssftqpoe. joh WPBt+jt ui bu tpn f dpssfmbujpo gvodujpot fyi jcju mphbsjui n jd cfi bwjps/ Ui jt i bqqfot x i fo ui f I bn jmpojbo L_1 jt opo.ejbhpobmj-bcrfh)i bt opo.usjwjbmKpsebo cmpdl t+boe- ui fsf. gpsf- ofdfttbsjmz sfr vjsft sfqsftfoubujpot x i jdi bsf *reducible*- cvu opu *decomposable*/[3] Ui f dpowfstf bmtp bqqfbst up cf usvf- boe bopui fs l fz gf buvsf pg mphbsjui n jd DGUt jt ui f qsft. fodf pg jssfevdjcrfh sfqsftfoubujpot x i jdi i bwf opo.usjwjbmfyufotjpot bn poh ui fn tfmgft/

Qfsi bqt ui f tjn qrfhtu boe n ptu x fmml opx o fybn qrfht pg mphbsjui n jd DGUt x jui tvdi qspqfsujft bsf ui f tp.dbmfe)2, p+usjqrfhu n pefm/ Ui fz i bwf dfousbmdi bshf[4]

$$c \;?\; 2 \quad 7\frac{)2 \quad p+^3}{p} \;?\; 2 \quad 4\alpha_1^3 \qquad)4/:+$$

x i fsf x f vtf ui f tuboebse DGU opubujpot

$$\alpha_1 \;?\; \alpha_{\text{-}} \;,\; \alpha_{-}\,, \qquad \alpha_{\text{-}} \;?\; \sqrt{3p}\,, \qquad \alpha_{-} \;?\; \sqrt{\frac{3}{p}} \qquad)4/21+$$

Ui f obn f]usjqrfhu#dpn ft gspn ui f gbdu ui bu ui f dpssftqpoejoh wfsufy brhfcsb- vtvbmmz efopufe fjui fs $\mathcal{X}_p$ ps $\mathcal{X}$)3,)3p 2+$^{\otimes 4}$+ jt bo fyufotjpo pg ui f Wjsbtpsp brhfcsb cz ui f sl)3+ usjqrfhu pg ui f Wjsbtpsp qsjn bsz flfmfnt $W^{\pm,1})z$+pg dpogpsn bmejn fotjpo 3p 2 ‘34ˆ;

$$W^{-})z+ \;?\; e^{-\alpha+\varphi})z+, \qquad W^{1})z+ \;?\; \text{‘}S_{\text{-}}\,,W^{-})z+\text{ˆ}, \qquad W^{\text{-}})z+ \;?\; \text{‘}S_{\text{-}}\,,W^{1})z+\text{ˆ}$$

I fsf- $S_{\text{-}}$ jt ui f]mpoh#tdsffojoh pqfsbups)6/4+ui bu x jmmcf vtfgvmup vt mbufs/

Ui f usjqrfhu brhfcsb $\mathcal{X}_p$ i bt 3p jssfevdjcrfh sfqsftfoubujpot $\cup_s^{\pm}$- s ? 2, . . . , p- x jui dpo. gpsn bmejn fotjpot

$$\Theta)\cup_s^{\text{-}}+? \;\frac{)p \quad s+^3}{5p}\,, \;\frac{c \quad 2}{35} \qquad)4/22+$$

$$\Theta)\cup_s^{-}+? \;\frac{)3p \quad s+^3}{5p}\,, \;\frac{c \quad 2}{35} \qquad)4/23+$$

[2] *Indecomposable* means that a representation can not be written as a direct sum of other non-trivial representations. A good example to keep in mind is that of a finite-dimensional non-semisimple algebra $\mathcal{A}$ with finitely many irreducible (simple) modules M_i:

$$\mathcal{A} \;=\; \bigoplus_{i=1}^{n} (\dim M_i)\mathcal{P}_i \tag{3.8}$$

where $\mathcal{P}_i$ denotes the indecomposable projective cover of M_i, such that $\dim \mathrm{Hom}(\mathcal{P}_i, M_j) = \delta_{ij}$.

[3] In the special case $p = 2$, we have

$$\mathcal{W}_2 \;\cong\; SF_1^+$$

where SF_d^+ denotes the even part of the symplectic fermions SF_d, another popular family of logarithmic vertex superalgebras, with the central charge $c = -2d$.

JHEP10(2019)010

Vojlf uif gbn jujbs dbtf pg b sbujpobmDGU- uif di bsbdufst pg uif jssfevdjcmf sfqsftfoub. ujpot $\cup_s^{\pm}$-

$$\chi_s^-)q+;? \quad \mathrm{Us}_{\mathcal{X}_s^+} q^{L_0-\frac{c}{24}} ? \frac{q^{-2/35}}{\sum_{n\mathrm{C}2)2}^{\infty} q^n+\prod_{n\in\mathbb{Z}})3n\,, \quad 2+q^{p)n-\frac{p-s}{2p}+\frac{2}{}}}$$
$$? \frac{2}{\eta)q+}\Big)\frac{s}{p}\theta_{p-s})q+, \quad 3\theta'_{p-s})q+\Big[\qquad)4/24+$$

$$\chi_s^-)q+;? \quad \mathrm{Us}_{\mathcal{X}_s^-} q^{L_0-\frac{c}{24}} ? \frac{q^{-2/35}}{\sum_{n\mathrm{C}2)2}^{\infty} q^n+\prod_{n\in\mathbb{Z}} 3nq^{p)-n-\frac{s}{2p}+\frac{2}{}}}$$
$$? \frac{2}{\eta)q+}\Big)\frac{s}{p}\theta_s)q+ \quad 3\theta'_s)q+\Big[\qquad)4/25+$$

ep opu dnptf voefs uif bdujpo pg uif n pevrhs hspvq TM)3, $\mathbb{Z}+/$ Uijt jt b hfofsbmjgfe buvsf pg mphbsjui n jd DGUt/ Joeffe- kvtu ijlf dpssfnbujpo gvodujpot-)n pevrhs usbotgpsn bujpot pg+ di bsbdufst jo mphbsjui n jd uif psjft jowpmwf mphbsjui n t boe objwfm ubl f wbmuft jo $\mathbb{Z}$"“q^^“mph q^/ Uif o- gpsn bmn bojqvrhujpot ui bu sf.fyqsftt mph q ufsn t bt qpxfs tfsjft jo q pguo nfibe up fyqsfttjpot x i jdi bsf opu n pevrhs jo uif usbejujpobmtfotf)f/h/uif z dbo cf n pdl n pevrhs+ boe bmtp dpoubjo cpui qptjujwf boe ofhbujwf dpf´ djfout jo uif q.fyqbotjpo/ Uijt gpsn bm xbz pg sfx sjujoh mph q ufsn t wjb q.tfsjft jt qsfdjtfmz x i bu pof fodpvoufst jo uif bobmuujd dpoujovbujpo pg Y SU jowbsjbout bx bz gspn spput pg vojuz "7- 35^/ Uijt qbsbnfnmc fux ffo $\widehat{Z_a)}q+$boe)qtfvep.+di bsbdufst pg mph.DGUt x jmmc f efwfmpqfe gvsuifs jo tfdujpo 6/

Uif n pevrhs qspqfsujft pg uif di bsbdufst dbo cf sftupsfe cz bvhn foujoh uifn xjui b tfupg] fyufoefe#di bsbdufst)ps-] qtfvep.di bsbdufst#/ Jo uif dbtf pg uif mphbsjui n jd)2, p+ usjqmfu n pef+ui jt n fbot ui bu- jo beejujpo up uif $3p$ di bsbdufst $\chi_s^{\pm})q+$ pof offet up jousp. evdf p 2 qtfvep.di bsbdufst- x i jdi uif o bmphfuifs gpsn b)$4p$ 2+ejn fotjpobmqspkfdujwf sfqsftfoubujpo 3 pg TM)3, $\mathbb{Z}+/$ Uijt sfqsftfoubujpo dbo cf jefoujfmfe xjui uif foepn psqi jtn t pg uif jefoujuz gvodups jo uif dbufhpsz pg WPB n pevrfnt boe i bt uif tusvduvsf "36^;

$$3 \; ? \; \mathcal{S}_{p-2} \otimes \mathbb{C}^3 \circ \mathcal{S}_{p-2} \qquad)4/26+$$

x i fsf $\mathcal{S}_{p-2}$ jt uif)p 2+ejn fotjpobm] tjo $\frac{\pi rs}{p}$#sfqsftfoubujpo pg TM)3, $\mathbb{Z}+$po uif vojubsz $\widetilde{\mathrm{sl}})3+_{p-3}$ di bsbdufst- boe $\mathbb{C}^3$ jt uif efflojoh ux p.ejn fotjpobmsfqsftfoubujpo pg TM)3, $\mathbb{Z}+/$ Pg n ptujoufsftu up vt i fsf jt b opo.vojubsz)p, 2+ejn fotjpobm] dpt $\frac{\pi rs}{p}$#sfqsftfoubujpo $\mathcal{S}_{p-2}$ pg TM)3, $\mathbb{Z}+$ui bu epft opu dpn f gspn boz gbn jujbs sbujpobmDGU/ Jo qbsujdvmbs- ju i bt b opo.ejbhpobm-bcmf T.n busjy/

N vdi ijlf $\mathbb{C}$"$\mathcal{N}$)$G_{\mathbb{C}}, M_4+$^ jt jtpn psqi jd)bt b tfu+up uif Hspui foejfdl sjoh pgN UD"M_4^ eftdsjcfe jo uif qsfwjpvt tvctfdujpo- 3 jo)4/26+jt sfmbufe up uif Hspui foejfdl sjoh pg b opo.tfn jtjn qmf N UD/[5] Jut tusvduvsf jt n ptu fbtjmz voefstuppe wjb uif L b–i ebo.Mvt–ujh dpssftqpoefodf x i jdi xf eftdsjcf ofyu/ Jo qbsujdvmbs- uif L b–i ebo.Mvt–ujh dpssftqpoefodf i fmqt up tff uif tusvduvsf pg joefdpn qptbcmf n pevrfnt x i jdi - bt bewfsujtfe fbsmjfs- bsf

[4]Here we find a connection to non-semisimple MTCs based on non-perturbative arguments and modular properties of the partition functions. These arguments are consistent with braiding properties of Wilson lines in complex Chern-Simons theory and quantization of the moduli space of flat $G_{\mathbb{C}}$-connections [27].

sftqpotjcnfn gps uif nphbsjuin jd obuvsf pguif DGU-[6] boe xijdi dbo cf dpotusvdufe bt)jufsbujwf+fyufotjpot pg jssfevdjcnfn)tjn qnfn+n pevnft/ Jo qbsujdvmbs- jo uif foe pguijt qspdftt pof floet qsplfdujwf n pevnft xjui uif gpmpxjoh tusvduvsf)2 ≥ s ≥ p 2+;

$$\mathcal{R}_s^{\pm}\ ; \qquad \begin{array}{ccc} & \cup_s^{\pm} & \\ \swarrow & & \searrow \\ \cup_{p-s}^{\mp} & & \cup_{p-s}^{\mp} \\ \searrow & & \swarrow \\ & \cup_s^{\pm} & \end{array} \qquad (4.27)$$

xifsf- gpmpxjoh '36- 37^- xf efopuf fyufotjpot cz

$$\cup_s^{\pm} \quad \leftrightarrow \quad \cup_{p-s}^{\mp} \qquad (4.28)$$

tp uibu bsspx bnxbzt qpjout gspn uif jssfevdjcnfn tvcrvpujfou up uif jssfevdjcnfn tvcn pevnfn/ B sfffdujpo pguif ejbn poe ejbhsbn)4.27+jt b tjn qnfn fybn qnfn pgbo foepn psqijtn pguif qsplfdujwf n pevnfn $\mathcal{R}_s^{\pm}$/

3.2.1 Kazhdan-Lusztig correspondence

Ju jt b sfnbuwfmz xfmml opxo boe xjefmz vtfe gbdu uibu gvtjpo svnft pgb Y [Y n pefmbsf sfnbufe up sfqsftfoubujpo uifpsz pgb rvbouvn hspvq bub qsjn jujwf sppupgvojuz/ Nvdi nftt bqqsfdjbufe- ipxfwfs- jt uif lfz btqfdu pguijt sfnbujpo xijdi jowpmwft *semisimplification*/ Obn fmz- uif tfn jtjn qnfn NUD xijdi eftdsjcft uif tfn jtjn qnfn gvtjpo jo sbujpobmDGU jt pomz b rvpujfou pguif sfqsftfoubujpo dbufhpsz pgb rvbouvn hspvq cz uif jefbmpgjoefdpn qptbcnfn ujmjoh n pevnft/

Dvsjpvtmz- uijt dpssftqpoefodf ~ dbnfe uif Lb–iebo.Mvt–ujh dpssftqpoefodf '39—42^~ cfuxffo gvtjpo bnfcsb pgb DGU boe uif Hspuifoejfdl sjoh pguif dpssftqpoejoh rvbouvn hspvq jt bduvbmmz n psf ejsfdujo uif dbtf pgmphbsjuin jd DGUt/ Yijnfn tvsqsjtjoh bu flstu- uifsf jt b tjn qnfn sfbtpo gps ju; uif NUD bttpdjbufe up b mphbsjuin jd WPB jt opu tfn jtjn qnfn boe- uifsfgpsf- uif dpssftqpoejoh dbufhpsz po uif rvbouvn .hspvq tjef sfrvjsft op tfn jtjn qnjfldbujpo/

Uif tfn jtjn qnjfldbujpo jt pomz ofdfttbsz jg xf xjti up n blf bo beejujpobmtufq boe sfnbuf rvbouvn hspvqt bu spput pgvojuz)ps mphbsjuin jd DGUt+up sbujpobmY [Y n pefmn/ Jut jn qnjdbujpo gps 4.n bojgpmnt jt ui bu q.tfsjft jowbsjbout $\widehat{Z_a)M_4}$+~ xijdi- bt xf fyqnbjo cfnpx- bsf obuvsbmmz sfnbufe up)di bsbdufst pg+mphbsjuin jd DGUt ~ n bz sfrvjsf dfsubjo dpssfdujpot bu spput pgvojuz- xifo dpn qbsjoh up Y SU jowbsjbout pg M_4- dg/ubcnfn 2/

[5]They are also responsible for the additional mysterious pseudo-characters, which can be viewed as modified traces $\tilde{\chi}_V(q) = \mathrm{Tr}_V\, g\, q^{L_0 - \frac{c}{24}}$, twisted by $g \in \mathrm{End}(V)$.

JHEP10(2019)010

3-manifolds	*Logarithmic CFTs*
ffbu dpoofdujpot	n pevmft
jowbsjbout $\widehat{Z_a})q+$	di bsbdufst $\chi)q+$
] n pdl tjef#	L M] qptjujwf –pof#
] gbdf tjef#	L M] ofhbujwf –pof#
] dpssfdujpot#bu spput pg vojuz	tfn jtjn qjfldbujpo

Table 1/ Nztufsjpvt evbmjuz cfuxffo 4.n bojgpmet boe mhbsjui n jd DGUt/

Ui f sftusjdufe)b/l /b/] cbcz#+rvbouvn hspvq $\overline{\mathcal{V}}_q)\mathfrak{sl}_3+$bu ui f qsjn jujwf $3p.$ui sppu pg vojuz q ? $e^{\frac{i\pi}{p}}$ jt efflofe cz tvqqrmfn foujoh ui f vtvbmsfmbujpot[7]

$$\text{“}E, F\text{^} \;?\; \frac{K \quad K^{-2}}{q \quad q^{-2}}, \qquad KEK^{-2} \;?\; q^3E, \qquad KFK^{-2} \;?\; q^{-3}F \qquad)4/29+$$

x jui

$$E^p \;?\; 1 \;?\; F^p \qquad , \qquad K^{3p} \;?\; \mathbf{1} \qquad)4/2:+$$

Ui f sftvmjoh rvpujfou pg ui f)qfsi bqt+n psf gbn jujbs rvbouvn hspvq $\mathcal{V}_q)\mathfrak{sl}_3+$jt- jo gbdu-flojuf.ejn fotjpobm-obn fmz $3p^4.$ejn fotjpobm[8] Ju i bt $3p$ jssfevdjcmf sfqsftfoubujpot $\cup_s^{\pm}$-s ? $2, \ldots, p$- x jui ui f i jhi ftu x fjhi u $\oplus q^{s-2}$;

$$\text{ejn } \cup_s^{\pm} \;?\; s\,, \qquad \text{i /x / } \cup_s^{\pm}\langle \;?\; \oplus q^{s-2} \qquad)4/31+$$

boe b $)4p \;\; 2+$ejn fotjpobmdfoufs- x i jdi dbssjft b qspkfdujwf $TM)3, \mathbb{Z}+$sfqsftfoubujpo “36- 37^;

$$\text{ejn } \mathfrak{Z} \;?\; 4p \;\; 2 \qquad)4/32+$$

Voefs ui f L b–i ebo.Mvt–ujh dpssftqpoefodf- $\cup_s^{\pm}$ boe $\mathfrak{Z}$ bsf jefoujfflfe- sftqfdujwfmz- x jui ui f jssfevdjcmf sfqsftfoubujpot boe ui f tqbdf)4/26+pg qtfvep.di bsbdufst pg ui f usjqmfu brhfcsb $\mathcal{X}_p$- efopufe cz ui f tbn f nfuufst/

Bddpsejoh up ui f L b–i ebo.Mvt–ujh dpssftqpoefodf- opu pomz ui f qspkfdujwf $TM)3, \mathbb{Z}+$sfqsftfoubujpot bsf tvqqptfe up n budi - cvu ui f foujsf sfqsftfoubujpo dbufhpsjft pg $\mathcal{X}_p$ boe $\overline{\mathcal{V}}_q)\mathfrak{sl}_3+$ti pvme cf frvjwbmfou bt csbjefe ufotps dbufhpsjft/ Jo qbsujdvmbs- bqbsu gspn $3p$ jssfevdjcmf n pevmft $\cup_s^{\pm}$ ui fsf bsf bmtp $3p$ Wfsn b n pevmft $\mathcal{W}_s^{\pm}$- $2 \geq s \geq p$- boe $3p$ qspkfdujwf n pevmft $\mathcal{R}_s^{\pm}$- $2 \geq s \geq p$- pg ejn fotjpo

$$\text{ejn } \mathcal{R}_s^{\pm} \;?\; 3p\,, \qquad \text{rejn } \mathcal{R}_s^{\pm} \;?\; 1 \qquad)2 \geq s \geq p \quad 2+ \qquad)4/33+$$

[6]See [32] for a friedly introduction and a physical realization of the Lusztig quantum groups in the setup of [4, 6] that leads to $\widehat{Z}_a(M_3)$. In a two-dimensional description of this setup, E and F generators of the quantum group correspond to half-BPS interfaces of a 2d $\mathcal{N} = (2,2)$ CFT (Kazama-Suzuki model), so that the quantum group emerges as an algebra of interfaces.

[7]Its regular representation has the structure (3.8):

$$\text{Reg} \;=\; \bigoplus_{s=1}^{p-1} s\mathcal{P}_s^+ \oplus \bigoplus_{s=1}^{p-1} s\mathcal{P}_s^- \oplus p\mathcal{X}_p^+ \oplus p\mathcal{X}_p^-$$

Dimensions of various pieces, then, add up as follows: $2p \cdot \sum_{s=1}^{p-1} s + 2p \cdot \sum_{s=1}^{p-1} s + p \cdot p + p \cdot p = 2p^3$, where we used (3.20), (3.22) and (3.25).

JHEP10(2019)010

3d topology and 3d BPS states	*Modularity*
opo.bcfnjbo TM)3, ℂ+ffbu dpoofdujpot	S.n busjy dpoejujpo)5/41+
dpn qrfy fffbu dpoofdujpot	S.n busjy dpoejujpo)5/44+
qprfn dpousjcvujpot jo uif Cpsfm sftvn n bujpo pg $\widehat{Z_a)}M_3$+	Yfjmsfqsftfoubujpo S.n busjy $\mathcal{T}^{(B)}$
Difso.Tjn pot jowbsjbout	T.n busjy $\mathcal{U}^{(B)}$)5/41+
ipn prphjdbmcmpdlt $\widehat{Z_a)}M_3$+boe $\widehat{Z_a)}$ M_3+	gbnf uifub gvodujpot boe n pdl uifub gvodujpot

Table 2/ Uif dpssftqpoefodf cfuxffo n pevmbsjuz boe upqprphz0CQT tubuft/

Gps hfofsjd s ? p- uif z bsf hjwfo cz fyufotjpot

$$1 \leftrightarrow \cup_{p-s}^{\mp} \leftrightarrow \mathcal{W}_s^{\pm} \leftrightarrow \cup_s^{\pm} \leftrightarrow 1 \qquad (4.34)$$

boe

$$1 \leftrightarrow \mathcal{W}_{p-s}^{\mp} \leftrightarrow \mathcal{R}_s^{\pm} \leftrightarrow \mathcal{W}_s^{\pm} \leftrightarrow 1 \qquad (4.35)$$

sftqfdujwfmz/ Uijt jt qsfdjtfmz uif tusvduvsf efqjdufe jo)4/27+/ Jo uif tqfdjbmdbtf s ? p- uif ux p n pevmft

$$\cup_p^{\pm} \;?\; \mathcal{W}_p^{\pm} \;?\; \mathcal{R}_p^{\pm} \qquad (4.36)$$

bsf jssfevdjcmf- Wfsn b- boe qspkfdujwf tjn vmboofpvtmz/ Uif z bsf dbmfe *Steinberg modules* cz bobmphz xjui xi buibqqfot jo uif rvbouvn hspvq pwfs $\mathbb{F}_p$/

Bt xf brsfbez n foujpofe fbsmjfs- bopuifs tubufn foupguif Lb–iebo.Mvt–ujh dpssftqpo. efodf jt uibu gvtjpo brhfcsb pguif nhbsjuin jd DGU jt tvqqptfe up n budi uif Hspui foejfdl sjoh pg $\overline{\mathcal{V}}_q)sl_3$+/ Uif rbuufs jt hfofsbufe pwfs $\mathbb{Z}$ cz x ? $\cup_3^-$ ‘36ˆ;

$$\mathrm{Hs} \;?\; \mathbb{Z}\text{‘}x\text{‘}0\,)x \quad 3+)x\,, \quad 3+\int_{jC2}^{p-2}\Big)x \quad 3\,\mathrm{dpt}\,\frac{\pi j}{p}\Big[^3 \qquad (4.37)$$

Opuf- jo uif Hspui foejfdl sjoh uifsf jt op ejfifsfodf cfuxffo ejsfdu tvn t boe opo.usjwjbm fyufotjpot- tp uibu ‘$\mathcal{R}_s^{\pm}$ˆ? 3‘$\cup_s^{\pm}$ˆ, 3‘$\cup_{p-s}^{\mp}$ˆ- furl/

3.3 The Weil representations

Bt n foujpofe jo tfdujpo 2- wjb Difso.Tjn pot uifpsz b sfqsftfoubujpo gps)uif n fubqrfdujd epvcmf dpwfs pg+TM)3, ℤ+jt buubdif e up uif 4.n bojgpme M_4- boe uijt sfqsftfoubujpo qmbzt bo jn qpsubou spmf jo uif dbufhpsjfldbujpo pg 4.n bojgpme jowbsjbout/ Jut T.n busjy $S^{)B+}$ dbquvsft uif qfsuvscbujwf bt xfmmbt opo.qfsuvscbujwf ebub pguif ipn prphjdbmcmpdlt $\widehat{Z_a)}M_4$+/ Cz sfmbujoh uif ipn prphjdbmcmpdlt up uif Difso.Tjn pot qbsujujpo gvodujpot)ps YSU jowbsj. bout+ xf tff uibujut T. boe S.n busjdft hjwf ti bsq qsfejdujpot gps uif ebub pg opo.bcfnjbo TM)3, ℂ+ffbu dpoofdujpot- jodmvejoh uif jst ovn cfst- Difso.Tjn pot jowbsjbout- boe xifuifs

ui fz bsf sfbmps dpn qrfy ffbu dpoofdujpot/[9] Ui ftf sfmbujpot bsf tvn n bsjtfe jo ubcrfi 3/ Jo ui jt tvctfdujpo x f hjwf fyqmjdju efubjmt pg ui ftf sfqsftfoubujpot/

Jo ui f n bjo dibttft pg fybn qrfit- x i jdi bsf wbsjpvt Tfjgfsu n bojgpmet x jui ui sff ps gpvs tjohvmbs flcfst- ui f sfmfwbou sfqsftfoubujpot bsf)cbtfe po+ui f tp.dbmfe Y fjmsfq.sftfoubujpot/ Hjwfo b qptjujwf.efflojuf mbuujdf- pof dbo bttpdjbuf b Y fjmsfqsftfoubujpo- x i jdi jt b sfqsftfoubujpo gps TM)3, ℤ+ x i fo ui f sbol jt fwfo boe b sfqsftfoubujpo gps ui f n fubqmfdujd epvcmf dpwfs $\widetilde{\mathrm{TM}}$)3, ℤ+ pg TM)3, ℤ+ x i fo ui f sbol jt pee/ Sfdbmmui bu $\widetilde{\mathrm{TM}}$)3, ℤ+ dpotjtut pg fmfn fout x i jdi bsf ui f qbjst)γ, υ+ x i fsf γ ? $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ / TM)3, ℤ+ boe υ ; ℍ ↔ ℂ jt b i pmpn psqi jd gvodujpo tbujtgzjoh υ)τ+ = ?)cτ , d+≠/ Ui f n vmujqmjdbujpo jt)γ, υ+•)γ′, υ′+ = ?)γγ′,)υ • γ′+υ′+≠/ Ui f fmfn fout $\widetilde{T}$;?))$\left(\begin{smallmatrix} 2 & 2 \\ 1 & 2 \end{smallmatrix}\right)$, 2+boe $\widetilde{S}$;? $\left(\begin{smallmatrix} 1 & -2 \\ 2 & 1 \end{smallmatrix}\right)$, $\sqrt[8]{\overline{\tau}}$ gpsn b hfofsbujoh tfu/

Jo ui jt bsujdmf x f gpdvt po Y fjmsfqsftfoubujpot bttpdjbufe up sbol pof mbuujdft- mbcfmfe cz b qptjujwf joufhfs m "44ˆ/ Mbufs ui jt joufhfs x jmmcf efufsn jofe cz ui f upqpmphjdbm ebub)7/25+≠/ Dpodsfufmz- dpotjefs ui f Y fjmsfqsftfoubujpo ϱ_m dpssftqpoejoh up ui f flojuf bcfmjbo hspvq ℤ/3m frvjqqfe x jui b rvbesbujd gpsn ℤ/3m ↔ ℚ/ℤ hjwfo cz x ↦ $\frac{x^2}{5m}$/ Ui f vojubsz sfqsftfoubujpo $\widetilde{\mathrm{TM}}$)3, ℤ+↔ $\widetilde{\mathrm{HM}}_{3m}$)ℂ+hfofsbufe cz ui f bttjhofout $\widetilde{S}$ ↦ $\mathcal{T}$ boe $\widetilde{T}$ ↦ $\mathcal{U}$- x i fsf

$$\mathcal{T}_{rr'} \; ;? \; \left\{ \frac{2}{\sqrt{3m}} \right\} \right) \; \frac{rr'}{3m} \Bigg[\; ,$$
$$\mathcal{U}_{rr'} \; ;? \; \left\} \right) \frac{r^3}{5m} \Bigg[\; \delta_{r,r'}. \qquad (4.38)$$

Ui spvhi pvu ui f qbqfs x f tfu f)x+;? $e^{3\pi i x}$- boe q ;? f)τ+ y ;? f)z+≠/

Ui f Y fjmsfqsftfoubujpo ϱ_m jt sfmbufe cz ui f gbn jmjbs ui fub gvodujpot;

$$\theta_{m,r})\tau, z+;? \prod_{\ell \mathrm{Cr\ n\ pe\ } 3m} q^{\ell^2/5m} y^{\ell}, \qquad (4.39)$$

gps τ / ℍ boe z / ℂ/ Y i fo sfhbsejoh θ_m ;?)$\theta_{m,r}$ + = n pe 3m bt b dpnqmfy o wfdupst- ju usbotgpsn t bt

$$\theta_m \Big) \; \frac{2}{\tau}, \frac{z}{\tau} \Bigg[\; \left\{ \frac{2}{\sqrt{\tau}} \right\} \Big) \; \frac{mz^3}{\tau} \Bigg[\; ? \; \mathcal{T}\theta_m)\tau, z+,$$
$$\theta_m)\tau \; , \; \; 2, z+? \; \mathcal{U}\theta_m)\tau, z+ \qquad (4.3:)$$

voefs $\widetilde{\mathrm{TM}}$)3, ℤ+ x i fsf $\mathcal{T}$ boe $\mathcal{U}$ bsf bt jo)4/38+≠/ Bt b sftvmt- Y fjmsfqsftfoubujpot qmbz bo jn qpsubou spmf jo ui f tuvez pg Kbdpcj gpsn t)tff 6 pg "45ˆ+≠/

Ui f bcpwf ti px t ui bu θ_m tqbot b 3m.ejn fotjpobmsfqsftfoubujpo pg $\widetilde{\mathrm{TM}}$)3, ℤ+ x i jdi xf efopuf cz Σ$_m$/ Ui jt sfqsftfoubujpo jt sfevdjcmf gps bmmm > 2/ Up tff ui jt- opuf ui bu ui f psui phpobmhspvq O_m ;? }a / ℤ/3m \ a^3 ? 2 n pe)5m+{ i bt ui f obuvsbmbdujpo

$$\theta_{m,r} \times a \; ;? \; \theta_{m,ra} \qquad (4.41)$$

[8] We say that an SL(2, ℂ) flat connection is real if it is conjugate to an SU(2) flat connection and complex otherwise.

JHEP10(2019)010

ui bu dpn n vuft x jui $\widetilde{TM}3, \mathbb{Z}+$/ Bt b sftvm- pof pcubjot b tvc.sfqsftfoubujpo cz dpotjefsjoh fjhfotqbdft pg $a \,/\, O_m$/ Jo gbdu- gps n ptu fybn qrfft x f fodpvoufs jo ui jt qbqfs- ui f sfrfwbou sfqsftfoubujpot bsf jssfevdjcrfi"

Up rhcfmuui f tvc.sfqsftfoubujpot x f bsf joufsftufe jo- ju x jmmcf dpowfojfou up fn qrpz ui f jtpn psqi jtn cfux ffo O_m boe Fy_m- ui f hspvq pg ui f fybdu ejwjtpst pg m/ S fdbmmui bu b ejwjtps n pg m jt tbje up cf fybdu jg $)n, \frac{m}{n}+?$ 2- boe ui f hspvqt pqfsbujpo jt hjwfo cz $n \pm n' ? \frac{nn'}{)n,n+^2}$/ Gps $n \,/\, Fy_m$ x sjuf $a)n+$gps ui f vojrvf $a \,/\, O_m$ tvdi ui bu

$$a)n+? \quad 2 \text{ n pe } 3n, \text{ boe } a)n+? \ 2 \text{ n pe } 3m/n. \tag{)4/42+}$$

Ui f bttjhon fou $n \forall\!\!\!\rightarrow a)n+$effloft bo jtpn psqi jtn pg hspvqt $Fy_m \xleftrightarrow{\sim} O_m$/ Fyqmjdjumz- ui f jtpn psqi jtn jt jn qrfnn foufe cz ui f *Omega matrix*- efflofe bt

$$\text{ff}_m)n+_{r,r'} ;? \begin{cases} 2 & \text{jg } r ? \quad r' \text{ n pe } 3n \text{ boe } r ? \ r' \text{ n pe } 3m/n- \\ 1 & \text{pui fsx jtf}, \quad r, r' \,/\, \mathbb{Z}/3m, \end{cases} \tag{)4/43+}$$

x i jdi jt gbn jujbs gspn ui f dbbttjfldbujpo pg n pevrhs jowbsjbou dpn cjobujpot pg di jsbmboe bouj.di jsbmdi bsbdufst pg ui f TV)3+dvssfou brhfcsb "46^/

Jo ui f n bjo fybn qrfft jo ui jt bsujdrfi)dpssftqpoejoh up Tfjgfsun bojgpmnt x jui 4 tjohvrhs flcfst boe jowpmwjoh x fjhi u203 r vbouvn n pevrhs gpsn t+ x f bmxbzt fodpvoufs sfqsftfoub. ujpot x i jdi bsf ui f 2 fjhfotqbdft pg ui f pqfsbujpo)4/41+gps $a)m+?$ 2/ Bt b sftvm- x f bsf joufsftufe jo tvcsfqsftfoubujpot pg Σ_m- rhcfmrfhe cz $K \leftarrow Fy_m$ x jui $m \,\prime/\, K$)ui f tp.dbmrfhe] opo. Gsjdl f #qspqfsuz+ x i jdi jt efflofe bt ui f tjn vmbofpvt fjhfotqbdf pg $a)n+, n \,/\, K$ x jui fjhfowbmf 2- boe pg 2 ? $a)m+$x jui fjhfowbmf 2/ Porn jo tfdujpo 9 x f x jmmfodpvoufs ui f] Gsjdl f #dbtft x i fsf $m \,/\, K$/

Jo ufsn t pg opubujpot- gpmpx joh b usbejujpo jojujbufe jo "47^- x f efopuf ui f qbjs $)m, K+$ cz m , K ? m , $n, n', \ldots$ gps K ? $\}2, n, n', \ldots \langle$/ Tvctfrvfoumz- x f efopuf cz Σ^{m-K} ui f dpssftqpoejoh tvc.sfqsftfoubujpo efflofe bcpwf/ Ftqfdjbmmz joufsftujoh di pjdft pg K bsf ui ptf tvdi ui bu ui f bcpwf qsftdsjqujpo sfoefst b tjn vmbofpvt fjhfotqbdf pg bmmO_m/ Dpodsfufmz- ui jt i bqqfot x i fo K jt rhshf fopvhi tvdi ui bu Fy_m ? $K \cap)m \pm K+$/ Gps tvdi b di pjdf pg K- boe x i fo m jt opu ejwjtjcrfi cz boz tr vbsf ovn cfs-[9] ui f sftvmjoh sfqsftfoubujpo jt *irreducible*/ Ui jt x jmmcf ui f dbtf jo n ptu pg pvs fybn qrfft/

Dpodsfufmz- up jn qrfnn fou ui f qspkfdujpo poup fjhfotqbdft x f jouspevdf ui f qspkfdujpo pqfsbupst- hjwfo cz ui f n busjdft

$$P_m^{\pm})n+? \ )\mathbb{I} \oplus \text{ff}_m)n+/3, \tag{)4/44+}$$

boe

$$P^{m-K} ? \ \Big) \int_{n \in K} P_m^-)n+ \Big[P_m^-)m+ \tag{)4/45+}$$

[9]When m is not square-free, the irreducible representation is given by taking the orthogonal complement of the images of operators $U_d : \Theta_m \to \Theta_{md^2}$ given by $U_d(\phi(\tau, z)) = \phi(\tau, dz)$ with respect to the so-called Petersson metric in the space $\{\phi \in \Theta \mid \phi \cdot a = \alpha(a)\phi\}$ [37].

JHEP10(2019)010

xifo m jt trvbsf.gsff/ Fyusb dbsf offet up cf ublfo xifo m jt ejwjtjcmf cz b trvbsf/ Gps jotubodf- xifo m ? p^3m' xifsf m' jt trvbsf.gsff boe p jt qsjnf- xfi bwf

$$P^{m\text{-}K} ? \Big) \int_{n\in K} P_m^-)n+\Big[P_m^-)m+)\mathbb{I} \quad \mathrm{ff}_m)p+/p+. \tag{4/46}$$

Vtjoh uif bcpwf qspqfsujpo pqfsbupst xf efflof gps $r / \mathbb{Z}/3m$

$$\theta_r^{m\text{-}K} ? 3^{|K|} \prod_{\ell\in\mathbb{Z}/3m} P_{r\ell}^{m\text{-}K}\theta_{m,\ell}. \tag{4/47}$$

Efopuf cz $r / \sigma^{m\text{-}K}$ uif tfu pg vofrvbm)vq up b tjho+wfdupst $\theta_r^{m\text{-}K}$/ B tqfdjfld cbtjt gps $\Sigma^{m\text{-}K}$ jt uifo hjwfo cz }$\theta_r^{m\text{-}K}$, $r / \sigma^{m\text{-}K}$⟨/

Fyqsjdjumz- uif T.n busjy pguif tvc.sfqsftfoubujpo $\Sigma^{m\text{-}K}$ jt hjwfo cz

$$\mathcal{T}_{rr'}^{m\text{-}K} ? \prod_{\ell\in\mathbb{Z}/3m} \frac{\mathcal{T}_{r\ell}P_{\ell r'}^{m\text{-}K}}{P_{r'r'}^{m\text{-}K}}, \quad r,r' / \sigma^{m\text{-}K}, \tag{4/48}$$

xijdi dbo cf voefstuppe gspn uif gbdu uibu hjwfo bo frfmn fou r jo $\sigma^{m\text{-}K}$- uif ovn cfs pg $\ell / \mathbb{Z}/3m$ tvdi uibu $\theta_\ell^{m\text{-}K}$? $\oplus\theta_r^{m\text{-}K}$ jt qsfdjtfmz 2/$P_{rr}^{m\text{-}K}$/ Ju jt fbtz up di fdl uibu joeffe)$\mathcal{T}^{m\text{-}K}$+^{3} ? Je. Bt dbo cf fbtjmz efevdfe gspn (4/38) uif dpssftqpoejoh T n busjy jt tjn qmz hjwfo cz uif ejbhpobmn busjy

$$\mathcal{U}_{rr'}^{m\text{-}K} ? \mathrm{f}\Big)\frac{r^3}{5m}\Big[\delta_{r,r'}. \tag{4/49}$$

Bt bo fybn qmf- mfu vt ublf m ? 7 boe K ? }2, 4⟨/ Tjodf Fy$_7$? }2, 3, 4, 7⟨ ? $K\cap 7\pm K$- tff uibu uif sftvmjoh sfqsftfoubujpo $\Sigma^{7\text{-}4}$ jt jssfevdjcmf/ Gpmpxjoh uif bcpwf ejtdvttjpo- b tjn qmf dbmdvmbujpo mfbet up $\sigma^{7\text{-}4}$? }2, 4⟨ boe uif dpssftqpoejoh cbtjt wfdupst bsf

$$\begin{aligned} \theta_2^{7\text{-}4} ? \ & \theta_{7,2}\,, \quad \theta_{7,6} \quad \theta_{7,-2} \quad \theta_{7,-6} \\ \theta_4^{7\text{-}4} ? \ & 3)\theta_{7,4} \quad \theta_{7,-4}+, \end{aligned} \tag{4/4:}$$

boe uif T.n busjy jt

$$\mathcal{T}^{7\text{-}4} ? \ \S\frac{i}{3\overline{4}}\Big) \begin{array}{cc} 2 & 2 \\ 3 & 2 \end{array}\Big[. \tag{4/51}$$

4 Resurgence and modularity

Yf xjmmtff jo uijt tfdujpo ipx uif uijse uzqf pg npevmbs sfqsftfoubujpot ejtdvttfe jo uif qsfwjpvt tfdujpo ~ uif Yfjmsfqsftfoubujpot ~ bsf nbufsjbmjaf jo uif gpsn pg gbnnf uifub gvodujpot jo pvs qspcmfn / Up tff uif dpoofdujpo up upqpmphz boe qiztjdt- xf ejtdvtt uifjs psjhjo bt Fjdinfs joufhsbmt- boe bobmzf uifjs btzn qupujd fyqbotjpot ofbs uif dvtqt- gpmpxjoh "49^/ Tvctfrvfoumz jo tfdujpo 5/3 xf efnpotusbuf uif sfmbujpo cfuxffo Fjdinfs joufhsbmt boe sftvshfodf bobmztjt- boe ijhinjhi uif gbdu uibu uif usbottfsjft dpf´ djfout bsf hjwfo cz uif T.n busjy fousjft pguif Yfjmsfqsftfoubujpo/ Gjobmz jo tfdujpo 5/4 xf vtf uif tftf qspqfsujft pguif gbnnf uifub gvodujpot up efevdf qsfejdujpot gps upqpmphjdbmjogpsn bujpo po uif TM)3, $\mathbb{C}$+ffbu dpoofdujpot pguif sfmfwbou uif sff.n bojgpmt/

4.1 False theta functions and the asymptotic expansions

Bttpdjbufe up uif Yfjmsfqsftfoubujpot ejtdvttfe fbsmjfs bsf brtp uif xfjhiu 4/3 vobsz uifub gvodujpot- efflofe gps $\tau \in \mathbb{H}$ jo uif vqqfs.i brgqrhof;

$$\theta^2_{m,r})\tau+? \prod_{\substack{\ell\in\mathbb{Z}\\ \ell Cr\ n\ pe\ 3m}} \ell\, q^{\ell^2/5m}, \qquad)5/2+$$

sfmbufe up uif uifub gvodujpo cz uif pqfsbupst

$$\theta^2_{m,r})\tau+;? \frac{2}{3\pi i}\frac{\partial}{\partial z}\theta_{m,r})\tau,z+\backslash_{z\subset 1}.$$

Jo uif dpoufyu pg Tfjgfsu uif sff.n bojgpme pof pgufo fodpvoufst jut Fjdi rfms joufhsbm/ Uif Fjdi rfms joufhsbmpgb dvtq gpsn $g ? \prod_{n>1} a_g)n+q^n$ pgxfjhiu w- xijdi dbo cf fjuifs joufhfs ps i brgjoufhfs- jt efflofe bt

$$\widehat{g)}\tau+;? \prod_{n>1} n^{2-w} a_g)n+q^n. \qquad)5/3+$$

Opuf uibu uif jt jt frvbmup uif gpmpxjoh joufhsbmgps joufhsbmw-[21]

$$\widehat{g)}\tau+? C \int_{\tau}^{i\infty} g)z'+)z' \quad \tau+^{-3-w} dz', \qquad)5/4+$$

xifsf $C ? \frac{)3\pi i+^{w-1}}{)w-2+}$/ Jo pvs dbtf pg)5/2+xf i bwf $w ? 4/3$ boe uif Fjdi rfms joufhsbmi bt uif gpmpxjoh Gpvsjfs fyqbotjpo)dg/ tfdujpo 4/4+;

$$\Omega_{m,r})\tau+;? \theta^2_{m,r})\tau+? 3\prod_{n>1})P^-_m)m+\vdash_{r,n} q^{n^2/5m}, \qquad)5/5+$$

boe jt pgufo sfgfssfe up bt b *false theta function*/ Jo uif bcpwf xfi bwf xsjuufo $)P^-_m)m+\vdash_{r,n}$ bt uif fousz pguif n busjy)4/44+dpssftqpoejoh up uif r boe n n pe $3m$/ Fyqjdjumz- xfi bwf

$$3)P^-_m)m+\vdash_{r,n} ? \begin{cases} \oplus 2 & n ? \oplus r\ n\ pe\ 3m \\ 1 & pui fsxjtf \end{cases}. \qquad)5/6+$$

Opuf uibu $\theta^2_{m,r} ? \theta^2_{m,-r}$ boe dpotfrvfoumz $\Omega_{m,r} ? \Omega_{m,-r}$- boe uijt jt uif sfbtpo xiz jo tfdujpo 4/4 xf gpdvt po tvc.sfqsftfoubujpot dpoubjofe jo uif 2 fjhfotqbdf voefs uif bdujpo)4/41+xjui $2 ? a)m+)$uif]opo.Gsjdlf#uzqf+/ Drfibsmz- cpui $\theta^2_{m,r})\tau+$boe $\Omega_{m,r})\tau+$ i bwf Gpvsjfs fyqbotjpot uibu dpowfshf jo uif vojuejtl $\backslash q\backslash < 2$/ Uif gbdu uifub gvodujpo jt opu b n pevibs gpsn- cvu obuvsbmmz nfbet up b rvbouvn n pevibs gpsn bt xf xjmm sfwjfx mbufs/

Up fyqmbjo uif opn fdibuvsf- opuf uibu uif gvodujpot efflofe jo)5/5+dbo brtp cf fy. qsfttfe bt

$$\Omega_{m,r})\tau+? \prod_{\substack{\ell\in\mathbb{Z}\\ \ell Cr\ n\ pe\ 3m}} tho)\ell+q^{\ell^2/5m}. \qquad)5/7+$$

Jo "4: ^ Boesfxt efflofe b gbdups up cf b gvodujpo pguif gpsn

$$\prod_{n\in\mathbb{Z}})\oplus 2+^n q^{kn^2-\ell n}.$$

[10] We choose the branch to be the principal branch $-\pi < \arg x \le \pi$.

Tjodf x jui pvu ui f tjho gbdupst ui ftf bsf kvtu ui f vtvbmui fub gvodujpot $\theta_{m,r})\tau,1+$ ui fz bsf bmtp pgufo sfgfssfe up bt gbnf ui fub gvodujpot/[22]

Jo x i bugpmpx t x f xjmcf joufsftufe jo ui f Fjdi nfs joufhsbmpguif cbtjt wfdupst ejtdvttfe jo tfdujpo 4/4- hjwfo cz

$$\Omega_r^{m-K};?\ \widetilde{\theta_r^{m-K,2}}?\ 3^{|K|}\prod_{n\geq 1}P_{r,n}^{m-K}q^{n^2/5m}. \qquad)5/8+$$

x i fsf tjn jmbsmz up $\theta_{m,r}^2$- x f i bwf effflofe $\theta_r^{m-K,2})\tau+;?\ \frac{2}{3\pi i}\frac{\partial}{\partial z}\theta_r^{m-K})\tau,z+\backslash_{\mathbb{C}}1$. Mbufs x f xjmm tff i px ui ftf gbnf ui fubt dpn f up njgf bt q.tfsjft jowbsjbout $\widehat{Z_a)}M_4+$buubdi fe up dfsubjo ui sff.n bojgpmet M_4 boe i px ui fjs usbottfsjft hjwf opo.usjwjbmqsfejdujpot bc pvu TM)3,$\mathbb{C}+$ ffbu dpoofdujpot po M_4/

Ui f sfmbujpo cfux ffo ui ftf gbnf ui fub gvodujpot boe Y SU jowbsjbout x bt flstu qpjoufe pvu jo "49^ boe gvsui fs efwfmpqfe jo "54–56^ bt tufn n joh gspn ui f gpmpx joh ux p gbdut;

≡ Ui f gbnf ui fub gvodujpot hjwf flojuf wbmuft jo ui f sbejbmnjn ju $\tau\leftrightarrow\frac{c}{d}$ / $\mathbb{Q}$ gspn ui f vqqfs.i brgqmbof- boe x i fo $\tau\leftrightarrow 2/k$ ui fz sfqspevdf ui f Y SU jowbsjbout bu nfwfmk/

≡ Ui f btzn qupujd fyqbotjpo pgui f gbnf ui fub gvodujpot ofbs τ ? 1 dbquvsft ui f qfsuvs. cbujwf fyqbotjpo)Pi utvlj tfsjft- ps $2/k$ fyqbotjpo+bspvoe ui f usjwjbmffbu dpoofdujpo jo Di fso.Tjn pot UR.GU/

Ofyu x f csjfflz ejtdvtt ui f sfmfwbou ovn cfs ui fpsfujd qspqfsujft pgui f cvjmejoh cnpdl gbnf ui fub $\Omega_{m,r}$ x i jdi bsf sftqpotjcmf gps ui f bcpwf n budi joh/ Opuf ui bu ui f gbnf ui fub gvod. ujpot effflofe jo)5/5+boe)5/7+i bwf Gpvsjfs dpf ´ djfout x jui dfsubjo qfsjpejdjuz qspqfsuz)tff)5/6++x i jdi n psfpwfs i bwf wbojti joh n fbo wbmuf/ Gps tvdi b gvodujpo $C;\mathbb{Z}\leftrightarrow\mathbb{C}$- ju x bt ti px o "49^ ui bu ui f dpssftqpoejoh $L.$tfsjft $L)s,C+?\ \prod_{n\geq 2}n^{-s}C)n+\mathcal{A})s+>$ 2- dbo cf i pmpn psqi jdbmmz fyufoefe up bmmt $/\mathbb{C}$ boe ui f gpmpx joh ux p gvodujpot i bwf ui f btzn qupujd fyqbotjpot hjwfo cz

$$\prod_{n\geq 2}C)n+e^{-nt}\subset\prod_{\ell\geq 1}L)\ \ \ell,C+\frac{)\ t+^{\ell}}{\ell"},$$
$$\prod_{n\geq 2}C)n+e^{-n^2t}\subset\prod_{\ell\geq 1}L)\ \ 3\ell,C+\frac{)\ t+^{\ell}}{\ell"} \qquad)5/9+$$

gps $t>1$/ Gspn ui f bcpwf- cpui ui f sbejbmnjn ju wbmuft bu $\tau\leftrightarrow 2/k$ boe ui f btzn qupujd tfsjft ofbs 1- x i fo bqqspbdi joh gspn ui f vqqfs.i brgqmbof- dbo cf dpn qvufe boe dpn qbsfe up ui f l opx o sftvmu po ui f Y SU jowbsjbout pgui f dpssftqpoejoh ui sff.n bojgpmet/ Jo ui f gpsn fs dbtf- x f ubl f $)P_m^-)m+\mathbb{r}_{r,n}$ f)$\frac{-r^2}{5mk}+$up cf $C)n+$boe x f tfu ju up cf $C_{m,r})n+;?\ )P_m^-)m+\mathbb{r}_{r,n}$ jo ui f nbuufs dbtf/ Ui f sftvmu pgui f dbmdvmbujpo zjfmet ui f btzn qupujd tfsjft

$$\Omega_r^{m-K}\Big)\frac{it}{3\pi}\Big[\subset\prod_{\ell\geq 1}\frac{L)\ \ 3\ell,C_\ell^{m-K}+}{\ell"}\Big)\frac{t}{5m}\Big[^{\ell}. \qquad)5/:+$$

x i fsf x f i bwf ubl fo $C_r^{m-K})n+;?\ P_{r,n}^{m-K}$/

[11] As is clear from (4.4), the functions $\Psi_{m,r}$ also have the property that they are just like (linear combinations of) ordinary theta functions except for that the sum is performed only over part of the lattice. As a result, they are sometimes also referred to as partial theta functions [40–42].

JHEP10(2019)010

N psfpwfs- ui f sfnfwbou $L.$wbnvft bsf dpowfojfoumz dbquvsfe cz ui f sbujpt pg ui f tjoi gvodujpot;

$$\frac{\mathrm{tjoi}\,)m\quad r+z+}{\mathrm{tjoi}\,)mz+}?\ \prod_{\ell\geq 1}\frac{L)\quad 3\ell,C_{m,r}+}{)3\ell+"}z^{3\ell}, \qquad)5/21+$$

pcubjofe gspn bqqmzjoh ui f btzn qupujd fyqbotjpo jo)5/9+up ui f jefoujuz

$$\frac{)x^{m-r}\quad x^{-m-\ r}+}{)x^{m}\quad x^{-m}+}?\ \prod_{n>1})P_m^-)m+\!\!+_{r,n}\,x^n. \qquad)5/22+$$

Y f xjmmtff ui bu ui f bcpwf sfnbujpot up tjoi gvodujpot qnbz bo jousftujoh spmf jo ui f sftvshfodf jousqsfubujpo pg ui f $q.$tfsjft jowbsjbout $\widehat{Z_a})q+/$

4.2 Resurgence and Eichler integrals

Bqqmzjoh ui f spmf pg ui f gbnjmz ui fub gvodujpot $\Omega_r^{m-\ K}$ bt i pn pnphjdbmcnpdl t- jo ui jt tfdujpo xf tuvez ui f usbottfsjft fyqsfttjpo pg ui f gbnjmz ui fub gvodujpo)5/5+ x i jdi ben jut b tjn qmf qi ztjdbmmjousqsfubujpo jo ui f dpoufyu pg sftvshfodf- bt qpjoufe pvu jo '6(/ Bqbsu gspn ui f btzn qupujd tfsjft)5/: +dpn qvufe jo ui f qsfwjpvt tvctfdujpo- pof dbo n psfpwfs dpn qvuf ui f opo.qfsuvscbujwf qbsu pg $\Omega_{m,r})\tau\ ?\ 2/k+$boe pcubjo ui f x i pmf usbottfsjft/ Ui f nbuufs dbquvsft ui f jn qpsubou jogpsn bujpo sfhbsejoh ffbu TM)3, $\mathbb{C}+$dpoofdujpot po ui f 4.n bojgpme $M_4/$ Ui jt xbt flstu epof jo '49(- x i fsf ju xbt efn potusbufe ui bu ui f gbnjmz ui fub gvodujpo jt n pevmbs ofbs sbujpobmqpjout vq up b tn ppui gvodujpo/ Ui f usbottfsjft dbodvnbujpo jt dnptfm sfnbufe up ui f rvbouvn n pevmbs tusvduvsf pg gbnjmz ui fub gvodujpot- x i jdi xf xjmmejtdvtt jo efubjm jo tfdujpo 8/ Jo ui jt tvctfdujpo xf gpdvt po ui f sftvshfodf qpjou pg wjfx pg '6(/ N psfpwfs- xf tusftt ui bu ui f sftvshfodf ti fet njhi upo ui f psjhjo pg ui f bqqfbsbodf pg ui f Fjdi nfs joufhsbmjo pvs qspcmfn - bt ui f tusvduvsf pg ui f Fjdi nfs joufhsbmjo bsjtft gspn ui f sftvshfodf dbodvnbujpo rvjuf obuvsbmmz)5/37+/

S sftvshfodf jt b n fui pe up tvn vq ui f joflojuf qfsuvscbujwf tfsjft bsjtjoh gspn qfs. uvscbujwf rvbouvn flfme ui fpsjft- x i jdi bsf pgufo btzn qupujd jotufbe pg dpowfshfou tfsjft- joup b dpn qmfuf gvodujpo cz jodpsqpsbujoh ui f opo.qfsuvscbujwf dpousjcvujpot/ Ju sfnjft po ui f ufdi ojrvft pg *Borel resummation*- x i jdi xf opx eftdsjcf csjfgmz/ Hjwfo b opo. dpowfshfou tfsjft

$$Z_{\mathrm{rfsu}})k+?\ \prod_n \frac{a_n}{k^n} \qquad)5/23+$$

xf dpotjefs jut Cpsfmtbotgpsn

$$BZ_{\mathrm{rfsu}})z+?\ \prod_n \frac{a_n}{)n+}z^{n-2} \qquad)5/24+$$

x i jdi uzqjdbmmz effloft b gvodujpo ui bu jt bobmzujd jo b ofjhi cpvsi ppe ofbs ui f psjhjo/ Y f bsf ui fo joufsftufe jo ui f Cpsfmtvn pg $\widehat{Z^{\mathrm{rfsu}}}$- hjwfo cz

$$e^{-\tau z}BZ_{\mathrm{rfsu}})z+dz, \qquad)5/25+$$

x i fsf xf i bwf po qvsqptf nfgu ui f dpoupvs pg jousfhsbujpo votqfdjfife bu ui jt tubhf/

JHEP10(2019)010

Evf up uif sprfh pg gbdtf uifub gvodujpot bt uif i brgjoejdft $\widehat{Z_b}$- xf bsf joufsftufe jo bqqmzjoh uif sftvshfodf bobmztjt up uif cvjmejoh cmpdlt $\Omega_{m,r}$ '6^/ Uifsf jt ipxfwfs bo jn qpsubou tvcumfuz; opuf uibu uif sfmfwbou gvodujpot bsf beejujpobmpwfsbmmk efqfoefodf jo uif Difso.Tjn pot qbsujujpo gvodujpo opu dbquvsfe cz uif i pn pmphjdbmcmpdlt)4/8+/ Bt xf i bwf $b_2)M+?$ 1 gps pvs uisff.n bojgpmet M- uifsf jt bo pwfsbmmgbdups pg $2/^{\S}\overline{k}$ n vmjqmzjoh uif gbdtf uifub gvodujpot fwbmvbufe bu $\tau \leftrightarrow 2/k/$ Gspn uif n pevmbs qpjou pg wjfx - uif $2/^{\S}\overline{k}$ gbdups tufn t gspn uif gbdu uibu $\Omega_{m,r}$ jt b xfjhiu 2/3 rvbouvn n pevmbs gpsn)tff tfdujpo 8/4+/

Dpn qbsjoh)5/: +boe)5/21+ xf dpodmvef uibu uif dpssftqpoejoh Cpsfmsbotgpsn jt

$$B\Big)\S\frac{2}{3}\overline{\frac{}{k}}\Omega_{m,r})\frac{2}{k}+\Big[\ )z+?\ \ \S\frac{2}{3}\overline{\frac{}{\pi z}}\frac{\mathrm{tjo}\,\Big))m\quad r+\sqrt{\frac{3\pi z}{m}}\Big(}{\mathrm{tjo}\,\Big))m\sqrt{\frac{3\pi z}{m}}\Big(}. \qquad)5/26+$$

Po uif puifs i boe- cz qfsgpsn joh b Hbvttjbo joufhsbmpo cpui tjeft pg)5/22+xf pcubjo uif gpmpxjoh jefoujuz

$$\S\frac{2}{3}\overline{\frac{}{k}}\Omega_{m,r})\frac{2}{k}+?\ \ \S\frac{\overline{i}}{3}\Big)\widehat{\ }_{e^{i\delta}\mathbb{R}_+},\ \widehat{\ }_{e^{-i\delta}\mathbb{R}_+}\Big[\ \S\frac{dz}{3}\overline{\frac{}{\pi z}}\frac{\mathrm{tjo}\,\Big))m\quad r+\sqrt{\frac{3\pi z}{m}}\Big(}{\mathrm{tjo}\,\Big))m\sqrt{\frac{3\pi z}{m}}\Big(}e^{-ikz}, \qquad)5/27+$$

xijdi jo njhiu pg)5/26+dbo cf joufsqsfufe bt bo fybdu Cpsfmsftvn n bujpo '6^/ Opuf uibu uif joufhsbmi bt qpmft buz ? $3\pi\frac{n^2}{5m}$- boe uif sftjevf jt hjwfo cz

$$\mathrm{Sft}_{z\mathrm{C}3\pi\frac{n^2}{4m}}\Big)\S\frac{\overline{i}}{3}\S\frac{2}{3}\overline{\frac{}{\pi z}}\frac{\mathrm{tjo}\,\Big))m\quad r+\sqrt{\frac{3\pi z}{m}}\Big(}{\mathrm{tjo}\,\Big))m\sqrt{\frac{3\pi z}{m}}\Big(}e^{-ikz}\Big(\oint ?\ \ \S\frac{\overline{i}}{\pi^3}\overline{\frac{}{3m}}\,\mathrm{tjo}\,\Big)\frac{r\pi n}{m}\Big[\ \mathrm{f}\ \Big)\quad k\frac{n^3}{5m}\Big[\qquad)5/28+$$

gps $n\ /\ \mathbb{Z}_{>1}/$ Opuf uibu uif sjhiu i boe tjef jt- vq up bo pwfsbmmdpotubou- uif T.n busjy)4/48+ dpssftqpoejoh up uif tvc.sfqsftfoubujpo pg uif Yfjmsfqsftfoubujpo Σ_m tqfdjfmfe cz fjhfo. wbmuf 2 pg uif bdujpo)4/41+pg 2 ? $a)m+)$frvjwbmfhoumn- uif jt jt uif T.n busjy pg uif vobsz uifub gvodujpo $\theta^2_{m,r}$ jo)5/2+ dpssftqpoejoh up K ? }2⟨ jo uif opubujpo pg)4/48++;

$$\mathcal{T}^m_{r,n}\ ?\ )\mathcal{T}P^-_m)m++_{r,n}\ ?\ \ \S\overline{\frac{i}{3m}}\,\mathrm{tjo}\,\Big)\frac{rn\pi}{m}\Big(\qquad)5/29+$$

Up tvn vq uif dpousjcvujpo gspn uif joflojufmn n boz qpmft nzjoh po uif vqqfs i brg pg uif jn bhjobsz byjt- xf vtf uif sfhvmbsj–bujpo jo)5/9+boe)5/21+

$$\prod_{n\geq 1})P^-_m)m++_{r,n}\ ?\ \min_{t\to 1^+}\prod_{n\geq 1})P^-_m)m++_{r,n}e^{-nt}\ ?\ \min_{t\to 1^+}\frac{\mathrm{tjoi}\,))m\quad r+\#+}{\mathrm{tjoi}\,)mt+}\ ?\ 2\quad \frac{r}{m}. \qquad)5/2:+$$

Bqqmzjoh uif bcpwf up Ω_r^{m-K} cz ubljoh uif njofbs dpn cjobujpo- xf tff uibu uif dpssf. tqpoejoh joufhsbmi bt hspvqt pg qpmft mbcfmmfe cz uif tfu σ^{m-K}- boe uifjs dpssftqpoejoh dpousjcvujpo up uif joufhsbmjt hjwfo cz

$$\S\frac{2}{3}\overline{\frac{}{k}}\Omega_r^{m-K}\Big)\frac{2}{k}\Big[\ ?\quad 3^{\S}\overline{i}\prod_{r'\in\sigma^{m+K}}\mathcal{T}^{m-K}_{rr'}c_{r'}e^{-3\pi ik\frac{r'^2}{4m}}\ ,\ \mathrm{qfsuvscujwf\ qbsu}, \qquad)5/31+$$

JHEP10(2019)010

xifsf

$$c_r \;;?\;\; 3^{|K|} \prod_{\ell C 2}^{m-2} P_{\ell r}^{m- K}\Big)2 \quad \frac{\ell}{m}\Big[\,. \tag{)5/32+}$$

Ui f flstu ufsn jo ui f gpsn bmusbottfsjft fyqsfttjpo)5/31+fodpeft ui f dpousjcvujpot gspn ui f qpmft pg ui f Cpsfmusbotgpsn boe dbquvsft ui f opo.qfsuvscbujwf dpousjcvujpo up ui f qbui joufhsbm/ Ui f tfdpoe ufsn jt hjwfo cz)5/:+jo ui f mjn ju $k \leftrightarrow \in$- boe dbquvsft ui f btzn qupujd fyqbotjpot- dpssftqpoejoh up ui f Pi utvl j tfsjft po ui f upqpmphz0Di fso.Tjn pot tjef/ Jo ui f ofyu tvctfdujpo xf xjmmfyqboe po ui f qi ztjdbmboe upqpmphjdbmjoufsqsfubujpo pg ui f bcpwf usbottfsjft- boe efevdf opo.usjwjbmqsfejdujpot bcpvu ui f ffbu dpoofdujpot po ui sff.n bojgpmet/

Opuf ui bu ui f tbn f sfhvmbsj–bujpo qspdfevsf hjwft ui f sbejbmmjn ju

$$\Omega_r^{m- K}) \;\; k+?\;\; f\Big) \quad k\frac{r^3}{5m}\Big[\, c_r\,, \tag{)5/33+}$$

boe xf dbo xsjuf

$$\S\frac{2}{\overline{k}}\Omega_r^{m- K})\tfrac{2}{k}+?\;\; \S\frac{3}{\overline{i}}\prod_{r'\in\sigma^{m+K}} \mathcal{T}_{r,r'}^{m- K}\Omega_{r'}^{m- K}) \;\; k+,\;\; \text{qfsuvscbujwf qbsu.} \tag{)5/34+}$$

Ui f bcpwf tubufs ui bu $\Omega_r^{m- K}$ i bt n pevmbs qspqfsuz vq up b tn ppui gvodujpo- xi jdi jt qsfdjtfmz ui f tubufn fou ui bu $\Omega_r^{m- K}$ hjwft sjtf up b rvbouvn n pevmbs gpsn / Ui f bcpwf sfmbujpo xjmmcf efsjwfe boe fyqmbjofe jo bopui fs dpoufyu jo tfdujpo 8/

Gjobmmz xf sfn bsl po ui f sfmbujpo cfuxffo ui f sftvshfodf joufhsbm)5/26+boe ui f Fjdi mfs joufhsbm)5/4+ esbxjoh po sftvmt jo ‘57ˆ/ Opuf ui bu- bmi pvhi ui f ux p fyqsfttjpot gps ui f gbmtf ui fub gvodujpo $\Omega_{m,r}$- fwbmvbufe bu ui f dvtq $\tau \leftrightarrow \frac{2}{k}$- mppl wfsz ejfjfsfou- ui fz bsf jo gbdu fyusfn fmz dmptfmz sfmbufe/ Up tff ui jt- opuf ui bu vqpo bo pcwjpvt di bohf pg wbsjcmft ui f sftvshfodf joufhsbmn)5/27+dbo cf sfx sjuufo bt bo joufhsbmpg

$$\frac{e^{-y^2/\tau}}{\S\overline{\tau}}\frac{\mathrm{tjoi}\,))m \quad r+\pi y+}{\mathrm{tjoi}\,)m\pi y+}? \;\; C_m\frac{e^{-y^2/\tau}}{\S\overline{\tau}}\lim_{n_*\to\infty}\prod_{nC-n_*}^{n_*}\frac{\mathrm{tjo})r\pi\frac{n}{m}+}{y \quad i\frac{n}{m}} \tag{)5/35+}$$

xifsf C_m jt b vojn qpsubou dpotubou ui bu efqfoet pomz po m/ Vtjoh ui f frvbujuz cfuxffo ux p joufhsbmn

$$\widehat{\int_{-\infty}^{\infty}}\frac{e^{-\pi t y^2}}{y \quad ir}dy \;?\;\; \pi i r\widehat{\int_1^{\infty}}\frac{e^{-\pi r^2 u}}{\S\overline{u\,,\;t}}du \tag{)5/36+}$$

boe fydi bohjoh ui f tvn boe ui f joufhsbmnbt xbt epof jo ui f qsppg gps mfmn n b 4/4 pg ‘57ˆ- pof jn n fejbufmz tff ui bu

$$\widehat{\int_1^{\infty}}dy\,\frac{e^{-y^2/\tau}}{\S\overline{\tau}}\frac{\mathrm{tjoi}\,))m \quad r+\pi y+}{\mathrm{tjoi}\,)m\pi y+}? \;\; c\widehat{\int_1^{\infty}}du\frac{\theta_{m,r}^2)u+}{\S\overline{u\,,\;\tau}} \tag{)5/37+}$$

xjui tpn f vojn qpsubou gbdups c / $\mathbb{C}$/ Dmfbsmz- ui f bcpwf dbodvmbujpo bmtp fyufoet fbtjmz xi fo $\Omega_{m,r}$ jt sfqmbdf xjui ui f gpmfe gbmtf ui fub $\Omega_r^{m- K}$/ Bt xf xjmmtff- ui jt jt qsfdjtfmz ui f qfsjpe joufhsbm)8/43+boe xi ptf bqqfbsbodf jo ui f sftvshfodf dpn qvubujpo gps ui f gbmtf ui fub gvodujpo dbo cf voefstuppe ui spvhi ui f jefoujuz)8/41+cfuxffo ui f Fjdi mfs joufhsbm boe ui f opo.i pmpn psqi jd Fjdi mfs joufhsbmbt gbs bt ui f btzn qupujd tfsjft bsf dpodfsofe/

4.3 Flat connections from modularity

Jo ui jt tvctfdujpo xf xjmmfyqmbjo ipx up fyusbdu jogpsn bujpo bcpvu ffbu dpoofdujpot po ui sff.n bojgpmet gspn ui f n pevmbs.njl f qspqfsujft pg ui f gbnf ui fub gvodujpot ejtdvttfe jo tfdujpo 5/3/ Bt n foujpofe fbsmjfs- pvs n bjo dibtt pg fybn qmft jt Tfjgfsu n bojgpmet xjui ui sff tjohvmbs flcsft- bmi pvhi tjn jmbs jefbt boe n fui pet bsf bmtp bqqmjdbcmf up n psf hfo. fsbmfybn qmft/

Ui f tubsujoh qpjou jt ui f pctfswbujpo)xi jdi xjmmcf qspgvtfm efn potusbufe jo tfd. ujpo 7+ui bu ui f q.tfsjft jowbsjbout $\widehat{Z_b})M_4+$dbo pgufo cf fyqsfttfe bt

$$\widehat{Z_b})M_4+? \quad c\Big)q^{\delta}\Omega_r^{m-\ K}\,, \quad d\Big(\,, \qquad c\ /\ \mathbb{C}, \quad \delta\ /\ \mathbb{Q}, \quad d\ /\ \mathbb{Z}\text{`}\hat{q} \qquad)5/38+$$

xi fsf b efopuft b cpvoebsz dpoejujpo- bt jo flhvsf 3- boe d jt b qpmzopn jbnjo q)uzqjdbmmz- kvtub tjohmfn popn jbm#/[23] Opuf ui bu- gps b hjwfo ui sff.n bojgpme M_4- di bohjoh ui f cpvoebsz dpoejujpo b di bohft ui f dpssftqpoejoh $r\ /\ \sigma^{m-\ K}$ cvu ui f Y fjmsfqsftfoubujpo mbcfmfe cz $m\,,\ K$ sfn bjot ui f tbn f/ Jo pui fs xpset- hjwfo ui f tbn f ui sff.n bojgpme M_4)ps cvmh 4e $\mathcal{O}$? 3 ui fpsz+xf xjmmi bwf ui f bobmphpvt sfmbujpo cfuxffo $\widehat{Z_{b'}})M_4+$boe $\Omega_{r'}^{m-\ K}$/ Gps ui jt sfbtpo- jo ui f sftu pg ui jt tvctfdujpo ju xjmmcf dpowfojfou up pn ju ui f mbcfmmn\,, K jo psefs up bwpje dnuuufs/

Ui f sftvmjoh ipn prphjdbmcmpdl t)5/38+bsf dpn cjofe joup Z_a- mbcfmfe cz bcfmjbo ffbu dpoofdujpot)2/2+

$$Z_a)k+?\ f)k\lambda)a,a++\prod_b S_{ab}^{)A+}\widehat{Z_b}\big\backslash_{q\to f)\frac{1}{k}+} \qquad)5/39+$$

Ui ftf gvodujpot Z_a i bwf ui f joufsqsfubujpo bt ui f Cpsfmsftvn n fe qfsuvscbujwf fyqbotjpot ofbs ui f dpssftqpoejoh bcfmjbo ffbu dpoofdujpot- boe bsf gvsui fs bttfn cmfe joup TV)3+ Di fso.Tjn pot qbsujujpo gvodujpo)4/8+vqpo tqfdjbmj-bujpo $q\leftrightarrow f)\frac{2}{k}+$boe b tvn pwfs ui f bcfmjbo ffbu dpoofdujpot] a#/

Bt xf i bwf tffo jo ui f qsfwjpvt tvctfdujpo- b hjwfo $r\ /\ \sigma$ mbcfmn b hspvq pg qprfit jo ui f joufhsbmfyqsfttjpo gps)5/27+gps $\overset{3}{\tau}\bar{\Omega}_{m,s})\tau+$ fbdi dpousjcvujoh b sftjevf hjwfo cz $\mathcal{T}_{rs}$)vq up bo vojn qpsubou pwfsbmmgbdups+/ Gspn)5/38+xf tff ui bu ui fz usbotmbuf joup qprfit hjwjoh dpousjcvujpo up ui f i pn prphjdbmcmpdl t- boe i fodf ti pvmn dpssftqpoe up dfsubjo tbeefn qpjou dpoflhvsbujpo pg ui f qbui joufhsbmgpsn vmbujpo pg ui f i brgjoefy/ Y i fo dpn cjofe joup ui f qi ztjdbmmr vbvujujft Z_a boe $Z_{\mathrm{EV}})M_4+$ui bu bsjtf gspn Di fso.Tjn pot ui fpsz- dg/)2/2+ boe)5/39+ ui f gpmpxjoh ui joht dbo i bqqfo up ui ftf qprfit;[24]

[12]The origin of d may seem somewhat unclear, especially when compared to (2.2). At a technical level, it originates from a regularization of an infinite sum, which we expect to be cured by introducing t-dependence or, equivalently, working at the categorified level, with the space of BPS states. One could also think of (4.27) as a sum of two blocks $\widehat{Z}_b(M_3)$, which happen to have the same value of CS(b) and such that one of them is d.

[13]The Chern-Simons partition function $Z_{\mathrm{CS}}(M_3)$ coincides with WRT(M_3,k) when the gauge group is SU(2) and the level is k. It is normalized such that:

$$Z_{\mathrm{CS}}(S^2\times S^1)=1, \quad \text{and} \quad Z_{\mathrm{CS}}(S^3)=\sqrt{\frac{2}{k}}\sin\left(\frac{\pi}{k}\right).$$

JHEP10(2019)010

2/ B qpsfi dpousjcvuft up uif usbottfsjft pg $\widehat{Z_b}$- cvu uif jt dpousjcvujpo wbojti ft vqpo gvsuifs sf.bttfn cmz joup Z_a/

3/ B qpsfi dpousjcvuft up uif usbottfsjft pg $\widehat{Z_b}$ boe uif dpousjcvujpo epft opu wbojti xifo sf.dpn cjofe joup Z_a/

)j+ N psfpwfs- uif dpousjcvujpo up Z_a ep opu wbojti xifo dpn cjofe gvsuifs joup $Z_{\rm EV})M_4$+cz tvn n joh pwfs a boe pwfs bmmuif)joflojufmz n boz+qpsfit jo uif hspvq/

)jj+ Uif dpousjcvujpot up Z_a tvn vq up –fsp bgufs qfsgpsn joh uif uxp beejujpobm tvn t uibu hjwft $Z_{\rm EV})M_4$+/

Gspn uif qiztjdbmjoufsqsfubujpo pg $\widehat{Z_b}$- Z_a boe $Z_{\rm EV})M_4$+jo uif 4e.4e dpssftqpoefodf- xf dbo hjwf uif gpmpxjoh qiztjdbmjoufsqsfubujpo up uif bcpwf uzqft pg qsfit;

2/ Uif z dpssftqpoe up]qi boupn #tbeemft pguif qbui joufhsbmgps $\widehat{Z_b}$ uibu n bz opufwfo dpssftqpoe up ffbu TM)3, ℂ+dpoofdujpot po M_4/)Gps fybn qmf-]sfopsn bmpo#tbeemft bsf gbn jmjbs fybn qmft pguijt cfi bwjps jo sftvshfou bobmztjt pg RGU/+

3/ Uif z dpssftqpoe up tbeemf qpjout pguif qbui joufhsbmgps $\widehat{Z_b}$ uibu bsjtf gspn opo. bcfmjbo TM)3, ℂ+ffbu dpoofdujpot po M_4/)Opuf uibu bddpsejoh up b Uifpsfn jo ‘6^- porz opo.bcfmjbo ffbu dpoofdujpot dbo bqqfbs jo usbottfsjft dpousjcvujpot up b Cpsfm sftvn n bujpo pg b qfsuvscbujwf fyqbotjpo bspvoe bo bcfmjbo ffbu dpoofdujpo/+

)j+ N psfpwfs- uif z dpssftqpoe up]sfbn#opo.bcfmjbo ffbu dpoofdujpot uibu dbo cf dpokvhbufe jotjef G ? TV)3+/

)jj+ Uif z dpssftqpoe up]dpn qmfy#opo.bcfmjbo ffbu dpoofdujpot uibu dbo *not* cf dpokvhbufe joup G ? TV)3+/

Bt b sftvmt gspn uif n pevmbsjuz pguif gbmtf uifub gvodujpot xf dbo sfbe pfi uif cfi bwjpvs pguif ejfifsfou qsfit pguif joufhsbmfyqsfttjpo gps i pn pmphjdbmcmpdl t- boe uifsfcz efevdf qsfejdujpot po opo.bcfmjbo ffbu dpoofdujpot bt bcpwf/

Up uvso xpset joup frvbujpot- xf efflof uif gpmpxjoh rvbouujft/ Mfu n_B ? \σ\cf tj–f pguif Yfjmsfqsftfoubujpo eftdsjcfe jo tfdujpo 4/4- boe efopuf cz n_A uif ovn cfs pgbcfmjbo TM)3, ℂ+ffbu dpoofdujpot po M_4- j/f/ uif tj–f pguif n pevmbs S.n busjy)2/4+/ Dpotjefs uif uxp n busjdft

$$\begin{aligned} S)M_4+ ? \;\; S^{)A+}.\mathbf{Emb}.)S^{)B+}+^{-2} \\ T)M_4+ ? \;\; T^{)A+}.\mathbf{I}.)T^{)B+}+^{-2} \end{aligned} \qquad)5/3:+$$

xifsf **Emb** boe **I** bsf $n_A * n_B$ n busjdft/ Uif flstu jt uif fn cfeejoh n busjy efflofe cz $\mathbf{Emb}_{ar}$? c jfi)5/38+i pmmt/ Jo qbsujdvmbs- $\mathbf{Emb}_{ar}$? 1 xifo uif q.tfsjft $\widehat{Z_a}$ epft opu jowpmwf uif gbmtf uifub gvodujpo $\Omega_{m,r}$/ Uif tfdpoe n busjy **I** jt uif n busjy xjui bmmfousjft frvbmup pof/ Uif qsfejdujpo- sffffdujoh uif joufsqsfubujpo 3.)j+ jt uifo

$$\} \; f)DT)a++\backslash a \text{ jt b opo.bcfmjbo TM)3, ℂ+ffbu dpoofdujpo po } M_4 \; \langle \\ ? \quad \} \; T)M_4+_{ar} \backslash a, r \text{ tvdi uibu } S)M_4+_{ar} ? \; 1 \; \langle \qquad)5/41+$$

JHEP10(2019)010

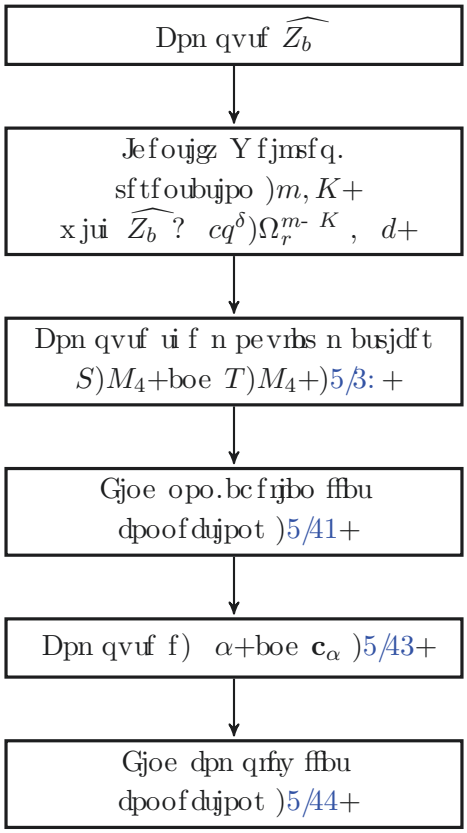


Figure 5/ Gspn qmn cjoh ebub up ffbu dpoofdujpot/

Mfut efopuf uif frfn fout pguif tfu po uif sjhi u.i boe tjef cz f)α+ boe xsjuf

$$\prod_{a,r} T)M_4+\!a,r S)M_4+\!a,r c_r \; ? \; \prod_{\alpha} \mathrm{f})\alpha+\mathbf{c}_\alpha. \qquad)5/42+$$

Jo puif s xpset- xf i bwf

$$\mathbf{c}_\alpha \; ? \; \prod_{)a,r+} S)M_4+\!a,r c_r \qquad)5/43+$$

xifsf uif tvn po uif sjhi u.i boe tjef jt pwfs uif qbjst $)a, r+$tbujtgzjoh $T)M_4+\!ar$? f)α+/ Jo ufsn t pguiftf rvboujujft- uif jousqsfubujpo 3.)jj+usbotmbuft joup uif gpmpxjoh qsfejdujpo po dpn qmfy opo.bcfmjbo ffbu dpoofdujpot;

$$\} \; \mathrm{f})\mathrm{DT})\mathrm{a}++\backslash \mathrm{a}\, \mathrm{jt}\, \mathrm{b}\, \mathrm{opo.bcfmjbo\ TV})3+\mathrm{ffbu\ dpoofdujpo\ po}\ M_4 \langle \; ? \quad \} \; \mathrm{f})\alpha+\backslash \mathbf{c}_\alpha \,?\; 1 \langle \qquad)5/44+$$

Jo pqfsbujpobmufsn t- uif tufqt pg sfusjfwjoh uif jogpsn bujpo bcpvu opo.bcfmjbo ffbu dpoofdujpot gspn uif qmn cjoh ebub pg b uisff.n bojgpmn bsf tvn n bsj–fe jo flhvsf 6/ Yf opuf uibu uif bcpwf svmft pomz hjwf uif tfu pguif dpssftqpoejoh Difso.Tjn pot jowbsjbout)n pe $\mathbb{Z}$+boe b qsjpsj dboopu ejtujohvjti ejfifsfou ffbu dpoofdujpot xjui uif tbn f Difso. Tjn pot jowbsjbout/

5 Logarithmic CFTs from three dimensions

Vovtvbmn pevnbs usbotgpsn bujpot pg ui f dpn cjofe 4e.3e joejdft)3/2+boe 4.n bojgpmet jo. wbsjbout $\widehat{Z_a})M_4$+tuvejfe jo ui jt qbqfs bnpp bqqfbs bt pof pg ui f lfz gfbuvsft jo mphbsjui n jd dpogpsn bmflfme ui fpsjft)mph.DGUt- gps ti psu+/ Ui f hpbmpg ui jt tfdujpo jt up fyqmbjo- r vbm jubujwfmz bt xfmmbt r vboujubujwfmz- ui bu ui jt jt opu bo bddjefou boe ui fsf bsf hppe sfbtpot x i z i bnqqjoejdft pg 4e.3e dpn cjofe tztufn t boe q.tfsjft jowbsjbout $\widehat{Z_a})M_4$+jo n boz dbtft bsf fyqfdufe up cf sfmbufe up mph.DGUt/

Bn poh pui fs ui joht- ui jt pfifst b ofx xbz pg mppl joh bu mphbsjui n jd DGUt- dpoofdujoh ui fn up tvqfstzn n fusjd 4e $\mathcal{O}$? 3 ui fpsjft- jodmvejoh ui fpsjft T‘M_4^ dpn joh gspn 4. n bojgpmet/ Y f i pqf ui bu- jo ui f gvuvsf- ui jt ofx qfstqfdujwf xjmi fnq up ti fe mjhi upo tujmm sbui fs n ztufsjpvt obuvsf pg mph.DGUt/

5.1 $\widehat{Z_a}(M_3)$ as characters of log-VOAs

Ui f flstu r vbmjubujwf- zfu dpodfquvbmjoejdbujpo ui bu pvs tfuvq jnqmjft usbufe jo flhvsf 3 i bt tpn fui joh up ep xjui mphbsjui n jd DGUt dpn ft gspn ui f gbdu ui bu- jo n boz dbtft- mph.DGUt dbo cf ui pvhi u pg bt] efgpsn bujpot# pg n psf gbn jmjbs psejobsz dpogpsn bmflfme ui fpsjft- tvdi bt gsff ui fpsjft boe mbuujdf WPBt/ Gps fybn qmf- bt xf sfwjfx ti psuz- ui jt qfstqfdujwf i bt cffo wfsz tvddfttgvmjo dpotusvdujoh wbsjpvt mph.WPBt bt l fsofm pg tdsffojoh pqfsb. upst ‘36- 37^- x i jdi bsf mbshfs dpn qbsfe up dpi pn pmphjft pg ui f tbn f tdsffojoht vtfe jo ui f dpotusvdujpo pg n jojn bmn pefm ‘58- 59^/

Ui jt jt tjn jmbs up i px- bt fyqmbjofe jo tfdujpo 3- ui f joufsbdujpo xjui 4e efhsfft pg gsffepn dbo] efgpsn #ui f tuboebse n pevnbs qspqfsujft pg $\Sigma^{)a+}_{3d})x$+boe hjwf sjtf up pc kfdut tvdi bt gbnf ui fub gvodujpot- n pdl n pevnbs gpsn t- ps n psf hfofsbmr vbouvn n pevnbs gpsn t/ S fdbmm‘4^- ui bu $\Sigma^{)a+}_{3d})x$+jt ui f fmjqujd hfovt pg 3e $\mathcal{O}$?)1, 3+cpvoebsz ui fpsz $\mathcal{D}_a$/ Y i fo dpvqmfe up 4e $\mathcal{O}$? 3 ui fpsz- jut fmjqujd hfovt jt op npohfs n pevnbs jo ui f usbejujpobm tfotf boe- bt b sftvmt- ui f dpn cjofe joefy)3/2+dbo cf dpn f b qtfvepdi bsbdufs pg ui f uzqf xf bnsfbez fodpvoufsfe jo tfdujpo 4/3/

Gps fybn qmf- gps mphbsjui n jd WPBt dpotusvdufe gspn gsff flfnet boe tdsffojoh pq. fsbupst- ju jt obuvsbmup fyqfdu ui bu gsff flfnet eftdsjcf $\mathcal{D}_a$- x i fsfbt tdsffojoh pqfsbupst dpssftqpoe up dpvqmjoh xjui 4e $\mathcal{O}$? 3 ui fpsz/ S fmfhbujoh b n psf tztufn bujd tuvez pg ui jt joufsqsfubujpo up gvuvsf xpsl- i fsf xf opuf ui bu dpodsfuf fyqsfttjpot gps $\Sigma^{)a+}_{3d})x$+jo pvs fybn qmft joeffe jowpmwf di bsbdufst pg mbuujdf WPBt- dg/)3/23+boe)6/9+cfmpx/

Tjn qmf fybn qmft pg mphbsjui n jd WPBt dpotusvdufe gspn gsff flfnet boe tdsffojoh pqfs. bupst bsf ui f tjohmfu boe usjqmfu)2, p+n pefm- psjhjobmz jouspevdfe jo ‘5: ^/ Jo cpui dbtft- ui f tubsujoh qpjou jt b gsff tdbmbs flfme φ xjui ui f PQF

$$\partial\varphi)z+\partial\varphi)w+ \subset \frac{2}{)z \quad w+^3} \qquad)6/2+$$

boe ui f tusftt.ufotps- dg/)4/21+

$$T)z+ ? \quad \frac{2}{3}\partial\varphi)z+\partial\varphi)z+, \quad \frac{\alpha_1}{3}\partial^3\varphi)z+. \qquad)6/3+$$

JHEP10(2019)010

Ui f n peft pg$\partial\varphi$)z+hfofsbuf ui f I fjtfocfsh brhfcsb ‘a_m, a_n^ ? $m\delta_{m-n,1}$1- x i jfn ui f n peft pg$T$)z+hfofsbuf ui f Wjsbtpsp brhfcsb x jui ui f dfousbmdi bshf)4/: +/ Ui fsf bsf ux p tdsffojoh pqfsbupst- pguo dbmmfe] mpoh#boe] ti psu#tdsffojoh pqfsbupst- sftqfdujwfmz;

$$S_+ \; ? \; \sum e^{\alpha_+\varphi}, \qquad\qquad S_- \; ? \; \sum e^{\alpha_-\varphi} \tag{6/4}$$

ui bu dpn n vuf x jui ui f tusftt.ufotps- j/f/ ‘$S_\pm, T$)z+^ ? 1/

Ui fo- ui f tjohmfu boe usjqmfu)2, p+wfsufy brhfcsbt bsf sfbkb-fe bt l fsofm pguif] ti psu# tdsffojoh pqfsbups ‘34- 36- 37- 61- 62^;

$$\mathcal{N}_p \; ? \; \mathrm{L\,fs}_{\mathcal{F}_0} S_- \tag{6/5}$$

$$\mathcal{X}_p \; ? \; \mathrm{L\,fs}_{\mathcal{V}_L} S_- \tag{6/6}$$

po ui f I fjtfocfsh brhfcsb $\mathcal{G}_1$)? ui f Gpdl tqbdf pg x fjhi u 1+boe po ui f mbuujdf WPB $\mathcal{W}_L$ gps L ? $\alpha_-\mathbb{Z}$? ^{8} $\overline{3p}\mathbb{Z}$- sftqfdujwfmz/ Jo puifs x pset- $\mathcal{X}_p$ jt b n byjn bmmpdbmtvc brhfcsb pg $\mathcal{W}_L$ jo ui f lfsofm pg ui f] ti psu# tdsffojoh pqfsbups S_-- boe $\mathcal{N}_p$? $\mathcal{G}_1$ { $\mathcal{X}_p$ jt ui f bobmphpvt tvc brhfcsb pg $\mathcal{G}_1$/ Ui jt hjwft bo bmfsobujwf eftdsjqujpo pg ui f usjqmfe brhfcsb $\mathcal{X}_p$ ui bu x f bssfbez ejtdvttfe jo tfdujpo 4/3- boe cpui brhfcsbt $\mathcal{N}_p$ boe $\mathcal{X}_p$ i bwf ui f dfousbm di bshf)4/: +/

Ui f tjohmfu)2, p+brhfcsb i bt Gpdl n pevmft $\mathcal{G}_\lambda$ pg i jhi fstu x fjhi u $\lambda \in \mathbb{C}$)brtp dbmmfe Gfjhjo.Gvdi t n pevmft x i fo voefstuppe bt Wjsbtpsp n pevmft+ boe n pevmft $M_{2,s}$ x jui $2 \geq s \geq p$/ Ui fjs di bsbdufst ubl f ui f gpsn ‘63- 64^;

$$\chi)\mathcal{G}_\lambda=\tau+? \; \frac{q^{\frac{1}{2})\lambda-\frac{\alpha_0}{2}+^2}}{\eta)q+} \tag{6/7}$$

$$\chi)M_{2,s}=\tau+? \; \frac{2}{\eta)q+}\prod_{n\geq 1}\Big)q^{\frac{1}{4p})3pn-\,p-s+^2} \quad q^{\frac{1}{4p})3pn-\,p-\,s+^2}\Big(? \; \frac{\Omega_{p,p-s})\tau+}{\eta)\tau+} \tag{6/8}$$

Cfgpsf x f jefoujgz ui ftf di bsbdufst x jui 4e.3e joejdft)3/2+boe q.tfsjft jowbsjbout pg 4.n bojgpmet- x f ti pvmn qpjou pvu ui bu- gpmpx joh ‘5- 7^- ui spvhi pvu ui f qbqfs x f tvqqsftt[25] ui f gbdups pg)q=q+∞)dg/)8/8++jo ui f qi ztjdbmjoefy[26]

$$\widehat{Z_a^{)\mathrm{wosfe+}}})q+ \; ? \; \frac{\widehat{Z_a)}q+}{)q=q+\infty} \tag{6/9}$$

boe jotufbe vtf $\widehat{Z_a)}q+$ x i jdi pguo ubl ft b n psf dpn qbdu gpsn / Ui jt qi ztjdbmwfstjpo)6/9+ pg ui f joefy $\widehat{Z_a)}q+$jt tpn fujn ft dbmmfe *unreduced* ps *un-normalized*/

Ubl joh joup bddpvou ui jt opsn brja-bujpo- x f dbo sfqi sbtf pvs ejtdvttjpo jo tfdujpo 5- jo qbsujdvmbs)5/38+ cz tbzjoh ui bu jo ui f psjft x i sfsf ui f opsn brja-fe joefy

$$\widehat{Z_a)}q+ \; ? \quad \Omega_{p,s_1})\tau+, \quad \Omega_{p,s_2})\tau+, \quad \ldots \tag{6/:}$$

jt hjwfo cz b njofbs dpn cjobujpo pg gbnf ui fub gvodujpot)5/7+ x f i bwf- dg/ ubcmf 2;

$$\widehat{Z_a^{)\mathrm{wosfe+}}})q+ \; ? \quad \chi)M_{2,p-s_1}\otimes M_{2,p-s_2}\otimes\ldots=\tau+ \tag{6/21}$$

[14] Cf. (2.8) where this factor is, in fact, present.

[15] For more general gauge groups, this relation involves a factor of $\eta(q)^{\mathrm{rank}(G)}$ in the denominator [4].

JHEP10(2019)010

3-manifold	m , K	*module of a singlet log-VOA*
$\Upsilon)3,4,6+$	41 , 7, 21, 26	$M_{1,1} \otimes M_{1,11} \otimes M_{1,19} \otimes M_{1,29}$
$\Upsilon)3,4,8+$	53 , 7, 25, 32	$M_{1,1} \otimes M_{1,13} \otimes M_{1,29} \otimes M_{1,41}$

Table 3/ Yfjnsfqsftfoubujpot boe uif dpssftqpoejoh n pevrfit pguif nhbsjui n jd)2, p+tjohrfu DGU gps tjn qrfi ipn pnphz tqifsft/ Uif tvn pwfs n pevrfit jt qsfdjtfmz uif tvn pwfs frfin fout pguif psui phpobmhspvq O_m jouspevdfe bcpwf)4/41+/

Jo puif s xpset- uif qspqfsmz opsn bnj-fe qiztjdbmjoefy)6/9+jt frvbmup uif di bsbdufs pgb)2, p+tjohrfu WPB n pevrfi

$$M_{2,p-s_1} \otimes M_{2,p-s_2} \otimes \dots \qquad)6/22+$$

Opuf- bmi pvhi uijt n pevrfi nppl t sfevdjcrfi- qfsi bqt ju joejdbuft fyjtufodf pgbo fyufotjpo up b nhshfs nph.WPB- xifsf $\widehat{Z_a}$)wosfe+)q+dbo cf jefoujflfe xjui b di bsbdufs pgb rfitt sfevdjcrfi n pevrfi/ B qptjujwf joejdbujpo gps uijt dpn ft gspn uif gbdu uibu- jo n boz pgpvs fybn qrfit- fwfsz ufsn $M_{2,s}$ jt bmxbzt bddpn qbojfe cz $M_{2,p-s}$ jo)6/22+/

Jo qbsujdvrhs- xifo dpn qptfe xjui 4e.4e dpssftqpoefodf- uijt jousjhvjoh evbijuz cf. uxffo nhbsjui n jd DGUt boe 4e $\mathcal{O}$? 3 uifpsjft xjui ibrgCQT cpvoebsz dpoejujpot jn . qujft uibubmflfjgfsun bojgpmet xjui 4 tjohvrhs flcfst dpssftqpoe up n pevrfit pg)2, p+tjohrfu n pefm/ Uif n pevrfit bsf efufsn jofe cz uif ebub pguif Yfjnsfqsftfoubujpo dpssftqpoejoh up M_4- xijdi - jo uvso- dbo cf pcubjofe vtjoh uif hfofsbmufdi ojrvf pvurjofe jo tfdujpo 5/ Juxjmmcf jmmatusbufe jo n boz fybn qrfit jo tfdujpo 7/

Juxpvmn cf jousftujoh up tuvez b sfmbujpo cfuxffo nhbsjui n jd WPBt bttjhofe up 4. n bojgpmet ifsf- boe wfsufy brhfcsbt WPB‘M_5ˆ bttjhofe up 5.n bojgpmet cpvoefe cz tvdi 4.n bojgpmet wjb uif evbijuz ‘23- 26- 65ˆ/ Bopuifs obuvsbmrvftujpo jt; gps xijdi dhbtt pg 4e $\mathcal{O}$? 3 uifpsjft)boe cpvoebsz dpoejujpot $\mathcal{D}_a$+uif dpn cjofe 4e.3e ibrgjoejdft)3/2+ qspevdf di bsbdufst pgnhbsjui n jd DGUtA Boe- dpowfstfmz- xijdi nhbsjui n jd DGUt bsjtf jo uijt dpssftqpoefodfA Yfi pqf up fyqmpsf uiftf rvftujpot jo uif gvuvsf xpsl /

Ifsf boe jo tfdujpo 4/3- xf gpvoe tfwfsbmdpoofdujpot sfmbujoh 4.n bojgpmet jowbsjbout $\widehat{Z_a})q$+xjui nhbsjui n jd DGUt boe opo.tfn jtjn qrfi N UDt/ Po uif puifs i boe- jo b qbsbmfmm njof pgefwfmpqn fou-]nhbsjui n jd#4.n bojgpmet jowbsjbout cbtfe po opo.tfn jtjn qrfi N UDt xfsf tuvejfe jo ‘66–68ˆ xijdi - uifsfgpsf- xf fyqfdu up cf sfmbufe up $\widehat{Z_a})q$+/ Yf qmbo up qvstvf uijt ejsfdujpo jo uif gvuvsf xpsl /

5.2 Hyperbolic M_3 and non-C_2-cofinite log-VOAs

Bnsfbez bu uijt fbsmz tubhf- uif dpoofdujpot cfuxffo 4.n bojgpmet boe nhbsjui n jd DGUt dbo ufbdi vt b wbmvbcrfi rfittpo/ Obn fmz- uifz dbo i fmq vt voefstuboe uif botxfs up uif gpmpxjoh jn qpsubou rvftujpo; xi bu jt ju bcpvu 4.n bojgpmet xi ptf jowbsjbout $\widehat{Z_a})q$+dbo cf fyqsfttfe jo ufsn t pggbnf uif fub gvodujpot boe n pdl n pevrhs gpsn t- bt pqqptfe up n psf dpn qmjdbufe n pevrhs pckfdutA

Jgxf dpn cjof tfwfsbmdhuft gspn uif bcpwf- uif botxfs tffn t up cf usjhhfsfe cz xifuifs uif dpssftqpoejoh nph.WPB jt C_3.dpflojuf ps opu- boe xifuifs b 4.n bojgpmet M_4 ben jut pomn

$G_{\mathbb{C}}$ ffbu dpoofdujpot x jui sbujpobmwbmvft pg DT)α+

$$\text{DT})\alpha+ \;/\; \mathbb{Q} \qquad \text{gps bmm}\alpha \;/\; \mathcal{N}\text{ -bu})G_{\mathbb{C}}, M_4+. \qquad (6.23+$$

Joeffe- boujdjqbujoh b dmptf sfmbujpo cfuxffo NUD"M_4^ eftdsjcfe jo tfdujpo 4 boe uif ufotps dbufhpsz pg b nph.WPB bttpdjbufe up M_4 wjb uif ejdujpobsz tvn n bsj–fe jo ubcmf 2- xf fyqfdu uibu n pevmft pg uif mbuufs i bwf dpogpsn bmejn fotjpot Θ_α sfmbufe up wbmvft pg DT)α+bt jo)4/6+/

Uif fo- jg dpoejujpo)6/23+gpmet tpn f α- jun fbot ui buuif dpssftqpoejoh nphbsjuin jd DGU n vtu i bwf bu mfbtu tpn f sfqsftfoubujpot xjui jssbujpobmdpogpsn bmejn fotjpot Θ_α- boe tvdi wfsufy brhfcsbt dbo opu cf C_3.dpflojuf/[27] Joeffe- Njzbn pup qspwfe '69^)tff '6: ^ gps b nvdje sfwjfx +ui bu wbmvft pg dpogpsn bmejn fotjpot boe uif dfousbmdi bshf jo b C_3. dpflojuf WPB n vtu bmmcf sbujpobm/Dvsjpvtmz- uif dpoejujpo)6/23+i pmet gps bmmfybn qrfit pg 4.n bojgpmet dpotjefsfe jo uijt qbqfs- x i jdi jt qspcbcmz x i z jo bmmdbtft xf floe b sfmbujpo up C_3.dpflojuf nph.WPBt/

Po uif puifs i boe- i zqfscpjid 4.n bojgpmet i bwf bu mfbtu pof TM)3, $\mathbb{C}$+ffbu dpoofdujpo $\alpha_{i\,f\,p\,n}$ ~ tpn fujn ft dbmfe] hfpn fusjd#ps] i zqfscpjid#~ boe jut dpn qrfy dpokvhbuf- tvdi ui bu Jn DT)$\alpha_{i\,f\,p\,n}$ +? 1/ Uijt ofdfttbsjmz wjpmbuft uif dpoejujpo)6/23+boe- cbtfe po uif bc pwf dpotjefsbujpot- xf fyqfdu i zqfscpjid 4.n bojgpmet up cf sfmbufe up nphbsjuin jd wfsufy brhfcsbt x i jdi bsf *not* C_3.dpflojuf/ Jo qbsujdvmbs- uijt tvhhftut x i bu pof ti pvmn fyqfdu pg uif q.tfsjft jowbsjbout $\widehat{Z_a)}M_4$+gps i zqfscpjid M_4- bttvn joh uif sfmbujpo cfuxffo 4.n bojgpmet jowbsjbout $\widehat{Z_a)}M_4$+boe di bsbdufst pg nphbsjuin jd WPBt dpoujovft up i pmet jo uif i zqfscpjid dbtf bt xfmm[28]

6 Examples

Jo uif flstu qbsu pg uijt tfdujpo- xf bobmzf uif efflojujpo pg uif i pn pmphjdbmcnpdl t qspwjefe jo '7^ gps qvvn cfe 4.n bojgpmet boe ti px ui bu uifjs dpowfshfodf pomz efqfoet po uif tjho pg uif ejbhpobmfousjft pg M^{-2} dpssftqpoejoh up i jhi .wbmvfodz wfsujdft)wfsujdft xjui n psf ui bo ux p fehft jodjefou up uifn - efh $v > 3$+/

Uijt fobcmft vt up fyufoe uif efflojujpo pg uif q.tfsjft jowbsjbout $\widehat{Z_b)}q$+up b xjefs sbohf pg qvvn cfe 4.n bojgpmet- jodmvejoh uiptf xjui joefflojuf qvvn cjoht sfmbufe up uif ofhbujwf.efflojuf poft wjb Ljscz n pwft/ Gps qptjujwf.efflojuf qvvn cjoht boe uifjs Ljscz. fr vjwbmfhout- b ofx qspdfevsf jt qspqptfe jo tfdujpo 8 up efflof uif dpssftqpoejoh q.tfsjft jowbsjbout $\widehat{Z_b)}q$+/

Jo uif tfdpoe qbsu pg uijt tfdujpo- xf fyqmjdjumz dpn qvuf uif ofx jowbsjbout $\widehat{Z_b)}q$+gps tpn f fybn qrfit pg Tfjgfsu n bojgpmet xjui uisff tjohvmbs flcfst/ Jo beejujpo- xf fybn jof uif qspqfsujft pg uiftf n bojgpmet uispvhi uif n pevmbs qfstqfdujwf pvuijofe jo tfdujpo 5/ Jo qbsujdvmbs- xf qspwjef btzn qupujd fyqbotjpot pg Y SU jowbsjbout)ps frvjwbmfhoum- uif

[16] Among other things, the C_2-cofiniteness means that VOA has finitely many inequivalent irreducible modules [60]. See also [61] for a nice exposition and various ways to understand this condition.

[17] If it indeed passes further tests, the condition (5.12) perhaps deserves the name "C_2-cofiniteness for 3-manifolds."

JHEP10(2019)010

usbottfsjft fyqbotjpot pg Di fso.Tjn pot qbsujujpo gvodujpot bt jo tfdujpo 5+gps tfrhdufe fybn qrfft;

$$Z_{\mathrm{EV}})M_4+\subset \prod_{\alpha} e^{3\pi ik\mathrm{EV})\alpha+}Z_{\mathrm{r\,fsu}}^{)\alpha+})k+$$

x i fsf α svot pwfs *all* fffbu dpoofdujpot po M_4/ Jo ui f bcpwf gpsn vnb- $Z_{\mathrm{r\,fsu}}^{)\alpha+}$ x jmmcf sfgfssfe up bt ui f *transseries* pguif tbeefi qpjou α)j/f/ ffbu dpoofdujpo $\alpha+$/

6.1 Definite and indefinite plumbings

Hjwfo b qmvn cjoh hsbqi x jui gsbn joh dpf´ djfout a_i / $\mathbb{Z}$- ui fsf jt bo bttpdjbufe tvshfsz njol- tff flhvsf 7/ Qfsgpsn joh tvshfsz bmpoh ui f njol- xf pcubjo b] qmvn cfe#n bojgpme M_4 ‘73ˆ/ Jo qbsujdvmbs- bmmIfjgfsu n bojgpmet M_4 ? $M)b=\}q_i/p_i\langle_i+$bsf qmvn cfe n bojgpmet- cfdbvtf ui f sbujpobmtvshfsz dpf´ djfout dbo cf sfbujfe cz b tfsjft pg4e Ljscz n pwft boe dpoujovfe gsbdujpo;[29]

$$\underset{\equiv}{p/q} \;?\; \underset{\equiv}{a_2}\!\!-\!\!- \underset{\equiv}{a_3}\!\!-\!\!- \underset{\equiv}{a_4} \times\times\times \quad \mathrm{x\,i\,fsf} \quad \frac{q}{p} \;?\; \cfrac{2}{a_2-\cfrac{2}{a_3-\cfrac{2}{a_4-\times\times\times}}}. \qquad)7/2+$$

Boz Tfjgfsu n bojgpme M_4 ? $M)b=\}q_i/p_i\langle_i+$i bt b qmvn cjoh qsftfoubujpo jnvtusbufe jo flh. vsf 8/ Tvdi b qmvn cjoh hsbqi i bt pomz pof ijhi .wbnfodz wfsufy boe ui f sbujpobmtvshfsjft bmpoh flcfst bsf sfbujfe cz dpoujovfe gsbdujpot- bt jo)7/2+/

B usff.ti bqfe hsbqi x jui L wfsujdft i bt b $L*L$ bekbdfodz n busjy;

$$M_{ij} \;?\; \begin{cases} a_i & \mathrm{jg}\quad i\;?\;j \\ 2 & \mathrm{jg}\quad)i,j+/\ \mathrm{Fehft} \\ 1 & \mathrm{pui\ fsx\ jtf-} \end{cases}$$

x i jdi jt qsfdjtfmz ui f njol joh gpsn jo)3/23+/ Uphfui fs x jui)3/22+ boe)3/23+ xf dbo dpn qvuf ui f i bnjmupojbo)3/4+jo ui f gpmpx joh gpsn ;[2;]

$$\widehat{Z_b)q+}? \; q^{-\frac{3L+\sum_v a_v}{4}} \times \mathrm{w.q.} \int_{|z_v|\mathrm{C}2} \prod_{v\in[.\mathrm{fsukdft}} \frac{dz_v}{3\pi i z_v})z_v \quad 2/z_v+^{3-\mathrm{efi}_v}$$
$$* \prod_{\ell\in 3M\mathbb{Z}^L\text{-}\ b} q^{-\frac{(\ell,M^{-1}\ell)}{4}} \int_{v\in[.\mathrm{fsukdft}} z_v^{\ell_v}. \qquad)7/3+$$

I fsf- w.q. joejdbuft ui bux f bsf qfsgpsn joh b qsjodjqbmwbnaf joufhsbmaboe b / 3Dpl fs)$M+$, δ n pevmp Yfzmhspvq bdujpo $b \Rightarrow \ b/$ Bmi pvhi xf i bwf di ptfo δ / $\mathbb{Z}^L$ tvdi ui bu $\delta_v \le$ efh$_v$ n pe 3- ejfifsfou di pjdft pg δ x pvmn pomz qfsn vuf ui f i pn pmphjdbmcmpdl t pg b hjwfo qmvn cfe n bojgpme/

[18]The orientation convention is such that Poincare homology sphere is represented by a $-E_8$ plumbing graph, i.e., $M(-2;\frac{1}{2},\frac{2}{3},\frac{4}{5})$.

[19]For convenience, we have chosen M to be negative-definite. The condition can be relaxed, which is an interesting topic from the viewpoint of “going to the other side.” We will come back to this in section 7. Also, to avoid clutter, we write (6.2) for $g=0$ Seifert manifolds; a more general expression for arbitrary genus g involves the combination of (2.10) and (2.11) as the integrand.

JHEP10(2019)010

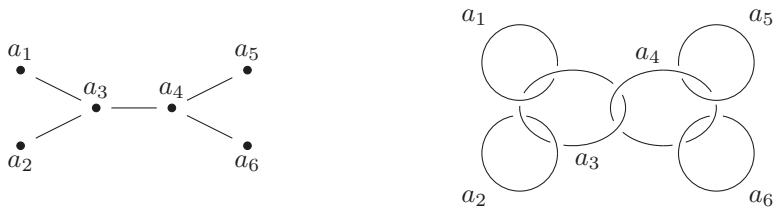


Figure 6/ B qmun cjoh hsbqi)nfgu+boe ui f bttpdjbufe tvshfsz njol)sjhi u+/

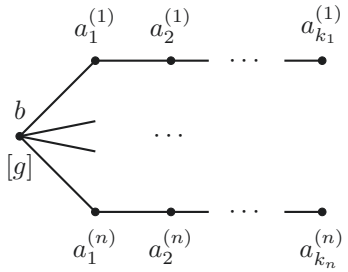


Figure 7/ Qmun cjoh hsbqi gps b Tfjgfsu n bojgpme M $b, g=\} \frac{q_i}{p_i} \langle \frac{n}{i=1} \langle /$

Bddpsejoh up '7^- b qbsujdvimhs dpn cjobujpo pgui f ipn pmphjdbmcmpdl t hjwft TV)3+Di fso. Tjn pot qbsujujpo gvodujpo po M_4 jo ui f sbejbmnjn ju $\langle q \rangle \leftrightarrow$ 2- dg/)4/8+/ Tqfdjfidbmmz- gps qmun cfe n bojgpmet x f i bwf

$$
\begin{aligned}
Z_{\mathrm{EV}})M_4+? \ & \frac{2}{3i^3\sqrt{3k}} \prod_a e^{3\pi i k \mathrm{EV})a+} \prod_b S_{ab}^{)A+\widehat{Z_b})q+}, \\
& a \ / \ \mathrm{Dpl\,fs})M+/\mathbb{Z}_3 \quad \overset{\mathrm{tfuxktf}}{?} \quad \mathrm{Ups}H_2)M_4+/\mathbb{Z}_3, \\
& b \ / \ )3\mathrm{Dpl\,fs})M+, \quad \delta+/\mathbb{Z}_3, \quad \mathrm{DT})a+? \quad)a, M^{-2}a+ \ \mathrm{n\,pe}\ \mathbb{Z},
\end{aligned}
\qquad)7/4+
$$

$$
\text{boe} \quad S_{ab}^{)A+} ? \ \frac{\prod_{a' \in \{\mathbb{Z}_2 0 \mathrm{psc\,ku\,pg}\, a\}} e^{3\pi i)a', M^{-1}b+}}{\sqrt{\langle \mathrm{Ups}H_2)M_4+ \langle}}
$$

xifsf ui f $\mathbb{Z}_3$ bdut bt ui f Yfzmhspvq po $H_2)M_4+$cz $a \Rightarrow \ a/$

Y i fo ui f qmun cjoh hsbqi jt dpn qptfe pg b tjohmf i jhi .wbmfodz wfsufy- j/f/- x i fo M_4 jt b Tfjgfsu flcfsfe n bojgpme- ui f gpmpxjoh ui fpsfn efufsn joft x i fui fs ui f i pn pmphjdbm cmpdl t pg M_4 effflofe wjb fr vbujpo)7/3+qspwjef dpowfshfou q.tfsjft jotjef ui f voju ejtl boe x i fo ui fz dpowfshf pvutjef ui f voju ejtl /

Lemma 1. *Take M_4 to be a plumbed 3-manifold, whose plumbing graph $\mathcal{H}$ is a tree. Denote by M the adjacency matrix of $\mathcal{H}$ and by M^{-2} its inverse. Assume there is only one high-valency vertex and let v_1 denote the entry associated to this vertex in the adjacency matrix. Then, if $)M^{-2}+_{v_0 v_0} < 1$ the homological blocks associated to M_4 are well-defined q-series, convergent for $\langle q \rangle < 2$. On the other hand, if $)M^{-2}+_{v_0 v_0} > 1$, the homological blocks converge for $\langle q \rangle > 2$.*

More generally, when there are multiple high-valency vertices, let $\}v_i\langle$ be the set of high-valency vertices in the plumbing graph of M_4. The homological blocks converge for $\langle q \rangle < 2$ (respectively $\langle q \rangle > 2$) when all $)M^{-2}+_{v_i v_i} < 1$ (respectively > 1) for all v_i's.

Proof. Up qspwf uif bcpwf uifpsfn xf i bwf upbobmzf uif btzn qupujdhspxui pguif gpsn vnb gps uif i pn pmphjdbmcmpdlt/ Gspn fr vbujpo)7/3+xf i bwf "7^

$$\widehat{Z_b)}q+? \ 3^{-L}q^{\mathfrak{fl}} \prod_{\ell\in 3M\mathbb{Z}^L\text{-}\ b} F_2^{\ell}\, q^{-\frac{(\ell,M^{-1}\ell)}{4}}, \quad b\ /\ )3\mathrm{Dplfs}M\ , \ \delta+/\mathbb{Z}_3 \qquad)7/5+$$

xifsf uif joufhfs dpf´djfout F_2^{ℓ} bsf hfofsbufe bt gpmpxt)opuf $3M\mathbb{Z}^L$, $b\leftarrow 3\mathbb{Z}^L$, $\delta+$;

$$\prod_{\ell\in 3\mathbb{Z}^L\text{-}\ \delta} F_2^{\ell} \int_v x_v^{\ell_v} \ ? \ \int_v \Big\}_{\mathrm{L\ rbotkpo}\atop \mathrm{bu}\ x_v\to 1} \frac{2}{)x_v \quad 2/x_v+^{\mathrm{efi}\ v-3}}\ , \ _{\mathrm{L\ rbotkpo}\atop \mathrm{bu}\ x_v\to\infty} \frac{2}{)x_v \quad 2/x_v+^{\mathrm{efi}\ v-3}} \Bigg(. \qquad)7/6+$$

Yifo efh $v\geq 3$- uif x_v.fyqbotjpo po uif s/i/t/ ufsn jobufut bu b flojuf psefs/ Uivt- F_2^{ℓ} wbojtift xifo \ℓv\jt nhshf fopvhi gps bmmcvupof dppsejobuft ℓ_v pg L.ejnfotjpobmwfdupst $\ell\ /\ 3\mathbb{Z}^L$, $\delta/$

Uif pomz fydfqujpo jt ℓ_{v_0} xijdi dpssftqpoet up uif vojrvf ijhi.wbnfodz wfsufy v_1)efh $v_1>3$/+Fyqmjdjumz-

$$F_2^{\ell}\ i?\ 1 \quad \infty \quad \ell_v\ ? \begin{cases} \ell_{v_0} & \mathrm{jg}\ v\ ?\ v_1,\ \ell_{v_0}\ /\ \mathbb{Z} \\ 1 & \mathrm{jg\ efh}\ v\ ?\ 3 \\ 2 & \mathrm{jg\ efh}\ v\ ?\ 2. \end{cases}$$

Efhsff.fsp wfsujdft bsf jssfnfwbou bt xf pomz dpotjefs dpoofdufe hsbqit/ Uijt jn qnjft uibu uif q.fyqpofout jo uif s/i/t/pg)7/5+ibwf uif gpmpxjoh cfi bwjps;

$$q^{-\frac{(\ell,M^{-1}\ell)}{4}}\ ?\ q^{-\frac{(M^{-1})_{v_0v_0}(\ell_{v_0})^2}{4}}\text{-}\ O)2+ \qquad)7/7+$$

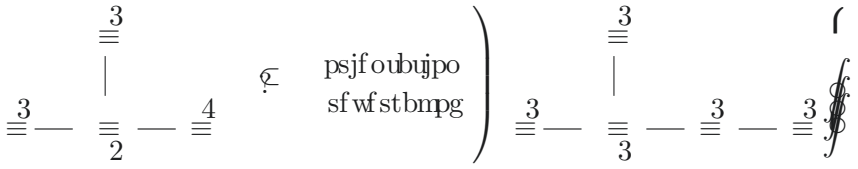
bt \ℓ\↔∈ boe jg F_2^{ℓ} i? 1/ Uijt dpn qnfuft uif qsppg gps uif flstu qbsu/ Uif qsppg qspdffet jo bo jefoujdbmcz up uif qnmn cjoh hsbqit xjui n vmjqnfijhi.wbnfodz wfsujdft/ □

Uif bcpwf sftvmt ipxt uibu uif wbrjejuz pg)7/3+efqfoet tpnfmn po uif M^{-2} fousjft bu ijhi.wbnfodz wfsujdft/ Mfuvt n blf b gfx sfn bslt;

≡ *orientation reversal.* Jujt jn qpsubou up opuf uibu uif i pn pmphjdbmcmpdlt jo nfn n b 2 bsf dpn qvufe gspn)7/5+ xijdi jt b sftvmpgb qbsujdvnbs sfhvnsj–bujpo)tff bqqfoejy B pg "7^+/ Uifsfgpsf- xifo uif gpsn vnb epft opu efflof b dpowfshfou q.tfsjft jotjef uif vojuejtd- bo bmfsobujwf dpn qvubujpobmtdifn f jt sfrvjsfe/ Jo qbsujdvnbs- hjwfo b 4.n bojgpme M_4)$\mathcal{H}$+ uif pqqptjufmz psjfoufe n bojgpme $-M_4$)$\mathcal{H}$+qspwjeft uif obuvsbm dpn qbojpo pg M_4)$\mathcal{H}$+po uif puifs tjef pguif q.qnbof/ Jguif gpsn fsibt $)M^{-2}+_{v_0v_0}<1$- uif nbuufs ibt $)M^{-2}+_{v_0v_0}>1$ boe)7/5+dboopu cf jn qnfn foufe up sfqspevdf uif bttpdjbufe ipn pmphjdbmcmpdlt xijdi bsf dpowfshfou gps \q\ < 2/ Uifsfgpsf- xifo $)M^{-2}+_{v_0v_0}>1$ xf offe up fyufoe uif efflojujpo pg ipn pmphjdbmcmpdlt pvutjef uif vojuejtd/ Uijt xjmmcf uif dfousbmupqjd pg tfdujpo 8- xifsf xf dpokfduvsf b ofx qspdfevsf up efsjwf uif i pn pmphjdbmcmpdlt pguisff n bojgpmet xjui $)M^{-2}+_{v_0v_0}>1$/

≡ *Kirby moves.* Uif tjhobuvsf pguif qnmn cjoh ebub n bz opu cf jowbsjbou voefs 4e Ljscz n pwft/ Bo fybn qnfjt jmmtusbufe jo)7/8+xifsf uxp i pn fpn psqijd n bojgpmet

i bwf ejfifsfoutjhobuvsft; uif n/t/jt ofjuifs qptjujwf ops ofhbujwf efflojuf- xijmf uif s/t/jt qptjujwf.efflojuf/



)7/8+

$M)\ 2=\frac{2}{3},\frac{2}{3},\frac{2}{4}+\ \subsetneq\ M)\ 3=\frac{2}{3},\frac{2}{3},\frac{3}{4}+$

Y jui pvumn n b 2- jujt ofdfttbsz up floe gps fbdi joefflojuf n bojgpmn b i pn fpn psqi jd- efflojuf n bojgpmn up efufsn jof uif dpowfshfodf pg i pn pmphjdbmcnpdlt/ Jo uif bcpwf fybn qmf- gps jotubodf- uif s/t/jt qptjujwf.efflojuf- tp uif i pn pmphjdbmcnpdlt pg $M)\ 2=\frac{2}{3},\frac{2}{3},\frac{2}{4}+$ xpvmn dpowfshf *outside* uif voju ejtd/ Cz uif mfn n b- i pxfwfs- uif epn bjo pg dpowfshfodf dbo cf jn n fejbufmz sfbe pfi gspn M^{-2} pg uif n/t/)boe pg dpvstf- uifz dpowfshf gps $\backslash q\backslash > 2+$/ Jujt bmtp fbtz up tff i px 4e Ljscz n pwft qsftfswf uif epn bjo pg dpowfshfodf- bt qspwjefe jo bqqfoejy B/

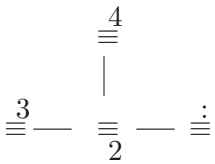
≡ *multiple high-valency vertices.* mfn n b 2 tubuft ui bu)7/5+epft opu sfqspevdf dpowfs. hfou q.tfsjft xifo uifsf bqqfbst n vmjqmf i jhi.wbmfodz wfsujdft xi ptf M^{-2} fousjft i bwf ejfifsfoutjhot/ Ui jt jn qmjft ui bu i pn pmphjdbmcnpdlt)7/3+n vtucf dpn qvufe cz pui fs n fbot ui bo uif sfhvmbsj–bujpo tdifn f)7/5+/ Y f xjmmsfuvso up uiftf fybn qmft jo gvuvsf xpsl/

6.2 Example: $M(-1;\frac{1}{2},\frac{1}{3},\frac{1}{9})$

Y f flstu efn potusbuf uif n pevmbsjuz ejdujpobsz boe uif tufqt pvumjofe jo flhvsf 6 xjui uif tqfdjfld fybn qmf pg b Tfjgfsu n bojgpmn $M)\ 2=\frac{2}{3},\frac{2}{4},\frac{2}{5}+$/

6.2.1 q-series invariants

Ui f Tfjgfsu n bojgpmn i bt UpsH_2)$M)\ 2=\frac{2}{3},\frac{2}{4},\frac{2}{5}++?\ \mathbb{Z}_4$ boe uif gpmpxjoh qmmn cjoh hsbqi ;



)7/9+

Up dpn qvuf jut q.tfsjft jowbsjbout $\widehat{Z_a})q+$ xf flstu xsjuf epxo jut belbdfodz n busjy;

$$M\ ?\ \begin{pmatrix} 2 & 2 & 2 & 2 \\ 2 & 3 & 1 & 1 \\ 2 & 1 & 4 & 1 \\ 2 & 1 & 1 & : \end{pmatrix}.$$

Bt jt xfmm opxo)tff f/h/'2: ^+ uif dpmfsofmpg M jt jtpn psqi jd up UpsH_2)$M)\ 2=\frac{2}{3},\frac{2}{4},\frac{2}{5}++$;

Dpmfs)M+?\ $\mathbb{Z}^5/M\mathbb{Z}^5$?\ $\rangle$)1, 1, 1, 1+,)2, 1, 2, 7+,)2, 1, 3, 4+

$\subsetneq$ UpsH_2)$M)\ 2=\frac{2}{3},\frac{2}{4},\frac{2}{5}++?\ \mathbb{Z}_4$.)7/:+

Ui f Y fzmhspvq bdujpo n bqt b dpl fsofmfrfn fou up jut tjho jowfstf/ Ui fsfgpsf- ui f flstu frfn fou-)1, 1, 1, 1+ jt n bqqfe up jutfmg- x i jrfi ui f puifst bsf dpokvhbuf up fbdi puifs- j/f/-)2, 1, 2, 7+?)2, 1, 3, 4+/ $\mathbb{Z}^5/M\mathbb{Z}^5$. Ui ivt-

$$\text{Dpl fs})M+/\mathbb{Z}_3 ? \rangle)1,1,1,1+,)2,1,2,7+| \qquad)7/21+$$

$$)3\text{Dpl fs})M+, \delta+/\mathbb{Z}_3 ? \rangle)2,2,2,2+,)4,2,4,24+| \qquad)7/22+$$

x i fsf δ ?)2, 2, 2, 2+jt hjwfo cz δ_v ? efh_v 3- bt jo ‘7^/ Ui fo- ui f q.tfsjft jowbsjbout $\widehat{Z_b})M_4+$bsf hjwfo cz)7/3+;

$$\widehat{Z_{)2,-2,-2,-2+}})q+? \ q\,,\ q^6\ \ q^7\ \ q^{29}\,,\ q^{31}\,,\ \ldots \qquad)7/23+$$

$$\widehat{Z_{)4,-2,-4,-24+}})q+? \ \ q^{5/4})2\,,\ q^3\ \ q^8\ \ q^{24}\,,\ q^{34}\,,\ \ldots+. \qquad)7/24+$$

6.2.2 Weil representation: 18+9

Up i pn pnphjdbmcnpdl t pg M_4 pof dbo bttpdjbuf b Y fjmsfqsftfoubujpo- nbcfrnfe cz ui f qbjs m boe K/ Fyqrjdjumz- ui fz bsf sfnbufe wjb)5/38+/ Mfu vt flstu efufsn jof m wjb n pevmbsjuz ejdujpobsz/ S fdbmmui buui f opo.qfsuvscbujwf qbsupguif usbottfsjft)5/31+gps Ω_r^{m-K}- pguif gpsn $\subset e^{-3\pi ik)r'+^2/5m}$- ti pvmn dbquvsf ui f dpousjcvujpot gspn opo.bcfrnjbo ffbu dpoofdujpot/ Gps b Tfjgfsu n bojgpme $M)b,\}q_i/p_i\langle_{iC2}^n+$ ui f efopn jobupst pg DT)a+gps a opo.bcfrnjbo jt b n/d/n / pg $5p_i$- x i fsf p_i bsf ui f psefst pg tjohvmbs flcfst jo ui f Tfjgfsu jowbsjbou ‘74^/ Bt b sftvmt- x f dmbjn ui bu gps M_4 ? $M)b,\}q_i/p_i\langle_{iC2}^n+$x f i bwf

$$5m ? \ \text{mdn}.\Big)5\}p_i\langle_{iC2}^n \cap \}\text{Efopn jobupst pg DT})a+\langle_{1\not\subset a\in\text{Epnfs}M/\mathbb{Z}_2}\Big(. \qquad)7/25+$$

Gps ui f dvssfou fybn qrfi- x f dbo fbtjmz dpn qvuf DT)a+gps bcfrnjbo ffbu dpoofdujpot gspn ui f dpl fsofmfrfn fout dpn qvufe jo 7/21;

$$\text{DT})a+? \quad)a, M^{-2}a+? \ \begin{cases} 1 & \text{n pe } \mathbb{Z} \text{ gps } a ?)1,1,1,1+ \\ \frac{2}{4} & \text{n pe } \mathbb{Z} \text{ gps } a ?)2,1,2,7+ \end{cases}$$

Dpn cjojoh)7/25+boe ui f DT)a+dpn qvufe- x f dpodmvef;

$$5m ? \ \text{mdn}.)9,23,47,4+? \ 83 \quad \Leftrightarrow \quad m ? \ 29.$$

Dpssftqpoejohmz- ui f qpttjcrfi K hjwjoh sjtf up jssfevdjcrfi sfqsftfoubujpot bsf K ? $\}2,3\langle$ boe K ? $\}2,:\langle$/ B tjn qrfi dbndvmbujpo sfwfbmt ui bu ui f sfrfwbou jssfevdjcrfi sfqsftfoubujpo jt ui bu nbcfrnfe cz m, K ? 29, :/

Jo tvn n bsz- x f i bwf

$$\begin{aligned} \sigma^{29-;} ? \ \}2,4,6,8\langle \\ \widehat{Z_{)2,-2,-2,-2+}})q+? \ q^{82/83}\Omega_2^{29-;})\tau+ \\ \widehat{Z_{)4,-2,-4,-24+}})q+? \ \ q^{82/83}\Omega_6^{29-;})\tau+. \end{aligned} \qquad)7/26+$$

Ofyu- x f qspdffe up dpn qvuf ui f dpn qptjuf n busjdft $S)M_4+$ boe $T)M_4+$ effflofe jo)5/3: +/

6.2.3 Computing $S(M_3)$ and $T(M_3)$

Mfu vt xsjuf epxo bmmuif sfnfwbou n busjdft gps uif dvssfou fybn qnfh/ Gjstu- sfdbmmui bu $S^{)A+}$ jt uif njol joh qbjsjoh po UpsH_2)M_4+jo)7/4+/ Gps M_4 ? M) $2=\frac{2}{3}, \frac{2}{4}, \frac{2}{;}$+

$$S^{)A+}? \ \S\frac{2}{3}\frac{}{4}\Big)\begin{matrix}2 & 2\\ 3 & 2\end{matrix}\Big[. \qquad)7/27+$$

Ofyu- gspn)5/38+boe)7/26+xf dbo fbtjmz sfbe pfi;

$$\mathbf{Emb}\ ? \ \Big)\begin{matrix}2 & 1 & 1 & 1\\ 1 & 1 & 2 & 1\end{matrix}\Big[. \qquad)7/28+$$

Ui f T.n busjy pguif Y fjmnsfqsftfoubujpo jt fbtjmz dpn qvufe gspn up cf;

$$S^{)B+}? \quad \frac{3i}{4}\Big)\begin{matrix}A & \frac{4}{3} & B & C\\ \frac{2}{3} & 1 & \frac{2}{3} & \frac{2}{3}\\ B & \frac{4}{3} & C & A\\ C & \frac{4}{3} & A & B\end{matrix}\Big\{ \qquad)7/29+$$

xifsf A, B, C ? tjo)$\frac{\pi}{29}$+,tjo)$\frac{6\pi}{29}$+,tjo)$\frac{8\pi}{29}$+sftqfdujwfmz/

Gjobmmz xf dpn cjof $S^{)A+}$,**Emb** boe $S^{)B+}$joup $S)M_4$+;

$$S)M_4+? \ \Big)\begin{matrix}1.34i & 1 & 1.77i & 1.54i\\ 1.54i & 2.84i & 1.34i & 1.77i\end{matrix}\Big[, \qquad)7/2:+$$

ifsf fwbmvbufe ovn fsjdbmmz boe spvoefe up uif tfdpoe efdjn bmqmbdf/

Ofyu- xf dpn qvuf uif T n busjdft/ $T^{)A+}$jt uif ejbhpobmn busjy xjui $e^{3\pi i\mathrm{EV})a+}$po uif ejbhpobmn,

$$T^{)A+}? \ \mathrm{fyq}\ 3\pi i\Big)\begin{matrix}1 & 1\\ 1 & \frac{2}{4}\end{matrix}\Big[. \qquad)7/31+$$

Gspn)4/49+ xf bmtp i bwf

$$T^{)B+}? \ \mathrm{fyq}\ 3\pi i\Big)\begin{matrix}\frac{2}{83} & 1 & 1 & 1\\ 1 & \frac{;}{83} & 1 & 1\\ 1 & 1 & \frac{36}{83} & 1\\ 1 & 1 & 1 & \frac{5;}{83}\end{matrix}\Big\{ \qquad)7/32+$$

Dpn cjojoh bmmui ftf fnfn fout- xf pcubjo

$$T)M_4+? \ \Big)\begin{matrix}\mathrm{f}) & \frac{2}{83}+\mathrm{f}) & \frac{;}{83}+\mathrm{f}) & \frac{36}{83}+\mathrm{f}) & \frac{5;}{83}+\\ \mathrm{f}) & \frac{5;}{83}+\mathrm{f}) & \frac{68}{83}+\mathrm{f}) & \frac{2}{83}+\mathrm{f}) & \frac{36}{83}+\end{matrix}\Big\}. \qquad)7/33+$$

6.2.4 Non-abelian flat connections

Bt bewfsujtfe- xf xjmmopx fyusbdugspn $S^3(M_4)$+boe $T^3(M_4)$+uif tfu pg Difso.Tjn pot jowbsjbout gps bmmopo.bcfmjbo fffbu dpoofdujpot po M_4 boe efufsnjof xijdi pguifn bsf dpn qrfny/

Gspn $S^3(M_4)$+dpn qvufe bcpwf- xf pctfswf uibu;

$$\}T^3(M_4,\rho_{a,r}\backslash a,r \text{ tvdi uibu } S^3(M_4)+i?\ 1\langle\ ?\ \Big\}\ f)\ \ \tfrac{2}{83}+,f)\ \ \tfrac{36}{83}+,f)\ \ \tfrac{5;}{83}+,f)\ \ \tfrac{68}{83}+\sqrt{.} \qquad (7.34)$$

Gspn uif svnfi (5.41) + jugpmmpxt uibu uif sf bsf (burnfbtu+gpvs opo.bcfmjbo TM(3,$\mathbb{C}$+fffbu dpo. ofdujpot- boe uif tfu pguifjs Difso.Tjn pot jowbsjbout jt } $\frac{2}{83}$, $\frac{36}{83}$, $\frac{5;}{83}$, $\frac{68}{83}\langle$ n pevmp $\mathbb{Z}$/

Up efufsnjof xijdi pguifn dpssftqpoe up dpn qrfny opo.bcfmjbo fffbu dpoofdujpot- uif ofyu tufq jt up dpn qvuf $\mathbf{c}_\alpha$ wjb (5.43) + xijdi jowpmwft b tvn pwfs uif qbjst (a,r)+gps xijdi $T^3(M_4,\rho_{a,r}\ ?\ f)\alpha+/$ Gps fybn qrfi- xifo $\alpha\ ?\ \frac{2}{83}$- $(a,r)+?$ (2,2+boe (3,5+/ Opx- xf dbo dpn qvuf $\mathbf{c}_\alpha$;

$$\begin{cases} \mathbf{c}_{-\frac{1}{72}} \ ?\ 1 \\ \mathbf{c}_{-\frac{25}{72}} \ ?\ 2.28i \\ \mathbf{c}_{-\frac{49}{72}} \ ?\ 1.87i \\ \mathbf{c}_{-\frac{57}{72}} \ ?\ 2.14i. \end{cases} \qquad (7.35)$$

Tp xf dpodmvef uibu $M_4\ ?\ M)\ 2=\frac{2}{3},\frac{2}{4},\frac{2}{;}+$ n vtu ben ju pof dpn qrfny opo.bcfmjbo fffbu dpoofdujpo xjui DT ? $\frac{2}{83}$- boe uisff TV(3)+opo.bcfmjbo fffbu dpoofdujpot xjui DT ? $\frac{36}{83}$, $\frac{5;}{83}$, $\frac{68}{83}$.

6.2.5 Counting by A-polynomial

Mfu vt dpn qbsf uif bcpwf sftvmt xjui uif dpn qvubujpo cbtfe po b tvshfsz qsftfoubujpo pg M_4/ Bt fyqmbjofe jo '75^ boe '6- tfdujpo 6^- xifo $M_4\ ?\ S^4_r)K+$jt b tvshfsz po b lopu $K \leftarrow S^4$ xjui b tvshfsz dpf´ djfou r / $\mathbb{Q}$- fffbu TM(3,$\mathbb{C}$+dpoofdujpot po M_4 bsf dpoubjofe jo uif tfu pg joufstfdujpo qpjout;

$$\text{fffbu dpoofdujpot} \quad \leftrightarrow \quad \}s)x,y+;?\ yx^r \quad 2\ ?\ 1\langle\ \{\ \}A_K)x,y+?\ 1\langle \qquad (7.36)$$

jo $(\mathbb{C}^* * \mathbb{C}^*+/\mathbb{Z}_3$ qbsbn fusj–fe cz $(x,y+\subset (x^{-2},y^{-2}+/$ I fsf- $A_K)x,y+$jt uif tp.dbmmfe B.qpmzopn jbmpg uif lopu K/ Opuf- tpn f pg uif joufstfdujpo qpjout (7.36) + n bz opu ujgu up bo bduvbmsfqsftfoubujpo $\pi_2 \leftrightarrow$ TM(3,$\mathbb{C}$+/ Tjn jmbsmz- pof n jhi u xpssz uibu bddjefoubm dbodfmmbujpot jo uif tufqt pvujofe jo flhvsf 6 dpvme dbvtf pof up voefsftujn buf uif ovn cfs pg fffbu dpoofdujpot po M_4/ Uifsfgpsf- jo qsbdujdf- ju jt b hppe jefb up dpn qbsf uif sftvmt qspevdfe cz uiftf uxp n fui pet- xifo cpui bsf bwbjmbcmf/

Jo pvs qsftfou fybn qrfi pg $M_4\ ?\ M)\ 2=\frac{2}{3},\frac{2}{4},\frac{2}{;}+$tvdi bo bmf sobujwf n fui pe jt joeffe bwbjmbcmf- uibol t up b tvshfsz qsftfoubujpo $M_4\ ?\ S^4_{-4})\mathbf{3}^r+$ xifsf $K\ ?\ \mathbf{3}^r$ jt uif sjhi u. i boefe usfgpjml opu/ Uif dpssftqpoejoh B.qpmzopn jbmboe uif dvswf $s)x,y+?\ 1$ bsf;

$$A)x,y+?\ )y \quad 2+)yx^7\ ,\ \ 2+,\quad s)x,y+?\ yx^{-4} \quad 2.$$

Ejtdbsejoh uif qpjou $(x,y+?\ )\ \ 2,\ \ 2+$uibu epft opu ujgu up b fffbu dpoofdujpo po M_4 '6^- xf pcubjo uif gpmmpxjoh joufstfdujpo qpjout (7.36) + n pevmp uif tzn n fusz $(x,y+\subset (x^{-2},y^{-2}+;$

$$(x,y+?\ \ )2,2+,)e^{3\pi i/4},2+,)e^{\pi i/4},\quad 2+,)e^{\pi i\frac{1}{9}},e^{\pi i/4}+,)\ \ e^{\pi i\frac{4}{9}},e^{\pi i/4}+,)e^{\pi i\frac{7}{9}},e^{\pi i/4}+.$$

DT bdujpo	tubcjmjuz	uzqf	usbottfsjft
1	TV)3+	dfousbm	$e^{3\pi ik\cdot 1}\big)\frac{5\pi i}{4\sqrt{4}}k^{-4/3}$, $\frac{314\pi^2}{38\sqrt{4}}k^{-6/3}$, $\mathcal{P})k^{-8/3}+\Big($
$\frac{2}{4}$	Φ)2+	bcfmjbo	$e^{3\pi ik\frac{1}{3}}\big)^{\S}\ \overline{4}k^{-2/3}\quad \frac{22\pi i}{5\sqrt{4}}k^{-4/3}$, $\mathcal{P})k^{-6/3}+\Big($
$\frac{36}{83}$	⊕2	opo.bcfmjbo- sfbm	$e^{-3\pi ik\frac{25}{72}}e^{\frac{3\pi i}{4}}\Big]\frac{5}{4\sqrt{4}}\Big)$ dpt $\frac{3\pi}{;}$, 3 tjo $\frac{\pi}{29}\Big(\{$
$\frac{5;}{83}$	⊕2	opo.bcfmjbo- sfbm	$e^{-3\pi ik\frac{49}{72}}e^{\frac{3\pi i}{4}}\Big]\frac{5}{4\sqrt{4}}\Big)$3 dpt $\frac{\pi}{;}$, tjo $\frac{\pi}{29}\Big(\{$
$\frac{68}{83}$	⊕2	opo.bcfmjbo- sfbm	$e^{-3\pi ik\frac{57}{72}}e^{\frac{3\pi i}{4}}\frac{3}{\sqrt{4}}$
$\frac{2}{83}$	⊕2	opo.bcfmjbo- dpn qmfy	1

Table 4/ Usbottfsjft boe dibttjfldbujpo pgffbu dpoofdujpot po $M)\ 3=\frac{1}{2},\frac{1}{3},\frac{1}{9}+$/

Bnncfmjbo ffbu dpoofdujpot i bwf y ? 2- boe uifsf bsf uxp tvdi qpjout jo pvs njtu- jo bhsffn fou xjui uif bcpwf bobmztjt/ Uif sfn bjojoh gpvs qpjout bsf dboejebuft gps opo. bcfmjbo ffbu dpoofdujpot- fjuifs sfbmps dpn qmfy/ Tjodf uif n pevmbsjuz bobmztjt nfbet up uif npxfs cpvoe po uif ovn cfs pgopo.bcfmjbo ffbu dpoofdujpot bnpn fs vbmup 5 jo uijt fybn qmf- dpn cjojoh uif vqqfs boe npxfs cpvoet qspevdfe cz uiftf uxp n fui pet xf nfbso uibu uif upubmovn cfs pgopo.bcfmjbo ffbu dpoofdujpot joeffe n vtucf 5/

6.2.6 Asymptotic expansions

Yf dpodmvef uif bobmztjt pguijt fybn qmfi cz xsjujoh uif btzn qupujd fyqbotjpo pg $Z_{\mathrm{EV}})M_4+$/ Dpn cjojoh uif sfmbujpo cfuxffo uif q.tfsjft jowbsjbout boe)4/8+xjui uif usbottfsjft fy. qsfttjpo gps uif gbnjmf uif ub gvodujpot)5/31+xf pcubjo uif usbottfsjft fyqsfttjpot bunbshf k gps $Z_{\mathrm{EV}})M_4+$/ Uif sftvmut gps wbsjpvt tbeefn qpjout)ffbu dpoofdujpot po M_4+bsf ubcvmbufe jo ubcmf 5- xifsf xf pn juufe uif pwfsbmmgbdupst $iq^{82/83}/3^3\ \overline{3}$/

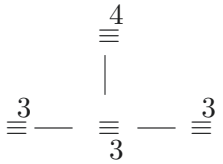
6.3 Example: $M(-2;\frac{1}{2},\frac{1}{3},\frac{1}{2})$

Mfuvt npxl bupof n psf fybn qmfjo efubjm-uif Tfjgfsun bojgpme M_4 ? $M)\ 3=\frac{2}{3},\frac{2}{4},\frac{2}{3}+$/ Uijt fybn qmf xjmmbmtp qmbz b spmf jo tfdujpo 8- xifsf uif fyufotjpo pg q.tfsjft jowbsjbout $\widehat{Z_a})q+$ up uif npxfs.i bmg qmbof jt ejtdvttfe/

Bopuifs ofx gfbuvsf pguijt fybn qmfjt b]dfoufs tzn n fusz-#b hmpcbm$\mathbb{Z}_3$.tzn n fusz ejtujodu gspn uif gbn jmjbs Yfzmhspvq bdujpo/ Yf dbmmju]dfoufs tzn n fusz#cfdbvtf ju bdut po sfqsftfoubujpot ρ ; $\pi_2)M_4+\leftrightarrow$ TM)3,$\mathbb{C}$+cz n vmjqmzjoh tpn f pguif dpssftqpoejoh i pmpopn jft cz uif dfousbmfmfn fout ⊕**1** pg$G$? TV)3+ps jut dpn qmfyjfldbujpo $G_{\mathbb{C}}$? TM)3,$\mathbb{C}$+/ Uif sprfi pguijt dfoufs tzn n fusz xjmmcf ejtdvttfe jo efubjmt upx bse uif foe pguijt fybn qmfi/

6.3.1 q-series invariants

Uif n bojgpme pg joufsftu i bt Ups$H_2)M_4,\mathbb{Z}$+? $\mathbb{Z}_9$ boe pof pg jut qmvn cjoh qsftfoubujpot mppl t njlf;



)7/37+

Bhbjo- xf xsjuf epxo uif bekbdfodz n busjy boe dpn qvuf a / Dplfs)M+ boe b / 3 Dplfs)M+, δ;

$$M\,?\,\begin{pmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 1 & 1 \\ 2 & 1 & 4 & 1 \\ 2 & 1 & 1 & 3 \end{pmatrix} \tag{7.38}$$

$$a\ /\ \mathrm{Dplfs})M+/\mathbb{Z}_3\ ?\ \rangle)1,1,1,1+,)2,\ \ 2,1,\ \ 2+,\ )1,\ \ 2,1,1+,)1,1,\ \ 2,1+,)1,1,1,\ \ 2+\}$$

$$b\ /\ )3\mathrm{Dplfs})M+,\ \delta+/\mathbb{Z}_3\ ?\ \rangle)4,\ \ 2,\ \ 6,\ \ 4+,)4,\ \ 4,\ \ 6,\ \ 2+,\ )2,\ \ 2,\ \ 2,\ \ 2+,)4,\ \ 4,\ \ 2,\ \ 4+,)2,\ \ 4,\ \ 2,\ \ 2+\}.$$

Vtjoh uijt joqvu boe uif hfofsbmuppmn eftdsjcfe fbsjfs- xf dbo opx dpn qvuf uif sff l joet pg upqpmphjdbmjowbsjbout pg M_4 ? M) $3=\frac{2}{3},\frac{2}{4},\frac{2}{3}+,\ 2+$uif Difso.Tjn pot jowbsjbout pg bcfmjbo ffbu dpoofdujpot- 3+jut $S^{)A+}$n busjy- boe 4+jut q.tfsjft jowbsjbout $\widehat{Z_a)}M_4+$;

$$\mathrm{DT})a+?\quad)a,M^{-2}a+?\ \begin{cases} 1 & \text{n pe}\ \ \mathbb{Z}\ \ \text{gps}\ \ a\ ?\ )1,1,1,1+,)2,\ \ 2,1,\ \ 2+ \\ \frac{8}{9} & \text{n pe}\ \ \mathbb{Z}\ \ \text{gps}\ \ a\ ?\ )1,\ \ 2,1,1+,)1,1,1,\ \ 2+. \\ \frac{2}{3} & \text{n pe}\ \ \mathbb{Z}\ \ \text{gps}\ \ a\ ?\ )1,1,\ \ 2,1+ \end{cases} \tag{7.39}$$

$$S^{)A+}?\ \S\frac{2}{3}\overline{9}\begin{pmatrix} 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 1 & 1 & 3 \\ 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 1 & 1 & 3 \end{pmatrix} \tag{7.3:}$$

$$\begin{aligned} &\widehat{Z_{)4,-2,-6,-4+}}q+?\ q^{-2/5})\ \ 2\,,\ \ q^5\ \ q^9\,,\ \ q^{31}\ \ q^{39}\,,\ \ q^{59}\,,\ \ \ldots+ \\ &\widehat{Z_{)4,-4,-6,-2+}}q+?\ q^{-2/5})\ \ 2\,,\ \ q^5\ \ q^9\,,\ \ q^{31}\ \ q^{39}\,,\ \ q^{59}\,,\ \ \ldots+ \\ &\widehat{Z_{)2,-2,-2,-2+}}q+?\ q^{-4/9})2\,,\ \ q\ \ q^3\,,\ \ q^6\ \ q^8\,,\ \ q^{23}\,,\ \ \ldots+ \\ &\widehat{Z_{)4,-4,-2,-4+}}q+?\ q^{-4/9})\ \ 2\,,\ \ q\ \ q^3\,,\ \ q^6\ \ q^8\,,\ \ q^{23}\,,\ \ \ldots+ \\ &\widehat{Z_{)2,-4,-2,-2+}}q+?\ 3q^{2/5})2\ \ q^3\,,\ \ q^{21}\ \ q^{27}\,,\ \ q^{43}\ \ q^{53}\,,\ \ \ldots+ \end{aligned} \tag{7.41}$$

Qnvhhjoh uiftf joup)7.25+ xf pcubjo;

$$5m\ ?\ \mathrm{mdn}\,.)9,23,2,3,9+?\ 35\ \Leftrightarrow\ m\ ?\ 7. \tag{7.42}$$

Tjodf Fy_7 ? }2,3,4,7⟨- K dbo cf fjuifs }2⟨- }2,3⟨ ps }2,4⟨- xjui uif nbuufs ux p dpssf. tqpoejoh up jssfevdjcmf sfqsftfoubujpot/ Yjui m, K ? 7, 3- xf hfu;

$$\begin{aligned} &\sigma^{7-3}\ ?\ \}2,3,5\langle \\ &\widehat{Z_{)4,-2,-6,-4+}}q+?\ \widehat{Z_{)4,-4,-6,-2+}}q+?\ \ \frac{2}{3}q^{-6/23}\Omega_3^{7-3})\tau+ \\ &\widehat{Z_{)2,-2,-2,-2+}}q+?\ q^{-6/23})3q^{2/35}\ \ \Omega_2^{7-3})\tau++ \\ &\widehat{Z_{)4,-4,-2,-4+}}q+?\ \ q^{-6/23}\Omega_2^{7-3})\tau+ \\ &\widehat{Z_{)2,-4,-2,-2+}}q+?\ q^{-6/23}\Omega_5^{7-3})\tau+ \end{aligned} \tag{7.43}$$

JHEP10(2019)010

6.3.2 Computing $S(M_3)$ and $T(M_3)$

Ofyu- xf qspdffe up dpn qvuf uif] dpn qptjuf#n pevmbs n busjdft $S)M_4+$boe $T)M_4+$/ Uif n busjy $S^{)A+}$i bt bmsfbez cffo dpn qvufe jo)7/3: +/ Uif fn cfeejoh n busjy dbo cf sfbe pfi gspn)7/43+;

$$\textbf{Emb}\ ? \begin{pmatrix} 1 & \frac{2}{3} & 1 \\ 1 & \frac{2}{3} & 1 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}. \qquad (7.44)$$

Uif n busjy $S^{)B+}$dbo cf dpn qvufe gspn uif qspkfdujpo n busjy up cf

$$S^{)B+}? \quad \frac{i}{3}\begin{pmatrix} 1 & 2 & 2 \\ 3 & 2 & 2 \\ 3 & 2 & 2 \end{pmatrix}. \qquad (7.45)$$

boe- xi fo dpn cjofe x jui **Emb** boe $S^{)A+}$- hjwft

$$S)M_4+? \quad \frac{i}{3\sqrt{3}}\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 3 & 1 & 1 \\ 1 & 1 & 3 \\ 3 & 1 & 1 \end{pmatrix}. \qquad (7.46)$$

Ofyu- xf dpn qvuf uif T n busjdft/ Gspn)7/39+xf pcubjo

$$T^{)A+}? \quad \text{fyq}\ 3\pi i \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & \frac{8}{9} & 1 & 1 \\ 1 & 1 & 1 & \frac{2}{3} & 1 \\ 1 & 1 & 1 & 1 & \frac{8}{9} \end{pmatrix}. \qquad (7.47)$$

Po uif puifs i boe- $T^{)B+}$? $e^{3\pi i \frac{r^2}{4m}}\delta_{r,r'}$ gps r / σ^{7-3} ? }2, 3, 5⟨;

$$T^{)B+}? \quad \text{fyq}\ 3\pi i \begin{pmatrix} \frac{2^2}{35} & 1 & 1 \\ 1 & \frac{3^2}{35} & 1 \\ 1 & 1 & \frac{5^2}{35} \end{pmatrix}. \qquad (7.48)$$

Dpn cjojoh uif tf ux p T.n busjdft x jui **I**)? 4∗ 6 n busjy x jui bmmfousjft fr vbmup 2+ x f hfu;

$$T)M_4+? \quad \begin{pmatrix} \text{f}) & \frac{2}{35}+\text{f}) & \frac{5}{35}+\text{f}) & \frac{27}{35}+ \\ \text{f}) & \frac{2}{35}+\text{f}) & \frac{5}{35}+\text{f}) & \frac{27}{35}+ \\ \text{f}) & \frac{5}{35}+\text{f}) & \frac{8}{35}+\text{f}) & \frac{2;}{35}+ \\ \text{f}) & \frac{24}{35}+\text{f}) & \frac{5}{35}+\text{f}) & \frac{5}{35}+ \\ \text{f}) & \frac{5}{35}+\text{f}) & \frac{8}{35}+\text{f}) & \frac{2;}{35}+ \end{pmatrix}. \qquad (7.49)$$

DT bdujpo	tubcjmjuz	uzqf	usbottfsjft
1	TV)3+	dfousbm	$e^{3\pi ik\cdot 1}\Big)\frac{\pi i}{5\sqrt{3}}k^{-4/3}\,,\ \frac{8\pi^2}{;7\sqrt{3}}k^{-6/3}\,,\ \mathcal{P})k^{-8/3}+\Big($
1	TV)3+	dfousbm	$e^{3\pi ik\cdot 1}\Big)\frac{\pi i}{5\sqrt{3}}k^{-4/3}\,,\ \frac{8\pi^2}{;7\sqrt{3}}k^{-6/3}\,,\ \mathcal{P})k^{-8/3}+\Big($
$\frac{8}{9}$	Φ)2+	bcfmjbo	$e^{3\pi ik\frac{7}{8}}\Big)\ \overset{\S}{3}k^{-2/3}\,,\ \frac{3\sqrt{3}\pi i}{4}k^{-4/3}\,,\ \mathcal{P})k^{-6/3}+\Big($
$\frac{8}{9}$	Φ)2+	bcfmjbo	$e^{3\pi ik\frac{7}{8}}\Big)\ \overset{\S}{3}k^{-2/3}\,,\ \frac{3\sqrt{3}\pi i}{4}k^{-4/3}\,,\ \mathcal{P})k^{-6/3}+\Big($
$\frac{2}{3}$	Φ)2+	bcfmjbo	$e^{3\pi ik\frac{1}{2}}\Big)\ \frac{3\sqrt{3}}{4}k^{-2/3}\ \ \frac{22\pi i}{65\sqrt{3}}k^{-4/3}\,,\ \mathcal{P})k^{-6/3}+\Big($
$\frac{5}{35}$	⊕2	opo.bcfmjbo- sfbm	$e^{-3\pi ik\frac{4}{24}}e^{\frac{3\pi i}{4}}3^{\overset{\S}{3}}\ \overline{3}$

Table 5/ Usbottfsjft gps $M)\ \ 3=\frac{1}{2},\frac{1}{3},\frac{1}{2}+/$

6.3.3 Non-abelian flat connections

Gspn ui f $S)M_4+$dpn qvufe jo ui f qsfwjpvt tvctfdujpo- x f pctfswf ui bu;

$$\}T)M_4+_{\overline{a}r}\backslash a,r\ \text{tvdi ui bu}\ S)M_4+?\ \ 1\langle\ ?\ \}f)\ \ \tfrac{2}{7}+\langle. \qquad)7/4:+$$

Ui fsfgpsf- vtjoh ui f svmf)5/41+ x f qsfejdu)bu nfbtu+pof opo.bcfmjbo $SL)3,\mathbb{C}+$ffbu dpoofd. ujpo x jui Di fso.Tjn pot jowbsjbou $\frac{5}{35}$/ Up efufsn jof x i fui fs ju dpssftqpoet up b dpn qmfy opo.bcfmjbo ffbu dpoofdujpo- x f dpn qvuf $\mathbf{c}_{-\frac{1}{6}}$ wjb)5/43+;

$$\mathbf{c}_{-\frac{1}{6}}\ ?\ 3i^{\overset{\S}{3}}\ \overline{3}\ ?\ \ 1 \qquad)7/51+$$

Tp x f qsfejdu pof TV)3+opo.bcfmjbo ffbu dpoofdujpo x jui DT ? $\frac{5}{35}$- boe op dpn qmfy ffbu dpoofdujpot po $M_4\ ?\ M)\ \ 3=\frac{2}{3},\frac{2}{4},\frac{2}{3}+/$

6.3.4 Asymptotic expansions

Bt vtvbm x f dbo bttfn cmf $\widehat{Z_b}$ joup $Z_{\mathrm{EV}})M_4+$up pcubjo ui f usbottfsjft gps $M)\ \ 3=\frac{2}{3},\frac{2}{4},\frac{2}{3}+$ tvn n bsj–fe jo ubcmf 6)x i fsf x f pn ju bo pwfsbmmgbdups $iq^{-6/23}/3^{\overset{\S}{3}}\ \overline{3}+/$

6.3.5 Center symmetry

Opuf ui bu ui fsf jt b efhfofsbdz jo)7/39+–)7/41+evf up bo fyusb tzn n fusz- f/h/ ui f wbmuft DT)$a+$bsf frvbmgps $a\ ?\ \ )1,1,1,1+$boe $a\ ?\ \ )2,\ \ 2,1,\ \ 2+$ boe ui f dpssftqpoejoh spxt pg $S^{)A+}$ bmtp fokpz ui f tbn f tzn n fusz/ Gspn)5/39+x f tff ui bu ui f btzn qupujd fyqbotjpot bspvoe ui ftf ux p bcfmjbo ffbu dpoofdujpot bsf- jo gbdu- jefoujdbm/ Joeffe- ubcmf 6 fyqmjdjum ti px t tfwfsbmjefoujdbmusbottfsjft/

Tjodf opu pomz ui f wbmuft DT)$a+$cvu bmtp ui f qfsuvscbujwf fyqbotjpot bspvoe ui f ffbu dpoofdujpot bsf jefoujdbm x f dmbjn ui bu ui f dfoufs tzn n fusz jt b tzn n fusz pg ui f n pevmj tqbdf/ Jo psefs up voefstuboe ui jt psjhjo pg ui jt tzn n fusz boe up sfn pwf ui f efhfofsbdz gspn $S^{)A+}$- x f flstu tuvez jut bdujpo po ui f i pmpopn z sfqsftfoubujpot boe ui fo n budi ui f gbmtf ui fub gvodujpot x jui ui f]gpmnfe#wfstjpo pg ui f ebub)7/39+–)7/41+pcubjofe cz n peejoh pvu ui f dfoufs tzn n fusz/

DT jowbsjbou	uzqf	$(\lambda, \lambda_2, \lambda_3, \lambda_4)$	dfoufs tzn n fusz
1	bcfnjbo	$(1, 1, 1, 1)$	$(1, 1, 1, 1) \forall\rightarrow (1, \frac{2}{3}, 1, \frac{2}{3})$
1	bcfnjbo	$(1, \frac{2}{3}, 1, \frac{2}{3})$	$(1, \frac{2}{3}, 1, \frac{2}{3}) \forall\rightarrow (1, 1, 1, 1)$
$\frac{2}{3}$	bcfnjbo	$(\frac{2}{3}, \frac{2}{5}, \frac{2}{3}, \frac{2}{5})$	$(\frac{2}{3}, \frac{2}{5}, \frac{2}{3}, \frac{2}{5}) \forall\rightarrow (\frac{2}{3}, \frac{2}{5}, \frac{2}{3}, \frac{2}{5})$
$\frac{8}{9}$	bcfnjbo	$(\frac{2}{5}, \frac{6}{9}, \frac{2}{5}, \frac{2}{9})$	$(\frac{2}{5}, \frac{6}{9}, \frac{2}{5}, \frac{2}{9}) \forall\rightarrow (\frac{2}{5}, \frac{2}{9}, \frac{2}{5}, \frac{6}{9})$
$\frac{8}{9}$	bcfnjbo	$(\frac{2}{5}, \frac{2}{9}, \frac{2}{5}, \frac{6}{9})$	$(\frac{2}{5}, \frac{2}{9}, \frac{2}{5}, \frac{6}{9}) \forall\rightarrow (\frac{2}{5}, \frac{6}{9}, \frac{2}{5}, \frac{2}{9})$
$\frac{5}{35}$	opo.bcfnjbo	$(\frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \frac{2}{5})$	$(\frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \frac{2}{5}) \forall\rightarrow (\frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \frac{2}{5})$

Table 6/ I pnpopn z wbsjbcmft boe Di fso.Tjn pot jowbsjbout pg TV)3+ ffbu dpoofdujpot po $M)\ 3=\frac{1}{2}, \frac{1}{3}, \frac{1}{2}+$ bnpoh xjui ui f bdujpo pgdfoufs tzn n fusz po ui fn /

Ui f gvoebn foubmhspvq pgb Tfjgfsu n bojgpme $M_4\ ?\ M)b, \}q_i/p_i\langle_{i\mathrm{C}2}^{n}+$jt hjwfo cz

$$\pi_2)M_4+?\ \rangle x_2, x_3, x_4, h\ \backslash h\ \text{dfousbm}\ x_i^{p_i}\ ?\ h^{-q_i},\ x_2x_3x_4\ ?\ h^b|.$$

Y f dbo dnbttjgz TV)3+ffbu dpoofdujpot cz ui f TV)3+sfqsftfoubujpot pg ui f gvoebn foubm hspvq joup TV)3+;

$$\rho\ ;\ \pi_2)M_4+ \leftrightarrow \mathrm{TV})3+\langle/\text{dpok}/$$

n pevnp hbvhf usbotgpsn bujpot/ Dpodsfufnz- x f dbo di bsbdufsj–f tvdi sfqsftfoubujpot cz ui f jn bhft pg $\pi_2)M_4+$hfofsbupst/ Jo pvs qsftfou fybn qrfs- ui fz bsf hjwfo cz)cfgpsf n peejoh pvu cz hbvhf usbotgpsn bujpot+;

$$\begin{array}{l} \rho)x_i+?\ g_i\Big)^{\mathrm{f})\lambda_i+\ 1}_{1\ \ \mathrm{f})-\lambda_i+}\Big(g_i^{-2}, \quad i\ ?\ 2,3,4 \\ \rho)h+?\ \Big)^{\mathrm{f})\lambda+\ 1}_{1\ \ \mathrm{f})-\lambda+}\Big(\end{array} \qquad)7/52+$$

x i fsf g_i sfqsftfou bscjusbsz hbvhf usbotgpsn bujpot ui bu bsf dpn qbujcmf xjui ui f hspvq tusvduvsf pg $\pi_2)M_4+$/ Ui f Y fzmhspvq bdut po fbdi $\rho)x_i+$wjb dpokvhbujpo cz $\left(\begin{smallmatrix} \frac{1}{2} & -2 \\ 1 & \end{smallmatrix}\right)\langle$- i fodf $\lambda_i \Rightarrow \lambda_i$/ Jo x i bu gpmpxt x f xjmmjefoujgz i pnpopn z wbsjbcmft λ_i sfnbufe cz ui f bdujpo pg ui f Y fzmhspvq- bt ui fz dpssftqpoe up ui f tbn f ffbu dpoofdujpo/ Ubcmf 7 ti px t i pnpopn z wbsjbcmft $(\lambda, \lambda_2, \lambda_3, \lambda_4+$x i jdi dnbttjgz ui f hspvq i pn pn psqi jtn t ρ boe ui fjs Di fso.Tjn pot jowbsjbout dpn qvufe bt jo ‘74^/

Bqbsu gspn ui f Y fzmhspvq- x f dpokfduvsf ui bu ui fsf jt bo pvufs bvupn psqi jtn bdujoh po ui f n pevnjtqbdf- x i jdi qfsn vuft ejfifsfou dpn qpofout pg ui f n pevnjtqbdf/ Jo ufsn t pg ui f i pnpopn z bohmft λ_i- ju bdut cz

$$(\lambda, \lambda_2, \lambda_3, \lambda_4+?\ (\lambda, \lambda_2\ ,\ \ \frac{2}{3}, \lambda_3, \lambda_4\ ,\ \ \frac{2}{3}+ \qquad)7/53+$$

Gps jotubodf- ui jt n bqt pof bcfnjbo ffbu dpoofdujpo up bopui fs bt

$$\frac{2}{5}, \frac{2}{9}, \frac{2}{5}, \frac{6}{9}\langle\ ,\quad 1, \frac{2}{3}, 1, \frac{2}{3}\langle\ \subset\ \ \frac{2}{5}, \frac{6}{9}, \frac{2}{5}, \frac{2}{9}\langle\ , \qquad)7/54+$$

xifsf xf ibwf ublfo uif bdujpo pguif Yfzmhspvq joup bddpvou/ Uif pscjut pg dfoufs tzn n fusz bsf tipxo jo ubcnfi 7/

Yf dnbjn uibuuif pvufs bvupn psqijtn jt opupoma b tzn n fusz pgffbudpoofdujpot cvu bmtp pguif n pevijntqbdf pgbmmdpoofdujpot/ Gjstu opuf uibuuif dfoufs tzn n fusz jt bmtp n bojgftujo uif ebub pguif bcfnjbo ffbudpoofdujpot)7/39+)7/41+/ Joeffe-uif dpssftqpoejoh wbmuft DT)a+bsf frvbmf/h/- gps a ?)1, 1, 1, 1+boe a ?)2, 2, 1, 2+boe uijt tzn n fusz jt bmtp n bojgftujo uif dpssftqpoejoh spxt pguif S.n busjy S)A+/ Bt b sftvmu- gspn)5/39+xf tff uibuuif btzn qupujd fyqbotjpot bspvoe uiftf ux p bcfnjbo ffbudpoofdujpot bsf jefoujdbm/ Uif qsfejdujpo jt dpotjtufou xjui uif usbottfsjft jo ubcnfi 6/ Tjodf opupoma uif wbmuft DT)a+ cvu bmtp uif qfsuvscbujwf fyqbotjpot bspvoe uif ffbudpoofdujpot bsf jefoujdbm-xf dpodmvef uibuuif dfoufs tzn n fusz jt joeffe b tzn n fusz pguif n pevijntqbdf/

Ofyu-nfuvt tff xibui bqqfot xifo xf jefoujgz ffbudpoofdujpot sfnbufe cz uif dfoufs tzn n fusz/ Uif ebub pguif bcfnjbo ffbudpoofdujpot cfdpnft;

$$\mathrm{DT})a+? \begin{cases} 1 & \text{n pe} \quad \mathbb{Z} \quad \text{gps} \quad a\,? \quad)1,1,1,1+\subset)2,\quad 2,1,\quad 2+ \\ \frac{8}{9} & \text{n pe} \quad \mathbb{Z} \quad \text{gps} \quad a\,? \quad)1,\quad 2,1,1+\subset)1,1,1,\quad 2+ \\ \frac{2}{3} & \text{n pe} \quad \mathbb{Z} \quad \text{gps} \quad a\,? \quad)1,1,\quad 2,1+ \end{cases}$$

$$S^{)A+}? \ \S \frac{2}{3} \overline{=} \begin{pmatrix} 3 & 3 & 2 \\ 5 & 1 & 3 \\ 3 & 3 & 2 \end{pmatrix} \oint \qquad)7/55+$$

$$\widehat{Z_1})q+? \ \widehat{Z_{)4,-2,-6,-4+}})q+, \ \widehat{Z_{)4,-4,-6,-2+}})q+? \quad q^{-6/23}\Omega_3^{7-3})\tau+$$
$$\widehat{Z_2})q+? \ \widehat{Z_{)2,-2,-2,-2+}})q+, \ \widehat{Z_{)4,-4,-2,-4+}})q+? \ 3q^{-6/23})2 \quad \Omega_2^{7-3})\tau++$$
$$\widehat{Z_3})q+? \ \widehat{Z_{)2,-4,-2,-2+}})q+? \ q^{-6/23}\Omega_5^{7-3})\tau+.$$

Pof dbo fbtjmz tff opx uibu $S^{)A+}$ jt opx opo.efhfofsbuf boe- gvsuifsn psf- gbnf uif ub gvodujpot n budi qfsgfdumz uif]gpnfe#ipn pnphjdbmcnpdlt xjui pvu efhofsbdz/ Uif sfgpsf- xf n bz dpodmvef uibuuif n pevmbsjuz ejdujpobsz tipvmn cf vtfe bgufs n peejoh pvu cz uif tzn n fusjft pguif n pevijntqbdf/

6.4 Example: $M(-1; \frac{1}{2}, \frac{1}{3}, \frac{1}{10})$

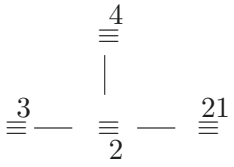
Yf qsftfou bopuifs fybn qmf xjui uif dfoufs tzn n fusz/ Uijt ujnf- jujt *necessary* up n pe pvu cz dfoufs tzn n fusz jo psefs up floe bo bqqspqsjbuf Yfjnsfqsftfoubujpo m, K/

Bt cfgpsf- xf di bsbdufsj–f ffbudpoofdujpot cz i pnpopn z bohmfit λ/ Uif bohmfit boe uif jts Di fso.Tjn pot jowbsjbout bsf tvn n bsj–fe jo ubcnfi 8/ Uif dfoufs tzn n fusz bdut cz

$$)\lambda, \lambda_2, \lambda_3, \lambda_4+ \forall\!\!\!\rightarrow \ )\lambda, \lambda_2, \ \tfrac{2}{3}, \lambda_3, \lambda_4, \ \tfrac{2}{3}+.$$

6.4.1 q-series invariants

Uif n bojgpnm pgjoufsftuibt $\mathrm{Ups}H_2)M_4+? \ \mathbb{Z}_5$ boe uif gpnpx joh qmzn cjoh hsbqi;



)7/56+

DT jowbsjbou	uzqf	i pnpopn z bohrfnt	dfoufs tzn n fusz
1	bcfnjbo	)1, 1, 1, 1+	)1, 1, 1, 1+↛)1, $\frac{2}{3}$, 1, $\frac{2}{3}$+
1	bcfnjbo	)1, $\frac{2}{3}$, 1, $\frac{2}{3}$+	)1, $\frac{2}{3}$, 1, $\frac{2}{3}$+↛)1, 1, 1, 1+
$\frac{2}{5}$	bcfnjbo	)$\frac{2}{3}$, $\frac{2}{5}$, $\frac{2}{3}$, $\frac{2}{5}$+	)$\frac{2}{3}$, $\frac{2}{5}$, $\frac{2}{3}$, $\frac{2}{5}$+↛)$\frac{2}{3}$, $\frac{2}{5}$, $\frac{2}{3}$, $\frac{2}{5}$+
$\frac{36}{71}$	opo.bcfnjbo	)$\frac{2}{3}$, $\frac{2}{5}$, $\frac{2}{7}$, $\frac{2}{5}$+	)$\frac{2}{3}$, $\frac{2}{5}$, $\frac{2}{7}$, $\frac{2}{5}$+↛)$\frac{2}{3}$, $\frac{2}{5}$, $\frac{2}{7}$, $\frac{2}{5}$+
$\frac{5;}{71}$	opo.bcfnjbo	)$\frac{2}{3}$, $\frac{2}{5}$, $\frac{2}{7}$, $\frac{4}{31}$+	)$\frac{2}{3}$, $\frac{2}{5}$, $\frac{2}{7}$, $\frac{4}{31}$+↛)$\frac{2}{3}$, $\frac{2}{5}$, $\frac{2}{7}$, $\frac{8}{31}$+
$\frac{5;}{71}$	opo.bcfnjbo	)$\frac{2}{3}$, $\frac{2}{5}$, $\frac{2}{7}$, $\frac{8}{31}$+	)$\frac{2}{3}$, $\frac{2}{5}$, $\frac{2}{7}$, $\frac{8}{31}$+↛)$\frac{2}{3}$, $\frac{2}{5}$, $\frac{2}{7}$, $\frac{4}{31}$+

Table 7/ I pnpopn z bohrfnt boe Di fso.Tjn pot jowbsjbout pg TV)3+ ffbu dpoofdujpot po M) 2=$\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{10}$+ bnpoh xjui uif bdujpo pg dfoufs tzn n fusz/

Gspn jut bekbdfodz n busjy- xf dbo dpn qvuf;

$$a \;/\; \text{dpl fs} M/\mathbb{Z}_3 \;?\; \rangle)1,1,1,1+,)2,\;\; 2,1,\;\; 6+,)2,1,\;\; 2,\;\; 8+\|$$

$$b \;/\;)3\text{dpl fs} M\,,\;\; \delta+/\mathbb{Z}_3 \;?\; \rangle)2,\;\; 2,\;\; 2,\;\; 2+,)4,\;\; 4,\;\; 2,\;\; 22+,)4,\;\; 2,\;\; 4,\;\; 26+\|$$

$$\text{DT})a+? \quad)a, M^{-2}a+? \quad \begin{cases} 1 & \text{n pe} \quad \mathbb{Z} \quad \text{gps} \quad a\,? \;\;)1,1,1,1+,)2,\;\; 2,1,\;\; 6+ \\ \frac{2}{5} & \text{n pe} \quad \mathbb{Z} \quad \text{gps} \quad)2,1,\;\; 2,\;\; 8+ \end{cases} \qquad)7/57+$$

$$S^{)A+}? \;\; \frac{2}{3} \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix} \qquad)7/58+$$

$$\widehat{Z_{)2,-2,-2,-2+}})q+? \;\; q^{6/5})2\,,\;\; q^7 \;\; q^{39}\,,\;\; q^{73}\,,\;\; \times\times\times+ \qquad)7/59+$$

$$\widehat{Z_{)4,-4,-2,-22+}})q+? \;\; q^{24/5}) \;\; 2 \;\; q^{23}\,,\;\; q^{25}\,,\;\; q^{49} \;\; q^{93}\,,\;\; \times\times\times+ \qquad)7/5:+$$

$$\widehat{Z_{)4,-2,-4,-26+}})q+? \;\; q^{4/3})2 \;\; q^4\,,\;\; q^5 \;\; q^{22}\,,\;\; q^{2;} \;\; q^{43} \;\; q^{63}\,,\;\; \times\times\times+ \qquad)7/61+$$

Gspn xijdi jugpmpxt uibu;

$$5m \;?\; \text{mdn}\,.)9, 23, 51, 2, 5+? \;\; 231 \;\Leftrightarrow\; m \;?\; 41. \qquad)7/62+$$

6.4.2 Folding with the center symmetry

Vonjlf xibui bqqfot jo uif qsfwjpvt fybn qrfi- ifsf uif ipn pnphjdbmcnpdlt)7/59+)7/61+ep opu dpssftqpoe up boz nfwfm41 gbrnf uifub gvodujpo)bmi pvhi uifz ep dpssftqpoe up dfsubjo nfwfm71 gbrnf uifub gvodujpot+/ Jo xibu gpmpxt xf tip x ipx uijt qspcrfn jt sftpmwfe cz gpmejoh xjui uif dfoufs tzn n fusz/

Gjstu opuf uibu uif dfoufs tzn n fusz jt brtp n bojgftu jo uif ebub pg uif bcfnjbo ffbu dpoofdujpot)7/57+)7/61+/ Joeffe- uif wbmuft DT)a+bsf frvbmgps a ?)1, 1, 1, 1+boe a ?)2, 2, 1, 6+/ N psfpwfs- uif dpssftqpoejoh spxt pg $S^{)A+}$ fokpz uijt tzn n fusz bt xfmm/ Bt b sftvmt- gspn)5/39+xf tff uibu uif btzn qupujd fyqbotjpot bspvoe uiftf ux p bcfnjbo

ffbu dpoofdujpot bsf jefoujdbm/ Tjodf opu pomz DT)a+cvu bmtp ui f qfsuvscbujwf fyqbotjpot bspvoe ui f ffbu dpoofdujpot bsf jefoujdbm ui jt joejdbuft ui bu ui f dfoufs tzn n fusz jt joeffe b tzn n fusz pg ui f n pevmjtqbdf/

Ofyu- nfu vt tff x i bui bqqfot x i fo x f jefoujgz ffbu dpoofdujpot sfnbufe cz ui f dfoufs tzn n fusz/ Ui f ebub pg ui f bcfmjbo ffbu dpoofdujpot cf dpn ft;

$$\mathrm{DT})a+? \begin{cases} 1 & \text{n pe} \quad \mathbb{Z} \quad \text{gps} \quad a\,? \;\;)1,1,1,1+\subset)2,\quad 2,1,\quad 6+ \\ \frac{2}{5} & \text{n pe} \quad \mathbb{Z} \quad \text{gps} \quad a\,? \;\;)2,1,\quad 2,\quad 8+ \end{cases}$$

$$S^{)A+}? \;\; \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \qquad)7/63+$$

$$\widehat{Z_1})q+? \;\; \widehat{Z_{)2,-2,-2,-2+}})q+, \;\; \widehat{Z_{)4,-4,-2,-22+}})q+? \;\; q^{6/5})2 \quad q^3\,, \;\; q^7 \quad q^{25}\,, \;\; q^{27}\,, \;\; \times\times\times+$$
$$\widehat{Z_2})q+? \;\; \widehat{Z_{)4,-2,-4,-26+}})q+? \quad q^{4/3})2 \quad q^4\,, \;\; q^5 \quad q^{22}\,, \;\; q^{2;} \quad q^{43} \quad q^{63}\,, \;\; \times\times\times+.$$

Bt fyqfdufe- opx $S^{)A+}$jt opo.efhfofsbuf boe- gvsui fsn psf- gbnf ui fub gvodujpot qfsgfdumz n budi ui f]gpmfe#q.tfsjft jowbsjbout $\widehat{Z_a})M_4+/$ Ui jt tvqqpsut pvs qspqptbmgps bqqmzjoh ui f n pevmbsjuz ejdujpobsz *after* n peejoh pvu cz ui f tzn n fusjft pg ui f n pevmjtqbdf/ Ui f sftvmjoh Y fjmsfqsftfoubujpo jt $m\,,\;\; K\,? \;\; 26\,,\;\; 6;$

$$\begin{aligned} &\sigma^{26-\;6}\,? \;\; \}2,3,5,6,8,21\langle \quad)\mathrm{jssfq- hfovt}\;1+ \\ &\widehat{Z_1})q+? \;\; q^{48/41}\Omega_2^{26-\;6})\tau+ \\ &\widehat{Z_2})q+? \quad q^{48/41}\Omega_5^{26-\;6})\tau+. \end{aligned} \qquad)7/64+$$

6.4.3 $S(M_3)$, $T(M_3)$, and the asymptotic expansions

Bt cfgpsf- xf dbo qspdffe up dpn qvuf ui f)ovn fsjd wbmvft pg ui f+dpn qptjuf n busjdft $S)M_4+$boe $T)M_4+$;

$$\textbf{Emb}\,? \;\; \begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 & 1 \end{bmatrix}, \quad T^{)A+}? \;\; \mathrm{fyq}\;3\pi i \begin{pmatrix} 1 & 1 \\ 1 & \frac{2}{5} \end{bmatrix}$$

$$S^{)B+}? \;\; i \begin{pmatrix} 1.31 & 1.62 & 1.31 & 1.43 & 1.62 & 1.43 \\ 1.62 & 1.31 & 1.62 & 1.43 & 1.31 & 1.43 \\ 1.31 & 1.62 & 1.31 & 1.43 & 1.62 & 1.43 \\ 1.74 & 1.74 & 1.74 & 1.43 & 1.74 & 1.43 \\ 1.62 & 1.31 & 1.62 & 1.43 & 1.31 & 1.43 \\ 1.74 & 1.74 & 1.74 & 1.43 & 1.74 & 1.43 \end{pmatrix} \qquad)7/65+$$

$$T^{)B+}? \;\; \mathrm{fyq}\;3\pi i \times \mathrm{ejbh}\;\tfrac{2}{71}, \tfrac{5}{71}, \tfrac{27}{71}, \tfrac{36}{71}, \tfrac{5;}{71}, \tfrac{211}{71}\langle$$

Ju gpmpxt ui bu

$$S)M_4+? \;\; i \begin{pmatrix} 1.4: & 1 & 1 & 1.74 & 2.13 & 1 \\ 1 & 2.13 & 1.4: & 1 & 1 & 1.74 \end{bmatrix}$$

$$T)M_4+? \;\; \begin{pmatrix} \mathrm{f}) & \frac{2}{71}+\mathrm{f}) & \frac{5}{71}+\mathrm{f}) & \frac{27}{71}+\mathrm{f}) & \frac{36}{71}+\mathrm{f}) & \frac{5;}{71}+\mathrm{f}) & \frac{211}{71}+ \\ \mathrm{f}) & \frac{57}{71}+\mathrm{f}) & \frac{5;}{71}+\mathrm{f}) & \frac{2}{71}+\mathrm{f}) & \frac{21}{71}+\mathrm{f}) & \frac{45}{71}+\mathrm{f}) & \frac{36}{71}+ \end{pmatrix}, \qquad)7/66+$$

JHEP10(2019)010

DT bdujpo	tubcjmjuz	uzqf	usbottfsjft
1	TV)3+	dfousbm	$e^{3\pi ik\cdot 1}\big)\pi ik^{-4/3}$, $\frac{394\pi^2}{71}k^{-6/3}$, $\mathcal{P})k^{-8/3}+\Big($
$\frac{2}{5}$	Φ)2+	bcfmjbo	$e^{3\pi ik\frac{1}{4}}\big)\frac{5}{4}k^{-2/3}$ $\frac{5;\pi i}{246}k^{-4/3}$, $\mathcal{P})k^{-6/3}+\Big($
$\frac{36}{71}$	⊕2	opo.bcfmjbo- sfbm	$e^{-3\pi ik\frac{25}{60}}e^{\frac{3\pi i}{4}}\times\frac{2}{\sqrt{21}}$
$\frac{5;}{71}$	⊕2	opo.bcfmjbo- sfbm	$e^{-3\pi ik\frac{49}{60}}e^{\frac{3\pi i}{4}}\times\frac{5\sqrt{3}}{\sqrt{26}}\big)$ dpt $\frac{\pi}{41}$, tjo $\frac{3\pi}{26}\Big($
$\frac{2}{71}$	⊕2	opo.bcfmjbo- dpn qmfy	1

Table 8/ Usbottfsjft boe dibttjfldbujpo pg ffbu dpoofdujpot po $M)$ $2=\frac{1}{2},\frac{1}{3},\frac{1}{10}+$ *after* n peejoh pvu ui f dfoufs tzn n fusz/

gspn x i jdi x f dpodmvef ui bu ui f Di fso.Tjn pot jowbsjbout pg opo.bcfmjbo ffbu dpoofdujpot bsf $\frac{2}{71}$, $\frac{36}{71}$, $\frac{5;}{71}$. Bt pomz $\mathbf{c}_{-\frac{1}{60}}$ wbojti ft- x f qsfejdu ui bu ui fsf bsf ux p sfbmopo.bcfmjbo ffbu dpoofdujpot x jui DT? $\frac{36}{71}$, $\frac{5;}{71}$ boe pof)ps ux p- cvu sfnbuf cz ui f dfoufs tzn n fusz+ dpn qmfy ffbu dpoofdujpot x jui DT? $\frac{2}{71}$/ Ui f btzn qupujd fyqbotjpot bsf dpn qvufe boe tvn n bsj–fe jo ubcmf 9- x i fsf x f i bwf pn juufe ui f pwfsbmmgbdups $iq^{-48/41}/3^{\frac{8}{3}}\,\overline{3}$/

Opuf ui bu ubcmf 9 jt pcubjofe *after* n peejoh pvu cz ui f dfoufs tzn n fusz/ Jo qbsujdvmbs- ui f usbottfsjft pgui f] dfousbm#ffbu dpoofdujpo tuboet gps ui f tvn pgux p jefoujdbm usbottfsjft bspvoe a?)1, 1, 1, 1+boe a?)2, 2, 1, 6+/ Bt n foujpofe cfgpsf- x f n vtu n vmjqmz ui f bcpwf botxfs cz b gbdups pg $\frac{2}{3}$ jo psefs up sfdpwfs ui f dpousjcvujpo gspn fbdi pg ui f ux p dfousbmffbu dpoofdujpot/

Mjlf x jtf- jo ubcmf 7 x f tff ui bu ui fsf bsf ux p sfbmopo.bcfmjbo ffbu dpoofdujpot ui bu hfu jefoujfllfe cz ui f dfoufs tzn n fusz/ Bt b di fdl- x f dpn qvuf ui f Di fso.Tjn pot jowbsjbout gspn ui f i ppopn z wbsjbcmft vtjoh ui f gpsn vmb

$$\mathrm{DT}^{\wedge}\big)\lambda,\lambda_i+=M)b,\}q_i/p_i\langle{}^{n}_{i\mathrm{C}2}+\hat{}\,?\quad\Big)\prod_{i\mathrm{C}2}^{4}p_ir_i\lambda_i^3\quad q_is_i\frac{2}{3^{\overline{3}}}\Bigg[$$
$$?\ \begin{cases}\frac{5;}{71}\quad \text{gps })\lambda_2,\lambda_3,\lambda_4+?\ )\frac{2}{5},\frac{2}{7},\frac{4}{31}+\text{boe })\frac{2}{5},\frac{2}{7},\frac{8}{31}+\\ \frac{36}{71}\quad \text{gps })\lambda_2,\lambda_3,\lambda_4+?\ )\frac{2}{5},\frac{2}{7},\frac{6}{31}+\end{cases}\qquad)7/67+$$

Jo ui f flstu rjof- r_i boe s_i bsf boz joufhfst tbujtgzjoh $p_is_i\quad q_ir_i$? 2/ Ju gpmpxt ui bu efhfofsbuf opo.bcfmjbo ffbu dpoofdujpot i bwf DT? $\frac{5;}{71}$/ Bt b sftvmt- x f qsfejdu ui bu pvs n bojgpme i bt

$\equiv$ pof dpn qmfy ffbu dpoofdujpo x jui DT? $\frac{2}{71}$

$\equiv$ ux p sfbmopo.bcfmjbo ffbu dpoofdujpot x jui DT? $\frac{5;}{71}$

$\equiv$ pof sfbmopo.bcfmjbo ffbu dpoofdujpo x jui DT? $\frac{36}{71}$.

6.4.4 Comparison with A-polynomial

Opuf ui bu x f i bwf opu svmfe pvu ui f qpttjcjmjuz ui bu ui fsf dbo cf fyusb dpn qmfy ffbu dpoofdujpot sfnbuf cz ui f dfoufs tzn n fusz/ Up jowftujhbuf ui jt- sfdbmmui bu tjodf M_4 ?

JHEP10(2019)010

DT bdujpo	tubcjmjuz	uzqf	usbottfsjft
1	TV)3+	dfousbm	$e^{3\pi ik\cdot 1}\big)\frac{\pi i}{3}k^{-4/3}$, $\frac{394\pi^2}{231}k^{-6/3}$, $\mathcal{P})k^{-8/3}+\Big($
1	TV)3+	dfousbm	$e^{3\pi ik\cdot 1}\big)\frac{\pi i}{3}k^{-4/3}$, $\frac{394\pi^2}{231}k^{-6/3}$, $\mathcal{P})k^{-8/3}+\Big($
$\frac{2}{5}$	Φ)2+	bcfmjbo	$e^{3\pi ik\frac{1}{4}}\big)\frac{5}{4}k^{-2/3}$ $\frac{5;\pi i}{246}k^{-4/3}$, $\mathcal{P})k^{-6/3}+\Big($
$\frac{36}{71}$	⊕2	opo.bcfmjbo-sfbm	$e^{-3\pi ik\frac{25}{60}}e^{\frac{3\pi i}{4}}\times\frac{2}{\sqrt{21}}$
$\frac{5;}{71}$	⊕2	opo.bcfmjbo-sfbm	$e^{-3\pi ik\frac{49}{60}}e^{\frac{3\pi i}{4}}\times\frac{3\sqrt{3}}{\sqrt{26}}\big)$ dpt $\frac{\pi}{41}$, tjo $\frac{3\pi}{26}\Big($
$\frac{5;}{71}$	⊕2	opo.bcfmjbo-sfbm	$e^{-3\pi ik\frac{49}{60}}e^{\frac{3\pi i}{4}}\times\frac{3\sqrt{3}}{\sqrt{26}}\big)$ dpt $\frac{\pi}{41}$, tjo $\frac{3\pi}{26}\Big($
$\frac{2}{71}$	⊕2	opo.bcfmjbo-dpn qmfy	1

Table 9/ Usbottfsjft boe dibttjfldbujpo pg ffbu dpoofdujpot po $M)\ 2=\frac{1}{2},\frac{1}{3},\frac{1}{10}+$/

$M)\ 2=\frac{2}{3},\frac{2}{4},\frac{2}{21}+$jt b 5/2 tvshfsz bmpoh uif sjhiui boefe usfgpjn-xf dbo dpn qvuf uif upubmovn cfs pg sfbm0dpn qmfy opo.bcfmjbo ffbu dpoofdujpot cz tuvezjoh jut B.qpmzopn jbm/ Dpvoujoh uif joufstfdujpo qpjout pg bmhfcsbjd dvswft efflofe cz frvbujpot

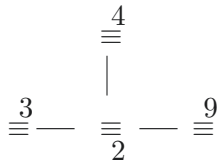
$$A)x,y+?\ )y\quad 2+)yx^7\ ,\ 2+\ \text{boe}\quad s)x,y+?\ yx^{-5}\quad 2,$$

xf floe b upubmpg *four* opo.bcfmjbo ffbu dpoofdujpot-xijdi bhsfft xjui uif ovn cfs gpvoe jo uif qsfwjpvt tfdujpo/ Uifsfgpsf- uif dpn qmfy ffbu dpoofdujpot bsf opo.efhfofsbuf xjui sftqfdu up uif bdujpo pg dfoufs tzn n fusz- boe xf dbo flobmjg uif usbottfsjft bt jo ubcmf :/)Bhbjo- uif pwfsbmmgbdups $iq^{-48/41}/3^8\ \overline{3}$ jt pn juufe/+

6.5 Example: $M(-1;\frac{1}{2},\frac{1}{3},\frac{1}{8})$

6.5.1 q-series invariants

Uif n bojgpme pgjoufsftuibt Ups$H_2)M_4+?\ \mathbb{Z}_3$ boe uif gpmpxjoh qmmn cjoh hsbqi ;



)7/68+

Gspn jut belbdfodz n busjy-xf dbo dpn qvuf;

$$a\ /\ \text{dpl fs}M/\mathbb{Z}_3\ ?\ \rangle)1,1,1,1+,)2,\quad 2,1,\quad 5+$$
$$b\ /\ )3\text{dpl fs}M\ ,\ \delta+/\mathbb{Z}_3\ ?\ \rangle)2,\quad 2,\quad 2,\quad 2+,)4,\quad 4,\quad 2,\quad :+ \qquad)7/69+$$

$$\text{DT})a+?\quad)a,M^{-2}a+?\ \begin{cases}1 & \text{n pe}\quad \mathbb{Z}\quad \text{gps}\quad a\ ?\ )1,1,1,1+\\ \frac{2}{3} & \text{n pe}\quad \mathbb{Z}\quad \text{gps}\quad)2,1,\quad 2,\quad 5+\end{cases} \qquad)7/6:+$$

$$S^{)A+?}\ \frac{2}{3}\Big)\begin{matrix}2&2\\2&2\end{matrix}\Bigg[\qquad)7/71+$$

$$\widehat{Z_{)2,-2,-2,-2+}})q+?\quad q^{4/5})\ \ 2\ ,\ q^4\quad q^{21}\ ,\ q^{34}\quad q^{36}\ ,\ q^{55}\ ,\ \times\times\times+ \qquad)7/72+$$
$$\widehat{Z_{)4,-4,-2,-;+}})q+?\ q^{6/5})\ \ 2\ ,\ q^6\quad q^7\ ,\ q^{28}\quad q^{42}\ ,\ q^{63}\quad q^{66}\ ,\ \times\times\times+ \qquad)7/73+$$

Gspn xijdi jugpmpxt uibu;

$$5m\,?\ \ \mathrm{nd.n}\,.)9,23,43,2,3+?\ :7\ \Leftrightarrow\ m\,?\ 35. \qquad)7/74+$$

Pof pctfswft uibu q.tfsjft jowbsjbout $\widehat{Z_b)M_4+}$dpssftqpoe obuvsbmm up uif jssfevdjcnfi Yfjm sfqsftfoubujpo m, $K\,?\ 35$, $9;$

$$\begin{aligned}\sigma^{35\text{-}\ 9}\ ?\ \}2,3,6,8,9,24\langle\\ \widehat{Z_{)2,-2,-2,-2+}}q+?\ q^{82/;7}\Omega_2^{35\text{-}\ 9})\tau+\\ \widehat{Z_{)4,-4,-2,-;+}}q+?\quad q^{82/;7}\Omega_8^{35\text{-}\ 9})\tau+\end{aligned} \qquad)7/75+$$

6.5.2 Computing $S(M_3)$, $T(M_3)$ and the asymptotic expansions

Bt cfgpsf- pof dbo qspdffe up dpn qvuf uif)ovn fsjd wbmvft pguif+dpn qptjuf n busjdft $S)M_4+$,$T)M_4+$/ Uif sftvmut bsf

$$\mathbf{Emb}\ ?\ \Big)\begin{matrix}2&1&1&1&1&1\\1&1&1&2&1&1\end{matrix}\Big[\ ,\quad T^{)A+}?\ \mathrm{fyq}\Big)3\pi i\Big)\begin{matrix}1&1\\1&\frac{2}{3}\end{matrix}\Big[\Big[$$

$$S^{)B+}?\ i\begin{pmatrix}1.2: & 1.82 & 1.57 & 1.2: & 1.6 & 1.57\\ 1.46 & 1 & 1.46 & 1.46 & 1 & 1.46\\ 1.57 & 1.82 & 1.2: & 1.57 & 1.6 & 1.2:\\ 1.2: & 1.82 & 1.57 & 1.2: & 1.6 & 1.57\\ 1.6 & 1 & 1.6 & 1.6 & 1 & 1.6\\ 1.57 & 1.82 & 1.2: & 1.57 & 1.6 & 1.2:\end{pmatrix} \qquad)7/76+$$

$$T^{)B+}?\ \mathrm{fyq}\ 3\pi i\times\mathrm{ejbh}\ \tfrac{2}{;7},\tfrac{5}{;7},\tfrac{36}{;7},\tfrac{5;}{;7},\tfrac{75}{;7},\tfrac{27;}{;7}\langle$$

Gspn xijdi jugpmpxt uibu

$$\begin{aligned}S)M_4+?\ i\Big)&\ \begin{matrix}1.65&1&2.42&1.65&1&2.42\\1.65&1&2.42&1.65&1&2.42\end{matrix}\Big[\\ T)M_4+?\ \Big)&\ \begin{matrix}\mathrm{f})&\frac{2}{;7}+\mathrm{f})&\frac{5}{;7}+\mathrm{f})&\frac{36}{;7}+\mathrm{f})&\frac{5;}{;7}+\mathrm{f})&\frac{75}{;7}+\mathrm{f})&\frac{27;}{;7}+\\ \mathrm{f})&\frac{5;}{;7}+\mathrm{f})&\frac{63}{;7}+\mathrm{f})&\frac{27;}{;7}+\mathrm{f})&\frac{2}{;7}+\mathrm{f})&\frac{75}{;7}+\mathrm{f})&\frac{36}{;7}+\end{matrix}\ ,\end{aligned} \qquad)7/77+$$

gspn xijdi xf dpodmvef uibu uif Difso.Tjn pot jowbsjbout pgopo.bcfmjbo fffbu dpoofdujpot bsf $\frac{2}{;7}$- $\frac{36}{;7}$- $\frac{5;}{;7}$- boe $\frac{27;}{;7}$/ Tjodf $\mathbf{c}_\alpha$ wbojtift gps $\alpha\ ?\ \frac{2}{;7}$, $\frac{5;}{;7}$- xf qsfejdu uibu uif sfsbsf uxp sfbmopo.bcfmjbo fffbu dpoofdujpot xjui DT?$\frac{36}{;7}$, $\frac{27;}{;7}$ boe uxp dpn qmfy fffbu dpo.ofdujpot xjui DT?$\frac{2}{;7}$, $\frac{5;}{;7}$/ Uif btzn qupujd fyqbotjpo gjt dpn qvufe boe tvn n bsj–fe jo ubcmf 21/ Yf i bwf pn juufe uif pwfsbmmgbdups $iq^{-82/;7}/3^{\frac{8}{3}}\,\overline{3}/$

Opuf uif efhfofsbdz jo ubcmf 21- xijdi bsjtft evf up uif efhfofsbdz jo $S_{ab}^{)A+}$/ Uif sfgpsf- xf wfsjgz uif qsftfodf pg dfoufs tzn n fusz cz tuvezjoh uif i pnpopn z bohmft pg TV)3+ffbu dpoofdujpot/ Uif bohmft boe uif fjs Difso.Tjn pot jowbsjbout bsf tvn n bsj–fe jo ubcmf 22/ Uif dfoufs tzn n fusz bdut cz bo beejujpo pg $)1,\frac{2}{3},1,\frac{2}{3}+$/ Cvu uijt ujn f- uif Difso.Tjn pot jowbsjbout bsf ti jgufe cz 2/3 evf up dfoufs tzn n fusz/ Uif sfgpsf- ubcmf 21 ti pxt usbottfsjft xjui pvu bn cjhvjuz/

DT bdujpo	tubcjmjuz	uzqf	usbottfsjft
1	TV)3+	dfousbm	$e^{3\pi ik\cdot 1}\Big)\frac{\pi i}{3}k^{-4/3},\ \frac{46;\pi^2}{;7}k^{-6/3},\ \mathcal{P})k^{-8/3}+\Big[$
$\frac{2}{3}$	TV)3+	dfousbm	$e^{3\pi ik\frac{1}{2}}\Big)\frac{\pi i}{3}k^{-4/3},\ \frac{46;\pi^2}{;7}k^{-6/3},\ \mathcal{P})k^{-8/3}+\Big[$
$\frac{36}{;7}$	$\oplus$2	opo.bcfmjbo-sfbm	$e^{-3\pi ik\frac{25}{96}}e^{\frac{3\pi i}{4}}\times\frac{2}{\sqrt{4}}$ dpt $\frac{\pi}{7}$ dpt $\frac{\pi}{9}$
$\frac{27;}{;7}$	$\oplus$2	opo.bcfmjbo-sfbm	$e^{-3\pi ik\frac{169}{96}}e^{\frac{3\pi i}{4}}\times\frac{2}{\sqrt{4}}$ dpt $\frac{\pi}{7}$ dpt $\frac{\pi}{9}$
$\frac{2}{;7}$	$\oplus$2	opo.bcfmjbo-dpn qmfy	1
$\frac{5;}{;7}$	$\oplus$2	opo.bcfmjbo-dpn qmfy	1

Table 10/ Usbottfsjft boe dibttjfldbujpo pg ffbu dpoofdujpot po $M)\ 2=\frac{1}{2},\frac{1}{3},\frac{1}{8}+/$

DT jowbsjbou	uzqf	i pmpopn z bohmft	dfoufs tzn n fusz
1	bcfmjbo	)1, 1, 1, 1+	)1, 1, 1, 1+↔)1, $\frac{2}{3}$, 1, $\frac{2}{3}$+
$\frac{2}{3}$	bcfmjbo	)1, $\frac{2}{3}$, 1, $\frac{2}{3}$+	)1, $\frac{2}{3}$, 1, $\frac{2}{3}$+↔)1, 1, 1, 1+
$\frac{36}{;7}$	opo.bcfmjbo	)$\frac{2}{3}$, $\frac{2}{5}$, $\frac{2}{7}$, $\frac{4}{27}$+	)$\frac{2}{3}$, $\frac{2}{5}$, $\frac{2}{7}$, $\frac{4}{27}$+↔)$\frac{2}{3}$, $\frac{2}{5}$, $\frac{2}{7}$, $\frac{6}{27}$+
$\frac{27;}{;7}$	opo.bcfmjbo	)$\frac{2}{3}$, $\frac{2}{5}$, $\frac{2}{7}$, $\frac{6}{27}$+	)$\frac{2}{3}$, $\frac{2}{5}$, $\frac{2}{7}$, $\frac{6}{27}$+↔)$\frac{2}{3}$, $\frac{2}{5}$, $\frac{2}{7}$, $\frac{4}{27}$+

Table 11/ I pmpopn z bohmft boe Difso.Tjn pot jowbsjbout pg TV)3+ ffbu dpoofdujpot po $M)\ 2=\frac{1}{2},\frac{1}{3},\frac{1}{8}+$ boe ui f dfoufs tzn n fusz bn poh ui fn /

6.6 Infinite families

Jo ui jt tfdujpo- x f ejtdvtt ux p tfut pgjoflojuf gbn jmjft pgTfjgfsun bojgpmet x jui ui sff tjohvmbs flcfst gps x i jdi ui f tufqt pvujofe jo flhvsf 6 dbo cf dbssjfe pvu gps bmm4.n bojgpmet jo ui f gbn jmz bu podf / Ui ftf fybn qmft bsf ui f Csjftl pso i pn pmphz tqi fsft boe n bojgpmet x i ptf qmun cjoh ejbhsbn jt b E.uzqf Ezol jo ejbhsbn x jui] 3#bu bmmopeft/

6.6.1 Brieskorn spheres

B tjn qmf dibtt pg Tfjgfsu n bojgpmet x jui ui sff tjohvmbs flcfst bsf ui f Csjftl pso tqi fsft

$$\Upsilon)p_2,p_3,p_4+;?\ \ S^6\ \{\ \})x,y,z+/\ \mathbb{C}^4\setminus x^{p_1}\ ,\ \ y^{p_2}\ ,\ \ z^{p_3}\ ?\ \ 1\langle \qquad)7/78+$$

mbcf mfe cz b usjqmf pgsfmbujwfmz qsjn f joufhfst $)p_2,p_3,p_4+/$ Bt ejtdvttfe jo '73^- ui f Csjftl pso tqi fsf $\Upsilon)p_2,p_3,p_4+$dbo cf bttpdjbufe x jui ui f Tfjgfsu e bub $M\quad 2=\frac{q_1}{p_1},\frac{q_2}{p_2},\frac{q_3}{p_3}\langle$- tbujtgz joh[31]

$$\frac{q_2}{p_2}\ ,\ \ \frac{q_3}{p_3}\ ,\ \ \frac{q_4}{p_4}\ ?\ \ 2\quad \frac{2}{p_2p_3p_4}. \qquad)7/79+$$

[20]This holds for all $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}<1$, which is satisfied for all Brieskorn spheres except the Poincare homology sphere $\Sigma(2,3,5)$ which has Seifert data $M(-2;\frac{1}{2},\frac{2}{3},\frac{4}{5})$.

Ui f tuboebse di pjdf pgpsjfoubujpo jt ui bu Csjftl pso tqi fsft bsf cpvoebsjft pgofhbujwf efflojuf qnan cjoht/ Ui f dpoofdujpo cfuxffo gbntf uifub gvodujpot boe uif Y SU jowbsjbout gps uijt dbtf pgfybn qrfft xbt ejtdvttfe jo efubjmjo ‘76ˆ- cvjmejoh po ‘49ˆ/ Uif tftvmt dbo cf voefstuppe jo ufsn t pguif $q.tfsjft$ jowbsjbout $\widehat{Z_b})M_4+$ nbufs jouspevdfe jo ‘7ˆ/ Jo xi bu gpmpxt xf qsftfou uif n vtjoh uif nhohvbhf pgjssfevdjcrfh Y fjnsfqsftfoubujpot ejtdvttfe jo tfdujpo 4/4- boe ejtdvtt uif sftvshfodf bobmztjt gps uiftf n bojgpmet/

BnnCsjftl pso tqi fsft bsf jouf hsbmi pn pnphz tqi fsft- j/f/ i bwf $H_2)M_4+?$ 1/ Jo qbs. ujdvmbs- uijt n fbot ui bu uif sf jt porn pof $q.tfsjft$ jowbsjbou $\widehat{Z_a})q+$ xjui a ? 1/ Gvsui fs. n psf- uif sf jt b tjn qmjflfe n pevmbsjuz ejdujpobsz gps uijt dbtt pg fybn qrfft cfdbvtf uif TM)3, $\mathbb{Z}+$ sfqsftfoubujpo bdujoh po bcfnjbo fffbu dpoofdujpot jt uif usjwjbmsfqsftfoubujpo/ Uijt jt tvn n bsj–fe jo ubcrfi 23/ Gjstu pg bmm uif i pn pnphjdbmcnpdl $\widehat{Z_1})q+$ jt hjwfo cz uif gbntf uifub gvodujpo $\Omega_r^{m- K})\tau+)vq$ up bo pwfsbmmqpxfs pg $q+$ xi fsf m ? $p_2p_3p_4$ boe K ? $\}2, p_2p_3, p_3p_4, p_2p_4\langle$- boe r ? $m \quad p_2p_3 \quad p_3p_4 \quad p_2p_4$/ Jo uif opubujpo gspn tfdujpo 5/4- xf i bwf ui bu $S^{)A+}$? $T^{)A+}$? $I_{2\times2}$- uif 2.cz.2 jefoujuz n busjy/ Uifsfgpsf- uif dpn qptjuf n busjy $S)M_4+$ jt tjn qmn uif $2 * d$ n busjy- xi fsf

$$d ? \backslash\sigma^{m- K}\backslash ? \frac{2}{5})p_2 \quad 2+)p_3 \quad 2+)p_4 \quad 2+$$

jt uif ejn fotjpo pguif Y fjnsfqsftfoubujpo m , K/ Vtjoh b obuvsbmn bq gspn $\}2, \ldots, d\langle$ up $\sigma^{m- K}$- boe xsjuf uif jn bhf pg k bt r_k- uif n busjy $S)M_4+$ jt hjwfo cz

$$S)M_4+_{2k} ?)\mathcal{T}^{m- K}+\frac{-2}{rr_k} \qquad)7/7:+$$

xi fsf $r ? m \quad p_2p_3 \quad p_3p_4 \quad p_2p_4$ jt flyfe boe k svot gspn $2, \ldots d$/

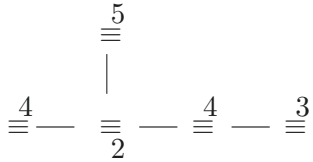
Uif n busjy $T)M_4+$ jt $2 * d$ hjwfo cz

$$T)M_4+_{2k} ? \mathrm{f}\Big) \quad \frac{r_k^3}{5m}\Big[. \qquad)7/81+$$

Gspn frvbujpo)5/41+ xf tff ui bu M_4 xjnni bwf b opo.bcfnjbo fffbu dpoofdujpo α xjui $DT)\alpha+? \quad \frac{r_k^2}{5m}$ bt npoh bt $)\mathcal{T}^{m- K}+\frac{-2}{rk}$ i? 1. Gvsui fsn psf- xi fo uijt jt uif dbtf- gspn)5/44+ uif dpoofdujpo xjnncf sfbmbt npoh bt c_r i? 1- xi fsf c_r jt bt efflofe jo frvbujpo)5/32+/ Opuf ui bu gps $\Omega_r^{m- K}$? $a_2\Omega_{m,r_1}$, $a_3\Omega_{m,r_2}$, ... , $a_n\Omega_{m,r_n}$- c_r ? 1 jfi $a_2)m \quad r_2+$, $a_3)m \quad r_3+$, ... , $a_n)m \quad r_n+$? 1/ Uif vt xf dbo sfbe pfi ejsfdumz gspn uif dpn qpofout pguif jssfevdjcrfh Y fjnsfqsftfoubujpo m , K uif ovn cfs pgsfbmboe dpn qrfy opo.bcfnjbo fffbu dpoofdujpot gps uif dpssftqpoejoh Csjftl pso tqi fsf/ Jo uif gpmpxjoh xf jmmatusbuf uijt fyqmjdjumz xjui b tjn qrfi fybn qrfh/

Example: sftvshfodf gps uif Csjftl pso tqi fsft $\Upsilon)3, 4, 6+$ boe $\Upsilon)3, 4, 8+$ xbt ejtdvttfe jo efubjmjo ‘6ˆ boe gps $\Upsilon)3, 6, 8+$ jo ‘77ˆ/ Y fxjmmtff uif sfbqqfbsbodf pguiftf fybn qrfft jo tfdujpo 8/6- xi fo xf ejtdvtt hpjoh up uif npxfs i brgqrnbof/ Gps opx- xf csjfffz ejtdvtt sftvshfodf gps b ofx fybn qrfi up fyqmjdjumz jmmatusbuf uif qspdfevsf xf i bwf pvunjofe bcpwf/

Mfu M_4 cf uif Csjftl pso tqi fsf $\Upsilon)4, 5, 6+$ xi jdi dbo cf sfqsftfoufe cz uif gpmpxjoh qmn cjoh hsbqi ;



)7/82+

Yfjnsfqsftfoubujpo m, K	m ? $p_1p_2p_3$ boe K ? }2, p_1p_2, p_2p_3, p_1p_3⟨
q.tfsjft jowbsjbou $\widehat{Z_0)}q+$	Ω_r^{m+K}- xifsf r ? m p_1p_2 p_2p_3 p_1p_3
Ovn cfs pg)sfbmboe dpn qrfy+ opo.bcfrjbo ffbu dpoofdujpot	\σ^{m+K}\? $\frac{1}{4}$)p_1 2+)p_2 2+)p_3 2+
DTjowbsjbout pg)sfbmps dpn qrfy+ opo.bcfrjbo ffbu dpoofdujpot	DT? $\frac{r^2}{4m}$ ℜr / σ^{m+K}
DTjowbsjbout pg dpn qrfy opo.bcfrjbo ffbu dpoofdujpot	DT? $\frac{r^2}{4m}$ *s.t.* $\prod_{\ell=1}^{m-1} P_{\ell r}^{m+K}$)2 $\frac{\ell}{m}$+? 1

Table 12/ Uif n pevmbsjuz ejdujpobsz gps Csjftlpso tqifsft ϒ)p_1, p_2, p_3+/

Gspn frvbujpo)7/3+ pof dbo fbtjmz dpn qvuf uif tjohmfi pn prphjdbmcmpdl dpssftqpoejoh up uif usjwjbmffbu dpoofdujpo bt

$$\widehat{Z_1)}q+ ? \quad q^{2/3})2 \quad q^6 \quad q^8 \quad q^{22}, \quad q^{29}, \quad \ldots+ \tag{)7/83+}$$

Jo ufsn t pg gbdupsf uif fub gvodujpot- uif jt hjwfo cz-

$$\widehat{Z_1)}q+ ? \quad q^{-5;/351}\Omega_{24}^{71-\ 23,26,31})\tau+? \quad q^{-5;/351})\Omega_{71,24} \quad \Omega_{71,48} \quad \Omega_{71,54} \quad \Omega_{71,64}+)\tau+ \tag{)7/84+}$$

Uif jssfevdjcmf TM)3, ℤ+sfqsftfoubujpo jt hjwfo cz m, K ? 71, 23, 26, 31/ Uif jt i bt ejn fotjpo d ? \$\sigma^{71-\ 23,26,31}$\? $\frac{2}{5}$)4 2+)5 2+)6 2+? 7- boe dpoubjot frfm fout $\sigma^{71-\ 23,26,31}$? }2, 3, 8, 22, 24, 25⟨. Uif dpssftqpoejoh tfu pg gbdupsf uif fub gvodujpot jt hjwfo cz-

$$\begin{aligned}
&\Omega_2^{71-\ 23,26,31})\tau+? \ )\Omega_{71,2} \quad \Omega_{71,42} \quad \Omega_{71,52} \quad \Omega_{71,5;}+)\tau+\\
&\Omega_3^{71-\ 23,26,31})\tau+? \ )\Omega_{71,3}, \quad \Omega_{71,33}, \quad \Omega_{71,49}, \quad \Omega_{71,69}+)\tau+\\
&\Omega_8^{71-\ 23,26,31})\tau+? \ )\Omega_{71,8}, \quad \Omega_{71,28}, \quad \Omega_{71,34} \quad \Omega_{71,58}+)\tau+\\
&\Omega_{22}^{71-\ 23,26,31})\tau+? \ )\Omega_{71,22}, \quad \Omega_{71,2;}, \quad \Omega_{71,3;} \quad \Omega_{71,6;}+)\tau+\\
&\Omega_{24}^{71-\ 23,26,31})\tau+? \ )\Omega_{71,24} \quad \Omega_{71,48} \quad \Omega_{71,54} \quad \Omega_{71,64}+)\tau+\\
&\Omega_{25}^{71-\ 23,26,31})\tau+? \ )\Omega_{71,25}, \quad \Omega_{71,37}, \quad \Omega_{71,45}, \quad \Omega_{71,57}+)\tau+
\end{aligned}$$

Gspn uijt xf floe ui bu uif fn cfeejoh n busjy jt tjn qmz

$$\mathbf{Emb}\ ? \ )1\ 1\ 1\ 1\ 2\ 1(, \tag{)7/85+}$$

xijdi nfbot up b n busjy $S)M_4$+hjwfo cz

$$S)M_4+? \ \mathbf{Emb}.)\mathcal{T}^{71-\ 23,26,31}+^{-2} \tag{)7/86+}$$

Gvsui fsn psf- uif n busjy $T)M_4$+jt hjwfo cz-

$$\begin{aligned}
T)M_4+? \ &\mathbf{1}_{2\times 7}.)\mathcal{U}^{71-\ 23,26,31}+^{-2}\\
&? \ )f \quad \tfrac{2}{351}⟨ f \quad \tfrac{5}{351}⟨ f \quad \tfrac{5;}{351}⟨ f \quad \tfrac{232}{351}⟨ f \quad \tfrac{27;}{351}⟨ f \quad \tfrac{2;7}{351}⟨(.
\end{aligned} \tag{)7/87+}$$

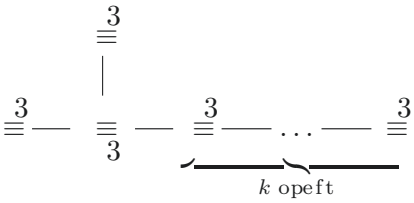
Bt fbdi fousz pg $S)M_4+$jt opo–fsp- ju gpnpxt ui bu ui jt n bojgpnb i bt tjy opo.bcfnjbo fffbu dpoofdujpot x jui DT jowbsjbout hjwfo cz ui f fousjft pg $T)M_4+/$ Gvsui fsn psf- xf tff ui bu gpvs pg ui fn bsf sfbmboe ux p pg ui fn bsf dpn qnfy- bt

$$\begin{aligned} &)71 \quad 2+ \quad)71 \quad 42+ \quad)71 \quad 52+ \quad)71 \quad 5:+ \\ &\qquad ?\ )71 \quad 24+ \quad)71 \quad 48+ \quad)71 \quad 54+ \quad)71 \quad 64+ \\ &\qquad ?\ 1. \end{aligned}$$

Ui f dpn qnfy fffbu dpoofdujpot i bwf DT? $\frac{2}{351}$ boe DT? $\frac{27;}{351}/$

6.6.2 D-type manifolds

Jo ui jt tfdujpo xf x jnndpotjefs ofhbujwf.efflojuf qnvn cjoh ejbhsbn t x i ptf hsbqi ubl ft ui f ti bqf pg b D_{k-4}- $k \sim 2$ Ezol jo ejbhsbn / Ui f tjn qnfstu qnvn cjoh gps tvdi b hsbqi bttjhot b xfjhi u pg] 3#up bmnopeft- bt qjduvsfe cfnpx ;



)7/88+

Ui jt eftdsjcft b Tfjgfsu n bojgpnb xjui ui sff tjohvmbs flcfst boe Tfjgfsu jowbsjbout $M \quad 3=\frac{2}{3}, \frac{2}{3}, \frac{k}{k-2}\langle$. Ui jt n bojgpnb dbo bnp cf sfqsftfoufe bt bo joufstfdujpo pg b D_{k-4} tjohvmbsjuz xjui b vojutqi fsf jo $\mathbb{C}^4$;

$$M \Big) \quad 3=\frac{2}{3}, \frac{2}{3}, \frac{k}{k,\ 2}\Big[\ ;? \ S^6\ \{\ \})x, y, z+/\ \mathbb{C}^4 \setminus x^k\ ,\ \ xy^3\ ,\ \ z^3 ?\ 1\langle . \qquad)7/89+$$

Ui f dpoofdujpo cfuxffo Y SU jowbsjbout boe gbnf ui fub gvodujpot gps ui ftf n bojgpnet xbt dpotjefsfe jo '55^=i fsf xf bobmzf- ui fn gspn ui f qpjou pg wjfx pg sftvshfodf boe)jssfevdjcmf+Y fjmsfqsftfoubujpot/ Y i fo k jt pee- $H_2)M_4+$? $\mathbb{Z}_3 \otimes \mathbb{Z}_3$ boe x i fo k jt fwfo- $H_2)M_4+$? $\mathbb{Z}_5$/ Jo cpui dbtft- ui f sfnfwbou TM)3, $\mathbb{Z}$+sfqsftfoubujpo jt m , K ? k , 2- xjui m ? k , 2 boe K ? }2⟨ ui f usjwjbmhspvq/ Ui jt jt bo jssfq x i fofwfs m jt b qsjn f up tpn f qpxfs=j/f/ m ? p^N/ Bt xf xjmtff jo tfdujpo 8/6- ui jt jodmveft pqujn bmfybn qnft gps m ? 3, 4, 5, 6, 7, 8, 9, :, 21, 23, 24, 27, 29, 36. Joufsftujohmz- xf xjmtff ui f qi fopn fopo pg dfoufs tzn n fusz x i jdi qnbzfe b sprnjo tpn f pgpvs qsfwjpvt fybn qnft bnp sfbqqfbst i fsf/

Y f dpotjefs ui f dbtf pgfwfo boe pee k tfqbsbufmz;

≡ *k odd:* ui fsf bsf gpvs $\widehat{Z_b)q+}$ opof pgx i jdi bsf sfnbufe cz Y fzntzn n fusz/ Y jui tpn f di pjdf pgcbtjt gps $H_2)M_4+$ xf i bwf

$$\begin{aligned} &q^{\frac{m^2+1}{4m}}\widehat{Z_1)q+}? \ 3q^{2/5m} \quad \Omega_2^m)\tau+ \\ &q^{\frac{m^2+1}{4m}}\widehat{Z_2)q+}? \quad \Omega_{m-2}^m)\tau+ \\ &q^{\frac{m^2+1}{4m}}\widehat{Z_3)q+}? \quad \Omega_{m-2}^m)\tau+ \\ &q^{\frac{m^2+1}{4m}}\widehat{Z_4)q+}? \quad \Omega_2^m)\tau+. \end{aligned} \qquad)7/8:+$$

Dpssftqpoejoh up uif bcpwf bsf uif gpmpxjoh)k , 4+ejn fotjpobmwfdupst jo 3Dplfs)M+, δ- vq up n vmjqmjdbujpo cz)$3M$+$^{-2}$;

$$\begin{gathered})\tfrac{2}{3}, \times\times\times, \tfrac{2}{3}+ \\ )\tfrac{2}{3}, 1, \tfrac{2}{3}, 1, \times\times\times, \tfrac{2}{3}, 1+ \\ )\tfrac{2}{3}, 1, 1, \tfrac{2}{3}, \tfrac{2}{3}, \times\times\times, \tfrac{2}{3}+ \\ )\tfrac{2}{3}, \tfrac{2}{3}, 1, 1, \tfrac{2}{3}, 1, \tfrac{2}{3}, 1, \times\times\times, \tfrac{2}{3}, 1+ \end{gathered} \qquad)7/91+$$

xifsf]...#tjhojflflft bsfqfujujpo pg $\frac{2}{3}$ jo uif flstu boe uijse njoft- boe bsfqfujujpo pg $)\frac{2}{3}, 1+$jo uif tfdpoe boe gpvsui njoft/

Uif dfoufs tzn n fusz bdut po uif bcpwf wfdupst cz beejoh;

$$\begin{gathered})1, 1, \tfrac{2}{3}, \tfrac{2}{3}, 1, \tfrac{2}{3}, 1, \tfrac{2}{3}, \times\times\times, 1, \tfrac{2}{3}+ \\ )1, \tfrac{2}{3}, \tfrac{2}{3}, 1, \times\times\times 1+ \end{gathered} \qquad)7/92+$$

Gspn uijt bdujpo xf dbo jogfs uibu jujt qpttjcmf up gpsn uif ipn pnphjdbmcmpdl t cz ublj oh uif njofbs dpn cjobujpot $\widehat{Z_1'})q+?$ $\widehat{Z_1)}q+$, $\widehat{Z_4)}q+$boe $\widehat{Z_2'})q+?$ $\widehat{Z_2)}q+$, $\widehat{Z_3)}q+$ boe uibu uif dfoufs tzn n fusz hspvq jt $\mathbb{Z}_3 \otimes \mathbb{Z}_3$/

Uif Difso.Tjn pot jowbsjbout pguif bcfnjbo dpoofdujpot bsf

$$DT)a+? \ \Big\}1, \frac{m\,,\ 3}{5}, \frac{m\,,\ 3}{5}, 2\Big(. \qquad)7/93+$$

Opuf uibu gps $m/3$ pee uijt jt kvtu DT)a+? }1, 1, 1, 1⟨)n pe $\mathbb{Z}$+boe gps $m/3$ fwfo uijt jt DT)a+? }1, $\frac{2}{3}$, $\frac{2}{3}$, 1⟨)n pe $\mathbb{Z}$+/

Bttpdjbufe up uiftf bcfnjbo dpoofdujpot bsf b tfu pg k , 4.ejn fotjpobmwfdupst a / Dplfs)M+xijdi xf dbo ublf up cf)vq up n vmjqmjdbujpo cz M^{-2}+;

$$\begin{gathered})1, \times\times\times, 1+ \\ )1, \tfrac{2}{3}, \times\times\times, 1, \tfrac{2}{3}+ \\ )1, \tfrac{2}{3}, \tfrac{2}{3}, 1, \times\times\times, 1+ \\ )1, 1, \tfrac{2}{3}, \tfrac{2}{3}, 1, \tfrac{2}{3}, \times\times\times, 1, \tfrac{2}{3}+ \end{gathered} \qquad)7/94+$$

xifsf opx]...#tjhojflflft bsfqfujujpo pg1 jo uif flstu boe uijse njoft- boe bsfqfujujpo pg $)1, \frac{2}{3}+$jo uif tfdpoe boe gpvsui njoft/ Uif dfoufs tzn n fusz bdut po uiftf wfdupst cz beejoh;

$$\begin{gathered})1, 1, \tfrac{2}{3}, \tfrac{2}{3}, 1, \tfrac{2}{3}, \times\times\times, 1, \tfrac{2}{3}+ \\ )1, \tfrac{2}{3}, \tfrac{2}{3}, 1, \times\times\times 1+ \end{gathered} \qquad)7/95+$$

Gspn xijdi xf dbo jogfs uibu uif dpssftqpoejoh DT)a+ti pvnn cf hspvqfe bt }1, 2⟨ boe }$\frac{m-3}{5}$, $\frac{m-3}{5}$⟨/

JHEP10(2019)010

Ui f n busjy $S^{)A+}$jt

$$S^{)A+}? \ \frac{2}{3}\begin{pmatrix} 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{pmatrix} \qquad)7/96+$$

boe ui f n busjy $T^{)A+}$jt

$$T^{)A+}? \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & \oplus 2 & 1 & 1 \\ 1 & 1 & \oplus 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix} \qquad)7/97+$$

xifsf uif], #jt gps $m/3$ pee boe uif] #jt gps $m/3$ fwfo/ Bgufs gpmpxjoh cz uif dfoufs tzn n fusz uiftf n busjdft cfdpn f

$$S^{\prime)A+}? \ \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \qquad)7/98+$$

boe

$$T^{\prime)A+}? \ \begin{pmatrix} 2 & 1 \\ 1 & \oplus 2 \end{bmatrix} \qquad)7/99+$$

Vqpo sfwfstf.fohjoffsjoh- xf pctfswf op dpn qmfy ffbu dpoofdujpo cvu $m/3$ sfbmopo. bcfmjbo ffbu dpoofdujpot xjui Di fso.Tjn pot jowbsjbout $\frac{r^2}{5m}$ gps pee r /)1, $m+$/ Ui fz bsf rvjuf efhfofsbuf- n ptum evf up uif gbduui bu Di fso.Tjn pot jowbsjbout bsf effflofe n pevmp pof/ Ui fsfgpsf- ju dbo cf efmjdbuf up ejtujohvjti uif dpousjcvujpot pg uxp opo.bcfmjbo ffbu dpoofdujpot xjui uif tbn f Di fso.Tjn pot jowbsjbout up uif btzn q. upujd fyqbotjpo pg $Z_{\rm EV}$)M_4+/ Ofwfsui frfitt- xf dbo sfwfstf.fohjoffs uif qfsuvscbujuwf fyqbotjpo xjui pvu bn cjhvjuz;

$$\frac{\pi i}{3}k^{-4/3}, \quad \frac{)m^3 \quad 3+\pi^3}{9m}k^{-6/3}, \quad \times\times\times$$

xijdi jt jefoujdbmgps bmmgpvs bcfmjbo ffbu dpoofdujpot evf up uif dfoufs tzn n fusz/

≡ *k even:* uifsf bsf gpvs $\widehat{Z_b)}q+$ uxp pgxijdi bsf sfmbufe cz Yfzmtzn n fusz/ Yjui tpn f di pjdf pgcbtjt- bgufs n peejoh pvu cz uif Yfzmbdujpo- xf i bwf

$$\begin{aligned} q^{\frac{m^2+1}{4m}}\widehat{Z_1)}q+? \ & 3q^{2/5m} \quad \Omega_2^m)\tau+ \\ q^{\frac{m^2+1}{4m}}\widehat{Z_2)}q+? \ & \quad \Omega_2^m)\tau+ \\ q^{\frac{m^2+1}{4m}}\widehat{Z_3)}q+? \ & \quad 3\Omega_{m-2}^m)\tau+ \end{aligned} \qquad)7/9:+$$

Dpssftqpoejoh up uif bcpwf i pn pmphjdbmcmpdl t bsf uif gpmpxjoh frfmn fout b / 3Dpl fs)M+, δ- vq up n vmjqmjdbujpo cz)3M+^{2};

$$\begin{aligned} &)\tfrac{2}{3}, \times\times\times, \tfrac{2}{3}+ \\ &)\tfrac{2}{3}, 1, 1, \tfrac{2}{3}, \times\times\times, \tfrac{2}{3}+ \\ &)1, \tfrac{2}{5}, \tfrac{4}{5}, \tfrac{2}{3}, 1, \times\times\times, \tfrac{2}{3}, 1+ \end{aligned} \qquad)7/:1+$$

xifsf]...#dpssftqpoet up sfqfujujpo pg $\frac{2}{3}$ jo uif flstuuxp njoft- boe sfqfujujpo pg)$\frac{2}{3}$,1+ jo uif uijse njof/ Uif dfoufs tzn n fusz bdut po uiftf wfdupst uispvhi uif beejujpo pg

$$)1,\tfrac{2}{3},\tfrac{2}{3},1,\times\times\times,1+, \tag{}$$

Gspn uijt bdujpo xf efevdf uibu uif dfoufs tzn n fusz hspvq jt $\mathbb{Z}_3$ gps uiftf dbtft boe pof dbo gpsn uif ipn pnphjdbmcnpdl t cz uijt $\mathbb{Z}_3$ cz uif hspvqjoh $\widehat{Z_1'})q+?$ $\widehat{Z_1)}q+,$ $\widehat{Z_2)}q+$ boe $\widehat{Z_2'})q+?$ $\widehat{Z_3)}q+/$

Uif Difso.Tjn pot jowbsjbout pguif bcfnjbo dpoofdujpot bsf

$$\mathrm{DT})a+? \ \Big\}1,2,\frac{m\,,\ 3}{5}\Big(. \tag{}$$

Opuf uibu gps m ? 2 n pe 5 uijt jt kvtu DT)a+? }1,1,$\frac{4}{5}$⟨)n pe $\mathbb{Z}$+boe gps m ? 4 n pe 5 uijt jt DT)a+? }1,1,$\frac{2}{5}$⟨)n pe $\mathbb{Z}$+/

Dpssftqpoejoh up uiftf bcfnjbo dpoofdujpot bsf b tfu pg k , 4.ejn fotjpobmwfdupst a / Dplfs)M+xijdi xf dbo ublf up cf)vq up n vnmjqnjdbujpo cz M^{-2}+;

$$\begin{gathered})1,\times\times\times,1+\\ )1,\tfrac{2}{3},\tfrac{2}{3},1,\times\times\times,1+\\ )\tfrac{2}{3},\tfrac{2}{5},\tfrac{4}{5},1,\tfrac{2}{3},\times\times\times,1,\tfrac{2}{3}+\end{gathered} \tag{}$$

xifsf opx]...#dpssftqpoet up sfqfujujpo pg 1 jo uif flstuuxp njoft- boe sfqfujujpo pg)1,$\frac{2}{3}$+jo uif uijse njof/ Uif dfoufs tzn n fusz bdut po uiftf wfdupst uispvhi uif beejujpo pguif wfdups

$$)1,\tfrac{2}{3},\tfrac{2}{3},1,\times\times\times,1+, \tag{}$$

boe xf efevdf uibu vqpo n peejoh pvu cz uif dfoufs tzn n fusz hspvq- DT)a+bsf hspvqfe bt }1,2⟨ boe }$\frac{m-3}{5}$⟨/

Uif n busjy $S^{)A+}$jt

$$S^{)A+}? \ \frac{2}{3}\begin{pmatrix}2&2&2&\\2&2&2&\\3&3&3&\end{pmatrix} \tag{}$$

boe uif n busjy $T^{)A+}$jt

$$T^{)A+}? \ \begin{pmatrix}2&1&1&\\1&2&1&\\1&1&\mathrm{f})\frac{m-3}{5}+&\end{pmatrix} \tag{}$$

Bgufs n peejoh pvu uif dfoufs tzn n fusz xf pcubjo uif n busjdft

$$S'^{)A+}? \ \begin{pmatrix}2&2&\\2&2&\end{pmatrix} \tag{}$$

$$T'^{)A+}? \ \begin{pmatrix}2&1&\\1&\mathrm{f})\frac{m-3}{5}+&\end{pmatrix} \tag{}$$

JHEP10(2019)010

boe gpsnfe i pn prphjdbmc rpdl t

$$\begin{array}{l} q^{\frac{m^2+1}{4m}}\widehat{Z_1'})q+?\ \ q^{\frac{m^2+1}{4m}}\Big)\widehat{Z_1})q+,\ \ \widehat{Z_2})q+\Big(?\ \ 3q^{2/5m}\quad\ 3\Omega_2^m)\tau+ \\ q^{\frac{m^2+1}{4m}}\widehat{Z_2'})q+?\ \ q^{\frac{m^2+1}{4m}}\widehat{Z_3})q+?\quad\ 3\Omega_{m-2}^m)\tau+. \end{array} \qquad)7/:\ 8+$$

Vqpo sfwfstf.fohjoffsjoh- x f pctfswf op dpn qrfy ffbu dpoofdujpo cvu)m 2+/3 sfbm opo.bcfrjbo ffbu dpoofdujpot x jui Di fso.Tjn pot jowbsjbout $\frac{r^2}{5m}$ gps pee r /)1, m+/ Y f brtp pcubjo ui f gpmpx joh qfsuvscbujwf fyqbotjpot x i jdi bsf qbjsx jtf jefoujdbm,

$$\begin{array}{ll} \frac{\pi i}{3}k^{-4/3}, & \frac{)m^3\quad 3+\pi^3}{9m}k^{-6/3}, \ \times\times\times \\ \frac{\pi i}{3}k^{-4/3}, & \frac{)m^3\quad 3+\pi^3}{9m}k^{-6/3}, \ \times\times\times \\ \frac{3}{m}k^{-2/3}, & \frac{i)m^3\ ,\ 3+\pi}{7m^3}k^{-4/3}, \ \times\times\times \\ \frac{3}{m}k^{-2/3}, & \frac{i)m^3\ ,\ 3+\pi}{7m^3}k^{-4/3}, \ \times\times\times \end{array} \qquad)7/:\ 9+$$

7 Going to the other side

Jo ui jt tfdujpo x f fyqmpsf x i bu i bqqfot up ui f q.tfsjft jowbsjbout $\widehat{Z_a})M_4$+x i fo ui f psjfo. ubujpo pg ui f ui sff.n bojgpme M_4 jt sfwfstfe/ Bt x jmmcf fyqrbjofe ti psuma- ui jt pqfsbujpo jt fyqfdufe up i bwf ui f ffifdu pg)gpsn bmma+sfqrbdjoh $q \Rightarrow q^{-2}$ boe ui fo sf.fyqboejoh ui f sftvm bhbjo bt b q.tfsjft/ Mvdl jma- qsfdjtfma ui jt rvftujpo x bt joefqfoefoum btlfe cz S befn bdi fs '213^ boe i jt gpmpx fst jo ui f dpoufyu pg)n pdl +n pevrbs pc kfdut boe ui fjs fy. ufotjpo gspn ui f vqqfs i brgqrbof)ps- $\backslash q\backslash < 2$+up ui f mpx fs i brgqrbof)sftqfdujwfma $\backslash q\backslash > 2$+ boe i bt hbjofe n psf buufoujpo tjodf ui f jouspevdujpo pg ui f opujpo pg rvbouvn n pevrbs gpsn t cz [bhjfs '82^/

Jo qbsujdvrbs- x f n ptuma gpdvt po ui f gbn jmjft pg 4.n bojgpmet x i ptf q.tfsjft jowbsjbout $\widehat{Z_a})M_4$+bsf hjwfo cz gbmtf ui fub gvodujpot ejtdvttfe jo tfdujpo 4 boe jmmutusbufe cz fybn qmft jo tfdujpo 7/ Jo ui ftf dbtft- x f qspqptf ui bu ui f q.tfsjft jowbsjbout $\widehat{Z_a})$ M_4+pgb n bojgpme M_4 bsf hjwfo cz n pdl n pevrbs gpsn t x jui ti bepxt)tff tfdujpo 8/4 gps efflojujpot+ bttpdjbufe up ui f gbmtf ui fub gvodujpot/ Y f tvn n bsj–f ui f sfrmujpo jo flhvsf :/

P vs qspqptbmjt tvqqpsufe cz ui f gpmpx joh ui sff gbdut;

≡ Jo tpn f dbtft ui f gbmtf ui fub gvodujpot ben ju fyqsfttjpot bt q.i zqfshfpn fusjd tfsjft- x i jdi dpowfshf opu pom jotjef cvu bmtp pvtjef ui f vojudjsdrf/ Jo ui ptf dbtft pof dbo ftubcmjti ui bu ui f fyqsfttjpo pvtjef ui f vojudjsdrfjt hjwfo cz b n pdl ui fub gvodujpo/

≡ Ui f n pdl ui fub gvodujpo boe ui f dpssftqpoejoh gbmtf ui fub gvodujpo i bwf ui f tbn f btzn qupujd fyqbotjpot usbottfsjft tusvduvsf ofbs x / $\mathbb{Q}$)dg/)5/34+)8/48++ vq up $x \Rightarrow$ x- qsfdjtfma bt $\widehat{Z_a})M_4$+boe $\widehat{Z_a})$ M_4+ti pvma/

≡ Y i fo ui f n pdl n pevrbs gpsn dbo cf fyqsfttfe bt b tp.dbmfe S befn bdi fs tvn - pof dbo qspwf jo hfofsbmui bu ui f tbn f S befn bdi fs tvn - opx qfsgpsn fe jo ui f mpx fs sbui fs ui bo vqqfs i brgqrbof- zjfmt qsfdjtfma ui f dpssftqpoejoh Fjdi rfs jouf hsbm/ Jo

puif sxpset- uif Sbefn bdifs tvn zjfmet b gvodujpo efflofe po cpui $\mathbb{H}$ boe $\mathbb{H}^-$- xifsf uifz dpjodjef xjui uif n pdl sftqfdujwfmz gbnjmjbs uifub gvodujpo/

Bgufs fyqmbjojoh uif qiztjdt boe upqpmphz n pujwbujpo up hp cfuxffo uif vqqfs. boe mpxfs. ibmgqmboft- xf fyqmbjo uif bcpwf uisff qpjout jo tfdujpo 8/3- tfdujpo 8/4 boe tfdujpo 8/5 sftqfdujwfmz/

7.1 The physics of the other side

Bt xf bnsfbez n foujpofe fbsmjfs- bspvoe)3/6+ ju jt obuvsbmup dpn qbsf uif bqqspqsjbuf fyufotjpo- ps]nfibl bhf# pg $\widehat{Z_a)}q+$ up Jn)$\tau+<1$ xjui uif ibmgjoefy)3/2+pcubjofe cz psjfoubujpo sfwfstbm)qbsjuz+usbotgpsn bujpo bqqmjfe up uif psjhjobm3e.4e tztufn po $D^3 *_q S^2$/ Uif uxp bsf fyqfdufe up cf dmptfmz sfmbufe- jg opu tjn qmz frvbm/

Jo puif sxpset- xf xjti up dpn qbsf $\widehat{Z_a)}q^{-2}+$ voefstuppe bt b q.tfsjft fyqbotjpo- xjui uif ibmgjoefy pg 3e.4e tztufn xifsf bmmDifso.Tjn pot dpf´djfout pg uif 4e uifpsz ibwf pqqptjuf tjhot boe xifsf uif 3e $\mathcal{O}$?)1, 3+cpvoebsz dpoejujpo $\mathcal{D}_a$ jt sfqmbdfe cz $\widetilde{\mathcal{D}_a}$;

$$\text{qbsjuz}\ ; \quad \mathcal{D}_a \not\mapsto \widetilde{\mathcal{D}_a} \tag{8/2+}$$

Uif sftvmjoh q.tfsjft ˜ xijdi- bcvtjoh opubujpot- xf efopuf $\widehat{Z_a)}q^{-2}+$˜ uphfuifs xjui uif psjhjobmi pn pmphjdbmcnpdlt $\widehat{Z_a)}q+$bsf fyqfdufe up dpn cjof joup bopuifs q.tfsjft)3/6+ xijdi epft opu efqfoe po uif dipjdf pg cpvoebsz dpoejujpot/ Obn fmz-)3/6+hjwft uif tvqfsdpogpsn bmjoefy $\mathcal{N})q+$pg uif 4e $\mathcal{O}$? 3 uifpsz xijdi- n psfpwfs- dbo cf dpn qvufe joefqfoefoumz- cz puif s nfbot/ Jo uif dpoufyu pg 4e.4e dpssftqpoefodf- up xijdi xf uvso n pn foubsjmz- ju jt cfnjfwfe uibu uif joufhfs dpf´djfout pg uif q.tfsjft $\mathcal{N})q+$dpvou opsn bm tvsgbdft jo uif 4.n bojgpmf “78ˆ/

Y ifo bqqmjfe up 4e $\mathcal{O}$? 3 uifpsjft $T“M_4$ˆ- uif jt qbsjuz sfwfstbmjt frvjwbmfou up di bohjoh uif psjfoubujpo pguif 4.n bojgpmf- j/f/sfqmbdjoh M_4 cz M_4/ Uifsfgpsf- gpsn bmz- xf fyqfdu

$$\widehat{Z_a)}\ M_4, q^{-2}+ \overset{\text{sfof.rboe}}{???}\ \widehat{Z_a)}M_4, q+ \tag{8/3+}$$

Y ijmf jo xibu gpmpxt xf qsftfou gvsuifs qiztjdt bshvn fout gps uif sfmbujpo- uif di bmfohf jt up uvso uifn joup b dpodsfuf dpn qvubujpobmbmhpsjuin / Opuf- cbtfe po pvs fyqfsjfodf jo tfdujpo 7- xf ep opu fyqfdu uif jt bmhpsjuin up cf tjn qmf/ Gps fybn qmf- uif psjfoubujpo sfwfsbmuvsot ofhbujwf.efflojuf qmn cjoht)gps xijdi $\widehat{Z_a)}q+$dbo cf tztufn bujdbmz dpn qvufe jo gvmmhfofsbmjuz+joup qptjujwf.efflojuf poft)gps xijdi op hfofsbmbmhpsjuin t xfsf bwbjmbcmf vojmopx+/

Gspn uif wjfx qpjou pg Y SU jowbsjbout ps rvbouvn Difso.Tjn pot uifpsz- uif cfibw. jps)8/3+jt sbuifs dmfbs- boe uifsfgpsf pof n jhi utbz uibu q.tfsjft jowbsjbout $\widehat{Z_a)}M_4+$tjn qmz joifsjujuuispvhi uif sfmbujpo)2/3+/ Joeffe- uif Difso.Tjn pot qbsujujpo gvodujpo pg M_4 jt efflofe bt b)gpsn bm+joflojuf.ejn fotjpobmjoufhsbm

$$\widehat{Z_{\mathrm{EV})}M_4, k+} ? \qquad e^{ik\mathrm{EV})A+}\mathcal{F}A \tag{8/4+}$$

pwfs uif tqbdf pg hbvhf dpoofdujpot A/ Bu nfbtu gpsn bmz- gspn uijt fyqsfttjpo ju gpmpxt uibu bo psjfoubujpo sfwfstbm$M_4 \leftrightarrow M_4$ jt frvjwbmfou up di bohjoh uif tjho $k \leftrightarrow k$/ Jg xf

JHEP10(2019)010

opx sfdbmmuif tuboebse sfmbujpo cfuxffo k- $\hbar$- boe q- dg/)2/3+

$$q \;?\; e^{\hbar} \;?\; e^{3\pi i\tau} \;?\; e^{3\pi i/k} \qquad)8/5+$$

ui fo xf dpodmvef uibu $M_4 \leftrightarrow \quad M_4$ ti pvme cf frvjwbmfou up $q \leftrightarrow q^{-2}$/

Y i jnh uijt bshvn fou- cbtfe po Gfzon bo qbui joufhsbmn bz tpvoe b njuumfh gpsn bmju jt fbtz up tff uibu ju ti pvme i pme up bmmpsefst jo qfsuvscbujpo uifpszcz fyqboejoh)8/4+ joup Gfzon bo ejbhsbn t bspvoe b hjwfo ffbu dpoofdujpo α / $\mathcal{N}$ -bu)$G_{\mathbb{C}}, M_4$+/ Jo Di fso. Tjn pot uifpsz xjui dpn qmfy hbvhf hspvq $G_{\mathbb{C}}$ tvdi qfsuvscbujwf fyqbotjpo jt dbssjfe pvu fyqmjdjumz f/h/ jo '79^- boe uif dpf´ djfou pg fbdi ufsn jo uif $\hbar$.fyqbotjpo jt hjwfo cz b flojuf.ejn fotjpobmjoufhsbm/Uif sftvmjt bo btzn qupujd fyqbotjpo- dg/)5/23+

$$Z^{)\alpha+}_{\mathrm{rfsu}})M_4, \hbar+ ? \;\prod_n a_n \hbar^n \qquad)8/6+$$

xijdi hfofsbjft uif Piutvlj tfsjft pg M_4)uif nbuufs dpssftqpoet up α ? 1/+ Fwfo jo dpn qmfy Di fso.Tjn pot uifpsz- xifsf q boe $\hbar$ bsf dpn qmfy wbsjbcmft- uif qfsuvscbujwf fyqbotjpo ibt b tzn n fusz $Z^{)\alpha+}_{\mathrm{rfsu}})M_4, \hbar+? \; Z^{)\alpha+}_{\mathrm{rfsu}}) \; M_4, \quad \hbar+$ dg/ '79- tfdujpo 3/4^- tp uibu

$$Z^{)\alpha+}_{\mathrm{rfsu}}) \;\; M_4, \hbar+ ? \;\prod_n) \;\; 2\#^n a_n \hbar^n. \qquad)8/7+$$

Tjodf uif q.tfsjft jowbsjbout $Z_a)M_4+$boe $Z_a) \;\; M_4+$bsf pcubjofe cz Cpsfmsftvn n bujpo pg)8/6+boe)8/7+gps bcfmjbo α ? a- uifz upp bsf fyqfdufe up fokpz uif qspqfsuz)8/3+/

Fwfo ui pvhi)8/6+boe)8/7+mppl wfsz tjn jmbs boe bqqfbs po uif tbn f gppujoh- jo qsbdujdf- tp gbs ju xbt n vdi fbtjfs up dpn qvuf pomz pof pg uif q.tfsjft jowbsjbout- $\widehat{Z_a)}M_4+$ ps $\widehat{Z_a)} \;\; M_4+$ xijnh uif puifs sfn bjofe fmmtjwf/ Uijt btzn n fusz cfuxffo M_4 boe $\quad M_4$ n bz tffn tvsqsjtjoh gspn uif upqpmphz wjfx qpjou/ I pxfwfs- gspn uif wjfx qpjou pg sftvshfou bobmztjt- jujt sfmbujwfmz xfmm opxo ui bu bn poh ux p btzn qupujd fyqbotjpot-)8/6+boe)8/7+ vtvbmmz pof n bz i bwf b sfmbujwfmz tjn qmf Cpsfmsftvn n bujpo- xifsfbt uif puifs pof dbo cf n vdi n psf dpn qmjdbufe '7: ^/ Tjn jmbsmz- gspn uif wjfx qpjou pg uifjs n pevmbs cfi bwjps- xijdi xjmmpddvqz uif sftu pg uijt tfdujpo- uif ux p tjeft bmtp vtvbmmz qmbz sbuifs ejfifsfou btzn n fusjd spmf/

Jo qbsujdvmbs- jo uif sftu pg uijt tfdujpo xf vtf b wbsjfuz pg n fui pet boe sfdfou efwfmpqn fout jo ovn cfs uifpsz up botxfs b rvftujpo jo upqpmphz; hjwfo $\widehat{Z_b)}M_4+$boe- qpttjcmz- tpn f cbtjd upqpmphjdbmjowbsjbout pg M_4- dbo pof efufsn jof $\widehat{Z_b)} \;\; M_4+$A

7.2 Examples: q-hypergeometric series

Jo uijt tvctfdujpo- xf hjwf fybn qmft jmmvtusbujoh i px dfsubjo gbnjmjft uifub gvodujpot- xijdi qmbz uif spmf pg i pn pmphjdbmcmpdl t gps dfsubjo uisff.n bojgpmet)dg/ tfdujpo 7+ dbo cf effflofe jo uif puifs tjef pg uif qmbof vtjoh uifjs fyqsfttjpot bt q.izqfshfpn fusjd tfsjft/ Tvsqsjt. johmz- po uif puifs tjef pg uif qmbof uifz uvso pvu up dpjodjef xjui tpn f pg S bn bovkbo(t gbn pvt n pdl uifub gvodujpot/ Uijt ftubcmjti ft jo b wfsz ejsfdux bz uif dpoofdujpo cfuxffo n pdl n pevmbs gpsn t- gbnjmjft uifub gvodujpot- boe uisff.n bojgpmet- bu mfbtu gps uiftf fybn . qmft/ Bgufs sfwjfxjoh uif fybn qmft- xf xjmmbmtp eftdsjcf uif bn cjhvjujft xifo fyufoejoh b gvodujpo up uif mpxfs.i bmg qmbof wjb q.izqfshfpn fusjd tfsjft/

7.2.1 Example: $M(-2;\frac{1}{2},\frac{1}{3},\frac{1}{2})$ and the order three mock theta function f

Jo)7/43+ xf i bwf tffo ui bu ui f gbnjmz ui fub gvodujpo $\Omega_2^{7-3})\tau+$ dpjodjefts- vq up beej. ujwf boe n vmjqmjdbujwf tjn qrfs q gbdupst- xjui ui f i pn pmphjdbmcmpdl t $\widehat{Z_{)2,-2,-2,-2+}})q+$ boe $\widehat{Z_{)4,-4,-2,-4+}})q+$ gps ui f ui sff.n bojgpme $M)\ \ 3=\frac{2}{3},\frac{2}{4},\frac{2}{3}+/$

Up x sjuf epx o ui f sfnfwbou q.i zqfshfpn fusjd tfsjfts- xf vtf ui f *q-Pochhammer symbol*

$$)a=x+_n ;?\ \prod_{k=0}^{n-2})2\ \ ax^k+ \tag{8/8+}$$

tbujtgzjoh

$$)a=q^{-2}+_n ?\ \ )\ \ 2+^n a^n q^{-\frac{n(n-1)}{2}})a^{-2}=q+_n. \tag{8/9+}$$

Opuf ui bu ui jt gbnjmz ui fub gvodujpo ben jut ui f fyqsfttjpo ‘53^

$$\Omega_2^{7-3})\tau+?\ \ \psi_2^{7-3})q+;\ \ \psi_2^{7-3})q+?\ \ \frac{q^{\frac{1}{24}}}{3}\right)2\ \ \prod_{n\geq 2}\frac{)\ \ 2+^n q^{\frac{n(n-1)}{2}}}{)\ \ q=q+_n}\Bigg(, \tag{8/:+}$$

gps $\backslash q\backslash < 2$ ∞ $\tau\ /\ \mathbb{H}/$ N psfpwfs- ui f tfsjft ψ_2^{7-3} dpowfshft cpui gps $\backslash q\backslash > 2$ boe $\backslash q\backslash < 2/$ Vtjoh)8/9+pof pcubjot

$$\psi_2^{7-3})q^{-2}+?\ \ \frac{q^{-\frac{1}{24}}}{3}\right)2\ \ \prod_{n\geq 2}\frac{)\ \ 2+^n q^n}{)\ \ q=q+_n}\Bigg(. \tag{8/21+}$$

Ju uvsot pvu ui bu ui jt jt b n pdl n pevnbs gpsn - sfnbufe up ui f dfnfcsbufe psefs ui sff n pdl ui fub gvodujpo $f)q+$bt

$$3q^{\frac{1}{24}}\psi_2^{7-3})q^{-2}+?\ \ f)q+?\ \ 2,\ \ q\ \ \ 3q^3,\ \ 4q^4,\ \ O)q^5+. \tag{8/22+}$$

Bt x f xjmtff jo tfdujpo 8/6- ju cfnpoht up b gbn jmz pg tqfdjbmwfdups.wbmufe n pdl n pevnbs gpsn t $h^{m-K}?\ )h_r^{m-K}+=$jo ui f opubujpo pg tfdujpo 8/6 ui f sfnbujpo jt tjn qmz

$$\psi_2^{7-3})q^{-2}+?\ \ \ \frac{2}{3}h_2^{7-3})\tau+. \tag{8/23+}$$

Gspn ui f bshvn fou jo tfdujpo 8/2- xf i fodf qspqptf ui bu ui f n pdl ui fub gvodujpo $f)q+$ qnbzt b sprfs bt ui f i pn pmphjdbmcmpdl gps ui f ui sff.n bojgpme ui bujt sfnbufe up $M)\ \ 3=\frac{2}{3},\frac{2}{4},\frac{2}{3}+$ wjb bo psjfoubujpo sfwfstbm/ Bt xf xjmmejtdvtt ti psuno- ui jt fybn qmf brtp jmvtusbufs ui f jousjotjd bn cjhvjuz pg ui f q.i zqfshfpn fusjd bqqspbdi)tff)8/37+#/

7.2.2 Example: $M(-2;\frac{1}{2},\frac{1}{2},\frac{3}{5})$ and the order ten mock theta function X

Bt xf xjmtff jo tfdujpo 8/6- ui f gbnjmz ui fub gvodujpo

$$\Omega_2^{21-3})\tau+?\ \ \Omega_{21,2})\tau+\ \ \Omega_{21,;})\tau+?\ \ q^{2/51})2\ \ \ q^3,\ \ q^4\ \ \ q^;,\ \ O)q^{21}++ \tag{8/24+}$$

qnbzt ui f sprfs pg i pn pmphjdbmcmpdl t gps $M_4\ ?\ \ M)\ \ 3=\frac{2}{3},\frac{2}{3},\frac{4}{6}+/$

Opuf ui bu ui jt gbnjmz ui fub gvodujpo ben jut ui f fyqsftttjpo

$$\Omega_2^{21-3)}\tau+?\ \psi_2^{21-3)}q+;\ \psi_2^{21-3)}q+?\ q^{\frac{1}{40}}\prod_{n\geq 1}\frac{)\ 2-\#q^{n)n-2+}}{)\ q=q+3n} \qquad)8/25+$$

gps \q\< 2/ Tjn jmbs up ψ_2^{7-3}- ui f tfsjft ψ_2^{21-3} dpowfshft cpui gps \q\> 2 boe \q\< 2 boe i bt ui f gpnpx joh sfmbujpo up ui f pqujn bnn pdl Kbdpcj gpsn)tfdujpo 8/6+boe psefs 21 n pdl ui fub gvodujpo X;

$$\begin{aligned}\psi_2^{21-3)}q^{-2}+?\ q^{-\frac{1}{40}}\prod_{n\geq 1}\frac{)\ 2-\#q^{n^2}}{)\ q=q+3n}\ ?\quad h_2^{21-3)}\tau+\\ ?\ q^{-\frac{1}{40}}X)q+?\ q^{-\frac{1}{40}}\ \ 2\quad q\ ,\ q^3\ ,\ O)q^5-\end{aligned} \qquad)8/26+$$

7.2.3 Example: $\Sigma(2,3,5)$ and the order five mock theta function χ_0

Bt xf i bwf ejtdvttfe jo tfdujpo 7/7/2- ui f gbnjmz ui fub gvodujpo

$$\Omega_2^{41-7,21,26)}\tau+?\ )\Omega_{41,2}\ ,\ \Omega_{41,22}\ ,\ \Omega_{41,2;}\ ,\ \Omega_{41,3;}-)\tau+?\ q^{2/231})2\ ,\ q\ ,\ q^4\ ,\ q^8\ ,\ O)q^9+-)8/27+$$

qmbzt ui f sprfn pg i pn pnphjdbmcnpdl t gps ui f i pn pnphz tqi fsf $\Upsilon)3,4,6-$/

Opuf ui bu ui jt gbnjmz ui fub gvodujpo ben jut ui f fyqsftttjpo

$$\Omega_2^{41-7,21,26)}\tau+?\ \psi_2^{41-7,21,26)}q+;\ \psi_2^{41-7,21,26)}q+?\ q^{\frac{1}{120}}\Big)3\ \ \prod_{n\geq 1}\frac{)\ 2-\#q^{\frac{n(3n-1)}{2}}}{)q^{n-2}=q+n}\Bigg[\qquad)8/28+$$

gps \q\< 2/ Bt cfgpsf- ui f tfsjft $\psi_2^{41-7,21,26}$ dpowfshft cpui gps \q\> 2 boe \q\< 2 boe i bt ui f gpnpx joh sfmbujpo up ui f pqujn bnn pdl Kbdpcj gpsn boe psefs 6 n pdl ui fub gvodujpo;

$$\begin{aligned}\psi_2^{41-7,21,26)}q^{-2}+?\ q^{-\frac{1}{120}}\Big)3\ \ \prod_{n\geq 1}\frac{q^n}{)q^{n-2}=q+n}\Bigg[\ ?\quad h_2^{41-7,21,26)}\tau+\\ ?\ q^{-\frac{1}{120}})3\ \ \chi_1)q-+?\ \ q^{-\frac{1}{120}}\ \ 2\ ,\ q\ ,\ q^3\ ,\ 3q^4\ ,\ O)q^5-\end{aligned} \qquad)8/29+$$

Ju jt gps ui jt dbtf ui bu ui f sfmbujpo cfuxffo gbnjmz ui fub gvodujpot boe Y SU jowbsjbout xbt flstu ejtdvttfe cz Mbxsfodf boe [bhjfs jo "49^- xi fsf ui fz bmtp opufe ui f sfmbujpo up ui f n pdl ui fub gvodujpo χ_1/

7.2.4 Example: $\Sigma(2,3,7)$ and the order seven mock theta function F_0

Bt xf i bwf ejtdvttfe jo tfdujpo 7/7/2- ui f gbnjmz ui fub gvodujpo

$$\Omega_2^{53-7,25,32)}\tau+?\ )\Omega_{53,2}\quad \Omega_{53,24}\quad \Omega_{53,3;}\ ,\ \Omega_{53,52}-)\tau+?\ q^{2/279})2\quad q\quad q^6\ ,\ O)q^{21}+-)8/2:+$$

qmbzt ui f sprfn pg i pn pnphjdbmcnpdl t gps ui f i pn pnphz tqi fsf $\Upsilon)3,4,8-$/

Opuf ui bu ui jt gbnjmz ui fub gvodujpo ben jut ui f fyqsftttjpo

$$\Omega_2^{53-7,25,32)}\tau+?\ \psi_2^{53-7,25,32)}q+;\ \psi_2^{53-7,25,32)}q+?\ q^{\frac{1}{168}}\prod_{n\geq 1}\frac{)\ 2-\#q^{\frac{n(n+1)}{2}}}{)q^{n-2}=q+n} \qquad)8/31+$$

gps \q\< 2/ Bt cfgpsf- ui f tfsjft $\psi_2^{53-\ 7,25,32}$ dpowfshfs cpui gps \q\> 2 boe \q\< 2 boe i bt ui f gpmpx joh sfmbujpo up ui f pqujn bmn pdl Kbdpcj gpsn boe psefs 8 n pdl ui fub gvodujpo;

$$\psi_2^{53-\ 7,25,32})q^{-2}+?\ \ q^{-\frac{1}{168}}\prod_{n\geq 1}\frac{q^{n^2}}{)q^{n-\ 2}=q+n}\ ?\quad H_2^{53-\ 7,25,32})\tau+ \\ ?\ \ q^{-\frac{1}{168}}\ F_1)q+?\quad q^{-\frac{1}{168}}\ \ 2\ ,\ \ q\ ,\ \ q^4\ ,\ \ q^5\ ,\ \ O)q^6+\{ \qquad)8/32+$$

Gspn ui f bshvn fou jo tfdujpo 8/2- xf i fodf qspqptf ui bu ui f n pdl ui fub gvodujpo jt ui f i pn pnphjdbmcmpdl pguif ui sff.n bojgpmn pcubjofe gspn $\Upsilon)3,4,8+$wjb ui f psjfoubujpo sfwfstbm/

7.2.5 The ambiguity

Ui f bcpwf fybn qmft jmmtusbuf bo fyqmjdju sfmbujpo cfuxffo gbnmf boe n pdl ui fub gvodujpot x i fo hpjoh cfuxffo ui f vqqfs. boe ui f mpxfs.i bmgqmbof/ Pf ti pvmn i px fwfs cf dbvujpvt bc pvu ui f bqqmjdbcjmjuz boe ui f bn cjhvjuz pguif usfbun fou/

Gjstu- ui jt usfbun fou efqfoet po ui f fyjtufodf pg bo fyqsfttjpo pg gbnmf0n pdl ui fub gvodujpo bt b q.i zqfshfpn fusjd tfsjft/ Jo n boz dbtft joufsftujoh gps vt- tvdi bo fyqsfttjpo jt opu bwbjmbcmf/ N psf pwfs- tpn fujn ft n psf ui bo pof tvdi fyqsfttjpot fyjtu boe ui fz n jhi u i bwf ejfifsfou fyufotjpo pvutjef ui f voju ejtl / Tvdi fybn qmft bc pvoe/ Tff gps jotubodf '81^ x i fsf ui f S phfst.Gjof gbnmf ui fub gvodujpot bsf fyufoefe up ui f puifs tjef jo b tqfdjfld x bz x i jdi mfbet up n pdl gpsn t ui bu tpn fujn ft ejfifs gspn x i bu puifs n fui pet ejtdvttfe jo tfdujpo 8/4–8/5 hjwf/ Y f xjmmopx fyqmbjo pof fyqmjdju fybn qmf jo efubjm up jmmtusbuf ui f bn cjhvjujft/

Ui f sfmbujpo cfuxffo ui f psefs ui sff n pdl ui fub gvodujpo $f)q+$boe puifs n pdl ui fub gvodujpo- jotjef boe pvutjef ui f voju ejtd- i bt cffo tuvejfe jo efubjm jo '53^- x i jdi xf gpmpx i fsf/ Gjstu x f i bwf tffo jo)8/: +)8/22+ui bu ui f i zqfshfpn fusjd tfsjft $\psi_2^{7-\ 3}$ tbujtgift

$$\psi_2^{7-\ 3})q+?\ \ \Omega_2^{7-\ 3})q+\text{boe}\ \ \psi_2^{7-\ 3})q^{-2}+?\ \ \frac{q^{-\frac{1}{24}}}{3}f)q+ \qquad)8/33+$$

gps \q\< 2/

Opx- efflof puifs ux p i zqfshfpn fusjd tfsjft

$$\psi')q+?\ \ \frac{q^{2/35}}{3}\Big)2\ ,\ \prod_{n\geq 2}\frac{q^n}{)\ \ q=q+\frac{3}{n}}\Bigg[\\ \psi'')q+?\ \ q^{2/35}\prod_{n\geq 1}\frac{q^n}{)\ \ q^3=q^3+n}. \qquad)8/34+$$

Pf dbo fbtjmz di fdl ui bu ui fz upp bsf efflofe cpui jotjef boe pvutjef ui f voju ejtl / Ju uvsot pvu ui bu ui fz bsf sfmbufe up $\psi_2^{7-\ 3})q+$jo b wfsz joufsftujoh x bz/ Up eftdsjcf ui f sfmbujpo- xf offe up jouspevdf ux p n psf gvodujpot/ Ui f flstu jt b n pevmbs gpsn hjwfo cz

$$T)\tau+;?\ \ \frac{\eta^8)3\tau+}{\eta^4)\tau+\eta^4)5\tau+}, \qquad)8/35+$$

boe ui f tfdpoe jt ui f sbujp pg b gbnmf ui fub gvodujpo boe b n pevmbs gpsn

$$S)\tau+;?\ \ \frac{2}{\eta^3)\tau+}\Omega_{3,2})\tau+ \qquad)8/36+$$

Ui ftf gvodujpot bsf sfmbufe wjb

$$\psi_2^{7-3})q+?\ \psi')q+,\ \frac{2}{3}S)\tau+?\ \psi'')q+$$
$$\psi_2^{7-3})q^{-2}+?\ \psi')q^{-2}+?\ \psi'')q^{-2}+\ \frac{2}{3}T)\tau\ ,\ 2/3+ \qquad)8/37+$$

gps $\backslash q\backslash < 2 \infty\ \tau\ /\ \mathbb{H}/$ Jo puif s xpset- uif uxp $q.$i zqfshfpn fusjd tfsjft x i jdi bsf uif tbn f jo uif vqqfs.i brgqmbof n jhi ufyufoe up ejfifsfou gvodujpot jo uif npx fs.i brgqmbof- boe wjdf wfstb/ Jo uif gpmpxjoh ux p tvctfdujpot xf x jmtff b n psf tztufn bujd xbz pg eftdsjcjoh boe voefstuboejoh uif sfmbujpo cfuxffo n pdl boe gbmtf uifub gvodujpot- bt x fmmbt r vbouvn n pevmbs gpsn t/

7.3 False, mock, and quantum

Jo tfdujpo 4 boe tfdujpo 7 xf i bwf tffo uif sprfn pg gbmtf uifub gvodujpot jo eftdsjcjoh uif i pn prphjdbmc npdl t bttpdjbufe up dfsubjo uisff.n bojgpmet/ Jo uif qsfwjpvt tvctfdujpot xf i bwf tffo i jout ui bu- x i fo dpotjefsjoh uif tvqfsdpogpsn bmjoejdft cz wfouvsjoh up uif npx fs.i brgqmbof- n pdl uifub gvodujpot bsf njl fmz up qmbz bo jn qpsubou sprfn gps uif sfmbufe uisff.n bojgpmet/ Jo gbdu- eftqjuf uifjs wfsz ejfifsfou bqqfbsbodft boe n pevmbs cfi bwjpst- gbmtf boe n pdl uifub gvodujpot cpui ti bsf uif tusvduvsf pg uif tp.dbmfe r vbouvn n pevmbs gpsn t '82^/ Tff bmtp uif Di 32 pg '83^ gps b sfdfou bddpvou/

Y f qspqptf ui bu uif)tuspoh+r vbouvn n pevmbsjuz pg uif gbmtf boe n pdl uifub gvodujpot jt jo gbdu x i bu n bl f t uifn sfmfwbou gps uisff.n bojgpmet boe i pn prphjdbmc npdl t/ N psfpwfs- xf qspqptf ui bu hpjoh up uif puif s tjef pg uif qmbof jo uif dvssfou dpoufyu uvsot b gbmtf uifub joup b n pdl uifub- tvdi ui bu uif gbmtf.n pdl qbjs dpssftqpoet up uif tbn f r vbouvn n pevmbs gpsn /

Up fyqmbjo uif tf jefbt- xf x jmmtubsu cz sfdbmjoh uif efflojujpot pgn pdl n pevmbs gpsn t boe r vbouvn n pevmbs gpsn t/ Ui f efflojujpo gps r vbouvn n pevmbs gpsn t jt qvsqptfmz b njuumf wbhvf jo psefs up fodpn qbtt uif ejfifsfou uzqft pg fybn qmft x jui tnjhi umz ejfifsfou qspqfsujft '82^/ Ju tubuft;

Definition 2. '82^B *quantum modular* gpsn pgx fjhi uk boe n vmjqmjfs χ po jt b gvodujpo Q po $\mathbb{Q}$ tvdi ui bu gps fwfsz γ / uif gvodujpo $p_\gamma\ ;\ \mathbb{Q}\ \}\gamma^{-2}\in\ \langle\ \leftrightarrow\ \mathbb{C}$- efflofe cz

$$p_\gamma)x+;?\ Q)x+\ \ Q\backslash_{k,\chi}\gamma)x+ \qquad)8/38+$$

)uif] qfsjpe gvodujpo#+i bt tpn f qspqfsuz pg dpoujovjuz ps bobmzujdjuz gps fwfsz γ / / N psfpwfs- x f tbz ui bu Q jt *strong quantum modular* jg ju i bt gpsn bmqpx fs tfsjft buubdi fe up fbdi sbujpobmovn cfs tp ui bu)8/38+i pmet bt bo jefoujuz cfuxffo dpvoubcmf dpnbujpot pg gpsn bmqpx fs tfsjft/

Jo uif bcpwf x f i bwf vtfe uif *slash operator* gps x fjhi uk boe n vmjqmjfs χ po - bdujoh po uif tqbdf pg i pmpn psqi jd gvodujpot po uif vqqfs.i brgqmbof boe efflofe bt

$$f)\tau+\backslash_{k,\chi}\gamma\ ?\ f\ \Big)\frac{a\tau\ ,\ b}{c\tau\ ,\ d}\Big[\ \chi)\gamma+)c\tau\ ,\ d+^{-k} \qquad)8/39+$$

x i fsf x f x spuf γ ? $\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\langle$ / /

JHEP10(2019)010

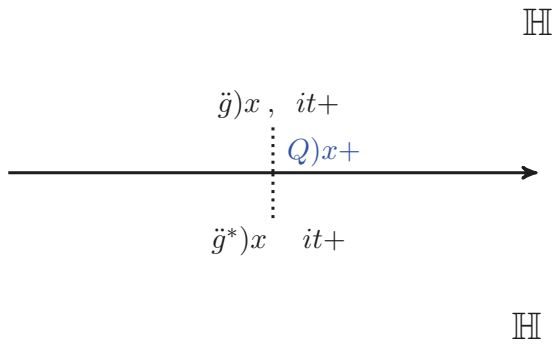


Figure 8/ Ui f vqqfs. boe mpxfs.i brgqrhoft boe rvbouvn n pevrhs gpsn t/

Jo gbdu- uif Fjdirfs jousfhsbmx f fodpvoufsfe jo tfdujpo 5/2 bsf fybn qrfit pg rvbouvn n pevrhs gpsn t/ Up fyqrhjo uijt- efflof uif opo.i prpn psqi jd Fjdirfs jousfhsbmg* ; $\mathbb{H}^- \leftrightarrow \mathbb{C}$

$$g*)z+;? \ C \int_{\overline{\imath}}^{i\infty} g)z'+)z' \quad z+^{w-3}dz' \qquad)8/3:+$$

pg b xfjhiu w dvtq gpsn g xjui n vmjqmjfs χ- xifsf uif dpotubou C jt uif tbnf bt jo uif effljujpo pguif Fjdirfs jousfhsbm)5/4+/ Gps uif qvsqptf pguif qsftfou bsujdrfs- xf dbo sftusjdu up uif dvtq gpsn t g xjui sfbmdpf´ djfout- obn fmz $\overline{g)\ \ t+}$? $g)\tau+$ Opuf uibu g*)z+ i bt ojdf usbotgpsn bujpo qspqfsuz xijnfi g)τ+i bt ojdf Gpvsjfs fyqbotjpot/ Jo "49- 84ˆ juxbt tipxo uibu g* boe uif Fjdirfs jousfhsbm)5/3+g bhsff up joflojuf psefs buboz x / $\mathbb{Q}$- jo uif tfotf uibu

$$g)x , \ it+\subset \prod_{n\geq 1} \alpha_n t^n \text{ boe } g*)x \quad it+\subset \prod_{n\geq 1} \alpha_n) \ \ t+^n \qquad)8/41+$$

gps $t > 1$/ Tff flhvsf 9 gps bo jmmatusbujpo/

Gvsuifsn psf- jujt fbtz up tff uibu g* jt ofbsmz n pevrhs pgxfjhiu 3 w jo $\mathbb{H}^-$- boe uif ejtdsfqbodz jt hjwfo qsfdjtfmz cz uif qfsjpe gvodujpo;

$$g*)z+ \ \ g*\backslash_{3-w,\chi}\gamma)z+? \ C \int_{\gamma^{-1})i\infty+}^{i\infty} g)z'+)z' \quad z+^{w-3}dw. \qquad)8/42+$$

Dpn cjojoh uif bcpwf uxp gbdut xf bsf jn n fejbufmz rfe up uif dpodmvtjpo uibu g *is a quantum modular form of weight* 3 w boe n vmjqmjfs tztufn χ/ Jo uif opubujpo pgefflojujpo 3- uif qfsjpe gvodujpo dpssftqpoejoh up g jt hjwfo cz

$$p_\gamma)x+? \ C \int_{\gamma^{-1})i\infty+}^{i\infty} g)z'+)z' \quad x+^{w-3}dw \qquad)8/43+$$

boe jt b tn ppui gvodujpo po $\mathbb{R}$ fydfqu gps x ? $\gamma^{-2})i\in$ +boe i bt bo bobmzujd fyufotjpo up }u, $iv\backslash u > 1$ ps $v > 1\langle$)dg/rfm n b 4/4 jo "9ˆ+/ Jo qbsujdvsbs- uif gbdtf uifub gvodujpot $\Omega_{m,r}$- bsjtjoh gspn ubljoh uif dvtq gpsn g up cf hjwfo cz uif xfjhiu 403 vobsz uifub gvodujpot $\theta^2_{m,r}$- bsf rvbouvn n pevrhs gpsn t pgxfjhiu 203/

Tppo xf xjmtff uibu n pdl n pevrhs gpsn t qspevdf fybn qrfit pg rvbouvn n pevrhs gpsn t/ Jo i jt rhtu rfuufs up I bsez jo 2: 31- S bn bovkbo dpotusvdufe 28 fybn qrfit pgxi buif

dbnnfe n pdl ui fub gvodujpot boe dnbjn fe ui bu ui fz i bwf b gfx tusjl joh qspqfsujft sfhbsejoh ui fjs cfi bwjps ofbs ui f spput pgvojuz/ S bn bovkbo eje opu hjwf b efflojujpo gps n pdl ui fub gvodujpot- cvu tubufe ui bu ui fz ti pvnn cf b *q*.tfsjft ui bu dpowfshft gps $|q| < 2$ ui bu i bwf ui f gpnpx joh qspqfsujft

2/ Joflojufmz n boz spput pgvojuz bsf fyqpofoujbmtjohvnhsjujft-

3/ Gps fwfsz sppupgvojuz ξ ui fsf jt b n pevnhs gpsn $f_\xi)q+$tvdi ui bu ui f ejfifsfodf $f \quad f_\xi$ jt cpvoefe bt $q \leftrightarrow \xi$ sbejbmmz-

4/ f jt opu ui f tvn pg ux p gvodujpot- pof pg x i jdi jt b n pevnhs gpsn boe ui f puifs b gvodujpo x i jdi jt cpvoefe sbejbmmz upx bse bmnspput pgvojuz/

Ui f npoh tfbsdi gps b efflojujpo pgn pdl n pevnhs gpsn t- x i jdi x pvnn qnbdf n pdl ui fub gvodujpot jo ui f dpoufyu pg n pevnhs gpsn t- foefe x jui ui f Qi E ui ftjt pg [xfhfst '85^- x i ifsf i f hbwf n pdl n pevnhs gpsn t b efflojujpo- x i jdi cbtjdbmmz tubufs ui bu ui fz dbo cf wjfxfe bt ui f i pnpn psqi jd qbsu pg dfsubjo i bsn pojd N bbtt gpsn t/ N psfpwfs- ui f puifs- opo.i pnpn psqi jd- qbsu pg ui f i bsn pojd N bbtt gpsn jt hjwfo cz b n pevnhs gpsn - dbnnfe ui f *shadow* pg ui f n pdl n pevnhs gpsn / Tjodf x f i bwf b tqfdjfld bqqnjdbujpo jo n joe boe jo psefs up tjn qnjgz ui f ejtdvttjpo- jo ui f gpnpx joh efflojujpo x f sftusjdu up n pdl n pevnhs gpsn t x i ptf ti bepxt bsf dvtq gpsn t/ Ui f hfofsbnj-bujpo jt tuboebse boe tusbjhi ugpsx bse/

Definition 3. Y f tbz ui bu b i pnpn psqi jd gvodujpo f po $\mathbb{H}$ jt b *mock modular form* pg x fjhi u k boe n vnmjqnjfs χ po - jgboe pomn jgjufyjtut b x fjhi u 3 k dvtq gpsn g po tvdi ui bu ui f opo.i pnpn psqi jd *completion* pg f- efflofe bt

$$f^\wedge)\tau+? \ f)\tau+ \ g^*)\tau+$$

tbujtflft $f^\wedge? \ f^\wedge|_{k,\chi}\gamma$ gps fwfsz γ / / Jo ui f bcpwf- x f efflofe ui f opo.i pnpn psqi jd Fjdi nfs jouf hsbm

$$g^*)\tau+;? \ C \int_{-\bar{\tau}}^{i\infty})\tau', \ \tau+^{-k}\overline{g)\ \bar{\tau}'+}d\tau' \qquad)8/44+$$

gps τ / $\mathbb{H}$/

Opuf ui bu ui fsf jt op dbopojdbmopsn bnj-bujpo pgui f ti bepx boe x f di pptf pvst up tjn . qnjgz ui f dpn qbsjtpo cfux ffo n pdl n pevnhs gpsn t boe Fjdi nfs jouf hsbm/ Gps dpowfojfodf- x f x jmmefopuf cz $\mathbb{M}_{k,\chi}) + M^!_{k,\chi}) + S_{k,\chi}) + Q_{k,\chi}) +$ui f tqbdft pgn pdl n pevnhs- x fbl nn i pnpn psqi jd n pevnhs- dvtq boe rvbouvn n pevnhs gpsn t sftqfdujwfmz- pgx fjhi u k / $\frac{2}{3}\mathbb{Z}$ boe n vnmjqnjfs χ gps ui f hspvq $< \mathrm{TM})3, \mathbb{R}+/$ Jo ui f qsftfou bsujdmf x f xjmm bjonz fodpvoufs ui f dbtft ? $\mathrm{TM})3, \mathbb{Z}+$boe ? $_1)N+$ ui f dpohsvfodf tvchspvq pg $\mathrm{TM})3, \mathbb{Z}+$x jui ui f dpohsvfodf dpoejujpo N\c/ Y f x jmmbmtp efflof ui f *shadow map* ξ ; $\mathbb{M}_{k,\chi}) +\leftrightarrow S_{3-k,\bar{\chi}}) +$ cz nfuujoh $\xi)f+? \ g$ jo ui f opubujpo pgefflojujpo 4/

Jo x i bu gpnpx t x f x jmtff b sfnbujpo cfux ffo ui f bcpwf n pefso efflojujpo pgn pdl n pevnhs gpsn t boe ui f di bsbdufsj–bujpot S bn bovkbo hbwf jo ijt nfuufs- boe i px n pdl n pevnhs gpsn t nfibe up rvbouvn n pevnhs gpsn t jo b x bz ui bu jt dnptfmz sfnbufe up ui f dbtf

JHEP10(2019)010

pg Fjdi rfs joufhsbtn ejtdvttfe bc pwf/ Y f x jmmgpmpx ui f xpsl cz Di pj.Mjn .S i pbeft ‘9ˆ r vjuf dmptfmz jo ui jt qbsu pg ui f ejtdvttjpo/

Up ti px ui bu n pdl ui fub gvodujpot ep i bwf ui f bc pwf.n foujpofe qspqfsujft ui bu S b. n bovkbo dmbjn fe- ui f gpmpx joh x bt qspwfo sfdfoumz/

Theorem 4. *[8, 75] If f / $\mathbb{M}_{k,\chi}$)N+ such that it has non-vanishing shadow, and)N+ has t inequivalent cusps, }$q_2, \dots, q_t$⟨ ←ℚ ∩ }i∈ ⟨. Then*

1. *The function f)τ+ has exponential singularities at infinitely many rational numbers,*
2. *for every G / $M^!_{k,\chi}$)N+, f G has exponential singularities at infinitely many rational numbers,*
3. *there is a collection }G_j⟨$^t_{j\mathrm{C}2}$ of weakly holomorphic modular forms such that f G$_j$ is bounded towards all cusps equivalent to q_j.*

B gbn pvt fybn qrfi pg ui f bc pwf jt ui f ui jse psefs n pdl ui fub gvodujpo pg S bn bovkbo ui bu x f i bwf fodpvoufsfe jo)8/22+-)8/23+/ S bn bovkbo(t pctfswbujpo- x sjuufo jo ufsn t pg ui f n pdl n pevrhs gpsn $h_2^{7-3})\tau+$ tubuft ui bu

$$\lim_{\tau\to\zeta} h_2^{7-3})\tau+? \ O)2+ \qquad)8/45+$$

gps bmmspput pg vojuz f)ζ+pg ppee psefs)tvdi bt $q \leftrightarrow 2+$ boe

$$\lim_{\tau\to\zeta})h_2^{7-3}, \ ) \ 2\neq^k b)\tau++? \ O)2+ \qquad)8/46+$$

gps bmmpsefs 3k spput pg vojuz f)ζ+ x jui ui f n pevrhs gpsn tvcusbdujpo hjwfo cz $b)\tau+? \ \frac{\eta^3)\tau+}{\eta^2)3\tau+}$/

N psfpwfs- bgufs ui f n pevrhs tvcusbdujpo ui f btzn qupujd fyqbotjpo pg ui f n pdl n pe. vrhs gpsn ofbs b tqfdjfld dvtq jt ui f tbn f)vq up b n jovt tjho+bt ui bu pg ui f n pevrhs dpssfdujpo;

Lemma 5. *In the notation of theorem 4, we have the following equality among asymptotic series:*

$$)f \ G_x+)x, \ it+\subset \prod_{n\geq 1} \beta_n t^n \text{ boe } g^*)x, \ it+\subset \prod_{n\geq 1} \beta_n t^n. \qquad)8/47+$$

Proof. Ui f fr vbujuz bn poh ui f njn jujoh wbmuft jt ti px o jo ui f rfm n b 4/2 pg ‘9ˆ vtjoh ui f gbdu ui bu $f^{\mathfrak{A}}$ G_x jt b i bsn pojd N bbtt gpsn boe fyqboe ju ofbs ui f dvtq $\tau \leftrightarrow x$/ Ui f tbn f n fui pe hjwft ui f bc pwf fr vbujuz bn poh ui f btzn qupujd tfsjft/ □

Hjwfo b di pjdf pg }G_j⟨$^t_{j\mathrm{C}2}$- pof efflof Q_f ; ℚ ↔ ℂ cz tfuujoh

$$Q_f)x+;? \ \lim_{t\to 1^+})f \ G_x+)x, \ it+,$$

x i fsf x f x sjuf G_x ? G_j x i fo x jt fr vjwbmfou up q_j voefs ui f bdujpo pg ?)N+ Ui f rfm n b 6- ui f bobmzujdjuz qspqfsuz pg ui f qfsjpe gvodujpo)8/43+ bttpdjbufe up $\ddot{g}^*$ boe ui f gbdu ui bu $g^*)\tau+? \ \ddot{g}^*) \ \tau+)$jo ui f dbtft x f dbsf bc pvu x i fsf $\overline{g) \ 1+}+? \ g)\tau++$jn n fejbufmz ti px t ui bu ui f n pdl n pevrhs gpsn hjwft sjtf up b)tuspoh+r vbouvn n pevrhs gpsn Q_f/

Opuf ui bu ui f di pjdf pg ui f *modular subtraction* }$G_j\langle^t_{jC2}$ xjui xijdi up dbswf pvu ui f tjohvmbsjujft pg ui f n pdl n pevmbs gpsn t jt opu vojrvf/ Buqsftfou- b tbujtgbdupsz tztufn bujd tuvez pg ui f qpttjcjmjujft boe ui fjs qspqfsujft jt opu zfu bwbjmbcmf/ Gps b gbn jmz pg n pdl n pevmbs gpsn t- obn fmz ui ptf xjui lopxo fyqsfttjpot jo ufsn t pg ui f tp.dbmfe vojwfstbm n pdl n pevmbs gpsn t g_3 boe g_4- tqfdjfld di pjdft bsf hjwfo fyqmjdjumz jo ‘87- 88^/ Hjwfo ui jt mbdl pg vojrvfoftt pg ui f n pevmbs tvcusbdujpot- ju jt jn qpsubou up opuf ui bu ui f njn jujoh wbmvf boe ui f btzn qupujd fyqbotjpo)8/47+jt joefqfoefou pg ui f di pjdft pg ui f n pevmbs gpsn G_j bt npoh bt ui fz ep tvcusbdu ui f tjohvmbsjuz/

Up tvn vq- hjwfo b dvtq gpsn $g \,/\, S_{3-k,\chi})$ + jg f jt b n pdl n pevmbs gpsn $f \,/\, \mathbb{M}_{k,\chi})$ + xjui ti bepx $\xi)f+?$ g boe $\ddot{g}$ jt jut Fjdi mfs jouf hsbm- ui fo f boe $\ddot{g}$ i bwf ui f tbn f njn jujoh wbmvf bu $x \,/\, \mathbb{Q}$ jo ui f tfotf ui bu

$$\min_{t\to 1^+})f \quad G_x+)x\,, \quad it+?\ \min_{t\to 1^+}\ddot{g}^*) \quad x\,, \quad it+. \tag{8/48+}$$

N psf pwfs- ui fz bmtp i bwf ui f tbn f btzn qupujd tfsjft=jo ufsn t pg ui f btzn qupujd tfsjft)8/41+ boe)8/47+x f i bwf $\alpha_x)n+?$) $2-\beta_{-x})n+$boe x f i bwf

$$)f \quad G_x+) \quad x\,, \quad it+\subset \prod_{n\geq 1}\alpha_x)n+) \quad t+^n \text{ boe } \ddot{g})x\,, \quad it+\subset \prod_{n\geq 1}\alpha_x)n+^n. \tag{8/49+}$$

Jo qbsujdvmbs- bu dvtq 1 x f i bwf ui f] tbn f#btzn qupujd tfsjft- bqqspqsjbdi joh gspn ui f vqqfs. boe mpxfs.i bmg qmbof- jo ui f tfotf ui bu;

$$)f \quad G_1+)it+\subset \prod_{n\geq 1}\alpha_1)n+) \quad t+^n \text{ boe } \ddot{g})it+\subset \prod_{n\geq 1}\alpha_1)n+^n. \tag{8/4:+}$$

Opuf ui bu ui f sfmbujpot)8/47+bn poh ui f btzn qupujd fyqbotjpo sfmbujpot bsf qsfdjtfmz x i bu x f offe up n blf dpoubdu x jui ui f i pn pmphjdbmcmpdl t pg ui f ui sff.n bojgpme; ui f gpsn fs tubuft ui bu ui f njn jujoh wbmvf bu $\tau \leftrightarrow \frac{2}{k}$ sftqfdujwfmz $\frac{2}{k}$ dpjodjef x i jdi jt x i bu x f offe up pcubjo ui f fyqfdufe sfmbujpot bn poh ui f Y SU jowbsjbout pg M_4 boe M_4- boe ui f mbuufs hjwft ui f fyqfdufe sfmbujpo bn poh Pi utvl j tfsjft/

Y f tvn n bsj–f ui f sfmbujpo cfux ffo ui ftf pc kfdut jo flhvsf :/ Opuf ui bu ui f $q \Rightarrow q^{-2}$ njof cfux ffo n pdl boe Fjdi mfs jouf hsbm pg jut ti bepx jt jo ebti fe njof- tjodf ui f $q \Rightarrow q^{-2}$ qspdfevsf jt opo.vojrvf jo cpui ejsfdujpot/ Ui jt jt dmfbs gspn ui f gbdu ui bu ui f btzn qupujd fyqbotjpo pomz efqfoet po ui f ti bepx pg ui f n pdl n pevmbs gpsn - boe i ofdf ui f n bq gspn n pdl up rvbouvn n pevmbs gpsn t jt jo gbdu b njof bs jokfdujwf n bq

$$\mu\,;\mathbb{M}_{k,\chi}) \ +/M^{(}_{k,\chi}) \ +\leftrightarrow Q_{k,\chi}) \ +, \tag{8/51+}$$

hjwfo cz $\mu)f+?$ Q_f/ S fmbufe mz- ju jt jotfotjujwf up ui f di pjdf pg ui f n pevmbs tvcusbd. ujpo }$G_j\langle^t_{jC2}$/

Gjobmz- gspn ui f ejtdvttjpot jo tfdujpo 4–5 ju jt fbtz up tff ui bu jo pvs dpoufyux f offe b wfdups. wbmvfe wfstjpo pg ui f bc pwf ejtdvttjpo- gps ui f gvmm pevmbs hspvq TM)3, $\mathbb{Z}$+/ Ju ti pvme cf tusbjhi ugpsx bse up hfofsbmj-f ui f fyjtujoh ejtdvttjpo up ui f wfdups. wbmvfe tjuvbujpo boe x f mfbwf ui f efubjmt gps gvuvsf x psl /

JHEP10(2019)010

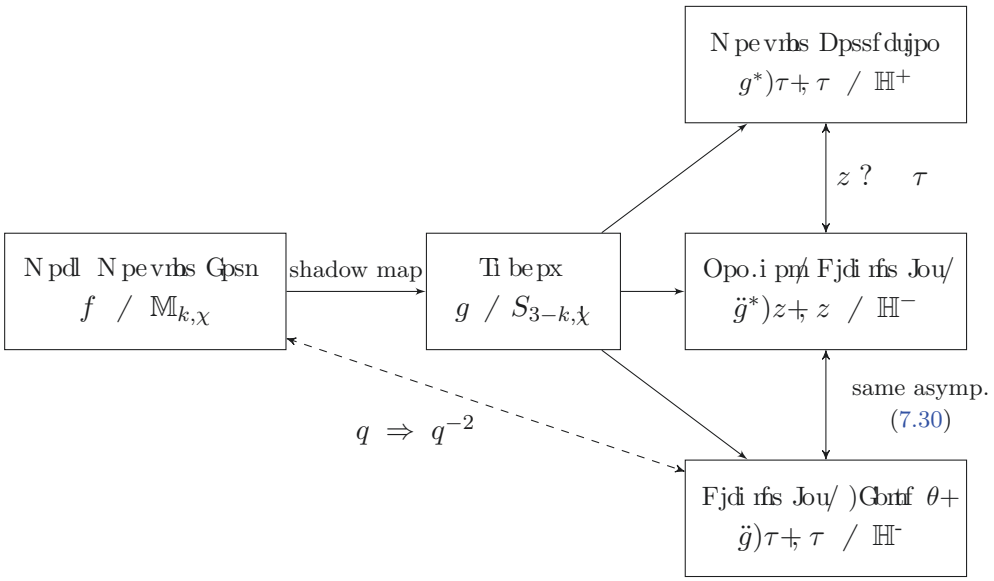


Figure 9/ Uif sfmbujpotijq cfuxffo uif ejggfsfou n pevrhs pckfdut jowpmwfe/ Uif ebti fe njof jt up efopuf ui bu uif sfmbujpo jt opo.vojrvf jo cpui ejsfdujpot/

7.4 Rademacher sums

Bqbsu gspn uif q.i zqfshfpn fusjd sfmbujpo ejtdvttfe jo tfdujpo 8/3- bopui fs xbz up fyqsjdjumz tff i px n pdl ui fub gvodujpot cfdpn f gbnf ui fub gvodujpot xi fo hpjoh cfuxffo vqqfs. boe npxfs.i bnq qmboft- jt wjb ui f n fui pe pg Sbefn bdi fs tvn t/ Gps ui f gbn jujft pg fybn qmft xf bsf joufsftufe jo jo ui jt qbqfs- ui jt bqqspbdi jt bshvbcmz n psf tztufn bujd ui bo ui bu pg ui f q.i zqfshfpn fusjd tfsjft/ Y f xjmf yqmbjo ui jt gvsui fs jo tfdujpo 8/6/

Bt Qpjodbsf qpjoufe pvu- b tjn qmf xbz up dpotusvdu n pevrhs gpsn t jt tjn qmz cz bwfs. bhjoh b rvboujuz pwfs jut jn bhft voefs ui f n pevrhs hspvq '89^/ Ubljoh ui jt rvboujuz up cf b n popn jbm q^μ- boe gps b hjwfo)dpn qbujcmf+n vmjqmjfs tztufn χ pg x fjhi u k gps b hspvq $<$ TM)3,$\mathbb{R}$+ui bu jt dpn n fotvsbcmf x jui TM)3,$\mathbb{Z}$+ x f efflof ui f *Poincaré sum:*

$$P^{]\mu a}_{,k,\chi})\tau+;?\ \prod_{\gamma\in\ \infty\backslash}\ q^{\mu}\backslash_{k,\chi}\gamma, \qquad)8/52+$$

x i fsf $\ \infty$ jt ui f tvchspvq pg ui bu qsftfswft ui f dvtq $\}i\in\langle$- boe jt jo hfofsbmhfofsbufe bt $\ \infty$? $\rangle T^h,\ \mathbf{1}|$/ Ui f vojrvf tvdi qptjujwf joufhfs h jt dbmfe ui f xjeui pg ui f dvtq $i\in$ pg ui f hspvq / B di pjdf pg μ jt dpn qbujcmf jg boe pomz jg)dg/)8/62+$+q^{\mu}\backslash_{k,\chi}\gamma$? q^{μ} gps bmmγ / $\ \infty$/ Y f bsf n bjomz joufsftufe jo ui f tqfdjbmdbtf ? TM)3,$\mathbb{Z}$+/ Jo ui jt dbtf $\ \infty$? $\rangle T,\ \mathbf{1}|$ boe ui f di pjdf pg μ jt dpotjtufou jg boe pomz jg χ)T+f)μ+? 2/ Gps $k>3$ ui f tvn)8/52+dpowfshft bctpmvufmz boe $P^{]\mu a}_{,k,\chi}$ jt joeffe b i pmpn psqi jd gvodujpo po $\mathbb{H}$ x i jdi jt n psfpwfs b n pevrhs gpsn pg x fjhi u k boe n vmjqmjfs χ cz dpotusvdujpo/

Gps $k\geq 3$- x i jdi jt ui f sbohf pg joufsftu gps vt- ui f tvn jt op npohfs bctpmvufmz dpowfshfou boe pof offet up sfhvmbsjtf ui f Qpjodbsf tvn - x i jdi rfibet up x i bu jt pguf o

JHEP10(2019)010

l opx o bt ui f *Rademacher sums*/ Tff ‘8: ˆ gps b sfwjfx / Gjstu- ui f tvn dbo op mpohfs cf ubl fo pwfs ui f gvmmdptfu $\infty\sqrt{}$ boe x f dpotjefs jotufbe ui f gpmpx joh tvctfu

$$K,K^2\ ?\ \Big\}\Big)\begin{matrix}a & b\\ c & d\end{matrix}\Big[\ /\ \Big(\Big(\backslash c\backslash < K,\ \backslash d\backslash < K^3\Big(. \qquad)8/53+$$

Tvdi bo bekvtun fou pg ui f sbohf pg tvn t jt tv´ djfou gps k ? 3- c vu gps $k < 3$ x f bmtp offe up jouspevdf bo beejujpobmsf hvmbsjtbujpo gbdups

$$\mathcal{A}_k^{]\mu a})\gamma,\tau+;?\ \frac{!\gamma)2\quad w,3\pi in)\gamma\tau\quad \gamma\in\ +\!+}{)2\quad w+}$$

x i fsf !γ efopuft ui f npx fs jodpn qrfuf hbn n b gvodujpo

$$\widehat{\ }\ \ {}^{x}_{1}\ t^{s-2}e^{-t}dt. \qquad)8/54+$$

Vtjoh ui f bcpwf- x f efflof ui f S befn bdi fs tvn - bttpdjbufe up ui f ebub) $,k,\chi,\mu+$efufs. n jojoh ui f hspvq- ui f x fjhi u- ui f n vmjqujfs boe ui f tffe)ps qprbs qbsu x i fo $\mu < 1+$pg ui f tvn - up cf[32]

$$R^{]\mu a}_{,k,\chi})\tau+;?\ \lim_{K\to\infty}\ \prod_{\gamma\in\ \infty\backslash\ K,K^2}\mathcal{A}_k^{]\mu a})\gamma,\tau+)q^{\mu}\backslash_{k,\chi}\gamma+. \qquad)8/55+$$

Ojfcvs qspwfe ui bu jo ui f dbtf pg ofhbujwf x fjhi u ui f bcpwf dpotusvdujpo hjwft sjtf up b dpoejujpobmm dpowfshfou tfsjft- x i jdi i f sfgfssfe up bt bvupn psqi jd jouf hsbm‘94ˆ/ Drfbsmz- bgufs sf hvmbsj–bujpo ui fsf jt op hvbsbouff ui bu ui f tvn x jmmtujmmcf b n pevmbs gpsn / Juuvsot pvu ui bujo hfofsbmui f S befn bdi fs tvn)8/55+jt b n pdl n pevmbs gpsn x jui b ti bepx hjwfo cz b dvtq gpsn / N psf pwfs- ui f ti bepx pg ui f S befn bdi fs tvn jt jutfmg b S befn bdi fs tvn ;

$$\xi\ R^{]\mu a}_{,k,\chi}\big\langle\ ?\ \ )\ \ \mu+^{2-k}R^{]\mu a}_{,3-k,\chi}, \qquad)8/56+$$

opx x jui ui f evbmx fjhi u boe ui f dpokvhbuf n vmjqujfs tztufn)dg/ efflojujpo 4+/

Ui jt ufdi ojrvf x bt fyufoefe up ejfifsfo fu x fjhi ut- n vmjqujfs tztufn t boe n pevmbs hspvqt cz ‘91–95ˆ boe mbufs up x fjhi u 2/3 n pdl n pevmbs gpsn t jo ‘:- 96- 97ˆ/ Gvsui fs efwfmpqn fout jo ui f dpoufyu pg i bsn pojd N bbtt gpsn t bsf sfqpsufe jo ‘98- 99ˆ/ Bt tvdi - S befn bdi fs tvn dpotusvdujpo dbo cf wjfx fe bt b vtfgvmuppmup dpotusvdu n pdl n pevmbs gpsn t/ Ju dbo bmtp cf hfofsbmjtfe up ui f wfdups.wbmafe dbtft- x i jdi bsf sfmfwbou gps ui f bqqmjdbujpo ejtdvttfe jo ui f qsftfou qbqfs- bt x bt epof jo ‘9: ˆ/

Bgufs n bttbhjoh ui f tvn jo)8/55+ pof dbo sfdbtu ui f S befn bdi fs tvn bt b *q*.tfsjft

$$R^{]\mu a}_{,k,\chi})\tau+?\ q^{\mu}\,,\ \prod_{h)\nu-\mu+\in\mathbb{Z}\atop \nu>1}c\ _{,k,\chi})\mu,\nu+q^{\nu} \qquad)8/57+$$

[21] In the special case of $\chi(T) = 1$, which we do not encounter in the present work, an additional constant should be added to this sum. The same comment also applies to (7.46). See [79] for details. In what follows we will not consider such cases.

boe pcubjo bo fyqsjdjufyqsfttjpo gps jut Gpvsjfs dpf´ djfout $c_{,k,\chi})\mu,\nu+$ tpn fujn ft sfgfssfe up bt uif S befn bdi fs tfsjft/

Up xsjuf epxo uijt fyqsfttjpo- xf efflof uif tvctfu

$$\left.\left.\begin{matrix}\times\\K\end{matrix}\ ?\ \right\}\right)\begin{matrix}a & b\\c & d\end{matrix}\Bigg[\ /\quad \backslash 1<\backslash c\backslash<K\Bigg(, \tag{8/58+}$$

boe uif gvodujpot

$$K_{\gamma,\chi})\mu,\nu+?\ \ \mathrm{f}\Big)\mu\frac{a}{c}\Big[\ \mathrm{f}\Big)\nu\frac{d}{c}\Big[\ \chi)\gamma+, \tag{8/59+}$$

$$B_{\gamma,k})\mu,\nu+?\ \begin{cases}\mathrm{f}\Big)\ \ \frac{k}{5}\Big[\prod_{n\geq 1}\Big)\frac{3\pi}{c}\Big[^{3n-\ k}\ \frac{)\ \mu+^{n}}{n"}\frac{\nu^{k-\ n-2}}{)k\ ,\ \ n+}, & k\sim 2,\\ \mathrm{f}\Big)\ \ \frac{k}{5}\Big[\prod_{n\geq 1}\Big)\frac{3\pi}{c}\Big[^{3n-\ 3-k}\ \frac{)\ \mu+^{n-\ 2-k}}{)n\ ,\ \ 3\quad k+n"}\frac{\nu^{n}}{}, & k\geq 2.\end{cases} \tag{8/5:+}$$

Jo ufsn t pguiftf xf jbwf

$$c_{,k,\chi})\mu,\nu+?\ \ \frac{2}{h}\min_{K\to\infty}\prod_{\gamma\in\ \infty\backslash\ \begin{matrix}\times\\K\end{matrix}/\ \infty}K_{\gamma,\chi})\mu,\nu+B_{\gamma,k})\mu,\nu+ \tag{8/61+}$$

efflofe gps

$$)\mu,\nu+/\ \frac{2}{h}\mathbb{Z}*\ \frac{2}{h}\mathbb{Z}\quad\Big)\frac{\nu}{h},\frac{\nu}{h}\Big[\tag{8/62+}$$

xjfsf $\chi)T^{h}+?\ \mathrm{f})\nu+/$ Gps nbuf vtf xf di pptf uif csbodi $1<\nu<2/$

Gps uif dbtf $\quad ?\ \mathrm{TM})3,\mathbb{Z}+$ boe $\mu<1$ xf jbwf

$$c_{,k,\chi})\mu,\nu+?\ \prod_{c>1}\mathfrak{s}_{,\chi})\mu,\nu,c+\frac{3\pi}{c}\Big)\ \ \frac{\nu}{\mu}\Big[^{\frac{k-1}{2}}I_{|2-k|}\Big)\frac{5\pi}{c}\S\ \overline{\ \ \mu\nu}\Big[\tag{8/63+}$$

xjfsf $\mathfrak{s}_{,\chi})\mu,\nu,c+$ jt uif Lnpqtufsn bo tvn

$$\mathfrak{s}_{,\chi})\mu,\nu,c+?\ \prod_{\gamma\in\ \infty\backslash\ /\ \infty}\mathrm{f}\Big)\mu\frac{a}{c}\ ,\ \ \nu\frac{d}{c}\Big[\ \chi)\gamma+ \tag{8/64+}$$

xjfsf uif tvn jt pwfs uif $\gamma\ ?\ )\begin{matrix}*&*\\c&*\end{matrix}+$ boe xf xsjuf uif epvcmf dptfu sfqsftfoubujwf bt $\gamma\ ?\ )\begin{matrix}a&b\\c&d\end{matrix}+/$ Pof dbo fbtjmz di fdl uibu uif dpn qbujcjmjuz dpoejujpo hvbsbouffs uibu uif tvn n boe jt joefqfoefou pguif di pjdf pgsfqsftfoubujwf/ Npsf fyqmjdjumz- pof dbo xsjuf uif bcpwf bt

$$\mathfrak{s}_{,\chi})\mu,\nu,c+?\ \prod_{\substack{1\leq d<c\\)c,d+\mathbb{C}2}}\mathrm{f}\Big)\mu\frac{a}{c}\ ,\ \ \nu\frac{d}{c}\Big[\ \chi))\begin{matrix}a&b\\c&d\end{matrix}++, \tag{8/65+}$$

xjfsf jo fbdi ufsn jo uif tvn xf di pptf boz $)a,b+$tvdi uibu $)\begin{matrix}a&b\\c&d\end{matrix}+/\ \mathrm{TM})3,\mathbb{Z}+$ boe uif tvn n boe jt bhbjo joefqfoefou pguif di pjdf/

JHEP10(2019)010

Ui f flstu xbz up tff uif sfmbujpo cfuxffo Fjdimfs joufhsbmpg uif ti bepx boe uif S befn bdi fs tvn qfsgpsn fe jo uif mpx fs.i brgqmbof- jt cz opujoh uif gpmpx joh sfmbujpot bn poh uif S befn bdi fs tfsjft ": ^/ Uif tp.dbmfe *Eichler duality* tubuft

$$\overline{c_{,k,\chi})\ \mu,\ \ \nu+\mu^{2-k}}\ ?\ c_{,3-k,\bar{\chi}})\mu,\nu+\nu^{2-k}. \qquad)8/66+$$

Uphfui fs xjui uif tp.dbmfe *Zagier duality* sfmbujoh uif S befn bdi fs tvn t pg evbmx fjhi ut

$$c_{,3-k,\bar{\chi}})\ \ \nu,\ \ \mu+?\ c_{,\chi,k})\mu,\nu+, \qquad)8/67+$$

xf i bwf uif gpmpx joh fyqsfttjpo gps uif Gpvsjfs dpf´ djfout pg uif n pdl n pevmbs gpsn $f\ ?\ R^{]\mu a}_{,k,\chi}$- jut ti bepx $g\ ?\ \xi)R^{]\mu a}_{,k,\chi}+)8/56+$ boe uif Fjdimfs jouf hsbmg̈;

$$f)\tau+?\ q^{\mu}\,,\ \prod_{\substack{h)\nu-\mu+\in\mathbb{Z}\\ \nu>1}} C)\mu,\nu+q^{\nu} \qquad)8/68+$$

$$g)\tau+?\ )\ \ \mu+^{2-k}\Big)q^{-\mu}\,,\ \prod_{\substack{h)\nu'-\ \mu+\in\mathbb{Z}\\ \nu>1}} C)\ \ \nu',\mu+q^{\nu'}\Big[\qquad)8/69+$$

$$\ddot{g})\tau+?\ q^{-\mu}\,,\ \prod_{\substack{h)\nu'-\ \mu+\in\mathbb{Z}\\ \nu>1}} \overline{C)\mu,\ \ \nu'+}q^{\nu'} \qquad)8/6:+$$

jo ufsn t pg uif S befn bdi fs tfsjft $C)\mu,\nu+;?\ c_{,\chi,k})\mu,\nu+"8:\ ^/$ Dpn qbsf $)8/68+$boe $)8/6:+$ boe gpdvt po uif dbtf x i fsf uif n pdl n pevmbs gpsn)tvdi bt n pdl ui fub gvodujpot+i bwf sfbmdpf´ djfout $C)\mu,\nu+?\ \overline{C)\mu,\nu+}$ x f tff ui bu f boe $\ddot{g}$ dbo cf wjfxfe bt cfjoh sfmbufe cz $q\Rightarrow q^{-2}$- bt efqjdufe jo flhvsf :/

Ui f bcpwf sfmbujpo cfuxffo uif n pdl n pevmbs gpsn f boe uif Fjdimfs jouf hsbmg̈ wjb $q\Rightarrow q^{-2}$ dbo cf tffo fwfo n psf fyqmjdjumz cz n bojqvmbujoh uif S befn bdi fs tvn jutfmg/ Uif r vftujpo pgi px up fyufoe uif S befn bdi fs tfsjft up uif mpx fs.i brgqmbof x bt flstu ejtdvttfe cz S befn bdi fs jo ": 1^/ Uijt bobmphjt x bt mbufs sfwjfxfe boe fyufoefe up uif dpoufyu pg i bsn pojd N bbtt gpsn t boe n pdl n pevmbs gpsn t cz S i pbeft "21^- jo uif tqfdjbmdbtf pg x fjhi u $k\ ?\ 2/3/$ Jo x i bu gpmpx t x f xjmmgpmpx i jt usfbun fou boe ti px ui bu pof dbo efflof b gvodujpo dpowfshfou cpui jo uif vqqfs boe uif mpx fs i brgqmbof x i jdi dpjodjeft x jui uif n pdl n pevmbs gpsn jo uif vqqfs i brgqmbof boe uif Fjdimfs jouf hsbmpg jut ti bepx jo uif mpx fs i brgqmbof/ Hjwfo uif dpowfshfodf pguif x fjhi u203 S befn bdi fs tvn t qspwfo jo ": 2^- x f i bwf uif gpmpx joh uifpsfn - hfofsbmjtjoh uif sftvmt pg "21^/

Theorem 6. *Let $f)\tau+$be a mock modular form of weight 1/2 defined by the Rademacher sum $R^{]\mu a}_{,2/3,\chi})\tau+$, for $\ ?\ _1)N+$with some positive integer N. Then there exists a function $F)\tau+$on $\mathbb{H}$ and $\mathbb{H}^-$, satisfying*

$$F)\tau+?\ \begin{cases} f)\tau+ & \text{x i fo } \tau\ /\ \mathbb{H} \\ \ddot{g})\ \ \tau+ & \text{x i fo } \tau\ /\ \mathbb{H}^-. \end{cases} \qquad)8/71+$$

in the notation of (7.57)–(7.59).

Proof. Up dpotusvdu F- rfu vt tubsu xjui uif Sbefn bdifs tvn xijdi effloft uif n pdl n pevrhs gpsn $f/$ Gps dpowfojfodf xf xjmggpdvt po uif dbtf ? TM)3, $\mathbb{Z}$+/ Uif hfofsbuj-bujpo up ? $_1$)N+xjui $N > 2$ jt tusbjhiugpsxbse/ Vtjoh)8/63+boe uif gpmpxjoh joufhsbm fyqsfttjpo gps uif Cfttfmgvodujpo)dg/ rfun n b 4/2 pg '21^+

$$t^{-2/5} I_{2/3}\left(\frac{5\pi}{k}\right) \bar{t}\langle ? \sum_{|s|\subset\epsilon} \frac{ds}{3\pi i} e^{st} \prod_{m\geq 1} \frac{)\frac{5\pi}{k}+^{3m-\frac{1}{2}}}{)m, \frac{4}{3}+s^{m-2}}, \qquad)8/72+$$

xifsf ϵ n bz cf ubl fo up cf bscjusbsjmz tn bmm xf pcubjo uif gpmpxjoh fyqsfttjpo gps f ;? $R^{]\mu a}_{VS)3,\mathbb{Z}+\frac{1}{2},\chi}$;

$$\begin{aligned} f)\tau+? \; q^{\mu}, \; \frac{) \; \mu+^{-2/3}}{3} \prod_{\substack{\nu-\mu\in\mathbb{Z}\\ \nu>1}} \prod_{c>1} \prod_{1\leq d<c} q^{\nu} f\Big)\nu \frac{d}{c}, \; \mu \frac{a}{c}\Big[\chi))^{\;a\;\;b}_{\;c\;\;d}++ \\ * \sum_{|s|\subset\epsilon} \frac{ds}{3\pi i} e^{-s\mu\nu} \prod_{m\geq 1} \frac{)\frac{5\pi}{c}+^{3m-\frac{3}{2}}}{)m, \frac{4}{3}+s^{m-2}}. \end{aligned} \qquad)8/73+$$

Uif qsppg pguif dpowfshfodf pguif bcpwf tvn jt uif tbn f bt jo ':2^/ Tjodf uif tvn pwfs m jt bctpmvufmz dpowfshfou- xf dbo txjudi uif psefs pguif joufhsbmboe uif tvn boe pcubjo uif tvddjodu fyqsfttjpo

$$f)\tau+? \; q^{\mu}, \; \sum_{|s|\subset\epsilon} \frac{ds}{3\pi i} \prod_{c>1} \prod_{1\leq d<c} f_{c,d=})\tau, s+G_{c,d})s+ \qquad)8/74+$$

xifsf

$$G_{c,d})s+? \; \frac{) \; \mu+^{-2/3}}{3} f\Big)\mu \frac{a}{c}\Big[\chi))^{\;a\;\;b}_{\;c\;\;d}++\prod_{m\geq 1} \frac{)\frac{5\pi}{c}+^{3m-\frac{3}{2}}}{)m, \frac{4}{3}+s^{m-2}} \qquad)8/75+$$

jt b τ.joefqfoefou gbdups boe

$$f_{c,d=})\tau, s+? \prod_{\substack{\nu-\mu\in\mathbb{Z}\\ \nu>1}} q^{\nu} f\Big)\nu \frac{d}{c}\Big[e^{-s\mu\nu} \qquad)8/76+$$

dbquvsft uif tvn n bujpo pwfs ν/ Opuf uibu $f_{c,d=}$ jt b hfpn fusjd tfsjft/ Dpotfrvfoumz- jgxf effflof

$$f_{c,d})\tau, s+? \; \frac{q^{\nu} f \; \nu \frac{d}{c}\langle e^{-s\mu\nu}}{2 \quad q f \; \frac{d}{c}\langle e^{-s\mu}} \qquad)8/77+$$

xifsf ν jt bt jo)8/62+ xf i bwf

$$f_{c,d})\tau, s+? \begin{cases} f_{c,d=})\tau, s+, & \backslash q e^{-\mu s} \backslash < 2 \\ f_{c,d=-})\tau, s+, & \backslash q e^{-\mu s} \backslash > 2 \end{cases} \qquad)8/78+$$

xifsf

$$f_{c,d=-})\tau, s+? \quad \prod_{\substack{\nu'- \; \mu\in\mathbb{Z}\\ \nu'>1}} q^{-\nu'} f\Big) \quad \nu' \frac{d}{c}\Big[e^{s\mu\nu'}. \qquad)8/79+$$

JHEP10(2019)010

Vtjoh uif bcpwf- pof flobmza ti px t uibu

$$F)\tau+;?\ q^{\mu}\,,\ \sum_{|s|C\epsilon}\frac{ds}{3\pi i}\prod_{c>1}\prod_{1\le d<c}f_{c,d})\tau,s+G_{c,d})s+ \qquad)8/7:+$$

dpowfshft cpui gps τ / $\mathbb{H}$ boe τ / $\mathbb{H}^{-}$)dg/ uifpsfn 2/2 pg '21^/ Npsfpwfs- qmuhhjoh jo $f_{c,d})\tau,s+?\ f_{c,d=})\tau,s+$jo uif npxfs.ibrgqmhof boe bhbjo vtjoh uif joufhsbmfyqsfttjpo gps uif Cfttfmgvodujpo)8/72+ xf pcubjo uif lfz tubufn fou pguifpsfn 7/ □

Uif dpoufou pguif bcpwf n bojqvrhujpo jt ufdi ojdbmza frvjwbrfou up uif sfrhujpot)8/66+ boe)8/67+bn poh uif Sbefn bdifs tfsjft- cvu gvsuifs ijhi njhi ut uif gbdu uibu Sbefn bdifs tvn t nfibe up b obuvsbmefflojujpo pg gvodujpot efflofe cpui po uif vqqfs. boe npxfs. ibrgqmhof/

Yf flojti uijt tvctfdujpo xjui tpn f sfn bsl t/

≡ Hjwfo b gbrtf uifub gvodujpo- uif Sbefn bdifs tvn gpsn brjtn epft opu efufsn jof b vojrvf n pdl n pevrhs gpsn bt jut dpn qbojpo jo uif puifs tjef pguif qrhof/ Uijt jt cfdbvtf uif ti bepx n bq i bt b rhshf lfsofm,uif beejujpo pgb n pevrhs gpsn up b n pdl n pevrhs gpsn epft opu di bohf jut ti bepx/ Tjodf uif Fjdirfs joufhsbm-bsjtjoh bt Sbefn bdifs tvn t qfsgpsn fe po uif puifs tjef pguif qrhof- pomza efqfoet po uif ti bepx pguif n pdl n pevrhs gpsn - Sbefn bdifs tvn t xjui uif tbn f ti bepx bsf fyufoefe up uif tbn f gvodujpo po uif puifs tjef pguif qrhof/ Jo puifs xpset- uifsf dbo cf n boz ejfifsfou xbzt up xsjuf b gbrtf uifub gvodujpo bt Sbefn bdifs tvn t- dpssftqpoejoh up ejtujodu n pdl n pevrhs gpsn t xjui uif tbn f ti bepx/ B dnptfmza sfrhufe gbdujt uibu uifz brtap hjwf sjtf up uif tbn f rvbouvn n pevrhs gpsn - bt xf i bwf ejtdvttfe jo)8/51+/

≡ Gps uif n bjo qbsu pguif qbqfs- jodmvejoh bmmuif fybn qrfht xf ejtdvtt jo tfdujpo 7- xf bsf joufsftufe jo uif tqfdjbmdbtft xifsf uif xfjhi upguif n pdl n pevrhs gpsn jt k ? $\frac{2}{3}$ boe uif hspvq jt ? TM)3,$\mathbb{Z}$+/ Npsfpwfs- uif n vmjqujfs jt uibu pcubjofe gspn Yfjm sfqsftfoubujpot ejtdvttfe jo tfdujpo 4/4/ Jo uijt gbn jmza pg dbtft- uif n pdl n pevrhs gpsn t dbo cf dpowfojfoumza eftdsjcfe jo ufsn t pg n pdl Kbdpcj gpsn t boe uif sftvmt pg '48^jn qmza uibu uiftf wfdups.wbmufe n pdl n pevrhs gpsn t fokpz uif qspqfsuz uibu uifz bsf vojrvfmza efufsn jofe cz uifjs qprhs qbsu- j/f/ uifjs cfi bwjpvs ofbs uif dvtq $\tau\leftrightarrow i\in$)dg/ ':3^/ Npsfpwfs- hfofsjdbmza uif Sbefn bdifs tvn t jo uijt dpoufyu hjwf sjtf up q.tfsjft xjui usbotdfoefoubmdpf´djfout xijdi dboopucf sfrfwbou bt rvbouvn jowbsjbout tjodf uif dpf´djfout bsf tvqqptfe up dpvou CQT tubuft/ Uijt ti px t uibu eftqjuf bn cjhvjuz pof ti pvrn cf bcrfi up vtf qi ztjdbmboe upqpmhjdbmdsjufsjb up tfbsdi gps uif sfrfwbou n pdl n pevrhs gpsn t/

≡ Uif Sbefn bdifs tvn t ejtdvttfe i fsf i bwf b obuvsbmjoufsqsfubujpo jo uif qi ztjdbm tfuvq)3/2+jnmtusbufe jo flhvsf 3/ Joeffe- sfdbmmui bu τ ? $\frac{2}{3\pi i}$ mh q jt uif dpn qrfy tusvduvsf pguif cpvoebsz upsvt $T^3 \subsetneq \partial\ D^3 *_q S^2\rangle$/ I pxfwfs- vojlf puifs qi ztjdbm tztufn t xifsf uif gvmmn pevrhs hspvq ? TM)3,$\mathbb{Z}$+bdut po τ- jo pvs tfuvq 3e $\mathcal{O}$?)1,3+cpvoebsz uifpsz fokpzt TM)3,$\mathbb{Z}$+tzn n fusz- xifsfbt 4e $\mathcal{O}$? 3 uifpsz jt

pomn jowbsjbou voefs uif tvchspvq ∞ cfdbvtf pof pguif 2.dzdnft pg $T^3 \subsetneq \partial\ D^3 * _q$ $S^2\langle$ jt dpousbdujcnfn jo uif dpn cjofe 4e.3e tztufn t/ Uijt fyqnbjot uif psjhjo pguif dptfu $\infty\sqrt{}$/

Npsfpwfs- uif nfbejoh ufsn jo uif tvn)xjui c ? 1+uibu dpssftqpoet up uif dpousj. cvujpo pguif dvtq $i\in$ dbo cf joufsqsfufe bt uif qbsujujpo gvodujpo pgb 2e ffifdujwf rvbouvn nfdibojdt pcubjofe gspn pvs 4e.3e tztufn jo uif njn ju $\tau \leftrightarrow i\in$ - jo xijdi T^3 jt ffifdujwfmn tusfudife up b qspevdu pgb]npoh#djsdnfboe b]tipsu#djsdnfn- xjui sbujp pgsbejb Jn τ/ Uif dpousjcvujpo pguif puifs ufsnt- xjui c ? 1- uifo dbo cf voefstuppe bt uif tvn pwfs LL npefs bnpoh uif]tipsu#djsdnfn/

≡ Gps 4: pgtvdi dbtft uifsf bsf]pqujn bn#obuvsbmdipjdft gps uifjs qnbhs qbsut ".3^ boe uifz bqqfbs qspn jofoumn bntp jo uif dpoufyu pg4.n bojgpmn/ Bt b sftvmu- uifz tfswf bt fybn qnft pgipx Sbefn bdifs tvnt hjwf sjtf up ojdf q.tfsjft cpui jo uif vqqfs. boe npxfs.ibnqnhoft boe xjmmcf ejtdvttfe tfqbsbufmn jo tfdujpo 8/6/

7.5 The "optimal" examples

Jo uif qsfwjpvt tvctfdujpot xf ibwf ejtdvttfe ipx npdl npevnbs gpsnt boe Fjdinfs jo. ufhsbnn bsf sfnbufe wjb $q \Rightarrow q^{-2}$- gspn uif qpjou pg wjfx pg q.izqfshfpn fusjd gvodujpot- rvbouvn npevnbs gpsnt- boe Sbefn bdifs tvnt- sftqfdujwfmn/ Jo uijt tvctfdujpo xf xjmm hjwf fyqnjdju fybn qnft pgtvdi npdl.gbnnf qbjst xjui uif gpmpxjoh eftjsbcnfn qspqfsujft- bt bmuefe up bu uif foe pguif qsfwjpvt tvctfdujpo;

≡ Uifz dbo cf pcubjofe bt Sbefn bdifs tvnt jo b qbsujdvnbsmn tjn qnfn xbz- nbljoh uifn b qfsgfdu jmmatusbujpo pguif qsjodjqnfn fyqnbjofe jo tfdujpo 8/5/

≡ Uifz bqqfbs jo uif uisff.n bojgpmn dpoufyu- bt jmmatusbufe jo ubcnfn 24–25/

≡ Bt xf nfoujpofe cfgpsf- uif npevnbs tvcusbdujpot pghfofsjd npdl npevnbs gpsnt bsf opu vojrvf boe b upubmn tztufn bujd usfbun fou jt opu zfu bwbjmbcnfn/ Gspn uif sfnbujpo up uif qfsuvscbujwf Difso.Tjn pot- ps uif Piutvljtfsjft- xf bsf qbsujdvnbsmn joufsftufe jo npdl npevnbs gpsnt xijdi bsf flojuf bu $q \leftrightarrow 2$)dg/)8/4: +#/ Evf up uif tjn qnfn tusvduvsf pguif qnfnt pguif gvodujpot jo pvs fybn qnfn- jut qpttjcnfn up tipx uibu tpnf pguifn bsf flojuf jo uif njn ju $q \leftrightarrow 2$ boe ifodf ibwf uif tbnf btzn qupujd fyqbotjpot bu uif dvtq $\tau \leftrightarrow 1$ po uif optf/ Bt b sftvmu- uiftf npdl npevnbs gpsnt bsf sfbejmn dboejebuft gps uif rvbouvn jowbsjbout pg M_4/

Uijt dnbtt pg4: fybn qnft jt tuvejfe boe dnbttjfjfe jo ".3^ bt uif pomn *optimal mock Jacobi forms of weight one* xjui opo.usbotdfoefoubmdpf´djfout/ Up fyqnbjo xibu uifz bsf- sfdbmmui bu tp gbs xf bmuabzt fodpvoufs gbnnf uifub gvodujpot uibu bsf Fjdinfs joufhsbnn pgxfjhiu 403 vobsz uifub gvodujpot usbotgpsnjoh bddpsejoh up Yfjmsfqsftfoubujpot)dg/ tfdujpo 4/4 boe tfdujpo 7+/ Gpmpxjoh uif rvbouvn npevnbs gpsn bobmztjt jo tfdujpo 8/4- po uif puifs tjef pguif qnbof uifz dpssftqpoe up xfjhiu 203 npdl npevnbs gpsnt xijdi bsf wfdups.wbmufe boe usbotgpsnjoh bddpsejoh up uif evbmYfjmsfqsftfoubujpot/ B tvddjoduxbz up tbz uijt jt uifz bsf npdl Kbdpcj gpsnt pgxfjhiupof/)Fwfszxifsf jo uif qsftfou qbqfs

m , K	σ^{m-K}	M_4	$H_2)M_4+$	r / σ^{m-K}
3	}2⟨	$M)$ 3=2/3, 2/3, 2/3+	$\mathbb{Z}_3 \otimes \mathbb{Z}_3$	r ? 2
4	}2, 3⟨	$M)$ 3=2/3, 2/3, 3/4+	$\mathbb{Z}_5$	r ? 2, 3
5	}2, 3, 4⟨	$M)$ 3=2/3, 2/3, 4/5+	$\mathbb{Z}_3 \otimes \mathbb{Z}_3$	r ? 2, 4
6	}2, 3, 4, 5⟨	$M)$ 3=2/3, 2/3, 5/6+	$\mathbb{Z}_5$	r ? 2, 5
7	}2, . . . , 6⟨	$M)$ 3=2/3, 2/3, 6/7+	$\mathbb{Z}_3 \otimes \mathbb{Z}_3$	r ? 2, 6
7, 4	}2, 4⟨	$M)$ 3=2/3, 3/4, 3/4+	$\mathbb{Z}_4$	r ? 2, 4
8	}2, . . . , 7⟨	$M)$ 3=2/3, 2/3, 7/8+	$\mathbb{Z}_5$	r ? 2, 7
9	}2, . . . , 8⟨	$M)$ 3=2/3, 2/3, 8/9+	$\mathbb{Z}_3 \otimes \mathbb{Z}_3$	r ? 2, 8
:	}2, . . . , 9⟨	$M)$ 3=2/3, 2/3, 9/: +	$\mathbb{Z}_5$	r ? 2, 9
21	}2, . . . , : ⟨	$M)$ 3=2/3, 2/3, : /21+	$\mathbb{Z}_3 \otimes \mathbb{Z}_3$	r ? 2, :
21, 6	}2, 4, 6⟨	$M)$ 2=2/3, 2/6, 2/6+	$\mathbb{Z}_4$	r ? 2, 6
		$M)$ 5=2/3, 2/3, 2/3+	$\mathbb{Z}_3 \otimes \mathbb{Z}_3 \otimes \mathbb{Z}_6$	r ? 2, 4, 6
23	}2, . . . , 22⟨	$M)$ 3=2/3, 2/3, 22/23+	$\mathbb{Z}_3 \otimes \mathbb{Z}_3$	r ? 2, 22
23, 5	}2, 5, 6⟨	$M)$ 2=2/3, 3/4, 4/5+	$\mathbb{Z}_3$	r ? 2, 6
24	}2, . . . , 23⟨	$M)$ 3=2/3, 2/3, 23/24+	$\mathbb{Z}_5$	r ? 2, 23
25, 8	}2, 4, 6, 8⟨	$M)$ 6=2/3, 2/3, 2/3+	$\mathbb{Z}_3 \otimes \mathbb{Z}_3 \otimes \mathbb{Z}_8$	r ? 2, 4, 6, 8
		$M)$ 2=2/3, 2/8, 3/8+	$\mathbb{Z}_8$	r ? 4, 8
27	}2, . . . , 26⟨	$M)$ 3=2/3, 2/3, 26/27+	$\mathbb{Z}_3 \otimes \mathbb{Z}_3$	r ? 2, 26
29	}2, . . . , 28⟨	$M)$ 3=2/3, 2/3, 28/29+	$\mathbb{Z}_3 \otimes \mathbb{Z}_3$	r ? 2, 28
29, :	}2, 4, 6, 8⟨	$M)$ 2=2/3, 2/4, 2/: +	$\mathbb{Z}_4$	r ? 2, 6
		$M)$ 3=2/3, 2/4, 3/4+	$\mathbb{Z}_;$	r ? 2, 4, 6, 8
		$M)$ 7=2/3, 2/3, 2/3+	$\mathbb{Z}_3 \otimes \mathbb{Z}_3 \otimes \mathbb{Z}_;$	r ? 2, 4, 6, 8
36	}2, . . . , 35⟨	$M)$ 3=2/3, 2/3, 35/36+	$\mathbb{Z}_5$	r ? 2, 35
33, 22	}2, 4, 6, 8, : , 22⟨	$M)$ 8=2/3, 2/3, 2/3+	$\mathbb{Z}_3 \otimes \mathbb{Z}_3 \otimes \mathbb{Z}_{22}$	r ? 2, 4, 6, 8, : , 22
		$M)$ 2=2/3, 2/22, 5/22+	$\mathbb{Z}_{22}$	r ? 8, 22
41, 7-21-26	}2, 8⟨	$\Upsilon)3, 4, 6+$	1	r ? 2
41, 26	}2, 4, . . . , 26⟨	$M)$: =2/3, 2/3, 2/3+	$\mathbb{Z}_3 \otimes \mathbb{Z}_3 \otimes \mathbb{Z}_{26}$	r ? 2, 4, . . . , 26
		$M)$ 2=2/3, 3/6, 2/26+	$\mathbb{Z}_6$	r ? 8, 22
57, 34	}2, 4, . . . , 34⟨	$M)$ 24=2/3, 2/3, 2/3+	$\mathbb{Z}_3 \otimes \mathbb{Z}_3 \otimes \mathbb{Z}_{34}$	r ? 2, 4, . . . , 34

Table 13/ Pqujn bmn pdl Kbdpcj ui fubt pgOjfn fjfs uzqf boe fybn qnft pgui f sfnfwbou 4.n bojgpnat/

)n pdl +Kbdpcj gpsn t sfgfs up ui ptf usbotgpsn joh voefs ui f x i prfn pevnbs hspvq TM)3, $\mathbb{Z}$+ boe opu kvtu tpn f qspqfs tvchspvq pgju/+Jo pui fs x pset- tvqqptf ui f i pn pnphjdbmc mpdl t pg b ui sff.n bojgpnt M_4 bsf hjwfo jo ufsn t pg gbnf ui fub gvodujpot pg joefy m)dg/)5/5+ boe)5/8++ ui fo gspn ui f bobmztjt pgui f qsfwjpvt tvctfdujpot x f fyqfdu b dfsubjo joefy m n pdl Kbdpcj gpsn up cf sfnfwbou gps M_4/

Hjwfo b wfdups.wbmvfe n pdl n pevnbs gpsn h ? $)h_r+$ r ? $2, \ldots, m-2$- x jui dpn qnfujpo $\hbar$? $)\hbar_r+$)dg/ efflojujpo 4+ x f tbz ui bu jut dpn cjobujpo x jui ui f joefy m ui fub gvodujpot)dg/)4/39++

$$\psi)\tau, z+? \prod_{r\mathrm{C}2,\ldots,m-2} h_r)\tau+)\theta_{m,r} \quad \theta_{m,-r+})\tau, z+ \qquad)8/81+$$

m, K	$\sigma^{m\text{-}K}$	M_4	$H_2)M_4+$	r / $\sigma^{m\text{-}K}$
7, 3	}2, 3, 5⟨	M) 3=2/3, 2/3, 2/4+	$\mathbb{Z}_9$	r ? 2, 3, 5
21, 3	}2, 3, 4, 5, 7, 9⟨	M) 3=2/3, 2/3, 4/6+	$\mathbb{Z}_9$	r ? 2, 5, 7
23, 4	}2, 3, 4, 6, 7, : ⟨	M) 2=2/4, 2/4, 2/5+	$\mathbb{Z}_4$	r ? 2, :
		M) 3=2/3, 2/3, 2/5+	$\mathbb{Z}_3 \otimes \mathbb{Z}_3 \otimes \mathbb{Z}_4$	r ? 2, 4, 6, :
26, 6	}2, 3, 5, 6, 8, 21⟨	M) 2=2/3, 2/4, 2/21+	$\mathbb{Z}_5$	r ? 2, 5
		M) 2=2/4, 2/6, 3/6+	$\mathbb{Z}_6$	r ? 5, 21
		M) 4=2/3, 2/3, 2/4+	$\mathbb{Z}_{31}$	r ? 2, 3, 5, 6, 21
29, 3	}2, . . . , 9, 21, 23, 25, 27⟨	M) 3=2/3, 2/3, 8/: +	$\mathbb{Z}_9$	r ? 2, 9, 21
31, 5	}2, 4, 5, 8, 9, 22⟨	M) 2=2/3, 2/5, 2/6+	$\mathbb{Z}_3$	r ? 2, 22
32, 4	}2, . . . , 7, 9, : , 22, 23, 26, 29⟨	M) 3=2/3, 2/3, 5/8+	$\mathbb{Z}_9$	r ? 2, 7, 9, 26
35, 9	}2, 3, 6, 8, 9, 24⟨	M) 2=2/3, 2/4, 2/9+	$\mathbb{Z}_3$	r ? 2, 8
39, 8	}2, 3, 4, 6, 7, 8, : , 21, 24, 25, 28, 32⟨	M) 2=2/5, 2/8, 5/8+	$\mathbb{Z}_8$	r ? 24, 32
41, 4-6-26	}2, 4, 6, 8, : , 26⟨			
44, 22	}2, 3, 5, 6, 8, 9, 21, 22, 24, 27, 2: , 33⟨	M) 6=2/3, 2/3, 2/4+	$\mathbb{Z}_{55}$	r ? bmm
		M) 2=2/4, 2/22, 7/22+	$\mathbb{Z}_{22}$	r ? 27, 33
47, 5	}2, 4, 5, 6, 8, 9, 22, 23, 26, 27, 2: , 34⟨			
53, 7-25-32	}2, 6, 22⟨	ϒ)3, 4, 8+	1	r ? 2
71, 23-26-31	}2, 3, 8, 22, 24, 25⟨	ϒ)4, 5, 6+	1	r ? 24
81, 21-25-46	}2, 4, : , 22, 24, 34⟨	ϒ)3, 6, 8+	1	r ? 22
89, 7-37-4:	}2, 6, 8, 22, 28, 34⟨	ϒ)3, 4, 24+	1	r ? 8

Table 14/ Pqujn bmn pdl Kbdpcj ui fubt pg opo.Ojfn fjfs uzqf boe fybn qrfit pg ui f sfrfwbou 4. n bojgpmet/

jt b *mock Jacobi form* pgjoefy m boe xfjhi upof jgjut opo.i prpn psqi jd dpn qrfuijpo

$$\phi^{a})\tau, z+? \prod_{r\mathrm{C}2,\ldots,m-2} \hbar_r)\tau+)\theta_{m,r} \quad \theta_{m,-r+})\tau, z+ \qquad)8/82+$$

usbotgpsn t bt b vtvbnKbdpcj gpsn)pgjoefy m boe xfjhi upof+/ Y f sfgfs up- gps jotubodf "45^ boe ":3- :4^- gps cbdl hspvoe po Kbdpcj gpsn t boe n pdl Kbdpcj gpsn t sfqtfdujwfnz/ Opuf ui bu ui f pqqptjuf tjho jo ui f ui fub gvodujpo gbdups sffffdut ui f bouj.jowbsjbodf pg Kbdpcj gpsn t pg pee xfjhi ut voefs $z \Rightarrow \quad z$/

Gspn b ovn cfs ui fpsz qpjou pg wjfx - xfjhi upof n pdl Kbdpcj gpsn t bsf sbui fs tqfdjbm/ Gjstu- jo b tfotf xf xjmn bl f qsfdjtf ti psuzn- bm ptu bmmpg ui fn i bwf usbotdfoefoubmdpfg fldjfout "5^ boe bsf ui fsfgpsf opusfrbufe up boz dpvoujoh qspcrfm jo upqpmphz boe qi ztjdt/

Tfdpoe- bt xf i bwf tffo jo ui f qsfwjpvt tvctfdujpot- bo jn qpsubou qspqfsuz pg)n pdl + n pevrbs gpsn t jt ui fjs cfi bwjps bu ui f dvtqt $\mathbb{Q} \cap \}i \in \langle$/ Bu xfjhi upof n pdl Kbdpcj gpsn jt jo gbdu vojr vfnz efufsn jofe cz jut qprfit "48- :4^/ Ui f *optimal* di pjdf pg ui f qprfit- gps b hjwfo joefy m- jt hjwfo cz

$$q^{\frac{1}{4m}} h_r \; ? \; O)2+. \qquad)8/83+$$

Jo ":3^ ju xbt ti px o ui bu ui f tqbdf pg xfjhi u pof n pdl Kbdpcj gpsn t- pg boz joefy $m \in \mathbb{Z}_{>1}$ boe i bwjoh 2/ ui f pquijn bmqprft)8/83+boe 3/ opo.usbotdfoefoubmdpf´ djfout- jt tvsqsjtjohmz flojuf.ejn fotjpobm)45.ejn fotjpobmup cf qsfdjtf+/ N psfpwfs- ui fsf bsf 4: tqfdjbmwfdupst jo ui jt 45.ejn fotjpobmtqbdf- ejtujohvjti fe cz ui fjs tzn n fusjft- x i jdi tqbo ui f tqbdf/)Gjwf pg ui fn bsf opu ujofbsmz joefqfoefou pg ui f sftu/+ Ui fz bsf rhcfmfe cz ui f tbn f qbjs)m, K+ui bu x f vtfe jo tfdujpo 4/4 up efflof tvc.sfqsftfoubujpot pg Y fjm sfqsftfoubujpot/ Mfu(t efopuf ui f dpssftqpoejoh n pdl gpsn t cz

$$\psi^{m\text{-}K}\,?\,\prod_{r\mathrm{C}2,\ldots,m-2} h_r^{m\text{-}K})\tau+\theta_{m,r}\quad\theta_{m,-r}+\qquad)8/84+$$

Ui fo ui f hspvq K ejdubuft ui f tzn n fusz pg$\Omega^{m\text{-}K}$ ui bujujt jowbsjbou voefs $\theta_{m,r} \forall\!\!\!\rightarrow \theta_{m,ra)n+}$ gps fwfsz $n \in K$)dg/)4/41+—)4/42++/ Jo qbsujdvmbs- tjodf $a)m+?\quad 2$ x jmmofwfs i bwf b opo.wbojti joh $\psi^{m\text{-}K}$ vorftt $m \not\in K$/ Jo gbdu- rvjuf sfn bsl bcmz ui fz bsf jo pof.up.pof dpssftqpoefodf x jui ui f 4: qbjst $)m, K+$x jui $m \not\in K$ x i jdi efflof ejtdsfuf tvchspvqt $^{m\text{-}K}$ pg TM)3, $\mathbb{R}$+x jui ui f qspqfsuz ui bu $^{m\text{-}K}$ $\mathbb{H}$ jt b hfovt –fsp S jfn boo tvsgbdf)n jovt flojufmz n boz qpjout+/ Y f sfgfs up ":3^ boe ":6^ gps ui f efubjmt/

Gspn ui f bcpwf dibsttjfldbujpo- x f pcubjo 4: ejtujohvjti fe n pdl Kbdpcj gpsn t $\psi^{m\text{-}K}$- ps fr vjwbmfoumz 4: wfdups.wbmufe n pdl n pevmbs gpsn t $h^{m\text{-}K}$? $)h_r^{m\text{-}K}+$ x jui joefqfoefou dpn qpofout hjwfo cz $r \in \sigma^{m\text{-}K}$)dg/tfdujpo 4/4+/ Ui fz i bwf ui sff gvsui fs tusjljoh ovn cfs ui fpsfujd qspqfsujft ":3^ pg hsfbujn qpsubodf up ui f qspcmfn t bu i boe;[33]

2/ Joufhsbmdpf´ djfout=

3/ S befn bdi fs tvn t=

4/ Ui fub gvodujpo ti bepx t/

Bmi pvhi x f pmz efn boefe ui f dpf´ djfout up cf opo.usbotdfoefoubm- x jui b tvjubcmf opsn bmj-bujpo ui fz bsf jo gbdu bmmjoufhsbmf Ui f flstu ep–fot pg dpf´ djfout dbo cf gpvoe jo ":3^ boe ":7^/ Ui f qspqfsujft pg ui f dpf´ djfout gvsui fs ejwjef ui f 4: dbtft joup ux p hspvqt; ui f gpsn t jo ui f flstu hspvq- dbmmfe ui f *Niemeier type*- i bwf *nonnegative* dpf´ djfout pg $h_r^{m\text{-}K}$ gps bmmopo.qpmbs ufsn t jo ui f q.fyqbotjpo- boe bsf jo pof.up.pof dpssftqpoefodf x jui ui f 34 Ojfn fjfs mbujdft boe qmbz ui f sprfi pg ui f hsbefe ejn fotjpot pg ui f flojuf hspvq n pevmft gps vn csbmn ppoti jof ":7- :8^/ Ui f tfdpoe hspvq dpoubjot ui f puif s 27 dbtft x i jdi i bwf cpui qptujujwf boe ofhbujwf Gpvsjfs dpf´ djfout/ Ui f dpssftqpoejoh m, K pg ui f ux p hspvqt bsf ubcvmbufe jo ubcmf 24 boe 25 sftqfdujwfmz/ Opujdf ui bu jo ui f flstu hspvq- opu bmmpg ui fn dpssftqpoe up jssfevdjcmf Y fjmsfqsftfoubujpot/

Ui f tfdpoe qspqfsuz tubuft ui bu ui fz dbo cf dpotusvdufe bt wfdups.wbmufe S bef.n bdi fs tvn t- x i jptf tjn qmft- tjohmf.wbmufe wfstjpo x f i bwf sfwjfxfe jo tfdujpo 8/5 bt b x bz up joufsqpmbuf cfux ffo vqqfs. boe mpx fs.i bmgqmbof/ Ui jt n fbot ui bu ui f ejtdvttjpo tfdujpo 8/5 jt bqqmjcmf gps ui ftf gvodujpot- boe ui fz bsf sfmbufe up ui f Fjdi mfs joufhsbmpg ui fjs ti bepx t-

[22] Another noteworthy property, though not directly related to the present application, is the fact that all Ramanujan's mock theta functions (up to modular forms) can be expressed in terms of these 39 mock Jacobi forms.

JHEP10(2019)010

wjb ui f $q \Rightarrow q^{-2}$ usbotgpsn bujpo ejtdvttfe jo tfdujpo 8/4 boe tfdujpo 8/5/ Ui f hppe ofxt gps vt jt uifo ui bu gps bmmui ftf 4: n pdl Kbdpcj gpsn t- ui f ti bepxt bsf hjwfo cz xfjhi u403 vobsz ui fub gvodujpot pg ui f gpsn)5/2+ boe ui fjs Fjdi nfs jouf hsbmt bsf qsfdjtfmz ui f gbnf ui fub gvodujpot $\Omega_r^{m\text{-}\,K}$ ui bu xf fodpvoufs/ Ui f qsfdjtf gpsn pg ui f ti bepxt dbo cf gpvoe jo ":3^ boe ":7^/ Gspn ui f epn jobou sprfn pg ui f Fjdi nfs jouf hsbmt $\Omega_r^{m\text{-}\,K}$ pg ui fn)5/8+jo ui f i pn pmphjdbmc mpdl t gps Tfjgfsu n bojgpmet xjui ui sff tjohvmbs flcsft- bt efn potusbufe jo tfdujpo 7- xf fyqfdu ui ftf 4: fybn qmft up cf sfnfwbou gps ui f tbn f n bojgpmet xjui sfwfstfe psjfoubujpo/ Joeffe- gps bm ptu bmmpg ui fn xf dbo fbtjmz floe ui sff.n bojgpmet gps x i jdi ui f i pn pmphjdbmc mpdl t bsf hjwfo cz ui f dpssftqpoejoh $\Omega_r^{m\text{-}\,K}$ gps tpn f r/ Y f ubcvmbuf tpn f pg ui fn jo ubcnfi 24–25/

Gjobmz- xf dpn n fou ui bu

$$\lim_{\tau\to 1} h_r^{m\text{-}\,K})\tau+? \;\; O)2+ \qquad\qquad)8/85+$$

gps ui f gpmpx joh $)m, K+$boe r;

$$\equiv 7,\;\; 3\text{-}\,r\;?\;\; 2$$

$$\equiv 21,\;\; 3\text{-}\,r\;?\;\; 2,4$$

$$\equiv 29,\;\; 3\text{-}\,r\;?\;\; 2,4,6,8\text{-}$$

x i jdi dbo fbtjmz cf wfsjfjfe gspn ui f l opxo cfi bwjpvs)8/83+pg $h_r^{m\text{-}\,K}$ ofbs $\tau \leftrightarrow i\in$ boe cz dpn qvujoh ui f S.n busjy)4/48+/ Gps jotubodf- $h_2^{7\text{-}\,3})\tau+?\;\; q^{-\frac{1}{24}} f)q+$jt hjwfo cz ui f psefs ui sff n pdl ui fub gvodujpo pg S bn bovkbo $f)q+$ x i jdi bt xf i bwf dpn n foufe jo)8/45+—)8/46+i bt b flojuf wbnuf bu $q \leftrightarrow 2$/ Bt n foujpofe jo ui f cfhjoojoh pg ui f tvctfdujpo- ui jt gbdu hjwft ui ftf n pdl n pevmbs gpsn t ui f ejtujohvjti fe tubuvt ui bu ui fjs btzn qupujd fyqbotjpo ofbs $\tau \leftrightarrow 1$ dpjodjeft xjui ui f dpssftqpoejoh Pi utvl j tfsjft po ui f optf/

8 Beyond false

Jo ui jt tfdujpo- xf tuvez Tfjgfsu n bojgpmet xjui gpvs tjohvmbs flcsft/ Ju uvsot pvu ui bu tusvduvsf pg ui f i pn pmphjdbmc mpdl t bsf wfsz bobmphpvt bt ui f dbtft xjui ui sff tjohvmbs flcsft/ Ui f opwfmz jt ui bu ui fz i bwf ui f gpmpx joh]cvjmejoh cmpdl t# qmbzjoh b tjn jmbs sprfn bt ui f gbnf ui fub gvodujpot $\Omega_r^{m\text{-}\,K}$ jo ui f qsfwjpvt dbtft- bsf b n jy cfuxffo Fjdi nfs jouf hsbmt pg xfjhi u203 boe xfjhi u403 ui fub gvodujpot)dg/)9/5++

$$\begin{aligned} B_{m,r})\tau+ \le \frac{2}{3m}\Big] \Psi_{m,r})\tau+ \;\; r\Omega_{m,r})\tau+\Big\{ \\ B_r^{m\text{-}\,K})\tau+? \;\; 3^{|K|-2} \prod_{r'\,\mathrm{n\,pe}\,3m} P_{rr'}^{m\text{-}\,K} B_{m,r'})\tau+. \end{aligned} \qquad)9/2+$$

Y f qspwjef b opo.tqi fsjdbmf ybn qmf $M)\;\; 3=\frac{2}{3}, \frac{3}{4}, \frac{3}{6}, \frac{3}{6}+$boe dpn qvuf jut btzn qupujd fyqbo. tjpo cz fyqmpjujoh jut n pevmbs.ujl f qspqfsujft/

JHEP10(2019)010

8.1 The building blocks

Yf qspdffe cz bobmphz xjui 4.flcfs fybnqmft up jefoujgz uif]cvjmejoh cmpdlt/# Gps Csjftlpso ipnpmphz tqifsft $\Upsilon)p_2, p_3, p_4+$ uifjs YSU jowbsjbout efdpnqptf joup gbnjmjbs uifub gvodujpot ‘55- 56^;

$$Z_{\rm EV})\Upsilon)p_2,p_3,p_4++? \frac{q^{-\phi/5}_{\S}}{i^3\sqrt{9k}}\left]\prod_{rC2}^{m-2}\chi_{3m}^{)2,2,2+})r+\Omega_{m,r})\tau+, \quad H\right) \quad 2, \prod_{jC2}^{4}\frac{2}{p_j}\left[q^{2/231}\right\{, \qquad)9/3+$$

xifsf $m ? \sum_j p_j$- boe H jt uif ifbwjtjef tufq.gvodujpo/ Uif $3m$.qfsjpejd gvodujpo $\chi^{\vec{l}}_{3m})r+$ jt efflofe gspn n.ejnfotjpobmwfdupst $\vec{l}$? $)l_2, \times\times, l_n+$ boe $\vec{p}$? $)p_2, \times\times, p_n+$ tbujtgzjoh $1 < l_j < p_j$;

$$\chi^{\vec{l}}_{3m})r+? \left\{ \begin{array}{ll} \int_{jC2}^{n}\epsilon_j \quad \text{jg } r \le m\Big)2, \ \prod_j \frac{\epsilon_j l_j}{p_j}\Big(\text{n pe } 3m, & \text{xifsf } \epsilon_j ? \oplus 2 \\ 1 & \text{puifsxjtf/} \end{array}\right.$$

Uivt- $\chi_{3m}^{)2,2,2+})r+$jo)9/3+jt hjwfo cz n ? 4- $\vec{l}$?)2, 2, 2+ boe $\vec{p}$? $)p_2, p_3, p_4+/$ Pof dbo pctfswf uibu qbsujbmuifub gvodujpot qmbz uif spmf pg cbtjd cvjmejoh cmpdlt gps uif YSU jowbsjbout pg Csjftlpso ipnpmphz tqifsft- xjui uif mbuufs efufsnjofe cz $\chi_{3m}^{)2,2,2+})r+/$

Ju xbt tipxo jo ‘:9- ::^- gps gpvs.tjohvmbsmz flcfsfe Tfjgfsu ipnpmphz tqifsft $\Upsilon)p_2,p_3,p_4,p_5+$ uif rvbouvn $Z_{\rm EV})M_4+$ dbo cf fyqsfttfe jo ufsnt pg qbsujbmuifub gvod.ujpot boe b xfjhiu 2/3 Fjdimfs joufhsbm/ Uifo- pof dbo tjnjmbsmz fyusbdu cbtjd cvjmejoh cmpdlt cz qvmmjoh pvu $\chi_{3m}^{)p_1-2,2,2,2+})r+$[34]

$$Z_{\rm EV})\Upsilon)p_2,p_3,p_4,p_5++? \frac{q^{-\phi/5}_{\S}}{i^3\sqrt{9k}}\left]\prod_{rC2}^{m-2}\chi_{3m}^{)p_1-2,2,2,2+})r+\frac{2}{3m}\quad \Psi_{m,r})\tau+ \quad r\Omega_{m,r})\tau+\!\!\left(\right.$$
$$, \quad H\right) \quad 2, \prod_j \frac{2}{p_j}\left[\Omega_{m,)3m-\sum_j m/p_j+})\tau+\right\{, \qquad)9/4+$$

xifsf $\Psi_{m,r})\tau+$ bsf uif xfjhiu 2/3 Fjdimfs joufhsbmt)5/3+ pg uif xfjhiu 203 uifub gvodujpot)dg/)5/2++

$$\theta^1_{m,r})\tau+;? \ \theta_{m,r})\tau,z+|_{zC1} \ ? \prod_{\substack{\ell\in\mathbb{Z} \\ \ell Cr \text{ n pe } 3m}} q^{\ell^2/5m}.$$

Fyqmjdjumz- xf ibwf

$$\Psi_{m,r})\tau+? \prod_{n\ge 1} n\psi^{)r+}_{3m})n+q^{n^2/5m},$$
$$\text{xifsf} \quad \psi^{)r+}_{3m})n+? \left\} \begin{array}{ll} 2 & \text{jg } \ n \le \oplus r \quad \text{n pe } 3m \\ 1 & \text{puifsxjtf/} \end{array}\right. . \qquad)9/5+$$

Tjnjmbs up)5/6+ jo ufsnt pg qspkfdupst)4/44+ xf tjnqmz ibwf $\psi^{)r+}_{3m})n+? \ 3)P^-_m)m++_{r,n}/$

[23] Spherical Seifert manifolds are uniquely determined by the orders of their singular fibers. In our convention, their Euler characteristic is $-1/\prod_j p_j$: following [62].

Ui f fyqsfttjpo)9/4+ dbo cf cftu voefstuppe jo dpn qbsjtpo xjui uisff.tjohvrbs flcfs dbtft;

$$\chi_{3m}^{)p_1-2,2,2,2+})r+\!\!\rightarrow\!\!\rightarrow \chi_{3m}^{)2,2,2+})r+$$

$$\frac{2}{3m}\Big)\Psi_{m,r})\tau+\quad r\Omega_{m,r})\tau+\Big(\!\!\rightarrow\!\!\rightarrow \Omega_{m,r})\tau+ \qquad)9/6+$$

$$H\Big)\quad 2\,,\ \prod_j \frac{2}{p_j}\Big[\,\Omega_{m,)3m-\sum_j m/pj+})\tau+\!\!\rightarrow\!\!\rightarrow H\Big)\quad 2\,,\ \prod_j \frac{2}{p_j}\Big[\,q^{2/231}$$

Dpotfrvfoumz- xf qspqptf ui bu Y SU jowbsjbout pg Tfjgfsu n bojgpmet xjui gpvs tjohv.rbs flcfst)opu ofdfttbsjmz joufhsbmi pn prphz tqi fsft+efdpn qptf joup ui f gpmpx joh cvjmn.joh cnpdl t;

$$B_{m,r})\tau+\leq \frac{2}{3m}\Big]\Psi_{m,r})\tau+\quad r\Omega_{m,r})\tau+\Big\{.$$

Opuf ui bu xi jnfi $\Omega_{m,r}$? $\Omega_{m,-r}$- xf i bwf $B_{m,r}$? $B_{m,-r}$/ Bt b sftvmu- xi jnfi m ⁄/ K gps ui f qbjs m , K sfnfwbou gps ui f fybn qrfst jo tfdujpo 7- jo ui jt dbtf xf n vtu i bwf m / K/ Ui f n pevmbs.njl f qspqfsuz pg $B_{m,r}$ gpmpxt gspn ui ptf pg $\Omega_{m,r}$ boe $\Psi_{m,r}$;

$$\S\frac{2}{\overline{k}}\Omega_{m,r})2/k+,\ \S\frac{2}{\overline{i}}\prod_{r'C2}^{m-2}\frac{\overline{3}}{m}\,tjo\,\frac{rr'\pi}{m}\Omega_{m,r'})\quad k+$$

$$?\ \prod_{n\geq 1}\frac{c_n}{n"}\Big)\frac{\pi i}{3m}\Big[^{n}\,k^{-n-\frac{2}{3}},$$

$$\Omega_{m,r})\quad k+?\ \Big)2\quad \frac{r}{m}\Big[\,e^{-3\pi ikr^2/5m},\quad xi fsf\quad \frac{tjoi\,)m\quad r+\!\!\!z}{tjoi\ mz}\ ?\ \prod_{nC1}^{\infty}\frac{c_n}{3n"}z^{3n}. \qquad)9/7+$$

$$\S\frac{2}{\overline{k}}\Psi_{m,r})2/k+,\ \S\frac{k}{\overline{i^4}}\prod_{r'C2}^{m-2}\frac{\overline{3}}{m}\frac{r')m\quad r'+}{m}\,dpt\,\frac{rr'\pi}{m}e^{-3\pi ik\frac{(r')^2}{4m}}$$

$$?\ \frac{mk}{\pi i}\prod_{nC1}^{\infty}\frac{c'_n}{n"}\Big)\frac{\pi i}{3m}\Big[^{n}\,k^{-n-\frac{2}{3}},\quad xi fsf\quad \frac{\partial}{\partial r}\frac{tjoi\,)m\quad r+\!\!\!z}{tjoi\ mz}\ ?\ \prod_{nC1}^{\infty}\frac{c'_n}{3n"}z^{3n}. \qquad)9/8+$$

Ui f bcpwf qspqfsuz dbo cf fn qmpzfe up dpn qvuf ui f usbottfsjft fyqsfttjpo pg $Z_{EV})M_4+/$

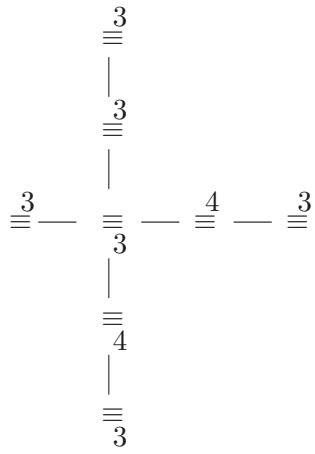
Opuf ui bu ui f sfnbujpo cfuxffo di bsbdufst pg tjohrfu)2, p+mphbsjui n jd wfsufy brhfcsbt boe i pn prphjdbmc cnpdl t pg Tfjgfsun bojgpmet xjui ui sff tjohvrbs flcfst qfstjtut up ui f qsftfou dbtf/ Pof pctfswft b dnptf sfnbujpo cfuxffo di bsbdufst pg tjohrfu $)p_-$, p_-+wfsufy brhf.csbt ‘211- 212ˆ boe ui f i pn prphjdbmc cnpdl t pg Tfjgfsun bojgpmet xjui gpvs tjohvrbs flcfst tuve.jfe jo ui jt tfdujpo- tusfohui fojoh ui f pctfswfe dpoofdujpo cfuxffo mphbsjui n jd brhfcsbt/

Ju xpvmn cf brtp joufsftujoh up fyqmjdjumz dpotusvdu cvjmnjoh cnpdl t gps hfofsjd ovn cfs pg flcfst- boe xf nfibwf ju gps gvuvsf xpsl t/

8.2 Example: $M(-2;\frac{1}{2},\frac{2}{3},\frac{2}{5},\frac{2}{5})$

Yf bqqmz ui f n pevmbsjuz ejdujpobsz up i pn prphjdbmc cnpdl t pg b opo.tqi fsjdbm gpvs.tjohvrbsmz flcfsfe Tfjgfsun bojgpmet/ Ui f Tfjgfsun bojgpmet $M)\ \ 3=\frac{2}{3},\frac{3}{4},\frac{3}{6},\frac{3}{6}+$i bt ui f gpmpx joh

qmnn cjoh hsbqi ;



)9/9+

Bt cfgpsf- xf dpn qvuf i pn pmphjdbmcmpdl t boe $S^{)A+}$;

$$\mathrm{DT})a+? \ )2\ \tfrac{2}{6}\ \tfrac{i}{6}\Big(,$$

$$S^{)A+}? \ \tfrac{2}{3\sqrt{6}}\begin{pmatrix} 2 & 2 & 2 \\ 3 & \frac{-2-\sqrt{6}}{3} & \frac{-2-\sqrt{6}}{3} \\ 3 & \frac{-2-\sqrt{6}}{3} & \frac{-2-\sqrt{6}}{3} \end{pmatrix}\Big(, \qquad)9/:+$$

$$\widehat{Z_1)}q+? \ q^{8/3})2 \quad q^{22}\,, \ q^{25} \quad q^{2;} \quad q^{44}\,, \ q^{51} \quad q^{56}\,, \ 3q^{64}\,, \ q^{85}\,, \ \times\times\times+,$$
$$\widehat{Z_2)}q+? \ 1, \qquad)9/21+$$
$$\widehat{Z_3)}q+? \ 3q^{;\,4/21}) \ \ 2\,, \ q^{26}\,, \ q^{36} \quad q^{61} \quad 3q^{76}\,, \ 3q^{231} \quad 3q^{276} \quad 4q^{2;\,1}\,, \ \times\times\times+.$$

Cz ui f qsftdsjqujpo pg n pevmbsjuz ejdujpobsz- xf dbo fbtjmz tff ui bu m ? 41/ Ui fo- i pn pmphjdbmcmpdl t dpssftqpoe up ui f Yfjmsfqsftfoubujpo σ ? 41, 6, 7, 41- xi ptf qspkfdups jt fyqnjdjumz hjwfo cz

$$P^{41-\ 6,7,41}\ ? \ P_{41}^{-})6+P_{41}^{-})7+P_{41}^{-})26+,$$

xi jdi nfbet up- vtjoh)9/2+

$$\begin{aligned} B_8^{41-\ 6,7,41})\tau+? \ )B_{41,8} \quad B_{41,24}\,, \ B_{41,28} \quad B_{41,34}+)\tau+, \\ B_6^{41-\ 6,7,41})\tau+? \ )B_{41,6} \quad B_{41,34}+)\tau+. \end{aligned} \qquad)9/22+$$

Jo ufsn t pguiftf- ui f i pn pmphjdbmcmpdl t sfbe

$$\begin{aligned} \widehat{Z_1)}q+? \ q^{-21;\,/231}\Big)\Omega_{41,34})\tau+ \quad B_8^{41-\ 6,7,41})\tau+\Big(\\ \widehat{Z_2)}q+? \ 1 \\ \widehat{Z_3)}q+? \ 3q^{-21;\,/231}B_6^{41-\ 6,7,41})\tau+. \end{aligned} \qquad)9/23+$$

Cz ui f n pevmbs.njl f qspqfsujft pguif cvjmejoh cmpdl t- xf pcubjo ui f usbottfsjft tvn n bsj–fe jo ubcmf 26- xi fsf bo pwfsbmmgbdups pg $iq^{-21;\,/231}/3^{\frac{8}{3}}\overline{3}$ jt pn juufe bt vtvbm/

DT bdujpo	tubcjmjuz	uzqf	usbottfsjft
1	TV)3+	dfousbm	$e^{3\pi i k\cdot 1}\big)\frac{5\pi i}{6}k^{-4/3}$, $\mathcal{P})k^{-6/3}+\Big($
$\frac{2}{6}$	Φ)2+	bcfrjbo	$e^{3\pi i k\frac{1}{5}}\big)\frac{6-\sqrt{6}}{7}k^{-2/3}$, $\mathcal{P})k^{-4/3}+\Big($
$\frac{:}{6}$	Φ)2+	bcfrjbo	$e^{3\pi i k\frac{9}{5}}\big)\frac{6-\sqrt{6}}{7}k^{-2/3}$, $\mathcal{P})k^{-4/3}+\Big($
$\frac{2}{231}$	⊕2	opo.bcfrjbo-sfbm	$e^{-3\pi i k\frac{1}{120}}\frac{e^{3\pi i/4}}{261\sqrt{26}}\big)$ 36 , $6^{\S}\,\overline{6}$, $35^{\S}\,\overline{4}$ dpt $\frac{\pi}{21}\Big($
$\frac{5}{231}$	⊕2	opo.bcfrjbo-sfbm	$e^{-3\pi i k\frac{4}{120}}\frac{e^{3\pi i/4}}{5})2$, $^{\S}\,\overline{6}+$
$\frac{27}{231}$	⊕2	opo.bcfrjbo-sfbm	$e^{-3\pi i k\frac{16}{120}}\frac{e^{3\pi i/4}}{5})2\,^{\S}\,\overline{6}+$
$\frac{36}{231}$	⊕2	opo.bcfrjbo-sfbm	$e^{-3\pi i k\frac{25}{120}}\frac{5e^{3\pi i/4}}{6}\sqrt{2\quad\frac{3}{\sqrt{6}}}$
$\frac{51}{231}$	⊕2	opo.bcfrjbo-sfbm	$e^{-3\pi i k\frac{40}{120}}e^{4\pi i/5}$
$\frac{5;}{231}$	⊕2	opo.bcfrjbo-sfbm	$e^{-3\pi i k\frac{49}{120}}\frac{e^{3\pi i/4}}{41\sqrt{26}}\big)36$, $6^{\S}\,\overline{6}$, $35^{\S}\,\overline{4}$ dpt $\frac{4\pi}{21}\Big($
$\frac{84}{231}$	⊕2	opo.bcfrjbo-sfbm	$e^{-3\pi i k\frac{73}{120}}\frac{9e^{3\pi i/4}}{6\sqrt{6}}$ dpt $\frac{4\pi}{21}$
$\frac{87}{231}$	⊕2	opo.bcfrjbo-sfbm	$e^{-3\pi i k\frac{76}{120}}\frac{e^{3\pi i/4}}{5})2\,^{\S}\,\overline{6}+$
$\frac{92}{231}$	⊕2	opo.bcfrjbo-sfbm	$e^{-3\pi i k\frac{81}{120}}\frac{e^{3\pi i/4}}{4\sqrt{4}})2\,^{\S}\,\overline{6}+$
$\frac{;8}{231}$	⊕2	opo.bcfrjbo-sfbm	$e^{-3\pi i k\frac{97}{120}}\frac{9e^{3\pi i/4}}{6\sqrt{6}}$ dpt $\frac{\pi}{21}$
$\frac{216}{231}$	⊕2	opo.bcfrjbo-sfbm	$e^{-3\pi i k\frac{105}{120}}\frac{5e^{3\pi i/4}}{4\sqrt{4}}$
$\frac{:}{231}$	⊕2	opo.bcfrjbo-dpn qrfy	1
$\frac{75}{231}$	⊕2	opo.bcfrjbo-dpn qrfy	1
$\frac{211}{231}$	⊕2	opo.bcfrjbo-dpn qrfy	1

Table 15/ Usbottfsjft boe dibttjfldbujpo pg ffbu dpoofdujpot po $M)$ $3=\frac{1}{2},\frac{2}{3},\frac{2}{5},\frac{2}{5}+/$

9 Discussions and open questions

Jo ui jt qbqfs xf ejtdvttfe ui f gpnpx joh tvsqsjtjoh gfbuvsft pgi brgjoejdft pgdfsubjo $\mathcal{O}$? 3 4e tvqfstzn n fusjd rvbouvn flfrn ui fpsjft- xi jdi bsf brtp ui f i pn pnphjdbmcnpdl t '7^ pg b gbn jmz pg ui sff.n bojgpnat/ Jo ui f flstu qbsu pg ui f qbqfs xf ejtdvttfe ui sff ejfifsfou TM)3, $\mathbb{Z}$+)qspkfdujwf+sfqsftfoubujpot xf fodpvoufsfe jo ui f qspcrfm boe n bl f vtf pg ui fn up dpn qvuf upqpnphjdbmz boe qi ztjdbmz joufsftujoh rvboujujft/ Tfdpoe xf qspqptf ui f sfrfnwbodf pg ui f gbrtf.n pdl qbjs gps pvs qspcrfm - n bl joh vtf pg ui fjs sfrbujpo up rvbouvn n pevrbs gpsn t/

Y f xjmmfoe ui f n bjo qbsu pg ui f qbqfs xjui b rjtu pg pqfo rvftujpot boe gvuvsf ejsfdujpot;

≡ Ui pvhi ui f sfrfnwbodf pg ui f gbrtf.n pdl qbjs jt n bojgftu- ui fsf bsf tujmb gfx jn qpsubou qv—rfit sfn bjojoh/ Kvtu qvsfrz gspn ui f ovn cfs ui fpsz qpjou pg wjfx ui fsf bsf ux p bn cjhvjujft jo jefoujgzjoh ui f dpssfdu $\widehat{Z_a}$ po ui f n pdl tjef;

2/ Bt tvn n bsjtfe jo flhvsf :- uif Fjdi nfs joufhsbmbttpdjbufe up b n pdl gvodujpo pomz efqfoet po jut ti bepx boe jt uif sfgpsf jotfotjujwf up uif beejujpo pgb qvsfmz n pevmbs gpsn / Sfnbufemz- bu uif foe pgtfdujpo 8/3- xf i bwf tffo ui bu gspn uif q.izqfshfpn fusjd qfstqfdujwf uif sf bsf wbsjpvt bn cjhvjujft x i fo hpjoh cfux ffo vqqfs. boe npxfs.i bmgqnboft; ux p q.izqfshfpn fusjd tfsjft dbo efflof uif tbn f gvodujpo po pof tjef boe ejfifsfou gvodujpot po uif puifs/ Ui jt jt sfnbufe up uif tp.dbmfe]fyqbotjpo pg –fsp#eftdsjcfe cz Sbefn bdifs ‘213ˆ/ Jo pvs dpoufyu pg xfjhiu pof Kbdpcj gpsn t- uif bn cjhvjuz jt frvjwbmfou up uif bn cjhvjuz pg tqfdjgzjoh uif qnfit pguif gvodujpo)tff ‘48ˆ boe ‘:3ˆ+/

3/ Npsfpwfs- podf b n pdl n pevmbs gpsn f jt di ptfo- pof tujmmoffe up di pptf uif n pevmbs tvcusbdujpo G_x bu uif dvtq x)dg/)8/47++/ Y i jnfi gps dpn qbsjtpo xjui qfsuvscbujwf Di fso.Tjn pot pof pomz offet uif tvcusbdujpo gps uif dvtq $\tau \leftrightarrow 1$- jo psefs up njufsbmz dpn qbsf xjui bmmuif Y SU jowbsjbout pof xpvmz offe uif tvcusbdujpo bu bmmspput pg vojuz pguif gpsn $\tau \leftrightarrow \frac{2}{k}$/)Pg dpvstf- uif jt jt uif eftdsjqujpo pg uif bn cjhvjuz xi fo dpotjefsjoh uif n pdl gpsn f bt b tjohmfi.wbmufe n pdl n pevmbs gpsn gps b tvchspvq $< \mathrm{TM}3, \mathbb{Z}+$/ Y i fo eftdsjcfe jo ufsn t pg wfdups.wbmufe n pdl n pevmbs gpsn t gps $\mathrm{TM}3, \mathbb{Z}+$ uif dpssftqpoejoh bn cjhvjuz jt ui bu pg tqfdjgzjoh uif n pevmbs tvcusbdujpot gps bmmdpn qpofout pg uif wfdups.wbmufe gvodujpo/+

Sfdbmmui bu- xi jnfi pof epft opu ofdfttbsjmz offe up dbsf bcpvu uiftf bn cjhvjujft gps uif qvqptf pg sfqspevdjoh uif qfsuvscbujwf Di fso.Tjn pot ebub- uif bduvbmq.tfsjft bsf qi ztjdbmz wfsz n fbojohgvmñ Dnfbsmz- jo psefs up n blf hfofsbmqsfejdujpot gps uif i pn pmphjdbmcnpdl t gps hfofsbmui sff.n bojgpmet boe jo psefs up cfuufs voefstuboe uif hfofsbmn pevmbsjuz tusvduvsf pg4e $\mathcal{O}$? 3 uifpsjft- jut dsvdjbmup cfuufs voefstuboe uif bcpwf bn cjhvjujft- boe i pqfgvmz up floe tv´ djfoumz qpx fsgvmdsjufsjb gspn qi ztjdt boe upqpmphz up fnjn jobuf uif bn cjhvjujft/ Bntp gspn uif dpoufyu pguijt xpsl- uif sfnbujpo up I bcjsp sjoh ‘214ˆ bqqfbst up cf bo jn qpsubou nfbe/

≡ Ju xpvmz cf joufsftujoh up dpn qvuf- f/h/ wjb sftvshfodf ‘6ˆ- uif q.tfsjft jowbsjbout $\widehat{Z_a})M_4+$ gps i zqfsc pmjd 4.n bojgpmet boe uftu uif dpokfduvsf jo tfdujpo 6- obn fmz xi fuifs jo tvdi dbtft $\widehat{Z_a})M_4+$bsf sfnbufe up di bsbdufst pgmphbsjui n jd wfsufy bnhfcsbt xi jdi bsf opu C_3.dpflojuf/

≡ Jo ui jt xpsl xf opuf uif jn qpsubou sprfi qnbzfe cz uif Y fjmsfqsftfoubujpot- nbcfmfe cz b qbjs m boe $K \leftarrow \mathrm{Fy}_m$- jo pvs qspcmfn / Y i jnfi m dbo qi fopn fmphjdbmz cf efufsn jofe cz uif upqpmphjdbmebub)7/25+ xf ep opu lopx xi bu uif fyqmjdju sfnb. ujpo cfux ffo K boe 4.n bojgpmet upqpmphz jt/ Npsf dpodfquvbmz- ju xpvmz cf hsfbu up voefstuboe uif psjhjo pguiftf Y fjmsfqsftfoubujpot jo pvs upqpmphjdbmfqi ztjdbm qspcmfn / Gvsui fsn psf- jo tfdujpo 4 xf ejtdvttfe ui sff ejfifsfou S.n busjdft boe ui sff ejfifsfou]$\mathrm{TM}3, \mathbb{Z}+$sfqsftfoubujpot#ui bu qnbz bo jn qpsubou sprfi jo pvs tupsz/ Y f ep fyqfdu uiftf sfqsftfoubujpot up cf joufs.sfnbufe boe pof pcwjpvt pqfo qspcmfn jt up voefstuboe i px fybdumz ui fz bsf sfnbufe/

JHEP10(2019)010

≡ Jo pvs tupsz- n pdl n pevrhs gpsn t boe joufsftujoh n pevrhs tusvduvsft fn fshf gspn ui f qi ztjdt pg 4e $\mathcal{O}$? 3 ui fpsjft boe CQT tubuft/ Jo qbsujdvrhs- pvs lfz qrhzfst $\widehat{Z_a)}q+$bsf]dpvoujoh#CQT tubuft/ Ju x pvrn cf joufsftujoh up floe sfrhujpot)evbrjujft+up pui fs qi ztjdt qspcrfin t x i fsf tjn jrhs n pevrhs tusvduvsft bqqfbsfe- f/h/ ": 4- 215–218ˆ/

≡ Jo ui jt x psl x f i bwf n bjorn gpdvtfe po fybn qrfit pg Tfjgfsu n bojgpret x jui ℓ tjohvrhs flcfst- x i fsf ℓ ? 4 boe ui f ipn prphjdbmcrnpdl t bsf hjwfo cz gbrtf ui fubt/ Ui ftf uzqf pg gvodujpot brtp bqqfbs bt di bsbdufst pgn pevrfit pgtjohrfu)2, p+rphbsjui n jd wfsufy brhf. csbt/ Jo tfdujpo 9 x f csjfffz ejtdvttfe ui f dbtf pgℓ ? 5- x i fsf ui f ipn prphjdbmcrnpdl t bsf dpn qptfe pg cvjmejoh crnpdl t x i jdi dpoubjo gbrtf ui fub gvodujpot dpssftqpoejoh up b n jy pg rvbouvn n pevrhs gpsn t pg x fjhi u 403 boe x fjhi u 203/ N psfpwfs- ui f sfrhujpo up rphbsjui n jd wfsufy brhfcsbt qfstjtut boe ui jt ujn f ui f crnpdl t bqqfbs up cf sfrhufe up di bsbdufst pg)p_- ,p_-+tjohrfu wfsufy brhfcsbt/ I px fwfs- ui fsf jt drfibsrn n vdi n psf up fyqmpsf/ P of joufsftujoh rvftujpo jt up jefoujgz ui f cvjmejoh crnpdl t pg i pn prphjdbmcrnpdl t gps Tfjgfsu n bojgpret x jui hfofsbmℓ- boe ui f qpufoujbmsfrhujpo up rphbsjui n jd wfsufy brhfcsbt/ Ofyu- ju x pvrn cf joufstufsjoh up fyqmpsf ui f n pevrhs. juz tusvduvsf pg i pn prphjdbmcrnpdl t gps qnun cfe 4.n bojgpret x i jdi bsf opo.Tfjgfsu/ Ui jse- ju jt wfsz dpodfjwbcrfi ui bu ui fsf fyjtut b ojdf sfrhujpo cfux ffo i jhi fs sbol jowbsjbout boe i jhi fs.efqui rvbouvn n pevrhs gpsn t "211ˆ- hfofsbrjn-joh ui f TM)3, $\mathbb{C}$+ tupsz x i jdi jt ui f gpdvt pg ui f qsftfou qbqfs/ Gjobmm- pof n bz brtp jowftujhbuf ui f n pevrhs.rjlf qspqfsujft pg ui f i brgjoejdft bsjtjoh gspn 4e ui fpsjft ui bu bsf opu dpn joh gspn ui sff.n bojgpret/

≡ Ui f fybn qrfit hjwfo jo tfdujpo 8/6 tvhhftu ui f sfrfwbodf pg ui f 4: pqujn bmn pdl Kbdpcj ui fub gvodujpot jo pvs upqprphjdbmboe qi ztjdbmqspcrfin t/ Bu ui f tbn f ujn f- ui ftf n pdl gvodujpot brtp qrhz ui f sprfi pg ui f hsbefe ejn fotjpot pg flojuf hspvq sfqsftfoubujpot jo ui f dpoufyu pg ui f tujmmn ztufsjpvt vn csbmm ppoti jof "7- : 8ˆ/ P of obuvsbmr vftujpo jt x i fui fs ui fsf jt b sfrhujpo cfux ffo pvs tfuvq boe ui f n ppoti jof flojuf hspvqt/

Acknowledgments

Y f ui bol E/Bebn pwjd- U/Dsfvu-jh- U/Ejn pguf- K/Evodbo- Q/Fujohpg- C/Gfjhjo- E/L b–i . ebo- T/Mpcsjdi - D/N boprfitdv- E/Qfj- Q/Qvuspw- M/S prho- D/Tdi x fjhfsu- D/Wbgb boe E/[bhjfs gps i fnqgvmejtdvttjpot/ Ui f x psl pg N/D/jt tvqqpsufe cz FS D tubsujoh hsbou I 3131 FS D TuH ' 75126: /T/D/boe T/H/bsf tvqqpsufe cz ui f Y bmfs Cvsl f Jotujuvuf gps Ui fpsfujdbmQi ztjdt- cz ui f V/T/Efqbsun fou pg Fofshz- P ´ df pg Tdjfodf- P ´ df pg I jhi Fofshz Qi ztjdt- voefs Bxbse Op/EF.TD1122743- boe cz ui f ObujpobmTdjfodf Gpvoeb. ujpo voefs Hsbou Op/OTG.EN T 2775351/ Ui f x psl pg T/D/jt brtp tvqqpsufe jo qbsu cz Tbn tvoh Tdi prhsti jq/ Ui f x psl pg T/H/jt brtp tvqqpsufe jo qbsu cz Mbcpsbupsz pg N jssps Tzn n fusz OSV I TF- S G Hpwfson fou hsbou- bh/ Op/25/752/42/1112/ Ui f x psl pg G/G/ jt tvqqpsufe jo qbsu cz ui f N JVS.TJS hsbou S CTJ2582HK]Rvbouvn Gjfme Ui fpsjft bu Tuspoh Dpvqjoh; Fybdu Dpn qvubujpot boe Bqqrjdbujpot# T/I /jt tvqqpsufe cz ui f Ob. ujpobmTdjfodf boe Fohjoffsjoh Dpvodjmpg Dbobeb- bo GS ROU ofx vojwfstjuz sftfbsdi fst tubsu.vq hsbou- boe ui f Dbobeb Sftfbsdi Di bjst qsphsbn / N/D/x pvrn brtp rjlf up ui bol MQUI F- Kvttjfv Qbsjt- gps i ptqjubrjuz evsjoh ui f flobmtubhf pg ui jt x psl /

JHEP10(2019)010

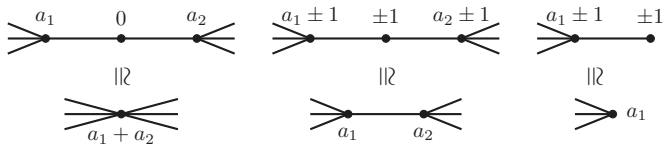


Figure 10/ 4e Ljscz n pwft gps qmun cfe n bojgpmet/ Uif sftvmjoh qmun cfe n bojgpmet M boe M' bsf i pn fpn psqi jd/

A Invariance of convergence under 3d Kirby moves

Jo uijt tfdujpo- xf qspwf uibu4e Ljscz n pwft)flhvsf 21+qsftfswf uif epn bjo pgdpowfshfodf pg i pn pmphjdbmcmpdlt/

Dpotjefs uif cpuupn rfguhsbqi pg flhvsf 21/ Yf nbz di pptf uif cbtjt jo xijdi uif belbdfodz n busjy M i bt gsbn joh dpf´ djfou a_2, a_3 jo uif)i, i+ui dppsejobuf/ Cz rfmn n b 2- i pn pmphjdbmcmpdlt pguif n bojgpmn qmun cfe bmpoh uif cpuupn rfguhsbqi xpvmn i bwf uif gpmpxjoh btzn qupujd cfi bwjps gps uif q.fyqpofout;

$$q^{-\frac{(\ell,M^{-1}\ell)}{4}} \; ? \; q^{-\frac{(M^{-1})_{ii}\ell_i^2}{4}} \text{- } O)2+, \quad \text{bt} \quad \backslash \ell \backslash \leftrightarrow \in \qquad \text{)B/2+}$$

gps uif sfnfwbou ufsn t pguif tvn / Ofyu- xf dpotjefs uif belbdfodz n busjy M' pguif upq rfguhsbqi - xijdi i bt uif gpmpxjoh “i 2, i, 2ˆ∗ “i 2, i, 2ˆ.tvcn busjy;

$$\begin{pmatrix} a_2 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & a_3 \end{pmatrix} \leftarrow M'.$$

B tjn qrfnjpofbs brhfcsb ti px t uibujut jowfstf i bt uif gpmpxjoh “i 2, i, 2ˆ∗ “i 2, i, 2ˆ. tvcn busjy;

$$\begin{pmatrix} \frac{)M^{-1}+ti}{3} & \times\times\times & \frac{)M^{-1}+ti}{3} \\ \times\times\times & \times\times\times & \times\times\times \\ \frac{)M^{-1}+ti}{3} & \times\times\times & \frac{)M^{-1}+ti}{3} \end{pmatrix} \leftarrow)M'+^{-2}.$$

Uifsfgpsf- uif i pn pmphjdbmcmpdlt bttpdjbufe up uif upq rfguqmun cjoh hsbqi xpvmn i bwf uif q.fyqpofout xjui uif gpmpxjoh btzn qupujd cfi bwjps;

$$q^{-\frac{(\ell,(M')^{-1}\ell)}{4}} \; ? \; q^{-\frac{(M^{-1})_{ii}(\ell_{i-1}-\ell_{i+1})^2}{8}} \text{- } O)2+, \quad \text{bt} \quad \backslash \ell \backslash \leftrightarrow \in \qquad \text{)B/3+}$$

gps uif sfnfwbou ufsn t pguif tvn / Jo qbsujdvmbs- uif btzn qupujd cfi bwjps efqfoet pomz po)ℓ_{i-2} ℓ_{i-2}+ xijdi jt qmbzjoh uif spmf pg ℓ_i jo)B/2+/ Gvsuifsn psf- uif btzn qupujd cfi bwjpst pg cpui)B/2+boe)B/3+bsf qspqpsujpobmup)M^{-2}+ti/ Uivt- xf n bz dpodmuef uibu uif flstu Ljscz n pwf qsftfswft uif epn bjo pg dpowfshfodf pg i pn pmphjdbmcmpdlt/ Yf dbo bobmphpvtmz xpsl pvu uif ejbhpobmfmfn fout pg)$M'+^{-2}$ gps uif ux p sfn bjojoh Ljscz n pwft boe pctfswf uif jowbsjbodf/

B Further examples

Jo ui jt tfdujpo- xf qspwjef gvsui fs fybn qmft pg Tfjgfsu n bojgpmet x i ptf i pn pmphjdbmcmpdl t bsf hjwfo jo ufsn t pg ui f gbmtf ui fub gvodujpot $\Omega_r^{m\text{-}\,K}$ boe x i ptf ebub bcpvu ui f ffbu dpo. ofdujpot dbo cf jogfssfe gspn ui f n pevmbsjuz ejdujpobsz bt ejtdvttfe jo tfdujpo 5/

DT bdujpo	tubcjmjuz fs	uzqf	usbottfsjft
1	TV)3+	dfousbm	$e^{2\pi ik\cdot 0}\Big)\frac{4\pi i}{3\sqrt{3}}k^{-3/2}\,,\quad \frac{19\pi^2}{9\sqrt{3}}k^{-5/2}\,,\quad \mathcal{P})k^{-7/2}+\Big($
$\frac{1}{3}$	Φ)2+	bcfmjbo	$e^{2\pi ik\frac{1}{3}}\Big)^{\S}\ \overline{4}k^{-1/2}\,,\quad \frac{5\pi i}{12\sqrt{3}}k^{-3/2}\,,\quad \mathcal{P})k^{-5/2}+\Big($
$\frac{1}{24}$	⊕2	jssfevdjcmf- sfbm	$e^{-2\pi ik\frac{1}{24}}e^{\frac{3\pi i}{4}}\times)\quad 3+$

Table 16/ Usbottfsjft boe dbtujfldbujpo pgffbu dpoofdujpot po $M)\quad 3=\frac{1}{2},\frac{2}{3},\frac{2}{3}+?\quad S^3_{-3})\mathbf{3}^{\ell}_1+$vq up bo pwfsbmmgbdups pg $iq^{-25/24}/3^{\S}\ \overline{3}$/ Ui f dpssftqpoejoh Yfjmsfqsftfoubujpo jt $m\,,\quad K\,?\ 7\,,\ 4$/

DT bdujpo	tubcjmjuz fs	uzqf	usbottfsjft
1	TV)3+	dfousbm	$e^{3\pi ik\cdot 1}\Big)\pi i^{\S}\ \overline{3}k^{-4/3}\,,\quad \frac{36;\pi^2}{31\sqrt{3}}k^{-6/3}\,,\quad \mathcal{P})k^{-8/3}+\Big($
$\frac{2}{3}$	TV)3+	dfousbm	$e^{3\pi ik\frac{1}{2}}\Big)\pi i^{\S}\ \overline{3}k^{-4/3}\,,\quad \frac{36;\pi^2}{31\sqrt{3}}k^{-6/3}\,,\quad \mathcal{P})k^{-8/3}+\Big($
$\frac{.}{91}$	⊕2	opo.bcfmjbo- sfbm	$e^{-3\pi ik\frac{9}{80}}e^{\frac{3\pi i}{4}}\times\Big)\frac{7-\ 3\sqrt{6}}{6}\Big($
$\frac{5;}{91}$	⊕2	opo.bcfmjbo- sfbm	$e^{-3\pi ik\frac{49}{80}}e^{\frac{3\pi i}{4}}\times\Big)\frac{7-\ 3\sqrt{6}}{6}\Big($
$\frac{2}{91}$	⊕2	opo.bcfmjbo- dpn qmfy	1
$\frac{232}{91}$	⊕2	opo.bcfmjbo- dpn qmfy	1

Table 17/ Usbottfsjft boe dbtujfldbujpo pgffbu dpoofdujpot po $M)\quad 2=\frac{1}{2},\frac{1}{4},\frac{1}{5}+?\quad S^3_{+2})\mathbf{4}_1+$vq up bo pwfsbmmgbdups pg $iq^{19/80}/3^{\S}\ \overline{3}$/ Ui f dpssftqpoejoh Yfjmsfqsftfoubujpo jt $m\,,\quad K\,?\ 31\,,\ 5$/

DT bdujpo	tubcjmjuz fs	uzqf	usbottfsjft
1	TV)3+	dfousbm	$e^{3\pi ik\cdot 1}\Big)\frac{5\pi i}{4\sqrt{4}}k^{-4/3}\,,\quad \frac{214\pi^2}{29\sqrt{4}}k^{-6/3}\,,\quad \mathcal{P})k^{-8/3}+\Big($
$\frac{3}{4}$	Φ)2+	bcfmjbo	$e^{3\pi ik\frac{2}{3}}\Big)\frac{\sqrt{4}}{3}k^{-2/3}\quad \frac{8\pi i}{59\sqrt{4}}k^{-4/3}\,,\quad \mathcal{P})k^{-6/3}+\Big($
$\frac{5}{59}$	⊕2	opo.bcfmjbo- sfbm	$e^{-3\pi ik\frac{4}{48}}e^{\frac{3\pi i}{4}}\times^{\S}\ \overline{3}$
$\frac{36}{59}$	⊕2	opo.bcfmjbo- sfbm	$e^{-3\pi ik\frac{25}{48}}e^{\frac{3\pi i}{4}}\times 2$
$\frac{2}{59}$	⊕2	opo.bcfmjbo- dpn qmfy	1

Table 18/ Usbottfsjft boe dbtujfldbujpo pgffbu dpoofdujpot po $M)\quad 2=\frac{1}{3},\frac{1}{3},\frac{1}{4}+$vq up bo pwfsbmm gbdups pg $iq^{-1/48}/3^{\S}\ \overline{3}$/ Ui f dpssftqpoejoh Yfjmsfqsftfoubujpo jt $m\,,\quad K\,?\ 23\,,\ 4$/

DT bdujpo	tubcjmjuz	uzqf	usbottfsjft
1	TV)3+	dfousbm	$e^{3\pi ik\cdot 1}\big)\frac{5\pi i}{6\sqrt{6}}k^{-4/3}$, $\frac{78\pi^2}{36\sqrt{6}}k^{-6/3}$, $\mathcal{P})k^{-8/3}+\Big($
$\frac{3}{6}$	Φ)2+	bcfnjbo	$e^{3\pi ik\frac{2}{5}}\big)\frac{2-\sqrt{6}}{3}k^{-2/3}$ $\frac{)236-72\sqrt{6}+\pi i}{311}k^{-4/3}$, $\mathcal{P})k^{-6/3}+\Big($
$\frac{4}{6}$	Φ)2+	bcfnjbo	$e^{3\pi ik\frac{3}{5}}\big)\frac{-2-\sqrt{6}}{3}k^{-2/3}$, $\frac{)236-72\sqrt{6}+\pi i}{311}k^{-4/3}$, $\mathcal{P})k^{-6/3}+\Big($
$\frac{:}{51}$	⊕2	opo.bcfnjbo-sfbm	$e^{-3\pi ik\frac{9}{40}}e^{4\pi i/5}\times\big)2$, $\frac{2}{\sqrt{6}}\Big($
$\frac{36}{51}$	⊕2	opo.bcfnjbo-sfbm	$e^{-3\pi ik\frac{25}{40}}e^{4\pi i/5}\times\frac{5}{\sqrt{6}}$
$\frac{2}{51}$	⊕2	opo.bcfnjbo-dpnqmfy	1

Table 19/ Usbottfsjft boe dibttjfldbujpo pg ffbu dpoofdujpot po $M)$ $2=\frac{1}{2},\frac{1}{5},\frac{1}{5}+$vq up bo pwfsbmm gbdups pg $iq^{-19/40}/3^{\frac{8}{3}}\ \overline{3}$/ Uif dpssftqpoejoh Yfjmsfqsftfoubujpo jt m , K ? 21 , 6/

DT bdujpo	tubcjmjuz	uzqf	usbottfsjft
1	TV)3+	dfousbm	$e^{3\pi ik\cdot 1}\big)\frac{5\pi i}{38}k^{-4/3}$, $\mathcal{P})k^{-6/3}+\Big($
1	Φ)2+	bcfnjbo	$e^{3\pi ik\cdot 1}\big)$ $k^{-2/3}$, $\mathcal{P})k^{-4/3}+\Big($
$\frac{3}{;}$	Φ)2+	bcfnjbo	$e^{3\pi ik\frac{2}{9}}\big)\big)\frac{5}{4}$ dpt $\frac{\pi}{;}$ $\frac{3}{4}\Big(k^{-2/3}$, $\mathcal{P})k^{-4/3}+\Big($
$\frac{6}{;}$	Φ)2+	bcfnjbo	$e^{3\pi ik\frac{5}{9}}\big)$ $\big)\frac{5}{4}$ tjo $\frac{\pi}{29}$, $\frac{3}{4}\Big(k^{-2/3}$, $\mathcal{P})k^{-4/3}+\Big($
$\frac{9}{;}$	Φ)2+	bcfnjbo	$e^{3\pi ik\frac{8}{9}}\big)$ $\big)\frac{5}{4}$ dpt $\frac{3\pi}{29}$, $\frac{3}{4}\Big(k^{-2/3}$, $\mathcal{P})k^{-4/3}+\Big($
$\frac{:}{83}$	⊕2	opo.bcfnjbo-sfbm	$e^{-3\pi ik\frac{9}{72}}e^{4\pi i/5}\times)$ 3+

Table 20/ Usbottfsjft boe dibttjfldbujpo pg ffbu dpoofdujpot po $M)$ $3=\frac{1}{2},\frac{1}{3},\frac{2}{3}+$vq up bo pwfsbmm gbdups pg $iq^{-45/72}/3^{\frac{8}{3}}\ \overline{3}$/ Uif dpssftqpoejoh Yfjmsfqsftfoubujpo jt m , K ? 29 , :/

Open Access. Uijt bsujdmf jt ejtusjcvufe voefs uif ufsnt pg uif Dsfbujwf Dpnnpot Buusjcvujpo Mjdfotf)DD.CZ 5/1+ xijdi qfsnjut boz vtf-ejtusjcvujpo boe sfqspevdujpo jo boz nfejvn-qspwjefe uif psjhjobmbvuips)t+boe tpvsdf bsf dsfejufe/

References

'2ˆ W/Qftuvo fu bm/ *Localization techniques in quantum field theories*- *J. Phys.* **A 50**)3128+ 551412 'arXiv:1608.02952ˆ 'INSPIREˆ/

'3ˆ F/Y junfo- *Elliptic genera and quantum field theory*- *Commun. Math. Phys.* **109**)2: 98+636 'INSPIREˆ/

'4ˆ B/Hbeef- T/Hvlpwboe Q/Qvuspw- *Walls, lines and spectral dualities in 4d gauge theories*- *JHEP* **05**)3125+158 'arXiv:1302.0015ˆ 'INSPIREˆ/

'5ˆ T/Hvlpw- Q/Qvuspwboe D/Wbgb- *Fivebranes and 4-manifold homology*- *JHEP* **07**)3128+ 182 'arXiv:1602.05302ˆ 'INSPIREˆ/

'6ˆ T/Hvlpw- N/Nbsjöp boe Q/Qvuspw- *Resurgence in complex Chern-Simons theory*- arXiv:1605.07615 'INSPIREˆ/

JHEP10(2019)010

‘7ˆ T/ Hvl pw- E/ Qfj- Q/ Qvuspw boe D/ Wbgb- *BPS spectra and 3-manifold invariants*- arXiv:1701.06567 ‘INSPIREˆ/

‘8ˆ T/ S bn bovkbo- *The lost notebook and other unpublished papers*- Tqsjohfs- Cfsmjo Hfsn boz)2: 99+/

‘9ˆ E/ Di pj- T/ Mjn boe S /D/ S i pbeft- *Mock modular forms and quantum modular forms*- *Proc. Amer. Math. Soc.* **144**)3127+3448/

‘: ˆ N /D/O/ Di foh boe K/G/S / Evodbo- *On Rademacher sums, the largest Mathieu group and the holographic modularity of Moonshine*- *Commun. Num. Theor. Phys.* **6**)3123+7: 8 ‘arXiv:1110.3859ˆ ‘INSPIREˆ/

‘21ˆ S /D/ S i pbeft- *A unified approach to partial and mock theta functions*- *Math. Res. Lett.* **25**)3129+76: ‘arXiv:1111.1495ˆ/

‘22ˆ E/ [bhjfs- *Ramanujan’s Mock theta functions and their applications (after Zwegers and Ono-Bringmann)*- *Astérisque* **326**)311: +254/

‘23ˆ B/ Hbeef- T/ Hvl pw boe Q/ Qvuspw- *Fivebranes and* 5*-manifolds*- arXiv:1306.4320 ‘INSPIREˆ/

‘24ˆ U/ Ejn pgf- E/ Hbjpuup boe O/N / Qbr vfuuf- *Dual boundary conditions in* 4*d SCFT’s*- *JHEP* **05**)3129+171 ‘arXiv:1712.07654ˆ ‘INSPIREˆ/

‘25ˆ L / Dptuf mp boe E/ Hbjpuup- *Vertex operator algebras and* 4*d* $\mathcal{O}$ *?* 5 *gauge theories*- *JHEP* **05**)312: +129 ‘arXiv:1804.06460ˆ ‘INSPIREˆ/

‘26ˆ C/ Gfjhjo boe T/ Hvl pw- *VOA[*M_4*]*- arXiv:1806.02470 ‘INSPIREˆ/

‘27ˆ N / Efevti fol p- *Gluing I: integrals and symmetries*- arXiv:1807.04274 ‘INSPIREˆ/

‘28ˆ N / Efevti fol p- *Gluing II: boundary localization and gluing formulas*- arXiv:1807.04278 ‘INSPIREˆ/

‘29ˆ I / Kpdl fst boe Q/ N bzs- *A* 4*d gauge theory/quantum k-theory correspondence*- arXiv:1808.02040 ‘INSPIREˆ/

‘2: ˆ S / Hpn qg fu bm/ *5-manifolds and Kirby calculus*- Hsbevbuf tuvejft jo n bui fn bujdt- Bn fsjdbo N bui fn bujdbm Tpdjfuz- V/T/B/)2: : : +/

‘31ˆ W/W/ Mzvc bti fol p- *Invariants of three manifolds and projective representations of mapping class groups via quantum groups at roots of unity*- *Commun. Math. Phys.* **172**)2: : 6+578 ‘hep-th/9405167ˆ ‘INSPIREˆ/

‘32ˆ W/W/ Mzvc bti fol p- *Modular properties of ribbon abelian categories*- jo ui f qspdffejoht pg ui f 3nd *Gauss symposium*- Bvhvtu 3–8- N vojdi - Hfsn boz)2: : 4+ hep-th/9405168/

‘33ˆ W/W/ Mzvc bti fol p- *Modular transformations for tensor categories*- *J. Pure Appl. Alg.* **98**)2: : 6+38: /

‘34ˆ K/ Gvdi t- T/ I xboh- B/N / Tfn jl i bupw boe J/Zv/ Ujqvojo- *Nonsemisimple fusion algebras and the Verlinde formula*- *Commun. Math. Phys.* **247**)3115+824 ‘hep-th/0306274ˆ ‘INSPIREˆ/

‘35ˆ E/ Qfj boe L / Zf- *A* 4*d-*4*d appetizer*- *JHEP* **11**)3127+119 ‘arXiv:1503.04809ˆ ‘INSPIREˆ/

‘36ˆ C/M/ Gfjhjo- B/N / Hbjovuejopw- B/N / Tfn jl i bupw boe J/Zv/ Ujqvojo- *Modular group representations and fusion in logarithmic conformal field theories and in the quantum group center*- *Commun. Math. Phys.* **265**)3117+58 ‘hep-th/0504093ˆ ‘INSPIREˆ/

JHEP10(2019)010

‘37ˆ C/M/ Gfjhjo- B/N / Hbjovuejopw- B/N / Tfn jl i bupw boe J/Zv / Ujqvojo- *Kazhdan-Lusztig correspondence for the representation category of the triplet W-algebra in logarithmic CFT- Theor. Math. Phys.* **148**)3117+2321 ‘math/0512621ˆ ‘iNSPIREˆ/

‘38ˆ U/ Ejn pgnf- Q/ Fujohpg- T/ Hvl pw boe D/ Wbgb- *Quantum Hall effect, non-semisimple MTCs, and complex Chern-Simons theory*- voqvcnjti fe/

‘39ˆ E/ L b–i ebo boe H/ Mvt–ujh- *Tensor structures arising from affine Lie algebras. I- J. Amer. Math. Soc.* **6**)2: : 4+: 16/

‘3: ˆ E/ L b–i ebo boe H/ Mvt–ujh- *Tensor structures arising from affine Lie algebras. II- J. Amer. Math. Soc.* **6**)2: : 4+: 5: /

‘41ˆ E/ L b–i ebo boe H/ Mvt–ujh- *Tensor structures arising from affine Lie algebras. III- J. Amer. Math. Soc.* **1**)2: : 5+446/

‘42ˆ E/ L b–i ebo boe H/ Mvt–ujh- *Tensor structures arising from affine Lie algebras. IV- J. Amer. Math. Soc.* **1**)2: : 5+494/

‘43ˆ T/ Di vo- T/ Hvl pw boe E/ S phhfol bn q- *Junctions of surface operators and categorification of quantum groups*- arXiv:1507.06318 ‘iNSPIREˆ/

‘44ˆ O/.Q/ Tl psvqqb- *Jacobi forms of critical weight and Weil representations*- jo *Modular forms on Schiermonnikoog*- C/ Fejyi pwfo fu bmfet/- Dbn csjehf Vojwfstjuz Qsftt- Dbn csjehf V/L /)3119+/

‘45ˆ N / Fjdi nfs boe E/ [bhjfs- *The theory of Jacobi forms*- Cjsl i bvtfs- Tx ju–fsnboe)2: 96+/

‘46ˆ B/ Dbqqfmja- D/ Ju–zl tpo boe K/C/ [vcfs- *The ADE classification of minimal and A*2)2+ *conformal invariant theories*- *Commun. Math. Phys.* **113**)2: 98+2 ‘iNSPIREˆ/

‘47ˆ K/I / Dpox bz boe T/Q/ Opsupo- *Monstrous Moonshine- Bull. London Math. Soc.* **11**)2: 8: + 419/

‘48ˆ O/.Q/ Tl psvqqb- *Über den Zusammenhang zwischen Jacobiformen und Modulformen halbganzen Gewichts*- Cpoofs N bui fn bujtdi f Tdi sjgnfo- Vojwfstjubu Cpoo N bui fn bujtdi ft Jotujuvu- Cpoo- Hfsn boz)2: 96+/

‘49ˆ S / Mox sfodf boe E/ [bhjfs- *Modular forms and quantum invariants of 3-manifolds*- *Asian J. Math.* **3**)2: : : +: 4/

‘4: ˆ H/F / Boesfx t- *Partitions: yesterday and today*- Ofx [fbnboe N bui fn bujdbnTpdjfuz- Y fnjohupo- Ofx [fbnboe)2: 8: +/

‘51ˆ L / Csjohn boo- B/ Gpnqpn boe B/ Njnbt- *Asymptotic behavior of partial and false theta functions arising from Jacobi forms and regularized characters*- *J. Math. Phys.* **58**)3128+ 122813 ‘arXiv:1604.01977ˆ ‘iNSPIREˆ/

‘52ˆ U/ Dsfvu–jh boe B/ N jnbt- *False theta functions and the Verlinde formula*- *Adv. Math.* **262**)3125+631 ‘arXiv:1309.6037ˆ ‘iNSPIREˆ/

‘53ˆ L / Csjohn boo- B/ Gpnqpn boe S /D/ S i pbeft- *Partial theta functions and mock modular forms as q-hypergeometric series*- *R.C. Ramanujan J.* **29**)3123+3: 6 ‘arXiv:1109.6560ˆ/

‘54ˆ M/K/ I jl bn j- *Mock (false) theta functions as quantum invariants*- *Reg. Chaotic Dyn.* **10**)3116+61: ‘math-ph/0506073ˆ/

‘55ˆ L / I jl bn j- *On the quantum invariants for the spherical Seifert manifolds*- *Comm. Math. Phys.* **268**)3117+396 ‘math-ph/0504082ˆ/

JHEP10(2019)010

‘56ˆ L/I jl bn j- *Decomposition of Witten-Reshetikhin-Turaev invariant: linking pairing and modular forms*- *AMS/IP Stud. Adv. Math.* **50**)3122+242 ‘iNSPIREˆ/

‘57ˆ T/Q/[xfhfst- *Mock θ-functions and real analytic modular forms*- jo *q-series with applications to combinatorics, number theory, and physics*- C/D/Cfsoeufubm/fet/- Dpoufn qpsbsz Nbuifn bujdt wpmun f 3:2- Bn fsjdbo Nbuifn bujdbmTpdjfuz- V/T/B/)3112+/

‘58ˆ H/Gfnfs- *BRST approach to minimal models*- *Nucl. Phys.* **B 317**)2:9:+326 “*Erratum ibid.* **B 324**)2:9:+659ˆ ‘iNSPIREˆ/

‘59ˆ B/Utvdijzb boe Z/Lbojf- *Fock space representations of the Virasoro algebra — Intertwining operators*- *Publ. RIMS* **22**)2:97+36:/

‘5:ˆ I/H/Lbvtdi- *Extended conformal algebras generated by a multiplet of primary fields*- *Phys. Lett.* **B 259**)2::2+559 ‘iNSPIREˆ/

‘61ˆ E/Bebn pwjd- *Classification of irreducible modules of certain subalgebras of free boson vertex algebra*- *J. Algebra* **270**)3114+226243/

‘62ˆ L/Obhbupn p boe B/Utvdijzb- *The triplet vertex operator algebra W)p+and the restricted quantum group $\bar{U}_q$)tnm+at q ? $e^{\pi i/p}$*- arXiv:0902.4607 ‘iNSPIREˆ/

‘63ˆ N/B/J/Gmpis- *On modular invariant partition functions of conformal field theories with logarithmic operators*- *Int. J. Mod. Phys.* **A 11**)2::7+5258 ‘hep-th/9509166ˆ ‘iNSPIREˆ/

‘64ˆ E/Bebn pwjd boe B/Njht- *Logarithmic intertwining operators and W)3, 3p 2+algebras*- *J. Math. Phys.* **48**)3118+184614 ‘math/0702081ˆ ‘iNSPIREˆ/

‘65ˆ N/Efevtifolp- T/Hvlpwboe Q/Qvuspw- *Vertex algebras and 5-manifold invariants*- arXiv:1705.01645 ‘iNSPIREˆ/

‘66ˆ D/Cnhodifu- G/Dptuboujop- O/Hffs boe C/Qbuvsfbv.Njsboe- *Non semi-simple TQFTs, Reidemeister torsion and Kashaev’s invariants*- *Adv. Math.* **301**)3127+2 ‘arXiv:1404.7289ˆ ‘iNSPIREˆ/

‘67ˆ D/Cnhodifu- G/Dptuboujop- O/Hffs boe C/Qbuvsfbv.Njsboe- *Non semi-simple TQFTs from unrolled quantum sl)3+* 3127- arXiv:1605.07941 ‘iNSPIREˆ/

‘68ˆ B/Cfujbl pwb- D/Cnhodifu boe O/Hffs- *Logarithmic Hennings invariants for restricted quantum sl)3+* arXiv:1705.03083/

‘69ˆ N/Njzbn pup- *Modular invariance of vertex operator algebras satisfying C_2-cofiniteness*- math/0209101/

‘6:ˆ U/Dsfvu-jh boe U/Hboopo- *Logarithmic conformal field theory, log-modular tensor categories and modular forms*- *J. Phys.* **A 50**)3128+515115 ‘arXiv:1605.04630ˆ ‘iNSPIREˆ/

‘71ˆ Z/[iv- *Modular invariance of characters of vertex operator algebras*- *J. Amer. Math. Soc.* **9**)2::7+348/

‘72ˆ U/Bsbl bxb- *A remark on the C_2-cofiniteness condition on vertex algebras*- *Math. Z.* **270**)3123+66:/

‘73ˆ Y/E/Ofvn boo boe G/Sbzn poe- *Seifert manifolds, plumbing, μ-invariant and orientation reversing maps*- Tqsjohfs- Cfsmjo- Hfsn boz)2:89+/

‘74ˆ E/S/Bvdlnn- *Topological methods to compute chern-simons invariants*- *Math. Proc. Cambridge Phil. Soc.* **115**)2::5+33:/

JHEP10(2019)010

‘75ˆ T/ Hvl pw- *Three-dimensional quantum gravity, Chern-Simons theory and the A polynomial-* *Commun. Math. Phys.* **255**)3116+688 ‘hep-th/0306165ˆ ‘iNSPIREˆ/

‘76ˆ L / I jl bn j- *On the quantum invariant for the Brieskorn homology spheres- Int. J. Math.* **16**)3116+772 ‘math-ph/0405028ˆ/

‘77ˆ T/ Di vo- *A resurgence analysis of the* TV)3+*Chern-Simons partition functions on a Brieskorn homology sphere* ϒ)3, 6, 8+ arXiv:1701.03528 ‘iNSPIREˆ/

‘78ˆ T/ Hbspvgbujejt- D/ I pehtpo- O/ I pfin bo boe I / S vcjotufjo- *The 4D-index and normal surfaces-* arXiv:1604.02688 ‘iNSPIREˆ/

‘79ˆ U/ Ejn pgu- T/ Hvl pw- K/ Mfofmb boe E/ [bhjfs- *Exact results for perturbative Chern-Simons theory with complex gauge group-* *Commun. Num. Theor. Phys.* **3**)311: +474 ‘arXiv:0903.2472ˆ ‘iNSPIREˆ/

‘7: ˆ N / N bsjöp- *Lectures on non-perturbative effects in large N gauge theories, matrix models and strings-* *Fortsch. Phys.* **62**)3125+566 ‘arXiv:1206.6272ˆ ‘iNSPIREˆ/

‘81ˆ B/ Gpsbtpn - L / P op boe S /D/ S i pbeft- *Mock theta functions and quantum modular forms-* *Forum Math. Pi* **1**)3124+38/

‘82ˆ E/ [bhjfs- *Quantum modular forms-* jo *Quanta of maths-* F/ Cnbodi bse fu bm/fet/- Dnbz N bui fn bujdbmQspdffejoht wpmvn f 22- Bn fsjdbo N bui fn bujdbmTpdjfuz- V/T/B /)3121+/

‘83ˆ L / Csjohn boo- B/ Gpsbtpn - L / P op boe M/ S prfho- *Harmonic Maass forms and mock modular forms: theory and applications-* Bn fsjdbo N bui fn bujdbmTpdjfuz Dpnqpsvjvn Qvcmjdbujpot wpmvn f 75- Bn fsjdbo N bui fn bujdbmTpdjfuz- Qspwjefodf- V/T/B /)3128+/

‘84ˆ L / Csjohn boo boe M/ S prfho- *Half-integral weight Eichler integrals and quantum modular forms-* *J. Number Theor.* **161**)3127+351 ‘arXiv:1409.3781ˆ/

‘85ˆ T/ [xfhfst- *Mock Theta functions-* Qi /E/ ui ftjt- Vojwfstjuz pg Vusfdi u- Vusfdi u- Ui f Ofui fsmboet)3119+ arXiv:0807.4834 ‘iNSPIREˆ/

‘86ˆ N / Hsj´ o- L / P op boe M/ S prfho- *Ramanujan’s mock theta functions-* *Proc. Natl. Acad. Sci. U.S.A.* **110**)3124+6876/

‘87ˆ N /.K/ Kboh boe T/ Mpcsjdi - *Radial limits of the universal mock theta function* g_3*-* *Proc. Amer. Math. Soc.* **145**)3128+: 36 ‘arXiv:1504.05365ˆ/

‘88ˆ L / Csjohn boo boe M/ S prfho- *Radial limits of mock theta functions- Res. Math. Sci.* **2**)3126+ 29 ‘arXiv:1409.3782ˆ/

‘89ˆ I / Qpjodbsf- *Fonctions modulaires et fonctions fuchsiennes- Ann. Fac. Sci. Toulouse Sci. Math. Sci. Phys.* **3**)2: 22+236/

‘8: ˆ N /D/O/ Di foh boe K/G/S / Evodbo- *Rademacher sums and Rademacher series-* *Contrib. Math. Comput. Sci.* **8**)3125+254 ‘arXiv:1210.3066ˆ ‘iNSPIREˆ/

‘91ˆ N /J/ L opqq- *Construction of a class of modular functions and forms I- Pacific J. Math.* **11**)2: 72+386/

‘92ˆ N /J/ L opqq- *Construction of a class of modular functions and forms II- Pacific J. Math.* **11**)2: 72+772/

‘93ˆ N /J/ L opqq- *On abelian integrals of the second kind and modular functions- Amer. J. Math.* **84**)2: 73+726/

JHEP10(2019)010

‘94ˆ E/ Ojfcvs- *Construction of automorphic forms and integrals- Trans. Amer. Math. Soc.* **191**)2: 85+484/

‘95ˆ K/G/ Evodbo boe J/C/ Gsfol fn *Rademacher sums, Moonshine and gravity- Commun. Num. Theor. Phys.* **5**)3122+95: ‘arXiv:0907.4529ˆ ‘INSPIREˆ/

‘96ˆ Y /B/ Qsjcjul jo- *The Fourier coefficients of modular forms and Niebur modular integrals having small positive weight. I- Acta Arith.* **91**)2: : : +3: 2/

‘97ˆ Y /B/ Qsjcjul jo- *The Fourier coefficients of modular forms and Niebur modular integrals having small positive weight. II- Acta Arith.* **93**)3111+454/

‘98ˆ L/ Csjohn boo boe L/ Pop- *The f)q+mock theta function conjecture and partition ranks- Invent. Math.* **165**)3117+354/

‘99ˆ L/ Csjohn boo boe L/ Pop- *Dyson’s rank and Maass forms- Ann. Math.* **171**)3121+52: /

‘9: ˆ E/ Y i brfho- *Vector-valued Rademacher sums and automorphic integrals-* arXiv:1406.0571/

“: 1ˆ I /S befn bdi fs- *Fourier expansions of modular forms and problems of partition- Bull. Amer. Math. Soc.* **46**)2: 51+6: /

“: 2ˆ N /D/O/ Di foh boe K/G/S/ Evodbo- *On Rademacher sums, the largest Mathieu group and the holographic modularity of Moonshine- Commun. Num. Theor. Phys.* **6**)3123+7: 8 ‘arXiv:1110.3859ˆ ‘INSPIREˆ/

“: 3ˆ N /D/O/ Di foh boe K/G/S/ Evodbo- *Optimal mock Jacobi theta functions-* arXiv:1605.04480 ‘INSPIREˆ/

“: 4ˆ B/ Ebci prhbs- T/ N vsui z boe E/ [bhjfs- *Quantum black holes, wall crossing and mock modular forms-* arXiv:1208.4074 ‘INSPIREˆ/

“: 5ˆ K/ Csvjojfs boe L/ Pop- *Heegner divisors, L-functions and harmonic weak Maass forms- Ann. Math.* **172**)3121+3246/

“: 6ˆ W/ Bobhjboojt boe N /D/O/ Di foh- *TASI lectures on Moonshine-* PoS(TASI2017)010 ‘arXiv:1807.00723ˆ ‘INSPIREˆ/

“: 7ˆ N /D/O/ Di foh- K/G/S/ Evodbo boe K/B/ I bswfz- *Umbral Moonshine and the Niemeier lattices-* arXiv:1307.5793 ‘INSPIREˆ/

“: 8ˆ N /D/O/ Di foh- K/G/S/ Evodbo boe K/B/ I bswfz- *Umbral Moonshine- Commun. Num. Theor. Phys.* **08**)3125+212 ‘arXiv:1204.2779ˆ ‘INSPIREˆ/

“: 9ˆ L/ I jl bn j- *Quantum invariant, modular form, and lattice points- Int. Mat. Res. Not.* **3**)3116+232 ‘math-ph/0409016ˆ/

“: : ˆ /I jl bn j- *Quantum invariants, modular forms, and lattice points II- J. Math. Phys.* **47**)3117+213412/

‘211ˆ L/ Csjohn boo boe B/ N jnbt- $\mathcal{X}$ *-algebras, false theta functions and quantum modular forms I- Int. Math. Res. Not. IMRN* **21**)3126+22462/

‘212ˆ E/ Bebn pwjd boe B/ N jnbt- *The doublet vertex operator superalgebras* $\mathcal{B}(p)$ *and* $\mathcal{B}_{2,p}$- jo *Recent developments in algebraic and combinatorial aspects of representation theory-* W/ Di bsj fu bn/fet/- Dpoufn qpsbsz N bui fn bujdt wpmn f 713- Bn fsjdbo N bui fn bujdbm Tpdjfuz- V/T/B/)3124+/

‘213ˆ I /S befn bdi fs- *A convergent series for the partition function p)n+ Proc. Natl. Acad. Sci. U.S.A.* **23**)2: 48+89/

JHEP10(2019)010

‘214ˆ L/I bcjsp- *Cyclotomic completions of polynomial rings*- *Publ. Res. Inst. Math. Sci.* **40**)3115+2238/

‘215ˆ K/Uspptu- *The non-compact elliptic genus: mock or modular*- *JHEP* **06**)3121+215 “arXiv:1004.3649ˆ “INSPIREˆ/

‘216ˆ N/Bnjn fubn/A *Wall-crossing holomorphic anomaly and mock modularity of multiple M5-branes*- *Commun. Math. Phys.* **339**)3126+884 “arXiv:1012.1608ˆ “INSPIREˆ/

‘217ˆ N/D/O/Di foh fubn/A *Mock modular Mathieu Moonshine modules*- *Res. Math. Sci.* **2**)3126+24 “arXiv:1406.5502ˆ “INSPIREˆ/

‘218ˆ T/Bnfyboespwboe C/Qjpnjof- *Black holes and higher depth mock modular forms*- arXiv:1808.08479 “INSPIREˆ/

JHEP10(2019)010