# Topology and the one-dimensional Kondo-Heisenberg model

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The Kondo-Heinsberg chain is an interesting model of a strongly correlated system which has a broad superconducting state with pair-density wave (PDW) order. Some of us have recently proposed that this PDW state is a symmetry-protected topological (SPT) state, and the gapped spin sector of the model supports Majorana zero modes. In this paper, we reexamine this problem using a combination of numeric and analytic methods. In extensive density-matrix renormalization group calculations, we find no evidence of a topological ground state degeneracy or the previously proposed Majorana zero modes in the PDW phase of this model. This result motivated us to reexamine the original arguments for the existence of the Majorana zero modes. A careful analysis of the effective continuum field theory of the model shows that the Hilbert space of the spin sector of the theory does not contain *any* single Majorana fermion excitations. This analysis shows that the PDW state of the doped 1D Kondo-Heisenberg model is not an SPT with Majorana zero modes.

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## I. INTRODUCTION

In recent years, evidence for nonuniform superconducting (SC) states has been found in certain high-temperature superconductors. An example of this appears to occur in the cuprate $La_{2-x}Ba_xCuO_4$ (LBCO) [1–3]. At $x = 1/8$, the critical temperature $T_c$ for the onset of the Meissner state of the uniform $d$-wave superconductivity is suppressed to near 4 K while the resistive transition onsets at 10 K. However, between 10 K and 16 K, where charge density wave (CDW) and spin density wave (SDW) orders are both present, there is a quasi-two-dimensional SC phase, where CuO planes are SC but the material remains insulating along the $c$ axis. This dynamical layer decoupling seen in LBCO near $x = 1/8$, as well as in $La_{2-x}Sr_xCuO_4$ (LSCO) and LBCO in magnetic fields, can be explained if the copper oxide planes have pair-density-wave (PDW) SC order. In the PDW state, the SC order parameter oscillates in space with a given wave vector. Further evidence for the existence of a PDW state has been found recently in scanning tunneling microscopy experiments in the "halo" of SC vortices in $Bi_2Sr_2CaCu_2O_{8+\delta}$. [4] A related state was proposed quite early on by Fulde and Ferrell [5] (FF) and independently by Larkin and Ovchinnikov [6] (LO), who showed that it is possible to have a SC state where the Cooper pairs have nonzero center-of-mass momentum in the presence of a uniform (Zeeman) magnetic field. In contrast, the PDW state preserves time-reversal symmetry and is generated by strong electron correlations instead of a BCS-like mechanism. This PDW state has also been proposed as a natural competing state of the uniform $d$-wave SC state in the pseudogap regime. An extensive review of the physics of PDW states and their experimental evidence is given by Agterberg *et al.* [7].

In previous work, it has been shown that a PDW state is supported in the doped Kondo-Heisenberg (KH) chain, [8] which consists of a 1D electron gas (1DEG) coupled to a quantum Heisenberg antiferromagnetic chain by a Kondo interaction [9], and in an extended Hubbard-Heisenberg model on a two-leg ladder at certain commensurate fillings [10]. In the PDW phase of the KH chain, the spin degrees of freedom are gapped, while its single charge mode decouples and remains gapless, and the PDW order parameter has quasi-long-range order. These results have been confirmed by using powerful numerical and analytic techniques such as the density-matrix renormalization group (DMRG) [9] and Abelian bosonization [8,11,12]. This PDW state is peculiar in that the only allowed order parameters with quasi-long-range order are composite operators such as $\mathcal{O}_{\rm PDW} \sim \boldsymbol{N}_h \cdot \boldsymbol{\Delta}$, where $\boldsymbol{N}_h$ is the Néel order parameter of the spin-1/2 Heisenberg spin chain and $\boldsymbol{\Delta}$ is the triplet SC order parameter of the 1DEG. All fermion bilinear observables decay exponentially with distance. Because of this feature, this PDW state cannot be described using the conventional Bogoliubov approximation, unlike the more conventional FFLO states.

Surprisingly, in a recent publication [13], three of us have put forth arguments that in the PDW phase, the spin sector of these systems is topological and supports Majorana zero modes (MZMs). MZMs have a long history in the study of topological superconductors. In particular, MZMs are believed to exist in vortex cores of two-dimensional $p_x + ip_y$ superconductors [14,15] in quantum wires proximate to superconductors [16] and in vortices on the SC surfaces of topological insulators [17]. In these examples, the superconductivity is encoded by use of a BCS mean-field term for the fermions of the system.

The MZMs proposed to exist in the doped KH chain are novel in that they originate from solitons of the spin sector of this strongly correlated system, localized at endpoints of the chain and at junctions with conventional phases. In particular, this model cannot be solved within the Bogoliubov mean-field theory, in which the phase mode of the superconductor is frozen as in the case of the Kitaev wire [16]. If the arguments for the topological character of the PDW state of the KH chain of Ref. [13] were correct, the KH chain would be a natural place to test for the existence of MZMs in a system with a dynamical massless charge mode. We should note that, after the publication of Ref. [13], Ruhman *et al.* have constructed a model with protected MZMs in a (uniform) 1D superconductor with a dynamical massless phase field [18].

In this paper, we reexamine the doped KH model in detail, using extensive DMRG simulations on long chains ($L = 128$) with various boundary conditions. We are able to identify the 1D PDW as was seen in Ref. [8] but do not find evidence of any MZMs in the PDW phase.

Motivated by the absence of evidence of MZMs in our numerical results, we turned to non-Abelian bosonization to reinvestigate analytically the original claims that the PDW wire is topological. In the non-Abelian bosonization approach, the effective field theory of this problem consists of four dynamical Majorana fermionic fields (see also Ref. [19]). As anticipated in Ref. [10], the effective field theory has two massive phases separated by a quantum phase transition in the $1 + 1$ dimensional Ising universality class in which just one Majorana fermion becomes massless. In the massive phases, all four Majorana fields are massive and are distinguished by the sign of the expectation value of the fermion bilinear of the light Majorana field. The massive phases are in the universality class of the $O(4) \simeq SU(2) \times SU(2)$ Gross-Neveu model investigated long ago by Witten [20] and by Shankar [21]. At the critical point, one Majorana fermion is massless and the remaining three Majoranas are massive and (with minor fine tuning) have an effective supersymmetry. [20].

By carefully examining the full Hilbert space of the spin sector of the theory, the non-Abelian bosonization results show explicitly that there are no states with odd-fermion parity in the physical spectrum (a necessary condition for the existence of MZMs). From this, we conclude that the previously proposed MZMs do not correspond to physical operators in the doped KH chain. Of course, this result does not prove that *a* PDW state cannot in principle be topological. A candidate topological PDW state is discussed qualitatively in the Conclusions of this paper. Whether or not a topological PDW state is possible in a non-mean-field model with a local Hamiltonian remains an open question.

This paper is organized as follows. In Sec. II, we present the model and discuss its phase diagram. In Sec. III, we present our numeric analysis of the doped KH model, and the lack of evidence of the MZMs. In Sec. IV, we present the previously proposed argument for the MZMs by using non-Abelian bosonization. In Sec. V, we reexamine these claims and show by careful analysis of the Hilbert space of the spin model that the MZMs are not physical operators. We also discuss the possibility of the doped KH model being a different symmetry-protected topological phase (SPT). We conclude with a discussion of our results in Sec. VI. Technical parts of our analysis are presented in several Appendices. In Appendix A, we determine the renormalization group (RG) equation for the KH model using non-Abelian bosonization. In Appendix B, we calculate the "fermion parity" of states that make up the Hilbert space of the spin sector of the theory. In Appendix C, we present the continuum limit of the model using Abelian bosonization. In Appendix D, we use Abelian bosonization to show that the proposed MZMs are not physical operators. In Appendix E, we discuss the order parameters that differentiate the trivial and PDW phases of the model.

## II. MODEL AND PDW STATES

In previous work, it has been shown that a PDW phase exists in the doped 1D KH ladder [8]. The KH ladder consists of a 1D electron gas (1DEG) coupled to a Heisenberg spin-1/2 chain via Kondo couplings. The Hamiltonian for this system is

$$\begin{aligned}
\mathcal{H} &= \mathcal{H}_e + \mathcal{H}_H + \mathcal{H}_K,\\
\mathcal{H}_e &= -t\sum_{j,\sigma} c^\dagger_{j,\sigma}c_{j+1,\sigma} + \text{H.c.} + U\sum_j n_\uparrow n_\downarrow - \mu\sum_{j,\sigma} n_{j,\sigma},\\
\mathcal{H}_H &= J_H\sum_j \boldsymbol{S}_{j,h}\cdot\boldsymbol{S}_{j+1,h} + J'_H\sum_j \boldsymbol{S}_{j,h}\cdot\boldsymbol{S}_{j+2,h},\\
\mathcal{H}_K &= J_K\sum_j \boldsymbol{S}_{j,h}\cdot\boldsymbol{S}_{j,e},
\end{aligned}\tag{1}$$

where $c^\dagger_{j,\sigma}$ are the electron creation operators, $\boldsymbol{S}_{j,h}$ are the Heisenberg spin operators, $\boldsymbol{S}_{j,e} = \frac{1}{2}c^\dagger_{j,\sigma}\boldsymbol{\tau}_{\sigma,\sigma'}c_{j,\sigma'}$ are the electron spin operators, and $\boldsymbol{\tau}$ are the Pauli matrices. We have included additional Hubbard $U$ interactions for the 1DEG and a next-nearest-neighbor spin coupling $J'_H$ in the Heisenberg chain. We will consider the case where the 1DEG electrons have been doped away from half filling. This model also arises naturally in two-leg Hubbard ladders, where the bonding band is at half filling [10]. In this case, the Umklapp process gaps out the charge degrees of freedom in the bonding band, and the Kondo and Heisenberg couplings for the spin degrees of freedom are generated perturbatively.

In terms of the spin and charge currents of the system, the continuum limit of Eqs. (1) is given by

$$\begin{aligned}\mathcal{H} &= \mathcal{H}_c + \mathcal{H}_s,\\ \mathcal{H}_c &= \frac{\pi v_c}{2}[J_{e,R}J_{e,R} + J_{e,L}J_{e,L}] + g_c J_{e,R}J_{e,L},\\ \mathcal{H}_s &= \frac{2\pi v_{s,e}}{3}\boldsymbol{J}_{e,R}\boldsymbol{J}_{e,R} + \frac{2\pi v_{s,h}}{3}\boldsymbol{J}_{h,R}\boldsymbol{J}_{h,R} + (R \leftrightarrow L)\\ &\quad - g_{s1}[\boldsymbol{J}_{e,R}\boldsymbol{J}_{e,L} + \boldsymbol{J}_{h,R}\boldsymbol{J}_{h,L}] - g_{s2}[\boldsymbol{J}_{e,R}\boldsymbol{J}_{h,L} + \boldsymbol{J}_{h,R}\boldsymbol{J}_{e,L}],\end{aligned} \tag{2}$$

where $J_e$ are the electron $U(1)$ charge currents, and $\boldsymbol{J}_{e/h}$ are the 1DEG and Heisenberg chain $SU(2)$ spin currents, respectively. The Abelian bosonization of this model and weak coupling analysis are discussed in Appendix C. The phase diagram for this system has been previously determined using Abelian bosonization [10] and is rederived here using non-Abelian bosonization [22] in Appendix A. Equations (2) have three fixed points corresponding to $(g_{s1}, g_{s2}) = (0, 0)$, $(-\infty, 0)$, $(0, -\infty)$. When $(g_{s1}, g_{s2}) = (0, 0)$, the system is a Luttinger liquid with one charge degree of freedom and two spin degrees of freedom (a C1S2 Luttinger liquid in the terminology of Ref. [23].)

At the $(g_{s1}, g_{s2}) = (0, -\infty)$ fixed point, the system is in a PDW phase, since the PDW order parameter

$$O_{\mathrm{PDW}} = \boldsymbol{\Delta} \cdot \boldsymbol{N}_h$$

has quasi-long-range order. Here, $\boldsymbol{\Delta}$ is the triplet superconductivity order parameter of the 1DEG and $\boldsymbol{N}_h$ is the staggered (Néel) component of the magnetization of the Heisenberg spins. In addition, the singlet SC order parameter decays exponentially fast. In the PDW phase, the charge sector remain gapless, while the spin sector acquires a gap and the magnetization vanishes in the ground state, $S^z \equiv \sum_j S^z_{j,e} + S^z_{h,e} = 0$. At the $(g_{s1}, g_{s2}) = (-\infty, 0)$ fixed point, the system is in a conventional SC phase, since the the singlet SC order parameter has quasi-long-range order, while the PDW order parameter decays exponentially fast. In this conventional SC phase, the charge sector is also free and the spin sector is gapped with vanishing magnetization. The line $g_{s1} = g_{s2} < 0$ marks a quantum phase transition between the PDW and trivial SC phases that is in the Ising universality class and can be described in terms of a free Majorana fermion.

In previous works, it has been argued that in the PDW phase, the gapped spin sector of the model is topological and hosts Majorana zero edge modes [13]. The "fermion parity" associated with a pair of these MZMs corresponds to the relative spin-parity of the lattice model:

$$(-1)^{Q^z}, \quad Q^z \equiv \sum_j S^z_{j,e} - S^z_{j,h}. \tag{3}$$

## III. NUMERICS

In this section, we will use DMRG [24] to search for evidence of the Majorana edge modes, eventually concluding that the numerics do not support the existence of Majorana edge modes in the PDW phase.

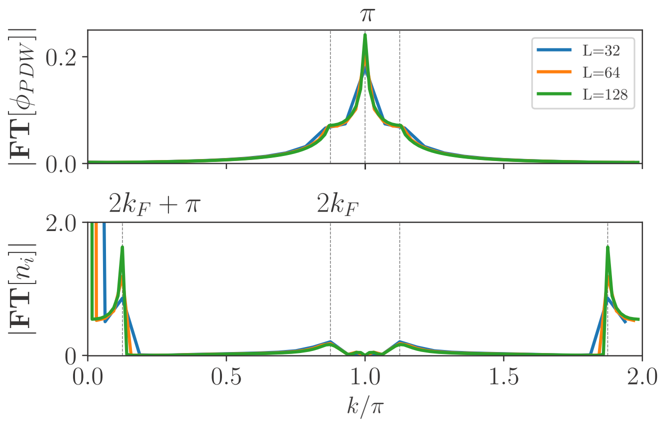


FIG. 1. Magnitude of the Fourier transform of the PDW correlation function (*top*) and charge density (*bottom*) of the ground state. $\phi_{\mathrm{PDW}}$ was averaged over all possible $|i-j|$.

We start by considering the Hamiltonian in Eqs. (1) on a finite ladder with $L = 32 - 128$ rungs, $n = 0.875$ filling, and open boundary conditions. We primarily consider the parameters $t = 1, J_H = J_K = 2, J'_H = U = 0$ which correspond to those used in Ref. [8]. We obtain a ground state and first excited state, keeping up to $m = 7200$ states with truncation errors $<10^{-8}$ and $\langle\psi_1|\psi_0\rangle < 10^{-7}$.

We first validate that we get the PDW in the ground state. We measure the order parameters:

$$\phi^\dagger_{B,i} = \tfrac{1}{2}(c^\dagger_{i\uparrow}c^\dagger_{i+1\downarrow} - c^\dagger_{i\downarrow}c^\dagger_{i+1\uparrow}), \tag{4}$$

$$\phi_{PDW} = \langle(-1)^{|i-j|}\phi^\dagger_B\phi_B(|i-j|)\rangle. \tag{5}$$

In Fig. 1, we see the salient features of the PDW quasi-long-range order—the oscillation of the $\phi_B$ bond singlet order and an accompanying charge-density wave. Thus, with open boundary conditions, we're able to obtain the proper phase.

There are a number of ways to establish the existence of Majorana zero edge modes. To begin, such a system will have degenerate energy eigenstates in the thermodynamic limit. The two degenerate eigenstates will be topological and naively should have different parity values [Eqs. (3)] as well as identical local reduced density matrices in the bulk. The edges of the two eigenstates would naturally show edge modes that should be visible in the spin order near the location of the Majoranas. An additional signature of these edge modes is the existence of degeneracy in the entanglement spectrum [25] of the ground state. While these attributes typically hold only for gapped systems, we presume that the gapless charge mode would sufficiently decouple and not affect these properties.

We begin by searching for the two degenerate states; Ref. [8] finds a spin gap to other $S_z$ sectors and therefore we would anticipate that the degenerate state should be in the $S^z = 0$ sector, although everything in this sector has parity 1. States which are degenerate in the thermodynamic limit will split in energy in any finite system. This energy splitting should (for large enough systems) decay exponentially with system size. Therefore, to search for the topological pair of states, we calculate the lowest two $S^z = 0$ eigenstates and look at the energy as a function of system size out to $L = 128$. Instead of

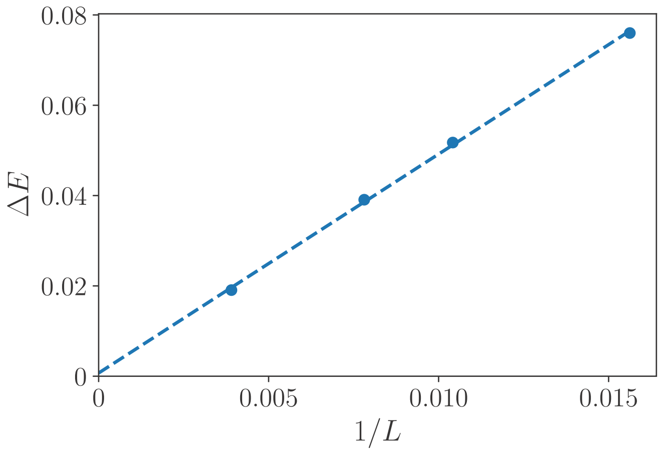


FIG. 2. Finite size scaling of the energy gap. For each point, we variance extrapolate near the end of DMRG optimization (see Figs. 7 and 8). The linear fit gives a thermodynamic gap of $\Delta E(L=\infty) \sim 0.0007 \approx 0$.

an exponentially decaying gap, we find a gap which is linear in $1/L$ extrapolating to zero in Fig. 2; this is exactly what is expected for the tower of states coming from a gapless charge-density wave.

In spite of this fact, we can compare these two eigenstates. We find that the charge density of the two eigenstates look very different (see Fig. 3) ruling out that they could be topological pairs.

It is clear then that we don't find the topological eigenstates out to this system size. We can also just look at the properties of only the ground state in the hope that the topological state is still too high in energy. Similar to a Haldane phase, one might find spin features localized near the edge/interface or spin-spin correlations peaked near the edge. In the ground state, the expectation values $\langle S^z_{j,e}\rangle$ and $\langle S^z_{j,h}\rangle$ are always very small (less than $10^{-8}$ in magnitude), indicating an absence of any edge mode. In addition, there are no significant edge-edge spin-spin correlations, as seen in Fig 4. We also can consider the entanglement spectrum (see Fig. 5) and find that the

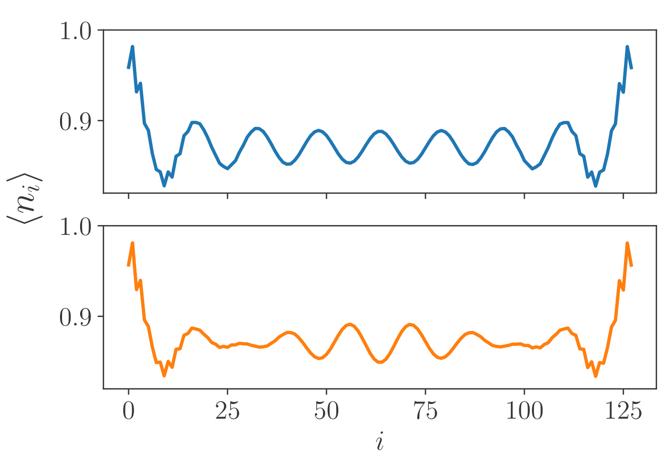


FIG. 3. Charge density in the ground state (*top*) and first excited state (*bottom*) for $L = 128$ and $n = 0.875$.

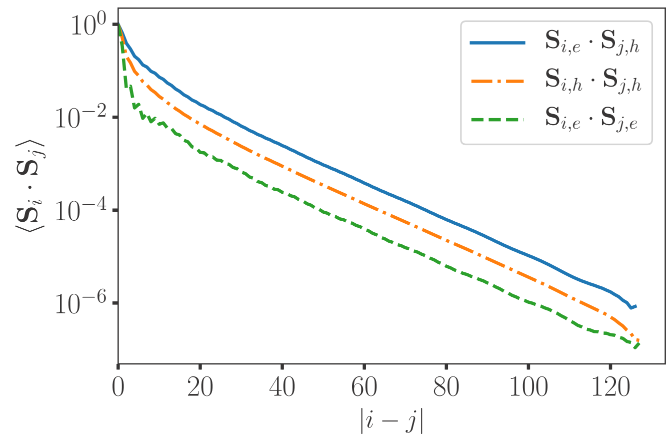


FIG. 4. Normalized spin-spin correlation between parts of the ladder for $L = 128$ ground state. $\langle \mathbf{S}_i \cdot \mathbf{S}_j\rangle$ was averaged over all possible $|i-j|$.

lowest entanglement eigenvalues are nondegenerate, unlike what would be anticipated for a topological system.

As a final search, we consider sandwiches, where we vary the value of $J_K$ in different sections of the ladder. Sandwiches have been found to be helpful in identifying nontopological zero modes in Ref. [26]. Here we considered a sandwich with PDW in the bulk ($J_K = 2$) and an insulator phase ($J_K = 10$) on the left and right 16 rungs (see Fig. 6). We maintain doping in the 1DEG such that the left and right insulators are half filled and the bulk maintains $\langle n\rangle \approx 0.875$. We do find PDW in the bulk as expected and explore for the presence of a Majorana mode in the interface of our sandwich. We again consider the ground and excited state. The gap is small ($\approx 0.0395t$, which we choose not to extrapolate for computational considerations), nearly the same as the open boundary condition gap ($\approx 0.0392t$). The charge-density, shown in Fig. 6, looks very different in the bulk, suggesting the states aren't topological. We also consider the entanglement entropy in Fig. 5(b), which

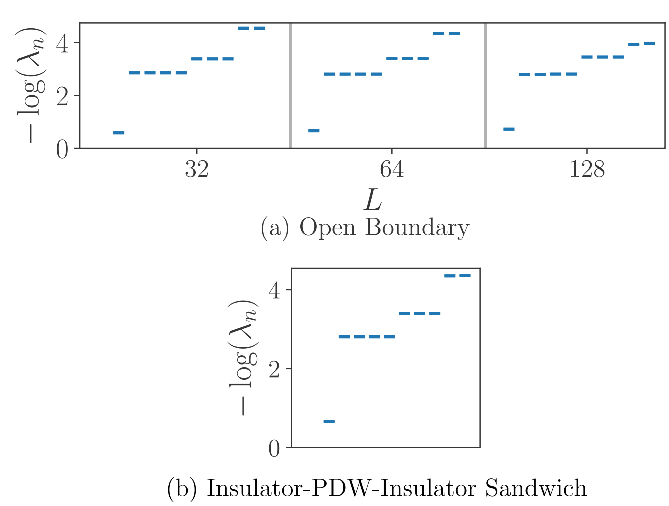


FIG. 5. Entanglement eigenvalue spectrum between the left and right half of the system for two boundary conditions: open (*top*) and sandwich (*bottom*).

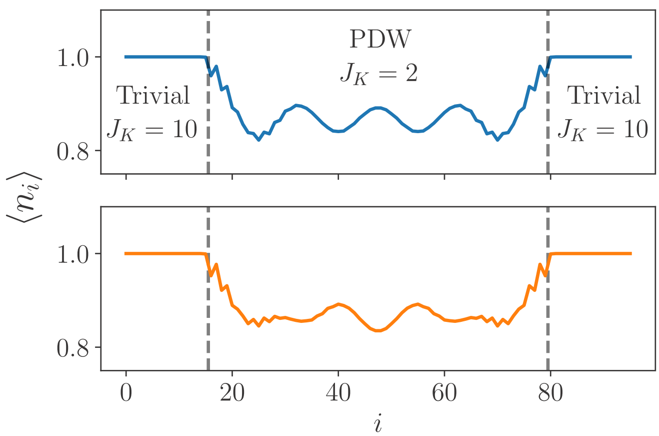


FIG. 6. Charge density in the ground state (*top*) and excited state (*bottom*) of the sandwich.

has a nearly identical entanglement spectrum to the open boundary system.

All the evidence presented does not provide any numerical evidence of MZMs. Despite clearly finding a PDW for both open and sandwich boundary conditions, neither the ground state nor excited state of those systems shows topological behavior.

## IV. PREVIOUSLY PROPOSED MAJORANA ZERO MODES

Due to the lack of numeric evidence of MZMs, we will reexamine the arguments that the PDW wire is topological. The original argument was made using Abelian bosonization and subsequent refermionization [13]. Here, we shall rederive these results using non-Abelian bosonization, since it is better suited to study the non-Abelian $SU(2)$ currents of the spin sector. Similar calculations have been previously done by Tsvelik [19,27].

There are three currents to study when considering the KH model. A $U(1)_2$ current describing the charge degrees of freedom of the 1DEG, a $SU(2)_1$ current describing the spin degrees of freedom of the 1DEG, and a second $SU(2)_1$ current describing the Heisenberg spins, leading to a total current structure of $U(1)_2 \times SU(2)_1 \times SU(2)_1$, as shown in Eqs. (2). Since, in the low-energy limit, the charge and spin sectors decouple (spin-charge separation), we will only focus on the spin sector, which corresponds to a $SU(2)_1 \times SU(2)_1$ Wess-Zumino-Witten (WZW) model [22]. It will also be useful to define the following currents:

$$\begin{aligned}\boldsymbol{J}_{\pm,R} &= \boldsymbol{J}_{e,R} \pm \boldsymbol{J}_{h,R},\\ \boldsymbol{J}_{\pm,L} &= \boldsymbol{J}_{e,L} \pm \boldsymbol{J}_{h,L}.\end{aligned} \tag{6}$$

Here the $\boldsymbol{J}_+$ fields describe the $SU(2)_2$ currents, and $\boldsymbol{J}_-$ describe the remaining $SU(2)_1 \times SU(2)_1/SU(2)_2$ currents. In terms of these fields, the spin Hamiltonian $H_s$ becomes (after setting the velocities of the spin modes to be equal to each other, $v_{s,t} = v_{s,b} = v_s$)

$$\begin{aligned}\mathcal{H}_s = &\frac{2\pi v_s}{6}[\boldsymbol{J}_{+,R}\boldsymbol{J}_{+,R} + \boldsymbol{J}_{-,R}\boldsymbol{J}_{-,R}]\\ &- g_+\boldsymbol{J}_{+,R}\boldsymbol{J}_{+,L} - g_-\boldsymbol{J}_{-,R}\boldsymbol{J}_{-,L},\end{aligned} \tag{7}$$

where $g_\pm = (g_{s1} \pm g_{s2})/2$.

Using the RG equations for Eq. (7) (see Appendix A), we can identify the four fixed points $(g_+, g_-) = (0, 0)$, $(-\infty, \infty)$, $(-\infty, -\infty)$, $(-\infty, 0)$. The $(g_+, g_-) = (0, 0)$ fixed point corresponds to the $C1S2$ Luttinger state, the $(g_+, g_-) = (-\infty, \infty)$ fixed point corresponds to the PDW phase and the $(g_+, g_-) = (-\infty, -\infty)$ fixed point corresponds to the trivial SC phase. The $(g_+, g_-) = (-\infty, 0)$ fixed point marks the Ising transition between the PDW and trivial SC phase.

To probe the existence of MZMs, we note that the two $SU(2)$ currents of the spin sector are equivalent to a single $SO(4)$ current since $SU(2) \times SU(2) \cong SO(4)$. The $SO(4)$ current algebra can naturally be expressed in terms of four Majorana fermions. With this in mind, let us now introduce the Majorana fermions $\eta_{0,R(L)}$ and $\eta_{a,R(L)}$, where $a = 1, 2, 3$. Using them, we can construct the left and right moving currents $\boldsymbol{J}_{\pm,R(L)}$ as

$$\begin{aligned}J^a_{+,R} &= \frac{i}{2}\epsilon^{abc}\eta_{b,R}\eta_{c,R},\\ J^a_{-,R} &= i\eta_{0,R}\eta_{a,R}.\end{aligned} \tag{8}$$

In terms of the Majorana fermions, the spin Hamiltonian becomes

$$\begin{aligned}\mathcal{H}_s = &\frac{iv_s}{2}(\eta_{0,L}\partial_x\eta_{0,L} - \eta_{0,R}\partial_x\eta_{0,R})\\ &+ \frac{iv_s}{2}\sum_a(\eta_{a,L}\partial_x\eta_{a,L} - \eta_{a,R}\partial_x\eta_{a,R}),\\ &- g_+\sum_{a>b}(\eta_{a,R}\eta_{a,L})(\eta_{b,R}\eta_{b,L})\\ &- g_-(\eta_{0,R}\eta_{0,L})\sum_a(\eta_{a,R}\eta_{a,L}).\end{aligned} \tag{9}$$

which, upon setting $g_+ = g_-$, is the Hamiltonian of the $O(4)$ Gross-Neveu model. Notice that in the full problem of Eqs. (9), the "light" Majorana field $\eta_0$ becomes massless at $g_- = 0$ and decouples from the rest. Due to the single free Majorana fermion, $g_- = 0$ marks an Ising critical point. We discuss the associated $\mathbb{Z}_2$ symmetry breaking that occurs at the phase transition in Appendix E.

In addition, this system also has a conserved fermion parity, which can be expressed as

$$(-1)^{N_f} = \exp(i\pi\int dx[i\eta_{0,R}\eta_{3,R} + i\eta_{1,R}\eta_{2,R} + (R \leftrightarrow L)]). \tag{10}$$

In terms of the lattice degrees of freedom, $(-1)^{N_f} = (-1)^{\sum_j 2S^z_{j,e}}$, which reduces to Eqs. (3) in the ground state, where $\sum_j[S^z_{j,e} + S^z_{j,h}] = 0$.

When $g_+$ is large, we expect that $i\eta_{a,R}\eta_{a,L}$ will gain an expectation value $\langle i\eta_{a,R}\eta_{a,L}\rangle = \Delta$. With this substitution, Eqs. (9) becomes

$$\mathcal{H}_s = \frac{iv_s}{2}(\eta_{0,L}\partial_x\eta_{0,L} - \eta_{0,R}\partial_x\eta_{0,R}) + ig_-\Delta(\eta_{0,R}\eta_{0,L}). \tag{11}$$

Between the PDW phase ($g_- > 0$) and the trivial SC phase ($g_- < 0$), the mass term for $\eta_0$ changes signs, and one would expect for there to be a localized MZM at the open ends of the system.

## V. ARGUMENTS AGAINST MAJORANA ZERO MODES

As we have shown in the previous section, the spin degrees of freedom of the doped KH model can be expressed in terms of four Majorana fermionic fields. Based on this, it is reasonable to conjecture, as was done in Ref. [13], that there may be MZMs at interfaces between the PDW and trivial SC phases. However, as we shall argue below, this is not the case here, and the doped KH model in the PDW phase does not host MZMs.

Let us first review several well-known features of SPTs. First, SPTs are short-range entangled gapped states of matter that cannot be smoothly deformed into a trivial state while preserving both symmetries and the bulk gap of the system. Second, at the interface between an SPT and a trivial state, there are localized zero energy degrees of freedom. This leads to a robust ground-state degeneracy for a system with symmetry-preserving boundaries.

In the case of the fermionized spin sector of the doped KH model, the localized zero energy modes are MZMs, and the ground-state degeneracy corresponds to the two fermion parity sectors. Acting on a ground state with a MZM changes the fermionic parity of the ground state from $\pm 1$ to $\mp 1$. Importantly, having two distinct fermion parity sectors is a necessary condition for the existence of MZMs. In reverse, if all states in a given theory have the same fermion parity, then a single MZM is not a physical operator.

The underlying question we are asking is if the Hilbert space of the spin sector of the original model, Eqs. (2), is the same as that of the fermionized model, Eqs. (9). Clearly, the Hilbert space of the fermionized model will consist of states with both even and odd fermion parity. In the following, we will discuss whether or not both of these fermion parity sectors exist in the Hilbert space of the original spin model. We find that all states in the Hilbert space of the spin model have even fermion parity. This means that the Hilbert space of the fermionic theory of Eqs. (9) is larger than that of the spin sector of the KH model. In particular, there are extra, unphysical states with odd fermion parity, that do not correspond to any state in the physical Hilbert space of the spin model. A similar situation is well known to happen in the quantum Ising chain which is described by the parity even sector of the fermionized version of the model.

To show this, it will be useful to define the system on a ring of length $L$. We are only interested in the topological features of the spin sector of the theory [Eq. (7)]. To have a pair of MZMs, we will put half of the ring in the PDW phase ($g_- > 0$ for $0 < x < L/2$) and the other half in the trivial SC phase ($g_- < 0$ for $L/2 < x < L$). From our earlier analysis, we expect that there will be two MZMs located at 0 and $L/2$. Since there are two MZMs in this system, we expect that there will be two degenerate ground states, one with fermion parity $+1$ and one with fermion parity $-1$.

With this system in mind, we now ask if the fermion parity odd states exist in the Hilbert space of the model described above. To probe this Hilbert space, it will actually be sufficient to just probe the Hilbert space of the unperturbed model ($g_- = g_+ = 0$), which is simply the $SU(2)_1 \times SU(2)_1$ WZW model. If all states in the Hilbert space of the unperturbed model have the same fermion parity, then all states in the Hilbert space of the perturbed model will also have the same fermion parity. This is because turning on a perturbation cannot add new states to the Hilbert space.

It is well known that the Hilbert space of a 1 + 1D conformal field theory (CFT) can be organized into Verma modules that are built off of a highest weight state [28]. These highest weight states are created by acting on the vacuum of the theory with a primary field. In Appendix B, we explicitly calculated the fermion parity [Eq. (10)] of all states in all Verma modules of the $SU(2)_1 \times SU(2)_1$ WZW CFT. We find that they all have even fermion parity and, as a result, all states in the perturbed model must also have even fermion parity. Individual MZM operators are therefore not physical operators since acting on an even fermion parity state with the MZM operator leads to an odd fermion parity state, the latter of which we know does not exist in the $SU(2)_1 \times SU(2)_1$ theory. Products of an even number of Majorana operators are physical, as can be seen from examining the $SU(2)$ currents of the model.

From this analysis, we can conclude that switching from the spin currents [Eq. (7)] to the fermion representation [Eqs. (9)] introduces new states into the Hilbert space of the system. In particular, the fermion parity-odd states are part of the unphysical fermionic Hilbert space, but not of the physical spin Hilbert space. So, to move from the expanded fermionic Hilbert space to the physical spin Hilbert space, the fermionic Hilbert space must be projected onto the fermion parity even states (known in string theory as a Gliozzi-Scherk-Olive (GSO) projection [29]). We present a similar argument using Abelian bosonization in Appendix D.

We can also consider the possibility that the spin sector of the doped KH model is another SPT protected by some other symmetry. The only other symmetry in the model is the total spin $SU(2) \cong SO(3)$ symmetry of the model. From cohomology classifications, it is known that there is one nontrivial SPT in $1d$ protected by the $SO(3)$ symmetry—the Haldane phase of the spin 1-chain. It is known that in the Haldane phase, the edge modes carry spin-1/2. In the Majorana representation, only the fermions $\eta_a$ ($a = 1, 2, 3$) carry spin. It is clear that there are no zero modes for $\eta_a$ in Eqs. (9) at a boundary between the PDW and trivial SC phases, since $g_+ < 0$ for both phases. This indicates that the spin sector of the model is not in the Haldane phase.

In addition, it is known that the $SU(2)_1 \times SU(2)_1$ WZW model enters the Haldane phase when the following interaction is added [30–33]:

$$H_{\text{int}} = \frac{\lambda}{2\pi} \sum_a \text{tr}(g_e \tau^a)\text{tr}(g_h \tau^a), \tag{12}$$

where $g_{e/h}$ are the WZW $g$ fields of the 1DEG and Heisenberg spins, respectively (see Appendix A), and $\lambda$ is negative. In terms of the fermionic representation, this interaction introduces a negative mass term for $\eta_a$ and, by extension, three MZMs at the boundaries of the system. These zero modes carry spin as expected in the Haldane phase. As shown in Appendix A, the interaction in Eq. (12) is not present in the doped KH model. Because of this, we can conclude that the doped KH model is not in the Haldane phase, and thereby is not an SPT.

## VI. CONCLUSION

In this paper, we have established, using both numeric and analytic methods, that the doped KH model does not host MZMs. Furthermore, it appears that the spin sector of the model is also not an SPT protected by the $SO(3)$ symmetry of the model. Based on this, we believe that the doped KH model is not an SPT of any kind. Our analysis does not rule out possible SPTs that exist beyond the cohomology classifications, however, there is no evidence for this, and we believe that this situation is extremely unlikely.

While our results do show that the PDW state of the KH model in 1D is not topological, it does not rule out a topological PDW state in principle. Indeed, it is easy to imagine a 1D toy model with a properly chosen PDW mean-field term that would have MZMs analogous to the Kitaev chain. Since in dimensions $d > 1$, PDW states generally have Fermi surfaces of Bogoliubov quasiparticles; in 1D one would expect that a PDW should have Majorana "zero-modes" along the length of the state. One such example is a paired $p$-wave state whose order parameter changes periodically its sign, i.e., a PDW relative of the uniform $p$-wave state. This state can be viewed as a sequence of regions with local uniform $p$-wave order with a periodic arrangement of domain walls where the sign changes occur. Then, a Jackiw-Rebbi type argument [34] implies the existence of (Majorana) zero modes at the location of each domain wall. A related topological two-dimensional state was recently studied by Santos and collaborators [35]. Actually, such a $p$-wave PDW is equivalent to a theory of massless Majorana fermions and is at a critical point. Subsequent breaking of inversion symmetry (by a uniform $p$-wave component) leads to a gapped topological state. It would be interesting to construct a 1D Hamiltonian with a state of this type (without resorting to a proximity effect mechanism).

Moving on to two dimensions, it is not difficult to imagine a weak-coupling 2D topological FFLO-type state. For example, if two spin-filtered Fermi surfaces exist away from the gamma point, as the Fermi surface of doped transition metal dichalcogenides, and if there is an intravalley triplet pairing channel, then its natural ground state should be an intravalley $p$-wave SC. Such a state is topological. The resulting topological content will be Ising $\times$ Ising. Note that this state can melt into the two distinct states, an isotropic $4e$ SC state and a CDW state without superconductivity. The topological nature of these states may be interesting to study in future work. On the other hand, since non-mean-field 2D models of PDW systems remain elusive, it is an open question whether topological PDW states may exist in higher dimensions. An effective field theory approach using a nonlinear sigma model may be a promising way to probe this question in future work.

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## APPENDIX A: NON-ABELIAN BOSONIZATION ANALYSIS OF THE KONDO-HEISENBERG MODEL

Here, we will now study the the problem of the KH model using non-Abelian bosonization. There are three currents to study when considering the KH model. A $U(1)_2$ current describing the charge degrees of freedom of the 1DEG, a $SU(2)_1$ current describing the spin degrees of freedom of the 1DEG, and a second $SU(2)_1$ current describing the spin degrees of freedom of the spin chain of the KH, leading to a total of current structure of $U(1)_2 \times SU(2)_1 \times SU(2)_1$.

The Hamiltonian for the charge degrees of freedom is given by

$$\mathcal{H}_c = \frac{v_c}{2}\left[\frac{1}{K_c}(\partial_t\theta_c)^2 + K_c(\partial_x\phi_c)^2\right]. \tag{A1}$$

These degrees of freedom are gapless and do not couple to the spin degrees of freedom.

It is known that the spin currents of this model can be expressed as a $SU(2)_1 \times SU(2)_1$ WZW model [37]. The spin currents, $\boldsymbol{J}_{e(h),R(L)}$, of the $SU(2)_1 \times SU(2)_1$ WZW model are defined as

$$J^a_{e,R} = -\frac{i}{2\pi}\mathrm{tr}\big(\partial_z g_e g_e^{-1}\tau^a\big), \quad J^a_{e,L} = \frac{i}{2\pi}\mathrm{tr}\big(g_e^{-1}\partial_{\bar{z}} g_e\tau^a\big), \tag{A2}$$

and similar for $e \leftrightarrow h$. Here, $g_{e/h}$ is a $SU(2)$ matrix valued field, $\partial_z$ $\partial_{\bar{z}}$ are the derivatives with respect to the holomophic and antiholophomic coordinates ($t \mp ix$), and $\tau^a$ are the Pauli matrices. The operator product expansion (OPE) for the spin currents are given by

$$J^a_{e,R}(z)J^b_{e,R}(w) \sim \frac{1}{(z-w)^2}\delta_{ab} + \frac{i}{(z-w)}\epsilon_{abc}J^c_{e,R}(w), \tag{A3}$$

and similarly for $e \leftrightarrow h$ and $R \leftrightarrow L$. The lattice spins of the system are defined as

$$\begin{aligned}\frac{\boldsymbol{S}_{j,e}}{a} &= \frac{1}{2\pi}[\boldsymbol{J}_{e,R}(x) + \boldsymbol{J}_{e,L}(x)] + e^{i2k_f x}\Theta_e\mathrm{tr}(g_e\boldsymbol{\tau}),\\ \frac{\boldsymbol{S}_{j,h}}{a} &= \frac{1}{2\pi}[\boldsymbol{J}_{h,R}(x) + \boldsymbol{J}_{h,L}(x)] + (-1)^{x/a}\Theta_h\mathrm{tr}(g_h\boldsymbol{\tau}),\end{aligned} \tag{A4}$$

where $\Theta_{e/h}$ are nonuniversal constants. The factor of $e^{i2k_fx}$ is due to the doping of the electron degrees of freedom.

Using the Sugawara construction and ignoring irrelevant operators, the Hamiltonian for the spin degrees of freedom of the doped KH model is given by

$$\begin{aligned}\mathcal{H}_{\mathrm{s}} = {}& \frac{2\pi v_{s,e}}{3}\boldsymbol{J}_{e,R}\boldsymbol{J}_{e,R} + \frac{2\pi v_{s,h}}{3}\boldsymbol{J}_{h,R}\boldsymbol{J}_{h,R} \\ &- g_{s1}[\boldsymbol{J}_{e,R}\boldsymbol{J}_{e,L} + \boldsymbol{J}_{h,R}\boldsymbol{J}_{h,L}] \\ &- g_{s2}[\boldsymbol{J}_{e,R}\boldsymbol{J}_{h,L} + \boldsymbol{J}_{h,R}\boldsymbol{J}_{e,L}]. \end{aligned} \tag{A5}$$

We note here that this model has a discrete symmetry that sends $(g_e, g_h) \to (-g_e, -g_h)$. If the electrons were at half filling ($k_f = \pi/2$), we would also be able to include the term $\sum_a \mathrm{tr}(g_e\tau^a)\mathrm{tr}(g_h\tau^a)$. However, due to the electron doping, this term oscillates as $e^{i(2k_f+\pi)x}$, and is thereby irrelevant.

We will now determine the RG flow for $\mathcal{H}_{\mathrm{s}}$. To do this, it will be useful to introduce new variables:

$$\boldsymbol{J}_{\pm,R} = \boldsymbol{J}_{e,R} \pm \boldsymbol{J}_{h,R}, \tag{A6}$$

$$\boldsymbol{J}_{\pm,L} = \boldsymbol{J}_{e,L} \pm \boldsymbol{J}_{h,L}. \tag{A7}$$

Here the $\boldsymbol{J}_+$ fields describe the $SU(2)_2$ currents, and $\boldsymbol{J}_-$ describe the remaining $SU(2)_1 \times SU(2)_1/SU(2)_2$ currents. In terms of these fields, the the spin Hamiltonian $H_{\mathrm{s}}$ becomes (setting $v_{s,t} = v_{s,b} = v_s$)

$$\begin{aligned}\mathcal{H}_{\mathrm{s}} = {}& \frac{2\pi v_s}{6}[\boldsymbol{J}_{+,R}\boldsymbol{J}_{+,R} + \boldsymbol{J}_{-,R}\boldsymbol{J}_{-,R}] \\ &- g_+\boldsymbol{J}_{+,R}\boldsymbol{J}_{+,L} - g_-\boldsymbol{J}_{-,R}\boldsymbol{J}_{-,L},\end{aligned}$$

where $g_\pm = (g_{s1} \pm g_{s2})/2$. The OPEs for the $\boldsymbol{J}_\pm$ fields are

$$\begin{aligned}
J^a_{+,R}(z)J^b_{+,R}(w) &\sim \frac{2}{(z-w)^2}\delta_{ab} + \frac{i}{(z-w)}\epsilon_{abc}J^c_{+,R}(w),\\
J^a_{-,R}(z)J^b_{-,R}(w) &\sim \frac{2}{(z-w)^2}\delta_{ab} + \frac{i}{(z-w)}\epsilon_{abc}J^c_{+,R}(w),\\
J^a_{-,R}(z)J^b_{+,R}(w) &\sim \frac{2}{(z-w)^2}\delta_{ab} + \frac{i}{(z-w)}\epsilon_{abc}J^c_{-,R}(w),\\
J^a_{+,R}(z)J^b_{-,R}(w) &\sim \frac{2}{(z-w)^2}\delta_{ab} + \frac{i}{(z-w)}\epsilon_{abc}J^c_{-,R}(w),
\end{aligned} \tag{A8}$$

and similar for $L \leftrightarrow R$. Using these OPEs for the $\boldsymbol{J}_\pm$ fields, we have the beta functions

$$\begin{aligned}\beta(g_+) &= -\frac{2}{\pi}(g_+^2 + g_-^2),\\ \beta(g_-) &= -\frac{4}{\pi}(g_+g_-).\end{aligned} \tag{A9}$$

Let us examine the $\beta$ functions near $(g_+, g_-) = (0, 0)$. For $g_- \neq 0$ or $g_+ < 0$, $g_+$ flows to $-\infty$. For $g_+ < 0$, we can rewrite the $\beta(g_-)$ as

$$\beta(g_-) = \frac{4}{\pi}|g_+|g_-. \tag{A10}$$

Rewriting $g_-$ as $\pm|g_-|$, we have that

$$\beta(|g_-|) = \frac{4}{\pi}|g_+||g_-|. \tag{A11}$$

So, for $g_+ < 0$, $g_- > 0$, $g_-$ flows to $\infty$ and for $g_+ < 0$, $g_- < 0$, $g_-$ flows to $-\infty$. Using this, we can identify the fixed points $(g_+, g_-) = (0, 0)$, $(-\infty, \infty)$, $(-\infty, -\infty)$, and $(-\infty, 0)$.

## APPENDIX B: FERMION NUMBER IN THE $SU(2)_1 \times SU(2)_1$ WZW MODEL

Let us consider the $SU(2)_1 \times SU(2)_1$ WZW model defined on a ring. This is equivalent to defining the WZW model on the complex plane where the radial direction is time and the polar angle is space. We can express the $SU(2)$ currents in terms of Majoranas using

$$J^a_{e,R/L} = \frac{i}{2}\left(\frac{\epsilon^{abc}}{2}\eta^b_{R/L}\eta^c_{R/L} + \eta^0_{R/L}\eta^a_{R/L}\right), \tag{B1}$$

$$J^a_{h,R/L} = \frac{i}{2}\left(\frac{\epsilon^{abc}}{2}\eta^b_{R/L}\eta^c_{R/L} - \eta^0_{R/L}\eta^a_{R/L}\right). \tag{B2}$$

Let us now define the following charge operator:

$$\begin{aligned}N_f &= \frac{2}{2\pi i}\oint dz\big(J^3_{e,R}(z) + J^3_{e,L}(\bar{z})\big)\\ &= \frac{1}{2\pi i}\oint dz(\eta_{1,R}(z)\eta_{2,R}(z) + \eta_{0,R}(z)\eta_{3,R}(z)\\ &\quad + \eta_{1,L}(\bar{z})\eta_{2,L}(\bar{z}) + \eta_{0,L}(\bar{z})\eta_{3,L}(\bar{z})),\end{aligned} \tag{B3}$$

where the contour integral is over a circle of constant radius in the complex plane, i.e., a constant time slice. The charge $q_A$ of a field $A(w, \bar{w})$ is given by

$$[N_f, A(w, \bar{w})] = q_A A(w, \bar{w}), \tag{B4}$$

where [...] is the radially ordered commutator. We find that the $\boldsymbol{J}_{e,R}$ currents have the following charges:

$$\big[N_f, J^3_{e,R}(w)\big] = 0, \tag{B5}$$

$$[N_f, J^\pm_{e,R}(w)] = \pm 2J^\pm_{e,R}(w). \tag{B6}$$

The charges of the $\boldsymbol{J}_{e,L}$ are identical. The charge of components of the matrix valued WZW field $g_e$ are

$$[N_f, g_e(w, \bar{w})_{00}] = 2g_e(w, \bar{w})_{00}, \tag{B7}$$

$$[N_f, g_e(w, \bar{w})_{01}] = [N_f, g_e(w, \bar{w})_{10}] = 0, \tag{B8}$$

$$[N_f, g_e(w, \bar{w})_{11}] = -2g_e(w, \bar{w})_{11}, \tag{B9}$$

where $g_e(w, \bar{w})_{ij}$ are the components of the matrix valued WZW field $g_e$. The charges of the (sum of) Majoranas are

$$[N_f, \eta_{1,R}(w) \pm i\eta_{2,R}(w)] = \pm(\eta_{1,R}(w) \pm i\eta_{2,R}(w)), \tag{B10}$$

$$[N_f, \eta_{0,R}(w) \pm i\eta_{3,R}(w)] = \pm(\eta_{0,R}(w) \pm i\eta_{3,R}(w)). \tag{B11}$$

The charges of the left-handed Majoranas are the same. Additionally,

$$\big[N_f, J^a_{h,R}(w)\big] = \big[N_f, J^a_{h,L}(\bar{w})\big] = [N_f, g_h(w, \bar{w})_{ij}] = 0, \tag{B12}$$

since all the OPEs disappear. From this, we can conclude that fields $\boldsymbol{J}_{e/h,R/L}$ and $g_{e,h}$ all have charge 0 mod(2). The Majorana fields $\eta_{\mu,R/L}$ have charge 1 mod(2). As such, $\boldsymbol{J}_{e/h,R/L}$ and $g_{e,h}$ all have even charge parity, $(-1)^{N_f}$, while $\eta_{\mu R/L}$ have odd charge parity.

We can also find the charges of the individual modes of spin currents $\boldsymbol{J}_{e/h,R/L}$ and Majorana currents $\eta_{\mu,R/L}$ analogously. The modes in radial quantization are, respectively,

$$J^a_{n,e,R} = \oint \frac{dw}{2\pi i} w^n J_{e,R}(w), \tag{B13}$$

$$\eta_{n,\mu,R} = \oint \frac{dw}{2\pi} w^{n-1/2} \eta_{\mu,R}(w), \tag{B14}$$

and similar for $e \to h$ and $R \to L$. Combining Eqs. (B13) and (B14) with Eqs. (B5)–(B12), we find that all modes $J^a_n$ have even charge 0 mod(2) and all modes $\eta_n$ have charge 1 mod(2) as expected.

Let us now consider the charge of various states in the Hilbert space of the $SU(2)_1 \times SU(2)_1$ WZW model. It is known that the Hilbert space of a CFT can be divided into Verma modules. The Verma modules are built off of a highest weight state. In the $SU(2)_1 \times SU(2)_1$ WZW model there are four highest weight states. First, there is the trivial vacuum state which we will label $|0\rangle$. Second, there are the highest weight states that correspond to inserting a primary field. For the $SU(2)_1 \times SU(2)_1$ WZW model in radial quantization, the primary fields that are inserted are $g_e(0,0)_{ij}$, $g_h(0,0)_{ij}$ and their product $g_e(0,0)_{ij} g_h(0,0)_{kl}$. We will label the corresponding highest weight states as $g_{e,ij}|0\rangle$, $g_{h,ij}|0\rangle$ and $g_{e,ij}g_{h,kl}|0\rangle$. The descendant states of these highest weight states are created by acting on the highest weight states with the operators $J^a_{-n,e/h,R/L}$.

Let us now consider the parity of a state in the Hilbert space. A general state built off the vacuum highest weight state $|0\rangle$ can be written as

$$J^a_{-n_1,e/h,L/R} J^b_{-n_2,e/h,L/R} \cdots |0\rangle. \tag{B15}$$

From our earlier analysis, we know that the modes $J^a_{-n_1,e/h,L/R}$ have charge 0 mod(2). Since the vacuum has charge 0 by definition, we can conclude that all states built off the vacuum have even charge, i.e., $(-1)^{N_f} = 1$ for all states in Eq. (B15).

We will now consider the other states in the Hilbert space that are built off the $g_{e,ij}|0\rangle$ and $g_{h,ij}|0\rangle$ and $g_{e,ij}g_{h,kl}|0\rangle$ highest weight states. In general, these states can be written as

$$\begin{gathered} J^a_{-n_1,e/h,L/R} J^b_{-n_2,e/h,L/R} \cdots g_{e,ij}|0\rangle, \\ J^a_{-n_1,e/h,L/R} J^b_{-n_2,e/h,L/R} \cdots g_{h,ij}|0\rangle, \\ J^a_{-n_1,e/h,L/R} J^b_{-n_2,e/h,L/R} \cdots g_{e,ij}g_{h,kl}|0\rangle. \end{gathered} \tag{B16}$$

As before, we know that the modes $J^a_{-n_1,e/h,L/R}$ have charge 0 mod(2). Our earlier analysis has also shown that $g_e(0,0)_{ij}$, $g_h(0,0)_{ij}$ and their product $g_e(0,0)_{ij} g_h(0,0)_{kl}$ all have charge 0 mod(2). Because of this, $(-1)^{N_f} = 1$ for all states in Eqs. (B16). From this, we can conclude that $(-1)^{N_f} = 1$ for all states in the Hilbert space of the $SU(2) \times SU(2)$ WZW model.

Let us now consider acting on a given even charge parity state $|\psi\rangle$ with a single Majorana fermion mode $\eta_{-n,\mu,R/L}$:

$$\eta_{-n,\mu,R/L}|\psi\rangle. \tag{B17}$$

From our earlier result, we know that $\eta_{-n,\mu,R/L}$ has charge 1 mod(2). Since the state $|\psi\rangle$ has $(-1)^{N_f} = 1$, the state in Eq. (B17) has $(-1)^{N_f} = -1$. However, we know that all states in Hilbert space of the $SU(2) \times SU(2)$ WZW model have $(-1)^{N_f} = 1$. So, the state in Eq. (B17) cannot be a physical state of the $SU(2) \times SU(2)$ WZW model. We can also consider acting the state $|\psi\rangle$ with two Majorana fermion modes $\eta_{-n,\mu,R/L}$ and $\eta_{-n',\nu,R/L}$:

$$\eta_{-n,\mu,R/L}\eta_{-n',\nu,R/L}|\psi\rangle. \tag{B18}$$

Since each of the modes have charge 1 mod(2), the state in Eq. (B18) has $(-1)^{N_f} = 1$. So this can be a physical state in the $SU(2) \times SU(2)$ WZW model. We can thereby conclude that a single Majorana mode operator is not a physical operator in the $SU(2) \times SU(2)$ WZW model. In other words, there are no single fermions modes in the spectrum of the $SU(2) \times SU(2)$ WZW model. The fermions only occur as bilinears.

## APPENDIX C: CONTINUUM LIMIT AND ABELIAN BOSONIZATION

Here, we will now discuss the continuum limit of the lattice model using Abelian bosonization. In the low-energy limit, the fermions and spins can be expressed in terms of continuum current operators:

$$\begin{aligned} \frac{1}{\sqrt{a}} c_{j,\sigma} &\to R_\sigma(x) e^{ik_f x} + L_{\sigma,t}(x) e^{-ik_f x} \\ \frac{\boldsymbol{S}_{j,h}}{a} &\to \boldsymbol{J}_{h,R}(x) + \boldsymbol{J}_{h,L}(x) + (-1)^{x/a} \boldsymbol{N}_h(x). \end{aligned} \tag{C1}$$

Here, $R_\sigma$ and $L_\sigma$ are the right- and left-moving components of the electron fields, $\boldsymbol{J}_{h,R}$ and $\boldsymbol{J}_{h,L}$are the slowly varying components of the spin field, $\boldsymbol{N}_h$ is the rapidly oscillating (Néel) component of the spin field, and $x = ja$ where $a$ is the lattice spacing.

The right- and left-moving continuum fields can be bosonized using the following identifications:

$$\begin{aligned} R_\sigma &= :\frac{1}{\sqrt{2\pi a}} e^{-i\sqrt{2\pi}[\phi_\sigma + \sigma\theta_\sigma]}:, \\ L_\sigma &= :\frac{1}{\sqrt{2\pi a}} e^{i\sqrt{2\pi}[\phi_\sigma + \sigma\theta_\sigma]}:, \\ J^z_{h,R} &= \frac{1}{\sqrt{2\pi}} \partial_x[\tilde{\phi}_s - \tilde{\theta}_s], \\ J^z_{h,L} &= \frac{1}{\sqrt{2\pi}} \partial_x[\tilde{\phi}_s + \tilde{\theta}_s], \\ J^\pm_{h,R} &= :\frac{1}{2\pi a} e^{\mp i\sqrt{2\pi}[\tilde{\phi}_s - \tilde{\theta}_s]}:, \\ J^\pm_{h,L} &= :\frac{1}{2\pi a} e^{\pm i\sqrt{2\pi}[\tilde{\phi}_s + \tilde{\theta}_s]}:, \end{aligned} \tag{C2}$$

where $\phi_\sigma$ and $\theta_\sigma$ are the field and dual field of the electrons, $\tilde{\phi}_s$ and $\tilde{\theta}_s$ are the field and dual field of the spins, and : ... : indicates normal ordering of the exponential. From here on, we will leave the normal ordering implicit. In this definition, we note that the the spin fields are defined such that $\tilde{\phi}_s \equiv \tilde{\phi}_s + \sqrt{2\pi}$.

It will be useful to decompose the field and dual fields into spin and charge degrees of freedom using

$$\phi_c = \frac{1}{\sqrt{2}}(\phi_\uparrow + \phi_\downarrow),$$
$$\phi_s = \frac{1}{\sqrt{2}}(\phi_\uparrow - \phi_\downarrow), \tag{C3}$$

and similarly for the $\theta$ fields.

The dominant interactions will be between the the 1DEG spins and the Heisenberg spins. In terms of the spin currents of the 1DEG $\boldsymbol{J}_{e,R} = \frac{1}{2}R_\sigma \boldsymbol{\tau}_{\sigma,\sigma'} R_{\sigma'}$ and $\boldsymbol{J}_{e,L} = \frac{1}{2}L_\sigma \boldsymbol{\tau}_{\sigma,\sigma'} L_{\sigma'}$, the most general interactions consistent with the $SU(2)$ symmetries of the model are

$$H_{int} = - g_{s1,e}\boldsymbol{J}_{e,R}\boldsymbol{J}_{e,L} - g_{s1,s}\boldsymbol{J}_{h,R}\boldsymbol{J}_{h,L}] \\ - g_{s2}[\boldsymbol{J}_{e,R}\boldsymbol{J}_{h,L} + \boldsymbol{J}_{h,R}\boldsymbol{J}_{e,L}]. \tag{C4}$$

At weak coupling, the relationship between the values of these coupling constants and those of the microscopic model Eqs. (1) are $g_{s1,e} = U$, $g_{s1,s} = J_H - 6J'_H$, and $g_{s2} = -J_K$. As noted before, the microscopic model Eqs. (1) can also arise as the effective description of a two-leg Hubbard ladder [10]. The relationship between the coupling constants in Eq. (C4) and the those of the two-leg Hubbard ladder are more complex and can be found in Ref. [38].

If we set $g_{s1,e} = g_{s1,s} \equiv g_{s1}$ and define $\phi_{s,\pm} \equiv \frac{1}{\sqrt{2}}(\phi_s \pm \tilde{\phi}_s)$, and similarly for $\theta_{s,\pm}$, we arrive at the continuum Hamiltonian:

$$\begin{aligned}
\mathcal{H} &= \mathcal{H}_c + \mathcal{H}_s,\\
\mathcal{H}_c &= \frac{v_c}{2}[K_c(\partial_t\theta_{c,e})^2 + \frac{1}{K_c}(\partial_x\phi_{c,e})^2],\\
\mathcal{H}_s &= \sum_{\epsilon=\pm}\frac{v_{s,\epsilon}}{2}[K_{s,\epsilon}(\partial_t\theta_{s,\epsilon})^2 + \frac{1}{K_{s,\epsilon}}(\partial_x\phi_{s,\epsilon})^2]\\
&\quad + \frac{g_{s1}}{2(\pi a)^2}\cos(\sqrt{4\pi}\phi_{s,+})\cos(\sqrt{4\pi}\phi_{s,-})\\
&\quad + \frac{g_{s2}}{2(\pi a)^2}\cos(\sqrt{4\pi}\phi_{s,+})\cos(\sqrt{4\pi}\theta_{s,-}).
\end{aligned} \tag{C5}$$

It is important to note that since $\tilde{\phi}_s \equiv \tilde{\phi}_s + \sqrt{2\pi}$, $(\phi_{s,+}, \phi_{s,-}) \equiv (\phi_{s,+} + \sqrt{\pi}, \phi_{s,-} - \sqrt{\pi})$.

## APPENDIX D: MAJORANA ZERO MODES USING ABELIAN BOSONIZATION

The original argument for the existence of MZMs [13] comes from considering a section of PDW wire ($(g_K, g_{\rm SC}) = (-\infty, 0)$) of length $L$ that is sandwiched in between between two sections of trivial SC wire ($(g_K, g_{\rm SC}) = (0, -\infty)$). So the wire is in a trivial SC state for $x < 0$ and $L < x$, and a PDW phase for $0 < x < L$. In the analysis of the topological features of the PDW wire, we will only be interested in the gapped spin sector of the wire [Eqs. (C5)], and not in the gapless charge sectors. We will also take $L$ to be much greater than the correlation length of the spin sector. To show the proposed MZMs which are localized at $x = 0$ and $x = L$, we will refermionize Eqs. (C5) around the $K_{s\pm} = 1$ point. Assuming that $\phi_{s+}$ is pinned to the same minimum throughout the entire system, the refermionized Hamiltonian is given by

$$\begin{aligned}
\mathcal{H}_s &= -iv_s(\mathcal{R}^\dagger\partial_x\mathcal{R} - \mathcal{L}^\dagger\partial_x\mathcal{L})\\
&\qquad + M_{USC}\mathcal{R}^\dagger\mathcal{L} + \Delta_{\rm PDW}\mathcal{R}^\dagger\mathcal{L}^\dagger + \text{H.c.}\\
\mathcal{R} &\sim e^{-i\sqrt{\pi}(\phi_{s,-}-\theta_{s,-})}\\
\mathcal{L} &\sim e^{i\sqrt{\pi}(\phi_{s,-}+\theta_{s,-})}
\end{aligned} \tag{D1}$$

where $M_{USC} \sim g_{s1}\langle\cos(\sqrt{4\pi}\phi_{s+})\rangle$ and $\Delta_{\rm PDW} \sim g_{s2}\langle\cos(\sqrt{4\pi}\phi_{s+})\rangle$. The fermion number is not conserved in Eq. (D1), but the fermion parity given by

$$(-1)^{N_f} = (-1)^{\int dx \mathcal{R}^\dagger\mathcal{R} + \mathcal{L}^\dagger\mathcal{L}} \tag{D2}$$

is conserved.

Decomposing the fermions into Majorana fermions using $\mathcal{R} = \frac{1}{\sqrt{2}}(\eta_{1,R} + i\eta_{2,R})$, $\mathcal{L} = \frac{1}{\sqrt{2}}(\eta_{2,L} + i\eta_{1,L})$ the potential term in Eq. (D1) becomes

$$\mathcal{V}_s = (M_{USC} - \Delta_{\rm PDW})i\eta_{1,R}\eta_{1,L} + (M_{USC} + \Delta_{\rm PDW})i\eta_{2,R}\eta_{2,L}. \tag{D3}$$

Since $M_{USC} \sim g_{s1}$ and $\Delta_{\rm PDW} \sim g_{s2}$, $M_{USC} - \Delta_{\rm PDW}$ changes sign when moving from the SC region to the PDW region at $x = 0$ and $x = L$. At these points there will be zero energy mode for the Majorana fermions $\eta_1 = (\eta_{1,R}, \eta_{1,L})$ due to the Jackiw-Rebbi mechanism. These MZMs imply that the spin sector of the doped KH model can be considered to be topological superconductor in class **D**, i.e., a Kiteav chain. Acting on a given state with a MZM operator changes the fermion parity of the state [Eq. (D2)] from $\pm1$ to $\mp1$. Naively, this would lead to two ground states, one with fermion parity even, and one with fermion parity odd.

It is at this point that we wish to ask if the MZMs from the refermionized model Eq. (D1) correspond to physical operators in the original spin model. To answer this in the Abelian bosonized framework, we first note that the bosonic fields describing the electron and Heisenberg spins are compact. This compactification means that the fields $\phi_\pm$ are defined such that $(\phi_{s+}, \phi_{s-}) \equiv (\phi_{s+} + \sqrt{\pi}, \phi_{s-} - \sqrt{\pi})$ (see Appendix C). Because of this, all physical operators in the theory must be invariant under sending $\phi_{s\pm} \to \phi_{s\pm} \pm \sqrt{\pi}$ simultaneously. However, if in the refermionization in Eq. (D1) we note that the fermions $\mathcal{R}$ and $\mathcal{L}$ are not not invariant under this transformation, but instead transform as $\mathcal{R} \to -\mathcal{R}$ and $\mathcal{L} \to -\mathcal{L}$.

Let us now consider the situation where there is a boundary between a section of PDW wire and a section of trivial SC wire. As noted before, one would expect there to be a pair of MZMs at either ends of the PDW wire. Let us consider the ground state of the system $|0\rangle$ that must be invariant under the transformation $\phi_{s\pm} \to \phi_{s\pm} \pm \sqrt{\pi}$. Furthermore, the ground state will have a well-defined fermion parity [Eq. (D2)], that we will take to be equal to $+1$. If one of the zero modes $\eta_1$ is physical, the state $\eta_1|0\rangle$ will be a degenerate ground state with fermion parity $-1$. However, as noted before, the zero mode $\eta_1$ is not a physical state, since under $\phi_{s\pm} \to \phi_{s\pm} \pm \sqrt{\pi}$, $\eta_1 \to -\eta_1$. So $\eta_1|0\rangle$ cannot be a physical state. From this, we can also conclude that only products of an even number of Majorana fermion operators leads to physical states. This means that all physical states will have fermion parity $+1$. From our earlier logic, we can then confirm that there is not ground-state degeneracy, and by extension no MZMs.

## APPENDIX E: $\mathbb{Z}_2$ ORDER PARAMETER

Here we discuss the a $\mathbb{Z}_2$ order parameter and the associated symmetry breaking that occurs between the trivial SC and PDW phases of the doped KH model. We expect this symmetry breaking to occur because of the phase transition between the two phases in the Ising universality class. The $\mathbb{Z}_2$ symmetry that we will need to consider sends $(g_e, g_h) \to (-g_e, -g_h)$.

To approach the problem of symmetry breaking, it will be useful to consider this problem with Abelian bosonization instead of non-Abelian bosonization, since in the former case, the order parameters can be read off by using a semiclassical analysis. In Abelian bosonization, the WZW fields $g_e$ and $g_h$ can be written as

$$g_e = \begin{bmatrix} e^{i\sqrt{2\pi}\phi_s} & e^{-i\sqrt{2\pi}\theta_s} \\ -e^{i\sqrt{2\pi}\theta_s} & e^{-i\sqrt{2\pi}\phi_s} \end{bmatrix}, \quad g_h = \begin{bmatrix} e^{i\sqrt{2\pi}\tilde{\phi}_s} & e^{-i\sqrt{2\pi}\tilde{\theta}_s} \\ -e^{i\sqrt{2\pi}\tilde{\theta}_s} & e^{-i\sqrt{2\pi}\tilde{\phi}_s} \end{bmatrix}. \tag{E1}$$

We note that combining Eqs. (E1) and (A2) reproduce Eq. (C2). The order parameters we are interested in will be $\mathrm{tr}(g_e) = 2\cos(\sqrt{2\pi}\phi_s)$, and $\mathrm{tr}(g_h) = 2\cos(\sqrt{2\pi}\tilde{\phi}_s)$, as well as their product $\mathrm{tr}(g_e)\mathrm{tr}(g_h)$. Clearly $\mathrm{tr}(g_e)$ and $\mathrm{tr}(g_h)$ are odd under $(g_e, g_h) \to (-g_e, -g_h)$ but their product is not.

Let us now determine when these order parameters have expectation values. In the trivial SC phase, $\langle\phi_{s+}\rangle = \langle\phi_{s-}\rangle = 0, \sqrt{\pi/2}$. Using that $\phi_{s\pm} = \frac{1}{\sqrt{2}}(\phi_s \pm \tilde{\phi}_s)$, we can determine that both $\langle\mathrm{tr}(g_e)\rangle = \pm 2$ and $\langle\mathrm{tr}(g_e)\rangle = \pm 2$, and so the $\mathbb{Z}_2$ symmetry is broken. In the PDW phase, $\langle\phi_{s+}\rangle = \langle\theta_{s-}\rangle = 0, \sqrt{\pi/2}$. In this phase, neither $\mathrm{tr}(g_e)$ or $\mathrm{tr}(g_e)$ have expectation values, but their product does have an expectation value $\langle\mathrm{tr}(g_e)\mathrm{tr}(g_h)\rangle = \pm 2$. So, the $(g_e, g_h) \to (-g_e, -g_h)$ symmetry is unbroken. We can thereby identify the phase transition between the PDW and trivial SC phase with breaking the $(g_e, g_h) \to (-g_e, -g_h)$ symmetry.

## APPENDIX F: ADDITIONAL NUMERICS

All energies shown in Fig. 2 were generated via extrapolation to 0 variance. For completeness, we show those energies in Figs. 7 and 8.

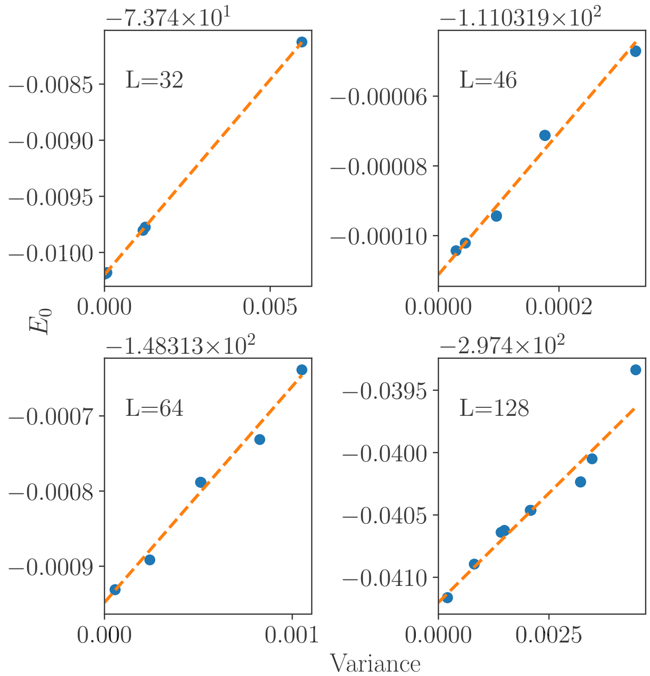


FIG. 7. Variance extrapolation of the ground state ($E_0$) for various system sizes.

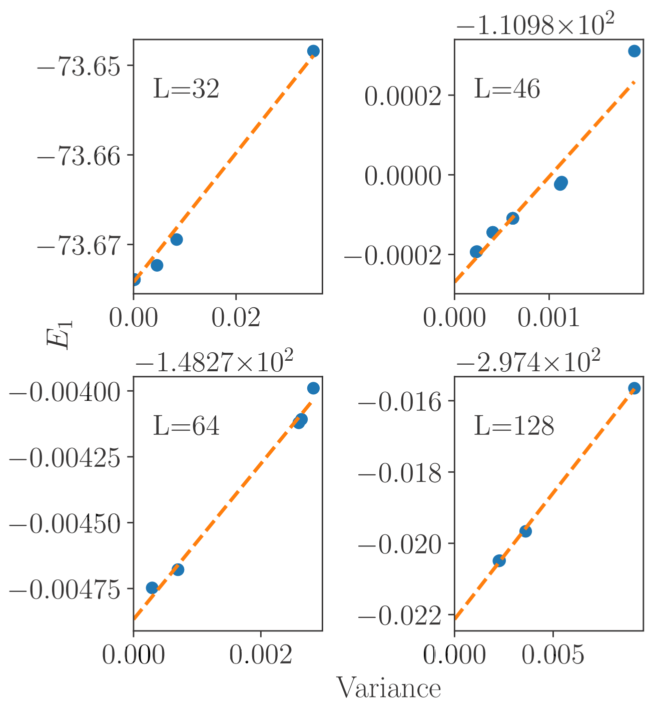


FIG. 8. Variance extrapolation of the first excited state ($E_1$) for various system sizes.

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