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# Differential polynomial rings in several variables over locally nilpotent rings

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We show that a differential polynomial ring over a locally nilpotent ring in several commuting variables is Behrens radical, extending a result by Chebotar.

*Keywords*: Behrens radical; differential polynomial ring; locally nilpotent ring.

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## 1. Introduction

Recall that a ring is called *Brown–McCoy radical* if it cannot be homomorphically mapped onto a simple ring with identity. Similarly, a ring is called *Behrens radical* if it cannot be homomorphically mapped onto a ring with a non-zero idempotent.

One of the equivalent statements of the Koethe problem is whether a polynomial ring over a nil ring is Jacobson radical [6]. Although many believe that the answer is negative, several positive approximations to a solution have been found. For instance, Puczyłowski and Smoktunowicz proved in 1998 that a polynomial ring over a nil ring is Brown–McCoy radical [8]. A few years later, Beidar *et al.* showed that a polynomial ring over a nil ring is Behrens radical [1]. It was not known for a long time whether a polynomial ring in several variables over a nil ring is Brown–McCoy radical. Then in 2018, the question was answered positively using techniques

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from Convex Geometry [4]. However, it is still unknown whether a polynomial ring in several variables over a nil ring is Behrens radical.

The study of differential polynomial rings over locally nilpotent rings is an active area of research [2, 3, 5, 7, 9, 10]. One of the most significant results occurred in 2014: Smoktunowicz and Ziembowski solved Shestakov's problem by proving that a differential polynomial ring over a locally nilpotent ring may not be Jacobson radical [10]. Furthermore, in a recent paper [5], Greenfeld *et al.* asked many questions about the properties of differential polynomial rings. Our result is related to one of their problems [5, Question 6.5]: does there exist a differential polynomial ring over a locally nilpotent ring that can be mapped onto a ring with a non-zero idempotent? This question was answered negatively by Chebotar [3]. We extend this result to differential polynomial rings in several commuting variables.

Let $\delta_1, \dots, \delta_p : R \to R$ be derivations of a ring $R$. The differential polynomial ring $R[X_1, \dots, X_p; \delta_1, \dots, \delta_p]$ is defined such that for all $r \in R$ and $1 \le j \le p$, $X_j r = rX_j + \delta_j(r)$. Our first result is the following:

**Theorem 1.** *Let $\delta_1, \delta_2, \dots, \delta_p$ be derivations of a locally nilpotent ring $R$. Then the differential polynomial ring $R[X_1, \dots, X_p; \delta_1, \dots, \delta_p]$ in commuting variables $X_1, \dots, X_p$ cannot be mapped onto a ring with a non-zero idempotent.*

Before stating our next result, we recall that a derivation $\delta$ over a ring $R$ is called *locally nilpotent* if for every $r \in R$, there exists a positive integer $n$ such that $\delta^n(r) = 0$.

**Theorem 2.** *Let $\delta$ be a derivation of a locally nilpotent ring $R$, and let $d$ be a derivation of $R[X;\delta]$ such that $d(R) \subseteq R$, $d\restriction_R$ is locally nilpotent, and $d^n(aX) - Xd^n(a) \in R$ for all $a \in R$ and positive integers $n$. Then the ring $R[X;\delta][Y;d]$ cannot be mapped onto a ring with a non-zero idempotent.*

**Remark 3.** Observe that any derivation $d$ such that $d(R) = 0$ and $d(RX) \subseteq R$ satisfies the conditions of Theorem 2. In particular, the derivative $d(p(X)) = \frac{d}{dX}(p(X))$ satisfies these conditions.

Furthermore, note that for any fixed $r \in R$, the inner derivation $d(p(X)) = [p(X), r]$ satisfies the conditions of Theorem 2.

We conclude this section with a question: Are the technical conditions on the derivation $d$ in Theorem 2 necessary?

## 2. Proofs

We follow Chebotar's approach in [3] to prove Theorem 1. Given elements $e$ and $x$ of a ring $R$, we define $[e,x]_0 = e$, $[e,x]_1 = [e,x] = ex - xe$, and $[e,x]_k = [[e,x]_{k-1}, x]$ for $k > 1$. Given elements $x_1, \dots, x_p \in R$ and non-negative integers $k_1, \dots, k_p$, we denote by $[e, \bar{x}]_{k_1,\dots,k_p}$ the expression $[\dots[e, x_1]_{k_1}, \dots, x_p]_{k_p}$ and denote by $\bar{x}^{k_1,\dots,k_p}$

the expression $x_1^{k_1} \dots x_p^{k_p}$. Our first lemma is a folklore result related to the general Leibniz rule:

**Lemma 4.** *Let $e, x_1, \dots, x_p$ be elements of a ring $R$ and $n_1, \dots, n_p$ be non-negative integers. Then*

$$e\bar{x}^{n_1,\dots,n_p} = \sum_{i_1=0}^{n_1} \cdots \sum_{i_p=0}^{n_p} \binom{n_1}{i_1} \cdots \binom{n_p}{i_p} \bar{x}^{i_1,\dots,i_p}[e,\bar{x}]_{n_1-i_1,\dots,n_p-i_p}.$$

Another useful result is the following:

**Lemma 5.** *Let $e, x_1, \dots, x_p$ be elements of a ring $R$ with $e^2 = e$. Then for any non-negative integers $k_1, \dots, k_p$, we have*

$$[e,\bar{x}]_{k_1,\dots,k_p} = \sum_{i_1=0}^{k_1} \cdots \sum_{i_p=0}^{k_p} r_{i_1,\dots,i_p} e[e,\bar{x}]_{i_1,\dots,i_p}$$

*for some $r_{i_1,\dots,i_p} \in R$.*

**Proof.** We use induction on $p \geq 1$. Note that when $p = 1$, the result follows from [3, Lemma 4].

Suppose the result holds for $p = l - 1$, and consider the case of $l$ variables. We induct on $k_l \geq 0$. Observe that the case $k_l = 0$ reduces to the case $p = l - 1$.

Suppose the statement holds for all $k_l < m$, where $m$ is some positive integer. Then

$$\begin{aligned}[][[e,\bar{x}]_{k_1,\dots,k_{l-1}}, x_l]_m &= \sum_{i_1=0}^{k_1} \cdots \sum_{i_{l-1}=0}^{k_{l-1}} [r_{i_1,\dots,i_{l-1}} e[e,\bar{x}]_{i_1,\dots,i_{l-1}}, x_l]_m \\ &= \sum_{i_1=0}^{k_1} \cdots \sum_{i_{l-1}=0}^{k_{l-1}} \sum_{i_l=0}^{m} \binom{m}{i_l} [r_{i_1,\dots,i_{l-1}} e, x_l]_{i_l} \; [[e,\bar{x}]_{i_1,\dots,i_{l-1}}, x_l]_{m-i_l}\end{aligned}$$

by the Leibniz rule. Note that for $i_l > 0$, we can write $[[e,\bar{x}]_{i_1,\dots,i_{l-1}}, x_l]_{m-i_l}$ in the desired form by our inductive hypothesis. When $i_l = 0$,

$$\begin{aligned}[][r_{i_1,\dots,i_{l-1}} e, x_l]_{i_l} \; [[e,\bar{x}]_{i_1,\dots,i_{l-1}}, x_l]_{m-i_l} &= [r_{i_1,\dots,i_{l-1}} e, x_l]_0 \; [[e,\bar{x}]_{i_1,\dots,i_{l-1}}, x_l]_m \\ &= r_{i_1,\dots,i_{l-1}} e[[e,\bar{x}]_{i_1,\dots,i_{l-1}}, x_l]_m,\end{aligned}$$

which is also of the desired form. Thus, the result follows. □

Let $\mathrm{End}_K(V)$ be the $K$-algebra of linear transformations of the $K$-vector space $V$. Our main lemma for our proof of Theorem 1 is the following:

**Lemma 6.** *Let $N$ be a locally nilpotent subalgebra of $\mathrm{End}_K(V)$. Suppose that $e = \sum_{i_1=0}^{n_1} \cdots \sum_{i_p=0}^{n_p} \bar{x}^{i_1,\dots,i_p} a_{i_1,\dots,i_p}$ is an idempotent of $\mathrm{End}_K(V)$ such that $x_1, \dots, x_p \in \mathrm{End}_K(V)$ are commuting endomorphisms and $[a_{i_1,\dots,i_p}, \bar{x}]_{k_1,\dots,k_p} \in N$, where $0 \leq i_j, k_j \leq n_j$ for all $1 \leq j \leq p$. Then $e = 0$.*

**Proof.** Let $S$ be the subalgebra of $N$ generated by elements of the form

$$[a_{i_1,\ldots,i_p}, \bar{x}]_{k_1,\ldots,k_p},$$

where $0 \le i_j, k_j \le n_j$ for all $1 \le j \le p$. Because $N$ is locally nilpotent and $S$ is a finitely generated subalgebra of $N$, $S$ is nilpotent. Then we can find subspaces $0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_M = V$ such that $S(V_l) = V_{l-1}$ for all $1 \le l \le M$.

We will show that if $k_j \le n_j$ for all $1 \le j \le p$, $e[e, \bar{x}]_{k_1,\ldots,k_p}(V_l) = 0$ for all $1 \le l \le M$.

We use induction on $l \ge 1$. Consider the case $l = 1$. Then we have

$$\begin{aligned} e[e, \bar{x}]_{k_1,\ldots,k_p}(V_1) &= e \sum_{i_1=0}^{n_1} \cdots \sum_{i_p=0}^{n_p} [\bar{x}^{i_1,\ldots,i_p} a_{i_1,\ldots,i_p}, \bar{x}]_{k_1,\ldots,k_p}(V_1) \\ &= e \sum_{i_1=0}^{n_1} \cdots \sum_{i_p=0}^{n_p} \bar{x}^{i_1,\ldots,i_p} [a_{i_1,\ldots,i_p}, \bar{x}]_{k_1,\ldots,k_p}(V_1) \\ &= 0 \end{aligned}$$

since $[a_{i_1,\ldots,i_p}, \bar{x}]_{k_1,\ldots,k_p} \in S$ and $S(V_1) = 0$.

Suppose $e[e, \bar{x}]_{k_1,\ldots,k_p}(V_{m-1}) = 0$, and consider any $v \in V_m$. We have

$$\begin{aligned} e[e, \bar{x}]_{k_1,\ldots,k_p}(v) &= e \sum_{i_1=0}^{n_1} \cdots \sum_{i_p=0}^{n_p} [\bar{x}^{i_1,\ldots,i_p} a_{i_1,\ldots,i_p}, \bar{x}]_{k_1,\ldots,k_p}(v) \\ &= \sum_{i_1=0}^{n_1} \cdots \sum_{i_p=0}^{n_p} e\bar{x}^{i_1,\ldots,i_p} [a_{i_1,\ldots,i_p}, \bar{x}]_{k_1,\ldots,k_p}(v) \\ &= \sum_{i_1=0}^{n_1} \cdots \sum_{i_p=0}^{n_p} e\bar{x}^{i_1,\ldots,i_p} (u_{i_1,\ldots,i_p}), \end{aligned}$$

where each $u_{i_1,\ldots,i_p} \in V_{m-1}$. By Lemma 4,

$$\begin{aligned} & e[e, \bar{x}]_{k_1,\ldots,k_p}(v) \\ &\quad = \sum_{i_1=0}^{n_1} \cdots \sum_{i_p=0}^{n_p} e\bar{x}^{i_1,\ldots,i_p} (u_{i_1,\ldots,i_p}) \\ &\quad = \sum_{i_1=0}^{n_1} \cdots \sum_{i_p=0}^{n_p} \sum_{i_1'=0}^{i_1} \cdots \sum_{i_p'=0}^{i_p} \binom{i_1}{i_1'} \cdots \binom{i_p}{i_p'} \bar{x}^{i_1',\ldots,i_p'} [e, \bar{x}]_{i_1-i_1',\ldots,i_p-i_p'}(u_{i_1,\ldots,i_p}). \end{aligned}$$

Using Lemma 5, we can rewrite each $[e, \bar{x}]_{i_1-i_1',\ldots,i_p-i_p'}(u_{i_1,\ldots,i_p})$ in the form $\sum_{i_1''=0}^{i_1-i_1'} \cdots \sum_{i_p''=0}^{i_p-i_p'} r_{i_1'',\ldots,i_p''} e[e, \bar{x}]_{i_1'',\ldots,i_p''}(u_{i_1,\ldots,i_p})$, where each $r_{i_1'',\ldots,i_p''} \in \mathrm{End}_K(V)$. By the inductive hypothesis, each of these terms is 0, so $e[e, \bar{x}]_{k_1,\ldots,k_p}(v) = 0$, as desired.

Thus, $e(V) = e(V_M) = 0$, so we conclude that $e = 0$. □

**Proof of Theorem 1.** To prove Theorem 1 we follow the proof of [3, Theorem 1], using several variables instead of a single variable and replacing Lemmas 3, 4, and 5 with our Lemmas 4, 5, and 6, respectively.

Suppose there is a locally nilpotent ring $R$ with derivations $\delta_1, \delta_2, \dots, \delta_p$ such that the differential polynomial ring $R[X_1, \dots, X_p; \delta_1, \dots, \delta_p]$ in commuting variables $X_1, \dots, X_p$ can be mapped onto a ring with a non-zero idempotent. Then there exists a surjective homomorphism $\varphi$ from $R[X_1, \dots, X_p; \delta_1, \dots, \delta_p]$ onto a subdirectly irreducible ring $A$ such that there is a non-zero idempotent $e$ in the heart of $A$. Note that $A$ must be a prime ring whose extended centroid $K$ is a field. Let $Q$ be the Martindale right ring of quotients of $A$.

For $j \in \{1, \dots, p\}$, let $x_j : A \to A$ be the map given by $x_j(\varphi(t)) := \varphi(X_j t)$ for all $t \in R[X_1, \dots, X_p; \delta_1, \dots, \delta_p]$. We claim that each $x_j$ is a well-defined map. Suppose $\varphi(t) = 0$ and $\varphi(X_j t) \neq 0$. Since $A$ is prime, there must be $t' \in R[X_1, \dots, X_p; \delta_1, \dots, \delta_p]$ such that $\varphi(t')\varphi(X_j t) \neq 0$. We also have

$$\begin{aligned}\varphi(t')\varphi(X_j t) &= \varphi(t' X_j t)\\ &= \varphi(t' X_j)\varphi(t)\\ &= 0,\end{aligned}$$

which is a contradiction.

Note that each $x_j : A_A \to A_A$ is an endomorphism of a right $A$-module $A_A$, so each $x_j$ is in $Q$. Let the subring of $Q$ generated by $A$ and $x_1, \dots, x_p$ be denoted by $A'$, and let the ring obtained by adjoining unity to $R$ be denoted by $R^{\#}$. Let $\psi : R^{\#}[X_1, \dots, X_p; \delta_1, \dots, \delta_p] \to A'$ be an additive map such that $\psi(X_j^i) = x_j^i$ for any $j \in \{1, \dots, p\}$ and any non-negative integer $i$, and such that $\psi(t) = \varphi(t)$ for all $t \in R[X_1, \dots, X_p; \delta_1, \dots, \delta_p]$. Note that $\psi$ is a homomorphism extending $\varphi$. We can write a non-zero idempotent $e \in A \subseteq A'$ as

$$\begin{aligned} e &= \varphi\left(\sum_{i_1=0}^{n_1} \cdots \sum_{i_p=0}^{n_p} X_1^{i_1} \dots X_p^{i_p} r_{i_1,\dots,i_p}\right)\\ &= \psi\left(\sum_{i_1=0}^{n_1} \cdots \sum_{i_p=0}^{n_p} X_1^{i_1} \dots X_p^{i_p} r_{i_1,\dots,i_p}\right)\\ &= \sum_{i_1=0}^{n_1} \cdots \sum_{i_p=0}^{n_p} x_1^{i_1} \dots x_p^{i_p} a_{i_1,\dots,i_p}\\ &= \sum_{i_1=0}^{n_1} \cdots \sum_{i_p=0}^{n_p} \bar{x}^{i_1,\dots,i_p} a_{i_1,\dots,i_p},\end{aligned}$$

where $n_1, \dots, n_p$ are non-negative integers, $r_{i_1,\dots,i_p} \in R$, and $\psi(r_{i_1,\dots,i_p}) = a_{i_1,\dots,i_p}$.

Let $D$ be the subring of $A'$ generated by $x_1, \dots, x_p$ and all $a_{i_1,\dots,i_p}$ with $0 \leq i_j \leq n_j$. Let $B = D \cap \psi(R)$. Note that $B$ and the subalgebra $BK$ of $Q$ are locally nilpotent. The subalgebra $DK$ of $A'K$ is finitely generated, so it can be embedded into $\mathrm{End}_K(V)$ for some $K$-vector space $V$. Then we can assume that $x_j \in \mathrm{End}_K(V)$ for $j \in \{1, \ldots, p\}$ and that these $x_j$ commute with each other. We have that $N = BK \subseteq \mathrm{End}_K(V)$ is locally nilpotent and that $e = \sum_{i_1=0}^{n_1} \cdots \sum_{i_p=0}^{n_p} \bar{x}^{i_1,\ldots,i_p} a_{i_1,\ldots,i_p} \in \mathrm{End}_K(V)$ is a non-zero idempotent. By applying Lemma 6, we conclude that $e = 0$, which is a contradiction. Therefore, we have proved the result. □

Before we prove our next theorem, we need an auxiliary result.

**Proposition 7.** *Let $\delta$ be a locally nilpotent derivation of a locally nilpotent ring $R$, and let $d$ be a derivation of $R[X;\delta]$. Then the ring $R[X;\delta][Y;d]$ cannot be mapped onto a ring with a non-zero idempotent.*

**Proof.** Observe that since $R$ is a locally nilpotent ring and $\delta$ is a locally nilpotent derivation of $R$, the differential polynomial ring $R[X;\delta]$ is locally nilpotent. Therefore, by [3, Theorem 1], $R[X;\delta][Y;d]$ cannot be mapped onto a ring with a non-zero idempotent. □

**Proof of Theorem 2.** Note that for any positive integer $m$ and any $a \in R$, we have

$$\begin{aligned}[][Y^m a, X] &= Y^m aX - XY^m a \\ &= \sum_{i=0}^{m} \binom{m}{i} d^i(aX) Y^{m-i} - X \sum_{i=0}^{m} \binom{m}{i} d^i(a) Y^{m-i} \\ &= \sum_{i=0}^{m} \binom{m}{i} (d^i(aX) - Xd^i(a)) Y^{m-i}.\end{aligned}$$

By assumption, $d^i(aX) - Xd^i(a) \in R$ for $1 \leq i \leq m$, and we know

$$d^0(aX) - Xd^0(a) = -\delta(a) \in R.$$

Observe that $d\restriction_R$ is a derivation on $R$ since $d(R) \subseteq R$, so $[Y^m a, X]$ is an element of the differential polynomial ring $R[Y; d\restriction_R]$.

Define the map $\delta' : R[Y; d\restriction_R] \to R[Y; d\restriction_R]$ by $\delta'(p(Y)) = -[p(Y), X]$ for all $p(Y) \in R[Y; d\restriction_R]$. Note that $\delta'$ is a derivation and that $\delta'(a) = \delta(a)$ for all $a \in R$. Then $R[X;\delta][Y;d] = R[Y; d\restriction_R][X;\delta']$. Since $d\restriction_R$ is locally nilpotent, the result follows from Proposition 7. □

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