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# On maps preserving Lie products equal to a rank-one nilpotent

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Let $\phi$ be a bijective linear map on the algebra of $n \times n$ complex matrices such that $\phi(e_{12}) = e_{12}$ and $[\phi(A), \phi(B)] = e_{12}$ whenever $[A, B] = e_{12}$. The purpose of this paper is to describe $\phi$. Surprisingly, $\phi$ has a different description from maps preserving zero Lie products.


## 1. Introduction

Let $\mathcal{A}$ be an algebra over a field $F$. It becomes a Lie algebra if we introduce the Lie product $[a, b]$ by $[a, b] = ab - ba$, $a, b \in \mathcal{A}$. Let $\mathcal{B}$ be another algebra over $F$. A map

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$\alpha : \mathcal{A} \to \mathcal{B}$ preserves zero Lie products if $[\alpha(a), \alpha(b)] = 0$ whenever $[a, b] = 0$. Equivalently, $\alpha$ preserves commutativity.

The first paper on commutativity preserving maps was published in 1976 by Watkins [10], who described bijective linear commutativity preserving maps on matrix algebras. Let $M_n(F)$ denote the algebra of matrices over $F$ and $I_n$ be its $n \times n$ identity matrix. It was shown under some mild technical restrictions that every commutativity preserving bijective linear map $L : M_n(F) \to M_n(F)$ is either of the form

$$L(X) = cS^{-1}XS + f(X)I_n, \quad X \in M_n(F),$$

or

$$L(X) = cS^{-1}X^TS + f(X)I_n, \quad X \in M_n(F),$$

where $c$ is a scalar, $S \in M_n(F)$ is an invertible matrix, $f$ is a linear functional on $M_n(F)$, and $X^T$ denotes the transpose of $X$. Maps of these forms are traditionally called standard commutativity preserving maps. For the case with general algebras $\mathcal{A}$ and $\mathcal{B}$, these forms can be stated in terms of automorphisms and anti-automorphisms.

Further developments went in two directions: analytic and algebraic. We will mention only two important results here. First, Omladič [8] extended Watkins's result to the infinite-dimensional case. He described bijective linear maps preserving commutativity in both directions on the algebra of bounded linear operators on an infinite-dimensional Banach space.

Second, Brešar [2] described bijective commutativity preserving additive maps on prime rings under some technical restrictions. This description was the key to his famous solution of Herstein's problem on Lie isomorphisms of prime rings. We refer the reader to Šemrl [9] for additional interesting results in the area of commutativity preservers.

In this paper, we will consider a seemingly similar problem. Let $M_n(\mathbb{C})$ denote the algebra of $n \times n$ complex matrices and $e_{ij}$ denote the matrix with 1 in the $(i, j)$-entry and zeros elsewhere. Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a bijective linear map such that

$$\phi(e_{12}) = e_{12} \text{ and } [\phi(A), \phi(B)] = e_{12} \text{ whenever } [A, B] = e_{12}, \tag{1.1}$$

where $A, B \in M_n(\mathbb{C})$ (this condition is referred to as "Property (1.1)" in the upcoming discussion). The purpose of the paper is to obtain a complete description of $\phi$.

Recently, similar questions were considered in the case of ordinary products [4] and Jordan products [5]. In both cases, the maps were shown to be of the standard form.

To our surprise, this is not the case in our situation. However, the description is still "nice" with the exception of one entry.

**Theorem 1.** *If $n \geq 5$ and $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a bijective linear map such that $\phi(e_{12}) = e_{12}$ and $[\phi(A), \phi(B)] = e_{12}$ whenever $[A, B] = e_{12}$, then there exists an invertible matrix $P$ and a linear functional $f$ on $M_n(\mathbb{C})$ such that, for $(i, j) \neq (2, 1)$,*

(1) $\phi(e_{ij}) = \quad P^{-1}e_{ij}P + f(e_{ij})I_n$, *or*
(2) $\phi(e_{ij}) = -P^{-1}e_{ji}P + f(e_{ij})I_n$.

*In either case,* $\phi(e_{21}) = xe_{21} + X$, *where* $x \in \mathbb{C}$ *is nonzero and* $X \in M_n(\mathbb{C})$ *has a zero* $(2,1)$-*entry.*

The assumption that $n \geq 5$ is imposed because the restriction of a map satisfying Property (1.1) on a subalgebra isomorphic to $M_{n-2}(\mathbb{C})$ will be a commutativity preserving map (see Theorem 6 below).

The fact that $\phi(e_{21})$ completely avoids description stems from the following key observation, which we present here with proof. Note that the symbol $(h_{ij})$ represents an $n \times n$ matrix with entries $h_{ij} \in \mathbb{C}, 1 \leq i, j, \leq n$.

**Theorem 2.** *If* $A = (a_{ij})$ *and* $B = (b_{ij})$ *are matrices such that* $[A, B] = e_{12}$, *then* $a_{21} = b_{21} = 0$.

**Proof.** Suppose for the sake of contradiction that there are matrices $A, B \in M_n(\mathbb{C})$ such that $[A, B] = e_{12}$ and $a_{21} \neq 0$ (resp. $b_{21} \neq 0$). Since rank$([A, B]) = 1$, it follows from Theorem 1 in Guralnick [6] (or Theorem 1.4 in Laffey [7]) that $A$ and $B$ are simultaneously triangularizable; that is, there exists an invertible matrix $P \in M_n(\mathbb{C})$ such that $PAP^{-1}$ and $PBP^{-1}$ are upper-triangular matrices. However, since tr$(A[A, B]) = a_{21} \neq 0$ (resp. tr$(B[A, B]) = b_{21} \neq 0$), it follows from Theorem 3.1 in Bourgeois [1] that $A$ and $B$ are not simultaneously triangularizable, a contradiction. □

**Example.** Maps satisfying Property (1.1) need not preserve commutativity. Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear map that acts as the identity map on all $e_{ij} \in M_n(\mathbb{C})$ except $e_{21}$, and let $\phi(e_{21}) = e_{21} + e_{12}$. Clearly such a map is bijective and satisfies Property (1.1), and is of the form (1) from Theorem 1. However, $[e_{21}, e_{23}] = 0$ while $[\phi(e_{21}), \phi(e_{23})] = e_{13}$.

We believe Theorem 1 is of interest for two reasons. First, we have not encountered a linear preserver problem whose solutions resemble those of the main theorem. Second, this shows that maps preserving nonzero Lie products cannot be expected *a priori* to have the same descriptions as maps preserving zero Lie products.

## 2. Preliminary results

Fix an $A \in M_n(\mathbb{C})$. Let $C(A)$ denote the set $\{X \in M_n(\mathbb{C}) : [A, X] = 0\}$. This set is called the centralizer of $A$. In Watkins [10], the author passes the commutativity preserver problem to the rank-one preserver problem using a matrix theoretical result concerning centralizers. While the commutator relation in Property (1.1) prevents us from repeating the argument in full, some information regarding specific centralizer subalgebras can be deduced.

**Lemma 3.** *If $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a bijective linear map that satisfies Property (1.1), then*

(1) $[\phi(A), e_{12}] = 0$ *whenever* $[A, e_{12}] = 0$*,*
(2) $[\phi(A), \phi(e_{11})] = 0$ *whenever* $[A, e_{11}] = 0$*, and*
(3) $[\phi(A), \phi(e_{22})] = 0$ *whenever* $[A, e_{22}] = 0$*.*

*In other words,* $\phi(C(e_{12})) \subseteq C(e_{12})$*,* $\phi(C(e_{11})) \subseteq C(\phi(e_{11}))$*, and* $\phi(C(e_{22})) \subseteq C(\phi(e_{22}))$*.*

**Proof.** If $[A, e_{12}] = 0$, then $[e_{11} + A, e_{12}] = e_{12}$. By Property (1.1), $[\phi(e_{11} + A), \phi(e_{12})] = e_{12}$, and so $[\phi(e_{11}), \phi(e_{12})] + [\phi(A), \phi(e_{12})] = e_{12}$ by linearity. The first summand is equal to $e_{12}$, hence $[\phi(A), \phi(e_{12})] = 0$. Statements (2) and (3) follow from a similar analysis of the Lie products $[e_{11}, e_{12} - A] = e_{12}$ and $[A + e_{12}, e_{22}] = e_{12}$, respectively. □

The preceding result is simply a consequence of linearity. Performing more delicate computations generates relations among (almost all) matrix units, as follows.

**Lemma 4.** *If $n \geq 3$ and $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a bijective linear map that satisfies Property (1.1), then*

(1) $[\phi(e_{ii}), \phi(e_{jk})] = 0$ if $i \notin \{j, k\}$ and $(j, k) \neq (2, 1)$,
(2) $[\phi(e_{ii} + e_{jj}), \phi(e_{ij})] = 0$ if $(i, j) \neq (2, 1)$,
(3) $\phi(I_n) = \lambda I_n$ for some nonzero $\lambda \in \mathbb{C}$.

**Proof.** Suppose that $i \notin \{j, k\}$. Since $e_{jk}$ commutes with $e_{ii}$, the cases when $i = 1$ or $i = 2$ follow directly from statements (2) and (3) of Lemma 3. In particular, if $j = k$, then $[\phi(e_{ii}), \phi(e_{jj})] = 0$ whenever $i = 1, 2$ and $i \neq j$. So we may assume that $i \geq 3$. Since $[e_{11} + e_{1k}, e_{12} + e_{ii}] = e_{12}$, we have that $[\phi(e_{11} + e_{1k}), \phi(e_{12} + e_{ii})] = e_{12}$. But $e_{1k}$ commutes with $e_{12}$ and $e_{11}$ commutes with $e_{ii}$, so we conclude by Lemma 3 that $[\phi(e_{1k}), \phi(e_{ii})] = 0$. Hence (1) holds if $j = 1$. Likewise, since $[e_{ii} + e_{22}, e_{j1} - e_{12}] = e_{12}$ whenever $j \neq 2$, we conclude that $[\phi(e_{ii}), \phi(e_{j1})] = 0$. Hence (1) also holds if $k = 1$.

Assume now that $j \neq 1$ and $k \neq 1$. Since $[e_{11} + e_{ii}, e_{12} + e_{jk}] = e_{12}$, we have that $[\phi(e_{11} + e_{ii}), \phi(e_{12} + e_{jk})] = e_{12}$. But $e_{jk}$ commutes with $e_{11}$ and $e_{ii}$ commutes with $e_{12}$, so it must be that $[\phi(e_{ii}), \phi(e_{jk})] = 0$. The proof of (1) is complete.

Now we will show that $[\phi(e_{ii} + e_{jj}), \phi(e_{ij})] = 0$. If $i = j$, there is nothing to prove, so assume $i \neq j$. Suppose first that $i = 1$ and $j = 2$. Since $[\phi(e_{11}), \phi(e_{12})] = e_{12}$ and $[\phi(e_{22}), \phi(e_{12})] = -e_{12}$, the linearity of the Lie product shows that $[\phi(e_{11}+e_{22}), \phi(e_{12})] = 0$ directly. Suppose that $i = 1$ and $j \geq 3$. Since $[e_{11} + e_{jj}, e_{1j} + e_{12}] = e_{12}$, we have that $[\phi(e_{11} + e_{jj}), \phi(e_{1j}) + e_{12}] = e_{12}$. But $e_{jj}$ commutes with $e_{12}$, so we conclude that $[\phi(e_{11} + e_{jj}), \phi(e_{1j})] = 0$. Replacing $e_{1j}$ with $e_{j1}$ and repeating this argument also shows that $[\phi(e_{11} + e_{jj}), \phi(e_{j1})] = 0$.

Assume that $i = 2$ and $j \geq 3$. Since $[e_{22} + e_{jj}, e_{2j} - e_{12}] = e_{12}$, it follows from linearity that $[\phi(e_{22} + e_{jj}), \phi(e_{2j})] = 0$. An analogous argument also shows that $[\phi(e_{22} + e_{jj}), \phi(e_{j2})] = 0$.

Lastly, if both $i \geq 3$ and $j \geq 3$, observe that $[e_{11} + e_{ii} + e_{jj}, e_{12} + e_{ij}] = e_{12}$. Since $e_{ij}$ commutes with $e_{11}$ and $e_{ii}, e_{jj}$ commute with $e_{12}$, we have by Lemma 3 that $[\phi(e_{ii} + e_{jj}), \phi(e_{ij})] = 0$. Thus statement (2) holds for all pairs $(i, j)$ except $(2, 1)$.

Consider $\phi(e_{ij})$, where $(i, j) \neq (2, 1)$. From linearity and statements (1) and (2), it follows that $[\phi(I_n), \phi(e_{ij})] = 0$. From bijectivity, we conclude that $\dim C(\phi(I_n)) \geq n^2 - 1$. The first lemma in Watkins [10] states that if $\dim C(A) > n^2 - 2n + 2$, then $A$ is a scalar multiple of the identity. Thus $\phi(I_n) = \lambda I_n$ for some nonzero $\lambda \in \mathbb{C}$, proving statement (3). The proof of the lemma is complete. □

**Lemma 5.** *If $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a bijective linear map that satisfies Property (1.1), then there exist rank-one matrices $E_{11}, E_{22}$ and scalars $\lambda_{11}, \lambda_{22}$ such that $\phi(e_{11}) = E_{11} + \lambda_{11} I_n$ and $\phi(e_{22}) = E_{22} + \lambda_{22} I_n$. Furthermore, $E_{11}$ is a matrix consisting of precisely one row vector (resp. column vector) and $E_{22}$ is a matrix consisting of precisely one column vector (resp. row vector).*

**Proof.** The first part of the proof occurs as special cases of the first lemma appearing in Watkins [10], which is reproduced here. From our Lemma 3, the centralizers of $e_{11}$ and $e_{22}$ are mapped into the centralizers of $\phi(e_{11})$ and $\phi(e_{22})$, respectively. Since $\phi$ is bijective and the dimension of the subalgebra $C(e_{11})$ is $n^2 - 2n + 2$, then $n^2 - 2n + 2 \leq \dim C(\phi(e_{11}))$. If $n^2 - 2n + 2 < \dim C(\phi(e_{11}))$, then $\phi(e_{11})$ must be a scalar matrix. This is impossible since $\phi$ also maps the identity to a scalar matrix by Lemma 4, contradicting the injectivity of $\phi$. Hence $\dim C(\phi(e_{11})) = n^2 - 2n + 2$, and so $\phi(e_{11}) = E_{11} + \lambda_{11} I_n$ where $E_{11}$ is a rank-one matrix. An analogous argument shows that $\phi(e_{22}) = E_{22} + \lambda_{22} I_n$, where $E_{22}$ is a rank-one matrix.

The forms of $E_{11}$ and $E_{22}$ will now be determined. If $\phi(e_{11}) = (a_{ij})$ and $\phi(e_{22}) = (b_{ij})$, then $[\phi(e_{11}), e_{12}] = e_{12}$ and $[e_{12}, \phi(e_{22})] = e_{12}$ by Property (1.1), so

$$\phi(e_{11}) = \begin{pmatrix} 1 + a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{11} & 0 & \cdots & 0 \\ 0 & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix},$$

$$\phi(e_{22}) = \begin{pmatrix} b_{22} & b_{12} & b_{13} & \cdots & b_{1n} \\ 0 & 1 + b_{22} & 0 & \cdots & 0 \\ 0 & b_{32} & b_{33} & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & b_{n2} & b_{n3} & \cdots & b_{nn} \end{pmatrix}.$$

By the previous remarks, we have that

$$E_{11} = \begin{pmatrix} 1+a_{11}-\lambda_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{11}-\lambda_{11} & 0 & \cdots & 0 \\ 0 & a_{32} & a_{33}-\lambda_{11} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & a_{n3} & \cdots & a_{nn}-\lambda_{11} \end{pmatrix}$$

and

$$E_{22} = \begin{pmatrix} b_{22}-\lambda_{22} & b_{12} & b_{13} & \cdots & b_{1n} \\ 0 & 1+b_{22}-\lambda_{22} & 0 & \cdots & 0 \\ 0 & b_{32} & b_{33}-\lambda_{22} & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & b_{n2} & b_{n3} & \cdots & b_{nn}-\lambda_{22} \end{pmatrix}.$$

However, since $E_{11}$ and $E_{22}$ are rank-one matrices, then either

$$E_{11} = \begin{pmatrix} 1 & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \text{or} \quad E_{11} = \begin{pmatrix} 0 & a_{12} & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & a_{32} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & 0 & \cdots & 0 \end{pmatrix} \tag{2.1}$$

and either

$$E_{22} = \begin{pmatrix} 0 & b_{12} & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & b_{32} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & b_{n2} & 0 & \cdots & 0 \end{pmatrix}, \quad \text{or} \quad E_{22} = \begin{pmatrix} -1 & b_{12} & b_{13} & \cdots & b_{1n} \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{2.2}$$

But $[\phi(e_{11}), \phi(e_{22})] = [E_{11}, E_{22}] = 0$, so if $E_{11}$ and $E_{22}$ are both row matrices, then $[E_{11}, E_{22}] = 0$ implies that $E_{22}$ is a multiple of $E_{11}$, which contradicts bijectivity of $\sigma_1$. Hence if $E_{11}$ is the row (resp. column) matrix in equation (2.1), $E_{22}$ must be the column (resp. row) matrix in equation (2.2). □

The calculations performed thus far have illustrated much of the structure of the map on the key elements $e_{11}, e_{22}$, and $e_{12}$. These matrix units have nontrivial multiplication with the first two rows and columns of matrices in $M_n(\mathbb{C})$, but have trivial multiplication with the subalgebra generated by matrices of the form

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & v_{33} & \cdots & v_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & v_{n3} & \cdots & v_{nn} \end{pmatrix}.$$

Therefore, in what follows, we compose several linear transformations with $\phi$ to pass to a map $\sigma_3$ that fixes $e_{11}, e_{22}$, and $e_{12}$ and satisfies Property (1.1). This ensures that $\sigma_3$, which will arise from strategically changing bases and adding scalar multiples of the identity, does not destroy the structure of $\phi$. However, the new map does preserve commutativity on the above subalgebra isomorphic to $M_{n-2}(\mathbb{C})$. This means a complete description is obtainable for the restriction of $\sigma_3$ (and therefore for $\phi$ as well, after reversing the composition).

**Theorem 6.** *Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a bijective linear map that satisfies Property (1.1) and let $\mathcal{V} \subseteq M_n(\mathbb{C})$ be the subalgebra of matrices of the form $(v_{ij})$ with $v_{ij} = 0$ whenever $i \leq 2$ or $j \leq 2$. The restriction of $\phi$ to $\mathcal{V}$ is a standard commutativity preserving map; that is, the restriction of $\phi$ to $\mathcal{V}$ is either of the form*

(1) $\phi(A) = dU^{-1}AU + h(A)I_{\mathcal{V}}$, *or*
(2) $\phi(A) = dU^{-1}A^TU + h(A)I_{\mathcal{V}}$

*where $d$ is a scalar, $U \in M_n(\mathbb{C})$ an invertible matrix, $h$ a linear functional on $M_n(\mathbb{C})$, and $I_{\mathcal{V}} = e_{33} + e_{44} + \cdots + e_{nn}$.*

**Proof.** Let $\sigma_1 : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be the linear map defined by $\sigma_1(A) = \phi(A) - g_1(A)I_n$, where $g_1$ is a linear functional on $M_n(\mathbb{C})$ chosen so that $\sigma_1$ is bijective with $g_1(e_{11}) = \lambda_{11}$, $g_1(e_{22}) = \lambda_{22}$, and $g_1(e_{12}) = 0$ (see Lemma 5 above); in other words,

$$\sigma_1(e_{11}) = E_{11}, \quad \sigma_1(e_{22}) = E_{22}, \quad \sigma_1(e_{12}) = e_{12},$$

respectively. It also follows immediately that $[\sigma_1(A), \sigma_1(B)] = [\phi(A), \phi(B)]$ for all $A, B \in M_n(\mathbb{C})$, so $\sigma_1$ certainly satisfies Property (1.1). We may also insist that $\sigma_1(I_n) = I_n$ by choosing $g_1(I_n) = \lambda - 1$ for $\lambda$ as in part (3) of Lemma 4.

Lemma 5 presents two possible definitions for the map $\sigma_1$ based on the forms of $E_{11}$ and $E_{22}$ Ideally, one would like to pass to a map that fixes $e_{11}, e_{22}$, and $e_{12}$ directly, which can be done as follows.

If $E_{11}$ is the row matrix in equation (2.1) and $E_{22}$ is the column matrix in equation (2.2), the condition $[E_{11}, E_{22}] = 0$ implies that $a_{12} + b_{12} + \sum_{k=3}^{n} a_{1k}b_{k2} = 0$ for the matrix entries. Using this relation, there is an invertible matrix $U_1$ such that

$$U_1E_{11}U_1^{-1} = e_{11}, \quad U_1E_{22}U_1^{-1} = e_{22} \quad \text{and} \quad U_1e_{12}U_1^{-1} = e_{12}, \tag{2.3}$$

given by

$$U_1 = \begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -b_{32} & 1 & 0 & \cdots & 0 \\ 0 & -b_{42} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -b_{n2} & 0 & 0 & \cdots & 1 \end{pmatrix},$$

$$U_1^{-1} = \begin{pmatrix} 1 & b_{12} & -a_{13} & -a_{14} & \cdots & -a_{1n} \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & b_{32} & 1 & 0 & \cdots & 0 \\ 0 & b_{42} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & b_{n2} & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

If $E_{11}$ is the column matrix in equation (2.1) and $E_{22}$ is the row matrix in equation (2.2), the condition $[E_{11}, E_{22}] = 0$ implies that $a_{12} + b_{12} - \sum_{k=3}^{n} b_{1k}a_{k2} = 0$. Using this relation, there is an invertible matrix $U_2$ such that

$$-U_2E_{11}U_2^{-1} = e_{11}, \quad -U_2E_{22}U_2^{-1} = e_{22}, \quad \text{and} \quad -U_2e_{12}U_2^{-1} = e_{21} \tag{2.4}$$

given by

$$U_2 = \begin{pmatrix} 0 & -1 & 0 & 0 & \cdots & 0 \\ 1 & -b_{12} & -b_{13} & -b_{14} & \cdots & -b_{1n} \\ 0 & a_{32} & 1 & 0 & \cdots & 0 \\ 0 & a_{42} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & 0 & 0 & \cdots & 1 \end{pmatrix},$$

$$U_2^{-1} = \begin{pmatrix} a_{12} & 1 & b_{13} & b_{14} & \cdots & b_{1n} \\ -1 & 0 & 0 & 0 & \cdots & 0 \\ a_{32} & 0 & 1 & 0 & \cdots & 0 \\ a_{42} & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n2} & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

If $U_3 = (U_2^{-1})^T$, taking the transpose through equation (2.4) yields

$$-U_3(E_{11})^TU_3^{-1} = e_{11}, \quad -U_3(E_{22})^TU_3^{-1} = e_{22}, \quad \text{and} \quad -U_3(e_{12})^TU_3^{-1} = e_{12}. \tag{2.5}$$

Therefore the automorphism $\sigma_2$ defined by $A \mapsto U_1\sigma_1(A)U_1^{-1}$ or the negative anti-automorphism $\sigma_2$ defined by $A \mapsto -U_3\sigma_1(A)^TU_3^{-1}$ satisfies

$$\sigma_2(e_{11}) = e_{11}, \quad \sigma_2(e_{22}) = e_{22}, \quad \text{and} \quad \sigma_2(e_{12}) = e_{12}. \tag{2.6}$$

With either determination of $E_{11}$ and $E_{22}$, equation (2.6) guarantees that $\sigma_2$ satisfies Property (1.1); in particular, that $\sigma_2$ satisfies the centralizer conclusion of Lemma 3. However, since $\sigma_2$ fixes the matrix units $e_{11}, e_{22}$, and $e_{12}$, the map $\sigma_2$ must preserve the respective centralizers as subalgebras. This means that if $A \in \mathcal{V}$, then $[\sigma_2(A), e_{11}] = [\sigma_2(A), e_{22}] = [\sigma_2(A), e_{12}] = 0$, which shows an inclusion $\sigma_2(\mathcal{V}) \subseteq C(e_{11}) \cap C(e_{22}) \cap C(e_{12})$. But $C(e_{11}) \cap C(e_{22}) \cap C(e_{12})$ is precisely the set $\mathcal{V} + \mathbb{C}I_n$. It is therefore possible to define a bijective linear map $\sigma_3 : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by

$$\sigma_3(A) = \sigma_2(A) - g_3(A)I_n,$$

where $g_3$ is a linear functional on $M_n(\mathbb{C})$ chosen so that $\sigma_3$ is bijective, fixes $e_{11}, e_{22}, e_{12}$, $\sigma_3(\mathcal{V}) \subseteq \mathcal{V}$, and $\sigma_3(I_n) = I_n$. With this definition, $\sigma_3$ has the property that $[\sigma_3(A), \sigma_3(B)] = [\sigma_2(A), \sigma_2(B)]$ for all $A, B \in M_n(\mathbb{C})$, so certainly $\sigma_3$ satisfies Property (1.1).

Choose $A, B \in \mathcal{V}$ such that $[A, B] = 0$. Since $[e_{11} + A, e_{12} + B] = e_{12}$, a direct application of Property (1.1) and linearity yields that $[\sigma_3(A), \sigma_3(B)] = 0$. Thus $\sigma_3$ is a commutativity preserver on a subalgebra $\mathcal{V}$ (which is isomorphic to $M_{n-2}(\mathbb{C})$, where $n-2 \geq 3$). By the first theorem of Watkins [10] (the $n-2 = 3$ case is handled by Brešar, Šemrl [3]), the restriction of $\sigma_3$ to $\mathcal{V}$ is a standard commutativity preserving map of the form

$$\sigma_3(A) = cS^{-1}AS + f(A)I_{\mathcal{V}}$$

or

$$\sigma_3(A) = cS^{-1}A^TS + f(A)I_{\mathcal{V}},$$

where $c$ is a nonzero scalar, $S \in \mathcal{V}$ is an invertible matrix, $f$ is a linear functional on $\mathcal{V}$, and $I_{\mathcal{V}}$ is the matrix $e_{33} + e_{44} + \cdots + e_{nn}$ (which acts as the identity on the subalgebra $\mathcal{V}$). With a complete description of the restriction of $\sigma_3$ to $\mathcal{V}$, the complete description of the restrictions of $\sigma_2, \sigma_1$, and $\phi$ to $\mathcal{V}$ as standard commutativity preserving maps is obtained by applying the inverse maps. The proof is complete. □

## 3. Proof of the main result

While the proof of Theorem 1 will technically be a continuation of the argument of Theorem 6, another technical lemma will be helpful.

**Lemma 7.** *Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a bijective linear map that satisfies Property (1.1), with $n \geq 4$. If $i \neq l$ and $j \neq k$, then $\phi$ has the following properties:*

(1) $[\phi(e_{ij}), \phi(e_{kl})] = 0$ for $i \neq 2$ and $l \neq 1$.
(2) $[\phi(e_{2j}), \phi(e_{2l})] = 0$ for $j, l \geq 3$.

(3) $[\phi(e_{i1}), \phi(e_{k1})] = 0$ for $i, k \geq 3$.

**Proof.** Note that in (1), the cases when $i = j$ or $k = l$ are covered by Lemma 4. We may therefore assume without loss of generality that $i \neq j$ and $k \neq l$.

Suppose $i \neq 2$ and $l \neq 1$. We prove (1) first under a stronger restriction that $1 \notin \{j, k\}$. Since $[e_{11} + e_{ij}, e_{12} + e_{kl}] = e_{12}$, by linearity we have that $[\phi(e_{11}), \phi(e_{12})] + [\phi(e_{11}), \phi(e_{kl})] + [\phi(e_{ij}), \phi(e_{12})] + [\phi(e_{ij}), \phi(e_{kl})] = e_{12}$. The first summand is equal to $e_{12}$ by hypothesis. Notice $[\phi(e_{11}), \phi(e_{kl})] = [\phi(e_{ij}), \phi(e_{12})] = 0$ by Lemma 3. We conclude that $[\phi(e_{ij}), \phi(e_{kl})] = 0$.

In fact, the assumption that $k \neq 1$ and $j \neq 1$ can be removed. Indeed, suppose $k = 1$. Since $i \neq l$ and $j \neq 1$ is assumed, the Lie product $[e_{ij} - e_{l2}, e_{1l}]$ is equal to $e_{12}$. By hypothesis and linearity, we conclude that $[\phi(e_{ij}), \phi(e_{1l})] = 0$.

Now suppose $j = 1$. Consider the Lie product $[e_{i1} + e_{1l}, e_{kl} + e_{l2}]$. Since $i \neq 2$ and $l \neq 1$, this Lie product is equal to $e_{12}$. Because $[e_{i1} + e_{1l}, e_{l2}] = [e_{1l}, e_{kl} + e_{l2}] = e_{12}$, we have that $[\phi(e_{i1}), \phi(e_{l2})] = [\phi(e_{1l}), \phi(e_{kl})] = 0$. It follows from linearity and the above that $[\phi(e_{i1}), \phi(e_{kl})] = 0$. The proof of (1) is complete.

We now show that (2) holds. Consider the equation $[\lambda e_{2j} + e_{12} - \lambda e_{1l}, \lambda e_{2l} - e_{11}] = e_{12}$ for $\lambda \in \mathbb{C}$. By linearity, we have

$$\lambda^2[\phi(e_{2j}), \phi(e_{2l})] - \lambda^2[\phi(e_{2l}), \phi(e_{1l})] + \lambda[\phi(e_{12}), \phi(e_{2l})] \\ + \lambda[\phi(e_{1l}), \phi(e_{11})] - \lambda[\phi(e_{11}), \phi(e_{2j})] + [\phi(e_{11}), \phi(e_{12})] = e_{12}.$$

By hypothesis, $[\phi(e_{11}), \phi(e_{12})] = e_{12}$. By Lemma 4 we can see that $[\phi(e_{11}), \phi(e_{2j})] = 0$. We also have that $[\phi(e_{1l}), \phi(e_{2l})] = 0$ as a special case of statement (1). Finally, adding together the two equations obtained by taking $\lambda = 1$ and $\lambda = -1$ yields that $[\phi(e_{2j}), \phi(e_{2l})] = 0$.

The proof of (3) uses the same technique. Consider the Lie product $[\lambda e_{i1} + e_{12} + \lambda e_{k2}, \lambda e_{k1} + e_{22}] = e_{12}$ for $\lambda \in \mathbb{C}$. As in (2), by adding equations for $\lambda = 1$ and $\lambda = -1$, we conclude that $[\phi(e_{i1}), \phi(e_{k1})] = 0$. □

**Remark 8.** Theorem 2 gives some explanation as to why $\phi$ is generally not a standard commutativity preserver map on $M_n(\mathbb{C})$. However, $\phi$ does have the standard form on an $(n^2 - 1)$-dimensional subspace of $M_n(\mathbb{C})$ and this is the best we can hope to achieve. We are now in a position to prove Theorem 1, but the proof is long and full of many details to check. It involves passing from $\sigma_4$ to $\sigma_6$ with several more computations in the intermediate steps. It suffices to show that $\sigma_6$ fixes all matrix units except $e_{21}$, and can be accomplished in three steps:

Step 1: Define $\sigma_4$ so that $e_{11}, e_{22}$, and $e_{12}$ are fixed, $\sigma_4$ acts as the identity on $\mathcal{V}$, and that $\sigma_4$ sends matrix units to multiples of themselves, up to the addition of a scalar matrix.

Step 2: Define $\sigma_5$ to subtract away the scalar matrix obtained in Step 1, so that $\sigma_5$ sends matrix units to themselves with a nonzero coefficient, and describe the relationship among all such coefficients.

Step 3: Use the coefficients to define $\sigma_6$ so that $\sigma_6(e_{ij}) = e_{ij}$ whenever $(i, j) \neq (2, 1)$.

Then the successive compositions $\phi \to \sigma_1 \to \sigma_2 \to \cdots \to \sigma_6$ can be reversed to obtain the final form of $\phi$.

**Proof of Theorem 1.** Recall the definition of $\sigma_3 : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ as in the proof of Theorem 6. The ultimate goal is to show that $\phi$ is either an automorphism or negative anti-automorphism, up to the addition of a scalar matrix. We may assume without loss of generality that the restriction of $\sigma_3$ to $\mathcal{V}$ is of the form $\sigma_3(A) = cS^{-1}AS + f(A)I_{\mathcal{V}}$, since the negative anti-automorphism case can still be obtained even under this assumption.

For all $A \in M_n(\mathbb{C})$, define $\sigma_4 : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ to be the bijective linear map given by

$$\sigma_4(A) = Q\sigma_3(A)Q^{-1},$$

where $Q = e_{11} + e_{22} + S$, which is clearly invertible because $S$ is. Since $\sigma_3$ fixes $e_{11}, e_{22}$ and $e_{12}$, so does $\sigma_4$. In order to show that $\sigma_4$ has the remaining desirable properties in Remark 8, the linear functional $f : \mathcal{V} \to \mathbb{C}$ must be identically zero and $c$ must be equal to 1.

For convenience, denote $\sigma := \sigma_4$. Let $j \geq 3$ and consider $\sigma(e_{1j}) = (a_{ij})$. Since $\sigma(e_{1j})$ commutes with $e_{22}$ and $e_{12}$, it follows that

$$\sigma(e_{1j}) = \begin{pmatrix} a_{11} & 0 & a_{13} & a_{14} & \cdots & a_{1n} \\ 0 & a_{11} & 0 & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & a_{34} & \cdots & a_{3n} \\ 0 & 0 & a_{43} & a_{44} & \cdots & a_{4n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & a_{n3} & a_{n4} & \cdots & a_{nn} \end{pmatrix}. \tag{3.1}$$

For any $p, q \geq 3$ with $j \neq p$, Lemma 4 and Lemma 7 imply that

$$[\sigma(e_{1j}), \sigma(e_{pq})] = [\sigma(e_{1j}), ce_{pq} + f(e_{pq})I_{\mathcal{V}}] = 0. \tag{3.2}$$

Suppose first that $p = q$. Using the form given by equation (3.1), expanding equation (3.2) reveals that the non-diagonal entries of $\sigma(e_{1j})$ in the $p$th row and $p$th column, except for those entries in the first row, are identically zero. This means $\sigma(e_{1j})$ is of the form

$$\sigma(e_{1j}) = \begin{pmatrix} a_{11} & 0 & a_{13} & a_{14} & \cdots & a_{1n} \\ 0 & a_{11} & 0 & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & 0 & \cdots & 0 \\ 0 & 0 & 0 & a_{44} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix}. \tag{3.3}$$

The centralizer of $e_{1j}$ is generated by the set $\{e_{kl} : j \neq k, l \neq 1\} \cup \{e_{11} + e_{jj}\}$ and has dimension $n^2 - 2n + 2$ over $\mathbb{C}$. Note that $e_{21}$ is not contained in the centralizer of $e_{1j}$; in other words, if $e_{kl}$ commutes with $e_{1j}$, then $\sigma(e_{kl})$ commutes with $\sigma(e_{1j})$ by Lemma 7. We have $\sigma(C(e_{1j})) \subseteq C(\sigma(e_{1j}))$ and, arguing as in Lemma 5, $\dim C(\sigma(e_{1j})) = n^2 - 2n + 2$. It follows that $\sigma(e_{1j}) = E_{1j} + \lambda_{1j} I_n$, where $\lambda_{1j}$ is a scalar and $E_{1j}$ is a rank-one matrix. This implies that $a_{11} = a_{33} = \cdots = a_{nn} = \lambda_{1j}$ and

$$E_{1j} = \begin{pmatrix} 0 & 0 & a_{13} & \cdots & a_{1n} \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Returning to equation (3.2), we now have

$$[E_{1j}, ce_{pq} + f(e_{pq}) I_{\mathcal{V}}] = 0. \tag{3.4}$$

Expanding the Lie product produces the system of equations

$$ca_{1p} + f(e_{pq}) a_{1q} = 0, \qquad f(e_{pq}) a_{1k} = 0, \quad \text{for } k \neq q. \tag{3.5}$$

Take $q = j$. If $a_{1j} = 0$, then the first equation implies that $a_{1p} = 0$ for all $p \neq j$. In this case, $E_{1j} = 0$, a contradiction. Take $k = j$. Since $n \geq 5$, we may choose $q \neq j$, and consider the second equation in (3.5), for if $a_{1j} \neq 0$, then $f(e_{pq}) = 0$. In the first equation, this implies that $a_{1p} = 0$ for all $p \neq j$. Hence $E_{1j} = a_{1j} e_{1j}$, and so

$$\sigma(e_{1j}) = a_{1j} e_{1j} + a_{11} I_n, \qquad a_{1j} \neq 0. \tag{3.6}$$

Let $k \geq 3$ and consider $\sigma(e_{k2}) = (b_{ij})$. Since $e_{21}$ also does not commute with $e_{k2}$, it must be that $\dim C(\sigma(e_{k2})) = n^2 - 2n + 2$. Employing the same constructions as above but adjusted for the second column, we conclude that

$$\sigma(e_{k2}) = b_{k2} e_{k2} + b_{11} I_n, \qquad b_{k2} \neq 0. \tag{3.7}$$

Since $a_{1j} \neq 0$ as above, it was shown that $f(e_{pq})$ was identically zero for all indices $p$ and $q$ such that $p \neq j$ and $q \neq j$. If $j' \geq 3$ is another index distinct to $j$, it follows that $f(e_{pq}) = 0$ for all indices $p$ and $q$ such that $p \neq j'$ and $q \neq j'$. Proceeding in this way across the row, it follows that the linear functional $f$ on $\mathcal{V}$ must be identically zero. Furthermore, since $\sigma(I_n) = I_n$ and $e_{11}, e_{22}$ are fixed by $\sigma$, we have $\sigma(I_{\mathcal{V}}) = I_{\mathcal{V}}$. Hence the calculation

$$\sigma(I_{\mathcal{V}}) = \sum_{p \geq 3} \sigma(e_{pp}) = cI_{\mathcal{V}}$$

shows that $c = 1$, so $\sigma$ acts as the identity on $\mathcal{V}$, as desired.

There are subtle differences in handling $\sigma(e_{2l})$ and $\sigma(e_{i1})$ because $e_{21}$ does commute with $e_{2l}$ and $e_{i1}$ for $i, l \geq 3$. We show the form of $\sigma(e_{2l}) = (c_{ij})$ for $l \geq 3$ directly.

First, for all $j \geq 3$, Lemma 7 implies that $[\sigma(e_{2l}), \sigma(e_{1j})] = 0$. By equation (3.6), this reduces to $[\sigma(e_{2l}), e_{1j}] = 0$. From this calculation, it follows readily that $c_{11} = c_{jj}$ for all $j \neq 2$. Also by the above, the fact that $\sigma$ fixes every diagonal matrix unit implies that every non-diagonal entry of $\sigma(e_{2l})$ except for the $(2, l)$-entry is zero. Therefore, $\sigma(e_{2l})$ is of the form

$$\sigma(e_{2l}) = c_{2l}e_{2l} + c_{11}(e_{11} + e_{33} + \cdots + e_{nn}) + c_{22}e_{22}. \tag{3.8}$$

We expect that $c_{11} = c_{22}$ and that $c_{2l} \neq 0$, but it requires justification. Consider also the matrices $\sigma(e_{2l'}) = (c'_{ij})$ and $\sigma(e_{2l''}) = (c''_{ij})$, with $l, l', l'' \geq 3$ all distinct. Each of these matrices is also of the form of equation (3.8). Assume that $c_{2l} = c'_{2l'} = c''_{2l''} = 0$; i.e., that $\sigma(e_{2l}), \sigma(e_{2l'}), \sigma(e_{2l''})$ are diagonal matrices in which only the (2,2)-entry differs from the rest in each matrix. Notice that $\{e_{2l}, e_{2l'}, e_{2l''}\}$ forms a basis of a 3-dimensional subspace of $M_n(\mathbb{C})$. However, $\{\sigma(e_{2l}), \sigma(e_{2l'}), \sigma(e_{2l''})\}$ under this assumption forms the basis of a 2-dimensional subspace, which contradicts bijectivity. Without loss of generality, we may conclude that $c_{2l} \neq 0$.

Now, by Lemma 7, we have that $[\sigma(e_{2l}), \sigma(e_{2l'})] = 0$. Since $c_{2l} \neq 0$, computing this Lie product shows that $c'_{11} = c'_{22} = c'_{33} = \cdots = c'_{nn}$ (and that $c_{11} = c_{22} = c_{33} = \cdots = c_{nn}$ when $c'_{2l'} \neq 0$). Now, if $c'_{2l'} = 0$, then $\sigma(e_{2l'})$ is a scalar matrix, which contradicts bijectivity (note that $\sigma$ fixes the identity matrix because $\sigma_3$ does). So $c'_{2l'} \neq 0$ as well, and the same argument shows that $c''_{2l''} \neq 0$.

Since $c'_{2l'} \neq 0$, it follows that $c_{11} = c_{22}$, as desired. Hence for $l \geq 3$,

$$\sigma(e_{2l}) = c_{2l}e_{2l} + c_{11}I_n, \qquad c_{2l} \neq 0. \tag{3.9}$$

We can also show that $\sigma(e_{i1}) = (d_{ij})$ for all $i \geq 3$ has the analogous form using the same argument, modified appropriately for the first column instead of the second row. Therefore, we have that for all $i \geq 2$,

$$\sigma(e_{i1}) = d_{i1}e_{i1} + d_{11}I_n, \qquad d_{i1} \neq 0. \tag{3.10}$$

At this point, Step 1 has been completed. Step 2 involves describing the nonzero coefficients appearing in equations (3.6) through (3.10).

Let $v \geq 3$ be fixed. For the two matrix units $e_{1v}$ and $e_{2v}$, let $\sigma(e_{1v}) = (a_{ij})$ and $\sigma(e_{2v}) = (c_{ij})$. Consider the expression $[e_{22} + e_{1v} + e_{2v}, e_{11} - e_{12}] = e_{12}$. By linearity and our hypothesis, we have that

$$\begin{aligned}[\sigma(e_{22}), \sigma(e_{11})] - [\sigma(e_{22}), \sigma(e_{12})] + [\sigma(e_{1v}), \sigma(e_{11})] \\ - [\sigma(e_{1v}), \sigma(e_{12})] + [\sigma(e_{2v}), \sigma(e_{11})] - [\sigma(e_{2v}), \sigma(e_{12})] = e_{12}.\end{aligned} \tag{3.11}$$

The first, fourth, and fifth summands are equal to zero by Lemma 3. The second summand is equal to $e_{12}$ by hypothesis, which simplifies equation (3.11) to:

$$[\sigma(e_{1v}), \sigma(e_{11})] - [\sigma(e_{2v}), \sigma(e_{12})] = 0.$$

Computing these Lie products shows that the first summand is equal to $-a_{1v}e_{v1}$, and the second is equal to $-c_{2v}e_{v1}$, so we conclude that $a_{1v} = c_{2v}$. The coefficients are equal across the columns in the first and second rows.

Let $u \geq 3$ be fixed. For the two matrix units $e_{u1}$ and $e_{u2}$, let $\sigma(e_{u2}) = (b_{ij})$ and $\sigma(e_{u1}) = (d_{ij})$. Observe that $[e_{12} + e_{u1} + e_{u2}, e_{22} - e_{12}] = e_{12}$, so

$$\begin{aligned}[\sigma(e_{12}), \sigma(e_{22})] - [\sigma(e_{12}), \sigma(e_{12})] + [\sigma(e_{u1}), \sigma(e_{22})] \\ - [\sigma(e_{u1}), \sigma(e_{12})] + [\sigma(e_{u2}), \sigma(e_{22})] - [\sigma(e_{u2}), \sigma(e_{12})] = e_{12}.\end{aligned} \tag{3.12}$$

The first summand is equal to $e_{12}$ and the second summand is trivially zero. The third and sixth summands are equal to zero by Lemma 3. This simplifies to

$$[\sigma(e_{u2}), \sigma(e_{22})] - [\sigma(e_{u1}), \sigma(e_{12})] = 0.$$

Using the definition of $\sigma$ appearing in equation (3.7) and equation (3.10), the first summand is equal to $b_{u2}e_{u2}$ and the second is equal to $d_{u1}e_{u2}$. Thus $d_{u1} = b_{u2}$, so the coefficients are equal across the rows in the first and second columns.

Now take $u = v$. Let $\sigma(e_{1u}) = (a_{ij}), \sigma(e_{u2}) = (b_{ij}), \sigma(e_{2u}) = (c_{ij})$, and $\sigma(e_{u1}) = (d_{ij})$. By hypothesis, $[\sigma(e_{1u}), \sigma(e_{u2})] = e_{12}$ implies that $a_{1u}b_{u2} = 1$ directly. But by the previous analysis, we have that $a_{1u} = c_{2u}$ and $b_{u2} = d_{u1}$. We conclude that

$$a_{1u} = c_{2u} = b_{u2}^{-1} = d_{u1}^{-1}. \tag{3.13}$$

The previous equation (3.13) says that the four nonzero coefficients sharing an index $u$ satisfy a certain inverse relationship. But we can say something stronger; in fact, every matrix $\sigma(e_{ij})$ with $i \leq 2$ and $j \geq 3$ has the same nonzero coefficient and every matrix $\sigma(e_{ij})$ with $i \geq 3$ and $j \leq 2$ has the same inverse coefficient to the above. Indeed, let $r, s \geq 3, r \neq s$. Let $\sigma(e_{1r}) = (a_{ij})$ and $\sigma(e_{1s}) = (x_{ij})$. Consider the expression $[e_{12} + e_{1r} - e_{1s}, e_{rs} + e_{ss} + e_{22}] = e_{12}$. By linearity and Property (1.1), we have that

$$\begin{aligned}[]
[\sigma(e_{12}), \sigma(e_{rs})] + [\sigma(e_{12}), \sigma(e_{ss})] + [\sigma(e_{12}), \sigma(e_{22})] + [\sigma(e_{1r}), \sigma(e_{rs})] + [\sigma(e_{1r}), \sigma(e_{ss})] \\
+ [\sigma(e_{1r}), \sigma(e_{22})] - [\sigma(e_{1s}), \sigma(e_{rs})] - [\sigma(e_{1s}), \sigma(e_{ss})] - [\sigma(e_{1s}), \sigma(e_{22})] \\
= e_{12}.
\end{aligned} \tag{3.14}$$

The third summand is equal to $e_{12}$ by hypothesis. The first, second, sixth, and ninth summands are equal to zero by Lemma 3, while the fifth and seventh summands are also equal to zero by Lemma 4, which simplifies equation (3.14) to

$$[\sigma(e_{1r}), \sigma(e_{rs})] - [\sigma(e_{1s}), \sigma(e_{ss})] = 0.$$

The first summand is equal to $a_{1r}e_{1s}$ while the second summand is equal to $x_{1s}e_{1s}$. It follows that $a_{1r} = x_{1s}$. Once the nonzero coefficients are identical across rows in the first two rows, by equation (3.13), this allows all nonzero coefficients to be determined. For instance, if $\sigma_4(e_{13}) = (a_{ij})$, denote $\beta := a_{13}$. Every nonzero coefficient of $\sigma_4(e_{ij})$ for $i \leq 2$ and $j \geq 3$ is $\beta$, while every nonzero coefficient of $\sigma_4(e_{ij})$ for $i \geq 3$ and $j \leq 2$ is $\beta^{-1}$. This motivates the definition of $\sigma_5$ as follows.

Let $g_5$ be a linear functional on $M_n(\mathbb{C})$ that is identically zero on $e_{11}, e_{22}$, and $e_{12}$. In addition, for all other $e_{ij}, (i,j) \neq (2,1)$, define $g_5$ to be the (1,1)-entry of $\sigma_4(e_{ij})$. In particular, $g_5$ vanishes on $\mathcal{V}$. Define

$$\sigma_5(e_{ij}) = \sigma_4(e_{ij}) - g_5(e_{ij})I_n.$$

Clearly this map is bijective, linear, and satisfies Property (1.1). In addition, for $(i,j) \neq (2,1)$ $\sigma_5$ has

$$\sigma_5(e_{ij}) = \begin{cases} \beta e_{ij}, & i \leq 2, j \geq 3 \\ \beta^{-1} e_{ij}, & i \geq 3, j \leq 2 \\ e_{ij}, & i,j \geq 3 \end{cases} \tag{3.15}$$

Now, Step 2 is complete. Proceeding to Step 3, we can now easily and confidently transform $\sigma_5$ into the identity map with a final change of basis. Let $\sigma_6 : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a bijective linear map defined by $\sigma_6(e_{ij}) = C^{-1}\sigma_5(e_{ij})C$, where $C = \beta(e_{11} + e_{22}) + e_{33} + \cdots + e_{nn}$. Then by equation (3.15), it follows that $\sigma_6(e_{ij}) = e_{ij}$ for $(i,j) \neq (2,1)$.

Since the sequential composition $\phi \to \sigma_1 \to \sigma_2 \to \cdots \to \sigma_6$ is bijective at every step, we may reverse the composition so that $\phi$ has the form (1) or (2) appearing in the statement of Theorem 1. The negative anti-automorphism case (2) is covered by the definition of $\sigma_2$ (see equation (2.3) and equation (2.5)). Finally, while $\phi(e_{21})$ cannot be determined, it must be the case that $\phi(e_{21}) = xe_{21} + X$, where $x \neq 0$ and the (2,1)-entry of $X$ is zero; otherwise, $\phi$ would fail to be bijective. The proof is complete. □

## Declaration of competing interest

There is no competing interest.

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## References

[1] Gerald Bourgeois, Pairs of matrices, one of which commutes with their commutator, Electron. J. Linear Algebra 22 (2011) 593–597.
[2] Matej Brešar, Commuting traces of biadditive mappings, commutativity-preserving mappings and Lie mappings, Trans. Am. Math. Soc. 335 (2) (1993) 525–546.
[3] Matej Brešar, Peter Šemrl, On bilinear maps on matrices with applications to commutativity preservers, J. Algebra 301 (2) (2006) 803–837.
[4] Louisa Catalano, Megan Chang-Lee, On maps preserving rank-one nilpotents, Linear Multilinear Algebra (2020), https://doi.org/10.1080/03081087.2019.1707753, in press.
[5] Louisa Catalano, Samuel Hsu, Regan Kapalko, On maps preserving products of matrices, Linear Algebra Appl. 563 (2019) 193–206.
[6] Robert M. Guralnick, A note on pairs of matrices with rank one commutator, Linear Multilinear Algebra 8 (2) (1979/1980) 97–99.
[7] Thomas J. Laffey, Simultaneous triangularization of matrices—low rank cases and the nonderogatory case, Linear Multilinear Algebra 6 (4) (1978/1979) 269–305.
[8] Matjaž Omladič, On operators preserving commutativity, J. Funct. Anal. 66 (1) (1986) 105–122.
[9] Peter Šemrl, Commutativity preserving maps, Linear Algebra Appl. 429 (5–6) (2008) 1051–1070.
[10] William Watkins, Linear maps that preserve commuting pairs of matrices, Linear Algebra Appl. 14 (1) (1976) 29–35.