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On maps preserving rank-one nilpotents

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ABSTRACT
In this paper, we show that a linear, bijective map $f : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ such that $f(X)f(Y) = N$ whenever $XY = M$, for M, N rank-one nilpotents, is of the form $f(X) = cU^{-1}XU$ for an invertible $U \in M_n(\mathbb{C})$ and a non-zero $c \in \mathbb{C}$. We extend this result to show in general that, given any M and N in the ring, $\mathcal{M}$, of $\mathbb{N} \times \mathbb{N}$ infinite matrices over a field $\mathbb{F}$ with finitely many non-zero entries, a linear, bijective map $f : \mathcal{M} \to \mathcal{M}$, such that $f(X)f(Y) = N$ whenever $XY = M$ must satisfy $f(XY) = cf(X)f(Y)$ for all $X, Y \in \mathcal{M}$ and $c \in \mathbb{F}$.

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1. Introduction

Let $\mathcal{R}$ be a ring. We say that the map $f : \mathcal{R} \to \mathcal{R}$ preserves zero products if, for all $x, y \in R$, $xy = 0$ implies $f(x)f(y) = 0$. Maps that preserve zero products have been well studied; for example, in [1–4], one can find characterizations of bijective, linear, zero product preserving maps on group algebras, von Neumann algebras, prime algebras, and matrices over division rings. Each of these results comes to the same conclusion: the map must be the product of a central element and a homomorphism. This description is known as the standard form.

From this field of study, two main branches have emerged. The first is zero product determined algebras, initiated in [5]. The other branch involves maps that preserve certain products (e.g. the Jordan product, $x \circ y = xy + yx$) or certain distinguished elements (other than zero), and this is the main focus of this paper.

A natural extension of zero product preserving maps are maps that preserve the (ordinary) product of non-zero elements; that is, a map $f : \mathcal{R} \to \mathcal{R}$ such that, for some fixed $a, b \in \mathcal{R}$, $f(x)f(y) = b$ whenever $xy = a$. Chebotar et al. studied this for $a = 1$ and $\mathcal{R} = \mathcal{D}$, a division ring [6]. Lin and Wong generalized this result to $\mathcal{R} = M_n(\mathcal{D})$ [7]. Each of these results describes the map as a standard solution. The first author was able to expand

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the result, studying the case when $a = k$ is invertible and $\mathcal{R} = \mathcal{D}$ [8]. However, in this case, the form of the map is slightly different from the standard solution; in particular, it was shown that $f(x) = \lambda\varphi(x)$, where φ is a homomorphism, but λ is not necessarily a central element. This was then generalized to $\mathcal{R} = M_n(\mathcal{D})$ by the first author, Hsu, and Kapalko [9]. Finally, the case where a, b are rank-one idempotents in the ring $\mathcal{R} = M_n(\mathcal{C})$ was studied by the first author. This result found that f is, again, a standard solution [10].

Let us say a few words on products other than the ordinary product. In [9], Catalano et al. classified maps preserving the Jordan product (i.e. maps such that $f(x) \circ f(y) = m$ whenever $x \circ y = k$ for some fixed m and k), and in this case, the map behaves in same way as zero Jordan product preserving maps. On the other hand, a very surprising result comes from considering maps that preserve the Lie product, $[x, y] = xy - yx$ for all $x, y \in \mathcal{R}$. In [11], Ginsburg et al. described the maps such that $[f(x), f(y)] = m$ whenever $[x, y] = k$ and proved in the case when $m = k$ is a rank-one nilpotent that the description is different from that of the zero Lie product preserving maps.

Taking into account the result by [11], it is now not obvious that ordinary product preserving maps will be of the standard form that appears in the zero product preserving case. We will show in this paper that the analogous case of a rank-one nilpotent product preserving map will indeed be of the standard form.

Theorem 1.1: *Let $n \geq 4$ and fix $N, M \in M_n(\mathbb{C})$ rank-one nilpotent matrices. Let $f : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear, bijective map such that*

$$f(X)f(Y) = N \text{ whenever } XY = M. \tag{1}$$

Then $f(X) = cU^{-1}XU$ for some invertible $U \in M_n(\mathbb{C})$ and non-zero $c \in \mathbb{C}$.

We can in fact extend this result using the same proof technique to show that equal product preserving maps of infinite matrices with finitely many non-zero coefficients can be classified.

Theorem 1.2: *Let $\mathcal{M}$ denote the ring of $\mathbb{N} \times \mathbb{N}$ infinite matrices over a field $\mathbb{F}$ with finitely many non-zero entries. Let $M, N \in \mathcal{M}$ be any matrices. Let $f : \mathcal{M} \to \mathcal{M}$ be a linear, bijective map such that*

$$f(X)f(Y) = N \text{ whenever } XY = M. \tag{2}$$

Then $f(XY) = cf(X)f(Y)$ for all $X, Y \in \mathcal{M}$ and non-zero $c \in \mathbb{F}$.

2. Proof of Theorem 1.1

We define e_{ij} to be the matrix unit with 1 in the (i, j)-entry and 0 elsewhere. We note that any rank-one nilpotent matrix is similar to the matrix e_{12}; that is, we may write $CMC^{-1} = e_{12} = DND^{-1}$ for some invertible matrices $C, D \in M_n(\mathbb{C})$. In particular, if we let $X' = CXC^{-1}$ for all $X \in M_n(\mathbb{C})$ and define $f'(X') = Df(X)D^{-1}$ (which is clearly a bijective linear map), then property (1) is equivalent to the property that $X'Y' = e_{12}$ implies

$$f'(X')f'(Y') = e_{12}.$$

However, we note that f is of the form $cU^{-1}XU$ if and only if f' is of the same form. Therefore, it suffices to prove the result for any bijective linear map f satisfying

$$f(X)f(Y) = e_{12} \text{ whenever } XY = e_{12}. \tag{3}$$

Let $\mathcal{X}$ be the set of all matrices with all rows zero except (possibly) one, and let $\mathcal{Y}$ be the set of all matrices with all columns zero except (possibly) one. Furthermore, let $\mathcal{X}_r$ be a subset of $\mathcal{X}$ such that each element in $\mathcal{X}_r$ has at most r non-zero entries, and analogously, let $\mathcal{Y}_r$ be a subset of $\mathcal{Y}$ such that each element of $\mathcal{Y}_r$ has at most r non-zero entries.

By a zero-r pair, we denote a pair $(X_r, Y_r) \in \mathcal{X}_r \times \mathcal{Y}_r$ such that $X_rY_r = 0$. Additionally, we say that f preserves zero products on zero-r pairs if

$$f(X_r)f(Y_r) = 0$$

for all zero-r pairs (X_r, Y_r).

Lemma 2.1: *If $X_rY = 0$ for $X_r \in \mathcal{X}_r$, $Y \in \mathcal{Y}, 2 < r \leq n$, then there exist $X_2 \in \mathcal{X}_2$ and $X_{r-1} \in \mathcal{X}_{r-1}$ such that $X_2Y = 0$, $X_{r-1}Y = 0$, and $X_r = X_2 + X_{r-1}$.*

Additionally, if $XY_r = 0$ for $X \in \mathcal{X}$, $Y_r \in \mathcal{Y}_r$, $2 < r \leq m$, then there exist $Y_2 \in \mathcal{Y}_2$ and $Y_{r-1} \in \mathcal{Y}_{r-1}$ such that $XY_2 = 0$, $XY_{r-1} = 0$, and $Y_r = Y_2 + Y_{r-1}$.

Proof: Let X_r be the row vector with r entries $x_1, \ldots, x_r$ and zeros elsewhere. Let $Y = (y_1 \ldots y_n)^T$. We can assume that all the x_i are non-zero, since r is arbitrary, so if any x_i is zero, we can reduce to the case of $r-1$ and disregard the entry x_i.

We assume x_1 is the first non-zero entry of X_r. Suppose first that $y_1 = 0$. We can take $X_2 = (x_1\ 0\ \ldots\ 0)$ and $X_{r-1} = X_r - X_2$ as our decomposition.

In the case when $y_1 \neq 0$, we can assume there exist some non-zero x_j and y_j, since this would otherwise contradict $X_rY = 0$. Let $X_2 = (x_1\ 0\ \ldots\ 0\ x_j'\ 0\ \ldots 0)$ be a vector with only 1st and jth entry non-zero, and let x_j' be such that $x_1y_1 + x_j'y_j = 0$. Then we can take $X_{r-1} = X_r - X_2$ as our decomposition.

The second statement can be proved identically. ■

Corollary 2.1: *Let $X_rY_r = 0$ for $X_r \in \mathcal{X}_r$, $Y \in \mathcal{Y}_r$, $2 \leq r \leq n$. Then*

(1) *there exist $X_{2_1}, \ldots, X_{2_k} \in \mathcal{X}_2$ such that $X_r = X_{2_1} + \cdots + X_{2_k}$ and $X_{2_i}Y_r = 0$ for all i.*
(2) *there exist $Y_{2_{i1}}, \ldots, Y_{2_{il_i}} \in \mathcal{Y}_2$ (depending on X_{2_i}) such that $Y_r = Y_{2_{i1}} + \cdots + Y_{2_{il_i}}$ and $X_{2_i}Y_{2_{ij}} = 0$ for all i, j.*
(3) *$X_rY_r = X_{2_1}Y_{2_{11}} + \cdots + X_{2_1}Y_{2_{1l_1}} + \cdots + X_{2_k}Y_{2_{k1}} + \cdots + X_{2_k}Y_{2_{kl_k}}$, and each $(X_{2_i}, Y_{2_{ij}})$ is a zero-2 pair.*

Proof: Using Lemma 2.1, we can write $X_r = X_{2_1} + X_{r-1}$, where $X_{2_1} \in \mathcal{X}_2$ and $X_{r-1} \in \mathcal{X}_{r-1}$. Furthermore, we know $X_{2_1}Y_r = 0$ and $X_{r-1}Y_r = 0$. Repeating this process at most $r-3$ additional times, we can find $X_{2_2}, \ldots, X_{2_k}$ where $X_r = X_{2_1} + \cdots + X_{2_k}, X_{2_i} \in \mathcal{X}_2$, and $X_{2_i}Y_r = 0$. Thus, we have proved (1).

We will now consider $X_{2_i}Y_r$ for $i \in \{1, \ldots, k\}$. As before, we may use Lemma 2.1 and induction to get that $Y_r = Y_{2_{i1}} + \cdots + Y_{2_{il_i}}$ where $Y_{2_{ij}} \in \mathcal{Y}_2$ and $X_{2_i}Y_{2_{ij}} = 0$; and so we have proved (2).

The last statement of the corollary follows directly from parts (1) and (2). ■

Corollary 2.2: *If f preserves zero products on zero-2 pairs, then f preserves zero products.*

Proof: Assume $f(X_2)f(Y_2) = 0$ for any zero-2 pair (X_2, Y_2). Suppose then (X_r, Y_r) is a zero-r pair. By Corollary 2.1, we may write

$$\begin{aligned} f(X_r)f(Y_r) &= f(X_{2_1} + \cdots + X_{2_k})f(Y_r) \\ &= \sum_{i=1}^{k} f(X_{2_i})f(Y_r) \\ &= \sum_{i=1}^{k} f(X_{2_i})f(Y_{2_{i1}} + \cdots + Y_{2_{il_i}}) \\ &= \sum_{i=1}^{k}\sum_{j=1}^{l_i} f(X_{2_i})f(Y_{2_{ij}}), \end{aligned}$$

where $(X_{2_i}, Y_{2_{ij}})$ is a zero-2 pair for all i, j. Then, since f preserves zero products on zero-2 pairs, we have $f(X_r)f(Y_r) = 0$.

We can pick $r = n$, which gives that f preserves zero products on the product of a single non-zero row with a single non-zero column. Since any matrix can be written as the sum of its rows (or alternatively, columns), it is straightforward to see that f preserving the zero product on the product of rows and columns will imply that f preserves the zero product in general. Thus, the lemma is proven. ■

We can then formulate a more general theorem on linear maps from $M_n(\mathbb{C})$ to itself.

Lemma 2.2: *If $f : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a linear, bijective map, $n \geq 4$, and f satisfies* (3), *then f preserves zero products on zero-2 pairs.*

Proof: Let (X_2, Y_2) be a zero-2 pair. Since X_2 is a row vector, the first index of the matrix units will be the same, so we may rewrite $X_2 = c_1e_{ij} + c_2e_{il}$. Similarly with Y_2, a column vector, $Y_2 = d_1e_{jk} + d_2e_{lk}$. Since we assume $n \geq 4$, we may pick some $m \neq j$, $m \neq l$. We can define $X_2' = X_2 + e_{1m}$, $Y_2' = Y_2 + e_{m2}$. Then

$$X_2'Y_2' = X_2e_{m2} + e_{1m}Y_2 + e_{1m}e_{m2} + X_2Y_2 = e_{12}.$$

From (3), it follows that

$$f(X_2')f(Y_2') = f(X_2)f(e_{m2}) + f(e_{1m})f(Y_2) + f(e_{1m})f(e_{m2}) + f(X_2)f(Y_2) = e_{12}.$$

By our choice of m, we know already that $e_{1m}Y_2 = 0$, so $f(e_{1m})f(e_{m2} + Y_2)$ must be equal to e_{12}. Using the linearity of f, we can expand this equation to $f(e_{1m})f(e_{m2}) + f(e_{1m})f(Y_2) =$

e_{12}. By property (3), since $e_{1m}e_{m2} = e_{12}$, we see that $f(e_{1m})f(e_{m2}) = e_{12}$, which gives that $f(e_{1m})f(Y_2) = 0$. We can do the same on the other side, multiplying $(e_{1m} + X_2)e_{m2} = e_{12}$, so

$$f(e_{1m})(f(e_{m2}) + f(X_2)) = f(e_{1m})f(e_{m2}) + f(X_2)f(e_{m2}) = e_{12}.$$

As before, $f(e_{1m})f(e_{m2}) = e_{12}$, and thus $f(X_2)f(e_{m2}) = 0$.

Combining with the above paragraph gives that

$$\begin{aligned} f(X_2')f(Y_2') &= f(X_2)f(e_{m2}) + f(e_{1m})f(Y_2) + f(e_{1m})f(e_{m2}) + f(X_2)f(Y_2) \\ &= e_{12} + 0 + 0 + f(X_2)f(Y_2) = e_{12}, \end{aligned}$$

so finally, $f(X_2)f(Y_2) = 0$. ∎

One can see that Corollary 2.2 and Lemma 2.2 together give us that the map f in Theorem 1.1 preserves the zero product. To obtain the complete form of f, we invoke the following result by Chebotar, et al.:

Theorem 2.1 ([12], Corollary 2.4): *Let $\theta : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a bijective linear map preserving zero products, where $n \geq 2$. Then there exist an invertible matrix $U \in M_n(\mathbb{C})$ and a non-zero $c \in C$ such that $\theta(X) = cU^{-1}XU$ for all $X \in M_n(\mathbb{C})$.*

Thus, the form described in Theorem 1.1 is obtained, and the proof is complete.

3. Proof of Theorem 1.2

We will now prove the second theorem.

Lemma 3.1: *A map f satisfying the conditions in Theorem* 1.2 *preserves zero products.*

Proof: Suppose $A, B \in \mathcal{M}$ such that $AB = 0$. Since A, B, and M have finitely many non-zero entries, there exists a positive integer n such that $A = E_{11}AE_{11}$, $B = E_{11}BE_{11}$, and $M = E_{11}ME_{11}$, where $E_{11} = \sum_{i=1}^{n} e_{ii}$. We then define $E_{12} = \sum_{i=1}^{n} e_{i,n+i}$ and $E_{21} = \sum_{i=1}^{n} e_{i+n,i}$. We will denote $M_{12} = E_{11}ME_{12}$.

Let X denote the matrix $A + M_{12}$; that is, the matrix with A in the upper left corner and M in the next $n \times n$ block to the right. In the same way, let Y denote the matrix $B + E_{21}$. We can observe that $XY = M$. Thus, by the defining property of f, $f(X)f(Y) = N$. By definition of Y, we may rewrite $f(X)f(Y) = f(X)f(B) + f(X)f(E_{21})$. We note that since $XE_{21} = M$, we have $f(X)f(E_{21}) = N$. Thus, $f(X)f(B) = 0$. Similarly, we may rewrite $f(X)f(Y) = f(A)f(Y) + f(M_{12})f(Y) = N$. But, as with the decomposition of Y, by observation, $M_{12}Y = M$, so $f(M_{12})f(Y) = N$. Thus, $f(A)f(Y) = 0$. Since

$f(A)f(Y) = f(X)f(B) = 0$, we may write

$$f(A)f(B) + f(A)f(E_{21}) = 0 \tag{4}$$

and

$$f(A)f(B) + f(M_{12})f(B) = 0. \tag{5}$$

It follows from (4) and (5) that $f(A)f(E_{21}) = f(M_{12})f(B)$. Since $M_{12}E_{21} = M, f(M_{12})f(E_{21}) = N$. We then note that

$$f(M_{12})f(Y) = f(M_{12})f(B) + f(M_{12})f(E_{21}) = N$$

implies $f(M_{12})f(B) = f(A)f(E_{12}) = 0$.

Thus, fully expanding the terms,

$$f(X)f(Y) = f(M_{12})f(E_{21}) + f(A)f(E_{21}) + f(M_{12})f(B) + f(A)f(B) = N.$$

By above, $f(M_{12})f(E_{21}) = N$, $f(A)f(E_{21}) = 0$, and $f(M_{12})f(B) = 0$, so $f(A)f(B) = 0$. Thus, f preserves zero products. ∎

We cite the following theorem to obtain the form of f given in the statement of Theorem 1.2:

Theorem 3.1 ([2], Theorem 1): *Let $\theta : \mathcal{M} \to \mathcal{M}$ be a bijective linear map preserving zero products. Then there exists a non-zero $c \in \mathbb{F}$ such that $\theta(XY) = c\theta(X)\theta(Y)$ for all $X, Y \in \mathcal{M}$.*

Thus, Theorem 1.2 is proven.

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