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On Almost Lyapunov Functions for Systems with Inputs

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***Abstract*— In this work an almost Lyapunov function theorem from our recent work is generalized to systems with inputs. It is shown that if for any inputs and initial conditions, the time that solutions of the system can stay in a "bad" region where the Lyapunov function does not decrease fast enough has a sufficiently small upper bound, then the system is globally exponentially stable uniformly with respect to the inputs. In our analysis, the almost Lyapunov function is directly expressed as a function of time along arbitrary solution and the upper bound of the ratio of this function at the time the solution trajectory leaves and enters the "bad" region is found to be less than 1. Consequently all solutions are shown to converge to the origin asymptotically with some careful justification. It is also concluded that a system with inputs is exponentially input-to-state stable if its auxiliary system satisfies all the hypotheses in our main theorem. The result is then applied on an example adopted and modified from our previous work and it shows an improvement in the sense that stability can still be verified even when there is stronger perturbation to the example's stable dynamics.**

I. Introduction

For general nonlinear systems, global asymptotic stability (GAS) is typically shown through Lyapunov's direct method (see, e.g., [1]), which involves constructing a Lyapunov function V whose time derivative along solutions is negative except at the equilibrium. It turns out not only such V tailored to the system dynamics is hard to construct, but also the property of strictly decreasing V is sufficient but not necessary to guarantee asymptotic stability. The gap is filled by the study of non-monotonic Lyapunov functions as in [2][3][4][5], etc.

In the literature, an interesting and systematic way to show GAS of a nonlinear system when V does not decrease "all the time" is to find a linear combination of higher order derivatives of V with positive coefficients and show that the linear combination is negative definite, as studied in [6] and [7]. This direction of work takes its roots in [8] and the intuition is that when $\dot{V}$ is positive, the negative linear combination guarantees that some higher order derivatives of V have to be sufficiently negative so V cannot grow too much; asymptotically V has to converge to 0 and hence the system is GAS.

Another approach is to bound the region Ω in the state space where V does not decrease fast enough, as studied in [9], [10] via the analysis of *almost Lyapunov* function. States where $\dot{V} > 0$ are inside Ω and because Ω is "small", the trajectories of solutions have to pass through Ω and the amount V can increase will be compensated by the later decrease. As a result, asymptotically V is decreasing and system is also shown to be GAS.

Nevertheless, all the aforementioned literature only deals with systems without inputs. We want to extend the method utilizing almost Lyapunov functions in our previous work [10] to systems with inputs. The main challenge here is that with the presence of inputs, the solution trajectory may be highly non-smooth in the sense that the curvature of the solution trajectories is ill-defined. As a result, our previous non-self-overlapping condition in [10] which is based on the curvature of the solution trajectories cannot be used. Nevertheless, we can bypass this difficulty by switching from bounding the length of solution trajectories in Ω to bounding the time spent in Ω. In this way we can directly express V evaluated along a solution as a function of time and regardless what inputs the system takes, the ratio between V evaluated at the time the solution trajectory enters and leaves Ω can be bounded. Under some assumptions this ratio is shown to be less than 1 and subsequently we are able to show that all solutions of the system will asymptotically converge to the origin. This result is then generalized via the study of auxiliary system and we propose hypotheses under which the system with inputs admits the well known input-to-state stability.

This work is divided into 6 Sections. Section II introduces the mathematical definitions of our dynamical system as well as different characterizations of system stability. Section III gives the main result of this work with some supporting lemmas. Section IV provides the proof of the main theorem. Section V contains an example modified from our earlier work on which our theorem is applicable. Finally Section VI concludes this paper.

II. Preliminaries

Our control system is given by

$$\dot{x} = f(x, u) \tag{1}$$

where $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ is locally Lipschitz jointly in (x, u). The input value set $U \subset \mathbb{R}^m$ is compact[1]. The input function $u(\cdot)$ is a locally essentially bounded function: $u(\cdot) \in L^\infty_{\text{loc}}(\mathbb{R}_{\geq 0} \to U) =: \mathcal{M}_U$. For a specific initial condition x_0 and control u, denote the solution state of (1) at time t by $x(t, x_0, u)$. We say that 0 is an equilibrium of the system (1) in the sense that $f(0, u) = 0$ for all $u \in U$. The system (1) is *globally uniformly asymptotically stable* (GUAS) if there

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[1] In this case the previous assumption is same as f being locally Lipschitz in x uniformly over all $u \in U$.

exists $\beta \in \mathcal{KL}$ [2] such that for any $x_0 \in \mathbb{R}^n, u \in \mathcal{M}_U, t \geq 0$,

$$|x(t, x_0, u)| \leq \beta(|x_0|, t).$$

Moreover, the system (1) is *globally uniformly exponentially stable* (GUES) if the β function used for defining GUAS is linear in the first argument and exponentially decaying with respect to the second argument[3]; that is, there exist $\bar{C} \geq 1, \bar{a} > 0$ such that for any $x_0 \in \mathbb{R}^n, u \in \mathcal{M}_U, t \geq 0$,

$$|x(t, x_0, u)| \leq \bar{C} e^{-\bar{a}t} |x_0| \tag{2}$$

GUES can also be shown via an exponentially decreasing Lyapunov function (see, e.g., [1]):

Lemma 1 *The system* (1) *is GUES if there exist constants* $a, a_1, a_2 > 0$ *and a positive definite function* $V \in C^1(\mathbb{R}^n \to \mathbb{R}_{\geq 0})$ *satisfying*

$$a_1|x|^2 \leq V(x) \leq a_2|x|^2 \quad \forall x \in \mathbb{R}^n \tag{3}$$

such that

$$\nabla V(x) \cdot f(x, u) \leq -aV(x) \quad \forall u \in U, x \in \mathbb{R}^n \tag{4}$$

where ∇V *is the gradient of* V.

Very often strictly decreasing V at exponential rate $-a$ at all time is hard to find; the parameter $\bar{C}$ in (2) also allows transient overshoots when it is larger than 1. We want to preserve this property and weaken the condition on V:

Lemma 2 *The system* (1) *is GUES if there exist* $C \geq 1$, $a_1, a_2, a > 0$ *and a positive definite function* $V \in C^1(\mathbb{R}^n \to \mathbb{R}_{\geq 0})$ *satisfying* (3) *such that for any* $x_0 \in \mathbb{R}^n, u \in \mathcal{M}_U$ *and* $t \geq 0$,

$$V(x(t, x_0, u)) \leq Ce^{-at}V(x_0) \tag{5}$$

The proof of Lemma 2 can be directly shown via comparison functions and hence omitted. Smoothness of V is not required but we include it here for consistency. In contract to the decreasing V as required by Lemma 1, Lemma 2 allows temporary rise in V. As long as the overshoot is uniformly bounded and asymptotically V converges to 0 exponentially fast, the system is GUES.

Another important stability definition is *input-to-state stability* (ISS) [11], which is defined via $\gamma \in \mathcal{K}_\infty$ [4], $\beta \in \mathcal{KL}$ such that for any $x_0 \in \mathbb{R}^n, u \in \mathcal{M}_U$,

$$|x(t, x_0, u)| \leq \beta(|x_0|, t) + \gamma(\|u\|_{[0,t]}).$$

Similarly when the β function used for defining ISS is linear in the first argument and exponentially decreasing with respect to the second argument, we say the system is *exponentially input-to-state stable* (exp-ISS) (see, e.g., [12]).

[2] $\beta \in \mathcal{KL}$ if $\beta(s, t) : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a continuous function such that $\beta(\cdot, t)$ is increasing, $\beta(s, \cdot)$ is decreasing and $\beta(0, t) = 0$ for all $t \geq 0$, $\lim_{t\to\infty} \beta(s, t) = 0$ for all $s \geq 0$.

[3] Note that the "U" in GUES and GUAS in this work is emphasized on the uniformity of convergence with respect to the inputs.

[4] $\gamma \in \mathcal{K}_\infty$ if $\gamma(s) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a continuous, strictly increasing function and $\gamma(0) = 0, \lim_{s\to\infty} \gamma(s) = \infty$.

The definitions of GUES and ISS are connected via auxiliary system as shown in the celebrated work [13]. In addition, since the class $\mathcal{KL}$ function β used for defining GUAS and ISS while showing their relations are the same, GUES and exp-ISS are also connected:

Lemma 3 *For a system* (1) *and some* $\rho \in \mathcal{K}_\infty$, *define the auxiliary system by*

$$\dot{x} = f(x, \rho(|x|)d) =: f'(x, d), \quad |d| \leq 1. \tag{6}$$

If the auxiliary system (6) *is GUAS, then the system* (1) *is ISS. Moreover, if* (6) *is GUES, then* (1) *is exp-ISS.*

III. Results via Ω, V' and T

As mentioned earlier, in order to show GUES, the condition (4) is very strong; it may not hold everywhere in $\mathbb{R}^n$. Suppose there exists a proper subset $\mathcal{W} \subset \mathbb{R}^n$ such that the inequality in (4) only holds or can only be verified for $x \in \mathcal{W}$. In this case we call V an **almost Lyapunov** function. Denote the "bad" set $\Omega = \mathbb{R}^n \backslash \mathcal{W}$ and

$$V'(x) := \sup_{u \in U} \{\nabla V(x) \cdot f(x, u)\}. \tag{7}$$

Then (4) being true for only $x \in \mathcal{W}$ is equivalent to

$$V'(x) \leq -aV(x) \quad \forall x \in \mathbb{R} \backslash \Omega. \tag{8}$$

Note that because the set in the supremum function in (7) may be unbounded when $x \in \Omega$, $V'(x)$ may not exist. Nevertheless, under the regularity assumptions of system (1) and the function V, the next lemma not only guarantees the existence of $V'(x)$ for all $x \in \mathbb{R}^n$, it also shows that V' is Lipschitz:

Lemma 4 *Let* $V \in C^1(\mathbb{R}^n \to \mathbb{R}_{\geq 0})$ *be a positive definite function and assume the system* (1) *has an equilibrium at* 0. *Then* V' *defined via* (7) *exists for all* $x \in \mathbb{R}^n$ *and is Lipschitz when both* $f, \nabla V$ *are Lipschitz.*

Lemma 5 *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded set,* $V \in C^1(\mathbb{R}^n \to \mathbb{R}_{\geq 0})$ *be a positive definite function satisfying* (3) *and assume the system* (1) *has an equilibrium at* 0. *Further assume that* $f, \nabla V$ *are Lipschitz. Then* $\nabla V'$ *defined via* (7) *exists almost everywhere in* Ω. *In addition, there exists* $c > 0$ *such that for all* $\xi \in \Omega$ *where* $\nabla V'(\xi)$ *exists and any* $u \in U$,

$$|\nabla V'(\xi) \cdot f(\xi, u)| \leq cV(\xi) \tag{9}$$

Due to space constraints, we refer the readers to our technical report [14] for the proofs of Lemma 4 and Lemma 5.

Finally define the Ω-*dwell time* T by

$$T := \sup_{x_0 \in \Omega, u \in \mathcal{M}_U} \inf_{t \geq 0} \{t : x(t, x_0, u) \notin \Omega\} \tag{10}$$

This is the longest time a solution of the system (1) can stay inside Ω. Intuitively if the solution of (1) never stays inside Ω for too long, V is still bounded and there is still a chance that the condition (5) will hold. The main objective of this paper is indeed to show that with some mild assumptions, when T is small enough, the system (1) is still GUES.

Theorem 1 *Consider a control system* (1) *with locally Lipschitz right hand side f and compact input value set U. Let $V \in C^1(\mathbb{R}^n \to \mathbb{R}_{\geq 0})$ be a positive definite function satisfying the condition* (3) *for some $a_1, a_2 > 0$ and assume ∇V is also locally Lipschitz. Define $V'(x)$ via* (7) *and for some $a > 0$, let $\Omega \subset \mathbb{R}^n$ be the set such that* (8) *holds. If there exists $c > 0$ such that* (9) *holds for all $x \in \Omega$ where $\nabla V'$ exists, then there exists an increasing function $\alpha : [0, 1) \to [0, \infty)$ with $\alpha(0) = 0, \lim_{t \to 1} \alpha(t) = \infty$ such that as long as the Ω-dwell time T defined in* (10) *satisfies*

$$T < \frac{1}{\sqrt{c}} \min\left\{\frac{\pi}{2}, \alpha\left(\frac{a}{\sqrt{c}}\right)\right\}, \tag{11}$$

the system (1) *is GUES.*

The explicit formula of α is given by

$$\alpha(t) = \ln\left(\frac{1+t}{1-t}\right) + 2\arccos\left((t^2+1)^{-\frac{1}{2}}\right). \tag{12}$$

Implied by the connection between GUES and exp-ISS as stated in Lemma 3, we also have the following corollary:

Corollary 1 *Consider a control system* (1)*. Let $V \in C^1(\mathbb{R}^n \to \mathbb{R}_{\geq 0})$ be a positive definite function satisfying the condition* (3) *for some $a_1, a_2 > 0$. If its auxiliary system defined via* (6) *with some $\rho \in \mathcal{K}_\infty$ satisfies all the hypotheses in Theorem 1, then the system* (1) *is exp-ISS.*

As a remark, the assumption on bounded Ω-dwell time can be further developed. A simple situation where this property can be shown is when Ω has finite size in some dimension, and the vector field f over the set Ω has a uniform lower bound on the norm of its projection onto this dimension. In this case the vector field is "transversal" to the set Ω and thus the solutions of the system will pass through it. In general, there is no systematic way to prove bounded Ω-dwell time; in other words, showing that Ω is not an invariant set requires not only extra knowledge of the shape and volume of Ω but also knowledge of the vector field f. This can be developed into a separate work and is an interesting future research direction.

IV. Proof of the Main Theorem

Proof: We start by making some direct observations of the hypotheses of Theorem 1 here. The definition (7) means that at any time t, the time derivative of V along any solution $x(\cdot, x_0, u)$ satisfies:

$$\dot{V}(x(t, x_0, u)) \leq V'(x(t, x_0, u)) \tag{13}$$

for all $t \geq 0$. Suppose there exists $t_2 > t_1 \geq 0$ such that the solution trajectory enters Ω at t_1 and leaves at t_2; that is, $x(t, x_0, u) \in \Omega$ for all $t \in (t_1, t_2)$ and $x(t_1, x_0, u), x(t_2, x_0, u) \in \partial\Omega$, the boundary of Ω. By definition (10) we have $t_2 - t_1 \leq T$. Thanks to Lemma 1 in [15], the properties that $\nabla V'(x)$ exists almost everywhere in Ω and (9) holds for all $x \in \Omega$ where $\nabla V'(x)$ exists implies that $V(x(\cdot, x_0, u))$ is absolutely continuous as long as $t \in (t_1, t_2)$ so that $x(t, x_0, u) \in \Omega$. In addition whenever $\dot{V}'(x(t, x_0, u))$ exists,

$$-cV(x(t, x_0, u)) \leq \dot{V}'(x(t, x_0, u)) \leq cV(x(t, x_0, u)) \tag{14}$$

Fix $x_0 \in \mathbb{R}^n, u \in \mathcal{M}_U$. Write $V(t), V'(t)$ for abbreviation of $V(x(t, x_0, u)), V'(x(t, x_0, u))$. A necessary condition for (13), (14) to hold is the existence of essentially non-negative functions $w_1(t), w_2(t), w_3(t)$ defined over (t_1, t_2) such that

$$\dot{V}(t) = V'(t) - w_1(t), \tag{15}$$

$$\dot{V}'(t) = cV(t) - w_2(t), \tag{16}$$

$$\dot{V}'(t) = -cV(t) + w_3(t). \tag{17}$$

Because V, V' are continuous, the above equations can be extended to $[t_1, t_2]$. In addition, because $x(t, x_0, u) \in \partial\Omega$ when $t = t_1$ or t_2, by continuity of V' and the property (8), we have

$$V'(t_1) \leq -aV(t_1), \tag{18}$$

$$V'(t_2) \leq -aV(t_2). \tag{19}$$

Our goal is to show that whenever the solution passes through Ω, $\frac{V(t_2)}{V(t_1)} < 1$. This can be achieved by picking a time $t \in (t_1, t_2)$ and bounding $\frac{V(t)}{V(t_1)}, \frac{V(t_2)}{V(t)}$ separately.

We bound $\frac{V(t)}{V(t_1)}$ first. From (15),(16), we have

$$\begin{pmatrix} \dot{V} \\ \dot{V}' \end{pmatrix} = A_1 \begin{pmatrix} V \\ V' \end{pmatrix} - \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix}.$$

Propagating the solutions from t_1 to t, we have

$$\begin{pmatrix} V(t) \\ V'(t) \end{pmatrix} = e^{A_1(t-t_1)} \begin{pmatrix} V(t_1) \\ V'(t_1) \end{pmatrix} - \int_{t_1}^{t} e^{A_1(t-s)} \begin{pmatrix} w_1(s) \\ w_2(s) \end{pmatrix} ds \tag{20}$$

where

$$e^{A_1 t} = \begin{pmatrix} \cosh\sqrt{c}t & \frac{1}{\sqrt{c}}\sinh\sqrt{c}t \\ \sqrt{c}\sinh\sqrt{c}t & \cosh\sqrt{c}t \end{pmatrix}$$

Notice that for any $s \in [t_1, t]$, the two elements in the first row of $e^{A_1(t-s)}$ are non-negative. In addition, recall that w_1, w_2 are non-negative; thus the integration in (20) gives a vector whose first element is always non-negative and because it is subtracted on the right, it implies that

$$\begin{aligned} V(t) &\leq \cosh\sqrt{c}(t-t_1)V(t_1) + \frac{1}{\sqrt{c}}\sinh\sqrt{c}(t-t_1)V'(t_1) \\ &\leq \left(\cosh\sqrt{c}(t-t_1) - \frac{a}{\sqrt{c}}\sinh\sqrt{c}(t-t_1)\right)V(t_1) \\ &=: R_1(t-t_1)V(t_1) \end{aligned}$$

where (18) and the fact that $\sinh\sqrt{c}(t-t_1)$ is non-negative are used for the second inequality. Thus we have $\frac{V(t)}{V(t_1)} \leq R_1(t-t_1)$ for all $t \in [t_1, t_2]$.

Some observations can be made on the function $R_1(t)$. Firstly,

$$\begin{aligned} \dot{R}_1(t) &= \sqrt{c}\sinh\sqrt{c}t - a\cosh\sqrt{c}t \\ &= -\frac{1}{2}\left((a-\sqrt{c})e^{\sqrt{c}t} + (a+\sqrt{c})e^{-\sqrt{c}t}\right) \end{aligned}$$

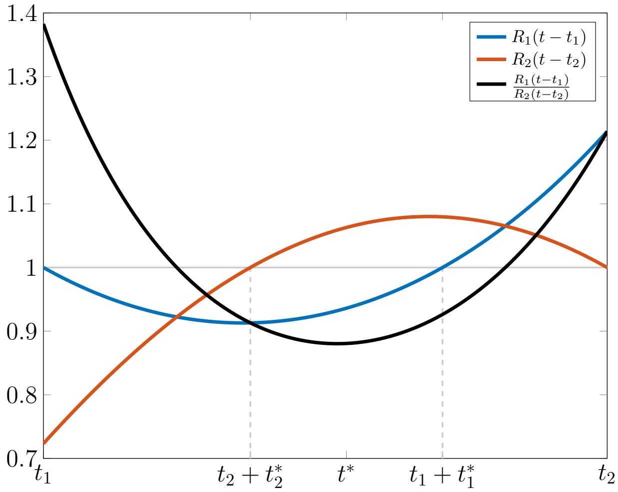

Fig. 1. Functions R_1, R_2 and R_1/R_2.

The case $c \le a^2$ is less interesting since then $\dot{R}_1(t) < 0$ for all $t \ge 0$ and $R_1(t)$ is a strictly decreasing function. By picking $t = t_2$ we directly conclude that $\frac{V(t_2)}{V(t_1)} < 1$. In the other case when $c > a^2$,

$$R_1(t) = \frac{1}{2}\left(\left(1 - \frac{a}{\sqrt{c}}\right)e^{\sqrt{c}t} + \left(1 + \frac{a}{\sqrt{c}}\right)e^{-\sqrt{c}t}\right)$$

and we have $R_1(t) > 0$ for all $t \ge 0$, $R_1(0) = 1$, $\dot{R}_1(0) = -a$, $\lim_{t\to\infty} R_1(t) = \infty$ and $\ddot{R}_1(t) = cR_1(t) > 0$. These properties imply that $R(t)$ is convex over $[0, \infty)$, and $R(t) = 1$ has two solutions, one at $t = 0$. Denote the other one t_1^*, $t_1^* > 0$. We also have

$$R_1(t) < 1 \quad \forall t \in (0, t_1^*). \tag{21}$$

The graphical illustration of function $R_1(t - t_1)$ with $a = 1, c = 6$ is shown as the blue curve in Fig. 1. In fact, t_1^* can be computed analytically:

$$t_1^* = \frac{1}{\sqrt{c}} \ln \frac{\sqrt{c}+a}{\sqrt{c}-a} = \frac{1}{\sqrt{c}} \ln\left(\frac{1+\frac{a}{\sqrt{c}}}{1-\frac{a}{\sqrt{c}}}\right) =: \frac{1}{\sqrt{c}}\alpha_1\left(\frac{a}{\sqrt{c}}\right) \tag{22}$$

and it is not hard to check that $\alpha_1(0) = 0, \lim_{t\to 1}\alpha_1(t) = \infty$.

Similarly when bounding $\frac{V(t_2)}{V(t)}$, consider the linear system given by (15),(17):

$$\begin{pmatrix}\dot{V}\\ \dot{V}'\end{pmatrix} = A_2\begin{pmatrix}V\\ V'\end{pmatrix} + \begin{pmatrix}-w_1\\ w_3\end{pmatrix}, \quad A_2 = \begin{pmatrix}0 & 1\\ -c & 0\end{pmatrix}.$$

Propagating the solutions backwards from t_2 to t, we have

$$\begin{pmatrix}V(t)\\ V'(t)\end{pmatrix} = e^{A_2(t-t_2)}\begin{pmatrix}V(t_2)\\ V'(t_2)\end{pmatrix} + \int_{t_2}^{t} e^{A_2(t-s)}\begin{pmatrix}-w_1(s)\\ w_3(s)\end{pmatrix} ds \tag{23}$$

where

$$e^{A_2 t} = \begin{pmatrix}\cos\sqrt{c}t & \frac{1}{\sqrt{c}}\sin\sqrt{c}t\\ -\sqrt{c}\sin\sqrt{c}t & \cos\sqrt{c}t\end{pmatrix}$$

This case is a bit more complicated compared with the previous case as the elements in $e^{A_2 t}$ are sign indefinite for all $t \le 0$. Nevertheless, our assumption (11) assures that $t_2 - t_1 < \frac{\pi}{2\sqrt{c}}$. As a result, for all $s \in [t, t_2]$, $\sqrt{c}(s - t_2) \in (-\frac{\pi}{2}, 0]$ and thus $\cos(\sqrt{c}(s-t_2)) > 0$, $\sin(\sqrt{c}(s-t_2)) \le 0$. In addition, notice the integration in (23) is backwards and recall that w_1, w_3 are non-negative, hence the first element of the vector obtained after integration is always non-negative. Because it is added on the right hand side in (23), we have

$$\begin{aligned} V(t) &\ge \cos\sqrt{c}(t-t_2)V(t_2) + \frac{1}{\sqrt{c}}\sin\sqrt{c}(t-t_2)V'(t_2)\\ &\ge \left(\cos\sqrt{c}(t-t_2) - \frac{a}{\sqrt{c}}\sin\sqrt{c}(t-t_2)\right)V(t_2)\\ &:= R_2(t-t_2)V(t_2)\end{aligned}$$

where (19) and the fact $\sin\sqrt{c}(t-t_2)$ is non-positive are used for the second inequality. Thus we have $\frac{V(t)}{V(t_2)} \ge R_2(t - t_2)$ for all $t \in [t_1, t_2]$.

Recall that we are only interested in the non-trivial case when $c > a^2$ (otherwise we already have $\frac{V(t_2)}{V(t_1)} < 1$ as in the discussion for R_1). Some observations can be made on $R_2(t)$: $R_2(0) = 1$, $\dot{R}_2(0) = -a$, $R_2(-\frac{\pi}{2\sqrt{c}}) = \frac{a}{\sqrt{c}} < 1$. In addition, for all $t \in [-\frac{\pi}{2\sqrt{c}}, 0]$, $R_2(t) > 0$ and $\ddot{R}_2(t) = -cR_2(t) < 0$. Hence $R_2(t)$ is concave over $[-\frac{\pi}{2\sqrt{c}}, 0]$, and $R_2(t) = 1$ have two solutions, one at $t = 0$. Denote the other by t_2^*, $t_2^* < 0$. We also have that

$$R_2(t) > 1 \quad \forall t \in (t_2^*, 0). \tag{24}$$

The graphical illustration of function $R_2(t - t_2)$ with $a = 1, c = 6$ is shown as the red curve in Fig. 1. Similarly to t_1^*, t_2^* can also be computed analytically:

$$\begin{aligned} t_2^* &= -\frac{2}{\sqrt{c}}\arccos\sqrt{\frac{c}{a^2+c}}\\ &= -\frac{2}{\sqrt{c}}\arccos\left(\left(\frac{a}{\sqrt{c}}\right)^2+1\right)^{-\frac{1}{2}} =: -\frac{1}{\sqrt{c}}\alpha_2\left(\frac{a}{\sqrt{c}}\right)\end{aligned} \tag{25}$$

We find that α_2 is an increasing function such that $\alpha_2(0) = 0, \alpha_2(1) = \frac{\pi}{2}$.

Define $\alpha := \alpha_1 + \alpha_2$. By this construction α satisfies the hypothesis in Theorem 1. The assumption (11) also implies that

$$\begin{aligned} t_2 - t_1 &\le T < \frac{1}{\sqrt{c}}\alpha\left(\frac{a}{\sqrt{c}}\right)\\ &= \left(\frac{1}{\sqrt{c}}\alpha_1\left(\frac{a}{\sqrt{c}}\right)\right) - \left(-\frac{1}{\sqrt{c}}\alpha_2\left(\frac{a}{\sqrt{c}}\right)\right) = t_1^* - t_2^*\end{aligned} \tag{26}$$

Thus $t_1 + t_1^* > t_2 + t_2^*$ and the interval $(\max\{t_1, t_2 + t_2^*\}, \min\{t_2, t_1 + t_1^*\}) = (t_1, t_1 + t_1^*) \cap (t_2 + t_2^*, t_2)$ is non-empty. Pick some point t^* in that interval by defining

$$t^* := (1-\zeta)\max\{t_1, t_2 + t_2^*\} + \zeta\min\{t_2, t_1 + t_1^*\} \tag{27}$$

for some $\zeta \in (0,1)$. Now write $s = t_2 - t_1$, then (27) becomes $t^* = t_1 + (1-\zeta)\max\{0, s+t_2^*\} + \zeta \min\{s, t_1^*\}$. When $s \le \min\{-t_2^*, t_1^*\}$, this reduces to $t^* = t_1 + \zeta s$. Define

$$h(s) := \frac{R_1(t^* - t_1)}{R_2(t^* - t_1 - s)}.$$

Since R_1, R_2 are smooth and positive, so is $h(s)$ over the domain $[0,T]$. We also have $h(0) = \frac{R_1(0)}{R_2(0)} = 1$. Recall $t^* = t_1 + \zeta s$ when s is small so

$$\begin{aligned} h'(0) = & \frac{d}{ds}\left(\frac{R_1(\zeta s)}{R_2((\zeta-1)s)}\right)\bigg|_{s=0} \\ &= \frac{\zeta \dot{R}_1(0)R_2(0) - (\zeta-1)R_1(0)\dot{R}_2(0)}{(R_2(0))^2} = -a \end{aligned}$$

In addition, $t^* \in (t_1, t_1+t_1^*) \cap (t_2+t_2^*, t_2)$ by its definition (27) so (21), (24) imply $R_1(t^*-t_1) < 1, R_2(t^*-t_1-s) > 1$; hence $h(s) < 1$ for all $s \in (0,T]$. This can also be seen from the black curve in Fig. 1 that $\frac{R(t^*-t_1)}{R_2(t^*-t_2)} < 1$. The next lemma, whose proof can also be found in our technical report [14], claims that $h(s)$ is in fact bounded from above by some exponentially decaying function:

Lemma 6 *Let $h : [0,T] \to \mathbb{R}$ be a continuous function such that $h(0) = 1$ and $h(s) \in (0,1)$ for all $s \in (0,T]$. Suppose $h(s)$ is differentiable at 0 and $h'(0) = -a$. Then there exists $\eta \in (0,1]$ such that for all $s \in [0,T]$, $h(s) \le e^{-\eta a s}$.*

As a result, as long as $t_2 - t_1 \le T$,

$$\begin{aligned} \frac{V(t_2)}{V(t_1)} \le \left(\frac{V(t^*)}{V(t_1)}\right)\left(\frac{V(t_2)}{V(t^*)}\right) \le \frac{R_1(t^*-t_1)}{R_2(t^*-t_2)} = \\ h(t_2-t_1) \le e^{-\eta a (t_2-t_1)}. \quad (28) \end{aligned}$$

Finally consider the solution $x(\cdot, x_0, u)$ of the system (1) from time 0 to t. At each time it either stays in $\mathbb{R}^n \backslash \Omega$ and according to (8) that $V(x(\cdot, x_0, u))$ decreases at exponential rate $-a$, or it will pass through Ω over some time interval (t_1, t_2), where $V(x(\cdot, x_0, u))$ is decreased by the ratio $e^{-\eta a(t_2-t_1)}$ from (28). Cascading them together we see that $V(x(\cdot, x_0, u))$ decreases at exponential rate bounded from above by $-\eta a$. There maybe overshoots in V, due to the possibilities that $x_0 \in \Omega$ or $x(t, x_0, u) \in \Omega$. Compared with the exponential decaying rate $-\eta a$, the overshoot in the first possibility is bounded by $(\min_{t\in[-T,0]} R_2(t))^{-1} e^{\eta a T}$ and the overshoot in the second possibility is bounded by $\max_{t \in [0,T]} R_1(t) e^{\eta a T}$. As a result, we will have

$$V(x(t, x_0)) \le C e^{-\eta a t} V(x_0)$$

for any $x_0 \in \mathbb{R}^n$ and $u \in \mathcal{M}_U$, where $C = \frac{\max_{t\in[0,T]} R_1(t)}{\min_{t\in[-T,0]} R_2(t)} e^{2\eta a T}$. Therefore the system (1) is GUES. ■

V. EXAMPLE

Consider the two dimensional system with inputs $u \in \mathbb{R}^2$:

$$\dot{x} = \begin{pmatrix} -\lambda(x) & -\mu \\ \mu & -\lambda(x) \end{pmatrix} x + u =: A(x)x + u \quad (29)$$

where

$$\lambda(x) = \left(\frac{a+b}{2}\right)\min\left\{\frac{|x-x_c|}{r}, 1\right\} - \frac{b}{2} + k \quad (30)$$

for some $a, b, k, r > 0, x_c \in \mathbb{R}^2$. This is modified from the autonomous system example in our earlier work [10] by adding inputs.

If λ is a constant and when $u = 0$, it is easy to see by changing into polar coordinates that the solution of the system (29) is converging to the origin along spiral trajectory. Moreover, the tangential velocity is μ counter-clockwise and the radial velocity is $\lambda|x|$ towards the origin. The dependence of λ on x as described in the definition (30) perturbs the spiral vector field in the region

$$\Omega = \{x \in \mathbb{R}^2 : |x - x_c| < r\}.$$

Pick the standard $V = \frac{1}{2}|x|^2$, define $\rho \in \mathcal{K}_\infty$ by $\rho(s) = ks$. The auxiliary system thus is

$$\dot{x} = A(x)x + k|x|d, \quad |d| \le 1$$

In this case,

$$\begin{aligned} \dot{V}(x) &= x \cdot (A(x)x + k|x|u) = -\lambda(x)|x|^2 + k|x|x \cdot u \\ &\le (-\lambda(x) + k)|x|^2 = 2(-\lambda(x)+k)V(x) =: V'(x). \end{aligned}$$

In addition, since $\lambda(x) = \frac{a}{2} + k$ for all $x \in \mathbb{R}^2 \backslash \Omega$, we have

$$V'(x) \le -aV(x) \quad \forall x \in \mathbb{R}^2 \backslash \Omega$$

exactly the same as the required assumption (8). Notice that when $x \in \Omega$, $V'(x) > -aV(x)$ and in particular when $x = x_c$, $V'(x) = bV(x) > 0$ so the classical Lyapunov theorem is not applicable here. In order to apply our Theorem 1, we need to compute the upper bound on $|\nabla V' \cdot (A(x)x + u)|$. We differentiate $\lambda(x)$ for $x \in \Omega$ first. Notice that in this case (30) implies $\lambda(x) = \frac{(a+b)|x-x_c|}{2r} - \frac{b}{2} + k \le \frac{a}{2} + k$ and

$$\nabla \lambda(x) = \frac{(a+b)(x-x_c)}{2r|x-x_c|},$$

which exists everywhere in Ω except for $x = x_c$. Hence

$$\begin{aligned} &|\nabla V' \cdot (A(x)x + k|x|d)| = 2|(-\nabla\lambda(x)V(x) \\ &\quad + (-\lambda(x)+k)\nabla V(x)) \cdot (A(x)x + k|x|d)| \\ &= 2\left|\left((-\frac{(a+b)(x-x_c)}{2r|x-x_c|}V(x) + (-\lambda(x)+k)x\right)\right. \\ &\quad \times (A(x)x + k|x|d)| \\ &\le \left|\frac{(a+b)(x-x_c)}{r|x-x_c|} \cdot (A(x)x + k|x|d)\right| V(x) \\ &\quad + 2|(-\lambda(x)+k)x \cdot (A(x)x + k|x|d)| \\ &\le \frac{(a+b)}{r}(\|A(x)\| + k)|x|V(x) \\ &\quad + 2|-\lambda(x)+k||x \cdot (A(x)x + k|x|d)| \\ &\le \frac{(a+b)}{r}(k + \sqrt{\lambda(x)^2 + \mu^2})(|x_c| + r)V(x) \\ &\quad + 4|-\lambda(x)+k|^2 V(x) \\ &\le \left(\frac{(a+b)}{r}\left(k + \sqrt{\left(\frac{a}{2}+k\right)^2 + \mu^2}\right)(|x_c|+r) + a^2\right)V(x) \end{aligned}$$

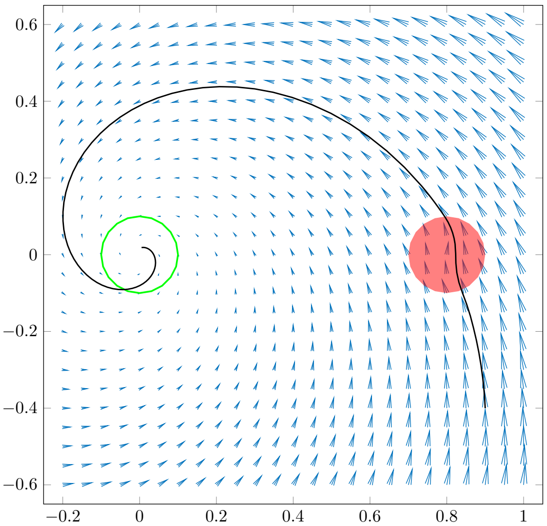

Fig. 2. An illustration of one system (29) solution. The vector field of the system is given by blue arrows. Although the solution trajectory is affected by the red region Ω and hence temporarily deviates, it eventually converges to the small ball indicated by the green circle.

We take $c = \frac{(a+b)}{r}\left(k + \sqrt{\left(\frac{a}{2}+k\right)^2 + \mu^2}\right)(|x_c|+r) + a^2$ so that (9) holds for all $x \in \Omega\backslash\{x_c\}$. Automatically we have $c > a^2$. For the Ω-dwell time, recall the solution of the system (29) is rotating at constant tangential velocity μ. Also Ω is bounded in a sector with center angle $2\arcsin\frac{r}{|x_c|}$ so $T \leq \frac{2}{\mu}\arcsin\frac{r}{|x_c|}$.

Take numerical values $a = 1, b = 0.5, k = 0.1, r = 0.1, \mu = 2$ and $x_c = \begin{pmatrix}0.8\\0\end{pmatrix}$. It is then computed $c \approx 30.54$ and $T < 0.125$. In addition, $\frac{\pi}{2\sqrt{c}} \approx 0.284$ and eventually by (26) and the formulas (12), we find $\frac{1}{\sqrt{c}}\alpha\left(\frac{a}{\sqrt{c}}\right) \approx 0.131$. Thus $T < \frac{1}{\sqrt{c}}\min\left\{\frac{\pi}{2}, \alpha\left(\frac{a}{\sqrt{c}}\right)\right\}$ and the system (29) is exp-ISS with linear ISS gain $\rho^{-1} = \frac{1}{k} = 10$. The vector field $A(x)x$ is shown in Fig. 2 by the blue arrows. Ω is the red shaded region. A solution generated with constant input $u = \begin{pmatrix}0.01\\0\end{pmatrix}$ and initial state $x_0 = \begin{pmatrix}0.9\\-0.4\end{pmatrix}$ is drawn by the black curve in the figure. Although it is temporarily affected by the distorted vector field in Ω, the solution passes through Ω and eventually converges to the ball $|x| \leq 10|u| = 0.1$, determined by the ISS gain and shown as the green circle in Fig. 2. Compared with the analysis of the example in [10], we observe that while all the other parameters are kept the same, the radius of Ω, $r = 0.1$ is much larger than the old one (which was 0.01) and the maximum increasing rate of V, $b = 0.5$ is also much larger than the old one (which was 0.01). Hence not only is the proposed Theorem 1 capable of dealing with the stability of systems with inputs, it is less conservative and able to address "worse behavior" systems.

VI. Conclusion

In this work we have generalized the method of almost Lyapunov functions from our previous work to systems with inputs. It was shown that if the time that solutions of the system can stay in a region where V does not decrease fast enough has a sufficiently small upper bound, then the system is GUES. In addition, if a system with inputs whose auxiliary system satisfies all the hypotheses in our main theorem, then this system is exp-ISS. The hypotheses are cleaner compared with our previous work; in addition, it is observed via the same example but with stronger perturbations to the stable vector field that the theorem in this paper is less conservative and it is believed to be applicable to a broader class of systems.

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