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Almost Lyapunov functions for nonlinear systems ☆

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ABSTRACT

We study convergence of nonlinear systems in the presence of an "almost Lyapunov" function which, unlike the classical Lyapunov function, is allowed to be nondecreasing – and even increasing – on a nontrivial subset of the phase space. Under the assumption that the vector field is free of singular points (away from the origin) and that the subset where the Lyapunov function does not decrease is sufficiently small, we prove that solutions approach a small neighborhood of the origin. A nontrivial example where this theorem applies is constructed.

1. Introduction

For general nonlinear systems, asymptotic stability is typically shown through Lyapunov's direct method (see, e.g., Khalil, 2002), which involves constructing a Lyapunov function V whose time derivative along solutions is negative except at the equilibrium. Even if this property holds for the nominal system, stability is not guaranteed when there is a perturbation because V might not necessarily decrease along solutions of the perturbed system. One natural way to address this issue is to find another Lyapunov function W for this perturbed system by perturbing V accordingly; this is known as the Zubov method (Driver, 1965) on which there are many recent results such as Camilli, Grüne, and Wirth (2001) and Dubljević and Kazantzis (2002). On the other hand, if it is desirable to use the same candidate Lyapunov function V, one may hope to establish stability, at least in some weaker sense, if the measure of the set where V is not decreasing along perturbed solutions is relatively small. We call such a candidate Lyapunov function "*almost Lyapunov*" in this paper.

Besides the above applications to perturbed systems, almost Lyapunov functions can be useful when computational complexity is the main difficulty. While it is straightforward to compute the derivative of an arbitrary Lyapunov function along solutions, it might be quite challenging to analytically check the sign of this derivative either for all states, or just for a region of interest. For example, in the case when both the differential equation and the Lyapunov function are polynomials of high degree, the derivative is also a polynomial and verifying stability reduces to checking whether a polynomial is negative definite. This problem is computationally hard, as it is related to Hilbert's 17th problem (Reznick, 2000) and is an important subject of current research (see, e.g., Blekherman, Parrilo, & Thomas, 2012; Chesi, 2011). Following existing techniques, we may be able to verify that the time derivative of V is negative only in a proper subset of the region of interest, while not in the entire region. This demonstrates the need for tools that would let one conclude stability if V is only an "almost Lyapunov" function, which is studied in this paper.

When a general candidate Lyapunov function is constructed, the sign of its derivative along solutions can also be checked by techniques based on random sampling (Tempo, Calafiore, & Dabbene, 2012) instead of deterministic methods. This approach only requires one to verify that the derivative is negative at a sequence of states picked randomly inside the region. One can use the Chernoff bound (see, e.g., Tempo et al., 2012; Vidyasagar, 1997) to characterize the number of such sample points needed to obtain a reliable upper bound on the relative measure of points in the region of interest for which the desired inequality can possibly fail. Hence the problem is again converted into finding an "almost Lyapunov" function. See also Kenanian, Balkan, Jungers, and Tabuada (2018) for some related recent work.

There is some relevant literature where stability is studied by generalizations of the Lyapunov function approach. In Gunderson (1971) and Meigoli and Nikravesh (2012), higher order derivatives of Lyapunov function are used and in Aeyels and Peuteman (1998) and Ahmadi and Parrilo (2008), the so-called finite-step

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Lyapunov function is used. There are significant difficulties associated with both approaches. Higher order derivatives are harder to compute, especially for nonlinear systems, and finding a linear combination of higher order derivatives so a negative definite function is obtained presents some challenges. In addition, as stated in Ahmadi and Parrilo (2011), the method through higher order derivatives of Lyapunov function approach is equivalent to finding a standard Lyapunov function of some special form. As for the finite-step Lyapunov functions, while the idea of "almost Lyapunov" functions used in this paper is similar in spirit, there is a conceptual difference. The finite-step Lyapunov function approach requires temporal information from the system because the difference between the values of Lyapunov functions over a finite time interval needs to be computed. Thus, the solutions would need to be traced in order to compute the difference. This is a cumbersome and sometimes impossible, in practice, task for a general nonlinear system. By contrast, the "almost Lyapunov" functions approach relies only on the spatial information of the system so that only some bounds on the vector field of the system and the Lyapunov function are needed.

When working with "almost Lyapunov" functions, we encounter regions in the state space where the system trajectories might temporarily diverge (in the sense of growth of Lyapunov function). Nevertheless, our main result shows that when the volume of the "bad" region where V does not decrease fast is sufficiently small, the system is stable in the following weaker sense as characterized by three properties: 1. Every solution starting within a region that is slightly smaller than the region of interest will remain in the region of interest; 2. All such solutions will converge to a small region containing the equilibrium, with a uniform bound in time; 3. Once they reach this small region around the equilibrium, solutions will remain there afterwards. The differences between the sizes of the respective regions depend on the measure of the bad set, and they compensate for possible temporary overshoots.

The first result of this type was obtained in Liberzon, Ying, and Zharnitsky (2014) by using a perturbation argument. In that paper, an arbitrary solution was compared with a solution that avoided "bad regions" and converged to the equilibrium. Then, using continuous dependence of solutions on initial conditions, it was found that this arbitrary solution will not end up too far from the equilibrium. In this paper we present a different approach, which is based on the geometry of curves in the Euclidean space. The basic idea here (following up on our preliminary work Liu, Liberzon, & Zharnitsky, 2016) is that in order to accumulate a net gain in V along a solution, the tubular neighborhood swept out by a ball of a certain radius moving along this solution trajectory needs to be contained inside the region where V does not decrease fast enough. Consequently, if such "bad" regions are not big enough, V cannot increase overall (even though a temporary gain is still possible). Since the criterion we are deriving is on the volume, not on the shape of the "bad" regions, some geometrical arguments on curvature and volume are needed in order to relate the size of the "bad" region to how long the solution of the system can stay inside it and how much V can increase. To illustrate this type of system behavior, we construct an example in which there is a small region where the time derivative of V is positive and to which our main result applies.

The paper is mainly organized in the following order: Frequently used terms and variables are defined in Section 2. Our main result (Theorem 1) is stated in Section 3. Its proof is given in Section 4. Section 5 presents a global result on system stability which can be derived from almost Lyapunov function and Section 6 contains a numerical example where our theorem is applied on with some discussion. After Section 7 concludes the paper, the previous result from Liberzon et al. (2014) is briefly mentioned in Appendix A and the proof of an auxiliary result (Proposition 11) is provided in Appendix B.

2. Preliminaries

Consider a general system

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \quad f(0) = 0, \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz function. Consider a function $V : \mathbb{R}^n \to [0, \infty)$ which is positive definite and C^1 with locally Lipschitz gradient, which we denote by V_x. We say it is a *Lyapunov function* for the system (1) if

$$\dot{V}(x) := V_x(x) \cdot f(x) < 0 \quad \forall x \neq 0 \tag{2}$$

The system (1) can be shown to be asymptotically stable if such a Lyapunov function exists (Khalil, 2002, Ch. 4). A stronger version of Lyapunov function is when V decays at a certain positive rate a:

$$\dot{V}(x) < -aV(x) \quad \forall x \neq 0$$

While this property does not need to hold on the entire region of interest D, we set

$$\Omega := \{x \in D : \dot{V}(x) \geq -aV(x)\} \tag{3}$$

and when the measure of Ω is "small", we informally say that this V is an almost Lyapunov function for the system (1) because now

$$\dot{V}(x) < -aV(x) \quad \forall x \in D \backslash \Omega.$$

Notice that the solution trajectory passing through Ω does not necessarily imply growth of V; it is only in the subset $\{x \in \Omega : \dot{V}(x) > 0\}$ that growth of V occurs. In this paper, we take the region D to be of the following form:

$$D := \{x \in \mathbb{R}^n : c_1 \leq V(x) \leq c_2\}, \quad c_2 > c_1 > 0 \tag{4}$$

We assume D to be compact.[1] We refer to f as "non-vanishing" when

$$f(x) \neq 0 \quad \forall x \in D. \tag{5}$$

The non-vanishing condition clearly requires the equilibrium at origin to be excluded from D. Next define

$$b := \max_{x \in D} \dot{V}(x). \tag{6}$$

Finally, let $B^n_\gamma(x)$ be the closed ball whose center is at x in $\mathbb{R}^n$ with radius γ. Also define the function vol$(\cdot)$ to be the standard volume function induced by the Euclidean metric. Recall that a general expression for the volume of a n-dimensional ball of radius γ is:

$$\text{vol}(B^n_\gamma) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}\gamma^n =: \chi(n)\gamma^n \tag{7}$$

where Γ is the standard gamma function (Courant & John, 1989, Ch. 4.11). Further notation will be introduced in the course of the proof.

3. Main result

We are now ready to state our main result:

Theorem 1. *Consider system (1) with a locally Lipschitz right-hand side f, and a function $V : \mathbb{R}^n \to [0, +\infty)$ which is positive definite and C^1 with locally Lipschitz gradient. Let the region D be defined via (4) with some fixed $c_1 < c_2$ and assume that it is compact.*

Let Ω be defined via (3) for some $a > 0$ and assume f is non-vanishing in D as defined in (5). Also assume

$$\max_{x \in D} \dot{V}(x) < a \min_{x \in D} V(x), \tag{8}$$

i.e. $b < ac_1$ where b is defined in (6).

[1] This is true when V is radially unbounded. Otherwise the results of our theorem are still applicable if the initial state of the system is inside a compact connected component of D. In this case we take this compact connected component as the region of interest D.

Then there exist constants $\bar{\epsilon} > 0$, $g > 0$, $h > 0$ such that for any $\epsilon \in [0, \bar{\epsilon})$, if $\text{vol}(\Omega^*) \le \epsilon$ for every connected component Ω^* of Ω, then there exists $T \ge 0$ so that for any initial state $x_0 \in D$ with $V(x_0) < c_2 - h\epsilon^{\frac{1}{n}} - g\epsilon$, we have

$$V(x(t)) \le c_2 \quad \forall t \ge 0$$

and

$$V(x(t)) \le c_1 + h\epsilon^{\frac{1}{n}} + g\epsilon \quad \forall t \ge T.$$

Remark 2. The results of Theorem 1 are illustrated in Fig. 1. As seen from the figure, the proof will actually give slightly sharper estimates than what is stated in the theorem, namely, $V(x(t)) \le V(x_0) + g\epsilon$ for all $t \ge 0$ and $V(x(T)) \le c_1 + h\epsilon^{\frac{1}{n}}$. The term $h\epsilon^{\frac{1}{n}}$ serves as a "buffer" ensuring that the solution is bounded while the term $g\epsilon$ is a threshold for possible transient overshoot. The exact formulas for g, h will be given by (21), (22) respectively and $\bar{\epsilon}$ will be explicitly found in Section 4.3. Later in the proof of the main theorem the reader will also see that the convergence before time T is in fact exponential, in the form of

$$V(x(t)) \le (V(x(0)) + \frac{g}{2}\epsilon)e^{-\lambda(\epsilon)t^*} + \frac{g}{2}\epsilon,$$

where $\lambda(\epsilon)$ is a positive, continuous and strictly decreasing function on $[0, \bar{\epsilon})$ with $\lambda(0) < a$ and some $t^* \in \left[\max\{0, t - \frac{2g\epsilon}{b}\}, t\right]$.

Remark 3. In the limit $\epsilon \to 0$, the almost Lyapunov function becomes the standard Lyapunov function and the theorem gives the usual conclusion that one could expect from the Lyapunov stability theory. In particular, any solution starting at the higher level set $V = c_2$ will converge to the lower level set $V = c_1$.

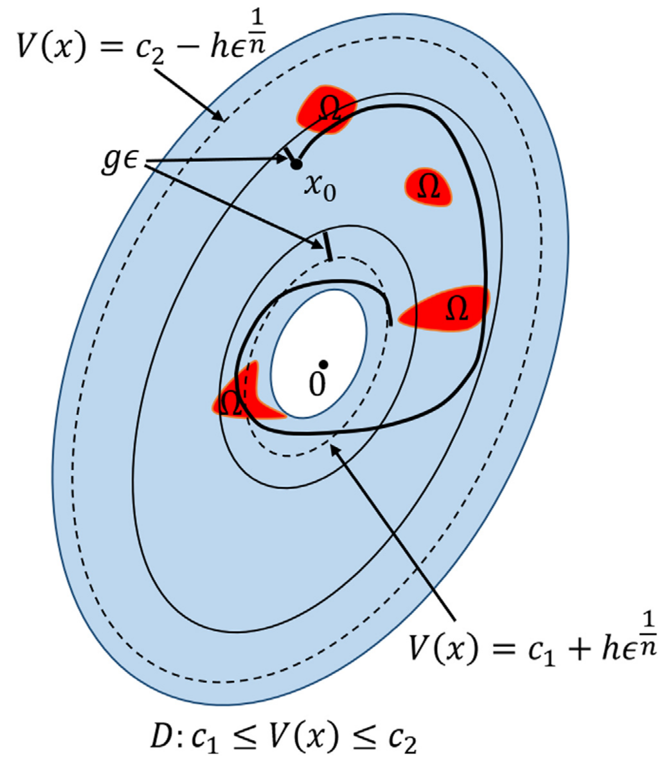

Fig. 1. Illustration of Theorem 1.

4. Proof of theorem

The main idea of the proof relies on the following observation: if the measure of Ω is small enough,[2] there will be too little time for a tube around the solution to stay inside Ω so the growth of V could not be accumulated. The proof contains 4 major steps:

(1) The first step is to show that when the time derivative of V is positive, the solution has to be in a subset of Ω and a tube around the solution segment is contained in Ω.
(2) The second step is to use a non-self-overlapping condition to compute an upper bound on the time that the solution stays in the above-mentioned subset based on the volume swept out by the solution tube.
(3) The next step is to find a bound on the change of V over the time estimated in the previous step. We will conclude that when the volume the connected component of Ω is sufficiently small, the change of V will be negative.
(4) The last step generalizes previously obtained estimates to the possible scenario of repeated passage of the solution through several, or even infinitely many, connected components of Ω. By connecting segments of the solution, we argue that although there might be temporary overshoots in V, overall the solution will converge to a smaller sub-level set.

4.1. Estimates on the solution tube

Since f is a Lipschitz function and D is compact, we can define the following bounds:

$$\bar{L}_0 := \max_{x \in D} |f(x)|, \tag{9}$$

$$\underline{L}_0 := \min_{x \in D} |f(x)|. \tag{10}$$

Note that the vector field f is non-vanishing in D if and only if $\underline{L}_0 > 0$. Let L_1 be the Lipschitz constant of f over D:

$$|f(x_1) - f(x_2)| \le L_1 |x_1 - x_2| \quad \forall x_1, x_2 \in D. \tag{11}$$

In addition, since V is assumed to be C^1 and has locally Lipschitz gradient, we also define some bounds on V_x:

$$M_1 := \max_{x \in D} |V_x(x)|, \tag{12}$$

and M_2 be the Lipschitz constant of V_x over D:

$$|V_x(x_1) - V_x(x_2)| \le M_2 |x_1 - x_2| \quad \forall x_1, x_2 \in D. \tag{13}$$

For $\eta \in [0, 1]$, Define

$$\Omega_\eta := \{x \in D : \dot{V}(x) \ge -\eta a V(x)\} \tag{14}$$

where a comes from the hypothesis of the theorem. By this definition Ω_1 is the same as Ω defined in (3). The next three lemmas establish existence of a disk of positive radius that is sweeping through Ω_1 along the solution forming a tube that is contained inside Ω_η:

Lemma 4. *For any $x_1, x_2 \in D$,*

$$|V(x_1) - V(x_2)| \le M_1 |x_1 - x_2|.$$

Proof. If the line segment between x_1, x_2 entirely lies in D, by Mean Value Theorem there exists x_3 on the segment such that $V(x_2) = V(x_1) + V_x(x_3) \cdot (x_2 - x_1)$. Now by (12),

$$\begin{aligned} |V(x_1) - V(x_2)| &= |V_x(x_3) \cdot (x_1 - x_2)| \\ &\le |V_x(x_3)|\,|x_1 - x_2| \le M_1 |x_1 - x_2|. \end{aligned}$$

In the case when the line segment is partially outside of D, let us say that $y_1, y_2 \in \partial D$ are two points on the segment connecting

[2] By definition Ω is closed and therefore, measurable.

x_1, x_2 such that the line segment between y_1, y_2 is outside D. Since y_1, y_2 are on the boundary of D, the V value must be either c_1 or c_2 at these two points. If $V(y_1) \neq V(y_2)$, say $V(y_1) = c_1$ and $V(y_2) = c_2$, then $V(x) \leq c_1$ or $V(x) \geq c_2$ for all x on the line segment from y_1 to y_2. This cannot happen since V is a continuous function. Therefore $V(y_1) = V(y_2)$. Hence using triangle inequality,

$$\begin{aligned}|V(x_1) - V(x_2)| &= |V(x_1) - V(y_1) + V(y_2) - V(x_2)| \\ &\leq |V(x_1) - V(y_1)| + |V(y_2) - V(x_2)| \\ &\leq M_1|x_1 - y_1| + M_1|y_2 - x_2| \\ &\leq M_1|x_1 - x_2|.\end{aligned}$$

The second to last inequality follows from the fact that the two segments x_1 to y_1 and x_2 to y_2 are contained in D so we can apply our earlier result. The last inequality is simply the fact that the sum of the lengths of the two segments is no longer than the total distance between x_1 and x_2. In the case when there are multiple segments between x_1 and x_2 that are outside of D, repeating the above analysis on each interval, we still get the same result.

Lemma 5. *For any $x_1, x_2 \in D$,*

$$|\dot{V}(x_1) - \dot{V}(x_2)| \leq \alpha|x_1 - x_2|, \tag{15}$$

where $\alpha := M_1L_1 + M_2\bar{L}_0$.

Proof. Estimate

$$\begin{aligned}|\dot{V}(x_1) - \dot{V}(x_2)| &= |V_x(x_1)f(x_1) - V_x(x_2)f(x_2)| \\ &\leq |V_x(x_1)|\,|f(x_1) - f(x_2)| + |f(x_2)|\,|V_x(x_1) - V_x(x_2)| \\ &\leq M_1|f(x_1) - f(x_2)| + \bar{L}_0|V_x(x_1) - V_x(x_2)| \\ &\leq M_1L_1|x_1 - x_2| + \bar{L}_0M_2|x_1 - x_2| \\ &= \alpha|x_1 - x_2|.\end{aligned}$$

Notice that we have used the definitions of M_1 from (12) and $\bar{L}_0$ from (9) in the second to last inequality and the two Lipschitz constants L_1, M_2 from (11), (13) in the last inequality.

Lemma 6. *If $x \in \Omega_\eta$ then $\left(B^n_{\gamma_\eta}(x) \cap D\right) \subseteq \Omega_1$, where*

$$\gamma_\eta := \frac{(1-\eta)ac_1}{\alpha + \eta a M_1} \tag{16}$$

with α as defined in Lemma 5.

Proof. Let $x \in \Omega_\eta, y \in D$ be such that $|x - y| \leq \gamma_\eta$. Since both of them are in D, by Lemma 4, $V(x) \leq V(y) + M_1|x - y| \leq V(y) + M_1\gamma_\eta$. Therefore

$$\begin{aligned}\dot{V}(y) &\geq \dot{V}(x) - |\dot{V}(x) - \dot{V}(y)| \geq -\eta a V(x) - \alpha|x - y| \\ &\geq -\eta a(V(y) + M_1\gamma_\eta) - \alpha\gamma_\eta \\ &= -\eta a V(y) - (1-\eta)ac_1 \geq -aV(y).\end{aligned}$$

In the second inequality we have used the fact that $x \in \Omega_\eta$ so $\dot{V}(x) \geq \eta a V(x)$. We also used the result from Lemma 5 for bounding the second term in this step. Lemma 4 is used in the third inequality. Across the second line the terms depending on γ_η are collected together and substituted with its definition (16). In the last inequality we have used the fact that $y \in D$ so $c_1 \leq V(y)$. Hence we have shown $y \in \Omega_1$ and $\left(B^n_{\gamma_\eta}(x) \cap D\right) \subseteq \Omega_1$.

Define the normal disk of radius γ centered at x to be

$$N_\gamma(x) = \{y \in B^n_\gamma(x) : (y - x) \cdot f(x) = 0\}, \tag{17}$$

which is a ball $B^{n-1}_\gamma(x)$ in the hyperplane

$\{y \in \mathbb{R}^n : (y - x) \cdot f(x) = 0\}$.

Define

$$S_{\eta,(s,t)} = \bigcup_{\tau \in (s,t)} N_{\gamma_\eta}(x(\tau)) \tag{18}$$

to be the tube of radius γ_η around the solution on the time interval s to t. We will often refer to it as the *solution tube*. We will say the tube is *non-self-overlapping* over time interval (s, t) if

$$N_{\gamma_\eta}(x(\tau_1)) \cap N_{\gamma_\eta}(x(\tau_2)) = \emptyset \quad \forall \tau_1, \tau_2 \in (s,t), \tau_1 \neq \tau_2. \tag{19}$$

In a non-self-overlapping tube all the states are swept out only once by such $N_{\gamma_\eta}(x(\tau))$ normal disk at some $\tau \in (s, t)$. There will be more discussion of non-self-overlapping condition in the next subsection. Let

$$\mathcal{L}^t_s := \int_s^t |f(x(\tau))|d\tau$$

be the length of the solution trajectory from time s to t. Using the bounds (9) and (10) on f, one has

$$\underline{L}_0(t - s) \leq \mathcal{L}^t_s \leq \bar{L}_0(t - s). \tag{20}$$

Define

$$g := \frac{b}{\underline{L}_0 \mathrm{vol}(B^{n-1}_{\gamma_\eta})}, \tag{21}$$

$$h := M_1\chi(n)^{-\frac{1}{n}}, \tag{22}$$

where $\chi(n)$ comes from (7). Define a shrunk domain

$$D^* := \{x \in \mathbb{R}^n : c_1 + h\epsilon^{\frac{1}{n}} \leq V(x) \leq c_2 - h\epsilon^{\frac{1}{n}}\}.$$

For any initial state $x(0) = x_0 \in D$ with $V(x_0) < c_2 - g\epsilon - h\epsilon^{\frac{1}{n}}$, by the standard theory of ODEs the solution can be continued either indefinitely or to the boundary of D^*. Define

$$T := \inf\{\tau \geq 0 : x(\tau) \notin D^*\} \tag{23}$$

By this definition, $T = 0$ if $V(x_0) < c_1 + h\epsilon^{\frac{1}{n}}$ and T is infinite if the solution stays in D^* forever. Eventually, in the proof we show that T has to be finite and it is impossible for the solution to reach the outer boundary of D^* with $V(x(T)) = c_2 - h\epsilon^{\frac{1}{n}}$. This T will be the one in the main theorem statement that we are looking for.

Define the subset of the time interval when the solution stays in Ω_η as

$$X_\eta = \{\tau \in [0, T) : x(\tau) \in \Omega_\eta\}. \tag{24}$$

While the set X_η might have a complicated structure, the relevant part for us is the interior which must be a union of intervals. When the solution is considered over a subset of X_η which has empty interior, the almost Lyapunov function will be decreasing with the rate a and hence less interesting. A *maximal interval* contained in X_η is an interval in X_η which cannot be enlarged without leaving X_η. We will also refer to such intervals as *connected components* of X_η. The sweeping tube $S_{\eta,(s,t)}$ generated over a connected component $(s, t) \subseteq X_\eta$ is illustrated in Fig. 2. Intuitively the volume of $S_{\eta,(s,t)}$ is the cross-section area times the trajectory length over (s, t). The next lemma proves this, under the assumption that there is no self-overlapping:

Lemma 7. *If the solution is non-self-overlapping over time interval (s, t), then*

$$\mathrm{vol}(S_{\eta,(s,t)}) = \chi(n-1)\gamma_\eta^{n-1}\mathcal{L}^t_s. \tag{25}$$

The proof of this lemma is a direct application of results from Courant and John (1989, Chapter 4.10) and Foote (2006). The conditions for non-self-overlapping will be discussed in the next section.

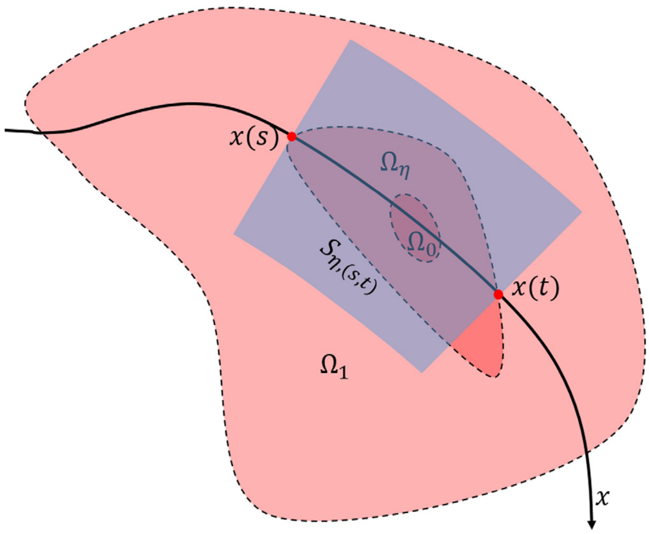

Fig. 2. A planar example showing the solution trajectory passing through Ω_η, generating a tubular neighborhood $S_{\eta,(s,t)}$. In higher dimension the set $S_{\eta,(s,t)}$ would look like a cylinder.

Remark 8. The formula in Foote (2006) yields a signed volume with multiplicity (which is a result of negative self-overlapping); nevertheless, the non-self-overlapping condition we have ensured that there are no negative or multiple counts of the integrated volume and the result is indeed the absolute volume that we want as a lower bound.

Lemma 9. *$S_{\eta,(s,t)} \subseteq \Omega_1$ for all $(s,t) \subset X_\eta$.*

Proof. By Lemma 6, the definition of $S_{\eta,(s,t)}$ in (18) and the definition of X_η in (24), it suffices to show that $B_{\gamma_\eta}(x) \subseteq D$ for any $x \in D^* \cap \Omega_\eta$. If this is not true, there exists $x \in D^* \cap \Omega_\eta$ such that $B_{\gamma_\eta}(x)$ is partially outside of D (cannot be completely outside of D as $x \in D^* \subset D$). In this case, we introduce the sets $S_{in} := \partial B_{\gamma_\eta}(x) \cap D$, $S_{out} := \partial B_{\gamma_\eta}(x) \backslash S_{in}$ and $S_D := \partial D \cap B_{\gamma_\eta}(x)$. None of these sets are empty and for any $y \in S_D$, $V(y) = c_1$ or c_2. By definition of D^* and Lemma 4 we have

$$h\epsilon^{\frac{1}{n}} \leq |V(x) - V(y)| \leq M_1|x-y| \Rightarrow |x-y| \geq \left(\frac{\epsilon}{\chi(n)}\right)^{\frac{1}{n}}.$$

Let $z \in S_{out}$. Then the line segment $[x,z]$ intersects with S_D at some point y so $|x-z| = |z-y| + |y-x|$ and then

$$\gamma_\eta \geq \delta + \left(\frac{\epsilon}{\chi(n)}\right)^{\frac{1}{n}}$$

for some $\delta > 0$. Denote the volume bounded by the surfaces S_{in}, S_D by A. Then $A \subseteq \Omega_1$ so $\mathrm{vol}(A) \leq \mathrm{vol}(\Omega_1) \leq \epsilon$. On the other hand, by the earlier analysis points on S_D are at least $\left(\frac{\epsilon}{\chi(n)}\right)^{\frac{1}{n}}$ away from x and points on S_{in} are at least $\delta + \left(\frac{\epsilon}{\chi(n)}\right)^{\frac{1}{n}}$ away from x. This means A contains a ball of radius $\left(\frac{\epsilon}{\chi(n)}\right)^{\frac{1}{n}}$ so $\mathrm{vol}(A) > \chi(n)\left(\frac{\epsilon}{\chi(n)}\right) = \epsilon$ (the positivity of δ and the continuity of the surface result in the strict inequality), which is a contradiction.

The result of Lemma 9 is illustrated in Fig. 2 that the sweeping tube is a subset of the "bad region" Ω_1. By the definition (18) $S_{\eta,(s,t)}$ is connected; also recall ϵ is the upper bound on the volume of every connected component of Ω and hence a direct consequence of Lemma 9 tells that $\mathrm{vol}(S_{\eta,(s,t)}) \leq \epsilon$. Now applying the formula (25) and the definition (21) here with the assumption that the solution is non-self-overlapping, we have

$$\begin{aligned} \epsilon &\geq \mathrm{vol}(S_{\eta,(s,t)}) = \mathrm{vol}(B^{n-1}_{\gamma_\eta})\mathcal{L}_s^t \\ &\geq \mathrm{vol}(B^{n-1}_{\gamma_\eta})\underline{L}_0(t-s) = \frac{b}{g}(t-s). \end{aligned} \quad (26)$$

Corollary 10. *Let $(s,t) \subset X_\eta$ and assume the solution over this time interval is non-self-overlapping. If the volume of the connected component of Ω is bounded from above by ϵ, then the length of the time interval (s,t) must satisfy*

$$t - s \leq \frac{g\epsilon}{b}.$$

4.2. On non-self-overlapping condition

The following proposition gives a geometric criterion of non-self-overlapping.

Proposition 11. *Consider a tube of radius ρ_0 around a space curve $\gamma(\tau)$ whose radius of curvature is bounded from below by ρ. If $\rho > \rho_0$ and if the length $\mathcal{L}$ of $\gamma(\tau)$ is bounded:*

$$\mathcal{L} < 2\rho\left(\pi - \sin^{-1}\left(\frac{\rho_0}{\rho}\right)\right) \quad (27)$$

then the tube is non-self-overlapping.

The value on the right hand side of (27) is the curve length of a circular arc with radius of curvature ρ and chord distance of $2\rho_0$ between the end points. The proof of this proposition makes use of two classical results of Fenchel's Theorem (Fenchel, 1951) and Schur's Comparison Theorem (Schur, 1921) (see also Sullivan, 2008 for the modified versions of the two theorems on curves with finite total curvature), and is provided in Appendix B.

At this point, the solution of our system can be viewed as a space curve $x = \gamma(t)$ in $\mathbb{R}^n$. Thus we have the curvature

$$\kappa(t) = \frac{[\gamma', \gamma'']}{|\gamma'|^3}(t), \quad (28)$$

where $[*,*]$ is a standard area form. This formula is a simple consequence of the definition of centripetal acceleration $a = v^2\kappa$. Indeed, $[\gamma', \gamma''] = |\gamma'||\gamma''|\sin\alpha$ where $\sin\alpha$ is the angle between the two vectors γ', γ''. When $[\gamma', \gamma'']$ is divided by $|\gamma'|^3$, we obtain $|\gamma''|\sin\alpha/|\gamma'|^2$, which is the projection of acceleration onto the normal vector to the curve (centripetal acceleration) divided by velocity squared. Hence if the solution trajectory is C^2, the bound to the curvature exists:

$$|\kappa(t)| \leq \frac{|\dot{x}||\ddot{x}|}{|\dot{x}|^3} \leq \frac{\|Df(x)\|\,|\dot{x}|}{|\dot{x}|^2} \leq \frac{\|Df(x)\|}{|f(x)|} \leq \frac{L_1}{\underline{L}_0}. \quad (29)$$

where Df is the Jacobian of f. Although in our case $\ddot{x}$ does not exist everywhere, it exists almost everywhere along a solution due to the fact that $f(x(t))$ is a Lipschitz function with respect to t and hence differentiable for almost all t by Rademacher's Theorem.[3] Thanks to Sullivan (2008, Section 5), where the arguments involving Fenchel's Theorem and Schur's Comparison Theorem are also applicable to curves not necessary C^2 but with Lipschitz tangent vectors, the upper bound $\frac{L_1}{\underline{L}_0}$ from (29) is still applicable.

Because radius of curvature is simply the reciprocal of curvature, Proposition 11 implies a sufficient condition for non-self-overlapping solution of our system:

[3] In fact $\ddot{x}$ can be bounded everywhere along the solution in a generalized sense, along the lines of the argument in Teel and Praly (2000, Section 2) based on Clarke's derivative (Clarke, 1990).

Corollary 12. *A tube of radius γ_η around the solution $x(\tau)$ is non-self-overlapping over the interval (s, t) if*

$$\gamma_\eta < \frac{L_0}{L_1} \tag{30}$$

and

$$\mathcal{L}_s^t < \frac{2L_0}{L_1}\left(\pi - \sin^{-1}(\frac{L_1\gamma_\eta}{L_0})\right). \tag{31}$$

Note that according to (16) γ_η is a decreasing function of η and $\gamma_1 = 0$, thus, the inequality (30) can always be satisfied by picking η close enough to 1.

Remark 13. Bounded curvature is an important feature for non-vanishing vector fields since bounded curvature prevents the system from some undesired behavior which will not generate new sweeping volume, such as revolving inside a small region.

Now we have found a criterion of non-self-overlapping (31) in terms of the constraint on the path length, but we need to reformulate this criterion in terms of the measure of the bad set. Suppose that (30) holds with the volume bound analogue of (31)

$$\epsilon < \epsilon_1 := \mathrm{vol}(B_{\gamma_\eta}^{n-1})\frac{2L_0}{L_1}\left(\pi - \sin^{-1}(\frac{L_1\gamma_\eta}{L_0})\right). \tag{32}$$

Then we have

Lemma 14. *Assume η satisfies the inequality (30) and $\epsilon < \epsilon_1$ as defined in (32). Then $S_{\eta,(s,t)}$ is non-self-overlapping for any $(s, t) \subseteq X_\eta$.*

Proof. Recall the direct consequence of Lemma 9 gives $\mathrm{vol}(S_{\eta,(s,t)}) \le \epsilon < \epsilon_1$. Let

$$\tilde{t} := \sup\{\tau \in (s, t] : x(t) \text{ is non-self-overlapping over } [s, \tau)\}.$$

The solution is always non-self-overlapping when τ is sufficiently close to s because of the inequality (30) so the above set is non-empty and the supremum exists. Our goal is to show $\tilde{t} = t$. Because (31) means any tube generated by any shorter curve will be non-self-overlapping, the solution is non-self-overlapping over $[s, \tau)$ for all $\tau \in (s, \tilde{t})$. Thus by the continuity of $\mathrm{vol}(S_{\eta,(s,\tau)})$ with respect to τ,

$$\begin{aligned}
\mathrm{vol}(B_{\gamma_\eta}^{n-1})\mathcal{L}_s^{\tilde{t}} &= \lim_{\tau\to\tilde{t}^-}\left(\mathrm{vol}(B_{\gamma_\eta}^{n-1})\mathcal{L}_s^{\tau}\right)\\
&= \lim_{\tau\to\tilde{t}^-}\mathrm{vol}(S_{\eta,(s,\tau)}) = \mathrm{vol}(S_{\eta,(s,\tilde{t})})\\
&\le \mathrm{vol}(S_{\eta,(s,t)}) < \epsilon_1\\
&= \mathrm{vol}(B_{\gamma_\eta}^{n-1})\frac{2L_0}{L_1}\left(\pi - \sin^{-1}(\frac{L_1\gamma_\eta}{L_0})\right)
\end{aligned}$$

Hence $\mathcal{L}_s^{\tilde{t}} < \frac{2L_0}{L_1}\left(\pi - \sin^{-1}(\frac{L_1\gamma_\eta}{L_0})\right)$. If $\tilde{t} \neq t$, then since $\mathcal{L}_s^{\tau}$ is a continuous and strictly increasing function of τ (because of non-vanishing vector field), we can always pick $t^* \in (\tilde{t}, t)$ such that

$$\mathcal{L}_s^{\tilde{t}} < \mathcal{L}_s^{t^*} < \frac{2L_0}{L_1}\left(\pi - \sin^{-1}(\frac{L_1\gamma_\eta}{L_0})\right).$$

Therefore by Corollary 12 we conclude that the solution is non-self-overlapping up to time t^*, which contradicts maximality of $\tilde{t}$. Thus $\tilde{t} = t$.

4.3. Change of V when passing through Ω_η

We now specify the threshold $\bar{\epsilon}$ in the statement of Theorem 1:

$$\bar{\epsilon} := \min\{\epsilon_1, \epsilon_2\},$$

where ϵ_1 is defined in (32) and

$$\epsilon_2 := \mathrm{vol}(B_{\gamma_\eta}^{n-1})\frac{L_0(b + \eta a c_1)^2}{\alpha\bar{L}_0 b}. \tag{33}$$

Note that when $\eta < 1$, we have $\gamma_\eta > 0$ and thus both ϵ_1, ϵ_2 are positive, which implies $\bar{\epsilon} > 0$. In addition, when (30) is satisfied and $\epsilon < \bar{\epsilon}$, $S_{\eta,(s,t)}$ is non-self-overlapping for any $(s, t) \in X_\eta$ by Lemma 14. Hence by Corollary 10 we have

$$t - s \le \frac{g\epsilon}{b} < \frac{g\bar{\epsilon}}{b} \le \frac{g\epsilon_2}{b} = \frac{(b + \eta a c_1)^2}{\alpha\bar{L}_0 b}. \tag{34}$$

These inequalities in (34) are essential and will be repeatedly used in the proofs of subsequent lemmas.

We now show that V will always decrease over any connected component of X_η excluding those containing boundary points 0 and T, if the latter exists. When the solution passes through the connected component containing the initial point $\tau = 0$ or $\tau = T$ then V may actually increase but is bounded by a fixed value. This is summarized in the next lemma:

Lemma 15. *Assume $\eta \in (0, 1)$ satisfies (30) and $\epsilon < \bar{\epsilon}$. For any connected component $(s, t) \subset X_\eta$, define $\Delta V_{(s,t)} := V(x(t)) - V(x(s))$. Then*

(1) If $s = 0$ and $V(x(0)) < c_2 - h\epsilon^{\frac{1}{n}} - g\epsilon$,

$$\Delta V_{(s,t)} \le \begin{cases} g\epsilon & \text{if } t = T,\\ \frac{g}{2}\epsilon & \text{if } t \neq T.\end{cases}$$

(2) If $s > 0$ and $V(x(s)) < c_2 - h\epsilon^{\frac{1}{n}} - \frac{g}{2}\epsilon$,

$$\Delta V_{(s,t)} \le \begin{cases} \frac{g}{2}\epsilon & \text{if } t = T,\\ \phi(t - s) & \text{if } t \neq T.\end{cases}$$

where

$$\phi(\tau) := \begin{cases} \frac{1}{4}\tau^2\alpha\bar{L}_0 - \tau\eta a c_1 & \text{if } \tau\alpha\bar{L}_0 < 2(b + \eta a c_1),\\ b\tau - \frac{(b+\eta a c_1)^2}{\alpha\bar{L}_0} & \text{if } \tau\alpha\bar{L}_0 \ge 2(b + \eta a c_1).\end{cases} \tag{35}$$

Remark 16. We observe that when $(t - s)\alpha\bar{L}_0 < 2(b + \eta a c_1)$, b does not appear in the bound for $\Delta V_{(s,t)}$. This corresponds to the case when the bound b is too loose, or the upper bound of $\dot{V}$ is unknown or not pre-determined. We have done studies of such less constrained almost Lyapunov functions previously and an example on which the theorem is applicable is not found yet.

Proof. The proof consists of four steps.

Case 1: ($s = 0$ **and** $t = T$).

Notice $\Delta V_{(s,t)} = \int_s^t \dot{V}(x(\tau))d\tau \le \int_s^t b d\tau = b(t - s) \le g\epsilon$ for any $(s, t) \subset X_\eta$. The last inequality comes from Corollary 10. Thus $g\epsilon$ is an upper bound for $\Delta V_{(s,t)}$ for any connected components (s, t) in X_η, in particular for the special case when both $s = 0$ and $t = T$.

Case 2: ($s = 0$ **and** $t \neq T$).

In this case t is finite. Since (s, t) is a maximal interval, either $x(t) \in \partial\Omega_\eta$ or $x(t) \in \partial D^*$, the boundary of D^*. If it is the latter one, we are only interested in the case when $\Delta V_{(0,t)} > 0$, that is, the case $V(x(t)) = c_2 - h\epsilon^{\frac{1}{n}}$. Notice that in this case $\Delta V_{(0,t)} = V(x(t)) - V(x(0)) > (c_2 - h\epsilon^{\frac{1}{n}}) - (c_2 - h\epsilon^{\frac{1}{n}} - g\epsilon) = g\epsilon$. This contradicts with the general upper bound of $g\epsilon$ on $\Delta V_{(s,t)}$ derived in Case 1. Thus we must have $x(t) \in \partial\Omega_\eta$ so $\dot{V}(x(t)) = -\eta a V(x(t)) \le -\eta a c_1$. Next we compute a tighter upper bound on $\Delta V_{(0,t)}$. It follows from (15) that for any $t_1, t_2 \in [s, t]$,

$$\begin{aligned}
|\dot{V}(x(t_1)) - \dot{V}(x(t_2))| &\le \alpha|x(t_1) - x(t_2)|\\
&\le \alpha\int_{t_1}^{t_2}|f(x(\tau))|d\tau \le \alpha\bar{L}_0|t_1 - t_2|.
\end{aligned} \tag{36}$$

Thus, $\dot{V}$, when considered as a function of time, is a Lipschitz function with Lipschitz constant $\alpha\bar{L}_0$. We can now estimate $\Delta V_{(0,t)} = \int_0^t \dot{V}(x(\tau))d\tau$ by collecting inequalities:

$$t < \frac{(b+\eta ac_1)^2}{\alpha\bar{L}_0 b}, \dot{V}(x(t)) \leq -\eta ac_1, \dot{V}(x(t_0)) \leq b,$$
$$|\dot{V}(x(t_1)) - \dot{V}(x(t_2))| \leq \alpha\bar{L}_0|t_1 - t_2| \ \forall t_0, t_1, t_2 \in [0, t]. \tag{37}$$

The first bound comes from (34) and the other bounds have been introduced earlier. We claim that a necessary condition for the inequalities in (37) to hold is:

$$\dot{V}(x(\tau)) \leq \min\{b, \alpha\bar{L}_0(t-\tau) - \eta ac_1\},$$

where the first bound b in the min function above is immediate. The second bound in the min function comes from $\dot{V}(x(t)) \leq -\eta ac_1$ and the Lipschitz bound on $\dot{V}$. Hence we conclude that its integration gives an upper bound for $\Delta V_{(0,t)}$:

$$\Delta V_{(0,t)} \leq \int_0^t \min\{b, \alpha\bar{L}_0(t-\tau) - \eta ac_1\}d\tau$$
$$= \int_0^t \min\{b, \alpha\bar{L}_0\tau - \eta ac_1\}d\tau$$

A change of variable is used for deriving the second line above. Notice that the minimum function switches value when $b = \alpha\bar{L}_0\tau - \eta ac_1$, that is, when $\tau = \frac{b+\eta ac_1}{\alpha\bar{L}_0}$. To estimate the integral, consider first the case when $t \geq \frac{b+\eta ac_1}{\alpha\bar{L}_0}$. In this case

$$\Delta V_{(0,t)} \leq \int_0^{\frac{b+\eta ac_1}{\alpha\bar{L}_0}} (\alpha\bar{L}_0\tau - \eta ac_1)d\tau + \int_{\frac{b+\eta ac_1}{\alpha\bar{L}_0}}^t b d\tau$$
$$= \frac{1}{2}\alpha\bar{L}_0\left(\frac{b+\eta ac_1}{\alpha\bar{L}_0}\right)^2 - \eta ac_1\frac{b+\eta ac_1}{\alpha\bar{L}_0}$$
$$+ b\left(t - \frac{b+\eta ac_1}{\alpha\bar{L}_0}\right)$$
$$= bt + \frac{(b+\eta ac_1)^2 - 2\eta ac_1(b+\eta ac_1) - 2b(b+\eta ac_1)}{2\alpha\bar{L}_0}$$
$$= bt - \frac{(b+\eta ac_1)^2}{2\alpha\bar{L}_0} < bt - \frac{bt}{2} \leq \frac{g}{2}\epsilon.$$

The two inequalities on the last line come from the inequalities in (34). Now, if $t < \frac{b+\eta ac_1}{\alpha\bar{L}_0}$, there is no switch and we only need to evaluate one integral:

$$\Delta V_{(s,t)} \leq \int_0^t (\alpha\bar{L}_0 s - \eta ac_1)ds = \frac{1}{2}\alpha\bar{L}_0 t^2 - \eta ac_1 t$$
$$= \left(\frac{1}{2}\alpha\bar{L}_0 t - \eta ac_1\right)t < \left(\frac{1}{2}\alpha\bar{L}_0\frac{b+\eta ac_1}{\alpha\bar{L}_0} - \eta ac_1\right)t$$
$$= \frac{1}{2}(b - \eta ac_1)t < \frac{b}{2}t \leq \frac{g}{2}\epsilon.$$

The last inequality above comes from (34). Thus we have shown that $\frac{g}{2}\epsilon$ is an upper bound for $\Delta V_{(s,t)}$ when $s = 0, t \neq T$.

Case 3: $(s \neq 0, t = T)$

We start by considering any connected component (s, t) such that $s \neq 0$. Again because it is maximal, we can only have $x(s) \in \partial\Omega_\eta$. This is because $x(s) \in \partial D^*$ is impossible as otherwise $x(\tau) \notin D^*$ for some $\tau < s$. Thus we should have $\dot{V}(x(s)) = -\eta aV(x(s)) \leq -\eta ac_1$. Similar to (37), we obtain the following inequalities for bounding $\Delta V_{(s,t)}$:

$$t - s < \frac{(b+\eta ac_1)^2}{\alpha\bar{L}_0 b}, \dot{V}(x(s)) \leq -\eta ac_1, \dot{V}(x(t_0)) \leq b,$$
$$|\dot{V}(x(t_1)) - \dot{V}(x(t_2))| \leq \alpha\bar{L}_0|t_1 - t_2| \ \forall t_0, t_1, t_2 \in [s, t], \tag{38}$$

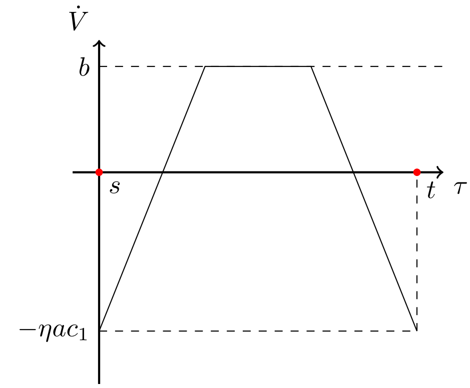

Fig. 3. Upper bound of $\dot{V}$ vs. τ on the trajectory passing through Ω_η.

where the first bound again comes from (34). The bounds are essentially the same as (37) but with the only difference that the boundary condition is $\dot{V}(x(s)) \leq -\eta ac_1$ instead of $\dot{V}(x(t)) \leq -\eta ac_1$. By symmetry considerations (change of variables $\tau' = t + s - \tau$ and then shift the time so $s = 0$), the upper bound will be the same and, thus, we have $\Delta V_{(s,t)} < \frac{g}{2}\epsilon$. This proves the special case when $\tau = T$, if $T < \infty$.

Case 4: $(s \neq 0, t \neq T)$

From the analysis in case 3 we see that $V(x(t)) = V(x(s)) + \Delta V_{(s,t)} < (c_2 - h - \frac{g}{2}\epsilon) + \frac{g}{2}\epsilon = c_2 - h$. Hence $x(t) \notin \partial D^*$. So by maximality of (s, t) we must have both $x(s), x(t) \in \partial\Omega_\eta$. Therefore, we have the following inequalities instead:

$$t - s < \frac{(b+\eta ac_1)^2}{\alpha\bar{L}_0 b}, \dot{V}(x(s)), \dot{V}(x(t)) \leq -\eta ac_1,$$
$$\dot{V}(x(\tau)) \leq b, |\dot{V}(x(t_1)) - \dot{V}(x(t_2))| \leq \alpha\bar{L}_0|t_1 - t_2|$$
$$\forall \tau, t_1, t_2 \in [s, t]. \tag{39}$$

By the same reasoning as we did for (37), we have the following bound as a necessary condition:

$$\dot{V}(\tau) \leq \min\{b, \alpha\bar{L}_0(\tau - s) - \eta ac_1, \alpha\bar{L}_0(t-\tau) - \eta ac_1\} \tag{40}$$

for all $\tau \in [s, t]$. Hence

$$\Delta V_{(s,t)} \leq \int_s^t \min\{b, \alpha\bar{L}_0(\tau - s) - \eta ac_1,$$
$$\alpha\bar{L}_0(t-\tau) - \eta ac_1\}d\tau$$
$$= \int_0^{t-s} \min\{b, \alpha\bar{L}_0\tau - \eta ac_1,$$
$$\alpha\bar{L}_0(t - s - \tau) - \eta ac_1\}d\tau.$$

An illustration of the upper bound of $\dot{V}$ over $[s, t]$ is plotted in Fig. 3, corresponding to the trajectory in Fig. 2. If $t - s \leq 2\frac{b+\eta ac_1}{\alpha\bar{L}_0}$ the functions to be minimized in (40) have only one switching point at $\frac{t-s}{2}$, and

$$\Delta V_{(s,t)} \leq \int_0^{\frac{t-s}{2}} (\alpha\bar{L}_0\tau - \eta ac_1)d\tau$$
$$+ \int_{\frac{t-s}{2}}^{t-s} (\alpha\bar{L}_0(t - s - \tau) - \eta ac_1)d\tau$$
$$= 2\int_0^{\frac{t-s}{2}} (\alpha\bar{L}_0\tau - \eta ac_1)d\tau$$
$$= \alpha\bar{L}_0(\frac{t-s}{2})^2 - 2\eta ac_1(\frac{t-s}{2})$$
$$= \frac{1}{4}\alpha\bar{L}_0(t-s)^2 - \eta ac_1(t-s).$$

If $(t-s) > 2\frac{b+\eta ac_1}{\alpha\bar{L}_0}$, there are two switching points: $\tau = \frac{b+\eta ac_1}{\alpha\bar{L}_0}$ and $\tau = t-s-\frac{b+\eta ac_1}{\alpha\bar{L}_0}$ so we have

$$\begin{aligned}\Delta V_{(s,t)} &\le \int_0^{\frac{b+\eta ac_1}{\alpha\bar{L}_0}} (\alpha\bar{L}_0\tau - \eta ac_1)d\tau + \int_{\frac{b+\eta ac_1}{\alpha\bar{L}_0}}^{t-s-\frac{b+\eta ac_1}{\alpha\bar{L}_0}} bd\tau \\ &+ \int_{t-s-\frac{b+\eta ac_1}{\alpha\bar{L}_0}}^{t-s} (\alpha\bar{L}_0(t-s-\tau) - \eta ac_1)d\tau \\ &= 2\int_0^{\frac{b+\eta ac_1}{\alpha\bar{L}_0}} (\alpha\bar{L}_0\tau - \eta ac_1)d\tau + \int_{\frac{b+\eta ac_1}{\alpha\bar{L}_0}}^{t-s-\frac{b+\eta ac_1}{\alpha\bar{L}_0}} b\,d\tau \\ &= \alpha\bar{L}_0(\frac{b+\eta ac_1}{\alpha\bar{L}_0})^2 - 2\eta ac_1(\frac{b+\eta ac_1}{\alpha\bar{L}_0}) \\ &+ b\left((t-s) - 2(\frac{b+\eta ac_1}{\alpha\bar{L}_0})\right) \\ &= b(t-s) - (\frac{b+\eta ac_1}{\alpha\bar{L}_0})^2.\end{aligned}$$

The two bounds are collected to be the ϕ function as stated in the lemma.

Now since we have assumed that $b < ac_1$ in the beginning, we can always pick an η sufficiently close to 1 to guarantee that

$$b < \eta ac_1. \tag{41}$$

From now on we will assume that η satisfies both (30) and (41). Notice that for the solution outside Ω_η, the almost Lyapunov function V clearly is decreasing; therefore, Lemma 15 also leads us to the following conclusion:

Corollary 17. *Consider a solution $x(\tau)$ with $V(x(0)) < c_2 - h\epsilon^{\frac{1}{n}} - g\epsilon$. Let (s,t) be a maximal connected component of X_η such that $s \neq 0, t \neq T$. Assume also $b < \eta ac_1$ and $\epsilon < \bar{\epsilon}$. Then $\Delta V_{(s,t)} \le \phi(t-s) < 0$.*

Proof. We first show that there are only finitely many connected components in X_η such that the corresponding arc of the solution enters Ω_0. (If the corresponding arc of the solution does not enter Ω_0, then V could only decrease and we declare this component for the purpose of this proof to be outside of X_η). This can be shown if the length of all such connected components is uniformly bounded from below. Note that except the cases $t_1 = 0$ or $t_2 = T$, if (t_1, t_2) is a connected component of X_η while there exists $t_3 \in (t_1, t_2)$ such that $x(t_3) \in \Omega_0$, we have $\dot{V}(x(t_1)) = -\eta aV(x(t_1)) \le -\eta ac_1$, $\dot{V}(x(t_3)) \ge 0$. Thus Eq. (15) in Lemma 5 gives

$$\begin{aligned}\eta ac_1 &\le |\dot{V}(x(t_1)) - \dot{V}(x(t_3))| \\ &\le \alpha|x(t_1) - x(t_3)| \le \alpha\bar{L}_0|t_1 - t_3|.\end{aligned}$$

Hence $t_2 - t_1 \ge t_3 - t_1 \ge \frac{\eta ac_1}{\alpha\bar{L}_0}$, which gives the lower bound. Thus, the number of connected components where V might increase has to be finite on a bounded time interval.

Next we prove Corollary 17 by induction. When $(t-s)\alpha\bar{L}_0 < 2(b+\eta ac_1)$, (41) implies $(t-s)\alpha\bar{L}_0 < 4\eta ac_1$ and hence the first line in (35) implies $\phi(t-s) = \frac{1}{4}(t-s)^2\alpha\bar{L}_0 - (t-s)\eta ac_1 < 0$. Otherwise, (34) implies $\phi(t-s) = b(t-s) - \frac{(b+\eta ac_1)^2}{\alpha\bar{L}_0} < 0$. Thus we always have $\phi(t-s) < 0$. Let (s,t) be the first connected component of X_η on the left with $s > 0$. If it is the first connected component on the left (i.e. there is no connected component starting at $\tau = 0$) then $V(x(s)) < V(x(0)) < c_2 - h\epsilon^{\frac{1}{n}} - g\epsilon$. If there is a connected component starting at $\tau = 0$, say the interval $(0, t_0)$, then still

$$\begin{aligned}V(x(s)) &\le V(x(0)) + \Delta V_{(0,t_0)} \\ &< (c_2 - g\epsilon - h\epsilon^{\frac{1}{n}}) + \frac{g}{2}\epsilon = c_2 - \frac{g}{2}\epsilon + h\epsilon^{\frac{1}{n}}.\end{aligned}$$

Either way, $V(x(s)) < c_2 - \frac{g}{2}\epsilon + h\epsilon^{\frac{1}{n}}$. Hence by Lemma 15, the base case is true and we have $\Delta V_{(s,t)} \le \phi(t-s) < 0$. Assume towards induction that at some connected component denoted also (s,t) we have $V(x(s)) < c_2 - \frac{g}{2}\epsilon - h$ and $\phi(t-s) < 0$. Then at the next connected component (s^+, t^+) we have

$$\begin{aligned}V(x(s^+)) &= \big(V(x(s^+)) - V(x(t))\big) + \Delta V_{(s,t)} + V(x(s)) \\ &\le \phi(t-s) + V(x(s)) < c_2 - \frac{g}{2}\epsilon - h\epsilon^{\frac{1}{n}}\end{aligned}$$

and by Lemma 15 we have $\Delta V_{(s^+,t^+)} \le \phi(t^+ - s^+) < 0$.

4.4. Exponential bound when repeatedly passing through Ω_η

Corollary 17 tells us that the Lyapunov function decreases each time the solution crosses Ω_η. This does not yet guarantee convergence to a smaller set. We now want to find an exponential type bound on V. Define $k(t) : \mathbb{R}_+ \to \mathbb{R}$ by

$$k(t) := \begin{cases} -\frac{1}{t}\ln\left(1 + \frac{1}{c_2}\phi(t)\right) & \text{if } \phi(t) > -c_2, \\ K & \text{if } \phi(t) \le -c_2. \end{cases}$$

where ϕ is defined in (35) and K is a sufficiently large positive constant. Note that $\phi(t)$ is continuous near 0 and $\phi(0) = 0$, so we can define $k(0) = \frac{\eta ac_1}{c_2}$ by extension via L'Hôpital's rule. In addition, define

$$\lambda(\epsilon) := \min_{0\le\delta\le\epsilon} k\left(\frac{g\delta}{b}\right).$$

By this definition, $\lambda(\epsilon)$ is a non-increasing function on $[0, \bar{\epsilon})$. On the one hand, we see from the proof of Corollary 17 that $\phi(t) < 0$ for all $t \in (0, \frac{(b+\eta ac_1)^2}{\alpha\bar{L}_0 b})$ and thus we have $k(t) > 0$ for all $t \in [0, \frac{(b+\eta ac_1)^2}{\alpha\bar{L}_0 b})$. In addition, because $\frac{g\bar{\epsilon}}{b} \le \frac{(b+\eta ac_1)^2}{\alpha\bar{L}_0 b}$ as in (34), $\lambda(\epsilon)$ is also positive on $[0, \bar{\epsilon})$. According to Corollary 10, $t-s \le \frac{g\epsilon}{b}$, which implies $k(t-s) \ge \min_{0\le\delta\le\epsilon} k\left(\frac{g\delta}{b}\right) = \lambda(\epsilon)$. Next, we have

$$\begin{aligned}V(x(t)) &= \Delta V_{(s,t)} + V(x(s)) = V(x(s))\left(1 + \frac{\Delta V_{(s,t)}}{V(x(s))}\right) \\ &\le V(x(s))\left(1 + \frac{\phi(t-s)}{c_2}\right) \\ &= V(x(s))e^{-k(t-s)(t-s)} \\ &\le V(x(s))e^{-\lambda(\epsilon)(t-s)}\end{aligned} \tag{42}$$

for any connected component of $(s,t) \subset X_\eta$ that does not contain the end points $\tau = 0$ or $\tau = t_{\max}$. From the second line to the third line the inequality $\Delta V_{(s,t)} \le \phi(t-s) < 0$ was used. We also have

$$\lambda(\epsilon) \le \lambda(0) = k(0) = \frac{\eta ac_1}{c_2} < \eta a$$

for all $\epsilon \in [0, \bar{\epsilon})$. Thus, when the solution is inside Ω_η, it has a decay rate slower than when the solution is in $D\backslash\Omega_\eta$, which has decay rate faster than ηa. We can modify $\lambda(\epsilon)$ so that it is a positive, continuous, strictly decreasing function on $[0, \bar{\epsilon})$ with $\lambda(0) < \eta a$ and so the inequality (42) still holds. As a result, for any $s, s' \in (0, T)\backslash\mathrm{int}X_\eta$, we have

$$V(x(s')) \le V(x(s))e^{-\lambda(\epsilon)(s'-s)}.$$

This exponential decaying bound suggests that T cannot be infinite, otherwise for $s' \in (0, T)\backslash\mathrm{int}X_\eta$ and large enough we will have $V(x(s')) < c_1 + h\epsilon^{\frac{1}{n}}$, implying $x(s') \notin D^*$, and such s' always exists when T is infinite because the possible connected component containing T has maximal length of $\frac{g\epsilon}{b}$.

Take an arbitrary $t \in [0, T]$. Recall that by Lemma 15 for any connected components of X_η, even those that contain the end points 0 and t, we still have the bound $\Delta V \leq \frac{g}{2}\epsilon$. Therefore, taking into account boundary components, we have

$$V(x(t)) \leq (V(x(0)) + \frac{g}{2}\epsilon)e^{-\lambda(\epsilon)(s'-s)} + \frac{g}{2}\epsilon \tag{43}$$

where $s' = t$ if $t \notin X_\eta$, or s' is the left boundary point of the connected component of X_η containing t otherwise; $s = 0$ if $s \notin X_\eta$, or s is the right boundary point of the connected component of X_η containing 0 otherwise. From (43) we directly see that

$$V(x(t)) \leq V(x(0)) + g\epsilon \quad \forall t \in [0, T]. \tag{44}$$

The first statement in the main Theorem follows from (44) up to time T. In addition, by Corollary 10,

$$s \leq \frac{g\epsilon}{b}, \quad t - s' \leq \frac{g\epsilon}{b} \Rightarrow s' - s \geq t - 2\frac{g\epsilon}{b}.$$

Substituting these expressions into (43), we have

$$V(x(t)) \leq e^{2\lambda(\epsilon)\frac{g\epsilon}{b}}(V(x(0)) + \frac{g}{2}\epsilon)e^{-\lambda(\epsilon)t} + \frac{g}{2}\epsilon. \tag{45}$$

This is also true for $t = T$. By definition of T in (23) we see that $x(T) \in \partial D^*$ and because of the exponential decaying bound in (45) so we must have $V(x(T)) = c_1 + h\epsilon^{\frac{1}{n}}$. The argument cannot proceed for $t > T$ because as $x(t)$ is outside of D^*, Lemma 9 cannot be applied and $B_{\gamma_\eta}(x(t))$ may not be contained in D even if $\dot{V}(x(t)) \leq -\eta a V(x(t))$; consequently the estimation of the sweeping volume, based on the bounds $\underline{L}_0, L_1$ etc. defined over D is no longer valid. Nevertheless, once the solution returns to the lower boundary of D^* such that $V(x(t)) = c_1 + h\epsilon^{\frac{1}{n}}$, it can be again treated as a new solution starting from $x(0) \in D$ with $V(x(0)) < c_2 - h\epsilon^{\frac{1}{n}} - g\epsilon$ and by the same analysis above we know that it can have an overshoot of $g\epsilon$ at most. This proves the second statement in the main theorem.

5. Global uniform asymptotic stability result by almost Lyapunov function

Our Theorem 1 gives a local convergence property so that any solution in the domain converges to a lower level set. It is often desirable to establish a global convergence property so the solutions converge to a stable equilibrium. One typical stability property for autonomous systems is *Global Uniform Asymptotic Stability* (GUAS), which means that the system is *globally stable* in the sense that for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $|x(0)| \leq \delta$, $|x(t)| \leq \varepsilon$ for all $t \geq 0$ and *uniformly attractive* in the sense that for any $\delta > 0, \kappa > 0$, there exists $T = T(\delta, \kappa)$ such that whenever $|x(0)| \leq \kappa$, $|x(t)| \leq \delta$ for all $t \geq T$. We now try to transfer our study to a global result. To do that, instead of a fixed region D defined by two constants c_1, c_2, we let the band-shaped region be defined for any $c > 0$:

$$D(c) := \{x \in \mathbb{R}^n : c \leq V(x) \leq 2c\}. \tag{46}$$

Following the definitions of $b, \bar{L}_0, \underline{L}_0, L_1, M_1, M_2$ from (6), (9), (10), (11), (12), (13) over the region $D(c)$, we see that now all of them are functions of c. We present a global uniform asymptotic stability result derived using an almost Lyapunov function:

Theorem 18. *Consider system (1) with a globally Lipschitz right-hand side f, and a function $V : \mathbb{R}^n \to [0, +\infty)$ which is positive definite and C^1 with globally Lipschitz gradient. In addition assume $V(x) \geq k_0|x|^2$ for some $k_0 > 0$ and all $x \in \mathbb{R}^n$. For any $c > 0$, let the region $D(c)$ be defined via (46) and assume that all of them are compact. Let $\Omega := \{x \in \mathbb{R}^n : \dot{V}(x) \geq -aV(x)\}$ for some $a > 0$. Assume $\sup_{c>0} \frac{b(c)}{ac} < 1$ where $b(c)$ is defined via (6) over $D(c)$. Let $\underline{L}_0(c)$ be defined via (10) over $D(c)$. Then there exist $K_1, K_2, K_3 > 0$ such that if $\mathrm{vol}(\Omega^*(c)) < \min\{K_1\underline{L}_0(c)^n, K_2 c^{\frac{n-1}{2}}\underline{L}_0(c), K_3 c^{\frac{n}{2}}\}$ for all $c > 0$ where $\Omega^*(c)$ is the largest connected component of $\Omega \cap D(c)$, the system (1) is GUAS.*

Before giving the proof of Theorem 18, let us discuss the validity and some variations of the assumptions of this theorem first. If we know that the system is globally stable or the working space is some compact set in $\mathbb{R}^n$ instead of $\mathbb{R}^n$ itself, then we can replace global Lipschitzness in f and V_x by local Lipschitzness as it is sufficient for the existence of uniform L_1, M_2, which will be used in the proof. The assumption $V(x) \geq k_0|x|^2$ is quite general since all quadratic Lyapunov function satisfies this assumption. Other assumptions are merely same as or the general versions of the assumptions in Theorem 1. The non-vanishing assumption is also reflected in the theorem statement that if f vanishes at any state which is different from the origin, $\underline{L}_0(c) = 0$ for some $c > 0$ and this theorem becomes inconclusive.

Proof. The idea of the proof is to repeatedly apply Theorem 1 over the region $D(c)$ for any $c > 0$ and show that $V(x(t))$ is bounded and will decrease by a factor of fixed factor each time.

First of all, globally Lipschitz f and V_x mean there exist $k_1, k_2 > 0$ such that $L_1 \leq k_1, \quad M_2 \leq k_2$, where L_1, M_2 are the global Lipschitz constants of f, V_x, respectively. In addition, if x^* is the maximizer of $|f(x)|$ in $D(c)$,

$$\bar{L}_0(c) = \max_{x \in D(c)} |f(x)| = |f(x^*)| = |f(x^*) - f(0)|$$

$$\leq L_1|x^* - 0| \leq k_1|x^*| \leq k_1\sqrt{\frac{V(x^*)}{k_0}} \leq k_1\sqrt{\frac{2c}{k_0}}.$$

By similar argument we also have $M_1 \leq k_2\sqrt{\frac{2c}{k_0}}$. Thus, $\alpha = M_1L_1 + M_2\bar{L}_0 \leq 2k_1k_2\sqrt{\frac{2c}{k_0}}$. Using $\eta \in (0, 1)$, (16) in Lemma 6 becomes

$$\gamma_\eta = \frac{(1-\eta)ac}{\alpha + \eta a M_1} \geq \frac{(1-\eta)ac}{2k_1k_2\sqrt{\frac{2c}{k_0}} + ak_2\sqrt{\frac{2c}{k_0}}} =$$

$$\frac{(1-\eta)a\sqrt{k_0}}{\sqrt{2}(2k_1 + a)k_2}c^{\frac{1}{2}} = (1-\eta)Kc^{\frac{1}{2}} =: \gamma^*,$$

where $K := \frac{a\sqrt{k_0}}{\sqrt{2}(2k_1+a)k_2}$ is a constant. For each $c > 0$, pick $\eta(c) \in (\frac{1}{2}, 1)$ such that

$$1 - \eta(c) < \min\left\{\frac{\underline{L}_0(c)}{2k_1K\sqrt{c}}, 1 - \sup_{c>0}\frac{b(c)}{ac}\right\}, \tag{47}$$

This can be done as the arguments in the min function on the right side of (47) are always positive (the positiveness of the second argument is given by the theorem assumption). This also means that,

$$\gamma^* < \min\left\{\left(1 - \sup_{c>0}\frac{b(c)}{ac}\right)Kc^{\frac{1}{2}}, \frac{\underline{L}_0}{2k_1}\right\}, \tag{48}$$

which tells us that by a proper choice of $\eta(c)$ satisfying (47), γ^* will be bounded by the minimum of two increasing functions of $c, \underline{L}_0$, respectively. Also by definition we know $\gamma^* \leq \gamma_\eta$, so the result in Lemma 9 holds for γ^* as well. In addition, the inequality between γ^* and $\frac{\underline{L}_0}{2k_1}$ in (48) tells that $\gamma^* < \frac{\underline{L}_0}{2k_1} \leq \frac{\underline{L}_0}{2L_1} < \frac{\underline{L}_0}{L_1}$ and the inequality between $1 - \eta(c)$ and $1 - \sup_{c>0}\frac{b(c)}{ac}$ in (47) tells that $\eta(c) > \sup_{c>0}\frac{b(c)}{ac}$ and thus $b(c) < \eta(c)ac$ for all $c > 0$. Therefore the bound (48) guarantees that both (30) and (41) are satisfied; γ^* is indeed a valid sweeping tube radius and hence all the subsequent results still follow if we replace every γ_η by γ^*. Now define

$$\epsilon_3 := \frac{\underline{L}_0(c)\mathrm{vol}(B^{n-1}_{\gamma^*})}{4b}c, \tag{49}$$

$$\epsilon_4 := \mathrm{vol}(B^n_{r(c)}), \quad r(c) = \sqrt{\frac{k_0 c}{32k_2^2}}. \tag{50}$$

Then $\epsilon < \epsilon_3$ with g substituted by its definition (21) implies

$$g\epsilon < \frac{b\epsilon_3}{\underline{L}_0\mathrm{vol}(B^{n-1}_{\gamma^*})} < \frac{1}{4}c.$$

On the other hand, $\epsilon < \epsilon_4$ with h substituted by its definition (22) implies

$$h\epsilon^{\frac{1}{n}} = M_1\left(\frac{\epsilon}{\chi(n)}\right)^{\frac{1}{n}} < k_2\sqrt{\frac{2c}{k_0}}\left(\frac{\epsilon_4}{\chi(n)}\right)^{\frac{1}{n}}$$
$$= k_2\sqrt{\frac{2c}{k_0}}r(c) < \frac{1}{4}c.$$

So we have both $g\epsilon$ and $h\epsilon^{\frac{1}{n}}$ bounded from above by $\frac{1}{4}c$ when ϵ is small enough.

Now for any initial state $x(0) \in \mathbb{R}^n$, we let $c = \frac{2}{3}V(x(0))$. Then $x_0 \in D(c)$ and we try to apply Theorem 1 on it. Notice that $V(x_0) = \frac{3}{2}c < 2c - h\epsilon^{\frac{1}{n}} - g\epsilon$, thus the initial state satisfies the hypothesis. Hence we conclude from Theorem 1 that for ϵ small enough, $V(x(t)) \le V(x(0)) + g\epsilon \le \frac{7}{4}c$ for all $t \ge 0$ and $V(x(t)) \le c + h\epsilon^{\frac{1}{n}} < \frac{5}{4}c$ for some $t \le T(c,\epsilon)$. The global stability part is given by the first conclusion by letting $\delta = \frac{7}{6}\varepsilon$. The second conclusion tells that

$$\frac{V(x(t))}{V(x(0))} < \frac{\frac{5}{4}c}{\frac{3}{2}c} = \frac{5}{6}$$

Thus over each iteration $|x(t)|$ is decreased at least by a factor of $\frac{5}{6}$, in time at most T. We then reset time t to be the initial time and can repeat the same argument. Thus while given δ and κ, the total number of iterations is $\lceil\frac{\ln\kappa-\ln\delta}{\ln 6-\ln 5}\rceil$ for a solution that starts from $\bar{B}^n_\kappa(0)$ and converges to $\bar{B}^n_\delta(0)$. The total time needed is bounded by the summation of $T(c,\epsilon)$'s of each iteration and hence for given ϵ, it only depends on κ,δ.

It remains to find how small ϵ needs to be; that is, find an expression of $\bar{\epsilon}$, which is the common lower bound of $\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4$, in terms of $c,\underline{L}_0$. Recall from (32) and (33) that we have

$$\epsilon_1 = \mathrm{vol}(B^{n-1}_{\gamma^*})\frac{2\underline{L}_0}{L_1}\left(\pi - \sin^{-1}(\frac{L_1\gamma^*_\eta}{\underline{L}_0})\right)$$
$$\ge \mathrm{vol}(B^{n-1}_{\gamma^*})\underline{L}_0\frac{2}{k_1}\left(\pi - \sin^{-1}(\frac{1}{2})\right),$$

$$\epsilon_2 = \mathrm{vol}(B^{n-1}_{\gamma_\eta})\frac{\underline{L}_0(b+\eta ac)^2}{\alpha\bar{L}_0 b}$$
$$\ge \mathrm{vol}(B^{n-1}_{\gamma^*})\underline{L}_0\frac{4b\eta ac}{2k_1^2k_2\frac{2c}{k_0}b} > \mathrm{vol}(B^{n-1}_{\gamma^*})\underline{L}_0\frac{a}{2k_1^2k_2},$$

where on the second line the assumption $\eta > \frac{1}{2}$ is used. Meanwhile, from (49) we have

$$\epsilon_3 = \frac{\underline{L}_0(c)\mathrm{vol}(B^{n-1}_{\gamma^*})}{4b}c > \mathrm{vol}(B^{n-1}_{\gamma^*})\underline{L}_0\frac{1}{4a}.$$

It is observed from the above inequalities that a common lower bound of $\epsilon_1,\epsilon_2,\epsilon_3$ is of the form $K_0\mathrm{vol}(B^{n-1}_{\gamma^*})\underline{L}_0$ with some constant $K_0 > 0$. Recall from (48) that γ^* is chosen to be the minimum between two linear increasing functions of $\underline{L}_0, c^{\frac{1}{2}}$, respectively. Thus $\mathrm{vol}(B^{n-1}_{\gamma^*})$ is the minimum between two linear increasing functions of $\underline{L}_0^{n-1}, c^{\frac{n-1}{2}}$, respectively. As a result,

$$\min\{\epsilon_1,\epsilon_2,\epsilon_3\} \ge \min\{K_1\underline{L}_0^n, K_2c^{\frac{n-1}{2}}\underline{L}_0\}$$

In addition, (50) means that ϵ_4 is a linear function of $c^{\frac{n}{2}}$. Put them together, we have

$$\bar{\epsilon} := \min\{K_1\underline{L}_0^n, K_2c^{\frac{n-1}{2}}\underline{L}_0, K_3c^{\frac{n}{2}}\} \le \min\{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4\} \tag{51}$$

This $\bar{\epsilon}$ is the upper bound of ϵ in Theorem 1. As a result, as long as $\mathrm{vol}(\Omega^*(c)) < \bar{\epsilon}$ for all $c > 0$ where $\Omega^*(c)$ is the largest connected component of $\Omega \cap D(c)$, the system (1) is GUAS.

6. Example and discussion

6.1. Example

The system (1) is explicitly defined as follows:

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = f(x) = \begin{pmatrix}-\lambda(x) & \mu\\ -\mu & -\lambda(x)\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} \tag{52}$$

where $\lambda(x) = 1.01\min\left\{\frac{|x-x_c|}{\rho}, 1\right\} - 0.01$, $x_c = (0.8, 0)^\top$, $\mu = 2$, $\rho = 0.01$. The relevant part of the phase portrait for the vector field $f(x)$ with a solution $x(t)$ passing through is shown in Fig. 4. Notice that the spiral-shaped vector field is distorted in the region of $B_\rho(x_c)$. The solution $x(t)$ passing through this region will temporarily move away from the origin when passing through $B_\rho(x_c)$. More explicitly, we consider the function

$$V(x) = |x|^2 = x_1^2 + x_2^2$$

as a candidate Lyapunov function. Then

$$\dot{V}(x) = 2(x_1\dot{x}_1 + x_2\dot{x}_2) = -2\lambda(x)(x_1^2 + x_2^2). \tag{53}$$

Notice that $\lambda(x) = 1$ everywhere except in $B_\rho(x_c)$. Outside this ball $B_\rho(x_c)$ the system is linear and satisfies the decay condition $\dot{V} = -2V$. When $x(t)$ is very close to x_c, $\lambda(x)$ becomes negative and $\dot{V}$ becomes positive. Hence for this system $\Omega_0 \neq \emptyset$ and V is not a Lyapunov function for this system but only an almost Lyapunov function. Nevertheless, we will show by our theorem that convergence to 0 takes place as the effect of Ω is not strong. To do so, choose $d_1 = 0.7, d_2 = 1, c_1 = d_1^2, c_2 = d_2^2$. We find that

$$|f(x)| = \sqrt{f(x)^\top f(x)}$$
$$= \sqrt{\begin{pmatrix}x_1 & x_2\end{pmatrix}\begin{pmatrix}-\lambda(x) & -\mu\\ \mu & -\lambda\end{pmatrix}\begin{pmatrix}-\lambda(x) & \mu\\ -\mu & -\lambda\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}}$$
$$= \sqrt{(\lambda^2(x)+\mu^2)(x_1^2+x_2^2)}$$
$$= |x|\sqrt{(\lambda^2(x)+\mu^2)}$$

Hence on the set $D = \{x : d_1 \le |x| \le d_2\}$, $\bar{L}_0 = d_2 \times \sqrt{\max\lambda(x)^2 + \mu^2} = \sqrt{5}$, $\underline{L}_0 = d_1 \times \sqrt{\min\lambda(x)^2 + \mu^2} = 1.4$.

The parameter L_1 was computed numerically to be 90.78. Since $V_x(x) = 2(x_1, x_2)$, $M_1 = 2d_2 = 2$, $M_2 = 2$. In addition, from (53) we see that $b = -2\min_{x\in D}\lambda(x)|x|^2$. The minimum is achieved at $x = x_c$ and it is computed to be $b = 0.0128$.

Naturally pick $a = 2$ so that $\Omega = B_\rho(x_c)$. Thus, $\epsilon = \mathrm{vol}(\Omega) = \pi\rho^2 \approx 3.14\times 10^{-4}$. Also note that this Ω is completely inside D.

Pick $\eta = 0.6$. It can be calculated that $\alpha = M_1L_1 + \bar{L}_0M_2 \approx 186$, $\gamma_\eta = \frac{(1-\eta)ac_1}{\alpha+\eta aM_1} \approx 0.0021 \le 0.0154 = \frac{\underline{L}_0}{L_1}$. Thus (30) is satisfied. In addition, $\eta ac_1 = 0.588 > b$ so (41) is also satisfied. Hence $\eta = 0.6$ is large enough. We can then compute $\bar{\epsilon}$: $\epsilon_1 = 4\gamma_\eta\frac{\underline{L}_0}{L_1}\left(\pi - \sin^{-1}(\frac{L_1\gamma_\eta}{\underline{L}_0})\right) \approx 3.86\times 10^{-4}$, $\epsilon_2 = \frac{2\gamma_\eta\underline{L}_0(b+\eta ac_1)^2}{\alpha\bar{L}_0 b}$

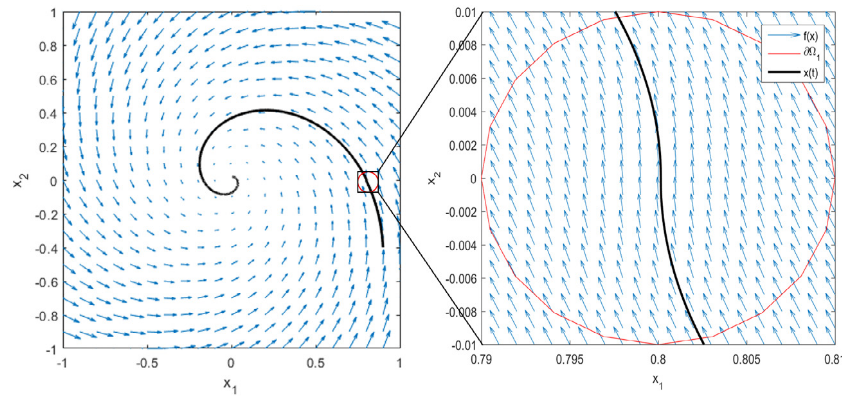

Fig. 4. Left: overview of the solution trajectory. Right: zoom in view of the local behavior. The region inside the red circle is Ω, where $\dot{V} \geq -aV$.

$\approx 3.95 \times 10^{-4}$. Therefore $\bar{\epsilon} = \max\{\epsilon_1, \epsilon_2\} = 3.95 \times 10^{-4}$. Indeed we have $\epsilon < \bar{\epsilon}$ so all the hypothesis in Theorem 1 hold. Meanwhile, $h = M_1\gamma_\eta \approx 0.0042$, $g\epsilon = \frac{b\mathrm{vol}(\Omega_1)}{2\gamma_\eta \underline{L}_0} \approx 6.9 \times 10^{-4} \ll c_2 - c_1$. The conclusions in Theorem 1 tell us that the system will converge to the set $\{x : V(x) \leq c_1 + h + g\epsilon\} \approx B_{0.7044}(0)$ if it starts at x_0 with $V(x_0) \leq c_2 - h - g\epsilon \approx 0.9951$.

6.2. Discussion of the example

Firstly, because our V is chosen to be quadratic and we know from the earlier discussion in Section 4.4 that the convergence of V is exponential, we can further conclude that the convergence of the solution to the ball $B_{0.7044}(0)$ is exponentially fast. In addition, since $\dot{V}(x) = -2V(x)$ for all $x \in B_{0.7044}(0) \cup \{x : V(x) > 0.9951\}$, the system is in fact globally exponentially stable.

It is important to note, as discussed earlier, that in this example $\Omega_0 \neq \emptyset$. By continuity of $\dot{V}$ as a function of states, we know that there will be $x' \in \Omega$ such that $V_x(x') \cdot f(x') = \dot{V}(x') = 0$ (which is in fact on $\partial\Omega_0$). If we do not require the vector field to be non-vanishing, then since $V_x(x) = 2x \neq 0$ for all $x \in D$, we either have $f(x') = 0$ or $V_x(x')$ is orthogonal to $f(x')$. In the first case x' is an equilibrium of the system and we will have a solution $x(t) \equiv x'$, which would not converge to a smaller set and hence the conclusion in Theorem 1 is no longer true. This indicates that the additional assumption of non-vanishing (which results in the positive bound $\underline{L}_0$) is indeed crucial to establishing the convergence result.

Recall that the significance of our main theorem appears when there are multiple "bad regions" with the volume of each of them bounded above. For instance, by modifying the vector field of the above example such that Ω consists of multiple $B_\rho(x^i)$ regions distributed in D with $|x^i| = 0.8$ for all i, our main theorem is still applicable and will lead to the same conclusion.

Nevertheless, the obtained $\bar{\epsilon}$ appears to be rather conservative. One can observe in the above example that the radius of the sweeping ball is quite small as $\gamma_\eta \approx \frac{1}{5}\rho$; as a result, $\bar{\epsilon}$ which is proportional to $\mathrm{vol}(B^{n-1}_{\gamma_\eta})$ becomes very small. It is not hard to see from the proofs of Lemmas 4–6 that γ_η is a very coarse bound on the radius of the largest ball that is contained in Ω. More careful analysis can be done on tightening γ_η; however, this may require additional information about system dynamics. Our current assumptions on the system, on the other hand, are rather general.

In addition, once η is chosen, a sweeping ball of constant radius is employed for the analysis. We can make γ_η time-varying based on the level set of Ω_η that x is in. Since it is known that the radius of the sweeping ball becomes larger when $\dot{V}$ becomes positive, $\bar{\epsilon}$ will be larger and this modification should yield a better result. However, difficulties arise in converting the bound (31) on the length of a particular trajectory to a (32)-like bound on the volume of Ω_1.

7. Conclusion

We presented a result (Theorem 1) which establishes convergence of system trajectories from a given set to a smaller set, based on an almost Lyapunov function which is known to decrease along solutions on the complement of a set of small enough volume. We have also developed Theorem 18 saying that under mild assumptions on the system, the result of Theorem 1 can be iterated so that when the volume where $\dot{V}$ is not negative enough is small, the system can still be shown to be GUAS. The study of almost Lyapunov functions provides an alternative way to study stability related properties of nonlinear systems, potentially for perturbations of stable systems, polynomial systems or systems where $\dot{V}$ can only be checked discretely and numerically.

Appendix A. A previous result

We provide a slightly different result in this section. In this case the region of interest is defined as:

$$D := \{x \in \mathbb{R}^n : V(x) \leq c\} \tag{A.1}$$

Notice that in this case D is defined with the origin included, in contrast to the one defined for Theorem 1 which excludes a neighborhood of origin. Here is the theorem statement:

Theorem 19 (*Liberzon et al., 2014*)**.** *Let $\rho : (0, +\infty) \to (0, +\infty)$ be the relation such that*

$$\mathrm{vol}(B_{\rho(\epsilon)}) = \epsilon$$

Consider the system (1) with a locally Lipschitz right-hand side f, and a function V which is positive definite and $\mathcal{C}^1$ with locally Lipschitz gradient. Let the region D be defined via (A.1) and assume that it is compact. Assume that (3) holds. Then there exist a constant $\bar{\epsilon} > 0$ and a continuous, strictly increasing function $\bar{R}$ on $[0, \bar{\epsilon}]$ with $\bar{R}(0) = 0$ such that for every $\epsilon \in (0, \bar{\epsilon})$, if $\mathrm{vol}(\Omega) < \epsilon$, then for every initial condition $x_0 \in D$ with

$$V(x_0) < c - 2M_1\rho(\epsilon)$$

where M_1 is defined by (12), the corresponding solution $x(\cdot)$ of (1) with $x(0) = x_0$ has the following properties:

(1) $V(x(t)) \leq V(x_0) + 2M_1\rho(\epsilon)$ for all $t \geq 0$ (and hence $x(t) \in D$ for all $t \geq 0$).
(2) $V(x(T)) \leq \bar{R}(\epsilon)$ for some $T \geq 0$.
(3) $V(x(t)) \leq \bar{R}(\epsilon) + 2M_1\rho(\epsilon)$ for all $t \geq T$.

The proof of Theorem 19 is established by a perturbation argument which compares a given system trajectory with nearby trajectories that lie entirely in $D\backslash\Omega$ and trades off convergence speed of these trajectories against the expansion rate of the distance to them from the given trajectory. For more details of the proof of Theorem 19, please refer to Liberzon et al. (2014). Notice that in the special case when $\dot{V}(x) \le -aV(x)$ for all $x \in D$ (which implies that ϵ can be any arbitrarily small positive number), Theorem 19 reduces to Lyapunov's classical asymptotic stability theorem. On the other hand, we cannot recover asymptotic stability from Theorem 1 when $\mathrm{vol}(\Omega) = 0$ simply because a neighborhood of origin is taken away from D. At first sight one may think the main Theorem 1 in this paper has some drawbacks as it requires extra conditions (existence of positive $\underline{L}_0, b$) to hold than Theorem 19; meanwhile, the result of Theorem 1 seems to be weaker than that of Theorem 19 due to the existence of gap h in all three statements, which unlike $g\epsilon$ in Theorem 1 or $R(\epsilon)$ in Theorem 19 and does not vanish as ϵ goes to 0. Nevertheless, we need to point out that the two $\bar{\epsilon}$'s in both theorems are very different; in fact the $\bar{\epsilon}$ in Theorem 19 is very conservative compared with that of Theorem 1. In order to fulfill the condition in Theorem 19, we need $\mathrm{vol}(\Omega) < \bar{\epsilon}$. However, we failed to construct a non-trivial example with $\dot{V}(x) > 0$ for some $x \in D$ while maintaining that inequality. This is left as an open question in Liberzon et al. (2014). An interesting observation is that by perturbing the system dynamics without increasing the Lipschitz constant, which is used in computing $\bar{\epsilon}$, an unstable equilibrium can be constructed away from the origin. There will be contradiction if Theorem 19 is applicable to such a system because a solution starting at that unstable equilibrium will not move, contrary to what is concluded from the theorem that the solution will be attracted to a neighborhood of the origin. On the other hand, if we try to apply Theorem 19 to the example in Section 6, through the procedure in Liberzon et al. (2014) we find that $\bar{\epsilon} < \pi\rho^2$, thus Theorem 19 is inconclusive. Hence we prefer to apply Theorem 1 with a modified region D.

Appendix B. Proof of Proposition 11

If a space curve $x^*(s), s \in [0, \mathcal{L}]$ is closed ($x^*(0) = x^*(\mathcal{L})$) and piecewise C^2, we set $I := \{s \in [0, \mathcal{L}) : \frac{d}{ds}x^*(s)$ does not exist$\}$. For each $s \in I$, we define the turning angle $\varphi_t(s)$ to be the oriented angle from the vector $\frac{d}{ds}x^*(s^-)$ (or $\frac{d}{ds}x^*(\mathcal{L}^-)$ if $s = 0$) to the vector $\frac{d}{ds}x^*(s^+)$. Then total curvature is defined as

$$K = \int_{s\in[0,\mathcal{L})\backslash I} \kappa(s)ds + \sum_{s\in I} \varphi_t(s)$$

In order to prove Proposition 11, two geometrical results are needed:

Lemma 20 (*Fenchel's Theorem Sullivan, 2008, Theorem 2.4*)**.** *For any closed space curve $x(s)$,*

$$K \ge 2\pi$$

and equality holds if and only if $x(s)$ is a convex planar curve.

Lemma 21 (*Schur's Comparison Theorem Sullivan, 2008, Theorem 5.1*)**.** *Suppose $C(s)$ is a plane curve with curvature $\kappa(s)$ which makes a convex curve when closed by the chord connecting its endpoints, and $C^*(s)$ is an arbitrary space curve of the same length with curvature $\kappa^*(s)$. Let d be the distance between the endpoints of C and d^* be the distance between the endpoints of C^*. If $\kappa^*(s) \le \kappa(s)$ then $d^* \ge d$.*

Suppose self-overlapping occurs between $N_{\rho_0}(x(t))$ and $N_{\rho_0}(x(s))$ for some $t > s$. We prove the proposition by showing that contradictions arise if $\mathcal{L}_s^t < 2\rho\left(\pi - \sin^{-1}(\frac{\rho_0}{\rho})\right)$.

Rewrite $\mathcal{L}_s^t = 2\rho\theta$ for some $\theta \in \left(0, \pi - \sin^{-1}(\frac{\rho_0}{\rho})\right)$. Let $z \in N_{\rho_0}(x(t)) \cap N_{\rho_0}(x(s))$. Denote the angle between vector $z \to x(t)$ and vector $z \to x(s)$ by ϕ_z. Notice that the curve $x(\tau)$ over $[s, t]$ and the two vectors $z \to x(t), z \to x(s)$ form a closed curve. Evaluating the total curvature alone this closed curve and applying Fenchel's Theorem and realizing that the turning angles at $x(t)$, $x(s)$ are both $\frac{\pi}{2}$ because they are on the normal disks, and the fact that the turning angle at z is the complement of ϕ_z, we have

$$\begin{aligned} 2\pi \le K &= \left(\int_{x(t)}^{x(s)} \kappa(x)dx\right) + \varphi_t(x(s)) + \varphi_t(x(t)) + \varphi_t(z) \\ &\le \int_{x(t)}^{x(s)} \frac{1}{\rho}dx + \varphi_t(x(s)) + \varphi_t(x(t)) + \varphi_t(z) \\ &= \frac{\mathcal{L}_s^t}{\rho} + \frac{\pi}{2} + \frac{\pi}{2} + (\pi - \phi_z). \end{aligned}$$

Therefore

$$\phi_z \le 2\theta. \tag{B.1}$$

Now we establish the contradiction in 3 different cases, based on the value of θ:

Case 1. $\theta < \frac{\pi}{4}$. Notice that because $f(x(t)), f(x(s))$ are normal vectors of $N_{\rho_0}(x(t))$, $N_{\rho_0}(x(s))$, the angle between them is the same as the dihedral angle between the two hyperplanes that contain the two normal disks, which is the maximal value of ϕ_z over all possible z along the intersection of the two hyperplanes. Because (B.1) always holds for such ϕ_z, it also holds for the maximum, hence in this case the angle between $f(x(t))$ and $f(x(s))$ is acute. Now because $\rho_0 < \rho$, the velocity of each point on the normal disk $N(x(\cdot))$ is in the same direction as $f(x(\cdot))$ when $N(x(\cdot))$ "sweeps" with respect to time. Thus renaming t by τ and using the earlier result of acute angle between $f(x(\tau))$ and $f(x(s))$, we see that the velocity of each point on the normal disk $N(x(\tau))$ has positive component in the $f(x(s))$ direction for all $\tau \in [s, t]$. In other words, the disk $N(x(t))$ moves away from $N(x(s))$ so self-overlapping is impossible.

Case 2. $\theta \in [\frac{\pi}{4}, \frac{\pi}{2})$. In this case, compare the solution $x(\cdot)$ to a circular arc with constant curvature $\frac{1}{\rho}$ and same arc length of $2\rho\theta$. Notice that such a circular arc has central angle 2θ and therefore the chord length is $2\rho\sin\theta$. By Schur's Comparison Theorem,

$$|x(t) - x(s)| \ge 2\rho\sin\theta \ge \sqrt{2}\rho.$$

In addition, $z \in N_{\rho_0}(x(t)) \cap N_{\rho_0}(x(s))$ means $|z - x(t)| \le \rho_0 < \rho$, $|z - x(s)| \le \rho_0 < \rho$. Thus $|z - x(t)|^2 + |z - x(s)|^2 < 2\rho^2 \le |x(t) - x(s)|^2$, which not only means that ϕ_z is obtuse, but also implies that

$$\begin{aligned} \cos\phi_z &= \frac{|z - x(t)|^2 + |z - x(s)|^2 - |x(t) - x(s)|^2}{2|z - x(t)|\,|z - x(s)|} \\ &< \frac{\rho^2 + \rho^2 - (2\rho\sin\theta)^2}{2\rho^2} = \cos 2\theta. \end{aligned}$$

Hence $\phi_z > 2\theta$, contradicting (B.1) so self-overlapping is impossible in this case.

Case 3. $\theta \in [\frac{\pi}{2}, \pi - \sin^{-1}(\frac{\rho_0}{\rho}))$. In this case we repeat the same procedure of comparing the solution $x(\cdot)$ to a circular arc. Again Schur's Comparison Theorem tells us that

$$|x(t) - x(s)| \ge 2\rho\sin\theta > 2\rho\sin(\pi - \sin^{-1}(\frac{\rho_0}{\rho})) = 2\rho_0.$$

Because $x(t)$ and $x(s)$ are separated by more than $2\rho_0$, self-overlapping is impossible.

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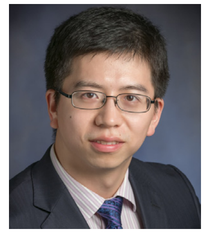

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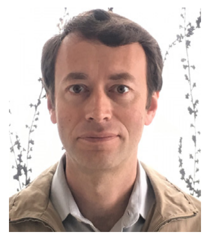

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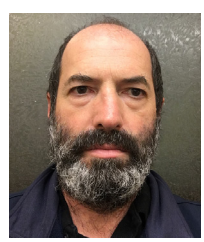

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