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On Classical Solutions for Viscous Polytropic Fluids with Degenerate Viscosities and Vacuum

Yachun Li, Ronghua Pan & Shengguo Zhu

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Abstract

In this paper, we consider the three-dimensional isentropic Navier–Stokes equations for compressible fluids allowing initial vacuum when viscosities depend on density in a superlinear power law. We introduce the notion of regular solutions and prove the local-in-time well-posedness of solutions with arbitrarily large initial data and a vacuum in this class, which is a long-standing open problem due to the very high degeneracy caused by a vacuum. Moreover, for certain classes of initial data with a local vacuum, we show that the regular solution that we obtained will break down in finite time, no matter how small and smooth the initial data are.

1. Introduction

In this paper, we investigate the local-in-time well-posedness and formation of singularities of classical solutions to the compressible Navier–Stokes equations for isentropic flows when viscosity coefficients, shear and bulk, are both degenerate and the initial data are arbitrarily large with possible vacuum states. The system of compressible isentropic Navier–Stokes equations in $\mathbb{R}^3$ reads as

$$\begin{cases}\rho_t+\operatorname{div}(\rho u)=0,\\ (\rho u)_t+\operatorname{div}(\rho u\otimes u)+\nabla P=\operatorname{div}\mathbb{T}.\end{cases}\tag{1.1}$$

We look for local classical solution with initial data

$$(\rho,u)|_{t=0}=(\rho_0(x),u_0(x)),\quad x\in\mathbb{R}^3,\tag{1.2}$$

and far field behavior

$$(\rho,u)\to(\overline{\rho},0)\quad \text{as}\quad |x|\to+\infty,\quad t>0,\tag{1.3}$$

where $\overline{\rho}\geq 0$ is a constant.

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In system (1.1), $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $t \geq 0$ are space and time variables, respectively. ρ is the density, and $u = \left(u^{(1)}, u^{(2)}, u^{(3)}\right)^\top \in \mathbb{R}^3$ is the velocity of the fluid. We assume that the pressure P satisfies

$$P = A\rho^\gamma, \quad \gamma > 1, \tag{1.4}$$

where A is a positive constant, γ is the adiabatic exponent. $\mathbb{T}$ denotes the viscosity stress tensor with the form

$$\mathbb{T} = \mu(\rho)(\nabla u + (\nabla u)^\top) + \lambda(\rho)\,\mathrm{div} u\, \mathbb{I}_3, \tag{1.5}$$

where $\mathbb{I}_3$ is the 3×3 identity matrix. We assume in this paper that

$$\mu(\rho) = \alpha\rho^\delta, \quad \lambda(\rho) = \beta\rho^\delta, \tag{1.6}$$

$\mu(\rho)$ is the shear viscosity coefficient, $\lambda(\rho)+\frac{2}{3}\mu(\rho)$ is the bulk viscosity coefficient, α and β are both constants satisfying the following physical constraints

$$\alpha > 0, \quad 2\alpha + 3\beta \geq 0. \tag{1.7}$$

Furthermore, in this paper, we require that the constant δ satisfies

$$1 < \delta \leqq \min\left\{\frac{\gamma+1}{2}, 3\right\}. \tag{1.8}$$

In the theory of gas dynamics, when one derives the compressible Navier–Stokes equations from the Boltzmann equations through the Chapman–Enskog expansion up to the second order, cf. CHAPMAN and COWLING [7] and LI and QIN [26], one finds that, under some proper physical assumptions, the viscosity coefficients, μ and λ, and the heat conductivity coefficient κ are not constants but functions of absolute temperature θ such as

$$\mu(\theta) = a_1\theta^{\frac{1}{2}}F(\theta), \quad \lambda(\theta) = a_2\theta^{\frac{1}{2}}F(\theta), \quad \kappa(\theta) = a_3\theta^{\frac{1}{2}}F(\theta) \tag{1.9}$$

for some constants a_i $(i = 1, 2, 3)$. For instance, it reads from [7] that for the cut-off inverse power force model, if the intermolecular potential varies as r^{-a} with $a > 0$, where r is intermolecular distance, then in (1.9), $F(\theta) = \theta^b$ with $b = \frac{2}{a} \in [0, +\infty)$. If we restrict the gas flow to be isentropic, such dependence is inherited through the laws of Boyle and Gay-Lussac:

$$P = R\rho\theta = A\rho^\gamma, \quad \text{for constant} \quad R > 0,$$

i.e., $\theta = AR^{-1}\rho^{\gamma-1}$, and one finds that the viscosity coefficients are functions of density as described in (1.7) with $\delta = (\frac{1}{2} + b)(\gamma - 1)$.

When $\inf_x \rho_0(x) > 0$, it is well-known that the local existence of classical solutions for (1.1)–(1.3) can be obtained by a standard Banach fixed point argument due to the contraction property of the solution operators of the linearized problem, c.f. NASH [34]. Many interesting developments have been carried out by various methods, we refer the readers to [12,20,38–40]. However, this approach does not work when $\inf_x \rho_0(x) = 0$, which occurs when some physical requirements are imposed, such as finite total initial mass, finite total initial energy, or vacuum appearing locally in some open sets.

When viscosity coefficients μ and λ are both constants, for the existence of solutions of the three-dimensional isentropic flows with generic data and a vacuum, the main breakthrough is due to LIONS [27], where he established the global existence of weak solutions provided that $\gamma > \frac{9}{5}$. This result is improved to the case $\gamma > \frac{3}{2}$ by FEIREISL ET AL. [14], and even to non-isentropic flows in [15,16]. However, the regularities of these weak solutions are fairly low, and the uniqueness problem is widely open. On the other hand, the local well-posedness problem in higher regularity class with possible vacuum initial data requires comprehensive treatment on the degeneracy in time evolution in momentum equations $(1.1)_2$. Since the leading coefficient of u_t is ρ, which vanishes at the vacuum, there are infinitely many ways to define velocity (if it exists) when a vacuum appears. Mathematically, this degeneracy brings forth an essential difficulty in determining the velocity when a vacuum occurs, since it is difficult to find a reasonable way to extend the definition of velocity into vacuum region. Physically, it is not clear how to define the fluid velocity when there is no fluid at vacuum. A reasonable framework was proposed by CHO ET AL. [8,10,11] through a *compatibility condition*

$$-\mathrm{div}\mathbb{T}_0 + \nabla P(\rho_0) = \sqrt{\rho_0} g \tag{1.10}$$

for some $g \in L^2$. In turn, a local theory of strong or classical solutions with initial vacuum was established successfully for three-dimensional case; see also DUAN ET AL. [13] and LUO [31] for two-dimensional case. More recently, HUANG ET AL. [22] extended this local classical solution [11] to a global one with small energy and vacuum for isentropic flow in $\mathbb{R}^3$. In these results, the uniform ellipticity of viscosity plays a key role in the a priori estimates of higher order terms of velocity u.

When viscosity coefficients μ and λ are both density-dependent, system (1.1) has received extensive attentions in recent years. Instead of the uniform elliptic structure, the viscosity degenerates at a vacuum, which raises the difficulty of the problem to another level. A remarkable discovery of a new mathematical entropy function was made by BRESCH AND DESJARDINS [2] for $\lambda(\rho)$ and $\mu(\rho)$ satisfying the relation

$$\lambda(\rho) = 2(\mu'(\rho)\rho - \mu(\rho)). \tag{1.11}$$

For example, in our problem, such an entropy structure (1.11) exists when $\beta = 2\alpha(\delta - 1) > 0$. This new entropy offers a nice estimate

$$\mu'(\rho)\frac{\nabla\rho}{\sqrt{\rho}} \in L^\infty([0, T]; L^2(\mathbb{R}^d)),$$

provided that $\mu'(\rho_0)\frac{\nabla\rho_0}{\sqrt{\rho_0}} \in L^2(\mathbb{R}^d)$ for any $d \geq 1$. This observation plays an important role in the development of the global existence of weak solutions with a vacuum for system (1.1) and related models, see BRESH AND DEJARDINS [3,4], MELLET AND VASSEUR [33], and VASSEUR AND YU [45], and some other interesting results, c.f. [5,6,25,28,30,43,44]. However, we remark that, in spite of these significant achievements mentioned above, the local well-posedness of classical solutions in multi-dimensions with vacuum is still open due to the high degeneracy of this system near vacuum region. Indeed, several leading research groups in the field have raised this important open problem, see for instance, page 9 in [6] and page 3 in [23]. Our existence result in this paper is a solid step toward this direction.

From (1.4)–(1.5), we notice that

$$\mathrm{div}\mathbb{T} = -\rho^\delta Lu + \nabla\rho^\delta \cdot \mathbb{S}(u),$$

where the so-called Lamé operator L and the operator $\mathbb{S}$ are given by

$$Lu = -\alpha\triangle u - (\alpha+\beta)\nabla\mathtt{div}u, \quad \mathbb{S}(u) = \alpha(\nabla u + (\nabla u)^\top) + \beta\,\mathtt{div}u\,\mathbb{I}_3. \tag{1.12}$$

When constructing the classical solutions with arbitrarily large data and vacuum, in addition to the issue shown above for the constant viscosity case on the degeneracy in time evolution in momentum equations, there are still two new significant difficulties due to the appearance of a vacuum. The first one lies in the strong degeneracy of the elliptic operator $\mathrm{div}\mathbb{T}$ caused by a vacuum. We note that under assumptions (1.6)–(1.8), viscosity coefficients vanish as density function connects to the vacuum continuously in some open sets. This degeneracy gives rise to some difficulties in our analysis because of the less regularizing effect of the viscosity on the solutions, and is one of the major obstacles preventing us from utilizing a similar remedy proposed by CHO ET. AL. [8,10,11] for the case of constant viscosity coefficients. The second one lies in the extra nonlinearity for the variable coefficients in $\mathrm{div}\mathbb{T}$ due to (1.6). We emphasize here that, unlike the constant viscosity case, the elliptic part $\mathrm{div}\mathbb{T}$ is not always a good term in the regularity analysis for the higher order terms of the velocity. For example, if we want to get the estimate on $\|\nabla^3 u\|_{L^\infty([0,T];L^2(\mathbb{R}^3))}$ independent of the lower bound of the initial density, we need to deal with an extra nonlinear term

$$\mathrm{div}\big(\partial_x^\zeta \rho^\delta\big(\alpha(\nabla u + (\nabla u)^\top) + \beta\mathtt{div}u\mathbb{I}_3\big)\big),$$

where $\zeta = (\zeta_1, \zeta_2, \zeta_3)$ is a multi-index with $|\zeta| = \zeta_1 + \zeta_2 + \zeta_3 = 3$. Therefore, much attention will be paid to control this strong nonlinearity.

In order to overcome these difficulties, some new ideas have to be introduced. In [29], we obtained the existence of the unique local classical solutions for system (1.1) with a vacuum far field under the assumption

$$\delta = 1, \quad \alpha > 0, \quad \alpha + \beta \geq 0 \tag{1.13}$$

for two-dimensional space, aiming at the application to shallow water models. In [29,46] we observed that, assuming $\rho > 0$, the momentum equations can be rewritten as

$$u_t + u\cdot\nabla u + \frac{2A\gamma}{\gamma-1}\rho^{\frac{\gamma-1}{2}}\nabla\rho^{\frac{\gamma-1}{2}} + Lu = \nabla\log\rho\cdot\mathbb{S}(u). \tag{1.14}$$

Then, (1.14) implies that if we can control the special source term $\nabla\log\rho\cdot\mathbb{S}(u)$ when vacuum appears, the velocity u of the fluid can be governed by a strong parabolic system. However, this result only allows vacuum at the far field. The corresponding problem with the vacuum appearing in some open sets, or even at a single point is still unsolved.

Motivated by this observation in [29], when $\rho > 0$, for our case under the assumption (1.8), instead of (1.14), $(1.1)_2$ can be rewritten as

$$\begin{aligned} & u_t + u \cdot \nabla u + \rho^{\delta-1} L u \\ & = -\frac{2A\gamma}{\delta-1} (\rho^{\frac{\delta-1}{2}})^{\frac{2\gamma-\delta-1}{\delta-1}} \nabla \rho^{\frac{\delta-1}{2}} + \frac{2\delta}{\delta-1} \rho^{\frac{\delta-1}{2}} \nabla \rho^{\frac{\delta-1}{2}} \cdot \mathbb{S}(u). \end{aligned} \tag{1.15}$$

Then if we pass to the limit as $\rho \to 0$ on both sides of (1.15), we formally have

$$u_t + u \cdot \nabla u = 0 \quad \text{when } \rho = 0. \tag{1.16}$$

Thus (1.15)–(1.16) imply that actually the velocity u can be governed by a nonlinear degenerate parabolic system when vacuum appears in some open sets or at the far field, which is essentially different from the parabolic system in (1.14) in the sense of mathematical structure. Based on this observation, we introduce a proper class of solutions and prove the local-in-time well-posedness of solutions with arbitrarily large data and vacuum in this class for system (1.1) using a new approach which bridges the parabolic system (1.15) when $\rho > 0$, and the hyperbolic system (1.16) when $\rho = 0$.

1.1. Main Results

In order to present our results clearly, we first introduce the following definition of regular solutions to (1.1)–(1.3):

Definition 1.1. (*Regular solutions*) Let $T > 0$ be a finite constant. $(\rho, u)(x, t)$ is called a regular solution in $\mathbb{R}^3 \times [0, T]$ to the Cauchy problem (1.1)–(1.3) if

(A) (ρ, u) satisfies the Cauchy problem (1.1)-(1.3) in the sense of distribution;

(B) $\rho \geq 0,\ \rho^{\frac{\delta-1}{2}} - \bar{\rho}^{\frac{\delta-1}{2}} \in C([0, T]; H^3),\ (\rho^{\frac{\delta-1}{2}})_t \in C([0, T]; H^2)$;

(C) $u \in C([0, T]; H^{s'}) \cap L^\infty([0, T]; H^3),\ \rho^{\frac{\delta-1}{2}} \nabla^4 u \in L^2([0, T]; L^2),$
$u_t \in C([0, T]; H^1) \cap L^2([0, T]; D^2)$;

(D) $u_t + u \cdot \nabla u = 0$ holds when $\rho(t, x) = 0,$

for any constant $s' \in [2, 3)$.

Here and throughout this paper, we adopt the following simplified notations, most of them are for the standard homogeneous and inhomogeneous Sobolev spaces:

$$\begin{aligned} & |f|_p = \|f\|_{L^p(\mathbb{R}^3)}, \quad \|f\|_s = \|f\|_{H^s(\mathbb{R}^3)}, \quad |f|_2 = \|f\|_0 = \|f\|_{L^2(\mathbb{R}^3)}, \\ & D^{k,r} = \{f \in L^1_{loc}(\mathbb{R}^3) : |\nabla^k f|_r < +\infty\}, \quad D^k = D^{k,2}, \\ & |f|_{D^{k,r}} = \|f\|_{D^{k,r}(\mathbb{R}^3)} \ (k \geq 2), \\ & D^1 = \{f \in L^6(\mathbb{R}^3) : |\nabla f|_2 < \infty\}, \quad |f|_{D^1} = \|f\|_{D^1(\mathbb{R}^3)}. \end{aligned}$$

A detailed study of homogeneous Sobolev spaces can be found in [17].

Remark 1.1. This notion of regular solution for compressible Navier–Stokes equations was first introduced by YANG and ZHU [44] for one-dimensional case. We defined the regular solutions in [29] for the case $\delta = 1$. Compared with [29], a significant difference is that the admissible initial data in this paper are much broader. Actually in [29], in order to make sure that the source term $\nabla \log \rho \cdot \mathbb{S}(u)$ appearing in the Equation (1.14) is well defined in H^2 space, we need

$$\rho^{\frac{\gamma-1}{2}} \in H^3(\mathbb{R}^2), \quad \nabla \log \rho \in L^6 \cap D^1 \cap D^2(\mathbb{R}^2),$$

which means that the vacuum must and only appear in the far field. In this paper, we only need

$$\rho^{\frac{\delta-1}{2}} - \overline{\rho}^{\frac{\delta-1}{2}} \in H^3(\mathbb{R}^3),$$

so the vacuum can appear in any open set or in the far field.

Now we give the main existence result of this paper.

Theorem 1.1. *If the initial data* (ρ_0, u_0) *satisfy the following regularity conditions:*

$$\rho_0 \geq 0, \quad \left(\rho_0^{\frac{\delta-1}{2}} - \overline{\rho}^{\frac{\delta-1}{2}}, u_0\right) \in H^3(\mathbb{R}^3), \tag{1.17}$$

then there exists a positive time T_* *and a unique regular solution* $(\rho, u)(x, t)$ *in* $\mathbb{R}^3 \times [0, T_*]$ *to the Cauchy problem* (1.1)–(1.3).

Remark 1.2. In (1.8), the upper bound of δ is a technical assumption needed for the mathematical analysis carried out in this paper. The motivation of such an upper bound can be roughly described as follows: first, we hope that $\rho = \phi^{\frac{2}{\delta-1}} \in C^1$, which requires that $\delta \leqq 3$ near vacuum; second, $\delta \leqq \frac{\gamma+1}{2}$ is needed in our estimates (3.27) and (3.45) in order to deal with a pressure related term. From the earlier discussion right after (1.8), it is clear that our interval in (1.8) covers a significant part of physical regime, the bigger δ makes reasonable sense physically. It would be interesting to further generalize our result for larger δ beyond current upper bound in (1.8).

Remark 1.3. We remark that the higher regularity of regular solutions is allowed and propagated in Definition 1.1 and Theorem 1.1. However, from the explanation in the previous remark, the upper bound of δ depends on the order of regularity of the solutions obtained by our approach. If one wants higher regularity of the solutions, then the smaller interval of δ is allowed within current framework. See, for instance, our estimates in (3.27) and (3.45) for more details.

Remark 1.4. The weak smoothing effect of the velocity u in positive time $t \in [\tau, T_*]$, $\forall \tau \in (0, T_*)$, tells us that the regular solution obtained in Theorem 1.1 is indeed a classical one in $(0, T_*] \times \mathbb{R}^3$. The details can be found in the "Appendix".

Remark 1.5. We can also consider the flow with external force in the momentum equations $(1.1)_2$ such as

$$(\rho u)_t + \mathrm{div}(\rho u \otimes u) + \nabla P = \mathrm{div}\mathbb{T} + \rho h. \tag{1.18}$$

Assume that

$$h \in C([0,T]; H^1(\mathbb{R}^3)) \cap L^2([0,T]; H^3(\mathbb{R}^3)) \tag{1.19}$$

for some $T > 0$, then the same conclusion as in Theorem 1.1 still holds if we replace the condition (C) in Definition 1.1 with

$$u_t + u \cdot \nabla u = h \quad \text{when } \rho = 0.$$

To achieve this result, one only needs to make minor modifications on our proof for Theorem 1.1. We will not go into details on this matter.

As a direct consequence of Theorem 1.1 and by the standard theory of quasi-linear hyperbolic equations, we have the following additional regularities for some specially choices of power δ:

Corollary 1.1. *Let* $1 < \delta \leqq \min\left\{\frac{5}{3}, \frac{\gamma+1}{2}\right\}$, *or* $\delta = 2$ $(\gamma \geq 3)$. *If the initial data* (ρ_0, u_0) *satisfy* (1.17), *then the regular solution* (ρ, u) *obtained in Theorem* 1.1 *also satisfies*

$$\rho - \overline{\rho} \in C([0,T_*); H^3), \quad \rho_t \in C([0,T_*); H^2). \tag{1.20}$$

Compared with previous results for $\delta = 0$ [11] and $\delta = 1$ [29], our analysis in the proof of Theorem 1.1 is quite different. From the observations shown in (1.15)–(1.16), the behavior of the velocity u near vacuum under the assumption (1.8) is more hyperbolic than the cases in [11,29]. This phenomenon leads us to developing a different approach in the analysis on the regularity of u. As a matter of fact, in [8,10,11,29], the uniform ellipticity of the Lamé operator L defined in (1.12) plays an essential role. Therefore, one can use standard elliptic theory to estimate $|u|_{D^{k+2}}$ $(0 \leqq k \leqq 2)$ by the D^k-norm of all other terms in momentum equations $(1.1)_2$ for the case $\delta = 0$ in [8,10,11], or in (1.14) for the case $\delta = 1$ in [29,46]. However, for our case, it is clear from (1.15) that the standard parabolic theory only offers the following weighted estimate:

$$\rho^{\frac{\delta-1}{2}} \partial_x^\zeta u \in L^2([0,T]; L^2), \quad \text{for} \quad |\zeta| = 1,2,3,4.$$

This estimate alone is not enough for the regularity analysis on u and ρ. In order to obtain the desired estimate

$$u \in L^\infty([0,T], H^3) \cap C([0,T]; H^{s'}) \quad \text{for any } s' \in [2,3)$$

shown in Definition 1.1, we turn to the help of symmetric hyperbolic structure

$$u_t + u \cdot \nabla u.$$

These estimates are accomplished along with a vanishing viscosity limit argument.

Another challenging question in the theory of fluid dynamics is the problem of global regularity. This problem is extremely difficult for three-dimensional compressible Navier–Stokes equations, especially when the initial density contains vacuum. When viscosity coefficients are constants, global regularity was achieved when initial energy is small, see [21] for the case away from vacuum, and [22] for the case allowing possible initial vacuum for the solutions in the class of [8]. In particular, the results of [22] showed that no singularity will form in finite time for smooth solutions in the class of [8] if the initial energy is small. On the other hand, there are a lot of finite time blowup results obtained, say [9,31,35,41,42,44], for various solution classes or conditions. However, it is not clear yet if such solutions exist locally in time. For our case, when the viscosity coefficients satisfy the conditions (1.6)–(1.8), we will show that the solution we obtained will break down in finite time for certain classes of initial data with local vacuum, no matter how small the initial energy is. This is in sharp contrast to the case of constant viscosity as shown in [8,11,22]. This is because our problem behaves more closely to compressible Euler equations in vacuum region, due to strong degeneracy of the viscosity.

In fact, we will present two scenarios for singularity development in finite time from local vacuum initial data. The first one is motivated by Xin and Yan [42], called the initial data with isolated mass group.

Definition 1.2. (*Isolated mass group*) $(\rho_0(x), u_0(x))$ is said to have an isolated mass group (A_0, B_0), if there are two smooth, bounded and connected open sets $A_0 \subset \mathbb{R}^3$ and $B_0 \subset \mathbb{R}^3$ satisfying

$$\begin{cases} \overline{A}_0 \subset B_0 \subseteq B_{R_0} \subset \mathbb{R}^3; \\ \rho_0(x) = 0, \ \forall\, x \in B_0 \backslash A_0, \quad \displaystyle\int_{A_0} \rho_0(x)\,\mathrm{d}x > 0 \\ u_0(x)|_{\partial A_0} = \overline{u}_0 \end{cases} \tag{1.21}$$

for some positive constant R_0 and constant vector $\overline{u}_0 \in \mathbb{R}^3$, where B_{R_0} is the ball centered at the origin with radius R_0.

We remark that, in this definition, the assumption of constant velocity at the boundary is crucial. We will need it for the proof of Lemma 4.1 later; it cannot be relaxed using the current argument.

Our first blowup result shows that an isolated mass group in initial data guarantees the finite time singularity formation of regular solutions.

Theorem 1.2. *(*Blow-up by isolated mass group*) If the initial data $(\rho_0, u_0)(x)$ have an isolated mass group (A_0, B_0), then the regular solution $(\rho, u)(x, t)$ in $\mathbb{R}^3 \times [0, T_m]$ obtained in Theorem 1.1 with maximal existence time T_m blows up in finite time, i.e., $T_m < +\infty$.*

For the second scenario, we explore the hyperbolic structure for the system in vacuum region. For this purpose, we introduce the following concept:

Definition 1.3. (*Hyperbolic singularity set*) The non-empty open set $V \subset \mathbb{R}^3$ is called a hyperbolic singularity set of $(\rho_0, u_0)(x)$, if V satisfies

$$\begin{cases} \rho_0(x) = 0, \quad \forall\, x \in V; \\ Sp(\nabla u_0(\xi_0)) \cap \mathbb{R}^- \neq \emptyset, \quad \text{for some} \quad \xi_0 \in V, \end{cases} \tag{1.22}$$

where $Sp(\nabla u_0(\xi_0))$ denotes the spectrum of the matrix $\nabla u_0(\xi_0)$.

Remark 1.6. We note that the Hyperbolic Singular Set is a local property. It contains a large class of initial data. For example, one can choose V to be the ball $B_r(x_0)$ with some small positive r. Now, choose ρ_0 to be any C_0^∞ function such that $\rho_0(x)$ is identically zero in $B_{2r}(x_0)$. In V, we can choose a smooth velocity $u_0 \in C_0^\infty$ such that $\nabla u_0(x_0)$ has a negative eigenvalue, then it has a negative eigenvalue for all $x \in B_r(x_0)$ if r is sufficiently small due to continuity. A simple example can be constructed by setting $u_0 = x_0 - x$ in $B_{2r}(x_0)$, and $u_0 = 0$ outside $B_{2r+1}(x_0)$, then smooth it out using mollifier.

This definition is inspired by the global existence theory of classical solutions to the compressible Euler equations in [19,36], for initial data satisfying the following conditions:

- ρ_0 is small and compactly supported,
- $Sp(\nabla u_0) \cap \mathbb{R}^- = \emptyset, \quad \forall\, x \in \mathbb{R}^3$.

Our second blowup result confirms that hyperbolic singularity set does generate singularity from local regular solutions in finite time.

Theorem 1.3. *(*Blow-up by hyperbolic singularity set*) If the initial data* $(\rho_0, u_0)(x)$ *have a non-empty hyperbolic singularity set V, then the regular solution* $(\rho, u)(x, t)$ *on* $\mathbb{R}^3 \times [0, T_m]$ *obtained in Theorem 1.1 with maximal existence time* T_m *blows up in finite time, i.e.,* $T_m < +\infty$.

1.2. Challenges and Strategy of the Proof for Theorem 1.1

Due to its strong physical background, the local-in-time existence of classical solutions with vacuum initial data to the Cauchy problem of the 3D compressible Navier–Stokes equations with density dependent viscosities (in a power law manner) is indeed a well-known open problem in the mathematical analysis for the 3D compressible fluids, which has received extensive attentions in recent years. Several leading research groups in this field have raised this important open problem, such as, page 9 in [6] and page 3 in [23], and it has remained open for quite a few years. Our existence result, Theorem 1.1, is the first definite answer. The proof of this Theorem, given in Section 3 below, is rather technical. We now discuss in greater detail the major challenges and our main observations and ideas regarding this proof.

The presence of a vacuum in the initial data and in the solutions causes degeneracy in both time evolution (the coefficient of u_t in momentum equation is ρ) even when viscosity coefficients are positive constants, and in viscosity when the viscosity coefficients are superlinear power of density as handled in current paper. In fact, our momentum equation can be rewritten as

$$\underbrace{\rho\big(u_t+u\cdot\nabla u\big)}_{\text{degenerate time evolution}}+\nabla P=\underbrace{-\rho^\delta Lu}_{\text{degenerate Lamé operator}}+\nabla\rho^\delta\cdot\mathbb{S}(u). \tag{1.23}$$

Such double degeneracy makes the problem highly challenging and requires new ideas.

The first major challenge is to find an appropriate function space for the local classical solutions. As explained earlier, we will need to find a reasonable way to smoothly extend velocity into the vacuum region. When viscosity coefficients are positive constants, with the help of uniform elliptic structure of viscosity, [8,10,11] used initial compatibility conditions (1.10) to overcome the degeneracy in time evolution. The initial compatibility conditions (1.10) not only restrict the initial data, but also lead to the definition of the solutions in homogeneous space. For instance, for Cauchy problem, the velocity, u, is not in L^2, [8,10,11]. For our problem, the framework of [8,10,11] is no longer applicable since our viscosity also vanishes at vacuum. Instead, we observe that system (1.1) behaves more like compressible Euler equations near the vacuum, therefore we will rely more on the **hyperbolic** structure of the system near the vacuum. For this purpose, we made our first important and critical observation that the momentum equation should be rewritten as (1.15) and thus hoping our velocity u in H^3 space. The appropriate density variable is not yet clear at this step, which will be chosen so that we are able to conduct effective estimates to control the double degeneracy in our problem.

The second major challenge is how to derive effective estimates on density and velocity that are good enough to define local classical solutions. As explained in the previous paragraph, in view of equation (1.15), we hope to prove $u\in H^3$, and thus due to the **hyperbolic** nature of the system in vacuum region, the density or some appropriate density variable (such as *soundspeed* in Isentripic Euler case) should also be in H^3. Roughly speaking, from the continuity equation $(1.1)_1$, according to the standard theory for the transport equation, if we want to consider the estimate for density in H^3 space, the following information is necessary:

$$\mathrm{div}u\in L^1([0,T];L^\infty)\quad\text{and}\quad\rho\nabla^3\mathrm{div}u\in L^2([0,T];L^2). \tag{1.24}$$

When $\delta>1$ in (1.6), for the double degenerate equations (1.23), we will instead use our effective momentum equation (1.15), rewritten as

$$u_t+u\cdot\nabla u+\frac{A\gamma}{\gamma-1}\nabla\rho^{\gamma-1}-\frac{\delta}{\delta-1}\nabla\rho^{\delta-1}\cdot\mathbb{S}(u)=-\rho^{\delta-1}Lu. \tag{1.25}$$

Formally, under the assumption that the initial data are sufficiently smooth, first we see that the transport operator $u_t+u\cdot\nabla u$ should provide us with the feet that $u\in L^\infty([0,T];H^3)$. Second, the information that the degenerate elliptic operator $\rho^{\delta-1}Lu$ could give us

$$\int_{\mathbb{R}^3}\rho^{\delta-1}|\nabla^k u|^2\,\mathrm{d}x<\infty,$$

for $k = 1, \ldots, 4$. Under the assumption that $1 < \delta \leqq 3$, the above regularity of u seems good enough for the estimates on ρ in H^3 space. However, due to the appearance of terms like $\nabla^k \rho^{\delta-1} \cdot \nabla u$ $(k = 1, \ldots, 4)$, the regularity $\rho \in L^\infty([0, T]; H^3)$ cannot help us to close the energy estimates on u which is needed to ensure (1.24). We remark that a similar obstacle appeared in the theory of Isentropic Euler equations; it was solved by introducing the appropriate density variable, the sound speed, which symmetrizes the system. In order to explore the symmetry of the system while balance degeneracy in viscosity, our second important and critical observation is to introduce the **new** density variable $\phi = \rho^{\frac{\delta-1}{2}}$, satisfying the following equation:

$$\phi_t + u \cdot \phi + \frac{\delta - 1}{2} \phi \mathrm{div} u = 0. \tag{1.26}$$

Now,

$$u \in L^\infty([0, T]; H^3) \quad \text{and} \quad \int_{\mathbb{R}^3} \phi^2 |\nabla^4 u|^2 \, \mathrm{d}x < \infty$$

is exactly good enough for the estimate on the new variable ϕ in H^3 space, which is the desired condition that can help us to close the estimates on u. With the help of this new variable ϕ, we find the appropriate function space for our solutions, and give the definition of the regular solution in Definition 1.1. It further helps us to rewrite the momentum equations $(1.1)_2$ into (1.15), or

$$\underbrace{u_t + u \cdot \nabla u}_{\text{principle order}} + \underbrace{\frac{2A\gamma}{\delta - 1} \phi^{\frac{2\gamma-\delta-1}{\delta-1}} \nabla \phi - \frac{2\delta}{\delta - 1} \phi \nabla \phi \cdot \mathbb{S}(u)}_{\text{lower order source term}} = \underbrace{-\phi^2 L u.}_{\text{degenerate Lamé operator}} \tag{1.27}$$

Now, at least formally, we have obtained one carefully chosen effective structure (1.26)–(1.27) to control the behavior of the solution in the regions with or without a vacuum in a unified way. *We remark that, to the best of our knowledge, the variable ϕ, defined from viscosity coefficients, was not introduced before for the purpose of the local existence of classical solutions.*

The last, but not least, major challenge is to construct successful approximation solutions, carrying effective uniform estimates without derivative loss, converging to a solution. For nonliear degenerate systems, it is not rare that the approximation is missing even when the estimates are available. There are nearly infinitely many ways to linearize and approximate the problem; it is non-trivial to choose the good one. We will achieve this by choosing an elaborate linearization based on a careful analysis on the nonlinear structure. In (1.26)–(1.27), $u_t + u \cdot \nabla u$ will be regarded as the principle part to control the regularity of the velocity in H^3, and $\phi^2 L u$ is only used as a source term. The only contribution of $\phi^2 L u$ is to provide the L^2 estimate of $\phi \nabla^4 u$ that will be used in the L^2 estimate of $\nabla^3 \phi$. Therefore, we regard the system (1.26)–(1.27) as a hyperbolic one with some density-weighted higher order source terms even though it is actually a degenerate parabolic system. With this strategy, we have to choose a specially designed linearization with a well-balanced artificial viscosity (for instance, no viscosity is added to the mass equation) approach to construct the approximate solutions, which offer estimates independent of the lower bound of initial density and the artificial viscosity. It is not clear if other linearization approaches will work, but the following one succeeded:

$$\begin{cases} \phi_t + v \cdot \nabla \phi + \dfrac{\delta - 1}{2} \psi \mathrm{div} v = 0, \\ u_t + v \cdot \nabla u + \dfrac{A\gamma}{\gamma - 1} \nabla \phi^{\frac{2\gamma-2}{\delta-1}} + (\phi^2 + \eta^2) L u = \nabla \phi^2 \cdot Q(v), \end{cases} \tag{1.28}$$

where ψ and $v = \left(v^{(1)}, v^{(2)}, v^{(3)}\right)^\top \in \mathbb{R}^3$ are given functions satisfying $(\psi, v)(t = 0, x) = (\phi_0 \geq 0, u_0)$ and (3.9). It should be pointed out that there exists a carefully chosen compatibility between the last term $\frac{\delta-1}{2}\psi \mathrm{div} v$ on the left hand side of $(1.28)_1$ and the viscosity term $(\phi^2 + \eta^2) L u$ in $(1.28)_2$. Then, based on the uniform energy estimates of the linearized problem and some iteration scheme, we can offer the answer to the local-in-time well-posedness of such kind of degenerate system. We remark that our analysis did not solve the well-posedness problem for all possible $\delta > 1$ even with the help of our effective system and the effective estimates. The detailed analysis is presented in Section 3.

1.3. Outline of the Paper

We now outline the organization of the rest of this paper. In Section 2, we list some basic lemmas to be used later. Section 3 is devoted to proving Theorem 1.1 and Corollary 1.1. Based on the strategy mentioned in Section 1.2, we first reformulate our problem into a simpler form in terms of some new variables, and then give the proof of the local existence of strong solutions to this reformulated problem, which is achieved in four steps: (1) we construct global approximate solutions for a specially designed linearized problem with an artificial viscosity $\eta^2 L u$ in momentum equations; (2) we establish the a priori estimates independent of artificial viscosity coefficient η; (3) we then pass to the limit $\eta \to 0$ to recover the solution of this linearized problem allowing degeneracy in the elliptic operator appearing in momentum equations; (4) we prove the unique solvability of the reformulated problem through a standard iteration process. Section 4 is devoted to proving the finite time blowup shown in Theorems 1.2–1.3. Section 5 is the "Appendix" where we prove the fact that the regular solution we obtained in Theorem 1.1 is indeed a classical one in $(0, T_*]$.

Finally, we remark that our framework in this paper is applicable to other physical dimensions, say 1 and 2, with some minor modifications. This is clear from the analysis carried out in the sections to follow.

2. Preliminaries

In this section, we list some basic lemmas to be used later. The first one is the well-known Gagliardo–Nirenberg inequality.

Lemma 2.1. [24] *For $p \in [2, 6]$, $q \in (1, +\infty)$, and $r \in (3, +\infty)$, there exists some generic constant $C > 0$ that may depend on q and r such that for*

$$f \in H^1(\mathbb{R}^3) \quad and \quad g \in L^q(\mathbb{R}^3) \cap D^{1,r}(\mathbb{R}^3),$$

we have

$$\begin{aligned} |f|_p^p &\leqq C|f|_2^{(6-p)/2}|\nabla f|_2^{(3p-6)/2}, \\ |g|_\infty &\leqq C|g|_q^{q(r-3)/(3r+q(r-3))}|\nabla g|_r^{3r/(3r+q(r-3))}. \end{aligned} \tag{2.1}$$

Some commonly used versions of this inequality are

$$|u|_6 \leqq C|u|_{D^1}, \quad |u|_\infty \leqq C|u|_6^{\frac{1}{2}}|\nabla u|_6^{\frac{1}{2}}, \quad |u|_\infty \leqq C\|u\|_{W^{1,r}}. \tag{2.2}$$

The second lemma is a collection of compactness results obtained via the Aubin–Lions Lemma.

Lemma 2.2. [37] *Let $X_0 \subset X \subset X_1$ be three Banach spaces. Suppose that X_0 is compactly embedded in X and that X is continuously embedded in X_1. Then the following statements hold:*

(1) *If G is bounded in $L^p(0,T;X_0)$ for $1 \leqq p < +\infty$, and $\frac{\partial G}{\partial t}$ is bounded in $L^1(0,T;X_1)$, then G is relatively compact in $L^p(0,T;X)$.*

(2) *If G is bounded in $L^\infty(0,T;X_0)$ and $\frac{\partial G}{\partial t}$ is bounded in $L^p(0,T;X_1)$ for $p>1$, then G is relatively compact in $C(0,T;X)$.*

The next four lemmas contain some Sobolev inequalities on the product estimates, the interpolation estimates, the composite function estimates, etc., which can be found in many works, say Majda [32]. We omit the proofs of them.

Lemma 2.3. [32] *Let constants r, a and b satisfy the relation*

$$\frac{1}{r} = \frac{1}{a} + \frac{1}{b}, \quad for \quad 1 \leqq a,\ b,\ r \leqq +\infty.$$

$\forall s \geq 1$, if $f, g \in W^{s,a} \cap W^{s,b}(\mathbb{R}^3)$, then

$$|\nabla^s(fg) - f\nabla^s g|_r \leqq C_s\big(|\nabla f|_a|\nabla^{s-1}g|_b + |\nabla^s f|_b|g|_a\big), \tag{2.3}$$

$$|\nabla^s(fg) - f\nabla^s g|_r \leqq C_s\big(|\nabla f|_a|\nabla^{s-1}g|_b + |\nabla^s f|_a|g|_b\big), \tag{2.4}$$

where $C_s > 0$ is a constant depending only on s, and $\nabla^s f$ $(s>1)$ stands for the set of all partial derivatives $\partial_x^\zeta f$ of order $|\zeta| = s$.

Lemma 2.4. [32] *If functions $f,\ g \in H^s$ and $s > \frac{3}{2}$, then $fg \in H^s$, and there exists a constant C_s depending only on s such that*

$$\|fg\|_s \leqq C_s\|f\|_s\|g\|_s.$$

Lemma 2.5. [32] *If $u \in H^s$, then for any $s' \in [0,s]$, there exists a constant C_s depending only on s such that*

$$\|u\|_{s'} \leqq C_s\|u\|_0^{1-\frac{s'}{s}}\|u\|_s^{\frac{s'}{s}}.$$

Lemma 2.6. [32]

(1) *If* $f,\ g \in H^s \cap L^\infty$ *and* $|\zeta| \leqq s$, *then there exists a constant* C_s *depending only on* s *such that*

$$\|\partial_x^\zeta (fg)\|_s \leqq C_s(|f|_\infty |\nabla^s g|_2 + |g|_\infty |\nabla^s f|_2). \tag{2.5}$$

(2) *Let* $u(x)$ *be a continuous function taking its values in some open set* G *such that* $u \in H^s \cap L^\infty$, *and* $g(u)$ *be a smooth vector-valued function on* G. *Then for* $s \geq 1$, *there exists a constant* C_s *depending only on* s *such that*

$$|\nabla^s g(u)|_2 \leqq C_s \left\| \frac{\partial g}{\partial u} \right\|_{s-1} |u|_\infty^{s-1} |\nabla^s u|_2. \tag{2.6}$$

The following lemma is a useful tool to improve weak convergence to strong convergence:

Lemma 2.7. [32] *If the function sequence* $\{w_n\}_{n=1}^\infty$ *converges weakly to* w *in a Hilbert space* X, *then it converges strongly to* w *in* X *if and only if*

$$\|w\|_X \geq \lim sup_{n\to\infty} \|w_n\|_X.$$

3. Existence of Regular Solutions

In this section, we aim at proving Theorem 1.1 and Corollary 1.1. To this end, based on the strategy mentioned in Section 1.2, we first reformulated our main problem (1.1)–(1.3) into a more convenient form. Since we will repeat the integration over $\mathbb{R}^3$ for many times, we will adopt the simplified notation

$$\int f = \int_{\mathbb{R}^3} f \, \mathrm{d}x,$$

throughout this paper without further specification. All other integrals will be specified when they appear.

3.1. Reformulation

Setting $\phi = \rho^{\frac{\delta-1}{2}}$, system (1.1) can be rewritten as

$$\begin{cases} \phi_t + u\cdot\nabla\phi + \dfrac{\delta-1}{2}\phi \mathrm{div} u = 0,\\ u_t + u\cdot\nabla u + \dfrac{2A\gamma}{\delta-1}\phi^{\frac{2\gamma-\delta-1}{\delta-1}}\nabla\phi + \phi^2 Lu = \nabla\phi^2\cdot Q(u), \end{cases} \tag{3.1}$$

where

$$Q(u) = \frac{\delta}{\delta-1}\big(\alpha(\nabla u + (\nabla u)^\top) + \beta \mathrm{div} u \mathbb{I}_3\big) = \frac{\delta}{\delta-1}\mathbb{S}(u). \tag{3.2}$$

The initial data are given by

$$(\phi, u)|_{t=0} = (\phi_0, u_0)(x) = \left(\rho_0^{\frac{\delta-1}{2}}(x), u_0(x)\right), \quad x \in \mathbb{R}^3, \tag{3.3}$$

with the far field behavior

$$(\phi, u) \to (\overline{\phi}, 0) = \left(\overline{\rho}^{\frac{\delta-1}{2}}, 0\right), \quad \text{as} \quad |x| \to +\infty, \quad t > 0. \tag{3.4}$$

To prove Theorem 1.1, we first establish the following existence result for the reformulated problem (3.1)–(3.4), and then in Section 3.6 we will show that this result indeed implies Theorem 1.1:

Theorem 3.1. *If the initial data* $(\phi_0, u_0)(x)$ *satisfy*

$$\phi_0 \geq 0, \quad (\phi_0 - \overline{\phi}, u_0) \in H^3, \tag{3.5}$$

then there exists a positive time T_* *and a unique strong solution* (ϕ, u) *on* $\mathbb{R}^3 \times [0, T_*]$ *to Cauchy problem* (3.1)–(3.4)*, that is,* (ϕ, u) *is a solution of the Cauchy problem* (3.1)–(3.4) *in the sense of distribution and satisfies*

$$\begin{aligned}
&\phi - \overline{\phi} \in C([0, T_*]; H^3), \quad \phi_t \in C([0, T_*]; H^2),\\
&u \in C([0, T_*]; H^{s'}) \cap L^\infty([0, T]; H^3), \quad \phi\nabla^4 u \in L^2([0, T_*]; L^2),\\
&u_t \in C([0, T_*]; H^1) \cap L^2([0, T_*]; D^2),
\end{aligned} \tag{3.6}$$

for any constant $s' \in [2, 3)$.

We will prove this theorem in the subsequent four Sections 3.2–3.5 as explained in the introduction.

3.2. Linearization with an Artificial Viscosity

In order to construct the local strong solutions for the nonlinear problem (3.1)–(3.4), we first consider the following linearized problem:

$$\begin{cases}
\phi_t + v \cdot \nabla\phi + \dfrac{\delta - 1}{2}\psi \mathrm{div} v = 0,\\
u_t + v \cdot \nabla u + \dfrac{A\gamma}{\gamma - 1}\nabla\phi^{\frac{2\gamma-2}{\delta-1}} + (\phi^2 + \eta^2)Lu = \nabla\phi^2 \cdot Q(v),\\
(\phi, u)|_{t=0} = (\phi_0(x), u_0(x)), \quad x \in \mathbb{R}^3,\\
(\phi, u) \to (\overline{\phi}, 0), \quad \text{as} \quad |x| \to +\infty, \quad t > 0,
\end{cases} \tag{3.7}$$

where $\eta \in (0, 1)$ is a constant and

$$Q(v) = \frac{\delta}{\delta - 1}\left(\alpha(\nabla v + (\nabla v)^\top) + \beta \mathrm{div} v \mathbb{I}_3\right) = \frac{\delta}{\delta - 1}\mathbb{S}(v). \tag{3.8}$$

Here ψ and $v = \left(v^{(1)}, v^{(2)}, v^{(3)}\right)^\top \in \mathbb{R}^3$ are given functions satisfying the initial assumption $(\psi, v)(t = 0, x) = (\phi_0, u_0)$ and the following regularities:

$$\begin{aligned}
&\psi - \overline{\phi} \in C([0, T]; H^3), \quad \psi_t \in C([0, T]; H^2),\\
&v \in C([0, T]; H^{s'}) \cap L^\infty([0, T]; H^3), \quad \psi\nabla^4 v \in L^2([0, T]; L^2),\\
&v_t \in C([0, T]; H^1) \cap L^2([0, T]; D^2),
\end{aligned} \tag{3.9}$$

for any constant $s' \in [2, 3)$ and fixed time $T > 0$.

Such a linearization is specially designed to serve our purpose. Unlike [29], here we have to keep the positive and symmetric hyperbolic operator in the momentum equations. More precisely, we linearize

$$u_t + u \cdot \nabla u \rightarrow u_t + v \cdot \nabla u,$$

instead of $u_t + u \cdot \nabla u \rightarrow u_t + v \cdot \nabla v$ as in [29], which plays a key role in the analysis on the regularities of velocity u. Generally speaking, in the proof of the existence to come, we will make a full use of the transport property of the hyperbolic part $u_t + v \cdot \nabla u$. On the other hand, the strong diffusion term $(\phi^2 + \eta^2)Lu$ helps to ensure the global existence of the linearized problem. Moreover, we remark that there exists a carefully chosen compatibility between the last term $\frac{\delta-1}{2}\psi \mathrm{div} v$ on the left hand side of $(3.7)_1$ and the viscosity term $(\phi^2 + \eta^2)Lu$ in $(3.7)_2$. Because of the lack of positive lower bound for ϕ_0, it is not visible to show that there exists some constant C independent of η such that

$$\|u\|_{L^2([0,T];D^4)} \leqq C.$$

Therefore, in order to obtain a finite upper bound of $|\phi|_{D^3}$ independent of η from the continuity equation $(3.7)_1$, we need a good estimate like $\psi \nabla^4 v \in L^2([0, T]; L^2)$. For our solution (ϕ, u), $\|\phi \nabla^4 u\|_{L^2([0,T];L^2)} \leqq C$ can be obtained by the smoothing property of the artificial viscosity term $(\phi^2 + \eta^2)Lu$. It is not clear if other linearization approaches will achieve the same goal, this one (3.7) succeeded. The details can be found in Lemmas 3.2–3.4.

As mentioned above, a rather standard method gives the following global existence of a strong solution (ϕ, u) to (3.7) for each fixed $\eta > 0$:

Lemma 3.1. *For each fixed $\eta > 0$ and any $T > 0$, assume that the initial data (ϕ_0, u_0) satisfy (3.5). Then there exists a unique strong solution $(\phi, u)(x, t)$ in $\mathbb{R}^3 \times [0, T]$ to (3.7), that is, a solution satisfying the Cauchy problem (3.7) with the regularities*

$$\begin{aligned} &\phi - \overline{\phi} \in C([0, T]; H^3), \ \phi_t \in C([0, T]; H^2), \\ &u \in C([0, T]; H^{s'}) \cap L^\infty([0, T]; H^3), \quad \phi \nabla^4 u \in L^2([0, T]; L^2), \\ &u_t \in C([0, T]; H^1) \cap L^2([0, T]; D^2). \end{aligned} \tag{3.10}$$

Proof. From (3.7), we know that $\phi - \overline{\phi}$ satisfies the following Cauchy problem of a linear transport equation

$$\phi_t + v \cdot \nabla(\phi - \overline{\phi}) = f = -\frac{\delta - 1}{2}\psi \mathrm{div} v, \quad \phi_0 - \overline{\phi} \in H^3,$$

where the inhomogenous source term f satisfies

$$f \in L^\infty([0, T]; H^2) \cap L^2([0, T]; H^3)$$
$$\text{and} \quad f_t \in L^\infty([0, T]; L^2) \cap L^2([0, T]; H^1).$$

Then the existence and regularity of a unique solution $\phi - \overline{\phi}$ in $\mathbb{R}^3 \times [0, T]$ to the above Cauchy problem can be obtained by the standard theory of transport equation (see the proof of Lemma 6 in [10]) with the help of Lemma 2.2.

After establishing ϕ from $(3.7)_1$, for $\eta > 0$, $(3.7)_2$ becomes a linear parabolic system of u:

$$u_t + v \cdot \nabla u + (\phi^2 + \eta^2) L u = g = -\frac{A\gamma}{\gamma - 1} \nabla \phi^{\frac{2\gamma-2}{\delta-1}} + \nabla \phi^2 \cdot Q(v),$$

where the inhomogenous source term g satisfies

$$g \in L^\infty([0, T]; H^2) \quad \text{and} \quad g_t \in L^\infty([0, T]; L^2) \cap L^2([0, T]; H^1).$$

Then it is not difficult to solve u by the standard parabolic theory (see the proof of Lemma 2 in [11]). Here we omit the details. □

In the next two subsections, we first establish the uniform estimates independent of η, then we pass to the limit $\eta \to 0$.

3.3. Uniform a Priori Estimates Independent of η

The main task of this subsection is to establish some local (in time) a priori estimates for the solution (ϕ, u) to (3.7) obtained in Lemma 3.1, independent of the artificial viscosity coefficient η. For this purpose, we fix a $T > 0$ and choose a positive constant c_0 large enough such that

$$2 + \overline{\phi} + |\phi_0|_\infty + \|\phi_0 - \overline{\phi}\|_3 + \|u_0\|_3 \leqq c_0. \tag{3.11}$$

We now assume that there exist some time $T^* \in (0, T)$ and constants c_i ($i = 1, 2, 3, 4$) such that

$$1 < c_0 \leqq c_1 \leqq c_2 \leqq c_3 \leqq c_4,$$

and

$$\begin{aligned}
&\sup_{0 \leqq t \leqq T^*} \left(|\psi|_\infty^2 + \|\psi(t) - \overline{\phi}\|_3^2 + \|v(t)\|_1^2\right) \leqq c_1^2, \\
&\sup_{0 \leqq t \leqq T^*} \left(|\psi_t(t)|_2^2 + |v(t)|_{D^2}^2 + |v_t(t)|_2^2\right) + \int_0^{T^*} \left(|\psi \nabla^3 v|_2^2 + |v_t|_{D^1}^2\right) \mathrm{d}t \leqq c_2^2, \\
&\operatorname{ess}\sup_{0 \leqq t \leqq T^*} \left(|\psi_t(t)|_{D^1}^2 + |v(t)|_{D^3}^2 + |v_t(t)|_{D^1}^2\right) + \int_0^{T^*} \left(|\psi \nabla^4 v|_2^2 + |v_t|_{D^2}^2\right) \mathrm{d}t \leqq c_3^2, \\
&\sup_{0 \leqq t \leqq T^*} |\psi_t(t)|_{D^2}^2 \leqq c_4^2.
\end{aligned} \tag{3.12}$$

We shall determine T^* and c_i ($i = 1, 2, 3, 4$) later, see (3.64) below, so that they depend only on c_0 and the fixed constants $\overline{\rho}$, α, β, γ, A, δ and T.

Let $(\phi, u)(x, t)$ be the unique strong solution to (3.7) in $\mathbb{R}^3 \times [0, T]$. We start from the estimates for ϕ. Hereinafter, we use $C \geq 1$ to denote a generic positive constant depending only on fixed constants $\overline{\rho}$, α, β, γ, A, δ and T, which may vary each time when it appears.

Lemma 3.2. *Let $(\phi, u)(x, t)$ be the unique strong solution to (3.7) on $\mathbb{R}^3 \times [0, T]$. Then*

$$1 + \overline{\phi}^2 + |\phi(t)|_\infty^2 + \|\phi(t) - \overline{\phi}\|_3^2 \leqq Cc_0^2,$$
$$|\phi_t(t)|_2^2 \leqq Cc_1^4, \quad |\phi_t(t)|_{D^1}^2 \leqq Cc_2^4, \quad |\phi_t(t)|_{D^2}^2 \leqq Cc_3^4,$$

for $0 \leqq t \leqq T_1 = \min(T^*, (1 + c_3)^{-2})$.

Proof. Let $\tilde{\phi} = \phi - \overline{\phi}$. Then from $(3.7)_1$, we have

$$\tilde{\phi}_t + v \cdot \nabla \tilde{\phi} + \frac{\delta - 1}{2} \psi \mathrm{div} v = 0. \tag{3.13}$$

Applying the operator ∂_x^ζ $(0 \leqq |\zeta| \leqq 3)$ to (3.13), we obtain

$$(\partial_x^\zeta \tilde{\phi})_t + v \cdot \nabla \partial_x^\zeta \tilde{\phi} = -(\partial_x^\zeta (v \cdot \nabla \tilde{\phi}) - v \cdot \nabla \partial_x^\zeta \tilde{\phi}) - \frac{\delta - 1}{2} \partial_x^\zeta (\psi \mathrm{div} v). \tag{3.14}$$

Multiplying both sides of (3.14) by $\partial_x^\zeta \tilde{\phi}$, and integrating over $\mathbb{R}^3$, we get

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} |\partial_x^\zeta \tilde{\phi}|_2^2 \leqq C \left(|\mathrm{div} v|_\infty |\partial_x^\zeta \tilde{\phi}|_2^2 + \Lambda_1^\zeta |\partial_x^\zeta \tilde{\phi}|_2 + \Lambda_2^\zeta |\partial_x^\zeta \tilde{\phi}|_2 \right), \tag{3.15}$$

where

$$\Lambda_1^\zeta = |\partial_x^\zeta (v \cdot \nabla \tilde{\phi}) - v \cdot \nabla \partial_x^\zeta \tilde{\phi}|_2, \quad \Lambda_2^\zeta = |\partial_x^\zeta (\psi \mathrm{div} v)|_2.$$

From Lemma 2.3, letting $r = b = 2, a = +\infty, f = v, g = \nabla \tilde{\phi}$ in (2.3), we obtain

$$\Lambda_1^\zeta \leqq C_\zeta \left(|\nabla v|_\infty |\nabla^{|\zeta|} \tilde{\phi}|_2 + |\nabla \tilde{\phi}|_\infty |\partial_x^\zeta v|_2 \right). \tag{3.16}$$

Now we consider the term Λ_2^ζ. When $|\zeta| \leqq 2$, it is easy to show that

$$\Lambda_2^\zeta \leqq C_\zeta \big(|\psi|_\infty + \|\nabla \psi\|_2 \big) \|v\|_3; \tag{3.17}$$

when $|\zeta| = 3$, we have

$$\begin{aligned} \Lambda_2^\zeta \leqq & C \left(|\psi \nabla^4 v|_2 + |\nabla \psi \cdot \nabla^3 v|_2 + |\nabla^2 \psi \cdot \nabla^2 v|_2 + |\nabla^3 \psi \cdot \nabla v|_2 \right) \\ \leqq & C \left(|\psi \nabla^4 v|_2 + \|\nabla \psi\|_2 \|v\|_3 \right). \end{aligned} \tag{3.18}$$

Combining (3.15)–(3.18), we arrive at

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \|\tilde{\phi}(t)\|_3 \leqq & C \left(\|v\|_3 \|\tilde{\phi}\|_3 + (|\psi|_\infty + \|\nabla \psi\|_2) \|v\|_3 + |\psi \nabla^4 v|_2 \right) \\ \leqq & C \left(c_3 \|\tilde{\phi}\|_3 + c_3^2 + |\psi \nabla^4 v|_2 \right). \end{aligned} \tag{3.19}$$

From Gronwall's inequality, we have

$$\|\tilde{\phi}(t)\|_3 \leqq \left(\|\phi_0 - \overline{\phi}\|_3 + c_3^2 t + \int_0^t |\psi \nabla^4 v|_2 \, \mathrm{d}s \right) \exp(Cc_3 t). \tag{3.20}$$

Therefore, observing that

$$\int_0^t |\psi \nabla^4 v(s)|_2 \,\mathrm{d}s \leqq t^{\frac{1}{2}} \left(\int_0^t |\psi \nabla^4 v(s)|_2^2 \,\mathrm{d}s \right)^{\frac{1}{2}} \leqq C c_3 t^{\frac{1}{2}},$$

we get

$$\|\phi(t) - \overline{\phi}\|_3 \leqq C c_0 \quad \text{for} \quad 0 \leqq t \leqq T_1 = \min(T^*, (1 + c_3)^{-2}).$$

The estimate for ϕ_t follows from the relation

$$\phi_t = -v \cdot \nabla \phi - \frac{\delta - 1}{2} \psi \mathrm{div} v.$$

For $0 \leqq t \leqq T_1$, the following estimates hold:

$$\begin{aligned}
|\phi_t(t)|_2 \leqq & C\big(|v(t)|_6 |\nabla \phi(t)|_3 + |\psi(t)|_\infty |\mathrm{div} v(t)|_2\big) \leqq C c_1^2, \\
|\phi_t(t)|_{D^1} \leqq & C\big(|v(t)|_\infty |\nabla^2 \phi(t)|_2 + |\nabla v(t)|_6 |\nabla \phi(t)|_3 + |\psi(t)|_\infty |\nabla^2 v(t)|_2 \\
& + |\nabla v(t)|_6 |\nabla \psi(t)|_3\big) \leqq C c_2^2, \\
|\phi_t(t)|_{D^2} \leqq & C\big(|v(t)|_\infty |\nabla^3 \phi(t)|_2 + |\nabla v(t)|_\infty |\nabla^2 \phi(t)|_2 + |\nabla^2 v(t)|_6 |\nabla \phi(t)|_3 \\
& + |\psi(t)|_\infty |\nabla^3 v(t)|_2 + |\nabla \psi(t)|_\infty |\nabla^2 v(t)|_2 + |\nabla^2 \psi(t)|_2 |\nabla v(t)|_\infty\big) \\
\leqq & C c_3^2.
\end{aligned} \tag{3.21}$$

These complete the proof. □

Now we turn to the estimates for the velocity u in the following two lemmas. In view of (1.8), we define

$$K = \frac{4\gamma - 4}{\delta - 1} \geq 8.$$

The following lemma gives some lower order estimates for the velocity u:

Lemma 3.3. *Let $(\phi, u)(x, t)$ be the unique strong solution to (3.7) on $\mathbb{R}^3 \times [0, T]$. Then*

$$\|u(t)\|_1^2 \leqq C c_0^2, \quad |u(t)|_{D^2}^2 + |u_t(t)|_2^2 + \int_0^t \Big(|\phi \nabla^3 u(s)|_2^2 + |u_t(s)|_{D^1}^2\Big) \mathrm{d}s \leqq C c_1^{2K}$$

for $0 \leqq t \leqq T_2 = \min(T^*, (1 + c_3)^{-2K})$.

Proof. The proof will be carried out in two steps.
Step 1 We estimate $|\partial_x^\zeta u|_2$ when $|\zeta| \leqq 2$. Taking ∂_x^ζ on equation $(3.7)_2$, we have

$$\begin{aligned}
& (\partial_x^\zeta u)_t + v \cdot \nabla(\partial_x^\zeta u) + (\phi^2 + \eta^2) L \partial_x^\zeta u \\
& = -\frac{A\gamma}{\gamma - 1} \partial_x^\zeta \nabla \phi^{\frac{2\gamma - 2}{\delta - 1}} + \nabla \phi^2 \cdot \partial_x^\zeta Q(v) - \big(\partial_x^\zeta (v \cdot \nabla u) - v \cdot \nabla(\partial_x^\zeta u)\big) \\
& \quad - \Big(\partial_x^\zeta ((\phi^2 + \eta^2) L u) - (\phi^2 + \eta^2) L \partial_x^\zeta u\Big) \\
& \quad + \Big(\partial_x^\zeta (\nabla \phi^2 \cdot Q(v)) - \nabla \phi^2 \cdot \partial_x^\zeta Q(v)\Big).
\end{aligned} \tag{3.22}$$

Multiplying (3.22) by $\partial_x^\zeta u$ on both sides and integrating over $\mathbb{R}^3$ by parts, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\partial_x^\zeta u|_2^2+\alpha|\sqrt{\phi^2+\eta^2}\nabla(\partial_x^\zeta u)|_2^2+(\alpha+\beta)|\sqrt{\phi^2+\eta^2}\mathrm{div}\partial_x^\zeta u|_2^2\\
&\quad=-\int(v\cdot\nabla(\partial_x^\zeta u))\cdot\partial_x^\zeta u-\frac{\delta-1}{\delta}\int\Big(\nabla(\phi^2+\eta^2)\cdot Q(\partial_x^\zeta u)\Big)\cdot\partial_x^\zeta u\\
&\qquad-\frac{A\gamma}{\gamma-1}\int\partial_x^\zeta\nabla\phi^{\frac{2\gamma-2}{\delta-1}}\cdot\partial_x^\zeta u+\int\nabla\phi^2\cdot\partial_x^\zeta Q(v)\cdot\partial_x^\zeta u\\
&\qquad-\int\Big(\partial_x^\zeta(v\cdot\nabla u)-v\cdot\nabla(\partial_x^\zeta u)\Big)\cdot\partial_x^\zeta u\\
&\qquad-\int\Big(\partial_x^\zeta((\phi^2+\eta^2)Lu)-(\phi^2+\eta^2)L\partial_x^\zeta u\Big)\cdot\partial_x^\zeta u\\
&\qquad+\int\Big(\partial_x^\zeta(\nabla\phi^2\cdot Q(v))-\nabla\phi^2\cdot\partial_x^\zeta Q(v)\Big)\cdot\partial_x^\zeta u=:\sum_{i=1}^7 I_i.
\end{aligned}\tag{3.23}$$

Now we estimate the last part of (3.23) term by term. Using Hölder's inequality, Lemma 2.1 and Young's inequality, we have

$$\begin{aligned}
I_1&=-\int(v\cdot\nabla(\partial_x^\zeta u))\cdot\partial_x^\zeta u\leqq C|\nabla v|_\infty|\partial_x^\zeta u|_2^2\leqq Cc_3|\partial_x^\zeta u|_2^2,\\
I_2&=-\frac{\delta-1}{\delta}\int\Big(\nabla(\phi^2+\eta^2)\cdot Q(\partial_x^\zeta u)\Big)\cdot\partial_x^\zeta u\\
&\leqq C|\phi\nabla(\partial_x^\zeta u)|_2|\nabla\phi|_\infty|\partial_x^\zeta u|_2\\
&\leqq C|\sqrt{\phi^2+\eta^2}\nabla(\partial_x^\zeta u)|_2|\nabla\phi|_\infty|\partial_x^\zeta u|_2\\
&\leqq\frac{\alpha}{20}|\sqrt{\phi^2+\eta^2}\nabla(\partial_x^\zeta u)|_2^2+Cc_0^2|\partial_x^\zeta u|_2^2.
\end{aligned}\tag{3.24}$$

For the term I_3, it is obvious that

$$I_3=-\frac{A\gamma}{\gamma-1}\int\nabla\partial_x^\zeta\phi^{\frac{2\gamma-2}{\delta-1}}\cdot\partial_x^\zeta u=\frac{A\gamma}{\gamma-1}\int\partial_x^\zeta\phi^{\frac{2\gamma-2}{\delta-1}}\mathrm{div}\partial_x^\zeta u.\tag{3.25}$$

Since $1<\delta\leqq\frac{\gamma+1}{2}$, $\frac{2\gamma-2}{\delta-1}\geq 4$. Then for $|\zeta|=0$, one has

$$\begin{aligned}
I_3&=-\frac{2A\gamma}{\delta-1}\int\phi^{\frac{2\gamma-\delta-1}{\delta-1}}\nabla\phi\cdot u\leqq C|\phi|_\infty^{\frac{2\gamma-\delta-1}{\delta-1}}|\nabla\phi|_2|u|_2\\
&\leqq C|\phi|_\infty^{\frac{4\gamma-2\delta-2}{\delta-1}}|\nabla\phi|_2^2+C|u|_2^2\leqq Cc_0^{\frac{4\gamma-4}{\delta-1}}+C|u|_2^2.
\end{aligned}\tag{3.26}$$

When $1\leqq|\zeta|\leqq 2$, we have

$$\begin{aligned}
I_3&=\frac{A\gamma}{\gamma-1}\int\partial_x^\zeta\phi^{\frac{2\gamma-2}{\delta-1}}\mathrm{div}\partial_x^\zeta u\\
&\leqq C\Big(|\phi|_\infty^{\frac{2\gamma-2}{\delta-1}-2}|\nabla\phi|_2|\phi\mathrm{div}\partial_x^\zeta u|_2+|\phi|_\infty^{\frac{2\gamma-2}{\delta-1}-2}|\nabla^2\phi|_2|\phi\mathrm{div}\partial_x^\zeta u|_2\\
&\quad+|\phi|_\infty^{\frac{2\gamma-2}{\delta-1}-3}|\nabla\phi|_6|\nabla\phi|_3|\phi\mathrm{div}\partial_x^\zeta u|_2\Big)
\end{aligned}$$

$$
\begin{aligned}
&\leqq C\Big(|\phi|_\infty^{\frac{2\gamma-2}{\delta-1}-2}\|\nabla\phi\|_2+|\phi|_\infty^{\frac{2\gamma-2}{\delta-1}-3}|\nabla\phi|_6|\nabla\phi|_3\Big)|\phi\nabla(\partial_x^\zeta u)|_2\\
&\leqq Cc_0^{\frac{2\gamma-\delta-1}{\delta-1}}|\phi\nabla(\partial_x^\zeta u)|_2\leqq Cc_0^{\frac{4\gamma-2\delta-2}{\delta-1}}+\frac{\alpha}{20}|\sqrt{\phi^2+\eta^2}\nabla(\partial_x^\zeta u)|_2^2,
\end{aligned}
\tag{3.27}
$$

which, combining with (3.26) and $\delta>1$, implies that

$$
I_3\leqq\frac{\alpha}{10}|\sqrt{\phi^2+\eta^2}\nabla(\partial_x^\zeta u)|_2^2+C|u|_2^2+Cc_0^{\frac{4\gamma-4}{\delta-1}},\quad\text{for}\quad|\zeta|\leqq 2.
\tag{3.28}
$$

For the term I_4, it is not difficult to show that

$$
\begin{aligned}
I_4=&\int\nabla\phi^2\cdot\partial_x^\zeta Q(v)\cdot\partial_x^\zeta u\\
\leqq& C|\phi|_\infty|\nabla\phi|_\infty\|\nabla v\|_2|\partial_x^\zeta u|_2\leqq Cc_3^3|\partial_x^\zeta u|_2,\quad\text{for}\quad|\zeta|\leqq 2.
\end{aligned}
\tag{3.29}
$$

For the term I_5, we have

$$
\begin{aligned}
I_5=&-\int\Big(\partial_x^\zeta(v\cdot\nabla u)-v\cdot\nabla(\partial_x^\zeta u)\Big)\cdot\partial_x^\zeta u\\
\leqq&|\partial_x^\zeta(v\cdot\nabla u)-v\cdot\nabla(\partial_x^\zeta u)|_2|\partial_x^\zeta u|_2\\
\leqq& C(|\nabla v|_\infty\|\nabla u\|_1+|\partial_x^\zeta v|_6|\nabla u|_3)|\partial_x^\zeta u|_2\\
\leqq& Cc_3\|u\|_2^2+Cc_3|\nabla u|_3|\partial_x^\zeta u|_2\leqq Cc_3\|u\|_2^2,\quad\text{for}\quad 1\leqq|\zeta|\leqq 2.
\end{aligned}
\tag{3.30}
$$

For the term I_6, when $|\zeta|=1$, we have,

$$
\begin{aligned}
I_6=&-\int\Big(\partial_x^\zeta((\phi^2+\eta^2)Lu)-(\phi^2+\eta^2)L\partial_x^\zeta u\Big)\cdot\partial_x^\zeta u\\
\leqq& C|\partial_x^\zeta((\phi^2+\eta^2)Lu)-(\phi^2+\eta^2)L\partial_x^\zeta u|_2|\partial_x^\zeta u|_2\\
\leqq& C|\nabla(\phi^2+\eta^2)|_\infty|Lu|_2|\partial_x^\zeta u|_2\\
\leqq& C|\phi|_\infty|\nabla\phi|_\infty|u|_{D^1}|u|_{D^2}\leqq Cc_0^2\|\nabla u\|_1^2.
\end{aligned}
\tag{3.31}
$$

When $|\zeta|=2$, we have

$$
\begin{aligned}
I_6=&-\int\Big(\partial_x^\zeta((\phi^2+\eta^2)Lu)-(\phi^2+\eta^2)L\partial_x^\zeta u\Big)\cdot\partial_x^\zeta u\\
\leqq& C\int\Big(|\nabla^2\phi||\phi Lu|+|\nabla\phi|^2|Lu|+|\phi\nabla Lu||\nabla\phi|\Big)|\partial_x^\zeta u|\\
\leqq& C\Big(|\nabla^2\phi|_3|\phi\nabla^2u|_6|\nabla^2u|_2+|\nabla\phi|_\infty^2|u|_{D^2}^2+|\nabla\phi|_\infty|\phi\nabla^3u|_2|\nabla^2u|_2\Big)\\
\leqq& C\|\nabla\phi\|_2^2|u|_{D^2}^2+\frac{\alpha}{20}|\phi\nabla^3u|_2^2\\
\leqq& Cc_0^2|u|_{D^2}^2+\frac{\alpha}{20}|\sqrt{\phi^2+\eta^2}\nabla^3u|_2^2,
\end{aligned}
\tag{3.32}
$$

where we used the fact that

$$
|\phi\nabla^2u|_6\leqq C|\phi\nabla^2u|_{D^1}\leqq C(|\nabla\phi|_\infty|\nabla^2u|_2+|\phi\nabla^3u|_2).
\tag{3.33}
$$

Combining (3.31)–(3.32), we then have

$$I_6 \leqq \frac{\alpha}{20}|\sqrt{\phi^2+\eta^2}\nabla^3 u|_2^2 + Cc_0^2\|\nabla u\|_1^2, \quad \text{for} \quad 1 \leqq |\zeta| \leqq 2. \tag{3.34}$$

Term I_7 can be controlled as follows:

$$\begin{aligned} I_7 =& \int \Big(\partial_x^\zeta(\nabla\phi^2 \cdot Q(v)) - \nabla\phi^2 \cdot \partial_x^\zeta Q(v)\Big) \cdot \partial_x^\zeta u \\ \leqq& |\partial_x^\zeta(\nabla\phi^2 Q(v)) - \nabla\phi^2 \partial_x^\zeta Q(v)|_2 |\partial_x^\zeta u|_2. \end{aligned} \tag{3.35}$$

When $|\zeta| = 1$,

$$\begin{aligned} &|\partial_x^\zeta(\nabla\phi^2 \cdot Q(v)) - \nabla\phi^2 \cdot \partial_x^\zeta Q(v)|_2 \\ &\leqq C(|\nabla\phi|_\infty^2|\nabla v|_2 + |\phi|_\infty|\nabla^2\phi|_2|\nabla v|_\infty) \leqq Cc_3^3; \end{aligned} \tag{3.36}$$

when $|\zeta| = 2$,

$$\begin{aligned} &|\partial_x^\zeta(\nabla\phi^2 \cdot Q(v)) - \nabla\phi^2 \cdot \partial_x^\zeta Q(v)|_2 \\ &\leqq C\Big(|\nabla\phi|_\infty^2|\nabla^2 v|_2 + |\phi|_\infty|\nabla^2\phi|_6|\nabla^2 v|_3 \\ &\quad + |\phi|_\infty|\nabla^3\phi|_2|\nabla v|_\infty + |\nabla\phi|_\infty|\nabla^2\phi|_2|\nabla v|_\infty\Big) \leqq Cc_3^3, \end{aligned} \tag{3.37}$$

which, combined with (3.35), implies that

$$I_7 \leqq Cc_3^3|\partial_x^\zeta u|_2, \quad \text{for} \quad 1 \leqq |\zeta| \leqq 2. \tag{3.38}$$

Therefore, from (3.23)–(3.24), (3.28)–(3.30), (3.34) and (3.38), we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_2^2 + |\sqrt{\phi^2+\eta^2}\nabla^3 u|_2^2 \leqq Cc_3^2\|u\|_2^2 + Cc_3^K. \tag{3.39}$$

Now Gronwall's inequality implies that

$$\|u(t)\|_2^2 + \int_0^t |\sqrt{\phi^2+\eta^2}\nabla^3 u|_2^2 \,\mathrm{d}s \leqq \big(Cc_3^K t + \|u_0\|_2^2\big)\exp(Cc_3^2 t) \leqq Cc_0^2 \tag{3.40}$$

for $0 \leqq t \leqq T_2 = \min\big(T^*, (1+c_3)^{-2K}\big)$.

Step 2 We estimate $|\partial_x^\zeta u_t|_2$ when $|\zeta| \leqq 1$. From the momentum equations $(3.7)_2$, we have

$$\begin{aligned} |u_t|_2 =& \Big|v\cdot\nabla u + \frac{A\gamma}{\gamma-1}\nabla\phi^{\frac{2\gamma-2}{\delta-1}} + (\phi^2+\eta^2)Lu - \nabla\phi^2\cdot Q(v)\Big|_2 \\ \leqq& C\Big(|v|_6|\nabla u|_3 + |\phi|_\infty^{\frac{2\gamma-2}{\delta-1}-1}|\nabla\phi|_2 + |\phi^2+\eta^2|_\infty|u|_{D^2} + |\phi|_\infty|\nabla\phi|_\infty|\nabla v|_2\Big) \\ \leqq& Cc_1^K. \end{aligned} \tag{3.41}$$

Similarly, for $|u_t|_{D^1}$, we have

$$\begin{aligned}|u_t|_{D^1} =& \Big|v\cdot\nabla u+\frac{A\gamma}{\gamma-1}\nabla\phi^{\frac{2\gamma-2}{\delta-1}}+(\phi^2+\eta^2)Lu-\nabla\phi^2\cdot Q(v)\Big|_{D^1}\\ \leqq& C\Big(\|v\|_2\|\nabla u\|_1+|\phi|_\infty^{\frac{2\gamma-2}{\delta-1}-1}|\nabla^2\phi|_2+|\phi|_\infty^{\frac{2\gamma-2}{\delta-1}-2}|\nabla\phi|_2|\nabla\phi|_\infty\\ &+|\sqrt{\phi^2+\eta^2}|_\infty|\sqrt{\phi^2+\eta^2}\nabla^3 u|_2+|\phi|_\infty|\nabla\phi|_\infty|u|_{D^2}\\ &+(|\phi|_\infty^2+\|\nabla\phi\|_2^2)\|\nabla v\|_1\Big)\\ \leqq& C\Big(c_2^K+c_0|\sqrt{\phi^2+\eta^2}\nabla^3 u|_2\Big),\end{aligned} \tag{3.42}$$

which implies that

$$\int_0^t |u_t|_{D^1}^2\,\mathrm{d}s \leqq C\int_0^t \big(c_2^{2K}+c_0^2|\sqrt{\phi^2+\eta^2}\nabla^3 u|_2^2\big)\,\mathrm{d}s\leqq Cc_0^4$$

for $0\leqq t\leqq T_2=\min(T^*,(1+c_3)^{-2K})$. □

Finally, we give the higher order estimates for velocity u such as $|u|_{L^\infty([0,T];D^3)}$ and $|\phi\nabla^4 u|_{L^2([0,T];L^2)}$ through an accurate analysis on the artificial viscosity $(\phi^2+\eta^2)Lu$.

Lemma 3.4. *Let $(\phi,u)(x,t)$ be the unique strong solution to (3.7) on $\mathbb{R}^3\times[0,T]$. Then*

$$|u(t)|_{D^3}^2+|u_t(t)|_{D^1}^2+\int_0^t\Big(|\phi\nabla^4 u(s)|_2^2+|u_t(s)|_{D^2}^2\Big)\,\mathrm{d}s\leqq Cc_2^{2K}$$

for $0\leqq t\leqq T_3=\min(T^*,(1+c_3)^{-2K-4})$.

Proof. The proof is divided into two steps.
Step 1 The estimate of $|\partial_x^\zeta u|_2$ for $|\zeta|=3$. Multiplying (3.22) by $\partial_x^\zeta u$ on both sides and integrating over $\mathbb{R}^3$ by parts, we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\partial_x^\zeta u|_2^2+\alpha|\sqrt{\phi^2+\eta^2}\nabla(\partial_x^\zeta u)|_2^2+(\alpha+\beta)|\sqrt{\phi^2+\eta^2}\mathrm{div}\partial_x^\zeta u|_2^2\\ &=-\int(v\cdot\nabla(\partial_x^\zeta u))\cdot\partial_x^\zeta u-\frac{\delta-1}{\delta}\int(\nabla(\phi^2+\eta^2)\cdot Q(\partial_x^\zeta u))\cdot\partial_x^\zeta u\\ &\quad-\frac{A\gamma}{\gamma-1}\int\partial_x^\zeta\nabla\phi^{\frac{2\gamma-2}{\delta-1}}\cdot\partial_x^\zeta u+\int\nabla\phi^2\cdot\partial_x^\zeta Q(v)\cdot\partial_x^\zeta u\\ &\quad-\int\Big(\partial_x^\zeta(v\cdot\nabla u)-v\cdot\nabla(\partial_x^\zeta u)\Big)\cdot\partial_x^\zeta u\\ &\quad-\int\Big(\partial_x^\zeta((\phi^2+\eta^2)Lu)-(\phi^2+\eta^2)L\partial_x^\zeta u\Big)\cdot\partial_x^\zeta u\\ &\quad+\int\Big(\partial_x^\zeta(\nabla\phi^2\cdot Q(v))-\nabla\phi^2\cdot\partial_x^\zeta Q(v)\Big)\cdot\partial_x^\zeta u=:\sum_{i=8}^{14}I_i.\end{aligned}\tag{3.43}$$

Now we estimate I_i $(i = 8, \cdots, 14)$ term by term. Similarly to the derivation of (3.24), the first two terms can be treated as follows:

$$\begin{aligned} I_8 &= -\int (v \cdot \nabla(\partial_x^\zeta u)) \cdot \partial_x^\zeta u \leqq C c_3 |\partial_x^\zeta u|_2^2, \\ I_9 &= -\frac{\delta-1}{\delta}\int (\nabla(\phi^2+\eta^2)\cdot Q(\partial_x^\zeta u))\cdot \partial_x^\zeta u \\ &\leqq \frac{\alpha}{20}|\sqrt{\phi^2+\eta^2}\nabla(\partial_x^\zeta u)|_2^2 + C c_0^2 |\partial_x^\zeta u|_2^2. \end{aligned} \tag{3.44}$$

For the term I_{10}, when $|\zeta| = 3$, one has

$$\begin{aligned} I_{10} &= -\frac{A\gamma}{\gamma-1}\int \nabla \partial_x^\zeta \phi^{\frac{2\gamma-2}{\delta-1}} \cdot \partial_x^\zeta u = \frac{A\gamma}{\gamma-1}\int \partial_x^\zeta \phi^{\frac{2\gamma-2}{\delta-1}} \mathrm{div} \partial_x^\zeta u \\ &\leqq C\int \Big(|\phi|^{\frac{2\gamma-2}{\delta-1}-4}|\nabla\phi|^3 + |\phi|^{\frac{2\gamma-2}{\delta-1}-3}|\nabla\phi||\nabla^2\phi| + |\phi|^{\frac{2\gamma-2}{\delta-1}-2}|\nabla^3\phi|\Big)|\phi \mathrm{div}\partial_x^\zeta u| \\ &\leqq C\Big(|\nabla\phi|_2|\nabla\phi|_\infty^2|\phi|_\infty^{\frac{2\gamma-2}{\delta-1}-4}|\phi\nabla(\partial_x^\zeta u)|_2 + |\nabla^2\phi|_2|\nabla\phi|_\infty|\phi|_\infty^{\frac{2\gamma-2}{\delta-1}-3}|\phi\nabla(\partial_x^\zeta u)|_2 \\ &\quad + |\nabla^3\phi|_2|\phi|_\infty^{\frac{2\gamma-2}{\delta-1}-2}|\phi\nabla(\partial_x^\zeta u)|_2\Big) \\ &\leqq C\Big(|\nabla\phi|_2^2|\nabla\phi|_\infty^4|\phi|_\infty^{\frac{4\gamma-4}{\delta-1}-8} + |\nabla^2\phi|_2^2|\nabla\phi|_\infty^2|\phi|_\infty^{\frac{4\gamma-4}{\delta-1}-6} + |\nabla^3\phi|_2^2|\phi|_\infty^{\frac{4\gamma-4}{\delta-1}-4}\Big) \\ &\quad + \frac{\alpha}{20}|\phi\nabla(\partial_x^\zeta u)|_2^2 \\ &\leqq C c_0^{\frac{4\gamma-2\delta-2}{\delta-1}} + \frac{\alpha}{20}|\sqrt{\phi^2+\eta^2}\nabla(\partial_x^\zeta u)|_2^2. \end{aligned} \tag{3.45}$$

For the term I_{11}, from integration by parts, we have

$$\begin{aligned} I_{11} &= \int \nabla\phi^2 \cdot \partial_x^\zeta Q(v) \cdot \partial_x^\zeta u \\ &\leqq C\int \Big(|\nabla^2\phi||\nabla^3 v||\phi\partial_x^\zeta u| + |\nabla\phi|^2|\nabla^3 v||\partial_x^\zeta u| + |\nabla\phi||\nabla^3 v||\phi\nabla\partial_x^\zeta u|\Big) \\ &\leqq C\Big(|\nabla^2\phi|_3|\nabla^3 v|_2|\phi\partial_x^\zeta u|_6 + |\nabla\phi|_\infty^2|\nabla^3 v|_2|\partial_x^\zeta u|_2 + |\nabla\phi|_\infty|\nabla^3 v|_2|\phi\nabla(\partial_x^\zeta u)|_2\Big) \\ &\leqq C\Big(|\nabla^2\phi|_3|\nabla^3 v|_2\big(|\phi\nabla(\partial_x^\zeta u)|_2 + |\nabla\phi|_\infty|\partial_x^\zeta u|_2\big) + c_3^3|\partial_x^\zeta u|_2 + c_3^2|\phi\nabla(\partial_x^\zeta u)|_2\Big) \\ &\leqq C\Big(c_3^3|\partial_x^\zeta u|_2 + c_3^4\Big) + \frac{\alpha}{20}|\sqrt{\phi^2+\eta^2}\nabla(\partial_x^\zeta u)|_2^2, \end{aligned} \tag{3.46}$$

where we used the fact that

$$|\phi\nabla^3 u|_6 \leqq C|\phi\nabla^3 u|_{D^1} \leqq C\big(|\nabla\phi|_\infty|\nabla^3 u|_2 + |\phi\nabla^4 u|_2\big). \tag{3.47}$$

For the term I_{12}, letting $r = b = 2$, $a = +\infty$, $f = v$, $g = \nabla u$ in (2.3) of Lemma 2.3, from (3.40) we obtain

$$
\begin{aligned}
I_{12} =& -\int \left(\partial_x^\zeta (v\cdot\nabla u) - v\cdot\nabla(\partial_x^\zeta u)\right)\cdot \partial_x^\zeta u\\
\leqq& |\partial_x^\zeta (v\cdot\nabla u) - v\cdot\nabla(\partial_x^\zeta u)|_2|\partial_x^\zeta u|_2\\
\leqq& C\left(|\nabla v|_\infty|\nabla^3 u|_2 + |\nabla^3 v|_2|\nabla u|_\infty\right)|\partial_x^\zeta u|_2\\
\leqq& C\left(c_3|\nabla^3 u|_2^2 + c_3\|\nabla u\|_2|\nabla^3 u|_2\right)\\
\leqq& C\left(c_3|\nabla^3 u|_2^2 + c_3(c_0+|\nabla^3 u|_2)|\nabla^3 u|_2\right)\\
\leqq& C\left(c_3^2|\nabla^3 u|_2^2 + c_3^2\right). \qquad (3.48)
\end{aligned}
$$

For the term I_{13}, from $|\zeta|=3$ and (3.47) we have

$$
\begin{aligned}
I_{13} =& -\int \left(\partial_x^\zeta((\phi^2+\eta^2)Lu) - (\phi^2+\eta^2)L\partial_x^\zeta u\right)\cdot\partial_x^\zeta u\\
\leqq& C\int\left(|\nabla^3\phi||Lu||\phi\partial_x^\zeta u| + |\nabla\phi||\nabla^2\phi||Lu||\partial_x^\zeta u| + |\nabla\phi|^2|\nabla Lu||\partial_x^\zeta u|\right)\\
&+C\int\left(|\nabla^2\phi||\phi\nabla Lu||\partial_x^\zeta u| + |\phi\nabla^2 Lu||\nabla\phi||\partial_x^\zeta u|\right)\\
\leqq& C\Big(|\nabla^3\phi|_2|\nabla^2 u|_3|\phi\nabla^3 u|_6 + |\nabla\phi|_\infty|\nabla^2\phi|_3|\nabla^2 u|_6|\nabla^3 u|_2 + |\nabla\phi|_\infty^2|\nabla^3 u|_2^2\\
&+|\nabla^2\phi|_3|\nabla^3 u|_2|\phi\nabla^3 u|_6 + |\nabla\phi|_\infty|\phi\nabla^4 u|_2|\nabla^3 u|_2\Big)\\
\leqq& C\Big(c_0|\nabla^2 u|_2^{\frac12}|\nabla^3 u|_2^{\frac12}\left(|\phi\nabla^4 u|_2 + |\nabla\phi|_\infty|\nabla^3 u|_2\right)\\
&+c_0|u|_{D^3}\left(|\phi\nabla^4 u|_2 + |\nabla\phi|_\infty|\nabla^3 u|_2\right) + c_0^2|u|_{D^3}^2\Big) + \frac{\alpha}{20}|\phi\nabla^4 u|_2^2\\
\leqq& C\left(c_3^4|u|_{D^3}^2 + c_3^{2K+4}\right) + \frac{\alpha}{20}|\sqrt{\phi^2+\eta^2}\nabla^4 u|_2^2. \qquad (3.49)
\end{aligned}
$$

For the term I_{14}, we notice that

$$
\begin{aligned}
&\partial_x^\zeta(\nabla\phi^2\cdot Q(v)) - \nabla\phi^2\cdot\partial_x^\zeta Q(v)\\
&=\sum_{1\le i,j,k\le 3} l_{ijk}\left(C_{1ijk}\nabla\partial_x^{\zeta^i}\phi^2\cdot\partial_x^{\zeta^j+\zeta^k}Q(v) + C_{2ijk}\nabla\partial_x^{\zeta^j+\zeta^k}\phi^2\cdot\partial_x^{\zeta^i}Q(v)\right)\\
&+\nabla\partial_x^\zeta\phi^2\cdot Q(v),
\end{aligned}
$$

where $\zeta=\zeta^1+\zeta^2+\zeta^3$ for three multi-indexes $\zeta^i\in\mathbb{R}^3$ $(i=1,2,3)$ satisfying $|\zeta^i|=1$; C_{1ijk} and C_{2ijk} are all constants. The summation is through all permutation on i,j,k. $l_{ijk}=1$ if i, j and k are different from each other, and otherwise $l_{ijk}=0$ since there is no duplicated differentiation with respect to such decomposition on ζ. Then, we deduce that

$$
\begin{aligned}
I_{14} =& \int\left(\partial_x^\zeta(\nabla\phi^2\cdot Q(v)) - \nabla\phi^2\cdot\partial_x^\zeta Q(v)\right)\cdot\partial_x^\zeta u\\
=& \int\left(\sum_{i,j,k} l_{ijk}C_{1ijk}\nabla\partial_x^{\zeta^i}\phi^2\cdot\partial_x^{\zeta^j+\zeta^k}Q(v)\right)\cdot\partial_x^\zeta u
\end{aligned}
$$

$$
\begin{aligned}
&+\int\Big(\sum_{i,j,k} l_{ijk}C_{2ijk}\nabla\partial_x^{\zeta^j+\zeta^k}\phi^2\cdot\partial_x^{\zeta^i}Q(v)\Big)\cdot\partial_x^{\zeta}u\\
&+\int\big(\nabla\partial_x^{\zeta}\phi^2\cdot Q(v)\big)\cdot\partial_x^{\zeta}u\\
&=: I_{141}+I_{142}+I_{143}.
\end{aligned}\tag{3.50}
$$

It follows that

$$
\begin{aligned}
I_{141}=&\int\Big(\sum_{i,j,k} l_{ijk}C_{1ijk}\nabla\partial_x^{\zeta^i}\phi^2\partial_x^{\zeta^j+\zeta^k}Q(v)\Big)\cdot\partial_x^{\zeta}u\\
\leqq& C\left(|\nabla\phi|_\infty^2|\nabla^3u|_2|\nabla^3v|_2+|\nabla^3v|_2|\phi\nabla^3u|_6|\nabla^2\phi|_3\right)\\
\leqq& C\left(c_3^3|\nabla^3u|_2+c_3^2\big(|\sqrt{\phi^2+\eta^2}\nabla^4u|_2+|\nabla\phi|_\infty|\nabla^3u|_2\big)\right)\\
\leqq& C\left(c_3^3|u|_{D^3}+c_3^4\right)+\frac{\alpha}{20}|\sqrt{\phi^2+\eta^2}\nabla^4u|_2^2,\\
I_{142}=&\int\Big(\sum_{i,j,k} l_{ijk}C_{2ijk}\nabla\partial_x^{\zeta^j+\zeta^k}\phi^2\cdot\partial_x^{\zeta^i}Q(v)\Big)\cdot\partial_x^{\zeta}u\\
\leqq& C\left(|\nabla^3\phi|_2|\nabla^2v|_3|\phi\nabla^3u|_6+|\nabla\phi|_\infty|\nabla^2\phi|_3|\nabla^2v|_6|\nabla^3u|_2\right)\\
\leqq& C\left(c_3^3|\nabla^3u|_2+c_3^2\big(|\sqrt{\phi^2+\eta^2}\nabla^4u|_2+|\nabla\phi|_\infty|\nabla^3u|_2\big)\right)\\
\leqq& C\left(c_3^3|u|_{D^3}+c_3^4\right)+\frac{\alpha}{20}|\sqrt{\phi^2+\eta^2}\nabla^4u|_2^2.
\end{aligned}\tag{3.51}
$$

For the term I_{143}, from integration by parts, we have

$$
\begin{aligned}
I_{143}=&\int\left(\nabla\partial_x^{\zeta}\phi^2Q(v)\right)\cdot\partial_x^{\zeta}u\\
=&-\int\sum_{i=1}^{3}\left(\partial_x^{\zeta-\zeta^i}\nabla\phi^2\cdot\partial_x^{\zeta^i}Q(v)\cdot\partial_x^{\zeta}u\right)\\
&-\int\sum_{i=1}^{3}\left(\partial_x^{\zeta-\zeta^i}\nabla\phi^2\cdot Q(v)\cdot\partial_x^{\zeta+\zeta^i}u\right)\\
=:& I_{1431}+I_{1432}.
\end{aligned}\tag{3.52}
$$

For simplicity, we only consider the case that $i=1$, the rest terms can be estimated similarly. When $i=1$, the corresponding term in I_{1431} is

$$
\begin{aligned}
I_{1431}^{(1)}=&-\int\partial_x^{\zeta^2+\zeta^3}\nabla\phi^2\cdot\partial_x^{\zeta^1}Q(v)\cdot\partial_x^{\zeta}u\\
\leqq& C\left(|\nabla^3\phi|_2|\nabla^2v|_3|\phi\nabla^3u|_6+|\nabla\phi|_\infty|\nabla^2\phi|_6|\nabla^2v|_3|\nabla^3u|_2\right)\\
\leqq& C\left(c_3^3|u|_{D^3}+c_3^2\big(|\phi\nabla^4u|_2+|\nabla\phi\nabla^3u|_2\big)\right)\\
\leqq& C\left(c_3^3|u|_{D^3}+c_3^4\right)+\frac{\alpha}{20}|\sqrt{\phi^2+\eta^2}\nabla^4u|_2^2,
\end{aligned}\tag{3.53}
$$

and the corresponding term in I_{1432} is

$$\begin{aligned} I_{1432}^{(1)} &= -\int \partial_x^{\zeta^2+\zeta^3}\nabla\phi^2 \cdot Q(v)\cdot \partial_x^{\zeta+\zeta^1} u \\ &= -2\int \Big(\partial_x^{\zeta^2+\zeta^3}\nabla\phi\phi + \partial_x^{\zeta^2}\nabla\phi\partial_x^{\zeta^3}\phi)\Big)\cdot Q(v)\cdot \partial_x^{\zeta+\zeta^1} u \\ &\quad - 2\int \Big(\partial_x^{\zeta^3}\nabla\phi\partial_x^{\zeta^2}\phi + \nabla\phi\partial_x^{\zeta^2+\zeta^3}\phi\Big)\cdot Q(v)\cdot \partial_x^{\zeta+\zeta^1} u \\ &=: I_A + I_B + I_C + I_D. \end{aligned} \tag{3.54}$$

It is not hard to show that

$$\begin{aligned} I_A &= -2\int \Big(\partial_x^{\zeta^2+\zeta^3}\nabla\phi\phi\Big)\cdot Q(v)\cdot \partial_x^{\zeta+\zeta^1} u \\ &\leqq C|\nabla^3\phi|_2|\nabla v|_\infty|\phi\nabla^4 u|_2 \leqq Cc_3^4 + \frac{\alpha}{20}|\sqrt{\phi^2+\eta^2}\nabla^4 u|_2^2. \end{aligned} \tag{3.55}$$

For the term I_B, from integration by parts, we deduce that

$$\begin{aligned} I_B &= -2\int \Big(\partial_x^{\zeta^2}\nabla\phi\partial_x^{\zeta^3}\phi)\Big)\cdot Q(v)\cdot \partial_x^{\zeta+\zeta^1} u \\ &= 2\int \partial_x^{\zeta^1}\Big[\Big(\partial_x^{\zeta^2}\nabla\phi\partial_x^{\zeta^3}\phi)\Big)\cdot Q(v)\Big]\cdot \partial_x^{\zeta} u \\ &\leqq C\int \Big(|\nabla^3 u||\nabla v|\big(|\nabla^2\phi|^2 + |\nabla\phi||\nabla^3\phi|\big) + |\nabla^3 u||\nabla^2 v||\nabla^2\phi||\nabla\phi|\Big) \\ &\leqq C\Big(|\nabla^2\phi|_3|\nabla^2\phi|_6|\nabla^3 u|_2|\nabla v|_\infty + |\nabla^3 u|_2|\nabla v|_\infty|\nabla^3\phi|_2|\nabla\phi|_\infty \\ &\quad + |\nabla^3 u|_2|\nabla^2 v|_3|\nabla^2\phi|_6|\nabla\phi|_\infty\Big) \\ &\leqq Cc_3^3|\nabla^3 u|_2. \end{aligned} \tag{3.56}$$

Similarly to I_B, we have

$$I_C + I_D \leqq Cc_3^3|\nabla^3 u|_2, \tag{3.57}$$

which, combined with (3.50)–(3.57), implies that

$$I_{14} \leqq \frac{\alpha}{10}|\sqrt{\phi^2+\eta^2}\nabla^4 u|_2^2 + Cc_3^4|u|_{D^3} + Cc_3^4. \tag{3.58}$$

Thus, from (3.43)–(3.49) and (3.58), we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}|u|_{D^3}^2 + \int (\phi^2+\eta^2)|\nabla^4 u|^2 \leqq Cc_3^4|u|_{D^3}^2 + Cc_3^{2K+4}. \tag{3.59}$$

Then from Gronwall's inequality, we have

$$|u(t)|_{D^3}^2 + \int_0^t |\sqrt{\phi^2+\eta^2}\nabla^4 u|_2^2\,\mathrm{d}s \leqq C\big(|u_0|_{D^3}^2 + c_3^{2K+4}t\big)\exp(Cc_3^4 t) \leqq Cc_0^2 \tag{3.60}$$

for $0 \leqq t \leqq T_3 = \min\{T^*, (1+c_3)^{-2K-4}\}$.

Step 2 The estimate for $|\partial_x^\zeta u_t|_2$ when $1 \leqq |\zeta| \leqq 2$. First, from (3.42) we have

$$|u_t|_{D^1} \leqq C\left(c_2^K + c_0|\sqrt{\phi^2+\eta^2}\nabla^3 u|_2\right) \leqq C\left(c_2^K + c_0^3\right) \leqq Cc_2^K. \tag{3.61}$$

Similarly, from the momentum equations, we also have

$$\begin{aligned}
|u_t|_{D^2} =& \left| v\cdot\nabla u + \frac{A\gamma}{\gamma-1}\nabla\phi^{\frac{2\gamma-2}{\delta-1}} + (\phi^2+\eta^2)Lu - \nabla\phi^2\cdot Q(v)\right|_{D^2}\\
\leqq& C\Big(\|v\|_3\|\nabla u\|_2 + |\phi|_\infty^{\frac{2\gamma-2}{\delta-1}-1}|\nabla^3\phi|_2 + |\phi|_\infty^{\frac{2\gamma-2}{\delta-1}-2}|\nabla\phi|_3|\nabla^2\phi|_6\\
&+ |\phi|_\infty^{\frac{2\gamma-2}{\delta-1}-3}|\nabla\phi|_\infty^2|\nabla\phi|_2 + |\sqrt{\phi^2+\eta^2}|_\infty|\sqrt{\phi^2+\eta^2}\nabla^4 u|_2\\
&+ (|\phi|_\infty + \|\nabla\phi\|_2)^2(\|u\|_3 + \|v\|_3)\Big)\\
\leqq& C\left(c_3^K + c_0|\sqrt{\phi^2+\eta^2}\nabla^4 u|_2\right),
\end{aligned} \tag{3.62}$$

which implies that

$$\int_0^{T_3} |u_t|_{D^2}^2\,\mathrm{d}s \leqq C\int_0^{T_3}\left(c_3^{2K} + c_0^2|\sqrt{\phi^2+\eta^2}\nabla^4 u|_2^2\right)\mathrm{d}s \leqq Cc_0^4.$$

□

Combining the estimates obtained in Lemmas 3.2–3.4, we have

$$\begin{aligned}
&1+\overline{\phi}^2 + |\phi(t)|_\infty^2 + \|\phi(t)-\overline{\phi}\|_3^2 \leqq Cc_0^2,\\
&|\phi_t(t)|_2^2 \leqq Cc_1^4,\quad |\phi_t(t)|_{D^1}^2 \leqq Cc_2^4,\quad |\phi_t(t)|_{D^2}^2 \leqq Cc_3^4,\\
&\|u(t)\|_1^2 \leqq Cc_0^2,\quad |u(t)|_{D^2}^2 + |u_t(t)|_2^2 + \int_0^t\left(|\phi\nabla^3 u|_2^2 + |u_t|_{D^1}^2\right)\mathrm{d}s \leqq Cc_1^{2K},\\
&|u(t)|_{D^3}^2 + |u_t(t)|_{D^1}^2 + \int_0^t\left(|\phi\nabla^4 u|_2^2 + |u_t|_{D^2}^2\right)\mathrm{d}s \leqq Cc_2^{2K}
\end{aligned} \tag{3.63}$$

for $0 \leqq t \leqq \min(T^*, (1+c_3)^{-2K-4})$. Therefore, if we define the constants c_i $(i = 1, 2, 3, 4)$ and T^* by

$$\begin{aligned}
&c_1 = C^{\frac{1}{2}}c_0,\quad c_2 = C^{\frac{1}{2}}c_1^K = C^{\frac{K+1}{2}}c_0^K,\quad c_3 = C^{\frac{1}{2}}c_2^K = C^{\frac{K^2+K+1}{2}}c_0^{K^2},\\
&c_4 = C^{\frac{1}{2}}c_3^2 = C^{K^2+K+\frac{3}{2}}c_0^{2K^2},\quad \text{and}\quad T^* = \min(T, (1+c_3)^{-2K-4}),
\end{aligned} \tag{3.64}$$

then we conclude that

$$1+\overline{\phi}^2 + \sup_{0\leqq t\leqq T^*}\left(|\phi(t)|_\infty^2 + \|\phi(t)-\overline{\phi}\|_3^2 + \|u(t)\|_1^2\right) \leqq c_1^2,$$

$$\sup_{0\leqq t\leqq T^*}\left(|\phi_t(t)|_2^2 + |u(t)|_{D^2}^2 + |u_t(t)|_2^2\right) + \int_0^{T^*}\left(|\phi\nabla^3 u|_2^2 + |u_t|_{D^1}^2\right)\mathrm{d}t \leqq c_2^2,$$

$$\operatorname{ess}\sup_{0\leqq t\leqq T^*}\left(|\phi_t(t)|_{D^1}^2+|u(t)|_{D^3}^2+|u_t(t)|_{D^1}^2\right)+\int_0^{T^*}\left(|\phi\nabla^4 u|_2^2+|u_t|_{D^2}^2\right)\mathrm{d}t\leqq c_3^2,$$
$$\sup_{0\leqq t\leqq T^*}|\phi_t(t)|_{D^2}^2\leqq c_4^2. \tag{3.65}$$

3.4. Passing to the Limit $\eta\to 0$

With the help of the η-independent estimates established in (3.65), we now establish the local existence result for the following linearized problem without artificial viscosity under the assumption $\phi_0\geq 0$:

$$\begin{cases}\phi_t+v\cdot\nabla\phi+\dfrac{\delta-1}{2}\psi\operatorname{div}v=0,\\ u_t+v\cdot\nabla u+\dfrac{A\gamma}{\gamma-1}\nabla\phi^{\frac{2\gamma-2}{\delta-1}}+\phi^2 Lu=\nabla\phi^2\cdot Q(v),\\ (\phi,u)|_{t=0}=(\phi_0(x),u_0(x)),\quad x\in\mathbb{R}^3\\ (\phi,u)\to(\overline{\phi},0),\quad \text{as}\quad |x|\to+\infty,\quad t>0.\end{cases} \tag{3.66}$$

Lemma 3.5. *Assume that the initial data satisfy* (3.5). *Then there exists a unique strong solution* $(\phi,u)(x,t)$ *to* (3.66) *such that*

$$\begin{aligned}&\phi-\overline{\phi}\in C([0,T^*];H^3),\ \phi_t\in C([0,T^*];H^2),\\ &u\in C([0,T^*];H^{s'})\cap L^\infty([0,T^*];H^3),\quad \phi\nabla^4u\in L^2([0,T^*];L^2)\\ &u_t\in C([0,T^*];H^1)\cap L^2([0,T^*];D^2)\end{aligned} \tag{3.67}$$

for any constant $s'\in[2,3)$. *Moreover,* (ϕ,u) *also satisfies the a priori estimates in* (3.65).

Proof. We shall prove the existence, uniqueness and time continuity in three steps. **Step 1** Existence. From Lemma 3.1, for every $\eta>0$, there exists a unique strong solution $(\phi^\eta,u^\eta)(x,t)$ to the linearized problem (3.7) satisfying the estimates in (3.65), which are independent of the artificial viscosity coefficient η.

By virtue or the uniform estimates in (3.65) independent of η and the compactness in Lemma 2.2 (see [?]), we know that for any $R>0$, there exists a subsequence of solutions (still denoted by) (ϕ^η,u^η), which converges to a limit (ϕ,u) in the following strong sense:

$$(\phi^\eta,u^\eta)\to(\phi,u)\quad\text{in}\ C([0,T^*];H^2(B_R)),\quad \text{as}\ \eta\to 0. \tag{3.68}$$

Again, using the uniform estimates in (3.65) independent of η, we also know that there exists a subsequence (of subsequence chosen above) of solutions (still denoted by) (ϕ^η,u^η), which converges to (ϕ,u) in the following weak or weak* sense:

$$\begin{aligned}(\phi^\eta,u^\eta)\rightharpoonup(\phi,u)\quad&\text{weakly* in } L^\infty([0,T^*];H^3(\mathbb{R}^3)),\\ \phi_t^\eta\rightharpoonup\phi_t\quad&\text{weakly* in } L^\infty([0,T^*];H^2(\mathbb{R}^3)),\\ u_t^\eta\rightharpoonup u_t\quad&\text{weakly* in } L^\infty([0,T^*];H^1(\mathbb{R}^3)),\\ u_t^\eta\rightharpoonup u_t\quad&\text{weakly in } L^2([0,T^*];D^2(\mathbb{R}^3)).\end{aligned} \tag{3.69}$$

Combining the strong convergence in (3.68) and the weak convergence in (3.69), we easily obtain that (ϕ, u) also satisfies the local estimates in (3.65) and

$$\phi^\eta \nabla^4 u^\eta \rightharpoonup \phi \nabla^4 u \quad \text{weakly in } L^2(\mathbb{R}^3 \times [0, T^*]). \tag{3.70}$$

Now we are going to show that (ϕ, u) is a weak solution in the sense of distribution to the linearized problem (3.66). Multiplying $(3.7)_2$ by test function $w(t, x) = (w^1, w^2, w^3) \in C_c^\infty(\mathbb{R}^3 \times [0, T^*))$ on both sides, and integrating over $\mathbb{R}^3 \times [0, T^*]$, we have

$$\begin{aligned} &\int_0^t \int \Big(u^\eta \cdot w_t - (v \cdot \nabla) u^\eta \cdot w + \frac{A\gamma}{\gamma - 1} (\phi^\eta)^{\frac{2\gamma-2}{\delta-1}} \operatorname{div} w\Big) \mathrm{d}x\, \mathrm{d}s \\ &= - \int u_0 \cdot w(0, x) \\ &\quad + \int_0^t \int \Big(((\phi^\eta)^2 + \eta^2) L u^\eta \cdot w - \nabla (\phi^\eta)^2 \cdot Q(v) \cdot w\Big) \mathrm{d}x\, \mathrm{d}s. \end{aligned} \tag{3.71}$$

Combining the strong convergence in (3.68) and the weak convergences in (3.69)–(3.70), and letting $\eta \to 0$ in (3.71), we have

$$\begin{aligned} &\int_0^t \int \Big(u \cdot w_t - (v \cdot \nabla) u \cdot w + \frac{A\gamma}{\gamma - 1} \phi^{\frac{2\gamma-2}{\delta-1}} \operatorname{div} w\Big) \mathrm{d}x\, \mathrm{d}s \\ &= - \int u_0 \cdot w(0, x) + \int_0^t \int \Big(\phi^2 L u \cdot w - \nabla \phi^2 \cdot Q(v) \cdot w\Big) \mathrm{d}x\, \mathrm{d}s. \end{aligned} \tag{3.72}$$

Thus it is obvious that (ϕ, u) is a weak solution in the sense of distribution to the linearized problem (3.66), satisfying the following regularities:

$$\begin{aligned} &\phi - \overline{\phi} \in L^\infty([0, T^*]; H^3), \ \ \phi_t \in L^\infty([0, T^*]; H^2), \\ &u \in L^\infty([0, T^*]; H^3), \quad \phi \nabla^4 u \in L^2([0, T^*]; L^2), \\ &u_t \in L^\infty([0, T^*]; H^1) \cap L^2([0, T^*]; D^2), \end{aligned} \tag{3.73}$$

where we used the lower semi-continuity of various norms in the weak or weak* convergence in (3.69)–(3.70). Therefore, this weak solutions (ϕ, u) of (3.66) is actually a strong one.

Step 2 Uniqueness. Let (ϕ_1, u_1) and (ϕ_2, u_2) be two solutions obtained in the above step. For $\varphi = \phi_1 - \phi_2$, we have from $(3.66)_1$ that

$$\varphi_t + v \cdot \nabla \varphi = 0, \tag{3.74}$$

which immediately implies that $\varphi = 0$ in $\mathbb{R}^3$ with zero initial data.

For $\overline{u} = u_1 - u_2$, from $(3.66)_2$, using the fact $\phi_1 = \phi_2$, it is clear that

$$\overline{u}_t + v \cdot \nabla \overline{u} - \phi_1^2 L \overline{u} = 0. \tag{3.75}$$

Multiplying (3.75) by $\overline{u}$ on both sides, and integrating over $\mathbb{R}^3$, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} |\overline{u}|_2^2 + |\phi_1 \nabla \overline{u}|_2^2 &\leqq C |\nabla v|_\infty |\overline{u}|_2^2 + C |\overline{u}|_2 |\nabla \phi_1|_\infty |\phi_1 \nabla \overline{u}|_2 \\ &\leqq \frac{1}{10} |\phi_1 \nabla \overline{u}|_2^2 + C(|\nabla v|_\infty + |\nabla \phi_1|_\infty^2) |\overline{u}|_2^2. \end{aligned} \tag{3.76}$$

Now, the Gronwall's inequality, along with zero initial data of $\overline{u}$ implies that $\overline{u} = 0$ in $\mathbb{R}^3$. This completes the proof of uniqueness.
Step 3 Time continuity. For ϕ, the regularities in (3.73) and the classical Sobolev imbedding theorem infer that

$$\phi - \overline{\phi} \in C([0, T^*]; H^2) \cap C([0, T^*]; \text{weak} - H^3). \tag{3.77}$$

Using the same arguments as in the proof of Lemma 3.2, we have

$$\begin{aligned} &\|\phi(t) - \overline{\phi}\|_3^2 \\ &\leqq \Big(\|\phi_0 - \overline{\phi}\|_3^2 + C\int_0^t \big(\|\nabla\psi\|_2^2\|v\|_3^2 + |\phi\nabla^4 v|_2^2\big)\,\mathrm{d}s\Big) \\ &\quad \times \exp\Big(C\int_0^t \big(\|v(s)\|_3 + 1\big)\,\mathrm{d}s\Big), \end{aligned} \tag{3.78}$$

which implies that

$$\limsup_{t\to 0} \|\phi(t) - \overline{\phi}\|_3 \leqq \|\phi_0 - \overline{\phi}\|_3. \tag{3.79}$$

From Lemma 2.7 and (3.77), we know that ϕ is right continuous at $t = 0$ in H^3 space. The time reversibility of the equation $(3.66)_1$ yields

$$\phi - \overline{\phi} \in C([0, T^*]; H^3). \tag{3.80}$$

For ϕ_t, we note that

$$\phi_t = -v \cdot \nabla\phi - \frac{\delta - 1}{2}\psi \operatorname{div} v. \tag{3.81}$$

On the other hand, since

$$\psi\nabla v \in L^2([0, T^*]; H^3), \quad (\psi\nabla v)_t \in L^2([0, T^*]; H^1), \tag{3.82}$$

using the Sobolev embedding theorem with time, we have

$$\psi\nabla v \in C([0, T^*]; H^2), \tag{3.83}$$

which implies that

$$\phi_t \in C([0, T^*]; H^2).$$

For velocity u, from the regularities shown in (3.73) and Sobolev imbedding theorem, we know that

$$u \in C([0, T^*]; H^2) \cap C([0, T^*]; \text{weak} - H^3). \tag{3.84}$$

Then from Lemma 2.5, for any $s' \in [2, 3)$, we have

$$\|u\|_{s'} \leqq C_3\|u\|_0^{1-\frac{s'}{3}}\|u\|_3^{\frac{s'}{3}}.$$

This, together with the upper bound estimates shown in (3.65) and the time continuity (3.84), yields

$$u \in C([0, T^*]; H^{s'}). \tag{3.85}$$

Finally, we consider u_t. Noting that

$$u_t = -v \cdot \nabla u - \frac{2A\gamma}{\delta - 1}\phi^{\frac{2r-\delta-1}{\delta-1}} \nabla\phi + \phi^2 Lu + \nabla\phi^2 \cdot Q(v), \tag{3.86}$$

where

$$Q(v) = \frac{\delta}{\delta - 1}\left(\alpha(\nabla v + (\nabla v)^\top) + \beta \mathrm{div} v \mathbb{I}_3\right) \in L^2([0, T^*]; H^2),$$

we then have from (3.73) that

$$\phi^2 Lu \in L^2([0, T^*]; H^2), \quad (\phi^2 Lu)_t \in L^2([0, T^*]; L^2), \tag{3.87}$$

which means that

$$\phi^2 Lu \in C([0, T^*]; H^1). \tag{3.88}$$

Combining (3.9), (3.80), (3.85) and (3.88), we deduce that

$$u_t \in C([0, T^*]; H^1).$$

This completes the proof of this lemma. □

3.5. Proof of Theorem 3.1

Now we turn to the nonlinear problem (3.1)–(3.4). Our proof is based on the classical iteration scheme and the existence results for the linearized problem obtained in Section 3.4. We first define constants c_0 and c_1, c_2, c_3, c_4 as in Section 3.3. Assume that

$$2 + \overline{\phi} + |\phi_0|_\infty + \|(\phi_0 - \overline{\phi}, u_0)\|_3 \leqq c_0.$$

Let (ϕ^0, u^0) be the solution to the system

$$\begin{cases} Y_t + u_0 \cdot \nabla Y = 0 \quad \text{in} \quad (0, +\infty) \times \mathbb{R}^3, \\ Z_t - Y^2 \triangle Z = 0 \quad \text{in} \quad (0, +\infty) \times \mathbb{R}^3, \\ (Y, Z)|_{t=0} = (\phi_0, u_0) \quad \text{in} \quad \mathbb{R}^3, \\ (Y, Z) \to (\overline{\phi}, 0) \quad \text{as} \quad |x| \to +\infty, \quad t > 0, \end{cases} \tag{3.89}$$

with the regularities

$$\begin{aligned} &\phi^0 - \overline{\phi} \in C([0, T^*]; H^3), \quad \phi^0 \nabla^4 u^0 \in L^2([0, T^*]; L^2), \\ &u^0 \in C([0, T^*]; H^{s'}) \cap L^\infty([0, T^*]; H^3) \quad \text{for any } s' \in [2, 3). \end{aligned} \tag{3.90}$$

Due to the regularity of $(\phi^0, u^0)(x)$, there is a positive time $T^{**} \in (0, T^*]$ such that

$$1 + \overline{\phi}^2 + \sup_{0 \leqq t \leqq T^{**}} \left(|\phi^0(t)|_\infty^2 + \|\phi^0(t) - \overline{\phi}\|_3^2 + \|u^0(t)\|_1^2\right) \leqq c_1^2,$$

$$\sup_{0 \leqq t \leqq T^{**}} \left(|\phi_t^0(t)|_2^2 + |u^0(t)|_{D^2}^2 + |u_t^0(t)|_2^2\right) + \int_0^{T^{**}} \left(|\phi^0 \nabla^3 u^0|_2^2 + |u_t^0|_{D^1}^2\right) \mathrm{d}t \leqq c_2^2,$$

$$\operatorname*{ess\,sup}_{0\leqq t\leqq T^{**}} \left(|\phi^0_t|^2_{D^1}+|u^0|^2_{D^3}+|u^0_t|^2_{D^1}\right)(t)+\int_0^{T^{**}}\left(|\phi^0\nabla^4 u^0|^2_2+|u^0_t|^2_{D^2}\right)\mathrm{d}t\leqq c_3^2,$$
$$\sup_{0\leqq t\leqq T^{**}} |\phi^0_t|^2_{D^2}\leqq c_4^2. \tag{3.91}$$

We now give the proof of Theorem 3.1.

Proof. We prove the existence, uniqueness and time continuity in three steps.

Step 1 Existence. Letting $(\psi, v)=(\phi^0,u^0)$, we first define (ϕ^1,u^1) as a strong solution to problem (3.66). Then we construct approximate solutions (ϕ^{k+1},u^{k+1}) inductively, as follows: assuming that (ϕ^k,u^k) was defined for $k\geq 1$, let (ϕ^{k+1},u^{k+1}) be the unique solution to problem (3.66) with (ψ,v) replaced by (ϕ^k,u^k), i.e., (ϕ^{k+1},u^{k+1}) is the unique solution of the following problem:

$$\begin{cases}\phi^{k+1}_t+u^k\cdot\nabla\phi^{k+1}+\dfrac{\delta-1}{2}\phi^k\mathrm{div}u^k=0,\\ u^{k+1}_t+u^k\cdot\nabla u^{k+1}+\dfrac{2A\gamma}{\delta-1}\Phi^{k+1}\nabla\phi^{k+1}+(\phi^{k+1})^2Lu^{k+1}=\nabla(\phi^{k+1})^2\cdot Q(u^k),\\ (\phi^{k+1},u^{k+1})|_{t=0}=(\phi_0,u_0),\quad x\in\mathbb{R}^3,\\ (\phi^{k+1},u^{k+1})\to(\overline{\phi},0),\quad \text{as}\quad |x|\to+\infty,\quad t>0,\end{cases} \tag{3.92}$$

where $\Phi^{k+1}=(\phi^{k+1})^{\frac{2\gamma-\delta-1}{\delta-1}}$.

From the estimates shown in Section 3.4, we know that the sequence (ϕ^k,u^k) satisfies the uniform a priori estimates in (3.65) for $0\leqq t\leqq T^{**}$.

Now we prove the convergence of the whole sequence (ϕ^k,u^k) of approximate solutions to a limit (ϕ,u) in some strong sense. Let

$$\overline{\phi}^{k+1}=\phi^{k+1}-\phi^k,\quad \overline{u}^{k+1}=u^{k+1}-u^k,$$
$$\overline{\Phi}^{k+1}=\Phi^{k+1}-\Phi^k=(\phi^{k+1})^{\frac{2\gamma-\delta-1}{\delta-1}}-(\phi^k)^{\frac{2\gamma-\delta-1}{\delta-1}}.$$

Then, from (3.92), we have

$$\begin{cases}\overline{\phi}^{k+1}_t+u^k\cdot\nabla\overline{\phi}^{k+1}+\overline{u}^k\cdot\nabla\phi^k+\dfrac{\delta-1}{2}(\overline{\phi}^k\mathrm{div}u^{k-1}+\phi^k\mathrm{div}\overline{u}^k)=0,\\ \overline{u}^{k+1}_t+u^k\cdot\nabla\overline{u}^{k+1}+\overline{u}^k\cdot\nabla u^k+\dfrac{2A\gamma}{\delta-1}(\Phi^{k+1}\nabla\overline{\phi}^{k+1}+\overline{\Phi}^{k+1}\nabla\phi^k)\\ =-(\phi^{k+1})^2L\overline{u}^{k+1}-\overline{\phi}^{k+1}(\phi^{k+1}+\phi^k)Lu^k\\ \quad+\nabla(\overline{\phi}^{k+1}(\phi^{k+1}+\phi^k))\cdot Q(u^k)+\nabla(\phi^k)^2\cdot(Q(u^k)-Q(u^{k-1})).\end{cases} \tag{3.93}$$

First, multiplying $(3.93)_1$ by $2\overline{\phi}^{k+1}$ and integrating over $\mathbb{R}^3$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|\overline{\phi}^{k+1}|^2_2&=-2\int\Big(u^k\cdot\nabla\overline{\phi}^{k+1}+\overline{u}^k\cdot\nabla\phi^k+\frac{\delta-1}{2}(\overline{\phi}^k\mathrm{div}u^{k-1}+\phi^k\mathrm{div}\overline{u}^k)\Big)\overline{\phi}^{k+1}\\ &\leqq C\Big(|\nabla u^k|_\infty|\overline{\phi}^{k+1}|^2_2+|\overline{\phi}^{k+1}|_2|\overline{u}^k|_2|\nabla\phi^k|_\infty\\ &\quad+|\overline{\phi}^k|_2|\nabla u^{k-1}|_\infty|\overline{\phi}^{k+1}|_2+|\phi^k\mathrm{div}\overline{u}^k|_2|\overline{\phi}^{k+1}|_2\Big),\end{aligned}$$

which yields that (for $0 < \nu \leqq \frac{1}{10}$ is a constant)

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}|\overline{\phi}^{k+1}(t)|_2^2 \leqq A_\nu^k(t)|\overline{\phi}^{k+1}(t)|_2^2 + \nu\Big(|\overline{u}^k(t)|_2^2 + |\overline{\phi}^k(t)|_2^2 + |\phi^k \mathrm{div}\overline{u}^k(t)|_2^2\Big), \\ A_\nu^k(t) = C\Big(|\nabla u^k|_\infty + \frac{1}{\nu}|\nabla u^{k-1}|_\infty^2 + \frac{1}{\nu}|\nabla \phi^k|_\infty^2 + \frac{1}{\nu}\Big). \end{cases} \tag{3.94}$$

From (3.65), we know that

$$\int_0^t A_\nu^k(s)\,\mathrm{d}s \leqq C_\nu t, \quad \text{for} \quad t \in [0, T^{**}],$$

where C_ν is a positive constant depending on ν and constant C.

Second, we multiply $(3.93)_2$ by $2\overline{u}^{k+1}$ and integrate over $\mathbb{R}^3$ to find

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}|\overline{u}^{k+1}|_2^2 + 2\alpha|\phi^{k+1}\nabla\overline{u}^{k+1}|_2^2 + 2(\alpha+\beta)|\phi^{k+1}\mathrm{div}\overline{u}^{k+1}|_2^2 \\ &= -2\int \Big(u^k\cdot\nabla\overline{u}^{k+1} + \overline{u}^k\cdot\nabla u^k\Big)\cdot\overline{u}^{k+1} \\ &\quad -2\int \frac{2A\gamma}{\delta-1}(\Phi^{k+1}\nabla\overline{\phi}^{k+1} + \overline{\Phi}^{k+1}\nabla\phi^k)\cdot\overline{u}^{k+1} \\ &\quad -4\alpha\int \phi^{k+1}\nabla\phi^{k+1}\cdot\nabla\overline{u}^{k+1}\cdot\overline{u}^{k+1} \\ &\quad -4(\alpha+\beta)\int \phi^{k+1}\nabla\phi^{k+1}\cdot\overline{u}^{k+1}\mathrm{div}\overline{u}^{k+1} \\ &\quad -2\int \overline{\phi}^{k+1}(\phi^{k+1}+\phi^k)Lu^k\cdot\overline{u}^{k+1} \\ &\quad +2\int \nabla(\overline{\phi}^{k+1}(\phi^{k+1}+\phi^k))\cdot Q(u^k)\cdot\overline{u}^{k+1} \\ &\quad +2\int \nabla(\phi^k)^2\cdot(Q(u^k)-Q(u^{k-1}))\cdot\overline{u}^{k+1} =: \sum_{i=1}^9 J_i. \end{aligned} \tag{3.95}$$

We now estimate J_i $(i = 1, \cdots, 9)$ term by term. For the term J_1, we see from integration by parts that

$$J_1 = -2\int u^k\cdot\nabla\overline{u}^{k+1}\cdot\overline{u}^{k+1} \leqq C|\nabla u^k|_\infty|\overline{u}^{k+1}|_2^2. \tag{3.96}$$

For J_2, it is easy to show that

$$\begin{aligned} J_2 &= -2\int \overline{u}^k\cdot\nabla u^k\cdot\overline{u}^{k+1} \\ &\leqq C|\nabla u^k|_\infty|\overline{u}^k|_2|\overline{u}^{k+1}|_2 \leqq \frac{C}{\nu}|\nabla u^k|_\infty^2|\overline{u}^{k+1}|^2 + \nu|\overline{u}^k|_2^2. \end{aligned} \tag{3.97}$$

Applying integration by parts for J_3, we have

$$
\begin{aligned}
J_3 &= -2\int \frac{2A\gamma}{\delta-1}\Phi^{k+1}\nabla\overline{\phi}^{k+1}\cdot\overline{u}^{k+1}\\
&= \frac{4A\gamma}{\delta-1}\int \left(\Phi^{k+1}\overline{\phi}^{k+1}\mathrm{div}\overline{u}^{k+1}+\overline{\phi}^{k+1}\nabla\Phi^{k+1}\cdot\overline{u}^{k+1}\right)\\
&\leqq C\Big(|\phi^{k+1}|_\infty^{\frac{2\gamma-2\delta}{\delta-1}}|\overline{\phi}^{k+1}|_2|\phi^{k+1}\mathrm{div}\overline{u}^{k+1}|_2+|\overline{\phi}^{k+1}|_2|\nabla\Phi^{k+1}|_\infty|\overline{u}^{k+1}|_2\Big)\\
&\leqq C\Big(|\phi^{k+1}|_\infty^{\frac{4\gamma-4\delta}{\delta-1}}|\overline{\phi}^{k+1}|_2^2+|\overline{\phi}^{k+1}|_2^2+|\nabla\Phi^{k+1}|_\infty^2|\overline{u}^{k+1}|_2^2\Big)+\frac{\alpha}{20}|\phi^{k+1}\nabla\overline{u}^{k+1}|_2^2.
\end{aligned}
\tag{3.98}
$$

J_4 is estimated directly as follows:

$$
\begin{aligned}
J_4 &= -2\int \frac{2A\gamma}{\delta-1}\overline{\Phi}^{k+1}\nabla\phi^k\cdot\overline{u}^{k+1}\\
&\leqq C|\overline{\Phi}^{k+1}|_2|\nabla\phi^k|_\infty|\overline{u}^{k+1}|_2\\
&\leqq C\Big(\big||\phi^{k+1}|+|\phi^k|\big|_\infty^{\frac{4\gamma-4\delta}{\delta-1}}|\overline{\phi}^{k+1}|_2^2+|\nabla\phi^k|_\infty^2|\overline{u}^{k+1}|_2^2\Big),
\end{aligned}
\tag{3.99}
$$

where we used the fact that

$$
|\overline{\Phi}^{k+1}|_2=|(\phi^{k+1})^{\frac{2\gamma-\delta-1}{\delta-1}}-(\phi^k)^{\frac{2\gamma-\delta-1}{\delta-1}}|_2\leqq C\big||\phi^{k+1}|+|\phi^k|\big|_\infty^{\frac{2\gamma-2\delta}{\delta-1}}|\overline{\phi}^{k+1}|_2.
$$

Similarly, we are able to treat terms J_5-J_7 in the following way:

$$
\begin{aligned}
J_5 &= -4\alpha\int \phi^{k+1}\nabla\phi^{k+1}\cdot\nabla\overline{u}^{k+1}\cdot\overline{u}^{k+1}\\
&\leqq C|\nabla\phi^{k+1}|_\infty|\phi^{k+1}\nabla\overline{u}^{k+1}|_2|\overline{u}^{k+1}|_2\\
&\leqq C|\nabla\phi^{k+1}|_\infty^2|\overline{u}^{k+1}|_2^2+\frac{\alpha}{20}|\phi^{k+1}\nabla\overline{u}^{k+1}|_2^2,\\
J_6 &= -4(\alpha+\beta)\int \phi^{k+1}\nabla\phi^{k+1}\cdot\overline{u}^{k+1}\mathrm{div}\overline{u}^{k+1}\\
&\leqq C|\nabla\phi^{k+1}|_\infty|\phi^{k+1}\nabla\overline{u}^{k+1}|_2|\overline{u}^{k+1}|_2\\
&\leqq C|\nabla\phi^{k+1}|_\infty^2|\overline{u}^{k+1}|_2^2+\frac{\alpha}{20}|\phi^{k+1}\nabla\overline{u}^{k+1}|_2^2,\\
J_7 &= 2\int \overline{\phi}^{k+1}(\phi^{k+1}+\phi^k)Lu^k\cdot\overline{u}^{k+1}\\
&\leqq C\Big(|\overline{\phi}^{k+1}|_2|\phi^{k+1}\overline{u}^{k+1}|_6|Lu^k|_3+|\overline{\phi}^{k+1}|_2|\phi^kLu^k|_\infty|\overline{u}^{k+1}|_2\Big)\\
&\leqq C\Big(|\overline{\phi}^{k+1}|_2(|\phi^{k+1}\nabla\overline{u}^{k+1}|_2+|\nabla\phi^{k+1}|_\infty|\overline{u}^{k+1}|_2)|Lu^k|_3\\
&\quad+|\overline{\phi}^{k+1}|_2(\|\nabla\phi^k\|_2\|u^k\|_3+|\phi^k\nabla^4u^k|_2)|\overline{u}^{k+1}|_2\Big)\\
&\leqq C\Big(|Lu^k|_3^2|\overline{\phi}^{k+1}|_2^2+|\overline{\phi}^{k+1}|_2^2+|\nabla\phi^{k+1}|_\infty^2|Lu^k|_3^2|\overline{u}^{k+1}|_2^2\\
&\quad+(\|\nabla\phi^k\|_2\|\nabla^2u^k\|_1+|\phi^k\nabla^4u^k|_2)^2|\overline{u}^{k+1}|_2^2\Big)+\frac{\alpha}{20}|\phi^{k+1}\nabla\overline{u}^{k+1}|_2^2,
\end{aligned}
\tag{3.100}
$$

where we used the fact (see Lemma 2.1) that

$$
\begin{aligned}
&|\phi^k\nabla^2 u^k|_\infty\\
&\quad\leqq C|\phi^k\nabla^2 u^k|_6^{\frac{1}{2}}|\nabla(\phi^k\nabla^2 u^k)|_6^{\frac{1}{2}}\\
&\quad\leqq C|\phi^k\nabla^2 u^k|_{D^1}^{\frac{1}{2}}|\nabla(\phi^k\nabla^2 u^k)|_{D^1}^{\frac{1}{2}}\\
&\quad\leqq C\|\nabla(\phi^k\nabla^2 u^k)\|_1\\
&\quad\leqq C\Big(|\nabla\phi^k|_\infty\|\nabla^2 u^k\|_1+|\phi^k|_\infty|\nabla^3 u^k|_2+|\phi^k\nabla^4 u^k|_2+|\nabla^2\phi^k|_6|\nabla^2 u^k|_3\Big)\\
&\quad\leqq C\Big(\|\nabla\phi^k\|_2\|\nabla^2 u^k\|_1+|\phi^k\nabla^4 u^k|_2\Big).
\end{aligned}
\tag{3.101}
$$

Now we turn to the tricky term J_8. First we have

$$
\begin{aligned}
J_8=&\,2\int\nabla(\overline{\phi}^{k+1}(\phi^{k+1}+\phi^k))\cdot Q(u^k)\cdot\overline{u}^{k+1}\\
=&-2\int\sum_{i,j}\overline{\phi}^{k+1}(\phi^{k+1}+\phi^k)\partial_i a_k^{ij}\overline{u}^{k+1,j}\\
&-2\int\sum_{i,j}\overline{\phi}^{k+1}(\phi^{k+1}+\phi^k)a_k^{ij}\partial_i\overline{u}^{k+1,j}\\
=:&\,J_{81}+J_{82}+J_{83}+J_{84},
\end{aligned}
\tag{3.102}
$$

where $u^{k,j}$ represents the j-th component of u^k $(k\geq 1)$,

$$
\overline{u}^{k,j}=u^{k,j}-u^{k-1,j},\quad\text{for}\quad k\geq 1,\quad j=1,2,3,
$$

and the quantity a_k^{ij} is given by

$$
a_k^{ij}=\frac{\delta}{\delta-1}\Big(\alpha(\partial_i u^{k,j}+\partial_j u^{k,i})+\text{div}u^k\delta_{ij}\Big)\quad\text{for } i,\ j=1,2,3,
$$

where δ_{ij} is the Kronecker symbol satisfying $\delta_{ij}=1$, $i=j$, and $\delta_{ij}=0$, otherwise.

We are now ready to estimate terms $J_{81}-J_{84}$ one by one. First we have

$$
\begin{aligned}
J_{81}=&-2\int\sum_{i,j}\overline{\phi}^{k+1}\phi^{k+1}\partial_i a_k^{ij}\overline{u}^{k+1,j}\leqq C|\nabla^2 u^k|_3|\overline{\phi}^{k+1}|_2|\phi^{k+1}\overline{u}^{k+1}|_6\\
\leqq&\, C|\nabla^2 u^k|_3|\overline{\phi}^{k+1}|_2\left(|\phi^{k+1}\nabla\overline{u}^{k+1}|_2+|\nabla\phi^{k+1}|_\infty|\overline{u}^{k+1}|_2\right)\\
\leqq&\, C\left(|\nabla\phi^{k+1}|_\infty^2|\nabla^2 u^k|_3^2|\overline{\phi}^{k+1}|_2^2+|\overline{u}^{k+1}|_2^2+|\nabla^2 u^k|_3^2|\overline{\phi}^{k+1}|_2^2\right)\\
&+\frac{\alpha}{20}|\phi^{k+1}\nabla\overline{u}^{k+1}|_2^2.
\end{aligned}
\tag{3.103}
$$

Similarly, for the terms I_{82}-I_{83}, using (3.101), we get

$$
\begin{aligned}
J_{82} &= -2\int \sum_{i,j} \overline{\phi}^{k+1}\phi^k \partial_i a_k^{ij}\overline{u}^{k+1,j} \leqq C|\phi^k\nabla^2 u^k|_\infty|\overline{\phi}^{k+1}|_2|\overline{u}^{k+1}|_2\\
&\leqq C\left(\left(\|\nabla\phi^k\|_2\|\nabla^2 u^k\|_1+|\phi^k\nabla^4 u^k|_2\right)^2|\overline{\phi}^{k+1}|_2^2+|\overline{u}^{k+1}|_2^2\right),\\
J_{83} &= -2\int \sum_{i,j} \overline{\phi}^{k+1}\phi^{k+1} a_k^{ij}\partial_i\overline{u}^{k+1,j} \leqq C|\overline{\phi}^{k+1}|_2|\phi^{k+1}\nabla\overline{u}^{k+1}|_2|\nabla u^k|_\infty\\
&\leqq C|\nabla u^k|_\infty^2|\overline{\phi}^{k+1}|_2^2+\frac{\alpha}{20}|\phi^{k+1}\nabla\overline{u}^{k+1}|_2^2.
\end{aligned}
\tag{3.104}
$$

For J_{84}, we have

$$
\begin{aligned}
J_{84} &= -2\int \sum_{i,j} \overline{\phi}^{k+1}\phi^k a_k^{ij}\partial_i\overline{u}^{k+1,j}\\
&= -2\int \sum_{i,j} \overline{\phi}^{k+1}(\phi^k-\phi^{k+1}+\phi^{k+1}) a_k^{ij}\partial_i\overline{u}^{k+1,j}\\
&\leqq C\left(|\nabla u^k|_\infty|\nabla\overline{u}^{k+1}|_\infty|\overline{\phi}^{k+1}|_2^2+|\nabla u^k|_\infty|\phi^{k+1}\nabla\overline{u}^{k+1}|_2|\overline{\phi}^{k+1}|_2\right)\\
&\leqq C\left(|\nabla u^k|_\infty|\nabla\overline{u}^{k+1}|_\infty|\overline{\phi}^{k+1}|_2^2+|\nabla u^k|_\infty^2|\overline{\phi}^{k+1}|_2^2\right)+\frac{\alpha}{20}|\phi^{k+1}\nabla\overline{u}^{k+1}|_2^2.
\end{aligned}
\tag{3.105}
$$

Combining (3.102)–(3.105), we arrive at

$$
\begin{aligned}
J_8 \leqq\ & C\Big(|\overline{u}^{k+1}|_2^2+(|\nabla\phi^{k+1}|_\infty^2+1)|\nabla^2 u^k|_3^2|\overline{\phi}^{k+1}|_2^2+|\nabla u^k|_\infty^2|\overline{\phi}^{k+1}|_2^2\\
&+\left(\|\nabla\phi^k\|_2\|\nabla^2 u^k\|_1+|\phi^k\nabla^4 u^k|_2\right)^2|\overline{\phi}^{k+1}|_2^2+|\nabla u^k|_\infty|\nabla\overline{u}^{k+1}|_\infty|\overline{\phi}^{k+1}|_2^2\Big)\\
&+\frac{\alpha}{5}|\phi^{k+1}\nabla\overline{u}^{k+1}|_2^2.
\end{aligned}
\tag{3.106}
$$

As for the last term J_9, it is easy to show that

$$
\begin{aligned}
J_9 &= 2\int \nabla(\phi^k)^2\cdot(Q(u^k)-Q(u^{k-1}))\cdot\overline{u}^{k+1}\\
&\leqq C|\nabla\phi^k|_\infty|\phi^k\nabla\overline{u}^k|_2|\overline{u}^{k+1}|_2 \leqq \frac{C}{\nu}|\nabla\phi^k|_\infty^2|\overline{u}^{k+1}|_2^2+\nu|\phi^k\nabla\overline{u}^k|_2^2.
\end{aligned}
\tag{3.107}
$$

In summary, using (3.97)–(3.100) and (3.106)–(3.107), (3.95) implies that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}|\overline{u}^{k+1}|_2^2+\alpha|\phi^{k+1}\nabla\overline{u}^{k+1}|_2^2\\
&\quad\leqq B_\nu^k(t)|\overline{u}^{k+1}|_2^2+B^k(t)|\overline{\phi}^{k+1}|_2^2+\nu(|\phi^k\nabla\overline{u}^k|_2^2+|\overline{u}^k|_2^2),
\end{aligned}
\tag{3.108}
$$

for $0 < \nu \leqq \frac{1}{10}$ is a constant. Here $B^k_\nu(t)$ and $B^k(t)$ are given by

$$\begin{aligned} B^k_\nu(t) = & C\Big(1 + |\nabla u^k|_\infty + \frac{1}{\nu}|\nabla u^k|^2_\infty + |\nabla \Phi^{k+1}|^2_\infty + |\nabla \phi^k|^2_\infty + |\nabla \phi^{k+1}|^2_\infty \\ & + (\|\nabla \phi^k\|_2 \|u^k\|_3 + |\phi^k \nabla^4 u^k|_2)^2 + |\nabla \phi^{k+1}|^2_\infty |Lu^k|^2_3 + \frac{1}{\nu}|\nabla \phi^k|^2_\infty\Big) \\ B^k(t) = & C\Big(1 + |\nabla^2 u^k|^2_3 + \big||\phi^{k+1}| + |\phi^k|\big|^{\frac{4\gamma-4\delta}{\delta-1}}_\infty + (\|\nabla \phi^k\|_2 \|u^k\|_3 + |\phi^k \nabla^4 u^k|_2)^2 \\ & + |\nabla \phi^{k+1}|^2_\infty |\nabla^2 u^k|^2_3 + |\nabla u^k|^2_\infty + |\nabla u^k|_\infty |\nabla \overline{u}^{k+1}|_\infty\Big), \end{aligned}$$

satisfying the estimate

$$\int_0^t \big(B^k_\nu(s) + B^k(s)\big)\, \mathrm{d}s \leqq C + C_\nu t, \quad t \in [0, T^{**}].$$

Denote

$$\Gamma^{k+1}(t) = \sup_{s \in [0,t]} |\overline{\phi}^{k+1}(s)|^2_2 + \sup_{s \in [0,t]} |\overline{u}^{k+1}(s)|^2_2.$$

From (3.94) and (3.108), we finally have

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}t}(|\overline{\phi}^{k+1}(t)|^2_2 + |\overline{u}^{k+1}(t)|^2_2) + \alpha |\phi^{k+1} \nabla \overline{u}^{k+1}|^2_2 \\ & \quad \leqq E^k_\nu \left(|\overline{\phi}^{k+1}(t)|^2_2 + |\overline{u}^{k+1}(t)|^2_2\right) + \nu \left(|\phi^k \nabla \overline{u}^k(t)|^2_2 + |\overline{\phi}^k(t)|^2_2 + |\overline{u}^k(t)|^2_2\right) \end{aligned}$$

for some E^k_ν satisfying $\int_0^t E^k_\nu(s)\, \mathrm{d}s \leqq C + C_\nu t$. Applying Gronwall's inequality, we have

$$\begin{aligned} & \Gamma^{k+1} + \int_0^t \alpha |\phi^{k+1} \nabla \overline{u}^{k+1}|^2_2 \, \mathrm{d}s \\ & \quad \leqq C\nu \int_0^t \left(|\phi^k \nabla \overline{u}^k|^2_2 + |\overline{\phi}^k|^2_2 + |\overline{u}^k|^2_2\right) \mathrm{d}s \exp(C + C_\nu t) \\ & \quad \leqq C\nu \left(\int_0^t |\phi^k \nabla \overline{u}^k|^2_2 \, \mathrm{d}s + t \sup_{s \in [0,t]} |\overline{\phi}^k(s)|^2_2 + t \sup_{s \in [0,t]} |\overline{u}^k(s)|^2_2\right) \exp(C_\nu t). \end{aligned}$$

We choose $\nu > 0$ and $T_* \in (0, \min(1, T^{**}))$ small enough such that

$$4C\nu \leqq \min(\alpha, 1), \quad \text{and} \quad \exp(C_\nu T_*) \leqq 2.$$

Therefore, we achieve

$$\sum_{k=1}^{\infty} \Big(\Gamma^{k+1}(T_*) + \int_0^{T_*} \alpha |\phi^{k+1} \nabla \overline{u}^{k+1}|^2_2 \, \mathrm{d}t\Big) \leqq C < +\infty, \tag{3.109}$$

which implies that our approximate solution sequence (ϕ^k, u^k) is a Cauchy sequence under the topology of $L^\infty([0, T_*]; L^2(\mathbb{R}^3))$. Together with the uniform bounds (3.65), one has

$$(\phi^{k+1}, u^{k+1}) \to (\phi, u) \quad \text{in} \quad L^\infty([0, T_*]; H^2(\mathbb{R}^3))$$

for limit functions (ϕ, u). On the other hand, since the uniform estimates in (3.65) is independent of k, we know that there exists a subsequence of solutions (still denoted by) (ϕ^k, u^k), which converges to a limit (ϕ, u) in the following weak or weak* sense:

$$\begin{aligned}
(\phi^k, u^k) \rightharpoonup (\phi, u) \quad & \text{weakly* in } L^\infty([0, T_*]; H^3(\mathbb{R}^3)),\\
\phi^k_t \rightharpoonup \phi_t \quad & \text{weakly* in } L^\infty([0, T_*]; H^2(\mathbb{R}^3)),\\
u^k_t \rightharpoonup u_t \quad & \text{weakly* in } L^\infty([0, T_*]; H^1(\mathbb{R}^3)),\\
u^k_t \rightharpoonup u_t \quad & \text{weakly in } L^2([0, T_*]; D^2(\mathbb{R}^3)),\\
\phi^k \nabla^4 u^k \rightharpoonup \phi \nabla^4 u \quad & \text{weakly in } L^2([0, T_*] \times \mathbb{R}^3),
\end{aligned} \tag{3.110}$$

which, from the lower semi-continuity of norm for weak or weak* convergence, imply that the local estimates in (3.65) still hold for the limit function (ϕ, u).

Now, it is easy to show that (ϕ, u) is a weak solution of (3.1)–(3.4) in the sense of distribution with the following regularities:

$$\begin{aligned}
& \phi - \overline{\phi} \in L^\infty([0, T_*]; H^3), \quad \phi_t \in L^\infty([0, T_*]; H^2),\\
& u \in L^\infty([0, T_*]; H^3), \quad \phi \nabla^4 u \in L^2([0, T_*]; L^2),\\
& u_t \in L^\infty([0, T_*]; H^1) \cap L^2([0, T_*]; D^2).
\end{aligned} \tag{3.111}$$

The existence of strong solutions is proved.

Step 2 Uniqueness. Let (ϕ_1, u_1) and (ϕ_2, u_2) be two strong solutions to Cauchy problem (3.1)–(3.4) satisfying the uniform a priori estimates (3.65). Denote

$$\overline{\varphi} = \phi_1 - \phi_2, \quad \overline{u} = u_1 - u_2,$$
$$\overline{\Phi} = \Phi_1 - \Phi_2 = \phi_1^{\frac{2\gamma-\delta-1}{\delta-1}} - \phi_2^{\frac{2\gamma-\delta-1}{\delta-1}}.$$

It follows from (3.1) that $(\overline{\varphi}, \overline{u})$ satisfies the following system:

$$\begin{cases}
\overline{\varphi}_t + u_1 \cdot \nabla \overline{\varphi} + \overline{u} \cdot \nabla \phi_2 + \dfrac{\delta - 1}{2}(\overline{\varphi} \mathrm{div} u_2 + \phi_1 \mathrm{div} \overline{u}) = 0,\\
\overline{u}_t + u_1 \cdot \nabla \overline{u} + \overline{u} \cdot \nabla u_2 + \dfrac{2A\gamma}{\delta - 1}(\Phi_1 \nabla \overline{\varphi} + \overline{\Phi} \nabla \phi_2)\\
\quad = -(\phi_1^2 L \overline{u} + \overline{\varphi}(\phi_1 + \phi_2) L u_2)\\
\qquad + \nabla \phi_1^2 (Q(u_1) - Q(u_2)) + \nabla(\overline{\varphi}(\phi_1 + \phi_2)) Q(u_2),
\end{cases} \tag{3.112}$$

with zero initial data. Let

$$\Psi(t) = |\overline{\varphi}(t)|_2^2 + |\overline{u}(t)|_2^2.$$

Using the same arguments as in the derivation of (3.94)–(3.108), we have

$$\begin{cases}
\dfrac{\mathrm{d}}{\mathrm{d}t} \Psi(t) + C |\phi_1 \nabla \overline{u}(t)|_2^2 \leqq F(t) \Psi(t),\\
\displaystyle\int_0^t F(s)\, \mathrm{d}s \leqq C \quad \text{for} \quad 0 \leqq t \leqq T_*.
\end{cases} \tag{3.113}$$

From Gronwall's inequality, we conclude that $\overline{\varphi} = \overline{u} = 0$. Then the uniqueness is obtained.
Step 3 Time-continuity. It can be proved by the same arguments as in the proof of Lemma 3.5. We omit the details here. □

3.6. Proof of Theorem 1.1 and Corollary 1.1

Based on Theorem 3.1, we are now ready to prove the local existence of regular solution to the original Cauchy problem (1.1)–(1.3).

3.6.1. Proof of Theorem 1.1

Proof. For the initial data (1.17), we know from Theorem 3.1 that there exists a time $T_* > 0$ such that the problem (3.1)–(3.4) admits a unique strong solution (ϕ, u) satisfying the regularities in (3.6), which means that

$$\rho^{\frac{\delta-1}{2}} = \phi \in C^1(\mathbb{R}^3 \times [0, T_*]). \tag{3.114}$$

Noticing that $\rho = \phi^{\frac{2}{\delta-1}}$, and $\frac{2}{\delta-1} \geq 1$ for $1 < \delta \leqq \min\left(3, \frac{\gamma+1}{2}\right)$, it is easy to show that

$$\rho(x, t) \in C^1(\mathbb{R}^3 \times [0, T_*]).$$

Now we verify that (ρ, u) satisfies the original equations (1.1). Multiplying both sides of $(3.1)_1$ by

$$\frac{\partial \rho}{\partial \phi}(x, t) = \frac{2}{\delta - 1}\phi^{\frac{3-\delta}{\delta-1}}(x, t) \in C(\mathbb{R}^3 \times [0, T_*]),$$

we get the continuity equation in $(1.1)_1$.

Multiplying both sides of $(3.1)_2$ by

$$\phi^{\frac{2}{\delta-1}} = \rho(x, t) \in C^1(\mathbb{R}^3 \times [0, T_*]),$$

we get the momentum equations in $(1.1)_2$. Therefore, (ρ, u) is a solution to (1.1)–(1.3) in the sense of distribution with the regularities shown in Definition 1.1.

Recalling that ρ can be represented by the formula

$$\rho(x, t) = \rho_0(U(0; x, t)) \exp\left(\int_0^t \mathrm{div} u(s, U(s; x, t))\, \mathrm{d}s\right),$$

where $U \in C^1([0, T_*] \times \mathbb{R}^3 \times [0, T_*])$ is the solution to the initial value problem

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}s} U(s; x, t) = u(s, U(s; x, t)), & 0 \leqq s \leqq T_*, \\ U(t; x, t) = x, \quad x \in \mathbb{R}^3, & 0 \leqq t \leqq T_*, \end{cases} \tag{3.115}$$

it is obvious that

$$\rho(x, t) \geq 0, \ \forall (x, t) \in \mathbb{R}^3 \times [0, T_*].$$

In summary, the Cauchy problem (1.1)–(1.3) has a unique regular solution (ρ, u). □

3.6.2. Proof of Corollary 1.1

Proof. First, from $1 < \delta \leqq \frac{5}{3}$ we know that $\frac{2}{\delta-1} \geq 3$. Since

$$\phi - \overline{\phi} \in C([0, T_*]; H^3) \cap C^1([0, T_*]; H^2),$$

and

$$\rho(x, t) = \phi^{\frac{2}{\delta-1}}(x, t),$$

we have

$$\rho - \overline{\rho} \in C([0, T_*]; H^3).$$

Second, due to the fact that

$$\begin{aligned} &\rho^{\frac{\delta-1}{2}} - \overline{\rho}^{\frac{\delta-1}{2}} \in C([0, T_*]; H^3), \quad u \in C([0, T_*]; H^{s'}) \cap L^\infty([0, T_*]; H^3), \\ &\rho^{\frac{\delta-1}{2}} \nabla^4 u \in L^2([0, T_*]; L^2), \quad u_t \in C([0, T]; H^1) \cap L^2([0, T]; D^2), \end{aligned} \tag{3.116}$$

for any constant $s' \in [2, 3)$, and the same arguments as in Lemma 3.5 for the time continuity, we deduce that

$$\rho \mathrm{div} u \in L^2([0, T_*]; H^3) \cap C([0, T_*]; H^2). \tag{3.117}$$

At last, with the aid of the continuity equation $\rho_t + u \cdot \nabla \rho + \rho \mathrm{div} u = 0$ and (3.116)–(3.117), it is clear that

$$\rho - \overline{\rho} \in C([0, T_*]; H^3) \cap C^1([0, T_*]; H^2).$$

Furthermore, when $\delta = 2$ and $\gamma \geq 3$, by the same token, the regularity of ρ in these cases can be achieved. □

4. Formation of Singularities

In this section, we consider the formation of singularities of regular solutions obtained in Section 3. Two classes of initial data that lead to finite time blow-up will be given. We assume that $(\rho, u)(x, t)$ is the regular solution in $\mathbb{R}^3 \times [0, T_m)$ obtained in Theorem 1.1, with T_m the maximal existence time.

4.1. Blow-up by Isolated Mass Group

The first kind of singularity formation is driven by isolated mass group, defined in Definition 1.2. Assume that the initial data (ρ_0, u_0) have an isolated mass group (A_0, B_0), the following definition helps to track the evolution of A_0 and B_0.

Definition 4.1. (*Particle path and flow map*) Let $x(t; x_0)$ be the particle path starting from x_0 at $t = 0$, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t; x_0) = u(x(t; x_0), t), \quad x(0; x_0) = x_0. \tag{4.1}$$

Let $A(t)$, $B(t)$, $B(t)\backslash A(t)$ be the images of A_0, B_0, and $B_0\backslash A_0$, respectively, under the flow map of (4.1), i.e.,

$$\begin{aligned} A(t) &= \{x(t; x_0)|x_0 \in A_0\}, \\ B(t) &= \{x(t; x_0)|x_0 \in B_0\}, \\ B(t)\backslash A(t) &= \{x(t; x_0)|x_0 \in B_0\backslash A_0\}. \end{aligned}$$

The following lemma confirms the invariance of the volume $|A(t)|$ for regular solutions:

Lemma 4.1. *Suppose that the initial data* $(\rho_0, u_0)(x)$ *have an isolated mass group* (A_0, B_0)*, then for the regular solution* $(\rho, u)(x, t)$ *on* $\mathbb{R}^3 \times [0, T_m)$ *to the Cauchy problem* (1.1)–(1.3)*, we have*

$$|A(t)| = |A_0|, \quad t \in [0, T_m).$$

Proof. Since

$$\rho(x(t; x_0), t) = \rho_0(x_0) \exp\Big(\int_0^t \mathrm{div}u(x(s; x_0), s)\,\mathrm{d}s\Big),$$

it is clear that

$$\rho \equiv 0, \quad \text{in} \quad B(t)\backslash A(t).$$

From the definition of regular solutions, we have

$$u_t + u \cdot \nabla u = 0, \quad \text{in} \quad B(t)\backslash A(t). \tag{4.2}$$

Therefore, u is invariant along the particle path $x(t; x_0)$ with $x_0 \in B_0\backslash A_0$.

For any $x_0^1,\ x_0^2 \in \partial A_0$, we define

$$\frac{\mathrm{d}}{\mathrm{d}t}x^i(t; x_0^i) = u(x^i(t; x_0^i), t), \quad x^i(0; x_0^i) = x_0^i, \quad \text{for} \quad i = 1, 2. \tag{4.3}$$

Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(x^1(t; x_0^1) - x^2(t; x_0^2)) = u(x^1(t; x_0^1), t) - u(x^2(t; x_0^2), t) = \bar{u}_0 - \bar{u}_0 = 0, \tag{4.4}$$

which implies that

$$|A(t)| = |A_0|, \quad t \in [0, T_m].$$

□

We point out that, although the volume of $A(t)$ is invariant, the vacuum boundary $\partial A(t)$ travels with constant velocity $\bar{u}_0$. The following well-known Reynolds transport theorem (c.f. [18]) is useful.

Lemma 4.2. *For any $G(x,t) \in C^1(\mathbb{R}^3 \times \mathbb{R}^+)$, one has*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{A(t)} G(x,t)\,\mathrm{d}x = \int_{A(t)} G_t(x,t)\,\mathrm{d}x + \int_{\partial A(t)} G(x,t)(u(x,t)\cdot\vec{n})\,\mathrm{d}S,$$

where $\vec{n}$ is the outward unit normal vector to $\partial A(t)$, and u is the velocity of the fluid.

In the rest part of this section, we will use the following useful physical quantities on the fluids in $A(t)$:

$$\begin{aligned}
m(t) &= \int_{A(t)} \rho(x,t)\,\mathrm{d}x \quad \text{(total mass)},\\
M(t) &= \int_{A(t)} \rho(x,t)|x|^2\,\mathrm{d}x \quad \text{(second moment)},\\
F(t) &= \int_{A(t)} \rho(x,t)u(x,t)\cdot x\,\mathrm{d}x \quad \text{(radial component of momentum)},\\
\varepsilon(t) &= \int_{A(t)} \Big(\frac{1}{2}\rho|u|^2 + \frac{P}{\gamma-1}\Big)(x,t)\,\mathrm{d}x \quad \text{(total energy)}.
\end{aligned}$$

From the continuity equation, it is clear that the mass is conserved.

Lemma 4.3. *Suppose that the initial data $(\rho_0, u_0)(x)$ have an isolated mass group (A_0, B_0), then for the regular solution $(\rho, u)(x,t)$ on $\mathbb{R}^3 \times [0, T_m)$ to the Cauchy problem (1.1)–(1.3), we have*

$$m(t) = m(0), \quad \textit{for} \quad t \in [0, T_m).$$

Proof. From $(1.1)_1$ and Lemma 4.2, direct computation shows that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}m(t) &= \int_{A(t)} \rho_t\ \mathrm{d}x + \int_{\partial A(t)} \rho u\cdot\vec{n}\,\mathrm{d}S\\
&= \int_{A(t)} -\mathrm{div}(\rho u)\,\mathrm{d}x = \int_{\partial A(t)} -\rho u\cdot\vec{n}\ \mathrm{d}S = 0,
\end{aligned}$$

which implies that $m(t) = m(0)$. □

Motivated by [41], we define the following functional:

$$\begin{aligned}
I(t) &= M(t) - 2(t+1)F(t) + 2(t+1)^2\varepsilon(t)\\
&= \int_{A(t)} |x-(t+1)u|^2\rho\,\mathrm{d}x + \frac{2}{\gamma-1}(t+1)^2\int_{A(t)} P\,\mathrm{d}x.
\end{aligned} \tag{4.5}$$

We now follow the arguments of [41] with some proper modifications to prove Theorem 1.2. One of the key observations in the following proof is that the viscosity tensor $\mathbb{T} = 0$ in vacuum region due to (1.5).

Proof of Theorem 1.2. From system (1.1), it is clear that

$$\left(\frac{1}{2}\rho|u|^2 + \frac{P}{\gamma-1}\right)_t = -\mathrm{div}\left(\frac{1}{2}\rho|u|^2 u\right) - \frac{\gamma}{\gamma-1}\mathrm{div}(Pu) + u\cdot\mathrm{div}\mathbb{T}. \tag{4.6}$$

From the continuity equation $(1.1)_1$, momentum equations $(1.1)_2$, relation (4.6), Lemma 4.2 and integration by parts, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}I(t) &= \frac{\mathrm{d}}{\mathrm{d}t}M(t) - 2(t+1)\frac{\mathrm{d}}{\mathrm{d}t}F(t) + 2(t+1)^2\frac{\mathrm{d}}{\mathrm{d}t}\varepsilon(t) - 2F(t) + 4(t+1)\varepsilon(t)\\ &= \frac{2}{\gamma-1}(2-3(\gamma-1))(t+1)\int_{A(t)} P\,\mathrm{d}x + \overline{J}_1 + \overline{J}_2,\end{aligned} \tag{4.7}$$

where $\overline{J}_1$ and $\overline{J}_2$ are given by

$$\overline{J}_1 = -2(t+1)\int_{A(t)} x\cdot\mathrm{div}\mathbb{T}\,\mathrm{d}x,\quad \overline{J}_2 = 2(t+1)^2\int_{A(t)} u\cdot\mathrm{div}\mathbb{T}\,\mathrm{d}x,$$

since

$$\mathrm{div}(x\cdot\mathbb{T}) = x\cdot(\mathrm{div}\mathbb{T}) + \sum_{i=1}^{3}\mathbb{T}_{ii} = x\cdot(\mathrm{div}\mathbb{T}) + 3\left(\beta+\frac{2}{3}\alpha\right)\rho^{\delta}\mathrm{div}u. \tag{4.8}$$

Integrating (4.8) by parts, with help of the fact $\mathbb{T}=0$ on $\partial A(t)$, we have

$$\overline{J}_1 = -2(t+1)\int_{A(t)} x\cdot\mathrm{div}\mathbb{T}\,\mathrm{d}x = 6(t+1)\int_{A(t)}\left(\beta+\frac{2}{3}\alpha\right)\rho^{\delta}\mathrm{div}u\,\mathrm{d}x. \tag{4.9}$$

Now we turn to $\overline{J}_2$. From (1.6) and Cauchy's inequality, we have

$$\begin{aligned}\mathrm{div}(u\mathbb{T}) &= u\cdot\mathrm{div}\mathbb{T} + 2\mu(\rho)\sum_{i=1}^{3}(\partial_i u_i)^2 + \mu(\rho)\sum_{i\neq j}^{3}(\partial_i u_j)^2\\ &\quad + 2\mu(\rho)\sum_{i>j}(\partial_i u_j)(\partial_j u_i) + \lambda(\rho)\left(\sum_{i=1}^{3}\partial_i u_i\right)^2\\ &\geq u\cdot\mathrm{div}\mathbb{T} + \left(\lambda(\rho)+\frac{2}{3}\mu(\rho)\right)(\mathrm{div}u)^2\\ &= u\cdot\mathrm{div}\mathbb{T} + \left(\beta+\frac{2}{3}\alpha\right)\rho^{\delta}(\mathrm{div}u)^2.\end{aligned} \tag{4.10}$$

We integrate (4.10) over $A(t)$ to get

$$\int_{A(t)} u\cdot\mathrm{div}\mathbb{T}\,\mathrm{d}x \leqq -\int_{A(t)}\left(\beta+\frac{2}{3}\alpha\right)\rho^{\delta}(\mathrm{div}u)^2\,\mathrm{d}x, \tag{4.11}$$

from which we have

$$\overline{J}_2 \leqq -2(t+1)^2\int_{A(t)}\left(\beta+\frac{2}{3}\alpha\right)\rho^{\delta}(\mathrm{div}u)^2\,\mathrm{d}x. \tag{4.12}$$

From (4.7), (4.9) and (4.12), for $0 \leqq t \leqq T_m$, we get

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} I(t) \leqq \frac{2}{\gamma-1}(2-3(\gamma-1))(t+1) \int_{A(t)} P \,\mathrm{d}x \\
- 2(t+1)^2 \int_{A(t)} \left(\beta + \frac{2}{3}\alpha\right) \rho^\delta (\operatorname{div} u)^2 \,\mathrm{d}x \\
+ 6(t+1) \int_{A(t)} \left(\beta + \frac{2}{3}\alpha\right) \rho^\delta \operatorname{div} u \,\mathrm{d}x.
\end{aligned} \tag{4.13}
$$

Since $\delta \leq \gamma$, with the help of Lemma 4.1, Cauchy's inequality, and Young's inequality, we have

$$
\begin{aligned}
& - 2(t+1)^2 \int_{A(t)} \rho^\delta (\operatorname{div} u)^2 \,\mathrm{d}x + 6(t+1) \int_{A(t)} \rho^\delta \operatorname{div} u \,\mathrm{d}x \\
& \leqq -2(t+1)^2 \int_{A(t)} \rho^\delta (\operatorname{div} u)^2 \,\mathrm{d}x + 2(t+1)^2 \int_{A(t)} \rho^\delta (\operatorname{div} u)^2 \,\mathrm{d}x + 18 \int_{A(t)} \rho^\delta \,\mathrm{d}x \\
& \leqq 18 \int_{A(t)} \rho^\delta \,\mathrm{d}x \leqq \frac{18\delta}{\gamma} \int_{A(t)} \rho^\gamma \,\mathrm{d}x + \frac{18(\gamma-\delta)}{\gamma} |A_0|.
\end{aligned} \tag{4.14}
$$

We deduce from (4.13) that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} I(t) \leqq \frac{2}{\gamma-1}(2-3(\gamma-1))(t+1) \int_{A(t)} P \,\mathrm{d}x \\
+ 18 \left(\beta + \frac{2}{3}\alpha\right) \frac{\delta}{\gamma} \int_{A(t)} \rho^\gamma \,\mathrm{d}x + 18 \left(\beta + \frac{2}{3}\alpha\right) \frac{\gamma-\delta}{\gamma} |A_0|.
\end{aligned} \tag{4.15}
$$

From the second expression of $I(t)$ in (4.5), one has

$$
\begin{aligned}
\frac{2-3(\gamma-1)}{t+1} I(t) = \frac{2-3(\gamma-1)}{t+1} \int_{A(t)} |x-(t+1)u|^2 \rho \,\mathrm{d}x \\
+ \frac{2}{\gamma-1}(2-3(\gamma-1))(t+1) \int_{A(t)} P \,\mathrm{d}x.
\end{aligned} \tag{4.16}
$$

In the case when $1 < \gamma < \frac{5}{3}$, from (4.15)–(4.16), for $0 \leqq t < T_m$, we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} I(t) \leqq \frac{2-3(\gamma-1)}{t+1} I(t) + 18 \left(\beta + \frac{2}{3}\alpha\right) \frac{\delta(\gamma-1)}{2A\gamma(t+1)^2} I(t) \\
+18 \left(\beta + \frac{2}{3}\alpha\right) \frac{\gamma-\delta}{\gamma} |A_0|.
\end{aligned} \tag{4.17}
$$

Solving (4.17) directly, we get

$$
I(t) \leqq (t+1)^{2-3(\gamma-1)} e^{-\frac{a_1}{t+1}} \left(e^{a_1} I(0) + a_2 \int_0^t (\tau+1)^{3(\gamma-1)-2} e^{\frac{a_1}{\tau+1}} \,\mathrm{d}\tau \right), \tag{4.18}
$$

where

$$a_1 = 18\left(\beta + \frac{2}{3}\alpha\right)\frac{\delta(\gamma-1)}{2A\gamma}, \quad a_2 = 18\left(\beta + \frac{2}{3}\alpha\right)\frac{\gamma-\delta}{\gamma}|A_0|.$$

If $3(\gamma-1)-2 \neq -1$, then from (4.18) we get

$$\begin{aligned} I(t) &\leqq (t+1)^{2-3(\gamma-1)}e^{-\frac{a_1}{t+1}}\left(e^{a_1}I(0) - \frac{a_2e^{a_1}}{3(\gamma-1)-1}\right) \\ &\quad + \frac{a_2(t+1)}{3(\gamma-1)-1}e^{-\frac{a_1}{t+1}}e^{a_1} \\ &\leqq C\left(t^{2-3(\gamma-1)} + t + 1\right), \quad \text{for } t \in [0, T_m). \end{aligned} \tag{4.19}$$

If $3(\gamma-1)-2 = -1$, from (4.18) we get

$$\begin{aligned} I(t) &\leqq (t+1)^{2-3(\gamma-1)}e^{-\frac{a_1}{t+1}}\left(e^{a_1}I(0) + a_2e^{a_1}\ln(t+1)\right) \\ &\leqq C\left((t+1)\ln(t+1) + t + 1\right), \quad \text{for } t \in [0, T_m). \end{aligned} \tag{4.20}$$

On the other hand, from the definition of $I(t)$, Jensen's inequality and Lemma 4.1, we show that

$$\begin{aligned} I(t) &\geq \frac{2(t+1)^2}{\gamma-1}|A_0|\int_{A(t)} A\rho^\gamma(x,t)\frac{\mathrm{d}x}{|A(t)|} \\ &\geq \frac{C(t+1)^2}{\gamma-1}|A_0|^{1-\gamma}m(0)^\gamma \geq C_0(1+t)^2, \end{aligned} \tag{4.21}$$

where $C_0 > 0$ is a constant and we used the fact in Lemma 4.3 that

$$m(t) = \int_{A(t)} \rho(x,t)\,\mathrm{d}x = \int_{A_0} \rho_0(x)\,\mathrm{d}x = m(0).$$

Then $T_m < +\infty$ follows immediately, otherwise a contradiction forms between (4.21) and (4.19) or (4.20).

In the case when $\frac{5}{3} \leqq \gamma < +\infty$, and thus $2-3(\gamma-1) \leqq 0$, from (4.15) we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}I(t) &\leqq 18\left(\beta + \frac{2}{3}\alpha\right)\frac{\delta}{\gamma}\int_{A(t)} \rho^\gamma\,\mathrm{d}x + 18\left(\beta + \frac{2}{3}\alpha\right)\frac{(\gamma-\delta)}{\gamma}|A_0| \\ &\leqq 18\left(\beta + \frac{2}{3}\alpha\right)\frac{\delta(\gamma-1)}{2\gamma A(t+1)^2}I(t) + 18\left(\beta + \frac{2}{3}\alpha\right)\frac{\gamma-\delta}{\gamma}|A_0| \\ &= \frac{a_1}{(t+1)^2}I(t) + a_2, \end{aligned} \tag{4.22}$$

and therefore

$$\begin{aligned} I(t) &\leq e^{a_1}I(0)e^{-\frac{a_1}{t+1}} + e^{a_1}e^{-\frac{a_1}{t+1}}a_2t \\ &\leq e^{a_1}(I(0) + a_2t). \end{aligned} \tag{4.23}$$

Again, this and (4.21) imply that $T_m < \infty$. This completes the proof of Theorem 1.2. □

4.2. Blow-Up by Hyperbolic Singularity Set

The mechanism for our second finite time blow-up result comes from the nonlinear hyperbolic structure (see (1.16)) which controls the behavior of the velocity u in the vacuum region. Assume that the initial data $(\rho_0, u_0)(x)$ have a hyperbolic singularity set V, see Definition 1.3.

Proof of Theorem 1.3. Let $V(t)$ be the image of V under the flow map of (4.1), i.e.,

$$V(t) = \{x(t;\xi_0)|\xi_0 \in V\}. \tag{4.24}$$

It follows from the continuity equation $(1.1)_1$ that the density is simply transported along the particle path, so

$$\rho(x,t) = 0, \quad \text{when} \quad x \in V(t).$$

From the Definition 1.1 for regular solutions, we have

$$u_t + u \cdot \nabla u = 0, \quad \text{when} \quad x \in V(t), \tag{4.25}$$

which means that u is a constant vector along the particle path $x(t;\xi_0)$ and

$$\xi_0 = x - tu(x,t) \in V.$$

Then, for any $x \in V(t)$, we have

$$u(x,t) = u_0(\xi_0) = u_0(x - tu(x,t)),$$

which implies that

$$\begin{aligned} \nabla u(x,t) &= \left(\mathbb{I}_3 + t\nabla u_0(x - tu(x,t))\right)^{-1} \nabla u_0(x - tu(x,t)) \\ &= \left(\mathbb{I}_3 + t\nabla u_0(\xi_0)\right)^{-1} \nabla u_0(\xi_0), \quad \text{for} \quad x \in V(t). \end{aligned} \tag{4.26}$$

According to the definition of the hyperbolic singularity set, there exists some

$$\xi_0 \in V, \quad \text{and} \quad l_{\xi_0} \in Sp(\nabla u_0(\xi_0)) \quad \text{satisfying} \quad l_{\xi_0} < 0.$$

Let $w \in \mathbb{R}^3$ be the eigenvector of $\nabla u_0(\xi_0)$ with respect to l_{ξ_0}, that is,

$$\nabla u_0(\xi_0) w = l_{\xi_0} w.$$

It is clear that

$$\left(\mathbb{I}_3 + t\nabla u_0(\xi_0)\right)^{-1} w = \left(1 + t l_{\xi_0}\right)^{-1} w.$$

Thus we know that the matrix $\nabla u(x,t)$ has an eigenvector w with the eigenvalue

$$\frac{l_{\xi_0}}{1 + t l_{\xi_0}},$$

which, along with $l_{\xi_0} < 0$, implies that the quantity ∇u will blow up in finite time, i.e.,

$$T_m < +\infty.$$

This completes the proof of Theorem 1.3. □

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5. Appendix: Proof for the Remark 1.4

In this section, we will show that the regular solution that we obtained in Theorem 1.1 is indeed a classical one in $(0, T_*]$. The following lemma will be used in our proof:

Lemma 5.1. [1] *If* $f(x,t) \in L^2([0,T]; L^2)$*, then there exists a sequence* s_k *such that*

$$s_k \to 0, \quad \textit{and} \quad s_k|f(x,s_k)|_2^2 \to 0, \quad \textit{as} \quad k \to +\infty.$$

From the definition of regular solution and the classical Sobolev embedding theorem, it is clear that

$$(\rho, \nabla\rho, \rho_t, u, \nabla u) \in C(\mathbb{R}^3 \times [0, T_*]),$$

so it remains to prove that

$$(u_t, \mathrm{div}\mathbb{T})(x,t) \in C(\mathbb{R}^3 \times (0, T_*]).$$

From the proof of Theorem 1.1 in Section 3, we know that (through a change of variable $\phi = \rho^{\frac{\delta-1}{2}}$), system (1.1) can be written as

$$\begin{cases} \phi_t + u\cdot\nabla\phi + \dfrac{\delta-1}{2}\phi\mathrm{div}u = 0, \\ u_t + u\cdot\nabla u + \dfrac{2A\gamma}{\delta-1}\phi^{\frac{2r-\delta-1}{\delta-1}}\nabla\phi + \phi^2 Lu = \nabla\phi^2\cdot Q(u). \end{cases} \tag{5.1}$$

The solution (ϕ, u) satisfies the regularities in (3.6) and $\phi \in C^1(\mathbb{R}^3 \times [0, T_*])$.

Step 1 The continuity of u_t. We differentiate (5.1)$_2$ with respect to t to get

$$u_{tt} + \phi^2 L u_t = -(\phi^2)_t L u - (u \cdot \nabla u)_t - \frac{A\gamma}{\gamma - 1} \nabla \Big(\phi^{\frac{2\gamma-2}{\delta-1}} \Big)_t + (\nabla \phi^2 \cdot Q(u))_t, \tag{5.2}$$

which, along with (3.6), implies that

$$u_{tt} \in L^2([0, T_*]; L^2). \tag{5.3}$$

Applying the operator ∂_x^ζ ($|\zeta| = 2$) to (5.2), multiplying the resulting equations by $\partial_x^\zeta u_t$ and integrating over $\mathbb{R}^3$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\partial_x^\zeta u_t|_2^2 + \alpha|\phi\nabla\partial_x^\zeta u_t|_2^2 + (\alpha+\beta)|\phi \mathrm{div}\partial_x^\zeta u_t|_2^2 \\
&\quad = \int \Big(-\nabla\phi^2 \cdot \frac{\delta-1}{\delta} Q(\partial_x^\zeta u_t) - \big(\partial_x^\zeta(\phi^2 L u_t) - \phi^2 L \partial_x^\zeta u_t\big)\Big) \cdot \partial_x^\zeta u_t \\
&\qquad + \int \Big(-\partial_x^\zeta\big((\phi^2)_t L u\big) - \partial_x^\zeta (u \cdot \nabla u)_t - \frac{A\gamma}{\gamma-1}\partial_x^\zeta \nabla \Big(\phi^{\frac{2\gamma-2}{\delta-1}}\Big)_t \Big) \cdot \partial_x^\zeta u_t \\
&\qquad + \int \partial_x^\zeta (\nabla\phi^2 \cdot Q(u))_t \cdot \partial_x^\zeta u_t =: \sum_{i=10}^{15} J_i .
\end{aligned} \tag{5.4}$$

Now we analyze the terms J_i ($i = 10, \cdots, 15$). By Hölder's inequality, Lemma 2.1 and Young's inequality, we have

$$\begin{aligned}
J_{10} &= \int \Big(-\nabla\phi^2 \cdot \frac{\delta-1}{\delta} Q(\partial_x^\zeta u_t)\Big) \cdot \partial_x^\zeta u_t \\
&\leqq C|\phi\nabla^3 u_t|_2 |\nabla^2 u_t|_2 |\nabla\phi|_\infty \leqq C|u_t|_{D^2}^2 + \frac{\alpha}{20}|\phi\nabla^3 u_t|_2^2, \\
J_{11} &= \int -\big(\partial_x^\zeta(\phi^2 L u_t) - \phi^2 L\partial_x^\zeta u_t\big) \cdot \partial_x^\zeta u_t \\
&\leqq C\Big(|\phi\nabla^3 u_t|_2|\nabla^2 u_t|_2|\nabla\phi|_\infty + |\nabla\phi|_\infty^2 |u_t|_{D^2}^2 + |\nabla^2\phi|_3|\phi\nabla^2 u_t|_6 |u_t|_{D^2}\Big) \\
&\leqq C|u_t|_{D^2}^2 + \frac{\alpha}{20}|\phi\nabla^3 u_t|_2^2,
\end{aligned} \tag{5.5}$$

and

$$\begin{aligned}
J_{12} &= \int -\partial_x^\zeta\big((\phi^2)_t L u\big) \cdot \partial_x^\zeta u_t \\
&\leqq C\Big(|u_t|_{D^2}(|\phi_t|_\infty |\phi\nabla^4 u|_2 + |\nabla^2\phi|_3 |Lu|_6 |\phi_t|_\infty + |\nabla\phi|_\infty |\nabla\phi_t|_6 |Lu|_3) \\
&\quad + |u_t|_{D^2}(|\phi\nabla^3 u|_6 |\nabla\phi_t|_3 + |\nabla\phi|_\infty |\phi_t|_\infty |\nabla^3 u|_2) + |\phi\nabla^2 u_t|_6 |\phi_t|_{D^2} |Lu|_3\Big) \\
&\leqq C\Big(|u_t|_{D^2}^2 + |\phi\nabla^4 u|_2^2 + 1\Big) + \frac{\alpha}{20}|\phi\nabla^3 u_t|_2^2, \\
J_{13} &= \int -\partial_x^\zeta (u \cdot \nabla u)_t \cdot \partial_x^\zeta u_t \\
&\leqq C\|u_t\|_2 \|u\|_3 |u_t|_{D^2} + \int -\big(u \cdot \nabla\big)\partial_x^\zeta u_t \cdot \partial_x^\zeta u_t
\end{aligned}$$

$$
\begin{aligned}
&\leqq C\Big(1+|u_t|_{D^2}^2+|\nabla u|_\infty|\partial_x^\zeta u_t|_2^2\Big)\leqq C\Big(1+|u_t|_{D^2}^2\Big),\\
J_{14}=&\int -\frac{A\gamma}{\gamma-1}\partial_x^\zeta\nabla\Big(\phi^{\frac{2\gamma-2}{\delta-1}}\Big)_t\cdot\partial_x^\zeta u_t=\int\frac{A\gamma}{\gamma-1}\partial_x^\zeta\Big(\phi^{\frac{2\gamma-2}{\delta-1}}\Big)_t\mathrm{div}\partial_x^\zeta u_t\\
&\leqq C\Big(|\phi^{\frac{K}{2}-2}|_\infty|\nabla^2\phi_t|_2|\phi\nabla^3u_t|_2+|\phi^{\frac{K}{2}-3}|_\infty|\phi_t|_\infty|\nabla^2\phi|_2|\phi\nabla^3u_t|_2\\
&\quad+|\phi^{\frac{K}{2}-4}|_\infty|\phi_t|_\infty|\nabla\phi|_6|\nabla\phi|_3|\phi\nabla^3u_t|_2+|\phi^{\frac{K}{2}-3}|_\infty|\nabla\phi_t|_2|\nabla\phi|_\infty|\phi\nabla^3u_t|_2\Big)\\
&\leqq C+\frac{\alpha}{20}|\phi\nabla^3u_t|_2^2,\\
J_{15}=&\int\partial_x^\zeta(\nabla\phi^2\cdot Q(u))_t\cdot\partial_x^\zeta u_t\\
&\leqq C\Big(\|\nabla\phi\|_2^2|u_t|_{D^2}^2+\big(\|\nabla\phi\|_2|\nabla u_t|_3+\|u\|_3\|\phi_t\|_2\big)|\phi\nabla^2u_t|_6\\
&\quad+\big(\|\nabla\phi\|_2|\phi\nabla^3u_t|_2+\|\nabla\phi\|_2|\phi\nabla^2u_t|_6\big)|u_t|_{D^2}\\
&\quad+\big(\|\nabla\phi\|_2\|\phi_t\|_2\|u\|_3+\|\phi_t\|_2|\phi\nabla^3u|_6\big)|u_t|_{D^2}\Big)\\
&\quad+\int\partial_x^\zeta(\nabla\phi^2)_t\cdot Q(u)\cdot\partial_x^\zeta u_t\\
&\leqq C\Big(|u_t|_{D^2}^2+|\phi\nabla^4u|_2^2\Big)+\frac{\alpha}{20}|\phi\nabla^3u_t|_2^2+J_{151},
\end{aligned}\tag{5.6}
$$

where

$$
\begin{aligned}
J_{151}=&\int\partial_x^\zeta(\nabla\phi^2)_t\cdot Q(u)\cdot\partial_x^\zeta u_t\\
&\leqq C\|\nabla\phi\|_2\|\phi_t\|_2\|u\|_3|u_t|_{D^2}+\int\phi\partial_x^\zeta\nabla\phi_t\cdot Q(u)\cdot\partial_x^\zeta u_t.
\end{aligned}\tag{5.7}
$$

Using integration by parts, the last term in (5.7) is estimated as

$$
\begin{aligned}
&\int\phi\partial_x^\zeta\nabla\phi_t\cdot Q(u)\cdot\partial_x^\zeta u_t\\
&\quad\leqq C\Big(|\nabla\phi|_\infty|\phi_t|_{D^2}|\nabla u|_\infty|u_t|_{D^2}+|\phi_t|_{D^2}|\nabla^2u|_3|\phi\nabla^2u_t|_6\\
&\qquad+|\phi\nabla^3u_t|_2|\nabla u|_\infty\|\phi_t\|_2\Big)\\
&\quad\leqq C|u_t|_{D^2}^2+\frac{\alpha}{20}|\phi\nabla^3u_t|_2^2.
\end{aligned}\tag{5.8}
$$

Then (5.4) reduces to

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u_t|_{D^2}^2+\frac{\alpha}{2}|\phi\nabla^3u_t|_2^2\leqq C\Big(|u_t|_{D^2}^2+|\phi\nabla^4u|_2^2+1\Big).\tag{5.9}
$$

Multiplying both sides of (5.9) with s and integrating the resulting inequalities over $[\tau,t]$ for any $\tau\in(0,t)$, we have

$$
t|u_t|_{D^2}^2+\int_\tau^t s|\phi\nabla^3u_t|_2^2\,\mathrm{d}s\leqq C\tau|u_t(\tau)|_{D^2}^2+C(1+t).\tag{5.10}
$$

According to the definition of the regular solution, we know that

$$\nabla^2 u_t \in L^2([0, T_*]; L^2).$$

Using Lemma 5.1 to $\nabla^2 u_t$, there exists a sequence s_k such that

$$s_k \to 0, \quad \text{and} \quad s_k |\nabla^2 u_t(\cdot, s_k)|_2^2 \to 0, \quad \text{as} \quad k \to +\infty.$$

Choosing $\tau = s_k \to 0$ in (5.10), we have

$$t|u_t|_{D^2}^2 + \int_0^t s|\phi \nabla^3 u_t|_2^2 \, \mathrm{d}s \leqq C(1+t), \tag{5.11}$$

then

$$t^{\frac{1}{2}} u_t \in L^\infty([0, T_*]; H^2). \tag{5.12}$$

The classical Sobolev embedding theorem gives

$$L^\infty([0, T]; H^1) \cap W^{1,2}([0, T]; H^{-1}) \hookrightarrow C([0, T]; L^q) \tag{5.13}$$

for any $q \in (3, 6]$. From (5.3) and (5.12) we have

$$t u_t \in C([0, T_*]; W^{1,4}),$$

which implies that

$$u_t \in C(\mathbb{R}^3 \times (0, T_*]).$$

Step 2 The continuity of $\operatorname{div}\mathbb{T}$. Denote $\mathbb{N} = \phi^2 L u - \nabla \phi^2 \cdot Q(u)$. From equations $(5.1)_2$, regularities (3.6) and (5.12), it is easy to show that

$$t\mathbb{N} \in L^\infty([0, T_*]; H^2).$$

Due to

$$\mathbb{N}_t \in L^2([0, T_*]; L^2),$$

we obtain from (5.13) that

$$t\mathbb{N} \in C([0, T_*]; W^{1,4}),$$

which implies that

$$\mathbb{N} \in C(\mathbb{R}^3 \times (0, T_*]).$$

Since $\rho \in C(\mathbb{R}^3 \times [0, T_*])$ and $\operatorname{div}\mathbb{T} = \rho\mathbb{N}$, we immediately obtain the desired conclusion.

In summary, we have shown that the regular solution that we obtained is indeed a classical one in $\mathbb{R}^3 \times [0, T_*]$ to the Cauchy problem (1.1)–(1.3).

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Yachun Li
School of Mathematical Sciences, MOE-LSC, and SHL-MAC,
Shanghai Jiao Tong University,
Shanghai
200240 People's Republic of China.
e-mail: ycli@sjtu.edu.cn

and

Ronghua Pan
School of Mathematics,
Georgia Tech,
Atlanta
30332 USA.
e-mail: panrh@math.gatech.edu

and

Shengguo Zhu
School of Mathematical Sciences,
Shanghai Jiao Tong University,
Shanghai
200240 People's Republic of China.

and

Mathematical Institute,
University of Oxford,
Oxford
OX2 6GG UK.
e-mail: zhushengguo@sjtu.edu.cn

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