

ASYMPTOTIC PROFILES OF BASIC REPRODUCTION NUMBER  
FOR EPIDEMIC SPREADING IN HETEROGENEOUS  
ENVIRONMENT\*SHANSHAN CHEN<sup>†</sup> AND JUNPING SHI<sup>‡</sup>

**Abstract.** The effect of diffusion rates on the basic reproduction number of a general compartmental reaction-diffusion epidemic model in a heterogeneous environment is considered. It is shown that when the diffusion rates tend to zero, the limit of the basic reproduction number is the maximum value of the local reproduction number on the spatial domain. On the other hand, when the diffusion rates tend to infinity, the basic reproduction number tends to the spectral radius of the “average” next generation matrix. These asymptotic limits of basic reproduction number hold for a class of general spatially heterogeneous compartmental epidemic models, and they are applied to a wide variety of examples.

**Key words.** basic reproduction number, reaction-diffusion, heterogeneous environment, compartmental epidemic models

**AMS subject classifications.** 35K57, 92B05, 92D30

**DOI.** 10.1137/19M1289078

**1. Introduction.** In mathematical modeling of infectious diseases, the basic reproduction number  $R_0$  is a key indicator for disease transmission. When  $R_0 < 1$ , the disease declines and eventually vanishes, and when  $R_0 > 1$ , the disease spreads in the population, and an outbreak is possible [4]. Roughly speaking, the basic reproduction number  $R_0$  is the average number of healthy people infected by one contagious person over the course of the infectious period. In more mathematically rigorous terms, for ODE epidemic models that are nonspatial,  $R_0$  is defined as the spectral radius of the next generation matrix [13, 43], which is established in a general framework of compartmental disease transmission models. This definition is also generalized to epidemic models with infinite-dimensional state space [41].

As the environment in which the disease spreads is spatially heterogeneous, the transmission and spreading of the infectious disease is inevitably affected by the spatial structure and heterogeneity of the environment. These factors can be incorporated into underlying mathematical models to show the effect of spatial heterogeneity on the disease transmission. The spatial structure and heterogeneity can be modeled in a discrete space using an ODE patch model [1, 5, 29, 42], or they can be modeled in a continuous space using a reaction-diffusion-advection PDE model [2, 11, 45, 48]. The notion of the basic reproduction number is also extended to both classes of models. In particular, a theory of basic reproduction numbers for general reaction-diffusion compartmental disease transmission models was recently developed in [45].

\*Received by the editors September 23, 2019; accepted for publication (in revised form) March 24, 2020; published electronically May 19, 2020.

<https://doi.org/10.1137/19M1289078>

**Funding:** The work of the first author was supported by the NSFC through grant 11771109. The work of the second author was supported by the NSF through grants DMS-1715651 and DMS-1853598.

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For spatially heterogeneous reaction-diffusion epidemic models, the basic reproduction number  $R_0$  usually depends on the diffusion rates of populations. For example, in the reaction-diffusion SIS epidemic model considered in [2],

$$\begin{cases} I_t = d_I \Delta I + \beta(x) \frac{SI}{S+I} - \gamma(x)I, & x \in \Omega, t > 0, \\ S_t = d_S \Delta S - \beta(x) \frac{SI}{S+I} + \gamma(x)I, & x \in \Omega, t > 0, \\ \partial_\nu I = \partial_\nu S = 0, & x \in \partial\Omega, t > 0, \end{cases}$$

where  $\beta(x)$  is the transmission rate,  $\gamma(x)$  is the removal rate, and  $d_I, d_S$  are the diffusion rates of infectious and susceptible populations, respectively. It was shown that the basic reproduction number is defined as

$$(1.1) \quad R_0 = \sup \left\{ \frac{\int_\Omega \beta \phi^2 dx}{\int_\Omega (d_I |\nabla \phi|^2 + \gamma \phi^2) dx} : \phi \in H^1(\Omega), \phi \neq 0 \right\}.$$

Moreover, it was shown in [2] that  $R_0$  has the following asymptotic profiles with respect to the infectious population diffusion rate  $d_I$ :

$$(1.2) \quad \lim_{d_I \rightarrow 0} R_0 = \max_{x \in \bar{\Omega}} \frac{\beta(x)}{\gamma(x)}, \quad \lim_{d_I \rightarrow \infty} R_0 = \frac{\int_\Omega \beta dx}{\int_\Omega \gamma dx}.$$

Notice that the quantity  $\beta(x)/\gamma(x)$  is the local basic reproduction number at  $x$  when there is no spatial movement; hence, the global basic reproduction number tends to the maximum of local one as the diffusion rate tends to zero. On the other hand, the limit of the basic reproduction number for a large diffusion rate is the ratio of the average transmission rate and the average removal rate. Similar asymptotic profiles for  $R_0$  were also obtained in [32, 40] for several kinds of other spatially heterogeneous epidemic reaction-diffusion models. The results in [32] are based on the fact that  $R_0$  equals the spectral radius of the product of the local basic reproduction number and strongly positive compact linear operators with spectral radii one.

In this paper, we aim to characterize limiting profiles of the basic reproduction number  $R_0$  for general spatially heterogeneous reaction-diffusion compartmental epidemic models for small or large diffusion rates.

We consider the reaction-diffusion compartmental epidemic model

$$(1.3) \quad \begin{cases} \partial u_i / \partial t = d_i \Delta u_i + f_i(x, u), & x \in \Omega, t > 0, 1 \leq i \leq n, \\ \partial_\nu u_i = 0, & x \in \partial\Omega, t > 0, 1 \leq i \leq n, \end{cases}$$

which was proposed in [45]. Here  $u_i$  is the density of the population in the  $i$ th compartment,  $d_i > 0$  is constant and represents the diffusion coefficient of population  $u_i$ ,  $\Omega$  is a bounded domain in  $\mathbb{R}^N$  ( $N \geq 1$ ) with smooth boundary  $\partial\Omega$ ,  $\nu$  is the outward unit normal vector at  $x \in \partial\Omega$ , and  $f_i(x, u)$  is the reaction term in the  $i$ th compartment. Moreover,

$$f_i(x, u) = \mathcal{F}_i(x, u) - \mathcal{V}_i(x, u),$$

where  $\mathcal{F}_i(x, u)$  is the input rate of newly infected individuals in the  $i$ th compartment,  $\mathcal{V}_i(x, u) = \mathcal{V}_i^-(x, u) - \mathcal{V}_i^+(x, u)$ ,  $\mathcal{V}_i^+(x, u)$  is the rate of transfer of individuals into the  $i$ th compartment by all other means, and  $\mathcal{V}_i^-(x, u)$  is the rate of transfer of individuals out of the  $i$ th compartment. More biological explanations of model (1.3) can be found in [45]. In this paper, we will show the asymptotic profiles of  $R_0$  for model (1.3)

as  $(d_1, \dots, d_n) \rightarrow (0, \dots, 0)$  and  $(d_1, \dots, d_n) \rightarrow (\infty, \dots, \infty)$ . Our results indicate that the trend set in [2, 32] holds true for epidemic models in a much more general setting: In the small diffusion limit, the global basic reproduction number tends to the maximum of the local basic reproduction number, and in the large diffusion limit, the global basic reproduction number tends to some kind of spatial average of the local basic reproduction number.

There are extensive results on reaction-diffusion epidemic models. The asymptotic profiles of the endemic steady states were considered in [2, 35, 37, 48] and references therein, and the global dynamics of the epidemic models can be found in [8, 12, 23, 25, 27, 31, 36, 46]. The effect of diffusion and advection rates on  $R_0$  and the stability of the disease-free steady state for a reaction-diffusion-advection epidemic model was considered in [11]; see also [10, 18, 22, 33] for reaction-diffusion-advection epidemic models. The definition of  $R_0$  for time-periodic reaction-diffusion epidemic models was given in [6, 26, 49], and the global dynamics for a time-periodic or almost time-periodic reaction-diffusion SIS epidemic model was studied in [38, 44]. The reaction-diffusion epidemic models with free boundary conditions were investigated in [9, 16, 28] and references therein, and reaction-diffusion epidemic models with time delays were also studied extensively; see, e.g., [7, 30, 47].

Throughout the paper, we use the following notations. For  $n \geq 1$ ,

$$(1.4) \quad \begin{aligned} \mathbb{R}_+^n &= \{u = (u_1, \dots, u_n) : u_i \geq 0 \text{ for any } i = 1, \dots, n\}, \\ C(\bar{\Omega}, \mathbb{R}_+^n) &= \{(u_1(x), \dots, u_n(x)) : u_i(x) (\in C(\bar{\Omega}, \mathbb{R})) \geq 0 \text{ for } 1 \leq i \leq n\}. \end{aligned}$$

For a closed and linear operator  $A$ , we denote the spectral radius of  $A$  by  $r(A)$ , the spectral set of  $A$  by  $\sigma(A)$ , and the spectral bound of  $A$  by

$$s(A) := \sup\{\mathcal{R}e\lambda : \lambda \in \sigma(A)\}.$$

Let  $P = (P_{ij})_{1 \leq i, j \leq l}$  and  $Q = (Q_{ij})_{1 \leq i, j \leq l}$  be  $l \times l$  ( $l \geq 1$ ) real-valued matrices, and let  $Q(x) = (Q_{ij}(x))_{1 \leq i, j \leq l}$  be an  $l \times l$  matrix-valued function.

$P \geq Q$  means  $P_{ij} \geq Q_{ij}$  for each  $1 \leq i, j \leq l$ .

$P > Q$  means  $P_{ij} > Q_{ij}$  for each  $1 \leq i, j \leq l$ .

$\lim_{x \rightarrow x_0} Q(x) = Q$  means  $\lim_{x \rightarrow x_0} Q_{ij}(x) = Q_{ij}$  for each  $1 \leq i, j \leq l$ .

The matrix  $P$  is called positive if all entries of  $P$  are nonnegative and there exists at least one positive entry.

The matrix  $P$  is called zero if all entries of  $P$  are zero.

The matrix  $P$  is called cooperative (or quasi-positive) if all off-diagonal entries of  $P$  are nonnegative, i.e.,  $P_{ij} \geq 0$  for  $i \neq j$ .

Moreover,  $(d_1, \dots, d_n) \rightarrow (0, \dots, 0)$  means  $\max_{1 \leq j \leq n} d_j \rightarrow 0$ .

$(d_1, \dots, d_n) \rightarrow (\infty, \dots, \infty)$  means  $\min_{1 \leq j \leq n} d_j \rightarrow \infty$ .

The remaining part of the paper is organized as follows. In section 2, we show some preliminaries for further applications. In sections 2 and 3, we show the asymptotic profiles of  $R_0$  for model (1.3) as  $(d_1, \dots, d_n) \rightarrow (0, \dots, 0)$  and  $(d_1, \dots, d_n) \rightarrow (\infty, \dots, \infty)$ , respectively. In section 4, we apply the theoretical results to some concrete examples.

**2. Some preliminaries.** In this section, we recall the definition of basic reproduction number for reaction-diffusion epidemic models in [45]. Assume that the population  $u = (u_1, \dots, u_n)^T$  of model (1.3) is divided into two types: infected compartments, labeled by  $i = 1, 2, \dots, m$ , and uninfected compartments, labeled by

$i = m + 1, \dots, n$ . We set

$$(2.1) \quad \begin{aligned} u_I &= (u_1, \dots, u_m)^T, \quad u_S = (u_{m+1}, \dots, u_n)^T, \\ d_I &= (d_1, \dots, d_m)^T, \quad d_S = (d_{m+1}, \dots, d_n)^T, \\ d_I \Delta u_I &= (d_1 \Delta u_1, \dots, d_m \Delta u_m)^T, \quad d_S \Delta u_S = (d_{m+1} \Delta u_{m+1}, \dots, d_n \Delta u_n)^T, \\ f_I(x, u) &= (f_1(x, u), \dots, f_m(x, u))^T, \quad f_S(x, u) = (f_{m+1}(x, u), \dots, f_n(x, u))^T. \end{aligned}$$

Let

$$U_s := \{u \geq 0 : u_i = 0 \text{ for any } i = 1, \dots, m\},$$

denote the set of all disease-free states of (1.3), and assume that model (1.3) has a disease-free steady state

$$(2.2) \quad u^0(x) = (0, \dots, 0, u_{m+1}^0(x), \dots, u_n^0(x))^T,$$

where  $u_i^0(x) > 0$  for any  $i = m + 1, \dots, n$  and  $x \in \bar{\Omega}$ . Define the three matrices

$$(2.3) \quad \begin{aligned} F(x, u) &= (F_{ij}(x, u))_{1 \leq i, j \leq m} = \left( \frac{\partial \mathcal{F}_i(x, u)}{\partial u_j} \right)_{1 \leq i, j \leq m}, \\ V(x, u) &= (V_{ij}(x, u))_{1 \leq i, j \leq m} = \left( \frac{\partial \mathcal{V}_i(x, u)}{\partial u_j} \right)_{1 \leq i, j \leq m}, \\ M(x, u) &= (M_{ij}(x, u))_{1 \leq i, j \leq n-m} = \left( \frac{\partial f_{i+m}(x, u)}{\partial u_{j+m}} \right)_{1 \leq i, j \leq n-m}, \end{aligned}$$

and let

$$(2.4) \quad Bu := d_I \Delta u - V(x, u^0(x)) u,$$

which is associated with the homogeneous Neumann boundary condition  $\partial_\nu u = 0$ . The following assumptions are imposed on model (1.3) (see assumptions (A1)–(A6) in [45]):

- (A1) For each  $1 \leq i \leq n$ , functions  $\mathcal{F}_i(x, u)$ ,  $\mathcal{V}_i^+(x, u)$ ,  $\mathcal{V}_i^-(x, u)$  are nonnegative and continuously differentiable on  $\bar{\Omega} \times \mathbb{R}_+^n$ .
- (A2) If  $u_i = 0$ , then  $\mathcal{V}_i^- = 0$ .
- (A3)  $\mathcal{F}_i = 0$  for  $i > m$ .
- (A4) If  $u \in U_s$ , then  $\mathcal{F}_i = \mathcal{V}_i^+ = 0$  for  $i = 1, \dots, m$ .
- (A5)  $M(x, u^0(x))$  is cooperative for any  $x \in \bar{\Omega}$  and

$$s(d_S \Delta + M(x, u^0(x))) < 0.$$

- (A6)  $-V(x, u^0(x))$  is cooperative for any  $x \in \bar{\Omega}$  and

$$s(B) = s(d_I \Delta - V(x, u^0(x))) < 0.$$

Assumptions (A1)–(A6) are satisfied for most reaction-diffusion epidemic models.

Denote

$$(2.5) \quad X = C(\bar{\Omega}, \mathbb{R}^m) \quad \text{and} \quad X_+ = C(\bar{\Omega}, \mathbb{R}_+^m).$$

$X$  is an ordered Banach space, and  $X_+$  is a positive cone with nonempty interior. Let  $T(t)$  be the semigroup generated by  $B$  on  $X$ ; i.e.,  $T(t)$  is the solution semigroup associated with the following linear reaction-diffusion system:

$$(2.6) \quad \begin{cases} \partial u_I / \partial t = d_I \Delta u_I - V(x, u^0(x)) u_I, & x \in \Omega, t > 0, \\ \partial_\nu u_I = 0, & x \in \partial\Omega, t > 0. \end{cases}$$

It follows from the comparison principle (see [41, Theorem 3.12]) and assumption (A6) that  $B$  is resolvent-positive,  $T(t)$  is positive (i.e.,  $T(t)X_+ \subset X_+$  for all  $t > 0$ ),  $s(B) < 0$ , and  $-B^{-1}\phi = \int_0^\infty T(t)\phi dt$  for  $\phi \in X$ . Note that  $F(x, u^0(x))$  is a positive matrix, and it can also be viewed as a positive operator on  $C(\bar{\Omega}, \mathbb{R}^m)$ :

$$\phi \in C(\bar{\Omega}, \mathbb{R}^m) \mapsto F(x, u^0(x))\phi.$$

Clearly, the linear operator  $B + F(x, u^0(x))$  is also resolvent-positive. Then the following holds from [45, section 3] (or [41, Theorem 3.5]):

**PROPOSITION 2.1.** *Assume that (A1)–(A6) hold. Then the basic reproduction number is defined by*

$$R_0 = r(-F(x, u^0(x))B^{-1}).$$

Moreover, the following statements hold:

- (i)  $R_0 - 1$  has the same sign as  $s(B + F(x, u^0(x)))$ .
- (ii) If  $R_0 < 1$ , then  $u^0(x)$  is locally asymptotically stable for system (1.3).

Next we recall several results which will be used later. First, we have the following comparison principle.

**LEMMA 2.2.** *Assume that  $P_i(x)$  ( $i = 1, 2$ ) are  $m \times m$  cooperative matrices for any  $x \in \bar{\Omega}$ , all entries of  $P_i(x)$  ( $i = 1, 2$ ) are continuous, and  $P_1(x) \geq P_2(x)$ . Let  $T_i(t)$  be the solution semigroup on  $X$  (defined in (2.5)) associated with the following linear reaction-diffusion system:*

$$(2.7) \quad \begin{cases} \partial u_I / \partial t = d_I \Delta u_I + P_i(x)u_I, & x \in \Omega, t > 0, \\ \partial_\nu u_I = 0, & x \in \partial\Omega, t > 0, \end{cases}$$

where  $d_I \Delta u_I$  is defined as in (2.1) and  $d_i > 0$  for  $i = 1, \dots, m$ . Then  $T_1(t)\phi \geq T_2(t)\phi$  for any  $\phi \in X_+$  and  $t > 0$ .

*Proof.* Denote  $U_i(x, t) = T_i(t)\phi$  for  $\phi \in X_+$ , and it follows from the comparison principle of cooperative parabolic systems that  $U_i(x, t) \geq 0$  for any  $(x, t) \in \bar{\Omega} \times (0, \infty)$  and  $i = 1, 2$ . Let  $W(x, t) = U_1(x, t) - U_2(x, t)$ , and then  $W(x, t)$  satisfies

$$(2.8) \quad \begin{cases} W_t = d_I \Delta W + P_2(x)W + (P_1(x) - P_2(x))U_1, & x \in \Omega, t > 0, \\ \partial_\nu W = 0, & x \in \partial\Omega, t > 0, \\ W(x, 0) = 0, & x \in \Omega. \end{cases}$$

Note that  $P_1(x) \geq P_2(x)$  and  $U_1(x, t) \geq 0$  for any  $(x, t) \in \bar{\Omega} \times (0, \infty)$ . Again it follows from the comparison principle of cooperative parabolic systems that  $W(x, t) \geq 0$  for any  $(x, t) \in \bar{\Omega} \times (0, \infty)$ . This completes the proof.  $\square$

Second, we recall the Krein–Rutmann theorem (see [3, Theorems 3.1 and 3.2] or [32, Theorem 2.5]).

LEMMA 2.3.

- (i) Suppose that  $T : X \rightarrow X$  is a positive compact linear operator with positive spectral radius  $r(T)$ . Then  $r(T)$  is an eigenvalue of  $T$  with an eigenvector in  $X_+ \setminus \{0\}$ .
- (ii) Suppose that  $T : X \rightarrow X$  is a strongly positive compact linear operator. Then  $r(T)$  is positive and is a simple eigenvalue of  $T$  with an eigenvector in  $\text{Int}(X_+)$ , and there is no other eigenvalue with a nonnegative eigenvector. Moreover, if  $S : X \rightarrow X$  is a linear operator such that  $S - T$  is strongly positive, then  $r(S) > r(T)$ .

Based on the Krein–Rutmann theorem in Lemma 2.3, we have the following two results.

LEMMA 2.4. Let  $L_1, L_2$  be bounded, positive linear operators on  $X$  (defined in (2.5)). Assume that  $L_1\phi \geq L_2\phi$  for any  $\phi \in X_+$  and that  $L_2$  is compact with positive spectral radius  $r(L_2)$ . Then  $r(L_1) \geq r(L_2)$ .

*Proof.* It follows from Lemma 2.3 that  $r(L_2)$  is an eigenvalue of  $L_2$ , and there exists  $\phi \in X_+ \setminus \{0\}$  such that  $\|\phi\|_\infty = 1$  and  $L_2\phi = r(L_2)\phi$ . Then  $L_1^n\phi \geq r^n(L_2)\phi$ , which implies that  $\|L_1^n\| \geq r^n(L_2)$ . Therefore,  $r(L_1) = \lim_{n \rightarrow \infty} \|L_1^n\|^{1/n} \geq r(L_2)$ .  $\square$

Consider the following eigenvalue problem:

$$(2.9) \quad \begin{cases} d_I \Delta \Phi - P(x)\Phi + aQ(x)\Phi = \lambda\Phi, & x \in \Omega, \\ \partial_\nu \Phi = 0, & x \in \partial\Omega, \end{cases}$$

where

$$(2.10) \quad \Phi = (\phi_1, \dots, \phi_m)^T, \quad d_I \Delta \Phi = (d_1 \Delta \phi_1, \dots, d_m \Delta \phi_m)^T,$$

$a > 0$ ,  $d_i > 0$  for  $i = 1, \dots, m$ , and  $P(x) = (P_{ij}(x))_{1 \leq i, j \leq m}$  and  $Q(x) = (Q_{ij}(x))_{1 \leq i, j \leq m}$  are  $m \times m$  matrices with continuous entries. Recall that an eigenvalue  $\lambda$  of (2.9) is called the principal eigenvalue if  $\lambda \in \mathbb{R}$  and that for any eigenvalue such that  $\tilde{\lambda} \neq \lambda$ , we have  $\mathcal{R}e\tilde{\lambda} < \lambda$ .

LEMMA 2.5. Assume that  $-P(x)$  is cooperative,  $Q(x)$  is positive for any  $x \in \bar{\Omega}$ , and for any  $a \in (0, \infty)$  there exists  $x_a \in \Omega$  such that  $-P(x_a) + aQ(x_a)$  is irreducible. Let  $\lambda(a)$  be the principal eigenvalue of (2.9). Then  $\lambda(a)$  is strictly increasing for  $a \in (0, \infty)$ .

*Proof.* Since  $-P(x) + aQ(x)$  is cooperative for any  $x \in \bar{\Omega}$  and  $a > 0$ , it follows from Lemma 2.3 that  $\lambda(a)$  is well defined and

$$\lambda(a) = \sup\{\mathcal{R}e\lambda : \lambda \text{ is an eigenvalue of problem (2.9)}\}.$$

Let  $T^a(t)$  be the solution semigroup associated with the linear parabolic system

$$(2.11) \quad \begin{cases} V_t = d_I \Delta V - P(x)V + aQ(x)V, & t > 0, x \in \Omega, \\ \partial_\nu V = 0, & t > 0, x \in \partial\Omega, \\ V(x, 0) = V_0(x), & x \in \Omega. \end{cases}$$

Then it follows from [39, Theorem 7.4.1] that  $T^a(t)$  is strongly positive and compact for any  $a > 0$  and  $t > 0$ . Let  $a_1 > a_2$ ,  $\Phi \in X_+ \setminus \{0\}$ , and

$$(2.12) \quad \begin{aligned} U_1(x, t) &= \left( U_1^{(1)}(x, t), \dots, U_1^{(m)}(x, t) \right)^T = T^{a_1}(t)\Phi, \\ U_2(x, t) &= \left( U_2^{(1)}(x, t), \dots, U_2^{(m)}(x, t) \right)^T = T^{a_2}(t)\Phi. \end{aligned}$$

Then  $U_1(x, t), U_2(x, t) > 0$  for any  $x \in \bar{\Omega}$  and  $t > 0$ . It follows from Lemma 2.2 that  $U_1(x, t) \geq U_2(x, t)$  for any  $x \in \bar{\Omega}$  and  $t > 0$ . Let  $W(x, t) = U_1(x, t) - U_2(x, t)$ , and we see that  $U(x, t)$  satisfies

$$(2.13) \quad \begin{cases} W_t = d_I \Delta W - P(x)W + a_2 Q(x)W + (a_1 - a_2)Q(x)U_1, & t > 0, x \in \Omega, \\ \partial_\nu W = 0, & t > 0, x \in \partial\Omega, \\ W(x, 0) = 0, & x \in \Omega. \end{cases}$$

Note that  $Q(x)$  is positive for any  $x \in \bar{\Omega}$  and that  $U_1(x, t) > 0$  for any  $x \in \bar{\Omega}$  and  $t > 0$ . Then there exist  $1 \leq i_1 \leq n$  and  $x_0 \in \Omega$  such that  $\sum_{j=1}^n Q_{i_1 j}(x_0)U_1^{(j)}(x_0, t) > 0$  for any  $t > 0$ , and consequently  $W_{i_1}(x, t) > 0$  for any  $x \in \bar{\Omega}$  and  $t > 0$ . Note that there exists  $x_{a_2} \in \Omega$  such that  $-P(x_{a_2}) + a_2 Q(x_{a_2})$  is irreducible. Then there exists  $i_2 \neq i_1$  such that  $-P_{i_2 i_1}(x_{a_2}) + a_2 Q_{i_2 i_1}(x_{a_2}) > 0$ , which implies that  $W_{i_2}(x, t) > 0$  for any  $x \in \bar{\Omega}$  and  $t > 0$ . Following the above process, we could obtain that  $W(x, t) > 0$  for any  $x \in \bar{\Omega}$  and  $t > 0$ , which implies that  $T_{a_1}(t) - T_{a_2}(t)$  is strongly positive for any  $t > 0$ . It follows from Lemma 2.3 that

$$(2.14) \quad r(T_{a_1}(t)) > r(T_{a_2}(t)) \quad \text{for any } t > 0,$$

It follows from [21, section 2] that  $\lambda(a)$  is an eigenvalue of problem (2.9). Then we see from the spectral mapping theorem [34, Theorem 2.2.4] that  $r(T_a(t)) = e^{\lambda(a)t}$ . This, combined with (2.14), implies that  $\lambda(a_1) > \lambda(a_2)$ . This completes the proof.  $\square$

**3. The effect of diffusion rates.** In this section, we show the asymptotic profile of  $R_0$  for model (1.3) when all the diffusion rates are large or small.

**3.1. Small diffusion rates.** In this subsection, we consider the asymptotic profile of  $R_0$  when  $(d_1, \dots, d_n) \rightarrow (0, \dots, 0)$ . We first impose an additional assumption for this case:

(A7) The disease-free steady state  $(0, \dots, 0, u_{m+1}^0(x), \dots, u_n^0(x))$  (defined in (2.2)) satisfies

$$(3.1) \quad \begin{aligned} &\lim_{(d_{m+1}, \dots, d_n) \rightarrow (0, \dots, 0)} (u_{m+1}^0(x), \dots, u_n^0(x)) \\ &= (c_{m+1}(x), \dots, c_n(x)) \quad \text{in } C(\bar{\Omega}, \mathbb{R}^{n-m}), \end{aligned}$$

where  $c_k(x) > 0$  for any  $x \in \bar{\Omega}$  and  $k = m+1, \dots, n$ .

In the next section, we will show that this assumption is not restrictive, and it is satisfied for many kinds of epidemic models. Denote

$$(3.2) \quad c(x) = (0, \dots, 0, c_{m+1}(x), \dots, c_n(x)) \in C(\bar{\Omega}, \mathbb{R}^n),$$

and denote, for sufficiently small  $\epsilon$  ( $0 < \epsilon < \min\{c_i(x) : i = m+1, \dots, n, x \in \bar{\Omega}\}$ ),

$$\begin{aligned}
\mathcal{D}_\epsilon^c &= \{(x, u_1, \dots, u_n) : x \in \bar{\Omega}, u_i = 0 \text{ for } i = 1, \dots, m, \\
&\quad u_i \in [c_i(x) - \epsilon, c_i(x) + \epsilon] \text{ for } i = m+1, \dots, n\}, \\
\mathcal{V}_\epsilon^c &= \left( \min_{(x,u) \in \mathcal{D}_\epsilon^c} V_{ij}(x, u) \right)_{1 \leq i, j \leq m} = \left( \min_{(x,u) \in \mathcal{D}_\epsilon^c} \frac{\partial \mathcal{V}_i(x, u)}{\partial u_j} \right)_{1 \leq i, j \leq m}, \\
(3.3) \quad \mathcal{V}_\epsilon^c &= \left( \max_{(x,u) \in \mathcal{D}_\epsilon^c} V_{ij}(x, u) \right)_{1 \leq i, j \leq m} = \left( \max_{(x,u) \in \mathcal{D}_\epsilon^c} \frac{\partial \mathcal{V}_i(x, u)}{\partial u_j} \right)_{1 \leq i, j \leq m}, \\
\mathcal{F}_\epsilon^c &= \left( \min_{(x,u) \in \mathcal{D}_\epsilon^c} F_{ij}(x, u) \right)_{1 \leq i, j \leq m} = \left( \min_{(x,u) \in \mathcal{D}_\epsilon^c} \frac{\partial \mathcal{F}_i(x, u)}{\partial u_j} \right)_{1 \leq i, j \leq m}, \\
\mathcal{F}_\epsilon^c &= \left( \max_{(x,u) \in \mathcal{D}_\epsilon^c} F_{ij}(x, u) \right)_{1 \leq i, j \leq m} = \left( \max_{(x,u) \in \mathcal{D}_\epsilon^c} \frac{\partial \mathcal{F}_i(x, u)}{\partial u_j} \right)_{1 \leq i, j \leq m}.
\end{aligned}$$

Since  $\mathcal{D}_{\epsilon_1}^c \subset \mathcal{D}_{\epsilon_2}^c$  for  $0 \leq \epsilon_1 < \epsilon_2$ , it follows that  $\mathcal{F}_\epsilon^c$  and  $\mathcal{V}_\epsilon^c$  are monotone increasing for  $\epsilon \geq 0$  and that  $\mathcal{F}_\epsilon^c$  and  $\mathcal{V}_\epsilon^c$  are monotone decreasing for  $\epsilon \geq 0$ . We will show that these functions  $\mathcal{F}_\epsilon^c$ ,  $\mathcal{V}_\epsilon^c$ ,  $\mathcal{F}_\epsilon^c$ , and  $\mathcal{V}_\epsilon^c$  are also continuous for  $\epsilon \geq 0$  in the appendix.

Clearly, for  $\epsilon = 0$ , we have

$$\begin{aligned}
\mathcal{V}_0^c &= \left( \min_{x \in \bar{\Omega}} V_{ij}(x, c(x)) \right)_{1 \leq i, j \leq m} = \left( \min_{x \in \bar{\Omega}} \frac{\partial \mathcal{V}_i(x, c(x))}{\partial u_j} \right)_{1 \leq i, j \leq m}, \\
\mathcal{V}_0^c &= \left( \max_{x \in \bar{\Omega}} V_{ij}(x, c(x)) \right)_{1 \leq i, j \leq m} = \left( \max_{x \in \bar{\Omega}} \frac{\partial \mathcal{V}_i(x, c(x))}{\partial u_j} \right)_{1 \leq i, j \leq m}, \\
\mathcal{F}_0^c &= \left( \min_{x \in \bar{\Omega}} F_{ij}(x, c(x)) \right)_{1 \leq i, j \leq m} = \left( \min_{x \in \bar{\Omega}} \frac{\partial \mathcal{F}_i(x, c(x))}{\partial u_j} \right)_{1 \leq i, j \leq m}, \\
\mathcal{F}_0^c &= \left( \max_{x \in \bar{\Omega}} F_{ij}(x, c(x)) \right)_{1 \leq i, j \leq m} = \left( \max_{x \in \bar{\Omega}} \frac{\partial \mathcal{F}_i(x, c(x))}{\partial u_j} \right)_{1 \leq i, j \leq m}.
\end{aligned}$$

Now we show the asymptotic profile of  $R_0$  as  $(d_1, \dots, d_n) \rightarrow (0, \dots, 0)$ , and the method is motivated by the one in [32].

**THEOREM 3.1.** *Assume that (A1)–(A5) and (A7) hold,*

$$(3.4) \quad s(-\mathcal{V}_0^c) < 0, \quad s(-\mathcal{V}_0^c) < 0 \quad \text{and} \quad r\left(\left(\mathcal{V}_0^c\right)^{-1} \mathcal{F}_0^c\right) > 0,$$

and that there exists  $\epsilon_0 > 0$  such that  $-\mathcal{V}_{\epsilon_0}^c$  is cooperative and  $\mathcal{F}_{\epsilon_0}^c$  is positive, where  $\mathcal{V}_\epsilon^c$ ,  $\mathcal{V}_\epsilon^c$ , and  $\mathcal{F}_\epsilon^c$  are defined in (3.3). If the matrix  $-V(x, c(x)) + aF(x, c(x))$  is irreducible for any  $a > 0$  and  $x \in \bar{\Omega}$ , where  $c(x)$  is defined in (3.2), then

$$(3.5) \quad \lim_{(d_1, \dots, d_n) \rightarrow (0, \dots, 0)} R_0 = R_0^c := \max_{x \in \bar{\Omega}} [r(-V^{-1}(x, c(x))F(x, c(x)))].$$

*Proof.* *Step 1.* We show that there exist positive constants  $R_0$ ,  $\bar{R}_0$ , and  $C_2$  such that  $R_0 \in [R_0, \bar{R}_0]$  for any  $d_1, \dots, d_m > 0$  and  $d_{m+1}, \dots, d_n \in (0, C_2)$ .

Since  $-\mathcal{V}_{\epsilon_0}^c$  is cooperative and  $\mathcal{F}_{\epsilon_0}^c$  is positive for any  $x \in \bar{\Omega}$ , it follows from the monotonicity of  $\mathcal{F}_\epsilon^c$ ,  $\mathcal{V}_\epsilon^c$ ,  $\mathcal{F}_\epsilon^c$ , and  $\mathcal{V}_\epsilon^c$  that  $-\mathcal{V}_\epsilon^c$  and  $-\mathcal{V}_\epsilon^c$  are cooperative and that  $\mathcal{F}_{\epsilon_0}^c$  and  $\mathcal{F}_\epsilon^c$  are positive for any  $\epsilon \in [0, \epsilon_0]$ . Note that  $\mathcal{V}_\epsilon^c$  and  $\mathcal{V}_\epsilon^c$  are continuous with respect to  $\epsilon$  (see Proposition A.1) and

$$\lim_{\epsilon \rightarrow 0} \mathcal{V}_\epsilon^c = \mathcal{V}_0^c \quad \text{and} \quad \lim_{\epsilon \rightarrow 0} \mathcal{V}_\epsilon^c = \mathcal{V}_0^c.$$

It follows from [20, Theorem 2.5.1] that there exists  $\epsilon_1 \in (0, \epsilon_0)$  such that

$$s(-\underline{V}_\epsilon^c) < 0, \quad s(-\bar{V}_\epsilon^c) < 0 \quad \text{for any } \epsilon \in (0, \epsilon_1].$$

Similarly,  $(\bar{V}_\epsilon^c)^{-1} \underline{F}_\epsilon^c$  is continuous with respect to  $\epsilon$  for  $\epsilon \in (0, \epsilon_1)$ , and there exists  $\epsilon_2 \in (0, \epsilon_1)$  such that  $r((\bar{V}_\epsilon^c)^{-1} \underline{F}_\epsilon^c) > 0$  for any  $\epsilon \in (0, \epsilon_2]$ . It follows from (A7) that, for the above given  $\epsilon_2 > 0$ , there exists  $C_2 > 0$  such that

$$c_i(x) - \epsilon_2 \leq u_i^0(x) \leq c_i(x) + \epsilon_2$$

for any  $x \in \bar{\Omega}$ ,  $d_{m+1}, \dots, d_n \in (0, C_2)$  and  $i = m+1, \dots, n$ . Denote by  $\bar{T}_{\epsilon_2}^c(t)$ ,  $\underline{T}_{\epsilon_2}^c(t)$ , and  $T(t)$  the semigroups generated by  $d_I \Delta - \bar{V}_{\epsilon_2}^c$ ,  $d_I \Delta - \underline{V}_{\epsilon_2}^c$ , and  $d_I \Delta - V(x, u^0(x))$ , respectively. Note that

$$(3.6) \quad -\bar{V}_{\epsilon_2}^c \leq -V(x, u^0(x)) \leq -\underline{V}_{\epsilon_2}^c$$

for any  $d_1, \dots, d_m > 0$  and  $d_{m+1}, \dots, d_n \in (0, C_2)$  and that  $-\bar{V}_{\epsilon_2}^c$  is cooperative for any  $x \in \bar{\Omega}$ . Then it follows from Lemma 2.2 that for any  $\phi \in X_+$  (defined in (2.5)),  $d_1, \dots, d_m > 0$  and  $d_{m+1}, \dots, d_n \in (0, C_2)$ ,

$$(3.7) \quad \bar{T}_{\epsilon_2}^c(t)\phi \leq T(t)\phi \leq \underline{T}_{\epsilon_2}^c(t)\phi.$$

Note that

$$s(d_I \Delta - \underline{V}_{\epsilon_2}^c) = s(-\underline{V}_{\epsilon_2}^c) < 0, \quad s(d_I \Delta - \bar{V}_{\epsilon_2}^c) = s(-\bar{V}_{\epsilon_2}^c) < 0.$$

This, combined with Lemma 2.3 and the spectral mapping theorem, implies that  $r(\bar{T}_{\epsilon_2}^c(t)), r(\underline{T}_{\epsilon_2}^c(t)) \in (0, 1)$ . Therefore, for any  $d_1, \dots, d_m > 0$  and  $d_{m+1}, \dots, d_n \in (0, C_2)$ ,  $s(d_I \Delta - V(x, u^0(x))) < 0$ , which implies that assumption (A6) is satisfied for any  $d_1, \dots, d_m > 0$  and  $d_{m+1}, \dots, d_n \in (0, C_2)$ . It follows from (3.7) that

$$\underline{F}_{\epsilon_2}^c \int_0^\infty \bar{T}_{\epsilon_2}^c(t)\phi dt \leq F(x, u^0(x)) \int_0^\infty T(t)\phi dt \leq \bar{F}_{\epsilon_2}^c \int_0^\infty \underline{T}_{\epsilon_2}^c(t)\phi dt.$$

It follows from [45, Theorem 3.4] that

$$r\left(\underline{F}_{\epsilon_2}^c \int_0^\infty \bar{T}_{\epsilon_2}^c(t)\phi dt\right) = r\left(\left(\bar{V}_{\epsilon_2}^c\right)^{-1} \underline{F}_{\epsilon_2}^c\right) > 0.$$

Note that  $\underline{F}_{\epsilon_2}^c$  is positive and not zero for any  $x \in \bar{\Omega}$ . Then we see from Lemma 2.4 that, for any  $d_1, \dots, d_m > 0$  and  $d_{m+1}, \dots, d_n \in (0, C_2)$ ,

$$r\left(\left(\bar{V}_{\epsilon_2}^c\right)^{-1} \underline{F}_{\epsilon_2}^c\right) \leq R_0 \leq r\left(\bar{F}_{\epsilon_2}^c \int_0^\infty \underline{T}_{\epsilon_2}^c(t)\phi dt\right) = r\left(\left(\underline{V}_{\epsilon_2}^c\right)^{-1} \bar{F}_{\epsilon_2}^c\right).$$

Let  $\underline{R}_0 = r((\bar{V}_{\epsilon_2}^c)^{-1} \underline{F}_{\epsilon_2}^c)$  and  $\bar{R}_0 = r((\underline{V}_{\epsilon_2}^c)^{-1} \bar{F}_{\epsilon_2}^c)$ . This completes the proof for Step 1.

Step 2. For any  $x \in \bar{\Omega}$ , denote

$$\begin{aligned}
 \mathcal{D}^x &= \{(u_1, \dots, u_n) : u_i = 0 \text{ for } i = 1, \dots, m, \\
 &\quad u_i \in [c_i(x) - \epsilon, c_i(x) + \epsilon] \text{ for } i = m + 1, \dots, n\}, \\
 \underline{V}_\epsilon^x &= \left( \min_{u \in \mathcal{D}^x} V_{ij}(x, u) \right)_{1 \leq i, j \leq m} = \left( \min_{u \in \mathcal{D}^x} \frac{\partial \mathcal{V}_i(x, u)}{\partial u_j} \right)_{1 \leq i, j \leq m}, \\
 (3.8) \quad \bar{V}_\epsilon^x &= \left( \max_{u \in \mathcal{D}^x} V_{ij}(x, u) \right)_{1 \leq i, j \leq m} = \left( \max_{u \in \mathcal{D}^x} \frac{\partial \mathcal{V}_i(x, u)}{\partial u_j} \right)_{1 \leq i, j \leq m}, \\
 \underline{F}_\epsilon^x &= \left( \min_{u \in \mathcal{D}^x} F_{ij}(x, u) \right)_{1 \leq i, j \leq m} = \left( \min_{u \in \mathcal{D}^x} \frac{\partial \mathcal{F}_i(x, u)}{\partial u_j} \right)_{1 \leq i, j \leq m}, \\
 \bar{F}_\epsilon^x &= \left( \max_{u \in \mathcal{D}^x} F_{ij}(x, u) \right)_{1 \leq i, j \leq m} = \left( \max_{u \in \mathcal{D}^x} \frac{\partial \mathcal{F}_i(x, u)}{\partial u_j} \right)_{1 \leq i, j \leq m}.
 \end{aligned}$$

We show that, for sufficiently small  $\epsilon > 0$ ,

$$\begin{aligned}
 \tilde{R}_0 &:= r \left( (d_I \Delta - \underline{V}_\epsilon^x)^{-1} \bar{F}_\epsilon^x \right) \rightarrow \tilde{R}_0^0 := \max_{x \in \bar{\Omega}} r \left( (\underline{V}_\epsilon^x)^{-1} \bar{F}_\epsilon^x \right), \\
 (3.9) \quad \check{R}_0 &:= r \left( (d_I \Delta - \bar{V}_\epsilon^x)^{-1} \underline{F}_\epsilon^x \right) \rightarrow \check{R}_0^0 := \max_{x \in \bar{\Omega}} r \left( (\bar{V}_\epsilon^x)^{-1} \underline{F}_\epsilon^x \right),
 \end{aligned}$$

as  $d_I = (d_1, \dots, d_m) \rightarrow (0, \dots, 0)$ .

We can view matrices  $-\underline{V}_\epsilon^x + a\bar{F}_\epsilon^x$  and  $-\bar{V}_\epsilon^x + a\underline{F}_\epsilon^x$  as matrix-valued functions of  $(x, \epsilon, a)$ . Then  $-\underline{V}_\epsilon^x + a\bar{F}_\epsilon^x$  and  $-\bar{V}_\epsilon^x + a\underline{F}_\epsilon^x$  are continuous and consequently uniformly continuous on  $\bar{\Omega} \times [0, \epsilon_2] \times [1/\bar{R}_0, 1/\underline{R}_0]$  (see Proposition A.1). This implies that

$$\begin{aligned}
 (3.10) \quad \lim_{\epsilon \rightarrow 0} (-\underline{V}_\epsilon^x + a\bar{F}_\epsilon^x) &= -V(x, c(x)) + aF(x, c(x)) \text{ and} \\
 \lim_{\epsilon \rightarrow 0} (-\bar{V}_\epsilon^x + a\underline{F}_\epsilon^x) &= -V(x, c(x)) + aF(x, c(x))
 \end{aligned}$$

uniformly for  $(x, a) \in \bar{\Omega} \times [1/\bar{R}_0, 1/\underline{R}_0]$ . Therefore, there exists  $\epsilon_3 < \epsilon_2$  such that for any  $\epsilon \in (0, \epsilon_3)$ , matrices  $-\underline{V}_\epsilon^x + a\bar{F}_\epsilon^x$  and  $-\bar{V}_\epsilon^x + a\underline{F}_\epsilon^x$  are irreducible for any  $a \in [1/\bar{R}_0, 1/\underline{R}_0]$  and  $x \in \bar{\Omega}$ . In this step, we always assume that  $\epsilon \in (0, \epsilon_3]$ . Clearly,

$$(3.11) \quad -\bar{V}_\epsilon^c \leq -\underline{V}_\epsilon^x \leq -\underline{V}_\epsilon^c.$$

Noticing that  $s(-\underline{V}_\epsilon^c), s(-\bar{V}_\epsilon^c) < 0$  and  $-\underline{V}_\epsilon^x$  is cooperative for any  $x \in \bar{\Omega}$ , we have  $s(d_I \Delta - \underline{V}_\epsilon^x) < 0$ .

Clearly,  $\tilde{R}_0 \in [\underline{R}_0, \bar{R}_0]$  and  $\tilde{R}_0 > 0$ . Let  $\tilde{\kappa} = 1/\tilde{R}_0$ , and it follows from Lemma 2.3 that  $\tilde{R}_0$  is an eigenvalue of  $(d_I \Delta - \underline{V}_\epsilon^x)^{-1} \bar{F}_\epsilon^x$  with a nonnegative eigenvector  $\tilde{\phi} = (\tilde{\phi}_1, \dots, \tilde{\phi}_m)$ . Clearly,  $\tilde{\kappa}$  can be viewed as a function of  $d_I$  (or, respectively,  $(d_1, \dots, d_m)$ ) and

$$d_I \Delta \tilde{\phi} - \underline{V}_\epsilon^x \tilde{\phi} + \tilde{\kappa}(d_1, \dots, d_m) \bar{F}_\epsilon^x \tilde{\phi} = 0.$$

Let  $\delta = \delta(d_1, \dots, d_m, a)$  be the principal eigenvalue of the auxiliary eigenvalue problem

$$(3.12) \quad \begin{cases} d_I \Delta \phi - \underline{V}_\epsilon^x \phi + a\bar{F}_\epsilon^x \phi = \delta \phi, & x \in \Omega, \\ \partial_\nu \phi = 0, & x \in \partial\Omega. \end{cases}$$

Note that  $\underline{V}_\epsilon^x + \tilde{\kappa}(d_1, \dots, d_m) \bar{F}_\epsilon^x$  is irreducible. Then  $\tilde{\phi} > 0$  and

$$\delta(d_1, \dots, d_m, \tilde{\kappa}(d_1, \dots, d_m)) = 0.$$

It follows from [24, Theorem 1.4] that

$$\lim_{(d_1, \dots, d_m) \rightarrow (0, \dots, 0)} \delta(d_1, \dots, d_m, a) = \max_{x \in \bar{\Omega}} \hat{\delta} \left( -\underline{V}_\epsilon^x + a \bar{F}_\epsilon^x \right).$$

Here  $\hat{\delta}(Q)$  represents the eigenvalue of matrix  $Q$  with greatest real part. Define

$$\delta(d_1, \dots, d_m, a) := \max_{x \in \bar{\Omega}} \hat{\delta} \left( -\underline{V}_\epsilon^x + a \bar{F}_\epsilon^x \right)$$

for  $(d_1, \dots, d_m) = (0, \dots, 0)$ . Then, for each  $a \in [1/\bar{R}_0, 1/\underline{R}_0]$ ,  $\delta(d_1, \dots, d_m, a)$  is a continuous function of  $(d_1, \dots, d_m)$  on  $\text{Int}(\mathbb{R}_+^m) \cup \{(0, \dots, 0)\}$ . It follows from Lemma 2.5 that  $\delta(d_1, \dots, d_m, a)$  is strictly increasing in  $a$  for each  $(d_1, \dots, d_m) > (0, \dots, 0)$ . Similarly, we see from Lemma 2.5 that, for each  $x \in \bar{\Omega}$ ,  $\hat{\delta}(-\underline{V}_\epsilon^x + a \bar{F}_\epsilon^x)$  is also strictly increasing in  $a$ . This implies that  $\delta(d_1, \dots, d_m, a)$  is also strictly increasing in  $a$  for  $(d_1, \dots, d_m) = (0, \dots, 0)$ . Since for any  $x \in \bar{\Omega}$ ,

$$\underline{V}_{\epsilon_2} \leq \underline{V}_\epsilon^x \leq \bar{V}_{\epsilon_2}, \quad \underline{F}_{\epsilon_2} \leq \bar{F}_\epsilon^x \leq \bar{F}_{\epsilon_2},$$

it follows from Step 1 that

$$\underline{R}_0 \leq r \left( (\underline{V}_\epsilon^x)^{-1} \bar{F}_\epsilon^x \right) \leq \bar{R}_0$$

for any  $x \in \bar{\Omega}$  and

$$(3.13) \quad \tilde{R}_0 = r \left( (d_I \Delta - \underline{V}_\epsilon^x)^{-1} \bar{F}_\epsilon^x \right) \in [\underline{R}_0, \bar{R}_0].$$

Noticing that, for each  $x \in \bar{\Omega}$ ,

$$\hat{\delta} \left( -\underline{V}_\epsilon^x + \frac{1}{r \left( (\underline{V}_\epsilon^x)^{-1} \bar{F}_\epsilon^x \right)} \bar{F}_\epsilon^x \right) = 0,$$

the monotonicity of  $\hat{\delta}(-\underline{V}_\epsilon^x + a \bar{F}_\epsilon^x)$  in  $a$  implies that, for any  $x \in \bar{\Omega}$ ,

$$\hat{\delta} \left( -\underline{V}_\epsilon^x + \frac{1}{\tilde{R}_0^0} \bar{F}_\epsilon^x \right) \leq 0,$$

where  $\tilde{R}_0^0$  is defined as in (3.9) and the equality holds if and only if  $x$  achieves the maximum point of  $r((\underline{V}_\epsilon^x)^{-1} \bar{F}_\epsilon^x)$ . Therefore, the monotonicity of  $\delta(0, \dots, 0, a)$  implies that the unique zero of

$$\delta(0, \dots, 0, a) = \max_{x \in \bar{\Omega}} \hat{\delta} \left( -\underline{V}_\epsilon^x + a \bar{F}_\epsilon^x \right) = 0$$

on  $[1/\bar{R}_0, 1/\underline{R}_0]$  is  $a = 1/\tilde{R}_0^0$ .

Now we claim that the first equation of (3.9) holds. If it is not true, then

$$\kappa(d_1, \dots, d_m) \not\rightarrow 1/\tilde{R}_0^0 \text{ as } (d_1, \dots, d_m) \rightarrow (0, \dots, 0).$$

Noticing that  $\kappa(d_1, \dots, d_m)$  is bounded from (3.13), we see that there exists a sequence  $\{(d_1^{(j)}, \dots, d_m^{(j)})\}_{j=1}^\infty$  and  $\kappa_0 (\neq 1/\tilde{R}_0^0) \in [1/\bar{R}_0, 1/\underline{R}_0]$  such that

$$(d_1^{(j)}, \dots, d_m^{(j)}) \rightarrow (0, \dots, 0), \quad \kappa_n := \kappa(d_1^{(j)}, \dots, d_m^{(j)}) \rightarrow \kappa_0 \text{ as } j \rightarrow \infty.$$

Without loss of generality, we assume that  $\kappa_0 < 1/\tilde{R}_0^0$ . Then there exist  $\tilde{\epsilon}$  and  $j_0$  such that  $\kappa_0 + \tilde{\epsilon} < 1/\tilde{R}_0^0$  and  $\kappa_j < \kappa_0 + \tilde{\epsilon}$  for any  $j > j_0$ . Then, for any  $j > j_0$ ,

$$0 = \delta \left( d_1^{(j)}, \dots, d_m^{(j)}, \kappa_j \right) < \delta \left( d_1^{(j)}, \dots, d_m^{(j)}, \kappa_0 + \tilde{\epsilon} \right),$$

which yields

$$0 \leq \lim_{j \rightarrow \infty} \delta \left( d_1^{(j)}, \dots, d_m^{(j)}, \kappa_0 + \tilde{\epsilon} \right) = \delta(0, \dots, 0, \kappa_0 + \tilde{\epsilon}) < 0.$$

This is a contradiction, and therefore the first equation of (3.9) holds. Similarly, we can prove that the second equation of (3.9) holds.

*Step 3.* We show that

$$\lim_{(d_1, \dots, d_n) \rightarrow (0, \dots, 0)} R_0 = \max_{x \in \bar{\Omega}} [r(-V^{-1}(x, c(x))F(x, c(x)))].$$

Clearly,  $(\underline{V}_\epsilon^x)^{-1} \bar{F}_\epsilon^x$  can be viewed as a matrix-valued function of  $(x, \epsilon)$ , where  $(x, \epsilon) \in \bar{\Omega} \times [0, \epsilon_3]$  and  $(\underline{V}_\epsilon^x)^{-1} \bar{F}_\epsilon^x$  is continuous on  $\bar{\Omega} \times [0, \epsilon_3]$  (see Proposition A.1). It follows from [20, section 2.5.7] that  $r((\underline{V}_\epsilon^x)^{-1} \bar{F}_\epsilon^x)$  is continuous on  $\bar{\Omega} \times [0, \epsilon_3]$  and consequently that  $r((\underline{V}_\epsilon^x)^{-1} \bar{F}_\epsilon^x)$  is uniformly continuous on  $\bar{\Omega} \times [0, \epsilon_3]$ . This implies that

$$\lim_{\epsilon \rightarrow 0} r \left( (\underline{V}_\epsilon^x)^{-1} \bar{F}_\epsilon^x \right) = r \left( -V^{-1}(x, c(x))F(x, c(x)) \right) \text{ in } C(\bar{\Omega}).$$

Then

$$\lim_{\epsilon \rightarrow 0} \tilde{R}_0^0 = \lim_{\epsilon \rightarrow 0} \max_{x \in \bar{\Omega}} r \left[ \left( (\underline{V}_\epsilon^x)^{-1} \bar{F}_\epsilon^x \right) \right] = R_0^c = \max_{x \in \bar{\Omega}} [r(-V^{-1}(x, c(x))F(x, c(x)))].$$

Similarly, we can prove that

$$\lim_{\epsilon \rightarrow 0} \check{R}_0^0 = R_0^c.$$

For any  $\epsilon \in (0, \epsilon_3)$ , there exists  $\delta > 0$  such that for any  $d_{m+1}, \dots, d_n < \delta$ ,

$$u_i^0(x) \in [c_i(x) - \epsilon, c_i(x) + \epsilon] \text{ for any } i = m+1, \dots, n \text{ and } x \in \bar{\Omega}.$$

Then

$$\check{R}_0 = r \left( \left( d_I \Delta - \bar{V}_\epsilon^x \right)^{-1} \bar{F}_\epsilon^x \right) \leq R_0 \leq \tilde{R}_0 = r \left( (d_I \Delta - \underline{V}_\epsilon^x)^{-1} \bar{F}_\epsilon^x \right)$$

for any  $d_1, \dots, d_m > 0$  and  $d_{m+1}, \dots, d_n < \delta$ . Therefore,

$$\check{R}_0^0 \leq \liminf_{(d_1, \dots, d_n) \rightarrow (0, \dots, 0)} R_0 \leq \limsup_{(d_1, \dots, d_n) \rightarrow (0, \dots, 0)} R_0 \leq \tilde{R}_0^0.$$

Taking  $\epsilon \rightarrow 0$ , we see that

$$\lim_{(d_1, \dots, d_n) \rightarrow (0, \dots, 0)} R_0 = R_0^c.$$

This completes the proof.  $\square$

*Remark 3.1.* In Theorem 3.1, we assume that there exists  $\epsilon_0 > 0$  such that, for any  $x \in \bar{\Omega}$ ,  $-\bar{V}_{\epsilon_0}^c$  is cooperative and  $\underline{F}_{\epsilon_0}^c$  is positive. In section 4, we will show that in some concrete examples, any off-diagonal entry in  $\bar{V}_\epsilon^c$ ,  $\bar{V}_\epsilon^c$  either equals zero or is strictly positive and that any entry of  $\bar{F}_\epsilon^c$  or  $\underline{F}_\epsilon^c$  is strictly positive. In that case we only need to assume that  $-\bar{V}_0^c$  is cooperative and  $\underline{F}_0^c$  is positive to obtain results in Theorem 3.1.

**3.2. Large diffusion rates.** In this subsection, we consider the asymptotic profile of  $R_0$  when  $(d_1, \dots, d_n) \rightarrow (\infty, \dots, \infty)$ . For this case, we impose an additional assumption:

(A8) The disease-free equilibrium  $(0, \dots, 0, u_{m+1}^0(x), \dots, u_n^0(x))$  (defined in (2.2)) satisfies

$$(3.14) \quad \begin{aligned} & \lim_{(d_{m+1}, \dots, d_n) \rightarrow (\infty, \dots, \infty)} (u_{m+1}^0(x), \dots, u_n^0(x)) \\ &= (\tilde{u}_{m+1}, \dots, \tilde{u}_n) \text{ in } C(\bar{\Omega}, \mathbb{R}^{n-m}), \end{aligned}$$

where  $\tilde{u}_k$  is a positive constant for  $k = m+1, \dots, n$ .

We will also show that this assumption is not restrictive and that it is satisfied for many kinds of epidemic models in the next section. Denote

$$(3.15) \quad \tilde{u} = (0, \dots, 0, \tilde{u}_{m+1}, \dots, \tilde{u}_n) \in \mathbb{R}^n,$$

and denote, for given sufficiently small  $\epsilon$  ( $0 < \epsilon < \min\{\tilde{u}_i : i = m+1, \dots, n\}$ ),

$$(3.16) \quad \begin{aligned} \mathcal{D} &= \{(u_1, \dots, u_n) : u_i = 0 \text{ for } i = 1, \dots, m, \\ &\quad u_i \in [\tilde{u}_i - \epsilon, \tilde{u}_i + \epsilon] \text{ for } i = m+1, \dots, n\}, \\ \underline{V}_\epsilon &= \left( \min_{x \in \bar{\Omega}, u \in \mathcal{D}} V_{ij}(x, u) \right)_{1 \leq i, j \leq m} = \left( \min_{x \in \bar{\Omega}, u \in \mathcal{D}} \frac{\partial \mathcal{V}_i(x, u)}{\partial u_j} \right)_{1 \leq i, j \leq m}, \\ \bar{V}_\epsilon &= \left( \max_{x \in \bar{\Omega}, u \in \mathcal{D}} V_{ij}(x, u) \right)_{1 \leq i, j \leq m} = \left( \max_{x \in \bar{\Omega}, u \in \mathcal{D}} \frac{\partial \mathcal{V}_i(x, u)}{\partial u_j} \right)_{1 \leq i, j \leq m}, \\ \underline{F}_\epsilon &= \left( \min_{x \in \bar{\Omega}, u \in \mathcal{D}} F_{ij}(x, u) \right)_{1 \leq i, j \leq m} = \left( \min_{x \in \bar{\Omega}, u \in \mathcal{D}} \frac{\partial \mathcal{F}_i(x, u)}{\partial u_j} \right)_{1 \leq i, j \leq m}, \\ \bar{F}_\epsilon &= \left( \max_{x \in \bar{\Omega}, u \in \mathcal{D}} F_{ij}(x, u) \right)_{1 \leq i, j \leq m} = \left( \max_{x \in \bar{\Omega}, u \in \mathcal{D}} \frac{\partial \mathcal{F}_i(x, u)}{\partial u_j} \right)_{1 \leq i, j \leq m}. \end{aligned}$$

Similar to subsection 3.1, we could also prove that  $\bar{F}_\epsilon$  and  $\bar{V}_\epsilon$  are monotone increasing for  $\epsilon \geq 0$  and that  $\underline{F}_\epsilon$  and  $\underline{V}_\epsilon$  is monotone decreasing for  $\epsilon \geq 0$ . Moreover, when  $\epsilon = 0$ ,

$$\begin{aligned} V_0 &= \left( \min_{x \in \bar{\Omega}} V_{ij}(x, \tilde{u}) \right)_{1 \leq i, j \leq m} = \left( \min_{x \in \bar{\Omega}} \frac{\partial \mathcal{V}_i(x, \tilde{u})}{\partial u_j} \right)_{1 \leq i, j \leq m}, \\ \bar{V}_0 &= \left( \max_{x \in \bar{\Omega}} V_{ij}(x, \tilde{u}) \right)_{1 \leq i, j \leq m} = \left( \max_{x \in \bar{\Omega}} \frac{\partial \mathcal{V}_i(x, \tilde{u})}{\partial u_j} \right)_{1 \leq i, j \leq m}, \\ F_0 &= \left( \min_{x \in \bar{\Omega}} F_{ij}(x, \tilde{u}) \right)_{1 \leq i, j \leq m} = \left( \min_{x \in \bar{\Omega}} \frac{\partial \mathcal{F}_i(x, \tilde{u})}{\partial u_j} \right)_{1 \leq i, j \leq m}, \\ \bar{F}_0 &= \left( \max_{x \in \bar{\Omega}} F_{ij}(x, \tilde{u}) \right)_{1 \leq i, j \leq m} = \left( \max_{x \in \bar{\Omega}} \frac{\partial \mathcal{F}_i(x, \tilde{u})}{\partial u_j} \right)_{1 \leq i, j \leq m}. \end{aligned}$$

Now we show the asymptotic profile of  $R_0$  as  $(d_1, \dots, d_n) \rightarrow (\infty, \dots, \infty)$ .

**THEOREM 3.2.** *Assume that (A1)–(A5) and (A8) hold,*

$$(3.17) \quad s(-\bar{V}_0) < 0, \quad s(-\underline{V}_0) < 0 \quad \text{and} \quad r(\bar{V}_0^{-1} \underline{F}_0) > 0,$$

and that there exists  $\epsilon_0 > 0$  such that  $-\bar{V}_{\epsilon_0}$  is cooperative and  $\underline{F}_{\epsilon_0}$  is positive, where  $\underline{V}_\epsilon$ ,  $\bar{V}_\epsilon$  and  $\underline{F}_\epsilon$  are defined in (3.16). Let

$$(3.18) \quad \begin{aligned} \check{V} &= \left( \int_{\Omega} V_{ij}(x, \tilde{u}) dx \right)_{1 \leq i, j \leq m} = \left( \int_{\Omega} \frac{\partial \mathcal{V}_i(x, \tilde{u})}{\partial u_j} dx \right)_{1 \leq i, j \leq m}, \\ \check{F} &= \left( \int_{\Omega} F_{ij}(x, \tilde{u}) dx \right)_{1 \leq i, j \leq m} = \left( \int_{\Omega} \frac{\partial \mathcal{F}_i(x, \tilde{u})}{\partial u_j} dx \right)_{1 \leq i, j \leq m}, \end{aligned}$$

where  $\tilde{u}$  is defined as in (3.15). If  $r(\check{V}^{-1}\check{F})$  is the unique eigenvalue of  $\check{V}^{-1}\check{F}$  with an eigenvector in  $\mathbb{R}_+^m \setminus \{\mathbf{0}\}$ , then

$$\lim_{(d_1, \dots, d_n) \rightarrow (\infty, \dots, \infty)} R_0 = r(\check{V}^{-1}\check{F}).$$

*Proof.* As in the Step 1 of Theorem 3.1, we could prove that there exist positive constants  $\underline{R}_0$ ,  $\bar{R}_0$ , and  $C_2$  such that  $R_0 \in [\underline{R}_0, \bar{R}_0]$  for any  $d_1, \dots, d_m > 0$  and  $d_{m+1}, \dots, d_n > C_2$ . Let  $\kappa = 1/R_0$ , and  $\kappa$  can be viewed as function of  $(d_1, \dots, d_n)$ . Since  $\kappa(d_1, \dots, d_n)$  is bounded for any  $d_1, \dots, d_m > 0$  and  $d_{m+1}, \dots, d_n > C_2$ , then, for any sequence  $\{(d_1^{(j)}, \dots, d_n^{(j)})\}_{j=1}^\infty$  satisfying  $(d_1^{(j)}, \dots, d_n^{(j)}) \rightarrow (\infty, \dots, \infty)$  as  $j \rightarrow \infty$ , there exists a subsequence  $\{(d_1^{(j_k)}, \dots, d_n^{(j_k)})\}_{j=1}^\infty$  such that  $\lim_{k \rightarrow \infty} \kappa(d_1^{(j_k)}, \dots, d_n^{(j_k)})$  exists and is positive, which is denoted by  $\kappa^*$ . For convenience, we denote  $d_i^{(j_k)}$  by  $d_i^{(k)}$  for each  $k \geq 1$  and  $i = 1, \dots, n$ . Without loss of generality, we assume that  $d_i^{(k)} \geq C_2$  for any  $k \geq 1$  and  $i = m+1, \dots, n$ .

Let  $\hat{\phi}^{(k)} = (\hat{\phi}_1^{(k)}, \dots, \hat{\phi}_m^{(k)})^T \geq (0, \dots, 0)^T$  be the corresponding eigenvector of operator

$$-(d_I \Delta - V(x, u^0(x))^{-1} F(x, u^0(x))$$

with respect to eigenvalue  $R_0(d_1^{(k)}, \dots, d_n^{(k)})$ , where  $\|\hat{\phi}^{(k)}\|_\infty = 1$  for each  $k \geq 1$ . That is, for  $i = 1, \dots, m$ ,

$$\Delta \hat{\phi}_i^{(k)} + \frac{1}{d_i^{(k)}} \left[ - \sum_{j=1}^m V_{ij}(x, u^0(x)) \hat{\phi}_j^{(k)} + \kappa(d_1^{(k)}, \dots, d_n^{(k)}) \sum_{j=1}^m F_{ij}(x, u^0(x)) \hat{\phi}_j^{(k)} \right] = 0,$$

where  $u^0(x)$  depends on  $(d_{m+1}^{(k)}, \dots, d_n^{(k)})$ . Then it follows from the  $L^p$  theory that there exists a subsequence  $\{k_l\}_{l=1}^\infty$  such that  $\lim_{l \rightarrow \infty} \hat{\phi}_i^{(k_l)} = c_i^\infty$  in  $C(\bar{\Omega}, \mathbb{R})$  for each  $i = 1, \dots, m$ , where  $c_i^\infty$  is a nonnegative constant and  $c^\infty := (c_1^\infty, \dots, c_m^\infty)^T$  satisfies

$$\|c^\infty\|_\infty = 1 \text{ and } \check{V}c^\infty = \kappa^* \check{F}c^\infty.$$

Then  $1/\kappa^* = r(\check{V}^{-1}\check{F})$ . This completes the proof.  $\square$

*Remark 3.2.* We remark that there always exists a decomposition

$$\{(\mathcal{F}_i(x, u), \mathcal{V}_i(x, u))\}_{i=1}^n$$

of  $\{f_i(x, u)\}_{i=1}^n$  such that

$$f_i(x, u) = \mathcal{F}_i(x, u) - \mathcal{V}_i(x, u) \text{ for } i = 1, \dots, n \text{ and rank } (F) = 1.$$

Consequently,  $r(\check{V}^{-1}\check{F})$  is the unique eigenvalue of  $\check{V}^{-1}\check{F}$  with an eigenvector in  $\mathbb{R}_+^n \setminus \{\mathbf{0}\}$ . Moreover, different decompositions of  $\{f_i(x, u)\}_{i=1}^n$  will not change the

portion of parameter space that the disease vanishes or spreads. Actually, if there exist two decompositions

$$\left\{ \left( \mathcal{F}_i^{(j)}(x, u), \mathcal{V}_i^{(j)}(x, u) \right) \right\}_{i=1}^n \quad (j = 1, 2),$$

then there exist two basic reproduction numbers  $R_0^{(1)}$  and  $R_0^{(2)}$ . It follows from [41, Theorem 3.5] that  $R_0^{(1)} - 1$  and  $R_0^{(2)} - 1$  have the same signs.

**Remark 3.3.** In Theorem 3.2, we assume that there exists  $\epsilon_0 > 0$  such that, for any  $x \in \bar{\Omega}$ ,  $-\bar{V}_{\epsilon_0}$  is cooperative and  $\underline{F}_{\epsilon_0}$  is positive. In section 4, we will show that in some concrete examples, any off-diagonal entry in  $\bar{V}_\epsilon$ ,  $\bar{V}_\epsilon$  equals zero or is strictly positive and that any entry of  $\bar{F}_\epsilon$  or  $\underline{F}_\epsilon$  is strictly positive. Therefore, we only need to show that  $-\bar{V}_0$  is cooperative and  $\underline{F}_0$  is positive to obtain results in Theorem 3.2.

**3.3. Discussion.** In this subsection, we show that if assumptions (A7) and (A8) are both satisfied, we can obtain an additional asymptotic profile for  $R_0$  other than the ones in Theorems 3.1 and 3.2. The proof is similar to the ones given in subsections 3.1 and 3.2, and here we omit the proof.

**PROPOSITION 3.3.** *Assume that (A1)–(A5), (A7)–(A8), and (3.17) hold and that there exists  $\epsilon_0 > 0$  such that  $-\bar{V}_{\epsilon_0}$  is cooperative and  $\underline{F}_{\epsilon_0}$  is positive, where  $\bar{V}_\epsilon$ ,  $\bar{V}_\epsilon$ , and  $\underline{F}_\epsilon$  are defined in (3.16). If the matrix  $-V(x, \tilde{u}) + aF(x, \tilde{u})$  is irreducible for any  $a > 0$  and  $x \in \bar{\Omega}$ , where  $\tilde{u}$  is defined in (3.15), then*

$$\lim_{(d_1, \dots, d_m, d_{m+1}, \dots, d_n) \rightarrow (0, \dots, 0, \infty, \dots, \infty)} R_0 = R_0^{\tilde{u}} := \max_{x \in \bar{\Omega}} [r(-V^{-1}(x, \tilde{u})F(x, \tilde{u}))],$$

where  $(d_1, \dots, d_m, d_{m+1}, \dots, d_n) \rightarrow (0, \dots, 0, \infty, \dots, \infty)$  means  $\max_{1 \leq j \leq m} d_j \rightarrow 0$  and  $\min_{m+1 \leq j \leq n} d_j \rightarrow \infty$ .

**PROPOSITION 3.4.** *Assume that (A1)–(A5), (A7)–(A8), and (3.4) hold and that there exists  $\epsilon_0 > 0$  such that  $-\bar{V}_{\epsilon_0}^c$  is cooperative and  $\underline{F}_{\epsilon_0}^c$  is positive, where  $\bar{V}_\epsilon^c$ ,  $\bar{V}_\epsilon^c$ , and  $\underline{F}_\epsilon^c$  are defined in (3.3). Let*

$$(3.19) \quad \begin{aligned} \check{V}^c &= \left( \int_{\Omega} V_{ij}(x, c(x)) dx \right)_{1 \leq i, j \leq m} = \left( \int_{\Omega} \frac{\partial \mathcal{V}_i(x, c(x))}{\partial u_j} dx \right)_{1 \leq i, j \leq m}, \\ \check{F}^c &= \left( \int_{\Omega} F_{ij}(x, c(x)) dx \right)_{1 \leq i, j \leq m} = \left( \int_{\Omega} \frac{\partial \mathcal{F}_i(x, c(x))}{\partial u_j} dx \right)_{1 \leq i, j \leq m}, \end{aligned}$$

where  $c(x)$  is defined in (3.2). If  $r((\check{V}^c)^{-1}\check{F}^c)$  is the unique eigenvalue of  $(\check{V}^c)^{-1}\check{F}^c$  with an eigenvector in  $\mathbb{R}_+^m \setminus \{\mathbf{0}\}$ , then

$$\lim_{(d_1, \dots, d_m, d_{m+1}, \dots, d_n) \rightarrow (\infty, \dots, \infty, 0, \dots, 0)} R_0 = r((\check{V}^c)^{-1}\check{F}^c),$$

where  $(d_1, \dots, d_m, d_{m+1}, \dots, d_n) \rightarrow (\infty, \dots, \infty, 0, \dots, 0)$  means  $\min_{1 \leq j \leq m} d_j \rightarrow \infty$  and  $\max_{m+1 \leq j \leq n} d_j \rightarrow 0$ .

Finally, we will consider the effect of diffusion rates on the basic reproduction number  $R_0$ . It follows from [2] that  $R_0$  is monotone decreasing with respect to the diffusion rates for an SIS epidemic model; that is, large dispersal could reduce the spread of the disease. It is of interest that  $R_0$  is not always monotone decreasing with respect to the diffusion rates; see [40] for an example of diffusive SEIRS model. Motivated by [40], we show that under certain conditions,  $R_0$  is not always monotone decreasing with respect to the diffusion rates for model (1.3).

PROPOSITION 3.5. *Assume that  $(d_1, d_2, \dots, d_n) = d(\epsilon_1, \epsilon_2, \dots, \epsilon_n)$  and that all the assumptions in Theorems 3.1 and 3.2 are satisfied. Then the following two statements hold:*

(i)

$$\lim_{d \rightarrow 0} R_0(d) = R_0^c \text{ and } \lim_{d \rightarrow \infty} R_0(d) = r(\check{V}^{-1} \check{F}),$$

where  $R_0^c$  and  $r(\check{V}^{-1} \check{F})$  are defined as in Theorems 3.1 and 3.2, respectively.

(ii) *If*

$$(3.20) \quad R_0^c < r(\check{V}^{-1} \check{F}),$$

then there exist  $M_2 > M_1 > 0$  such that for any  $\tilde{d}_2 > M_2$  and  $\tilde{d}_1 < M_1$ ,  $R_0(\tilde{d}_2) > R_0(\tilde{d}_1)$ .

We will show that (3.20) is satisfied for a concrete epidemic model in section 4.

**4. Applications.** In this section, we give some examples to show that the general results in Theorems 3.1 and 3.2 can be applied to many different reaction-diffusion epidemic models.

**4.1. Vector-host epidemic models.** We consider two vector-host epidemic models. The first is given by [15] to model the outbreak of Zika in Rio de Janeiro:

(4.1)

$$\begin{cases} \partial H_i / \partial t = \delta_1 \Delta H_i - \lambda(x) H_i + \sigma_1(x) H_u(x) V_i, & x \in \Omega, t > 0, \\ \partial V_i / \partial t = \delta_2 \Delta V_i + \sigma_2(x) V_u H_i - \mu(x) (V_u + V_i) V_i, & x \in \Omega, t > 0, \\ \partial V_u / \partial t = \delta_3 \Delta V_u - \sigma_2(x) V_u H_i + \beta(x) (V_u + V_i) - \mu(x) (V_u + V_i) V_u, & x \in \Omega, t > 0, \\ \partial_\nu H_i = \partial_\nu V_i = \partial_\nu V_u = 0, & x \in \partial\Omega, t > 0, \end{cases}$$

where  $H_u(x)$ ,  $H_i(x, t)$ ,  $V_i(x, t)$ , and  $V_u(x, t)$  are the densities of uninfected hosts, infected hosts, infected vectors, and uninfected vectors at space  $x$  and time  $t$ , respectively;  $\Omega$  is a bounded domain with smooth boundary  $\partial\Omega$ ;  $\nu$  is the outward unit normal vector on  $\partial\Omega$ ;  $\delta_1, \delta_2, \delta_3$  are positive constants; and  $\lambda(x)$ ,  $H_u(x)$ ,  $\sigma_i(x)$  ( $i = 1, 2$ ),  $\beta(x)$ , and  $\mu(x)$  are strictly positive and belong to  $C^\alpha(\overline{\Omega})$ . The asymptotic properties of  $R_0$  for this model have been investigated in [32]; see also [31] for the global dynamics. We revisit it to show that the main results in section 3 can be applied to this model to determine the asymptotic behavior of basic reproduction number  $R_0$ .

Letting

$$n = 3, m = 2 \text{ and } (u_1, u_2, u_3) = (H_i, V_i, V_u),$$

we could use the framework in section 3. It follows from [32] that model (4.1) has a unique disease-free steady state  $u^0(x) = (0, 0, \hat{V}(x))$ , where  $\hat{V}(x)$  satisfies

$$(4.2) \quad \lim_{\delta_3 \rightarrow 0} \hat{V}(x) = \frac{\beta(x)}{\mu(x)} \text{ and } \lim_{\delta_3 \rightarrow \infty} \hat{V}(x) = \frac{\int_\Omega \beta(x) dx}{\int_\Omega \mu(x) dx} \text{ in } C(\overline{\Omega}).$$

This implies that assumptions (A7) and (A8) are satisfied. For model (4.1),

$$(4.3) \quad V(x, u) = \begin{pmatrix} \lambda(x) & -\sigma_1(x) H_u(x) \\ 0 & \mu(x) u_3 \end{pmatrix}, \quad F(x, u) = \begin{pmatrix} 0 & 0 \\ \sigma_2(x) u_3 & 0 \end{pmatrix},$$

where  $u = (u_1, u_2, u_3)^T$ , and

$$(4.4) \quad B = (\delta_1 \Delta, \delta_2 \Delta)^T - V(x, u^0(x)).$$

Then the basic reproduction number is given by

$$(4.5) \quad R_0 = r \left( -F(x, u^0(x)) B^{-1} \right).$$

Moreover, for model (4.1),

$$\tilde{u} = \left( 0, 0, \frac{\int_{\Omega} \beta dx}{\int_{\Omega} \mu dx} \right) \text{ and } c(x) = \left( 0, 0, \frac{\beta(x)}{\mu(x)} \right),$$

and a direct computation implies that all the assumptions of Theorems 3.1 and 3.2 and Propositions 3.3 and 3.4 are satisfied. Then we have the following results.

**PROPOSITION 4.1.** *For model (4.1), the following statements hold:*

(i)

$$\lim_{(\delta_1, \delta_2, \delta_3) \rightarrow (\infty, \infty, \infty)} R_0 = \frac{\int_{\Omega} \sigma_1 H_u dx \int_{\Omega} \sigma_2 dx}{\int_{\Omega} \lambda dx \int_{\Omega} \mu dx},$$

(ii)

$$\lim_{(\delta_1, \delta_2, \delta_3) \rightarrow (0, 0, 0)} R_0 = \max_{x \in \bar{\Omega}} \frac{\sigma_1(x) \sigma_2(x) H_u(x)}{\lambda(x) \mu(x)},$$

(iii)

$$\lim_{(\delta_1, \delta_2, \delta_3) \rightarrow (\infty, \infty, 0)} R_0 = \frac{\int_{\Omega} \sigma_1 H_u dx \int_{\Omega} \frac{\sigma_2 \beta}{\mu} dx}{\int_{\Omega} \lambda dx \int_{\Omega} \beta dx},$$

(iv)

$$\lim_{(\delta_1, \delta_2, \delta_3) \rightarrow (0, 0, \infty)} R_0 = \max_{x \in \bar{\Omega}} \frac{\sigma_1(x) \sigma_2(x) H_u(x)}{\lambda(x) \mu(x)}.$$

Next we consider another vector-host epidemic model:

$$(4.6) \quad \begin{cases} I_t = d_1 \Delta I + \beta_s(x) SV - (b(x) + \gamma(x)) I, & x \in \Omega, t > 0, \\ V_t = d_2 \Delta V + \beta_m(x) MI - c(x) V, & x \in \Omega, t > 0, \\ S_t = d_3 \Delta S + \lambda_1(x) - b(x) S + \gamma(x) I - \beta_s(x) SV, & x \in \Omega, t > 0, \\ M_t = d_4 \Delta M + \lambda_2(x) - c(x) M - \beta_m(x) MI, & x \in \Omega, t > 0, \\ \partial_{\nu} I = \partial_{\nu} V = \partial_{\nu} S = \partial_{\nu} M = 0, & x \in \partial\Omega, t > 0, \end{cases}$$

where  $I(x, t)$ ,  $V(x, t)$ ,  $S(x, t)$ , and  $M(x, t)$  are the densities of infected hosts, infected vectors, susceptible hosts, and susceptible vectors at space  $x$  and time  $t$ , respectively;  $\Omega$  is a bounded domain with smooth boundary  $\partial\Omega$ ;  $\nu$  is the outward unit normal vector on  $\partial\Omega$ ;  $d_1, d_2, d_3, d_4$  are positive constants; and  $\lambda_i(x)$  ( $i = 1, 2$ ),  $\beta_s(x)$ ,  $\beta_m(x)$ ,  $b(x)$ ,  $\gamma(x)$ , and  $c(x)$  are strictly positive and belong to  $C^{\alpha}(\bar{\Omega})$ . The model was originally an ODE model (i.e.,  $d_1 = d_2 = d_3 = d_4 = 0$ ) proposed by Feng and Velasco-Hernández [14], and  $R_0$  of the ODE model was obtained in [14, 43].

Letting

$$n = 4, m = 2 \text{ and } (u_1, u_2, u_3, u_4) = (I, V, S, M),$$

we could use the framework in section 3. The model (4.6) has a unique disease-free steady state

$$u^0(x) = (0, 0, \hat{S}(x), \hat{M}(x)),$$

where  $(\hat{S}(x), \hat{M}(x))$  satisfies

$$(4.7) \quad \begin{aligned} \lim_{(d_3, d_4) \rightarrow (0, 0)} (\hat{S}(x), \hat{M}(x)) &= \left( \frac{\lambda_1(x)}{b(x)}, \frac{\lambda_2(x)}{c(x)} \right) \\ \lim_{(d_3, d_4) \rightarrow (0, 0)} (\hat{S}(x), \hat{M}(x)) &= \left( \frac{\int_{\Omega} \lambda_1 dx}{\int_{\Omega} b dx}, \frac{\int_{\Omega} \lambda_2 dx}{\int_{\Omega} c dx} \right) \text{ in } C(\bar{\Omega}, \mathbb{R}^2). \end{aligned}$$

This implies that assumptions (A7) and (A8) are satisfied. A direct computation implies that, for model (4.6),

$$(4.8) \quad F(x, u) = \begin{pmatrix} 0 & \beta_s(x)u_3 \\ \beta_m(x)u_4 & 0 \end{pmatrix}, \quad V(x, u) = \begin{pmatrix} b(x) + \gamma(x) & 0 \\ 0 & c(x) \end{pmatrix}$$

for  $u = (u_1, u_2, u_3, u_4)^T$ , and

$$B = (d_1 \Delta, d_2 \Delta)^T - V(x, u^0(x)).$$

Then the basic reproduction number is also given by (4.5). Finally, for model (4.6),

$$\tilde{u} = \left( 0, 0, \frac{\int_{\Omega} \lambda_1 dx}{\int_{\Omega} b dx}, \frac{\int_{\Omega} \lambda_2 dx}{\int_{\Omega} c dx} \right) \text{ and } c(x) = \left( 0, 0, \frac{\lambda_1(x)}{b(x)}, \frac{\lambda_2(x)}{c(x)} \right).$$

It is easy to check that all the assumptions of Theorems 3.1 and 3.2 and Propositions 3.3 and 3.4 are satisfied. Then we have the following results.

PROPOSITION 4.2. *For model (4.6), the following statements hold:*

(i)

$$\lim_{(d_1, d_2, d_3, d_4) \rightarrow (\infty, \infty, \infty, \infty)} R_0 = \sqrt{\frac{\int_{\Omega} \lambda_1 dx \int_{\Omega} \lambda_2 dx \int_{\Omega} \beta_s dx \int_{\Omega} \beta_m dx}{\int_{\Omega} b dx (\int_{\Omega} c dx)^2 \int_{\Omega} (b + \gamma) dx}},$$

(ii)

$$\lim_{(d_1, d_2, d_3, d_4) \rightarrow (0, 0, 0, 0)} R_0 = \max_{x \in \bar{\Omega}} \sqrt{\frac{\lambda_1(x)\lambda_2(x)\beta_s(x)\beta_m(x)}{b(x)c^2(x)(b(x) + \gamma(x))}},$$

(iii)

$$\lim_{(d_1, d_2, d_3, d_4) \rightarrow (\infty, \infty, 0, 0)} R_0 = \frac{1}{\sqrt{\int_{\Omega} c dx \int_{\Omega} (b + \gamma) dx}} \sqrt{\int_{\Omega} \frac{\beta_s \lambda_1}{b} dx \int_{\Omega} \frac{\beta_m \lambda_2}{c} dx},$$

(iv)

$$\lim_{(d_1, d_2, d_3, d_4) \rightarrow (0, 0, 0, 0)} R_0 = \sqrt{\frac{\int_{\Omega} \lambda_1 dx \int_{\Omega} \lambda_2 dx}{\int_{\Omega} b dx \int_{\Omega} c dx}} \max_{x \in \bar{\Omega}} \sqrt{\frac{\beta_s(x)\beta_m(x)}{c(x)(b(x) + \gamma(x))}}.$$

We remark that (3.20) can be satisfied for models (4.1) and (4.6). For example, we consider model (4.1) and choose  $\Omega = (\epsilon, 1 - \epsilon)$ , where  $0 < \epsilon \ll 1$ . Let  $\lambda(x) \equiv \tilde{\lambda}$ ,  $\mu(x) \equiv \tilde{\mu}$  and  $\sigma_1(x) = \tilde{\sigma}_1$ , where  $\tilde{\lambda}, \tilde{\mu}, \tilde{\sigma}_1$  are positive constants,  $H_u(x) = x^2$ , and  $\sigma_2(x) = 1 - x$ . Then, for sufficiently small  $\epsilon$ ,

$$\max_{x \in \bar{\Omega}} \frac{\sigma_1(x)\sigma_2(x)H_u(x)}{\lambda(x)\mu(x)} < \frac{\int_{\Omega} \sigma_1 H_u dx \int_{\Omega} \sigma_2 dx}{\int_{\Omega} \lambda dx \int_{\Omega} \mu dx}.$$

Let  $(\delta_1, \delta_2, \delta_3) = d(\epsilon_1, \epsilon_2, \epsilon_3)$ . Then we see from Proposition 3.5 that there exist  $M_2 > M_1 > 0$  such that for any  $\tilde{d}_2 > M_2$  and  $\tilde{d}_1 < M_1$ ,  $R_0(\tilde{d}_2) > R_0(\tilde{d}_1)$ . Therefore, large dispersal could promote the spread of the disease.

**4.2. Staged progression model.** In this subsection, we consider a staged progression model proposed in [19]. This model has a single uninfected compartment, and the infected individuals could pass through several stages of the disease with changing infectivity. It could be applied to model the transmission of many diseases, such as HIV/AIDS; see [19]. The original model was an ODE model, and the reproduction number was obtained in [17, 43]. Here we consider the associated reaction-diffusion case:

$$\begin{cases} \partial I_1 / \partial t = d_1 \Delta I_1 + h(N) (\sum_{k=1}^m \beta_k(x) S I_k) - (\nu_1(x) + \gamma_1(x)) I_1, & x \in \Omega, t > 0, \\ \partial I_i / \partial t = d_i \Delta I_i + \nu_{i-1}(x) I_{i-1} - (\nu_i(x) + \gamma_i(x)) I_i, & x \in \Omega, t > 0, 2 \leq i \leq m, \\ \partial I_{m+1} / \partial t = d_{m+2} \Delta I_{m+1} + \nu_m(x) I_m - \gamma_{m+1} I_{m+1}, & x \in \Omega, t > 0, \\ \partial S / \partial t = d_{m+1} \Delta S + \lambda(x) - b(x) S - h(N) (\sum_{k=1}^m \beta_k(x) S I_k), & x \in \Omega, t > 0, \\ \partial_\nu S = \partial_\nu I_i = 0, & x \in \partial\Omega, t > 0, 1 \leq i \leq m, \end{cases}$$

where  $N = S + \sum_{i=1}^m I_i$ ,  $h(N) = N^{-\alpha}$  with  $\alpha \in [0, 1]$ ,  $S(x, t)$  is the density of the susceptible individuals,  $I_i (i = 1, \dots, m+1)$  is the density of the infected individuals at stage  $i$ ,  $\Omega$  is a bounded domain with smooth boundary  $\partial\Omega$ ,  $\nu$  is the outward unit normal vector on  $\partial\Omega$ ,  $d_i (i = 1, \dots, m+2)$  are positive constants, and  $\lambda(x)$ ,  $b(x)$ ,  $\beta_i(x) (i = 1, \dots, m)$ ,  $\nu_i(x) (i = 1, \dots, m)$ , and  $\gamma_i(x) (i = 1, \dots, m+1)$  are strictly positive and belong to  $C^\alpha(\overline{\Omega})$ . Note that  $I_{m+1}$  decouples from the others, and consequently we could consider the following model:

$$(4.9) \quad \begin{cases} \partial I_1 / \partial t = d_1 \Delta I_1 + h(N) (\sum_{k=1}^m \beta_k(x) S I_k) - (\nu_1(x) + \gamma_1(x)) I_1, & x \in \Omega, t > 0, \\ \partial I_i / \partial t = d_i \Delta I_i + \nu_{i-1}(x) I_{i-1} - (\nu_i(x) + \gamma_i(x)) I_i, & x \in \Omega, t > 0, 2 \leq i \leq m, \\ \partial S / \partial t = d_{m+1} \Delta S + \lambda(x) - b(x) S - h(N) (\sum_{k=1}^m \beta_k(x) S I_k), & x \in \Omega, t > 0, \\ \partial_\nu S = \partial_\nu I_i = 0, & x \in \partial\Omega, t > 0, 1 \leq i \leq m. \end{cases}$$

Letting

$$n = m + 1, (u_1, \dots, u_m) = (I_1, \dots, I_m), \text{ and } u_{m+1} = S,$$

we could use the framework in section 3. The model (4.9) has a unique disease-free steady state

$$u^0(x) = (0, \dots, 0, \hat{S}(x)),$$

where  $\hat{S}(x)$  satisfies

$$(4.10) \quad \lim_{d_{m+1} \rightarrow 0} \hat{S}(x) = \frac{\lambda(x)}{b(x)} \text{ and } \lim_{d_{m+1} \rightarrow \infty} \hat{S}(x) = \frac{\int_\Omega \lambda dx}{\int_\Omega b dx} \text{ in } C(\overline{\Omega}, \mathbb{R}).$$

This implies that assumptions (A7) and (A8) are satisfied. For model (4.9),

$$V(x, u) = (V_{ij}(u))_{1 \leq i, j \leq m} \text{ and } F(x, u) = (F_{ij}(x, u))_{1 \leq i, j \leq m},$$

where for  $u = (u_1, \dots, u_m)^T$ ,

$$F_{ij}(x, u) = \begin{cases} h\left(\sum_{k=1}^{m+1} u_k\right) \beta_j(x) u_{m+1} \\ + h'\left(\sum_{k=1}^{m+1} u_k\right) \left(\sum_{k=1}^m \beta_k(x) u_k\right) u_{m+1} & i = 1, 1 \leq j \leq m, \\ 0 & \text{otherwise,} \end{cases}$$

$$V_{ij}(x, u) = \begin{cases} \nu_i(x) + \gamma_i(x) & 1 \leq i \leq m, j = i, \\ -\nu_{i-1}(x) & 2 \leq i \leq m, j = i-1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$B = (d_1 \Delta, \dots, d_m \Delta)^T - V(x, u^0(x)).$$

And the basic reproduction number is given by (4.5). Also, for model (4.9),

$$\tilde{u} = \left(0, \dots, 0, \frac{\int_{\Omega} \lambda dx}{\int_{\Omega} b dx}\right) \text{ and } c(x) = \left(0, \dots, 0, \frac{\lambda(x)}{b(x)}\right).$$

Since this model is more complex, we show that all the assumptions of Theorems 3.1 and 3.2 are satisfied.

LEMMA 4.3. *The following statements hold:*

- (i) *For any  $a > 0$  and  $x \in \bar{\Omega}$ ,  $-V(x, c(x)) + aF(x, c(x))$  is irreducible.*
- (ii)  *$r(\check{V}^{-1}\check{F})$  is the unique positive eigenvalue of  $\check{V}^{-1}\check{F}$ , where  $\check{V}^{-1}$  and  $\check{F}$  are defined as in (3.18).*

*Proof.* Let  $Q_{ij}(x) = -V_{ij}(x, c(x)) + aF_{ij}(x, c(x))$ . Then a direct computation implies that

$$(4.11) \quad Q_{ij}(x) = \begin{cases} a\beta_1(x) \frac{\lambda(x)}{b(x)} h\left(\frac{\lambda(x)}{b(x)}\right) - \nu_1(x) - \gamma_1(x) & i = 1, j = 1, \\ a\beta_j(x) \frac{\lambda(x)}{b(x)} h\left(\frac{\lambda(x)}{b(x)}\right) & i = 1, 2 \leq j \leq m, \\ -\nu_i(x) - \gamma_i(x) & 2 \leq i \leq m, j = i, \\ \nu_{i-1}(x) & 2 \leq i \leq m, j = i-1, \\ 0 & \text{otherwise.} \end{cases}$$

For  $i = 1, 2 \leq j \leq m$ ,

$$Q_{1j} = a\beta_j(x) \frac{\lambda(x)}{b(x)} h\left(\frac{\lambda(x)}{b(x)}\right) \neq 0,$$

and for any  $2 \leq i \leq m, j > i$ ,

$$Q_{i(i-1)} \cdots Q_{21} Q_{1j} = a\nu_{i-1}(x) \cdots \nu_1(x) \beta_j(x) \frac{\lambda(x)}{b(x)} h\left(\frac{\lambda(x)}{b(x)}\right) \neq 0.$$

Similarly, for  $1 \leq j \leq m, i > j$ ,

$$Q_{i(i-1)} Q_{(i-1)(i-2)} \cdots Q_{(j+1)j} = \nu_{i-1}(x) \cdots \nu_j(x) \neq 0.$$

Therefore,  $-V(x, c(x)) + aF(x, c(x))$  is irreducible for any  $a > 0$  and  $x \in \bar{\Omega}$ . This completes the proof of part (i).

Let  $\check{V}^{-1} = (\alpha_{ij})_{1 \leq i, j \leq m}$ . From [43] and a direct computation, we see that

$$(4.12) \quad \alpha_{ij} = \begin{cases} 0 & 1 \leq i \leq m, j > i, \\ \frac{1}{\int_{\Omega}(\nu_i + \gamma_i)dx} & 1 \leq i \leq m, j = i, \\ \frac{\prod_{k=j}^{i-1} \int_{\Omega} \nu_k dx}{\prod_{k=j}^i \int_{\Omega} (\nu_k + \gamma_k) dx} & 1 \leq i \leq m, j < i. \end{cases}$$

Let  $\check{F}\check{V}^{-1} = (\tilde{\alpha}_{ij})_{1 \leq i, j \leq m}$ . Then  $\tilde{\alpha}_{ij} = 0$  for any  $2 \leq i \leq m$  and  $1 \leq j \leq m$  and

$$(4.13) \quad \tilde{\alpha}_{11} = \left( \sum_{j=1}^m \frac{\int_{\Omega} \beta_j dx \prod_{k=1}^{j-1} \int_{\Omega} \nu_k dx}{\prod_{k=1}^j \int_{\Omega} (\nu_k + \gamma_k) dx} \right) \frac{\int_{\Omega} \lambda dx}{\int_{\Omega} b dx} h \left( \frac{\int_{\Omega} \lambda dx}{\int_{\Omega} b dx} \right).$$

Therefore,  $r(\check{V}^{-1}\check{F}) = \tilde{\alpha}_{11}$  is the unique positive eigenvalue of  $\check{V}^{-1}\check{F}$ . This completes the proof of part (ii).  $\square$

The other assumptions of Theorems 3.1 and 3.2 and Propositions 3.3 and 3.4 are easy to verify, and we omit the proof. Then we have the following results.

**PROPOSITION 4.4.** *Let  $R_0$  be the basic reproduction number of model (4.9). Then*

(i)

$$\lim_{(d_1, \dots, d_{m+1}) \rightarrow (0, \dots, 0)} R_0 = \max_{x \in \bar{\Omega}} \left( \sum_{j=1}^m \frac{\beta_j(x) \prod_{k=1}^{j-1} \nu_k(x)}{\prod_{k=1}^j (\nu_k(x) + \gamma_k(x))} \right) \frac{\lambda(x)}{b(x)} h \left( \frac{\lambda(x)}{b(x)} \right),$$

(ii)

$$\lim_{(d_1, \dots, d_{m+1}) \rightarrow (\infty, \dots, \infty)} R_0 = \left( \sum_{j=1}^m \frac{\int_{\Omega} \beta_j dx \prod_{k=1}^{j-1} \int_{\Omega} \nu_k dx}{\prod_{k=1}^j \int_{\Omega} (\nu_k + \gamma_k) dx} \right) \frac{\int_{\Omega} \lambda dx}{\int_{\Omega} b dx} h \left( \frac{\int_{\Omega} \lambda dx}{\int_{\Omega} b dx} \right),$$

(iii)

$$\lim_{(d_1, \dots, d_{m+1}) \rightarrow (0, \dots, 0, \infty)} R_0 = \frac{\int_{\Omega} \lambda dx}{\int_{\Omega} b dx} h \left( \frac{\int_{\Omega} \lambda dx}{\int_{\Omega} b dx} \right) \max_{x \in \bar{\Omega}} \left( \sum_{j=1}^m \frac{\beta_j(x) \prod_{k=1}^{j-1} \nu_k(x)}{\prod_{k=1}^j (\nu_k(x) + \gamma_k(x))} \right),$$

(iv)

$$\lim_{(d_1, \dots, d_{m+1}) \rightarrow (\infty, \dots, \infty, 0)} R_0 = \left( \sum_{j=1}^m \frac{\int_{\Omega} \beta_j \frac{\lambda}{b} h \left( \frac{\lambda}{b} \right) dx \prod_{k=1}^{j-1} \int_{\Omega} \nu_k dx}{\prod_{k=1}^j \int_{\Omega} (\nu_k + \gamma_k) dx} \right).$$

**Appendix A.** In this part, we prove a result that verifies the continuity of functions  $F_{\epsilon}^c, \bar{F}_{\epsilon}^c, V_{\epsilon}^c, \bar{V}_{\epsilon}^c, F_{\epsilon}^x, \bar{F}_{\epsilon}^x, V_{\epsilon}^x$ , and  $\bar{V}_{\epsilon}^x$ , which are defined in (3.3) and (3.8).

**PROPOSITION A.1.** *Let  $f(x, u) \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$  and  $c(x) \in C(\bar{\Omega}, \mathbb{R})$ , where  $\Omega$  is a bounded domain in  $\mathbb{R}^N$  ( $N \geq 1$ ). Denote*

$$\mathcal{D}_{\epsilon}^c = \{(x, u) : x \in \bar{\Omega}, u \in [c(x) - \epsilon, c(x) + \epsilon]\}, \quad \mathcal{D}_{\epsilon}^x = \{u : u \in [c(x) - \epsilon, c(x) + \epsilon]\}$$

and

$$H(\epsilon) = \max_{(x, u) \in \mathcal{D}_{\epsilon}^c} f(x, u), \quad G(x, \epsilon) = \max_{u \in \mathcal{D}_{\epsilon}^x} f(x, u).$$

Then  $H(\epsilon) \in C([0, 1], \mathbb{R})$  and  $G(x, \epsilon) \in C(\bar{\Omega} \times [0, 1], \mathbb{R})$ .

*Proof.* We first consider the continuity of  $H(\epsilon)$ . Let

$$C_1 := \min_{x \in \bar{\Omega}} c(x) - 2 \quad \text{and} \quad C_2 =: \max_{x \in \bar{\Omega}} c(x) + 2.$$

The continuity of  $f(x, u)$  implies that  $f(x, u)$  is uniformly continuous on  $\bar{\Omega} \times [C_1, C_2]$ . Then, for any given  $\gamma > 0$ , there exists  $\delta > 0$  such that, for any  $(x_1, u_1), (x_2, u_2) \in \bar{\Omega} \times [C_1, C_2]$  satisfying  $|x_1 - x_2| < \delta$  and  $|u_1 - u_2| < \delta$ ,

$$(A.1) \quad |f(x_1, u_1) - f(x_2, u_2)| < \gamma.$$

Assume that  $0 \leq \epsilon_1 < \epsilon_2 \leq 1$  and  $\epsilon_2 - \epsilon_1 < \delta$ . Clearly,  $H(\epsilon_1) \leq H(\epsilon_2)$ . Noticing that  $\mathcal{D}_{\epsilon_2}^c$  is compact, we see that there exists  $(x_0, u_0) \in \mathcal{D}_{\epsilon_2}^c$  such that  $H(\epsilon_2) = f(x_0, u_0)$ . Then there exists  $(x_0, u_1)$  such that  $(x_0, u_1) \in \mathcal{D}_{\epsilon_1}^c$  and  $|u_1 - u_0| < \delta$ . It follows from (A.1) that  $f(x_0, u_0) < f(x_0, u_1) + \gamma$ , which implies that  $H(\epsilon_2) < H(\epsilon_1) + \gamma$ . Then, exchanging the position of  $\epsilon_1$  and  $\epsilon_2$ , we can also obtain that, for any  $0 \leq \epsilon_2 < \epsilon_1 \leq 1$  and  $\epsilon_1 - \epsilon_2 < \delta$ ,

$$H(\epsilon_2) \leq H(\epsilon_1) \leq H(\epsilon_2) + \gamma.$$

Therefore, for any given  $\gamma > 0$ , there exists  $\delta > 0$  such that, for any  $\epsilon_1, \epsilon_2 \in [0, 1]$  satisfying  $|\epsilon_1 - \epsilon_2| < \delta$ ,

$$|H(\epsilon_1) - H(\epsilon_2)| < \gamma.$$

This implies that  $H(\epsilon) \in C([0, 1], \mathbb{R})$ .

Then we consider the continuity of  $G(x, \epsilon)$ . Note that  $c(x)$  is continuous. Then, for the above  $\delta$ , there exists  $\delta_1 \in (0, \delta)$  such that, for any  $x_1, x_2 \in \bar{\Omega}$  satisfying  $|x_1 - x_2| < \delta_1$ ,

$$|c(x_1) - c(x_2)| < \delta/2.$$

Clearly, if  $|\epsilon_1 - \epsilon_2| < \delta/2$  and  $|x_1 - x_2| < \delta_1$ , then

$$(A.2) \quad |c(x_2) + \epsilon_2 - c(x_1) - \epsilon_1| < \delta \quad \text{and} \quad |c(x_2) - \epsilon_2 - c(x_1) + \epsilon_1| < \delta.$$

Choose  $(x_1, \epsilon_1), (x_2, \epsilon_2) \in \bar{\Omega} \times [0, 1]$  satisfying

$$|x_1 - x_2|, |\epsilon_1 - \epsilon_2| < \delta_2,$$

where  $\delta_2 := \min\{\delta/2, \delta_1\}$ . Clearly, there exists  $u_1 \in [c(x_1) - \epsilon_1, c(x_1) + \epsilon_1]$  such that  $G(x_1, \epsilon_1) = f(x_1, u_1)$ . Then we claim that

$$G(x_1, \epsilon_1) < G(x_2, \epsilon_2) + \gamma,$$

and the proof is divided into two cases.

*Case 1.*  $u_1 \in [c(x_2) - \epsilon_2, c(x_2) + \epsilon_2]$ .

Since  $|x_1 - x_2| < \delta_2 < \delta$ , it follows from (A.1) that

$$G(x_1, \epsilon_1) = f(x_1, u_1) < f(x_2, u_1) + \gamma \leq G(x_2, \epsilon_2) + \gamma.$$

*Case 2.*  $u_1 \notin [c(x_2) - \epsilon_2, c(x_2) + \epsilon_2]$ .

Then  $u_1 > c(x_2) + \epsilon_2$  or  $u_1 < c(x_2) - \epsilon_2$ . We only consider the case of  $u_1 > c(x_2) + \epsilon_2$ , and the other case could be proved similarly. Then  $c(x_2) + \epsilon_2 < u_1 \leq c(x_1) + \epsilon_1$ , This,

combined with (A.2), implies that  $|c(x_2) + \epsilon_2 - u_1| < \delta$ . Then it follows from (A.1) that

$$G(x_1, \epsilon_1) = f(x_1, u_1) < f(x_2, c(x_2) + \epsilon_2) + \gamma \leq G(x_2, \epsilon_2) + \gamma.$$

Then, exchanging the positions of  $(x_1, \epsilon_1)$  and  $(x_2, \epsilon_2)$ , we also have  $G(x_2, \epsilon_2) < G(x_1, \epsilon_1) + \gamma$ . This implies that for any given  $\gamma > 0$ , there exists  $\delta_2 > 0$  such that, for any  $(x_1, \epsilon_1), (x_2, \epsilon_2) \in \bar{\Omega} \times [0, 1]$  satisfying  $|x_1 - x_2| < \delta_2$  and  $|\epsilon_1 - \epsilon_2| < \delta_2$ ,

$$(A.3) \quad |G(x_1, \epsilon_1) - G(x_2, \epsilon_2)| \leq \gamma.$$

This completes the proofs.  $\square$

**Acknowledgments.** We thank two anonymous referees for very helpful suggestions that helped us improve the initial draft of the paper. The first named author thanks the Department of Mathematics, William & Mary, for hospitality while this work was done when she visited William & Mary in 2018–2019.

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