A Deterministic Algorithm for Counting Colorings with 2Δ Colors

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***Abstract*—We give a polynomial time deterministic approximation algorithm (an FPTAS) for counting the number of q-colorings of a graph of maximum degree Δ, provided only that $q \geq 2\Delta$. This substantially improves on previous deterministic algorithms for this problem, the best of which requires $q \geq 2.58\Delta$, and matches the natural bound for randomized algorithms obtained by a straightforward application of Markov chain Monte Carlo. In the case when the graph is also triangle-free, we show that our algorithm applies under the weaker condition $q \geq \alpha\Delta + \beta$, where $\alpha \approx 1.764$ and $\beta = \beta(\alpha)$ are absolute constants. Our result applies more generally to list colorings, and to the partition function of the anti-ferromagnetic Potts model.**

The core of our argument is the establishment of a region in the complex plane in which the Potts model partition function (a classical graph polynomial) has no zeros. This result, which substantially sharpens previous work on the same problem, is of independent interest. Our algorithms follow immediately from zero-freeness via the "polynomial interpolation" method of Barvinok. Interestingly, our method for identifying the zero-free region leverages probabilistic and combinatorial ideas that have been used in the analysis of Markov chains.

***Keywords*-Approximate counting; Graph coloring; Potts model; Partition function; Stability theory; Derandomization;**

I. Introduction

A. Background and related work

Counting colorings of a bounded degree graph is a benchmark problem in approximate counting, due both to its importance in combinatorics and statistical physics, as well as to the fact that it has repeatedly challenged existing algorithmic techniques and stimulated the development of new ones.

Given a finite graph $G = (V, E)$ of maximum degree Δ, and a positive integer q, the goal is to count the number of (proper) vertex colorings of G with q colors. It is well known [7] that a greedy coloring exists if $q \geq \Delta + 1$. While counting colorings exactly is #P-complete, a long-standing conjecture asserts that approximately counting colorings is possible in polynomial time provided $q \geq \Delta + 1$. It is known that when $q \leq \Delta$, even approximate counting is NP-hard [17].

This question has led to numerous algorithmic developments over the past 25 years. The first approach was via Markov chain Monte Carlo (MCMC), based on the fact that approximate counting can be reduced to sampling a coloring (almost) uniformly at random. Sampling can be achieved by simulating a natural local Markov chain (or Glauber dynamics) that randomly flips colors on vertices: provided the chain is rapidly mixing, this leads to an efficient algorithm (a *fully polynomial randomized approximation scheme*, or *FPRAS*).

Jerrum's 1995 result [29] that the Glauber dynamics is rapidly mixing for $q \geq 2\Delta + 1$ gave the first non-trivial randomized approximation algorithm for colorings and led to a plethora of follow-up work on MCMC (see, e.g., [11, 12, 15, 21, 23–25, 36, 43] and [16] for a survey), focusing on reducing the constant 2 in front of Δ. The best constant known for general graphs remains essentially $\frac{11}{6}$, obtained by Vigoda [43] using a more sophisticated Markov chain, though this was very recently reduced to $\frac{11}{6} - \varepsilon$ for a very small ε by Chen et al. [9]. The constant can be substantially improved if additional restrictions are placed on the graph: e.g., Dyer et al. [12] achieve roughly $q \geq 1.49\Delta$ provided the girth is at least 6 and the degree is a large enough constant, while Hayes and Vigoda improve this to $q \geq (1 + \varepsilon)\Delta$ for girth at least 11 and degree $\Delta = \Omega(\log n)$, where n is the number of vertices.

A significant recent development in approximate counting is the emergence of *deterministic* approximation algorithms that in some cases match, or even improve upon, the best known MCMC algorithms.[1]

[1]In this case, the notion of an FPRAS is replaced by that of a *fully polynomial time approximation scheme*, or *FPTAS*. An FPTAS for q-colorings of graphs of maximum degree at most Δ is an algorithm that given the graph G and an error parameter δ on the input, produces a $(1 \pm \delta)$-factor multiplicative approximation to the number of q-colorings of G in time $\mathrm{poly}(|G|, 1/\delta)$ (the degree of the polynomial is allowed to depend upon the constants q and Δ).

These algorithms have made use of one of two main techniques: *decay of correlations*, which exploits decreasing influence of the spins (colors) on distant vertices on the spin at a given vertex; and *polynomial interpolation*, which uses the absence of zeros of the partition function in a suitable region of the complex plane. Early examples of the decay of correlations approach include [1,5,45], while for early examples of the polynomial interpolation method, we refer to the monograph of Barvinok [2] (see also, e.g., [3,13,22,26,28,33,37] for more recent examples). Unfortunately, however, in the case of colorings on general bounded degree graphs, these techniques have so far lagged well behind the MCMC algorithms mentioned above. One obstacle to getting correlation decay to work is the lack of a higher-dimensional analog of Weitz's beautiful algorithmic framework [45], which allows correlation decay to be fully exploited via strong spatial mixing in the case of spin systems with just two spins (as opposed to the q colors present in coloring). For polynomial interpolation, the obstacle has been a lack of precise information about the location of the zeros of associated partition functions (see below for a definition of the partition function in the context of colorings).

So far, the best algorithmic condition for colorings obtained via correlation decay is $q \geq 2.58\Delta + 1$, due to Lu and Yin [35], and this remains the best available condition for any deterministic algorithm. This improved on an earlier bound of roughly $q \geq 2.78\Delta$ (proved only for triangle-free graphs), due to Gamarnik and Katz [18]. For the special case $\Delta = 3$, Lu et al. [34] give a correlation decay algorithm for counting 4-colorings. Furthermore, Gamarnik, Katz and Misra [19] establish the related property of "strong spatial mixing" under the weaker condition $q \geq \alpha\Delta + \beta$ for any constant $\alpha > \alpha^\star$, where $\alpha^\star \approx 1.7633$ is the unique solution to $xe^{-1/x} = 1$ and β is a constant depending on α, and under the assumption that G is triangle-free (see also [20,21] for similar results on restricted classes of graphs). However, as discussed in [19], this strong spatial mixing result unfortunately does not lead to a deterministic algorithm.[2]

The newer technique of polynomial interpolation, pioneered by Barvinok [2], has also recently been brought to bear on counting colorings. In a recent paper, Bencs et al. [6] use this technique to derive a FPTAS for counting colorings provided $q \geq e\Delta + 1$. This result is of independent interest because it uses a different algorithmic approach, and because it establishes a new zero-free region for the associated partition function in the complex plane (see below), but it is weaker than those obtained via correlation decay.

In this paper, we push the polynomial interpolation method further and obtain a FPTAS for counting colorings under the condition $q \geq 2\Delta$:

Theorem I.1. *Fix positive integers q and Δ such that $q \geq 2\Delta$. Then there exists a fully polynomial time deterministic approximation scheme (FPTAS) for counting q-colorings in any graph of maximum degree Δ.*

This is the first deterministic algorithm (of *any* kind) that for all Δ matches the "natural" bound for MCMC, first obtained by Jerrum [29]. Indeed, $q \geq 2\Delta + 1$ remains the best bound known for rapid mixing of the basic Glauber dynamics that does not require either additional assumptions on the graph or a spectral comparison with another Markov chain: all the improvements mentioned above require either lower bounds on the girth and/or maximum degree, or (in the case of Vigoda's result [43]) analysis of a more sophisticated Markov chain. This is for good reason, since the bound $q \geq 2\Delta + 1$ coincides with the closely related Dobrushin uniqueness condition from statistical physics [40], which in turn is closely related [44] to the path coupling method of Bubley and Dyer [8] that provides the simplest currently known proof of the $q \geq 2\Delta + 1$ bound for the Glauber dynamics.

We therefore view our result as a promising starting point for deterministic coloring algorithms to finally compete with their randomized counterparts. In fact, as discussed later in section I-C, our technique is capable of directly harnessing strong spatial mixing arguments used in the analysis of Markov chains for certain classes of graphs. As an example, we can exploit such an argument of Gamarnik, Katz and Misra [19] to improve the bound on q in Theorem I.1 when the graph is triangle-free, for all but small values of Δ. (Recall that $\alpha^\star \approx 1.7633$ is the unique positive solution of the equation $xe^{-1/x} = 1$.)

Theorem I.2. *For every $\alpha > \alpha^\star$, there exists a $\beta = \beta(\alpha)$ such that the following is true. For all integers q and Δ such that $q \geq \alpha\Delta+\beta$, there exists a fully polynomial time deterministic approximation scheme (FPTAS) for counting q-colorings in any triangle-free graph of maximum degree Δ.*

We mention also that our technique applies without further effort to the more general setting of *list*

[2]The strong spatial mixing condition does imply fast mixing of the Glauber dynamics, and hence an FPRAS, but only when the graph family being considered is "amenable", i.e., if the size of the ℓ-neighborhood of any vertex does not grow exponentially in ℓ. This restriction is satisfied by regular lattices, but fails, e.g., for random regular graphs.

colorings, where each vertex has a list of allowed colors of size q, under the same conditions as above on q. Indeed, our proofs are written to handle this more general situation.

In the next subsection we describe our algorithm in more detail.

B. Our approach

Let $G = (V, E)$ be an n-vertex graph of maximum degree Δ, and $[q] := \{1, \dots, q\}$ a set of colors. Define the polynomial

$$Z_G(w) := \sum_{\sigma : V \to [q]} w^{|\{\{u,v\} \in E \,:\, \sigma(u) = \sigma(v)\}|}. \qquad (1)$$

Here σ ranges over arbitrary (not necessarily proper) assignments of colors to vertices, and each such coloring has a weight $w^{m(\sigma)}$, where $m(\sigma)$ is the number of monochromatic edges in σ. Note that the number of proper q-colorings of G is just $Z_G(0)$.

The polynomial $Z_G(w)$ is the partition function of the Potts model of statistical physics, and implicitly defines a probability distribution on colorings σ according to their weights in (1). The parameter w measures the strength of nearest-neighbor interactions. The value $w = 1$ corresponds to the trivial setting where there is no constraint on the colors of neighboring vertices, while $w = 0$ imposes the hard constraint that no neighboring vertices receive the same color. For intermediate values $w \in [0, 1]$, neighbors with the same color are penalized by a factor of w. Theorems I.1 and I.2 are in fact special cases of the following more general theorem.

Theorem I.3. *Suppose that the hypotheses of either Theorem I.1 or Theorem I.2 are satisfied, and fix $w \in [0, 1]$. Then there exists an FPTAS for the partition function $Z_G(w)$.*

Theorem I.3 of course subsumes Theorems I.1 and I.2, but the extension to other values of w is of independent interest as the computation of partition functions is a very active area of study in statistical physics and combinatorics.

To prove Theorem I.3, we view $Z_G(w)$ as a polynomial in the *complex* variable w and identify a region in the complex plane in which $Z_G(w)$ is guaranteed to have no zeros. Specifically, we will show that this holds for the open simply connected set $\mathcal{D}_\Delta \subset \mathbb{C}$ obtained by augmenting the real interval $[0, 1]$ with a ball of radius τ_Δ around each point, where τ_Δ is a (small) constant depending only on Δ.

Theorem I.4. *Fix a positive integer Δ. Then there exists a $\tau_\Delta > 0$ and a region $\mathcal{D}_\Delta$ of the above form containing the interval $[0, 1]$ such that the following is true. For any graph G of maximum degree Δ and integer q satisfying the hypotheses of either Theorem I.1 or Theorem I.2, $Z_G(w) \neq 0$ when $w \in \mathcal{D}_\Delta$.*

We remark that this theorem is also of independent interest, as the location of zeros of partition functions has a long and noble history going back to the Lee-Yang theorem of the 1950s [30, 46]. In the case of the Potts model, Sokal [41, 42] proved (in the language of the Tutte polynomial) that the partition function has no zeros in w in the entire unit disk centered at 0, under the strong condition $q \geq 7.964\Delta$; the constant was later improved to 6.907 by Fernández and Procacci [14] (see also [27]).[3] Much more recently, the work of Bencs et al. [6] referred to above gives a zero-free region analogous to that in Theorem I.4 above, but under the stronger condition $q \geq e\Delta + 1$. We note also that Barvinok and Soberón [4] (see also [2] for an improved version) established a zero-free region in a disk centered at $w = 1$.

Theorem I.4 immediately gives our algorithmic result, Theorem I.3, by appealing to the recent algorithmic paradigm of Barvinok [2]. The paradigm (see Lemma 2.2.3 of [2]) states that, for a partition function Z of degree m, if one can identify a simply connected, zero-free region $\mathcal{D}$ for Z in the complex plane that contains a τ-neighborhood of the interval $[0, 1]$, and a point on that interval where the evaluation of Z is easy (in our setting this is the point $w = 1$), then using the first $O\big(e^{\Theta(1/\tau)} \log(m/\varepsilon)\big)$ coefficients of Z, one can obtain a $1 \pm \varepsilon$ multiplicative approximation of $Z(x)$ at any point $x \in \mathcal{D}$. Barvinok's framework is based on exploiting the fact that the zero-freeness of Z in D is equivalent to $\log Z$ being analytic in D, and then using a carefully chosen transformation to deform D into a disk (with the easy point at the center) in order to perform a convergent Taylor expansion. The coefficients of Z are used to compute the coefficients of this Taylor expansion.

Barvinok's framework in general leads to a quasi-polynomial time algorithm as the computation of the $O\big(e^{\Theta(1/\tau)} \log(m/\varepsilon)\big)$ terms of the expansion may take quasiploynomial time $O\Big((m/\varepsilon)^{e^{\Theta(1/\tau)} \log m}\Big)$ for the partition functions considered here. However, additional insights provided by Patel and Regts [37] (see, e.g., the proof of Theorem 6.2 in [37]) show how to reduce this

[3]The results in these papers are in terms of the Tutte polynomial, and in fact extend to *complex* values of q.

computation time to $O\Big((m/\varepsilon)^{e^{\Theta(1/\tau)}\log\Delta}\Big)$ for many models on bounded degree graphs of degree at most Δ, including the Potts model with a bounded number of colors at each vertex. Hence we obtain an FPTAS. This (by now standard) reduction is the same path as that followed by Bencs et al. [6, Corollary 1.2]; for completeness, we provide a sketch in section II-C. We note that for each fixed Δ and q the running time of our final algorithm is polynomial in n (the size of G) and ε^{-1}, as required for an FPTAS. However, as is typical of deterministic algorithms for approximate counting, the exponent in the polynomial depends on Δ (through the quantity τ_Δ in Theorem I.4, which in the case where all lists are subsets of $[q]$, is inverse polynomial in q).

We end this introduction by sketching our approach to proving Theorem I.4, which is the main contribution of the paper.

C. Technical overview

The starting point of our proof is a simple geometric observation, versions of which have been used before for constructing inductive proofs of zero-freeness of partition functions (see, e.g., [2, 6]). Fix a vertex v in the graph G. Given $w \in \mathbb{C}$, and a color $k \in [q]$, let $Z_v^{(k)}(w)$ denote the *restricted partition function* in which one only includes those colorings σ in which $\sigma(v) = k$. Then, since $Z_G(w) = \sum_{k\in[q]} Z_v^{(k)}(w)$, the zero-freeness of Z_G will follow if the angles between the complex numbers $Z_v^{(k)}(w)$, viewed as vectors in $\mathbb{R}^2$, are all small, and provided that at least one of the $Z_v^{(k)}$ is non-zero. (In fact, this condition on angles can be relaxed for those $Z_v^{(k)}(w)$ that are sufficiently small in magnitude, and this flexibility is important when w is a complex number close to 0.) Therefore, one is naturally led to consider so-called *marginal ratios*:

$$R_{G,v}^{(i,j)}(w) := \frac{Z_v^{(i)}(w)}{Z_v^{(j)}(w)}.$$

(In the q-coloring problem, this ratio is 1 by symmetry. However, in our recursive approach, we have to handle the more general list-coloring problem, in which the ratio becomes non-trivial.)

We then require that for any two colors i, j for which $Z_v^{(k)}(w)$ is large enough in magnitude, the ratio $R_{G,v}^{(i,j)}(w)$ is a complex number with small argument. This is what we prove inductively in sections IV and V.

The broad contours of our approach as outlined so far are quite similar to some recent work [2, 6]. However, it is at the crucial step of how the marginal ratios are analyzed that we depart from these previous results. Instead of attacking the restricted partition functions or the marginal ratios directly for given $w \in \mathbb{C}$, as in these previous works, we crucially exploit the fact that for any $\tilde{w} \in [0, 1]$ close to the given w, these quantities have natural probabilistic interpretations, and hence can be much better understood via probabilistic and combinatorial methods. For instance, when $\tilde{w} \in [0, 1]$, the marginal ratio $R_{G,v}^{(i,j)}(w)$ is in fact a ratio of the marginal probabilities $\Pr_{G,\tilde{w}}[\sigma(v) = i]$ and $\Pr_{G,\tilde{w}}[\sigma(v) = j]$, under the natural probability distribution on colorings σ. In fact, our analysis cleanly breaks into two separate parts:

1) First, understand the behavior of true marginal probabilities of the form $\Pr_{G,\tilde{w}}[\sigma(v) = i]$ for $\tilde{w} \in [0, 1]$. This is carried out in section III.
2) Second, argue that, for complex $w \approx \tilde{w}$, the ratios $R_{G,v}^{(i,j)}(w)$ remain well-behaved. This is carried out separately for the two cases when w is close to 0 (in section IV) and when w is bounded away from 0 but still in the vicinity of $[0, 1]$ (in section V).

A key point in our technical analysis is the notion of "niceness" of vertices, which stipulates that the marginal probability $\Pr_{G,\tilde{w}}[\sigma(v) = i] \le \frac{1}{\deg_G(v)+2}$ where $\deg_G(v)$ is the degree of v in G (see Definition III.1). Note that this condition refers only to real non-negative $\tilde{w}$, and hence is amenable to analysis via standard combinatorial tools. Indeed, our proofs that the conditions on q and Δ in Theorems I.1 and I.2 imply this niceness condition are very similar to probabilistic arguments used by Gamarnik et al. [19] to establish the property of "strong spatial mixing" (in the special case $\tilde{w} = 0$). We emphasize that this is the only place in our analysis where the lower bounds on q are used. One can therefore expect that combinatorial and probabilistic ideas used in the analysis of strong spatial mixing and the Glauber dynamics with smaller number of colors in special classes of graphs can be combined with our analysis to obtain deterministic algorithms for those settings, as we have demonstrated in the case of [19].

The above ideas are sufficient to understand the real-valued case (part 1 above). For the complex case in part 2, we start from a recurrence for the marginal ratios $R_{G,v}^{(i,j)}$ that is a generalization (to the case $w \neq 0$) of a similar recurrence used by Gamarnik et al. [19] (see Lemma II.4). The inductive proofs in sections IV and V use this recurrence to show that, if $\tilde{w} \in [0, 1]$ is close to $w \in \mathbb{C}$, then all the relevant $R_{G,v}^{(i,j)}(w)$ remain close to $R_{G,v}^{(i,j)}(\tilde{w})$ throughout. The actual induction, especially in the case when w is close to 0, requires a delicate choice of induction hypotheses (see Lemmas IV.2 and V.3). The key technical idea is to use the "niceness" property of vertices established in part 1 to argue that the two recurrences (real and complex) remain close at every step of the induction. This in turn depends upon a careful application of the mean value theorem, *separately* to the real and imaginary parts (see Lemma II.5), of a function f_κ that arises in the analysis of the recurrence (see Lemma II.6).

D. Comparison with correlation-decay based algorithms

We conclude this overview with a brief discussion of how we are able to obtain a better bound on the number of colors than in correlation decay algorithms, such as [18, 35] cited earlier. In these algorithms, one first uses recurrences similar to the one mentioned above to *compute* the marginal probabilities, and then appeals to self-reducibility to compute the partition function. Of course, expanding the full tree of computations generated by the recurrence will in general give an exponential time (but exact) algorithm. The core of the analysis of these algorithms is to show that even if this tree of computations is only expanded to depth about $O(\log(n/\varepsilon))$, and the recurrence at that point is initialized with *arbitrary* values, the computation still converges to an ε-approximation of the true value. However, the requirement that the analysis be able to deal with arbitrary initializations implies that one cannot directly use properties of the actual probability distribution (e.g., the "niceness" property alluded to above); indeed, this issue is also pointed out by Gamarnik et al. [19]. In contrast, our analysis does not truncate the recurrence, and thus only has to handle initializations that make sense in the context of the graph being considered. Moreover, the exponential size of the recursion tree is no longer a barrier since, in contrast to correlation decay algorithms, we are using the tree only as a tool to establish zero-freeness; the algorithm itself follows from Barvinok's polynomial interpolation paradigm. Our approach suggests that this paradigm can be viewed as a method for using (complex-valued generalizations of) strong spatial mixing results to obtain deterministic algorithms.

We note also that our results in this paper are part of a wider program exploring the connections between correlation decay and zero-freeness of partition functions; see, e.g., [32, 38, 39]. Further discussion of these connections, along with applications to other models, including the Ising model and the hard-core model, can be found in the full version of this paper and in the first author's PhD thesis [31].

II. Preliminaries

A. Colorings and the Potts model

Throughout, we assume that the graphs that we consider are augmented with a list of colors for every vertex. Formally, a graph is a triple $G = (V, E, L)$, where V is the vertex set, E is the edge set, and $L : V \to 2^{\mathbb{N}}$ specifies a list of colors for every vertex. The partition function as defined in the introduction generalizes naturally to this setting: the sum is now over all those colorings σ which satisfy $\sigma(v) \in L(v)$.

We also allow graphs to contain *pinned* vertices: a vertex v is said to be *pinned* to a color c if only those colorings of G are allowed in which v has color c. Suppose that a vertex v of degree d_v in a graph G is pinned to a color c, and consider the graph G' obtained by replacing v with d_v copies of itself, each of which is pinned to c and connected to exactly one of the original neighbors of v in G. It is clear that $Z_{G'}(w) = Z_G(w)$ for all w. We will therefore assume that all pinned vertices in our graphs G have degree exactly one. The *size* of graph, denoted as $|G|$, is defined to be the number of unpinned vertices. It is worth noting that the above operation of duplicating pinned vertices does not change the size of the graph.

Let G be a graph and v an unpinned vertex in G. A color c in the list of v is said to be *good* for v if for every pinned neighbor u of v is pinned to a color different from c. The set of good colors for a vertex v in graph G is denoted $\Gamma_{G,v}$. We sometimes omit the graph G and write Γ_v when G is clear from the context. A color c that is not in Γ_v is called *bad* for v. Further, given a graph G with possibly pinned vertices, we say that the graph is *unconflicted* if no two neighboring vertices in G are pinned to the same color. Note that since all pinned vertices have degree exactly one, each conflicted graph is the vertex-disjoint union of an unconflicted graph and a collection of disjoint, conflicted edges.

We will assume throughout that all unconflicted graphs G we consider have at least one proper coloring: this will be guaranteed in our applications since we will always have $|L(u)| \geq \deg_G(u) + 1$ for every unpinned vertex u in G.

Definition II.1. For a graph G, a vertex v and a color $i \in L(v)$, the *restricted partition function* $Z^{(i)}_{G,v}(w)$ is the partition function restricted to colorings in which the vertex v receives color i.

Definition II.2. Let ω be a formal variable. For any G, a vertex v and colors $i, j \in L(v)$, we define the *marginal ratio* of color i to color j as $R^{(i,j)}_{G,v}(\omega) := \frac{Z^{(i)}_{G,v}(\omega)}{Z^{(j)}_{G,v}(\omega)}$. Similarly we also define formally the corresponding *pseudo marginal probability* as $\mathcal{P}_{G,\omega}[c(v) = i] := \frac{Z^{(i)}_{G,v}(\omega)}{Z_G(\omega)}$.

Remark 1. Note that when a numerical value $w \in \mathbb{C}$ is substituted in place of ω in the above formal definition, $R^{(i,j)}_{G,v}(w)$ is numerically well-defined as long as $Z^{(j)}_{G,v}(w) \neq 0$, and $\mathcal{P}_{G,w}[c(v) = i]$ is numerically well-defined as long as $Z_G(w) \neq 0$. In the proof of the main theorem in sections IV and V, we will ensure that the above definitions are numerically instantiated only in cases where the corresponding conditions for such an instantiation to be well-defined, as stated above, are satisfied. For instance, when $w \in [0, 1]$, this is the case for the first definition when either (i) $w \neq 0$; or (ii) $w = 0$, but G is unconflicted and $j \in \Gamma_{G,v}$; while for the second definition, this is the case when either (i) $w \neq 0$; or (ii) $w = 0$, but G is unconflicted.

Remark 2. Note also that when $w \in [0, 1]$, the pseudo probabilities, if well-defined, are actual marginal probabilities. In this case, we will also write $\mathcal{P}_{G,w}[c(v) = i]$ as $\Pr_{G,w}[c(v) = i]$. For arbitrary complex w, this interpretation as probabilities is of course not valid (since $\mathcal{P}_{G,w}[c(v) = i]$ can be non-real), but provided that $Z_G(w) \neq 0$ it is still true that $\sum_{i \in L(v)} \mathcal{P}_{G,w}[c(v) = i] = \frac{1}{Z_G(w)} \sum_{i \in L(v)} Z^{(i)}_{G,v}(w) = \frac{Z_G(w)}{Z_G(w)} = 1$. We also note that if v is pinned to color k, then $\mathcal{P}_{G,w}[c(v) = i]$ is 1 when $k = i$ and 0 when $k \neq i$.

Notation. For the case $w = 0$ we will sometimes shorten the notations $\mathcal{P}_{G,0}[c(v) = i]$ and $\Pr_{G,0}[c(v) = i]$ to $\mathcal{P}_G[c(v) = i]$ and $\Pr_G[c(v) = i]$ respectively.

Definition II.3 (The graphs $G^{(i,j)}_k$). Given a graph G and a vertex u in G, let $v_1, \cdots, v_{\deg_G(u)}$ be the neighbors of u. We define $G^{(i,j)}_k$ (the vertex u will be understood from the context) to be the graph obtained from G as follows:

- first we replace vertex u with $u_1, \cdots, u_{\deg_G(u)}$, and connect u_1 to v_1, u_2 to v_2, and so on;
- next we pin vertices $u_1, \cdots, u_{k-1}$ to color i, and vertices $u_{k+1}, \cdots, u_{\deg_G(u)}$ to color j;
- finally we remove the vertex u_k.

Note that the graph $G^{(i,j)}_k$ has one fewer unpinned vertex than G. Moreover, $u_1, \cdots, u_{\deg_G(u)}$ are of degree 1, so this construction maintains the property that pinned vertices have degree 1.

We now derive a recurrence relation between the marginal ratios of the graph G and pseudo marginal probabilities of the graphs $G^{(i,j)}_k$. This is an extension to the Potts model of a similar recurrence relation derived by Gamarnik, Katz and Misra [19] for the special case of colorings (that is, $w = 0$).

Lemma II.4. *Let ω be a formal variable. For a graph G, a vertex u and colors $i, j \in L(u)$, we have*

$$R^{(i,j)}_{G,u}(\omega) = \prod_{k=1}^{\deg_G(u)} \frac{1 - \gamma \cdot \mathcal{P}_{G^{(i,j)}_k,\omega}[c(v_k) = i]}{1 - \gamma \cdot \mathcal{P}_{G^{(i,j)}_k,\omega}[c(v_k) = j]},$$

where we define $\gamma := 1 - \omega$. In particular, when a numerical value $w \in \mathbb{C}$ is substituted in place of ω, the above recurrence is valid as long as the quantities $Z_{G^{(i,j)}_k}(w)$ and $1 - \gamma \cdot \mathcal{P}_{G^{(i,j)}_k,w}[c(v_k) = j]$ for $1 \leq k \leq \deg_G(u)$ are all non-zero.

Proof: Let $t := \deg_G(u)$. For $0 \leq k \leq t$, let H_k be the graph obtained from G as follows:

- first we replace vertex u with $u_1, \cdots, u_t$, and connect u_1 to v_1, u_2 to v_2, and so on;
- we then pin vertices $u_1, \cdots, u_k$ to color i, and vertices $u_{k+1}, \cdots, u_t$ to color j.

Note that H_k is the same as $G^{(i,j)}_k$, except that the last step of the construction of $G^{(i,j)}_k$ is skipped, i.e, the vertex u_k is not removed, and, further, u_k is pinned to color i. We can now write

$$R^{(i,j)}_{G,u}(\omega) = \frac{Z^{(i)}_{G,u}(\omega)}{Z^{(j)}_{G,u}(\omega)} = \frac{Z_{H_t}(\omega)}{Z_{H_0}(\omega)} = \prod_{k=1}^{t} \frac{Z_{H_k}(\omega)}{Z_{H_{k-1}}(\omega)}.$$

Next, for $1 \leq k \leq t$, let $Y_k := Z_{G^{(i,j)}_k}(\omega)$ and $Y^{(i)}_k := Z^{(i)}_{G^{(i,j)}_k,v_k}(\omega)$. We observe that

$$\begin{aligned} \mathcal{P}_{G^{(i,j)}_k,\omega}[c(v_k) = i] &= \frac{Y^{(i)}_k}{Y_k}; \\ Z_{H_k}(\omega) &= Y_k - (1 - \omega) \cdot Y^{(i)}_k; \\ Z_{H_{k-1}}(\omega) &= Y_k - (1 - \omega) \cdot Y^{(j)}_k. \end{aligned}$$

Therefore we have

$$\begin{aligned} R^{(i,j)}_{G,u}(\omega) &= \prod_{k=1}^{t} \frac{Y_k - (1 - \omega) \cdot Y^{(i)}_k}{Y_k - (1 - \omega) \cdot Y^{(j)}_k} \\ &= \prod_{k=1}^{t} \frac{1 - \gamma \cdot \mathcal{P}_{G^{(i,j)}_k,\omega}[c(v_k) = i]}{1 - \gamma \cdot \mathcal{P}_{G^{(i,j)}_k,\omega}[c(v_k) = j]}, \end{aligned}$$

where $\gamma = 1 - \omega$. The claim about the validity of the recurrence on numerical substitution then follows from the conditions outlined in Definition II.2. ∎

B. Complex analysis

In this subsection we collect some tools and observations from complex analysis. Throughout this paper, we use ι to denote the imaginary unit $\sqrt{-1}$, in order to avoid confusion with the symbol "i" used for other purposes. For a complex number $z = a + \iota b$ with $a, b \in \mathbb{R}$, we denote its real part a as $\Re z$, its imaginary part b as $\Im z$, its *length* $\sqrt{a^2 + b^2}$ as $|z|$, and, when $z \neq 0$, its *argument* $\sin^{-1}\left(\frac{b}{|z|}\right) \in (-\pi, \pi]$ as $\arg z$. We also generalize the notation $[x, y]$ used for closed real intervals to the case when $x, y \in \mathbb{C}$, and use it to denote the closed straight line segment joining x and y.

We start with a consequence of the mean value theorem for complex functions, specifically tailored to our application. Let D be any domain in $\mathbb{C}$ with the following properties.

- For any $z \in D$, $\Re z \in D$.
- For any $z_1, z_2 \in D$, there exists a point $z_0 \in D$ such that one of the numbers $z_1 - z_0, z_2 - z_0$ has zero real part while the other has zero imaginary part.
- If $z_1, z_2 \in D$ are such that either $\Im z_1 = \Im z_2$ or $\Re z_1 = \Re z_2$, then the segment $[z_1, z_2]$ lies in D.

We remark that a rectangular region symmetric about the real axis will satisfy all the above properties.

Lemma II.5 (Mean value theorem for complex functions). *Let f be a holomorphic function on a domain D as above, such that for $z \in D$, $\Im f(z)$ has the same sign as $\Im z$. Suppose further that there exist positive constants ρ_I and ρ_R such that*

- *for all $z \in D$, $|\Im f'(z)| \leq \rho_I$;*
- *for all $z \in D$, $\Re f'(z) \in [0, \rho_R]$.*

Then for any $z_1, z_2 \in D$, there exists $C_{z_1,z_2} \in [0, \rho_R]$ such that

$$\begin{aligned}|\Re(f(z_1) - f(z_2)) - C_{z_1,z_2} \cdot\ & \Re(z_1 - z_2)| \\ &\leq \rho_I \cdot |\Im(z_1 - z_2)|,\end{aligned}$$

and furthermore,

$$\begin{aligned}&|\Im(f(z_1) - f(z_2))| \\ &\leq \rho_R \cdot \begin{cases} |\Im(z_1 - z_2)|, & \textit{when } (\Im z_1) \cdot (\Im z_2) \leq 0; \\ \max\{|\Im z_1|, |\Im z_2|\}, & \textit{otherwise.} \end{cases}\end{aligned}$$

Proof: We write $f = u + \iota v$, where $u, v : D \to \mathbb{R}$ are seen as differentiable functions from $\mathbb{R}^2$ to $\mathbb{R}$ satisfying the Cauchy-Riemann equations

$$u^{(1,0)} = v^{(0,1)} \quad \text{and} \quad u^{(0,1)} = -v^{(1,0)}.$$

This implies in particular that $\Re f'(z) = u^{(1,0)}(z) = v^{(0,1)}(z)$ and $\Im f'(z) = v^{(1,0)}(z) = -u^{(0,1)}(z)$.

Let z_0 be a point in D such that $\Re(z_2 - z_0) = 0$ and $\Im(z_1 - z_0) = 0$ (by the conditions imposed on D, such a z_0 exists, possibly after interchanging z_1 and z_2). Now we have

$$\begin{aligned}&\Re(f(z_1) - f(z_2)) \\ &\quad = u(z_1) - u(z_0) + u(z_0) - u(z_2) \\ &\quad = u^{(1,0)}(z') \cdot \Re(z_1 - z_0) + u(z_0) - u(z_2),\end{aligned}$$

where z' is a point lying on the segment $[z_0, z_1]$, obtained by applying the standard mean value theorem to the function u along this segment (note that the segment is parallel to the real axis). On the other hand, since the segment $[z_0, z_2]$ is parallel to the imaginary axis, we apply the standard mean value theorem to the real valued function u to get (after recalling that $\left|u^{(0,1)}(z)\right| = |\Im f'(z)| \leq \rho_I$ for all $z \in D$)

$$|u(z_0) - u(z_2)| \leq \rho_I |\Im(z_2 - z_0)| = \rho_I |\Im(z_2 - z_1)|.$$

This proves the first part, once we set $C_{z_1,z_2} = u^{(1,0)}(z') = \Re f'(z')$, which must lie in $[0, \rho_R]$ since $z' \in D$.

For the second part, we note that since $\Im f(z) = 0$ when $\Im z = 0$, we have for $z \in D$,

$$\begin{aligned}\Im f(z) = \Im(f(z) - f(\Re z)) &= v(z) - v(\Re z) \\ &= v^{(0,1)}(z') \cdot \Im z,\end{aligned}$$

where z' is a point lying on the segment $[z, \Re z]$, obtained by applying the standard mean value theorem to the function v along this segment (note that the segment is parallel to the imaginary axis).

Since $v^{(0,1)}(z') = u^{(1,0)}(z') \in [0, \rho_R]$ for all $z' \in D$, there exist $a, b \in [0, \rho_R]$ such that

$$|\Im(f(z_1) - f(z_2))| = |a\Im z_1 - b\Im z_2|,$$

so that we get

$$\begin{aligned}&|\Im\ (f(z_1) - f(z_2))| = |a\Im z_1 - b\Im z_2| \\ &\quad \leq \rho_R \cdot \begin{cases} |\Im(z_1 - z_2)|, & \text{when } (\Im z_1) \cdot (\Im z_2) \leq 0; \\ \max\{|\Im z_1|, |\Im z_2|\}, & \text{otherwise.} \end{cases}\end{aligned}$$

■

We will apply the above lemma to the function

$$f_\kappa(x) := -\ln(1 - \kappa e^x), \tag{2}$$

which, as we shall see later, will play a central role in our proofs. (We note that here, and also later in the paper, we use $\ln$ to denote the principal branch of the complex logarithm; i.e., if $z = re^{\iota\theta}$ with $r > 0$

and $\theta \in (-\pi, \pi)$, then $\ln z = \ln r + \iota\theta$.) Below we verify that such an application is valid, and record the consequences.

Lemma II.6. *Consider the domain D given by*

$$D := \{z \mid \Re z \in (-\infty, -\zeta) \text{ and } |\Im z| < \tau\},$$

where $\tau < 1/2$ and ζ are positive real numbers such that $\tau^2 + e^{-\zeta} < 1$. Suppose $\kappa \in [0,1]$ and consider the function f_κ as defined in eq. (2). Then,

1) *The function f_κ and the domain D satisfy the hypotheses of Lemma II.5, if ρ_R and ρ_I in the statement of the theorem are taken to be $\frac{e^{-\zeta}}{1-e^{-\zeta}}$ and $\frac{\tau \cdot e^{-\zeta}}{(1-e^{-\zeta})^2}$, respectively.*
2) *If $\varepsilon > 0$ and κ' are such that $|\kappa' - \kappa| < \varepsilon$ and $(1+\varepsilon) < e^\zeta$, then for any $z \in D$,*

$$|f_{\kappa'}(z) - f_\kappa(z)| \le \frac{\varepsilon}{e^\zeta - 1 - \varepsilon}.$$

Proof: Note first that the domain D is rectangular and symmetric about the real axis, so it satisfies the properties listed before Lemma II.5. We also note that since $\kappa \le 1$, $f_\kappa(z)$ is well defined when $\Re z < 0$, and maps real numbers in D to real numbers. Further, a direct calculation shows that $\Im f_\kappa(z) = -\arg(1 - \kappa e^z)$ has the same sign as $\sin(\Im z)$ when $\Re z < 0$ (since $\kappa \in [0,1]$). Since $|\Im z| \le \tau < \pi$, we see therefore that $\Im f_\kappa(z)$ has the same sign as $\Im z$, and hence f_κ satisfies the hypothesis of Lemma II.5.

Note that $f'_\kappa(z) = \frac{\kappa e^z}{1-\kappa e^z}$. A direct calculation then shows that $\Re f'_\kappa(z) = \frac{\kappa \Re e^z - \kappa^2 |e^z|^2}{|1-\kappa e^z|^2}$ and $\Im f'_\kappa(z) = \frac{\kappa \Im e^z}{|1-\kappa e^z|^2}$. Now, for $z \in D$, $|\arg e^z| \le \tau$, so that $\Re e^z \ge |e^z| \cos \arg e^z \ge |e^z|(1-\tau^2)$. Thus, we see that $\kappa \Re e^z - \kappa^2 |e^z|^2 \ge \kappa |e^z| \left(1 - \tau^2 - \kappa |e^z|\right) \ge \kappa |e^z| \left(1 - \tau^2 - \kappa e^{-\zeta}\right)$. Since $\kappa \in [0,1]$ and $\tau^2 + e^{-\zeta} < 1$ by assumption, we therefore have $\Re f'_\kappa(z) \ge 0$. Further $\Re f'_\kappa(z) \le |f'_\kappa(z)| = \frac{\kappa |e^z|}{|1-\kappa e^z|} \le \frac{\kappa |e^z|}{1-\kappa |e^z|} \le \frac{\kappa e^{-\zeta}}{1-e^{-\zeta}}$, since $\kappa \in [0,1]$. Together, these show that $\Re f'_\kappa(z) \in \left[0, \frac{e^{-\zeta}}{1-e^{-\zeta}}\right]$ for $z \in D$, so that the claimed choice of the parameter ρ_R in Lemma II.5 is justified.

Similarly, for the imaginary part, we have $|\Im f'_\kappa(z)| = \frac{\kappa |\Im e^z|}{|1-\kappa e^z|^2}$, which in turn is at most $\frac{\kappa \cdot \tau \cdot e^{-\zeta}}{(1-\kappa e^{-\zeta})^2}$ for $z \in D$. Since $\kappa \in [0,1]$, this justifies the choice of the parameter ρ_I.

We now turn to the second item of the observation. The derivative of $f_x(z)$ with respect to x is $\frac{e^z}{1-xe^z}$, which for x within distance ε (satisfying $(1+\varepsilon) < e^\zeta$) of κ and $z \in D$ has length at most $\frac{1}{e^\zeta - 1 - \varepsilon}$. Thus, the standard mean value theorem applied along the segment $[\kappa, \kappa']$ (which is of length at most ε) yields the claim. ■

We will also need the following simple geometric lemma, versions of which have been used in the work of Barvinok [2] and also Bencs et al. [6].

Lemma II.7. *Let $z_1, z_2, \dots, z_n$ be complex numbers such that the angle between any two non-zero z_i is at most $\alpha \in [0, \pi/2)$. Then $|\sum_{i=1}^n z_i| \ge \cos(\alpha/2) \sum_{i=1}^n |z_i|$.*

Proof: Fix a non-zero z_i, and without loss of generality let z_1 and z_2 be the non-zero elements giving the maximum and minimum values, respectively, of the quantity $\arg(z_j/z_i)$, as z_j varies over all the non-zero elements (breaking ties arbitrarily). Consider the ray z bisecting the angle between z_1 and z_2. Then, by the assumption, the angle made by z and any of the non-zero z_i is at most $\alpha/2$, so that the projection of z_i on z is of length at least $|z_i| \cos(\alpha/2)$ and is in the same direction as z. Thus, denoting by S' the projection of $S = \sum_{i=1}^n z_i$ on z, we have

$$|S| \ge |S'| \ge \sum_{i=1}^n |z_i| \cos(\alpha/2).$$

■

C. Sketch of the algorithm

In this subsection we outline how to apply Barvinok's algorithmic paradigm to translate our zero-freeness result (Theorem I.4) into the FPTAS claimed in Theorem I.3. Let G be a graph with n vertices and m edges and maximum degree Δ. Recall that our goal is to obtain a $1 \pm \varepsilon$ approximation of the Potts model partition function $Z_G(w)$ at any point $w \in [0,1]$. Note that Z_G is a polynomial of degree m, and that computing Z_G at $w = 1$ is trivial since $Z_G(1) = q^n$. Recall also that Theorem I.4 ensures that Z_G has no zeros in the region $\mathcal{D}_\Delta$ of width τ_Δ around the real interval $[0,1]$. For technical convenience we will actually work with a slightly smaller zero-free region consisting of the rectangle

$$\mathcal{D}'_\Delta = \{w \in \mathbb{C} : -\tau'_\Delta \le \Re w \le 1 + \tau'_\Delta;\ |\Im w| \le \tau'_\Delta\},$$

where $\tau'_\Delta = \tau_\Delta/\sqrt{2}$. Note that $\mathcal{D}'_\Delta \subset \mathcal{D}_\Delta$ so $\mathcal{D}'_\Delta$ is also zero-free. In the rest of this section, we drop the subscript Δ from these quantities.

Now let $f(z)$ be a complex polynomial of degree d for which $f(0)$ is easy to evaluate, and suppose we wish to approximate $f(1)$. Barvinok's basic paradigm [2, Section 2.2] achieves this under the assumption that f has no zeros in the open disk $\mathcal{B}(0, 1+\delta)$ of radius $1+\delta$ centered at 0: the approximation simply consists of the first $k = O(\frac{1}{\delta}\log(\frac{d}{\varepsilon\delta}))$ terms of the Taylor expansion of

$\log f$ around 0. (Note that this expansion is absolutely convergent within $\mathcal{B}(0, 1+\delta)$ by the zero-freeness of f.) These terms can in turn be expressed as linear combinations of the first k coefficients of f itself. We now sketch how to reduce our computation of $Z_G(w)$ to this situation.

First, for any fixed $w \in [0, 1]$, define the polynomial $g(z) := Z_G(z(w-1)+1)$. Note that $g(0) = Z_G(1)$ is trivial, while $g(1) = Z_G(w)$ is the value we are trying to compute. Moreover, plainly $g(z) \neq 0$ for all $z \in \mathcal{D}'$. Next, define a polynomial $\phi : \mathbb{C} \to \mathbb{C}$ that maps the disk $\mathcal{B}(0, 1+\delta)$ into the rectangle $\mathcal{D}'$, so that $\phi(0) = 0$ and $\phi(1) = 1$; Barvinok [2, Lemma 2.2.3] gives an explicit construction of such a polynomial, with degree $N = \exp(\Theta(\tau^{-1}))$ and with $\delta = \exp(-\Theta(\tau^{-1}))$. Now we have reduced the computation of $Z_G(w)$ to that of $f(1)$, where $f(z) := g(\phi(z))$ is a polynomial of degree $\deg(g) \cdot \deg(\phi) = mN$ that is non-zero on the disk $\mathcal{B}(0, 1+\delta)$, so the framework of the previous paragraph applies. Note that the number of terms required in the Taylor expansion of $\log f$ is $k = O(\frac{1}{\delta}\log(\frac{mN}{\varepsilon\delta})) = \exp(O(\tau^{-1}))\log(\frac{n\Delta}{\varepsilon})$.

Naive computation of these k terms requires time $n^{\Theta(k)}$, which yields only a quasi-polynomial algorithm since k contains a factor of $\log n$. This complexity comes from the need to enumerate all colorings of subgraphs induced by up to k edges. However, a technique of Patel and Regts [37], based on Newton's identities and an observation of Csikvari and Frenkel [10], can be used to reduce this computation to an enumeration over subgraphs induced by *connected* sets of edges (see [37, Section 6] for details). Since G has bounded degree, this reduces the complexity to $\Delta^{O(k)} = (\frac{n\Delta}{\varepsilon})^{\log(\Delta)\exp(O(\tau^{-1}))}$. For any fixed Δ this is polynomial in (n/ε), thus satisfying the requirement of a FPTAS.

Note that the degree of the polynomial is exponential in τ^{-1}; since τ^{-1} in turn is exponential in Δ (see the discussion following the proof of Theorem I.4), the degree of the polynomial is doubly exponential in Δ. The same discussion explains how this can be improved to singly exponential for the case of uniformly large list sizes.

III. Properties of the real-valued recurrence

In this section we prove some basic properties of the real-valued recurrence established in Lemma II.4, that is, in the case where $w \in [0, 1]$ is real (and hence, $\gamma = 1 - w \in [0, 1]$).

We remark that in all graphs G appearing in our analysis, we will be able to assume that for any unpinned vertex u in G, $|L(u)| \geq \deg_G(u) + 1$. Thus, $Z_G(w) \neq 0$ whenever either (i) $w \in (0, 1]$; or (ii) $w = 0$, but G is unconflicted. As discussed in the previous section, this implies that the marginal ratios and the pseudo marginal probabilities are well-defined, and, further, the latter are actual probabilities. Moreover, if G is not connected, and G' is the connected component containing u, then we have $R^{(i,j)}_{G,u}(w) = R^{(i,j)}_{G',u}(w)$ and $\mathcal{P}_{G,w}[c(u) = i] = \mathcal{P}_{G',w}[c(u) = i]$. Thus without loss of generality, we will only consider connected graphs in this section.

We now formally state the conditions on the list sizes under which our main theorem holds.

Condition 1 (Large lists). *The graph G satisfies at least one of the following two conditions.*

1) $|L(v)| \geq \max\{2, 2 \cdot \deg_G(v)\}$ *for each unpinned vertex v in G.*
2) *The graph G is triangle-free and further, for each vertex v of G,*
$$|L(v)| \geq \alpha \cdot \deg_G(v) + \beta,$$
where α is any fixed constant larger than the unique positive solution $\alpha^\star$ of the equation $xe^{-\frac{1}{x}} = 1$ and $\beta = \beta(\alpha) \geq 2\alpha$ is a constant chosen so that $\alpha \cdot e^{-\frac{1}{\alpha}(1+\frac{1}{\beta})} \geq 1$. We note that $\alpha^\star$ lies in the interval $[1.763, 1.764]$, and β as chosen above is at least $7/2$.

Remark 3. Note that the condition $|L(v)| \geq 2$ imposed in item 1 above is without loss of generality, since any vertex with $|L(v)| = 1$ can be removed from G after removing the unique color in its list from the lists of its neighbors, without changing the number of colorings of G.

As stated in the introduction, an important element of our analysis is going to be the fact that under Condition 1, one can show that certain vertices are "nice" in the sense of the following definition. We emphasize that Condition 1 is ancillary to our main technical development: any condition under which the probability bounds imposed in the following definition can be proved (as is done in Lemma III.2 below) will be sufficient for the analysis.

Definition III.1. Given a graph G and an unpinned vertex u in G, let d be the number of unpinned neighbors of u. We say the vertex u is *nice* in G if for any $w \in [0, 1]$ and any color $i \in L(u)$, $\Pr_{G,w}[c(u) = i] \leq \frac{1}{d+2}$.

Remark 4. We adopt the convention that if G is a conflicted graph (so that it has no proper colorings) and

$w = 0$, then $\Pr_{G,w}[c(u) = i] = 0$ for every color i and every unpinned vertex u in G. This is just to simplify the presentation in this section by avoiding the need to explicitly exclude this case from the lemmas below. In the proof of our main result in sections IV and V, we will never consider conflicted graphs in a situation where w could be 0, so that this convention will then be rendered moot.

Lemma III.2. *If G satisfies Condition 1 then for any vertex u in G, and any unpinned neighbor v_k of u, we have that v_k is nice in $G_k^{(i,j)}$.*

We prove this lemma separately for each of the two cases in Condition 1.

A. Analysis for item 1 of Condition 1

Lemma III.3. *Let G be a graph that satisfies item 1 of Condition 1. Then for any unpinned vertex u in G, and any unpinned neighbor v_k of u, we have that v_k is nice in $G_k^{(i,j)}$.*

Proof: For ease of notation, we denote $G_k^{(i,j)}$ by H and v_k by v. Since G satisfies item 1 of Condition 1, and $\deg_H(v) = \deg_G(v_k) - 1$ (since the neighbor u of v_k in G is dropped in the construction of $H = G_k^{(i,j)}$), we have $|L_H(v)| = |L_G(v_k)| \geq 2\deg_G(v_k) \geq 2 \cdot \deg_H(v) + 2$.

Consider any valid coloring[4] σ' of the neighbors of v in H. For $k \in L_H(v)$, let n_k denote the number of neighbors of v that are colored k in σ'. Then for any $w \in [0, 1]$ and $i \in L_H(v)$,

$$\begin{aligned}\Pr_{H,w}[c(v) = i|\sigma'] &= \frac{w^{n_i}}{\sum_{j \in L_H(v)} w^{n_j}} \\ &\leq \frac{1}{|L_H(v)| - \deg_H(v)},\end{aligned}$$

since at most $\deg_H(v)$ of the n_j can be positive. Note in particular that if i is not a good color for v in H, then the probability is 0. Since this holds for any coloring σ', we have $\Pr_{H,w}[c(v) = i] \leq \frac{1}{|L_H(v)| - \deg_H(v)}$. Now, let d be the number of unpinned neighbors of v in H. Noting that $\deg_H(v) \geq d$, and recalling the observation above that $|L_H(v)| \geq 2\deg_H(v) + 2$, we thus have

$$\begin{aligned}\Pr_{G_k^{(i,j)},w}[c(v_k) = i] &= \Pr_{H,w}[c(v) = i] \\ &\leq \frac{1}{|L_H(v)| - \deg_H(v)} \leq \frac{1}{d+2}.\end{aligned}$$

Thus v_k is nice in $G_k^{(i,j)}$. ∎

[4]Here, we say that a coloring σ is *valid* if the color σ assigns to any vertex v is from $L(v)$, and further, in case $w = 0$, no two neighbors are assigned the same color by σ.

B. Analysis for item 2 of Condition 1

Notice that if G satisfies item 2 of Condition 1, then so does $G_k^{(i,j)}$. Thus in order to show that v_k is nice in $G_k^{(i,j)}$, it suffices to show the following more general fact.

Lemma III.4. *Let G be any graph that satisfies item 2 of Condition 1, and let u be any unpinned vertex in G, then u is nice in G.*

The proof of this lemma is almost identical to arguments that appear in the work of Gamarnik, Katz and Misra [19] on strong spatial mixing; we include a proof here for completeness.

Proof: We show first that $\Pr_{G,w}[c(u) = i] \leq \frac{1}{\beta}$ whenever $L_G(u) \geq \deg_G(u) + \beta$; this will be required later in the proof. To do so, we repeat the arguments in the proof of Lemma III.3 to see that $\Pr_{G,w}[c(u) = i] \leq \frac{1}{|L(u)| - \deg_G(u)}$. The claimed bound then follows since $|L(u)| - \deg_G(u) \geq \beta$.

Next we show that the upper bound of $\frac{1}{d+2}$, where d is the number of unpinned neighbors of u in G, holds conditioned on every coloring of the neighbors of the (unpinned) neighbors of u, by following a similar path as in [19]. Consider any valid coloring[5] σ' of the vertices at distance *two* from u. Since G is triangle free, we claim that conditional on σ' there is a tree T of depth 2 rooted at u, with all the leaves pinned according to σ', such that

$$\Pr_{G,w}[c(u) = i|\sigma'] = \Pr_{T,w}[c(u) = i]\,. \tag{3}$$

To see this, notice that once we condition on the coloring of the vertices at distance 2 from u, the distribution of the color at u becomes independent of the distribution of colors of vertices at distance 3 or more. Further, because of triangle freeness, no two neighbors of u have an edge between them, and hence any cycle in the distance-2 neighborhood, if one exists, must go through at least one pinned vertex. We then observe that such a cycle can be broken by replacing any pinned vertex v' in it with $\deg(v')$ copies, one for each of its neighbor: as discussed earlier, this operation cannot change the partition function or probabilities. This operation therefore ensures that every pinned vertex in the resulting graph is now a leaf of a tree T of depth 2 rooted at u. Further, in T, the root u has d unpinned children, and all vertices at depth 2 are pinned according to σ'.

[5]Here, we say that a coloring σ is *valid* if the color σ assign to any vertex v is from $L(v)$, and further, in case $w = 0$, no two neighbors are assigned the same color by σ.

Let $v_1, \cdots, v_d$ be the d unpinned neighbors of u in T, and let $T_1, \cdots, T_d$ be the subtrees rooted at $v_1, \cdots, v_d$ respectively. For each $k \in L_G(u)$, let n_k be the number of neighbors of u that are pinned to color k. Then by Lemma II.4,

$$R_{T,u}^{(j,i)}(w) = \frac{w^{n_j} \cdot \prod_{k=1}^{d}(1 - \gamma \cdot \mathcal{P}_{T_k,w}[c(v_k) = j]\,)}{w^{n_i} \cdot \prod_{k=1}^{d}(1 - \gamma \cdot \mathcal{P}_{T_k,w}[c(v_k) = i]\,)}.$$

Define $t_{kj} := \gamma \cdot \Pr_{T_k,w}[c(v_k) = j]$, and note that from the calculation at the beginning of the proof, we have $0 \leq t_{kj} \leq \frac{\gamma}{\beta} \leq \frac{1}{\beta} \leq 1/2$. Note also that $t_{kj} = 0$ if $j \notin L(v_k)$. Thus, we have

$$\sum_{j \in \Gamma_u} t_{kj} = \gamma \sum_{j \in \Gamma_u \cap L(v_k)} \Pr_{T_k,w}[c(v_k) = j] \leq \gamma \leq 1. \tag{4}$$

Therefore,

$$\begin{aligned} \Pr_{T,w}[c(u) = i] &= \frac{1}{\sum_{j \in L(v)} R_{T,v}^{(j,i)}(w)} \\ &= \frac{w^{n_i} \cdot \prod_{k=1}^{d}(1 - t_{ki})}{\sum_{j \in L(u)} w^{n_j} \prod_{k=1}^{d}(1 - t_{kj})} \\ &\leq \frac{1}{\sum_{j \in \Gamma_u} \prod_{k=1}^{d}(1 - t_{kj})}, \end{aligned} \tag{5}$$

where, in the last inequality we use that $n_j = 0$ when j is good for u in G, and also that $w \in [0, 1]$.

Since $\Pr_{G,w}[c(u) = i|\sigma'] = \Pr_{T,w}[c(u) = i]$, it remains to lower bound the denominator term $\sum_{j \in \Gamma_u} \prod_{k=1}^{d}(1 - t_{kj})$. We begin by recalling the following standard consequence of the Taylor expansion of $\ln(1 - x)$ around 0: when $0 \leq x \leq \frac{1}{\beta} < 1$, and β is such that $(1 - 1/\beta)^2 \geq 1/2$,

$$\begin{aligned} \ln(1 - x) &\geq -x - \frac{x^2}{2(1 - 1/\beta)^2} \\ &\geq -x - x^2 \\ &\geq -\left(1 + \frac{1}{\beta}\right)x. \end{aligned} \tag{6}$$

Note that the condition required of β is satisfied since $\beta \geq 2\alpha \geq 7/2$, as stipulated in item 2 of Condition 1. Since $0 \leq t_{kj} \leq 1/\beta$, we therefore obtain, for every $j \in \Gamma_u$,

$$\begin{aligned} \prod_{k=1}^{d}(1 - t_{kj}) &\geq \prod_{k=1}^{d} \exp\left(-\left(1 + \frac{1}{\beta}\right)t_{kj}\right) \\ &= \exp\left(-\left(1 + \frac{1}{\beta}\right)\sum_{k=1}^{d} t_{kj}\right). \end{aligned} \tag{7}$$

For convenience of notation, we denote $|\Gamma_u|$ by q_u. Note that since $|L(u)| \geq \alpha \deg(u) + \beta$, and u has $\deg(u) - d$ pinned neighbors, we have

$$\begin{aligned} q_u &\geq |L(u)| - (\deg(u) - d) \\ &\geq |L(u)| - \alpha(\deg(u) - d) \\ &\geq \alpha d + \beta, \end{aligned} \tag{8}$$

where in the second inequality we use $\alpha \geq 1$. Now, by the AM-GM inequality, we get

$$\begin{aligned} &\sum_{j \in \Gamma_u} \prod_{k=1}^{d}(1 - t_{kj}) \\ &\geq q_u \left(\prod_{j \in \Gamma_u} \prod_{k=1}^{d}(1 - t_{kj})\right)^{\frac{1}{q_u}} \\ &\geq q_u \exp\left(-\frac{1 + 1/\beta}{q_u} \cdot \sum_{k=1}^{d} \sum_{j \in \Gamma_u} t_{kj}\right), \text{ using eq. (7)} \\ &\geq (\alpha d + \beta) \exp\left(-\frac{d(1 + 1/\beta)}{\alpha d + \beta}\right), \text{ by eqs. (4) and (8)} \\ &\geq (d + 2)\alpha \cdot \exp\left(-\frac{(1 + 1/\beta)}{\alpha}\right), \text{ using } \beta \geq 2\alpha \\ &\geq (d + 2), \end{aligned}$$

where the last line uses the stipulation in item 2 of Condition 1 that α and β satisfy $\alpha \cdot \exp\left(-\frac{(1+1/\beta)}{\alpha}\right) \geq 1$. From eqs. (3) and (5) we therefore get

$$\Pr_{G,w}[c(u) = i|\sigma'] \leq \frac{1}{d + 2}.$$

Since this holds for any conditioning σ' of the colors of the neighbors of the neighbors of u in G, we then have

$$\Pr_{G,w}[c(u) = i] \leq \frac{1}{d + 2},$$

which concludes the proof.

■

The proof of Lemma III.2 is immediate from Lemmas III.3 and III.4.

Proof of Lemma III.2: If G satisfies item 1 of Condition 1 then we apply Lemma III.3. If G satisfies item 2 of Condition 1 then we apply Lemma III.4 after noting that if G satisfies item 2 of Condition 1, then so does $G_k^{(i,j)}$, and further that, as assumed in the hypothesis of Lemma III.2, v_k is unpinned in $G_k^{(i,j)}$. ■

We conclude this section by noting that, the niceness condition can be strengthened in the case when all the list sizes are uniformly large (e.g., as in the case of q-colorings).

Remark 5. In Condition 1, if we replace the degree of a vertex by the maximum degree Δ (e.g., in item 1 of the condition, if we assume $|L(v)| \geq 2\Delta$, instead of $2\deg_G(v)$, for each v), then for every vertex v in the graph G, it holds that $\Pr_{G,w}[c(v) = i] < \min\left\{\frac{4}{3\Delta}, 1\right\}$.

To see this, notice that the same calculation as in the proof of Lemma III.3 above gives $\Pr_{G,w}[c(v) = i] \leq \frac{1}{|L(v)|-\Delta} \leq \frac{1}{(\alpha-1)\Delta+\beta} \leq \frac{1}{(\alpha-1)\Delta} < \frac{4}{3\Delta}$. We will refer to this stronger condition on list sizes (which holds, in particular, in the case of q-colorings), as the *uniformly large list size* condition.

IV. Zero-free region for small $|w|$

As explained in the introduction, all our algorithmic results follow from Theorem I.4, which establishes a zero-free region for the partition function $Z_G(w)$ around the interval $[0, 1]$ in the complex plane. We split the proof of Theorem I.4 into two parts: in this section, we establish the existence of a zero-free disk around the endpoint $w = 0$ (see Theorem IV.1): this is the most delicate case because $w = 0$ corresponds to proper colorings. Then in section V (see Theorem V.1) we derive a zero-free region around the remainder of the interval, using a similar but less delicate approach. Taken together, Theorems IV.1 and V.1 immediately imply Theorem I.4, so this will conclude our analysis.

Theorem IV.1. *Fix a positive integer Δ. There exists a $\nu_w = \nu_w(\Delta)$ such that the following is true. Let G be a graph of maximum degree Δ satisfying Condition 1, and having no pinned vertices. Then, $Z_G(w) \neq 0$ for any w satisfying $|w| \leq \nu_w$.*

In the proof, we will encounter several constants which we now fix. Given the degree bound $\Delta \geq 1$, we define

$$\varepsilon_R := \frac{0.01}{\Delta^2},\ \varepsilon_I := \varepsilon_R \cdot \frac{0.01}{\Delta^2},\ \text{and } \varepsilon_w := \varepsilon_I \cdot \frac{0.01}{\Delta^3}. \quad (9)$$

We will then see that the quantity ν_w in the statement of the theorem can be chosen to be $0.2\varepsilon_w/2^\Delta$. (In fact, we will show that if one has the slightly stronger assumption of uniformly large list sizes considered in Remark 5, then ν_w can be chosen to be $\varepsilon_w/(300\Delta)$).

Throughout the rest of this section, we fix Δ to be the maximum degree of the graphs, and let $\varepsilon_w, \varepsilon_I, \varepsilon_R$ be as above.

We now briefly outline our strategy for the proof. Recall that, for a vertex u and colors i, j, the marginal ratio is given by $R^{(i,j)}_{G,u}(w) = \frac{Z^{(i)}_{G,u}(w)}{Z^{(j)}_{G,u}(w)}$. When G is an unconflicted graph, $R^{(i,j)}_{G,u}(0)$ is always a well-defined non-negative real number. Intuitively, we would like to show that $R^{(i,j)}_{G,u}(w) \approx R^{(i,j)}_{G,u}(0)$, independent of the size of G, when $w \in \mathbb{C}$ is close to 0. Given such an approximation one can use a simple geometric argument (see Consequence IV.3) to conclude that the partition function does not vanish for such w. In order to prove the above approximate equality inductively for a given graph G, we take an approach that exploits the properties of the "real" case (i.e., of $R^{(i,j)}_{G,u}(0)$) and then uses the notion of "niceness" of certain vertices described earlier to control the accumulation of errors. To this end, we will prove the following lemma via induction on the number of unpinned vertices in G. Theorem IV.1 will follow almost immediately from the lemma; see the end of this section for the details.

Lemma IV.2. *Let G be an unconflicted graph of maximum degree Δ satisfying Condition 1, and u be any unpinned vertex in G. Then, the following are true (with $\varepsilon_w, \varepsilon_I$, and ε_I as defined in eq. (9)):*

1) *For $i \in \Gamma_u$, $\left|Z^{(i)}_{G,u}(w)\right| > 0$.*
2) *For $i, j \in \Gamma_u$, if u has all neighbors pinned, then $R^{(i,j)}_{G,u}(w) = R^{(i,j)}_{G,u}(0) = 1$.*
3) *For $i, j \in \Gamma_u$, if u has $d \geq 1$ unpinned neighbors, then*
$$\frac{1}{d}\left|\Re \ln R^{(i,j)}_{G,u}(w) - \Re \ln R^{(i,j)}_{G,u}(0)\right| < \varepsilon_R.$$
4) *For any $i, j \in \Gamma_u$, if u has $d \geq 1$ unpinned neighbors, we have $\frac{1}{d}\left|\Im \ln R^{(i,j)}_{G,u}(w)\right| < \varepsilon_I$.*
5) *For any $i \notin \Gamma_u, j \in \Gamma_u$, then $\left|R^{(i,j)}_{G,u}(w)\right| \leq \varepsilon_w$.*

We will refer to items 1 to 5 as "items of the induction hypothesis". The rest of this section is devoted to the proof of this lemma via induction on the number of unpinned vertices in G.

We begin by verifying that the induction hypothesis holds in the base case when u is the only unpinned vertex in an unconflicted graph G. In this case, items 3 and 4 are vacuously true since u has no unpinned neighbors. Since all neighbors of u in G are pinned, the fact that all pinned vertices have degree at most one implies that G can be decomposed into two disjoint components G_1 and G_2, where G_1 consists of u and its pinned neighbors, while G_2 consists of a disjoint union of unconflicted edges (since G is unconflicted). Now, since G_1 and G_2 are disjoint components, we have $Z^{(i)}_{G,u}(w) = Z_{G_2}(w) = 1$ for all $i \in \Gamma_{G,u}$ and all $w \in \mathbb{C}$. This proves items 1 and 2. Similarly, when $i \notin \Gamma_{G,u}$, we have $Z^{(i)}_{G,u}(w) = w^{n_i}$, where $n_i \geq 1$ is the number of neighbors of u pinned to color i. This gives

$$\left|R^{(i,j)}_{G,u}(w)\right| \leq |w|^{n_i} \leq \varepsilon_w,$$

since $|w| \le \varepsilon_w \le 1$, and proves item 5.

We now derive some consequences of the above induction hypothesis that will be helpful in carrying out the induction. Throughout, we assume that G is an unconflicted graph satisfying Condition 1.

Consequence IV.3. *If* $|L(u)| \ge \deg_G(u) + 1$ *then*

$$|Z_G(w)| \ge 0.9 \min_{i \in \Gamma_u} \left| Z_{G,v}^{(i)}(w) \right| > 0.$$

Proof: Note that $Z_G(w) = \sum_{i \in L(u)} Z_{G,u}^{(i)}(w)$. From item 4, we see that the angle between the complex numbers $Z_{G,u}^{(i)}(w)$ and $Z_{G,u}^{(j)}(w)$, when $i, j \in \Gamma_u$, is at most $d\varepsilon_I$. Applying Lemma II.7 to the terms corresponding to the good colors and item 5 to the terms corresponding to the bad colors, we then have

$$\begin{aligned}
&\left| \sum_{i \in L(u)} Z_{G,u}^{(i)}(w) \right| \\
&\ge \left(|\Gamma_u| \cos \frac{d\varepsilon_I}{2} - |L(u) \setminus \Gamma_u| \, \varepsilon_w \right) \min_{i \in \Gamma_u} \left| Z_{G,u}^{(i)}(w) \right| \\
&\ge \left(\cos \frac{d\varepsilon_I}{2} - \deg_G(u) \cdot \varepsilon_w \right) \min_{i \in \Gamma_u} \left| Z_{G,u}^{(i)}(w) \right|,
\end{aligned}$$

where we use the fact that $|L(u) \setminus \Gamma_u| \le \deg_G(u)$ and $|L(u)| \ge \deg_G(u) + 1$ in the last inequality. Since $d\varepsilon_I \le 0.01$ and $\varepsilon_w \le 0.01/\Delta$, we then have $\left| \sum_{i \in L(u)} Z_{G,u}^{(i)}(w) \right| \ge 0.9 \min_{i \in \Gamma_u} \left| Z_{G,v}^{(i)}(w) \right|$, which in turn is positive from item 1. ■

Consequence IV.4. *The pseudo-probabilities approximate the real probabilities in the following sense:*

1) *for any* $i \notin \Gamma_u$, $|\mathcal{P}_{G,w}[c(u) = i]| \le 1.2\varepsilon_w$.
2) *for any* $j \in \Gamma_u$,

$$\begin{aligned}
\left| \Im \ln \frac{\mathcal{P}_{G,w}[c(u) = j]}{\mathcal{P}_G[c(u) = j]} \right| &= |\Im \ln \mathcal{P}_{G,w}[c(u) = j]| \\
&\le d\varepsilon_I + 2\Delta\varepsilon_w,
\end{aligned}$$

and,

$$\left| \Re \ln \frac{\mathcal{P}_{G,w}[c(u) = j]}{\mathcal{P}_G[c(u) = j]} \right| \le d\varepsilon_R + d\varepsilon_I + 2\Delta\varepsilon_w,$$

where d *is the number of unpinned neighbors of* u *in* G.

Proof: For part (1), by Consequence IV.3 we have

$$\begin{aligned}
|\mathcal{P}_{G,w}[c(u) = i]| &= \frac{\left| Z_{G,u}^{(i)}(w) \right|}{|Z_G(w)|} \\
&\le \frac{\left| Z_{G,u}^{(i)}(w) \right|}{0.9 \min_{j \in \Gamma_u} \left| Z_{G,u}^{(j)}(w) \right|} \le 1.2\varepsilon_w,
\end{aligned}$$

where the last inequality follows from induction hypothesis item 5.

For part (2), by items 2 to 4 of the induction hypothesis, there exist complex numbers ξ_i (for all $i \in \Gamma_u$) satisfying $|\Re \xi_i| \le d\varepsilon_R$ and $|\Im \xi_i| \le d\varepsilon_I$ such that

$$\begin{aligned}
&\frac{1}{\mathcal{P}_{G,w}[c(u) = j]} \\
&= \sum_{i \in L(u)} \frac{Z_{G,u}^{(i)}(w)}{Z_{G,u}^{(j)}(w)} \\
&= \underbrace{\sum_{i \in \Gamma_u} \frac{Z_{G,u}^{(i)}(0)}{Z_{G,u}^{(j)}(0)} e^{\xi_i}}_{:=A} + \underbrace{\sum_{i \in L(u) \setminus \Gamma_u} \frac{Z_{G,u}^{(i)}(w)}{Z_{G,u}^{(j)}(w)}}_{:=B}.
\end{aligned}$$

Next we show that $A \approx \frac{1}{\mathcal{P}_G[c(u)=j]}$ and B is negligible. From item 5 of the induction hypothesis we have

$$\mathcal{P}_G[c(u) = j] \cdot |B| \le \Delta\varepsilon_w. \tag{10}$$

Now, note that $\sum_{i \in \Gamma_u} \frac{Z_{G,u}^{(i)}(0)}{Z_{G,u}^{(j)}(0)} = \frac{1}{\mathcal{P}_G[c(u)=j]}$. Further, when $\varepsilon_I \le 0.1/\Delta$, we also have[6]

$$\Re e^{\xi_i} \in \left(e^{-d\varepsilon_R} - d^2\varepsilon_I^2,\ e^{d\varepsilon_R} \right), \text{ and } |\arg e^{\xi_i}| \le d\varepsilon_I. \tag{11}$$

The above will therefore be true also for any convex combination of the e^{ξ_i}. Noting that $\mathcal{P}_G[c(u) = j] \cdot A$ is just such a convex combination (as the coefficients of the e^{ξ_i} are non-negative reals summing to 1), we have

$$\mathcal{P}_G[c(u) = j] \cdot \Re A \in (e^{-d\varepsilon_R} - d^2\varepsilon_I^2,\ e^{d\varepsilon_R}), \tag{12}$$

$$|\arg(\mathcal{P}_G[c(u) = j] \cdot A)| \le d\varepsilon_I. \tag{13}$$

Together, eqs. (10), (12) and (13) imply that if $C := \frac{\mathcal{P}_G[c(u)=j]}{\mathcal{P}_{G,w}[c(u)=j]}$ then (using the values of $\varepsilon_R, \varepsilon_I$, and ε_w)[7]

$$\begin{aligned}
&\Re C \in \left(e^{-d\varepsilon_R} - d^2\varepsilon_I^2 - \Delta\varepsilon_w,\ e^{d\varepsilon_R} + \Delta\varepsilon_w \right), \text{ and} \\
&\arg C \in (-d\varepsilon_I - 2\Delta\varepsilon_w, d\varepsilon_I + 2\Delta\varepsilon_w).
\end{aligned}$$

Thus, since $\varepsilon_I, \varepsilon_R$ are small enough and $\varepsilon_w \le 0.01 \min\{\varepsilon_I, \varepsilon_R\}$, we have

$$\begin{aligned}
&|\Re \ln C| \le d\varepsilon_R + d\varepsilon_I + 2\Delta\varepsilon_w, \text{ and} \\
&|\Im \ln C| \le d\varepsilon_I + 2\Delta\varepsilon_w.
\end{aligned}$$

[6] Here, we also use the elementary facts that if z is a complex number satisfying $\Re z = r$ and $|\Im z| = \theta \le 0.1$ then $|\arg e^z| = |\Im z| = \theta$, and $e^r \ge \Re e^z = e^r \cos\theta = \exp(r + \ln\cos\theta) \ge \exp(r - \theta^2) \ge e^r - e^r\theta^2$. Hence if $r < 0$, we have $\Re e^z \ge e^r - \theta^2$.

[7] Here, for the second inclusion, we use the following elementary computation. Let z, s be complex numbers such that $\Re z = r \in [0.9, 1.1]$, $|\arg z| = \theta \le 0.1$ and $|s| \le 0.1$. Then, we have $\Re(z + s) \ge r - |s|$ and $|\Im(z + s)| \le r\theta + |s|$. Thus, $|\arg(z + s)| \le \frac{|\Im(z+s)|}{|\Re(z+s)|} \le \frac{r\theta + |s|}{r - |s|} = \theta + |s| \cdot \frac{1+\theta}{r - |s|} \le \theta + 2|s|$.

Here we use the elementary fact that for $z \in \mathbb{C}$, $\Re \ln z = \ln |z|$ and $\Im \ln z = \arg z$. Further, for z satisfying $\Re z = r \in [0.9, 1.1]$ and $|\arg z| = \theta \leq 0.1$, we also have $\ln r \leq \Re \ln z \leq \ln r + \ln \sec \theta \leq \ln r + \theta^2$. ■

In the next consequence, we show that the error contracts during the induction. We first set up some notation. For a graph G, a vertex u, and a color $i \in \Gamma_u$, we let $a^{(i)}_{G,u}(w) = \ln \mathcal{P}_{G,w}[c(u) = i]$. We also recall that $\gamma := 1 - w$, and the definition of the function $f_\gamma(x) := -\ln(1 - \gamma e^x)$ from eq. (2).

Consequence IV.5. *There exists a positive constant $\eta \in [0.9, 1)$ so that the following is true. Let d be the number of unpinned neighbors of u. Assume further that u is nice in G. Then, for any colors $i, j \in \Gamma_u$, there exists a real constant $c = c_{G,u,i} \in [0, \frac{1}{d+\eta}]$ such that*

$$\Big|\Re f_\gamma(a^{(i)}_{G,u}(w)) - f_1(a^{(i)}_{G,u}(0)) - c \cdot \Re(a^{(i)}_{G,u}(w) - a^{(i)}_{G,u}(0))\Big| \leq \varepsilon_I + \varepsilon_w; \quad (14)$$

$$\Big|\Im f_\gamma(a^{(i)}_{G,u}(w)) - \Im f_\gamma(a^{(j)}_{G,u}(w))\Big| \leq \frac{1}{d+\eta} \cdot (d\varepsilon_I + 4\Delta\varepsilon_w) + 2\varepsilon_w; \quad (15)$$

$$\Big|\Im f_\gamma(a^{(i)}_{G,u}(w))\Big| \leq \frac{1}{d+\eta} \cdot (d\varepsilon_I + 4\Delta\varepsilon_w) + \varepsilon_w. \quad (16)$$

Proof: Since u is nice in G, the bound $\mathcal{P}_{G,0}[c(u) = k] \leq \frac{1}{d+2}$ (for any $k \in \Gamma_{G,u}$) applies. Combining them with Consequence IV.4 we see that $a^{(i)}_{G,u}(w), a^{(i)}_{G,u}(0), a^{(j)}_{G,u}(w), a^{(j)}_{G,u}(0)$ lie in a domain D as described in Lemma II.6 (with the parameter κ therein set to 1), with the parameters ζ and τ in that observation chosen as

$$\begin{aligned} \zeta &= \ln(d+2) - d\varepsilon_R - d\varepsilon_I - 2\Delta\varepsilon_w \text{ , and} \\ \tau &= d\varepsilon_I + 2\Delta\varepsilon_w. \end{aligned} \quad (17)$$

Here, for the bound on ζ, we use the fact that for $j \in \Gamma_{G,u}$, $\mathcal{P}_G[c(u) = j] \leq \frac{1}{d+2}$, which is due to u being nice in G.

The bounds on $\varepsilon_w, \varepsilon_I$ and ε_R now imply $e^\zeta \geq (d + 2)\left(1 - \frac{0.02}{\Delta}\right) \geq d + 1.94$, and also that $\tau \leq 0.02/\Delta$. Thus, the conditions required on ζ and τ in Lemma II.6 (i.e. that $\tau < 1/2$ and $\tau^2 + e^{-\zeta} < 1$) are satisfied. Further, ρ_R and ρ_I as set in the observation satisfy $\rho_R \leq \frac{1}{d+\eta}$, where η can be taken to be 0.94, and $\rho_I < 3\varepsilon_I$.

Using Lemma II.5 followed by the value of ε_w, and noting that $a^{(i)}_{G,u}(0)$ is a real number, we then have

$$\begin{aligned} &\Big|\Re f_1(a^{(i)}_{G,u}(w)) - f_1(a^{(i)}_{G,u}(0)) - c \cdot \Re\Big(a^{(i)}_{G,u}(w) - a^{(i)}_{G,u}(0)\Big)\Big| \\ &\leq \rho_I \cdot \Big|\Im\Big(a^{(i)}_{G,u}(w) - a^{(i)}_{G,u}(0)\Big)\Big| \\ &\leq 3\varepsilon_I(d\varepsilon_I + 2\Delta\varepsilon_w) \leq 4d\varepsilon_I^2 \leq \varepsilon_I, \end{aligned} \quad (18)$$

for an appropriate positive $c \leq 1/(d + \eta)$. This is almost eq. (14), whose difference will be handled later.

Similarly, applying Lemma II.5 to the imaginary part we have

$$\begin{aligned} &\Big|\Im f_1(a^{(i)}_{G,u}(w)) - \Im f_1(a^{(j)}_{G,u}(w))\Big)\Big| \\ &\leq \rho_R \cdot \max\Big\{\Big|\Im\Big(a^{(i)}_{G,u}(w) - a^{(j)}_{G,u}(w)\Big)\Big|, \\ &\qquad \Big|\Im a^{(i)}_{G,u}(w)\Big|, \Big|\Im a^{(j)}_{G,u}(w)\Big|\Big\}, \end{aligned} \quad (19)$$

where, as noted above, $\rho_R \leq \frac{1}{d+\eta}$. Now, note that the first term in the above maximum is less than $d\varepsilon_I$ by item 4 of the induction hypothesis, while the other two terms are at most $d\varepsilon_I + 2\Delta\varepsilon_w$ from item 2 of Consequence IV.4. This is almost the bound in eq. (15), whose difference will be handled later.

To prove the bound in eq. (16), we first apply the imaginary part of Lemma II.5 along with the fact that $\Im a^{(i)}_{G,u}(0) = 0$ to get

$$\begin{aligned} \Big|\Im f_1(a^{(i)}_{G,u}(w))\Big| &= \Big|\Im f_1(a^{(i)}_{G,u}(w)) - f_1(a^{(i)}_{G,u}(0))\Big| \\ &\leq \rho_R \cdot \Big|\Im\Big(a^{(i)}_{G,u}(w)\Big)\Big| \\ &\leq \frac{1}{d+\eta}(d\varepsilon_I + \Delta\varepsilon_w). \end{aligned} \quad (20)$$

Finally, we use item 2 of Lemma II.6 (with the parameter κ' therein set to γ) to conclude the proofs of eqs. (14) to (16) . To this end, we note that γ satisfies $|\gamma - 1| \leq \varepsilon_w$, so that the condition $(1 + \varepsilon_w) < e^\zeta$ required for item 2 to apply is satisfied. Thus we see that for any $z \in D$,

$$|f_\gamma(z) - f_1(z)| \leq \varepsilon_w,$$

so that the quantities $|\Re f_\gamma(a^{(i)}_{G,u}(w)) - \Re f_1(a^{(i)}_{G,u}(w))|$, $|\Im f_\gamma(a^{(i)}_{G,u}(w)) - \Im f_1(a^{(i)}_{G,u}(w))|$, $|\Im f_\gamma(a^{(j)}_{G,u}(w)) - \Im f_1(a^{(j)}_{G,u}(w))|$, and $|\Im f_\gamma(a^{(j)}_{G,u}(w)) - \Im f_1(a^{(j)}_{G,u}(w))|$ are all at most ε_w. The desired bounds of eqs. (14) to (16) now follow from the triangle inequality and the bounds in eqs. (18) to (20) . ■

We set up some further notation for the next consequence. For a color $i \in L(u) \setminus \Gamma_u$ we let

$b^{(i)}_{G,u}(w) = \mathcal{P}_{G,w}[c(u) = i]$. We then consider the function $g_\gamma(x) := -\ln(1 - \gamma x)$.

Consequence IV.6. *For every color $i \notin \Gamma_u$, $\left|g_\gamma(b^{(i)}_{G,u}(w))\right| \le 2\varepsilon_w$.*

Proof: Item 1 of Consequence IV.4 implies that $\left|b^{(i)}_{G,u}(w)\right| \le 1.2\varepsilon_w$. Thus, recalling that $|\gamma - 1| \le \varepsilon_w$, we get that for all $\varepsilon_w < 0.01$, $\left|g_\gamma(b^{(i)}_{G,u}(w))\right| = \left|\ln(1 - \gamma b^{(i)}_{G,u}(w))\right| \le 2\varepsilon_w$. ■

Inductive proof of Lemma IV.2

We are now ready to see the induction step in the proof of Lemma IV.2; recall that the base case was already established following the statement of the lemma. Let G be any unconflicted graph which satisfies Condition 1 and had at least two unpinned vertices (the base case when $|G| = 1$ was already handled above). We first prove induction item 1 for any vertex $u \in G$. Consider the graph G' obtained from G by pinning vertex u to color i. Note that by the definition of the pinning operation, $Z^{(i)}_{G,u}(w) = Z_{G'}(w)$, and when $i \in \Gamma_{G,u}$, the graph G' is also unconflicted and satisfies Condition 1, and has one fewer unpinned vertex than G. Thus, from Consequence IV.3 of the induction hypothesis applied to G', we have that $\left|Z^{(i)}_{G,u}(w)\right| = |Z_{G'}(w)| > 0$.

We now consider item 2. When all neighbors of u in G are pinned, the fact that all pinned vertices have degree at most one implies that G can be decomposed into two disjoint components G_1 and G_2, where G_1 consists of u and its pinned neighbors, while G_2 is also unconflicted (when G is unconflicted) and has one fewer unpinned vertex than G. Now, since G_1 and G_2 are disjoint components, we have $Z^{(k)}_{G,u}(x) = Z_{G_2}(x)$ for all $k \in \Gamma_{G,u}$ and all $x \in \mathbb{C}$. Further, from Consequence IV.3 of the induction hypothesis applied to G_2, we also have that $Z_{G_2}(w)$ and $Z_{G_2}(0)$ are both non-zero. It therefore follows that when $i, j \in \Gamma_{G,u}$, $R^{(i,j)}_{G,u}(w) = R^{(i,j)}_{G,u}(0) = 1$.

We now consider items 3 and 4. Recall that by Lemma II.4, we have

$$R^{(i,j)}_{G,u}(w) = \prod_{k=1}^{\deg_G(u)} \frac{1 - \gamma \mathcal{P}_{G^{(i,j)}_k,w}[c(v_k) = i]}{1 - \gamma \mathcal{P}_{G^{(i,j)}_k,w}[c(v_k) = j]}. \tag{21}$$

For simplicity we write $G_k := G^{(i,j)}_k$. Note that when $i, j \in \Gamma_{G,u}$, and G is unconflicted, so are the G_k. Further, each G_k has exactly one fewer unpinned vertex than G, so that the induction hypothesis applies to each G_k. Note also that when $i, j \in \Gamma_{G,u}$, we can restrict the product above to the d unpinned neighbors of u, since for such i, j, the contribution of the factor corresponding to a pinned neighbor is 1, irrespective of the value of w. Without loss of generality, we relabel these unpinned neighbors as $v_1, v_2, \dots, v_d$.

Now, as before, for $s \in \Gamma_{G_k,v_k}$ we define $a^{(s)}_{G_k,v_k}(w) := \ln \mathcal{P}_{G_k,w}[c(v_k) = s]$; while for $t \in L(v_k) \setminus \Gamma_{G_k,v_k}$ we let $b^{(t)}_{G_k,v_k}(w) := \mathcal{P}_{G_k,w}[c(v_k) = t]$. For a graph G, a vertex u and a color s, we let $B_{G,u}(s)$ be the set of those neighbors of u for which s is a bad color in $G \setminus \{u\}$. For simplicity we will also write $B(s) := B_{G,u}(s)$ when it is clear from the context. As before, we have $\gamma = 1 - w$, $f_\gamma(x) = -\ln(1 - \gamma e^x)$, $g_\gamma(x) = -\ln(1 - \gamma x)$. From the above recurrence, we then have:

$$\begin{aligned}
&-\ln R^{(i,j)}_{G,u}(w) \\
&= \sum_{v_k \in \overline{B(i)} \cap \overline{B(j)}} \left(f_\gamma\left(a^{(i)}_{G_k,v_k}(w)\right) - f_\gamma\left(a^{(j)}_{G_k,v_k}(w)\right)\right) \\
&+ \sum_{v_k \in \overline{B(i)} \cap B(j)} f_\gamma\left(a^{(i)}_{G_k,v_k}(w)\right) \\
&- \sum_{v_k \in B(i) \cap \overline{B(j)}} f_\gamma\left(a^{(j)}_{G_k,v_k}(w)\right) \\
&- \sum_{v_k \in \overline{B(i)} \cap B(j)} g_\gamma\left(b^{(j)}_{G_k,v_k}(w)\right) \\
&+ \sum_{v_k \in B(i) \cap \overline{B(j)}} g_\gamma\left(b^{(i)}_{G_k,v_k}(w)\right) \\
&+ \sum_{v_k \in B(i) \cap B(j)} \left(g_\gamma\left(b^{(i)}_{G_k,v_k}(w)\right) - g_\gamma\left(b^{(j)}_{G_k,v_k}(w)\right)\right).
\end{aligned} \tag{22}$$

Note that the same recurrence also applies when w is replaced by 0 (and hence γ by 1), except in that case the last three sums are 0 (as, when i is bad for v_k in G_k, we have $b^{(i)}_{G_k,v_k}(0) := \Pr_{G_k}[c(v_k) = i] = 0$):

$$\begin{aligned}
&-\ln R^{(i,j)}_{G,u}(0) \\
&= \sum_{v_k \in \overline{B(i)} \cap \overline{B(j)}} \left(f_1\left(a^{(i)}_{G_k,v_k}(0)\right) - f_1\left(a^{(j)}_{G_k,v_k}(0)\right)\right) \\
&+ \sum_{v_k \in \overline{B(i)} \cap B(j)} f_1\left(a^{(i)}_{G_k,v_k}(0)\right) \\
&- \sum_{v_k \in B(i) \cap \overline{B(j)}} f_1\left(a^{(j)}_{G_k,v_k}(0)\right).
\end{aligned} \tag{23}$$

Further, by Consequence IV.6 of the induction hypothesis applied to the graph G_k at a vertex $v_k \in B(i)$ (respectively, $v_k \in B(j)$) we see that $\left|g_\gamma\left(b^{(i)}_{G_k,v_k}(w)\right)\right| \le$

$2\varepsilon_w$ (respectively, $g_\gamma\left(b^{(j)}_{G_k,v_k}(w)\right) \le 2\varepsilon_w$). Thus, applying the triangle inequality to the real part of the difference of the two recurrences, we get

$$\begin{aligned}\frac{1}{d}\left|\Re \ln R^{(i,j)}_{G,u}(0) - \ln R^{(i,j)}_{G,u}(w)\right| &\le 2\Delta\varepsilon_w + \max\Bigg\{\\ \max_{v_k \in \overline{B(i)}\cap\overline{B(j)}} &\Bigg\{\Bigg|\left(\Re f_\gamma\left(a^{(i)}_{G_k,v_k}(w)\right) - f_1\left(a^{(i)}_{G_k,v_k}(0)\right)\right)\\ &\quad -\left(\Re f_\gamma\left(a^{(j)}_{G_k,v_k}(w)\right) - f_1\left(a^{(j)}_{G_k,v_k}(0)\right)\right)\Bigg|\Bigg\},\\ \max_{v_k \in \overline{B(i)}\cap B(j)} &\left\{\left|\Re f_\gamma\left(a^{(i)}_{G_k,v_k}(w)\right) - f_1\left(a^{(i)}_{G_k,v_k}(0)\right)\right|\right\},\\ \max_{v_k \in \overline{B(j)}\cap B(i)} &\left\{\left|\Re f_\gamma\left(a^{(j)}_{G_k,v_k}(w)\right) - f_1\left(a^{(j)}_{G_k,v_k}(0)\right)\right|\right\}\Bigg\}.\end{aligned} \tag{24}$$

In what follows, we let v_k be the vertex that maximizes the above expression, and d_k be the number of unpinned neighbors of v_k in G_k. Before proceeding with the analysis, we note that the graphs G_k are unconflicted and satisfy Condition 1, and further that v_k is nice in G_k (this last fact follows from Lemma III.2 and the fact that G satisfies Condition 1). Thus, the preconditions of Consequence IV.5 apply to the vertex v_k in graph G_k. We now proceed with the analysis.

We first consider $v_k \in \overline{B(i)} \cap B(j)$. Note that this implies that $i \in \Gamma_{G_k,v_k}$. Thus, the conditions of Consequence IV.5 of the induction hypothesis instantiated on G_k apply to v_k with color i, and we thus have from eq. (14) that

$$\begin{aligned}&\left|\Re f_\gamma\left(a^{(i)}_{G_k,v_k}(w)\right) - f_1\left(a^{(i)}_{G_k,v_k}(0)\right)\right|\\ &\le \frac{1}{d_k+\eta}\left|\Re a^{(i)}_{G_k,v_k}(w) - a^{(i)}_{G_k,v_k}(0)\right| + \varepsilon_I + \varepsilon_w,\end{aligned}$$

where d_k is the number of unpinned neighbors of v_k and $\eta \in [0.9, 1)$ is as in the statement of Consequence IV.5. Applying item 2 of Consequence IV.4 (which, again, is applicable because $i \in \Gamma_{G_k,v_k}$), we then have $\left|\Re a^{(i)}_{G_k,v_k}(w) - a^{(i)}_{G_k,v_k}(0)\right| \le d_k(\varepsilon_R + \varepsilon_I) + 2\Delta\varepsilon_w$, so that

$$\begin{aligned}&\left|\Re f_\gamma\left(a^{(i)}_{G_k,v_k}(w)\right) - f_1\left(a^{(i)}_{G_k,v_k}(0)\right)\right|\\ &\le \frac{d_k}{d_k+\eta}\varepsilon_R + 2\varepsilon_I + 3\Delta\varepsilon_w.\end{aligned} \tag{25}$$

By interchanging the roles of i and j in the above argument, we see that, for $v_k \in \overline{B(j)} \cap B(i)$

$$\begin{aligned}&\left|\Re f_\gamma\left(a^{(j)}_{G_k,v_k}(w)\right) - f_1\left(a^{(j)}_{G_k,v_k}(0)\right)\right|\\ &\le \frac{d_k}{d_k+\eta}\varepsilon_R + 2\varepsilon_I + 3\Delta\varepsilon_w.\end{aligned} \tag{26}$$

We now consider $v_k \in \overline{B(i)} \cap \overline{B(j)}$. Note that both i and j are good for v_k in G_k, so that

$$\begin{aligned}&\Bigg|\left(\Re f_\gamma\left(a^{(i)}_{G_k,v_k}(w)\right) - f_1\left(a^{(i)}_{G_k,v_k}(0)\right)\right)\\ &\qquad -\left(\Re f_\gamma\left(a^{(j)}_{G_k,v_k}(w)\right) - f_1\left(a^{(j)}_{G_k,v_k}(0)\right)\right)\Bigg|\\ &\le \max_{i',j' \in \Gamma_{G_k,v_k}} \Bigg|\left(\Re f_\gamma\left(a^{(i')}_{G_k,v_k}(w)\right) - f_1\left(a^{(i')}_{G_k,v_k}(0)\right)\right)\\ &\qquad -\left(\Re f_\gamma\left(a^{(j')}_{G_k,v_k}(w)\right) - f_1\left(a^{(j')}_{G_k,v_k}(0)\right)\right)\Bigg|.\end{aligned}$$

Now, for any color $s \in \Gamma_{G_k,v_k}$, Consequence IV.5 of the induction hypothesis instantiated on G_k and applied to v_k and s shows that there exists a $C_s = C_{s,v_k,G_k} \in [0, 1/(d_k+\eta)]$ such that

$$\begin{aligned}&\Bigg|\Re f_\gamma\left(a^{(s)}_{G_k,v_k}(w)\right) - f_1\left(a^{(s)}_{G_k,v_k}(0)\right)\\ &\quad -C_s\left(\Re a^{(s)}_{G_k,v_k}(w) - a^{(s)}_{G_k,v_k}(0)\right)\Bigg| \le \varepsilon_I + \varepsilon_w.\end{aligned} \tag{27}$$

Substituting this in the previous display shows that

$$\begin{aligned}&\Bigg|\left(\Re f_\gamma\left(a^{(i)}_{G_k,v_k}(w)\right) - f_1\left(a^{(i)}_{G_k,v_k}(0)\right)\right)\\ &\qquad -\left(\Re f_\gamma\left(a^{(j)}_{G_k,v_k}(w)\right) - f_1\left(a^{(j)}_{G_k,v_k}(0)\right)\right)\Bigg|\\ &\le \max_{i',j' \in \Gamma_{G_k,v_k}} \Big|C_{i'}(\Re a^{(i')}_{G_k,v_k}(w) - a^{(i')}_{G_k,v_k}(0))\\ &\qquad -C_{j'}(\Re a^{(j')}_{G_k,v_k}(w) - a^{(j')}_{G_k,v_k}(0))\Big| + 2\varepsilon_I + 2\varepsilon_w\\ &= 2\varepsilon_I + 2\varepsilon_w + \max_{i',j' \in \Gamma_{G_k,v_k}} \left|C_{i'}\Re\xi_{i'} - C_{j'}\Re\xi_{j'}\right|\\ &= 2\varepsilon_I + 2\varepsilon_w + C_s\Re\xi_s - C_t\Re\xi_t,\end{aligned} \tag{28}$$

where $\xi_l := a^{(l)}_{G_k,v_k}(w) - a^{(l)}_{G_k,v_k}(0)$ for $l \in \Gamma_{G_k,v_k}$, and s and t are given by

$$s := \operatorname*{arg\,max}_{i' \in \Gamma_{G_k,v_k}} C_{i'}\Re\xi_{i'} \quad \text{and} \quad t := \operatorname*{arg\,min}_{i' \in \Gamma_{G_k,v_k}} C_{i'}\Re\xi_{i'}.$$

We now have the following two cases:

Case 1: $(\Re\xi_s)\cdot(\Re\xi_t) \le 0$**.** Recall that C_s, C_t are non-negative and lie in $[0, 1/(d_k+\eta)]$. Thus, in this case, we must have $\Re\xi_s \ge 0$ and $\Re\xi_t \le 0$, so that

$$\begin{aligned}C_s\Re\xi_s - C_t\Re\xi_t &= C_s\Re\xi_s + C_t\left|\Re\xi_t\right|\\ &\le \frac{\Re\xi_s + |\Re\xi_t|}{d_k+\eta} = \frac{|\Re\xi_s - \Re\xi_t|}{d_k+\eta}.\end{aligned} \tag{29}$$

Now, note that

$$\begin{aligned}&\Re\xi_s - \Re\xi_t\\ &= \Re\ln\frac{\mathcal{P}_{G_k,w}[c(v_k)=s]}{\mathcal{P}_{G_k}[c(v_k)=s]} - \Re\ln\frac{\mathcal{P}_{G_k,w}[c(v_k)=t]}{\mathcal{P}_{G_k}[c(v_k)=t]}\\ &= \Re\ln\frac{\mathcal{P}_{G_k,w}[c(v_k)=s]}{\mathcal{P}_{G_k,w}[c(v_k)=t]} - \Re\ln\frac{\mathcal{P}_{G_k}[c(v_k)=s]}{\mathcal{P}_{G_k}[c(v_k)=t]}\\ &= \Re\ln R^{(s,t)}_{G_k,v_k}(w) - \ln R^{(s,t)}_{G_k,v_k}(0).\end{aligned}$$

Note that all the logarithms in the above are well defined from Consequence IV.4 of the induction hypothesis applied to G_k and v_k (as $s, t \in \Gamma_{G_k,v_k}$). Further, from items 2 and 3 of the induction hypothesis, the last term is at most $d_k\varepsilon_R$ in absolute value. Substituting this in eq. (29), we get

$$C_s \Re\xi_s - C_t \Re\xi_t \le \frac{d_k}{d_k+\eta}\varepsilon_R. \tag{30}$$

This concludes the analysis of Case 1.

Case 2: $\Re\xi_{i'}$ for $i' \in \Gamma_{G_k,v_k}$ all have the same sign. Suppose first that $\Re\xi_{i'} \ge 0$ for all $i' \in \Gamma_{G_k,v_k}$. Then, we have

$$0 \le C_s\Re\xi_s - C_t\Re\xi_t \le \frac{\Re\xi_s}{d_k+\eta} \le \frac{d_k\cdot\varepsilon_R}{d_k+\eta} + \varepsilon_I + 4\Delta\varepsilon_w, \tag{31}$$

where the last inequality follows from item 2 of Consequence IV.4 of the induction hypothesis applied to G_k at vertex v_k with color s, which states that $|\Re\xi_s| \le d_k(\varepsilon_R+\varepsilon_I) + 4\Delta\varepsilon_w$. Similarly, when $\Re\xi_{i'} \le 0$ for all $i' \in \Gamma_{G_k,v_k}$, we have

$$\begin{aligned} 0 \le C_s\Re\xi_s - C_t\Re\xi_t &= C_t|\Re\xi_t| - C_s|\Re\xi_s| \\ &\le \frac{|\Re\xi_t|}{d_k+\eta} \\ &\le \frac{d_k\cdot\varepsilon_R}{d_k+\eta} + \varepsilon_I + 4\Delta\varepsilon_w, \end{aligned} \tag{32}$$

where the last inequality follows from item 2 of Consequence IV.4 of the induction hypothesis applied to G_k at vertex v_k with color t, which states that $|\Re\xi_t| \le d_k(\varepsilon_R+\varepsilon_I) + 4\Delta\varepsilon_w$. This concludes the analysis of Case 2.

Now, substituting eqs. (30) to (32) into eq. (28), we get

$$\begin{aligned} &\Big|\Big(\Re f_\gamma\Big(a^{(i)}_{G_k,v_k}(w)\Big) - f_1\Big(a^{(i)}_{G_k,v_k}(0)\Big)\Big) \\ &\quad -\Big(\Re f_\gamma\Big(a^{(j)}_{G_k,v_k}(w)\Big) - f_1\Big(a^{(j)}_{G_k,v_k}(0)\Big)\Big)\Big| \\ &\qquad \le \frac{d_k}{d_k+\eta}\varepsilon_R + 3\varepsilon_I + 5\Delta\varepsilon_w. \end{aligned} \tag{33}$$

Substituting eqs. (25), (26) and (33) into eq. (24), we get

$$\begin{aligned} &\frac{1}{d}\left|\Re\ln R^{(i,j)}_{G,u}(w) - \ln R^{(i,j)}_{G,u}(0)\right| \\ &\quad \le \frac{d_k\cdot\varepsilon_R}{d_k+\eta} + 3\varepsilon_I + 7\Delta\varepsilon_w < \varepsilon_R, \end{aligned}$$

where the last inequality follows since $\eta\varepsilon_R > (\Delta+1)(3\varepsilon_I + 7\Delta\varepsilon_w)$ (recalling that $0 \le d_k \le \Delta$ and $\eta \in [0.9, 1)$). This verifies item 3 of the induction hypothesis.

For item 4, we consider the imaginary part of eq. (22). As in the derivation of eq. (24), we use the fact that the induction hypothesis applied to the graph G_k at the vertex $v_k \in B(i)$ (respectively, $v_k \in B(j)$) implies that $\left|g_\gamma\Big(b^{(i)}_{G_k,v_k}(w)\Big)\right| \le 2\varepsilon_w$ (respectively, $g_\gamma\Big(b^{(j)}_{G_k,v_k}(w)\Big) \le 2\varepsilon_w$). This yields

$$\begin{aligned} &\frac{1}{d}\left|\Im\ln R^{(i,j)}_{G,u}(w)\right| \le 2\Delta\varepsilon_w + \max\Big\{ \\ &\quad \max_{v_k\in\overline{B(i)}\cap\overline{B(j)}}\left|\Im f_\gamma\Big(a^{(i)}_{G_k,v_k}(w)\Big) - \Im f_\gamma\Big(a^{(j)}_{G_k,v_k}(w)\Big)\right|, \\ &\quad \max_{v_k\in\overline{B(i)}\cap B(j)}\left|\Im f_\gamma\Big(a^{(i)}_{G_k,v_k}(w)\Big)\right|, \\ &\quad \max_{v_k\in\overline{B(j)}\cap B(i)}\left|\Im f_\gamma\Big(a^{(j)}_{G_k,v_k}(w)\Big)\right|\Big\}. \end{aligned} \tag{34}$$

Again, let v_k be the vertex that maximizes the above expression, and d_k be the number of unpinned neighbors of v_k in G_k. We first consider $v_k \in \overline{B(i)} \cap \overline{B(j)}$. Applying eq. (15) of Consequence IV.5 of the induction hypothesis to the graph G_k at vertex v_k with colors $i, j \in \Gamma_{G_k,v_k}$ gives

$$\begin{aligned} &\left|\Im f_\gamma\Big(a^{(i)}_{G_k,v_k}(w)\Big) - \Im f_\gamma\Big(a^{(j)}_{G_k,v_k}(w)\Big)\right| \\ &\qquad \le \frac{d_k}{d_k+\eta}\varepsilon_I + 6\Delta\varepsilon_w. \end{aligned} \tag{35}$$

Now consider $v_k \in \overline{B(i)} \cap B(j)$. For this case, eq. (16) of Consequence IV.5 of the induction hypothesis applied to G_k at vertex v_k with color $i \in \Gamma_{G_k,v_k}$ gives

$$\left|\Im f_\gamma\Big(a^{(i)}_{G_k,v_k}(w)\Big)\right| \le \frac{d_k}{d_k+\eta}\varepsilon_I + 5\Delta\varepsilon_w. \tag{36}$$

Similarly, for $v_k \in \overline{B(j)} \cap B(i)$, eq. (16) of Consequence IV.5 of the induction hypothesis applied to G_k at vertex v_k with color $j \in \Gamma_{G_k,v_k}$ gives

$$\left|\Im f_\gamma\Big(a^{(j)}_{G_k,v_k}(w)\Big)\right| \le \frac{d_k}{d_k+\eta}\varepsilon_I + 5\Delta\varepsilon_w. \tag{37}$$

Substituting eqs. (35) to (37) into eq. (34) we have

$$\frac{1}{d}\left|\Im\ln R^{(i,j)}_{G,u}(w)\right| \le \frac{d_k}{d_k+\eta}\varepsilon_I + 8\Delta\varepsilon_w < \varepsilon_I,$$

where the last inequality holds since $\eta\varepsilon_I > 8(\Delta+1)\Delta\varepsilon_w$ (recalling that $0 \le d_k \le \Delta$ and $\eta \in [0.9, 1)$). This completes the proof of item 4 of the induction hypothesis.

Finally, we prove item 5. Since $i \notin \Gamma_u$, there exist $n_i > 0$ neighbors of u that are pinned to color i. Let H be the graph obtained by removing these neighbors of u from G. Then, H is an unconflicted graph with the *same* number of unpinned vertices as G which also satisfies $i, j \in \Gamma_{H,u}$; we can therefore apply the already proved items 1 to 3 to H to conclude that

$$\left|R_H^{(i,j)}(w)\right| \le \left|R_H^{(i,j)}(0)\right| e^{d\varepsilon_R}. \tag{38}$$

Now, since $i, j \in \Gamma_{H,u}$, we can apply the recurrence of Lemma II.4 in the same way as in the derivation of eq. (21) above to get

$$R_{H,u}^{(i,j)}(w) = \prod_{k=1}^{\deg_H(u)} \frac{1 - \mathcal{P}_{H_k^{(i,j)},w}[c(v_k) = i]}{1 - \mathcal{P}_{H_k^{(i,j)},w}[c(v_k) = j]}, \tag{39}$$

where, for the reasons described in the discussion following eq. (21), the product can be restricted to unpinned neighbors of u in H. Renaming these unpinned neighbors as $v_1, v_2, \ldots, v_d$, we then have

$$0 \le R_H^{(i,j)}(0) = \prod_{k=1}^{d} \frac{(1 - \mathcal{P}_{H_k}[c(v_k) = i]\,)}{(1 - \mathcal{P}_{H_k}[c(v_k) = j]\,)}, \tag{40}$$

where, as before, $H_k := H_k^{(i,j)}$. Now, since G satisfies Condition 1, so does H. Thus, for $1 \le k \le d$, v_k is nice in H_k (Lemma III.2), and hence, $\mathcal{P}_{H_k}[c(v_k) = j] \le \frac{1}{d_k+2}$ for $1 \le k \le d$, where $d_k \ge 0$ is the number of unpinned neighbors of v_k in H_k. We then have

$$\begin{aligned} 0 \le R_H^{(i,j)}(0) &= \prod_{k=1}^{d} \frac{(1 - \mathcal{P}_{H_k}[c(v_k) = i]\,)}{(1 - \mathcal{P}_{H_k}[c(v_k) = j]\,)} \\ &\le \prod_{k=1}^{d} \frac{1}{1 - \frac{1}{d_k+2}} = \prod_{k=1}^{d} \frac{d_k+2}{d_k+1} \le 2^{\Delta}. \end{aligned}$$

(As an aside, we note that one could get a better bound under the slightly stronger assumption of uniformly large list sizes considered in Remark 5. Under the conditions of that remark, we have $\mathcal{P}_{H_k}[c(v_k) = j] < \min\left\{\frac{4}{3\Delta}, 1\right\}$, so that the above upper bound can be improved to $R_H^{(i,j)}(0) \le e^4$ for $\Delta > 1$.)

Combining the estimate with eq. (38), we get $\left|R_H^{(i,j)}(w)\right| \le 5 \cdot 2^{\Delta}$ since $d\varepsilon_R \le 1/2$. Now note that since $j \in \Gamma_{G,u}$,

$$Z_{G,u}^{(i)}(w) = w^{n_i} Z_{H,u}^{(i)}(w), \quad \text{and} \quad Z_{G,u}^{(j)}(w) = Z_{H,u}^{(j)}(w),$$

so that $\left|R_{G,u}^{(i,j)}(w)\right| = |w|^{n_i} \left|R_{H,u}^{(i,j)}(w)\right| \le 5 \cdot 2^{\Delta} \cdot |w|^{n_i}$. The latter is at most ε_w whenever $|w| \le 0.2\varepsilon_w/2^{\Delta}$. This proves item 5, and also completes the inductive proof of Lemma IV.2. (Note also that using the stronger upper bound above under the condition of uniformly large list sizes, we can in fact relax the requirement further to $|w| \le \varepsilon_w/(300\Delta)$.) □

We conclude this section by using Lemma IV.2 to prove Theorem IV.1.

Proof of Theorem IV.1: Let G be a graph satisfying Condition 1. Since G has no pinned vertices, G is unconflicted. Let u be an unpinned vertex in G. By Consequence IV.3 of the induction hypothesis (which we proved in Lemma IV.2), we then have $Z_w(G) \ne 0$ provided $\nu_w \le 0.2\varepsilon_w/2^{\Delta}$.

Furthermore, as discussed above, under a slightly stronger assumption of uniformly large list sizes considered in Remark 5, ν_w can be chosen to be $\varepsilon_w/(300\Delta)$. ■

V. Zero-free region around the interval (0, 1]

In this section, we consider the case of w close to $[0,1]$ but bounded away from 0. In particular, we prove the following theorem, which complements Theorem IV.1.

Theorem V.1. *Fix a positive integer Δ and let $\nu_w = \nu_w(\Delta)$ be as in Theorem IV.1. Then, for any w satisfying*

$$\Re w \in [\nu_w/2, 1 + \nu_w^2/8] \qquad \text{and} \qquad |\Im w| \le \nu_w^2/8, \tag{41}$$

and any graph G satisfying Condition 1, we have $Z_G(w) \ne 0$.

(Here, we recall that as described in the discussion following Theorem IV.1, ν_w can be chosen to be $\varepsilon_w/(300\Delta)$ when the uniformly large list size condition of Remark 5 is satisfied. However, as in that theorem, in the case of general list coloring, one chooses $\nu_w = 0.2\varepsilon_w/2^{\Delta}$.)

For w as in eq. (41), we define $\tilde{w}$ to be the point on the interval $[0,1]$ which is closest to w. Thus

$$\tilde{w} := \begin{cases} \Re w & \text{when } \Re w \in [\nu_w/2, 1]; \\ 1 & \text{when } \Re w \in (1, 1 + \nu_w^2/8]. \end{cases}$$

We also define, in analogy with the last section, $\gamma := 1 - w$ and $\tilde{\gamma} := 1 - \tilde{w}$. We record a few properties of these quantities in the following observation.

Observation V.2. *With $w, \gamma, \tilde{w}$ and $\tilde{\gamma}$ as above, we have*

1) $0 \le \tilde{\gamma}, |\gamma| < 1$.
2) $|\ln w - \ln \tilde{w}| \le \nu_w$.

Proof: We have $\tilde{\gamma} \in [0, 1 - \nu_w/2]$, $\Re\gamma \in [-\nu_w^2/8, 1 - \nu_w/2]$ and $|\Im\gamma| \le \nu_w^2/8$. Since $\nu_w \le 0.01$, these bounds taken together imply item 1. We also have $0 \le \tilde{w} \le |w| \le \tilde{w} + \nu_w^2/4$ and $\tilde{w} \ge \nu_w/2$. Thus

$$0 \le \Re(\ln w - \ln \tilde{w}) = \ln \frac{|w|}{\tilde{w}} \le \ln\left(1 + \frac{\nu_w^2}{4\tilde{w}}\right) \le \frac{\nu_w}{2}.$$

Similarly, $\Im(\ln w - \ln \tilde{w}) = \Im \ln w = \arg w$, so that

$$|\Im(\ln w - \ln \tilde{w})| \le |\arg w| \le \frac{|\Im w|}{\Re w} \le \frac{\nu_w}{4}.$$

Together, the above two bounds imply item 2. ∎

In analogous fashion to the proof of Theorem IV.1, we would like to show that $R^{(i,j)}_{G,u}(w) \approx R^{(i,j)}_{G,u}(\tilde{w})$ independent of the size of G. (Note that for positive $\tilde{w}$, $R^{(i,j)}_{G,u}(\tilde{w})$ is a well defined positive real number for any graph.) To this end, we will prove the following analog of Lemma IV.2 for any graph G satisfying Condition 1 and any vertex u in G, via an induction on the number of unpinned vertices in G. The induction is very similar in structure to that used in the proof of Lemma IV.2, except that the fact that w has strictly positive real part allows us to simplify several aspects of the proof. In particular, we do not need to consider good and bad colors separately, and do not require the underlying graphs to be unconflicted.

As in the previous section, we assume that all graphs in this section have maximum degree at most $\Delta \ge 1$, and define the quantities $\varepsilon_w, \varepsilon_R, \varepsilon_I$ in terms of Δ using eq. (9).

Lemma V.3. *Let G be a graph of maximum degree Δ satisfying Condition 1 and let u be any unpinned vertex in G. Then, the following are true (here, $\varepsilon_w, \varepsilon_I, \varepsilon_R$ are as defined in eq. (9)):*

1) *For $i \in L(u)$, $\left|Z^{(i)}_{G,u}(w)\right| > 0$.*
2) *For $i, j \in L(u)$, if u has all neighbors pinned, then $|\ln R^{(i,j)}_{G,u}(w) - \ln R^{(i,j)}_{G,u}(\tilde{w})| < \varepsilon_w$.*
3) *For $i, j \in L(u)$, if u has $d \ge 1$ unpinned neighbors, then*
$$\frac{1}{d}\left|\Re \ln R^{(i,j)}_{G,u}(w) - \Re \ln R^{(i,j)}_{G,u}(\tilde{w})\right| < \varepsilon_R.$$
4) *For any $i, j \in L(u)$, if u has $d \ge 1$ unpinned neighbors, then $\frac{1}{d}\left|\Im \ln R^{(i,j)}_{G,u}(w)\right| < \varepsilon_I$.*

We will refer to items 1 to 4 as "items of the induction hypothesis". The rest of this section is devoted to the proof of this lemma via an induction on the number of unpinned vertices in G.

We begin by verifying that the induction hypothesis holds in the base case when u is the only unpinned vertex in a graph G. In this case, items 3 and 4 are vacuously true since u has no unpinned neighbors. Since all neighbors of u in G are pinned, the fact that all pinned vertices have degree at most one implies that G can be decomposed into two disjoint components G_1 and G_2, where G_1 consists of u and its pinned neighbors, while G_2 consists of a disjoint union of edges with pinned end-points. Let m be the number of conflicted edges on G_2, and let n_k denote the number of neighbors of u pinned to color k. We then have $Z^{(k)}_{G,u}(x) = x^{n_k} Z_{G_2}(x) = x^{n_k+m}$ for all $x \in \mathbb{C}$. This already proves item 1 since $w, \tilde{w} \neq 0$. Item 2 follows via the following computation (which uses item 2 of Observation V.2):

$$\begin{aligned} |\ln R^{(i,j)}_{G,u}(w) - \ln R^{(i,j)}_{G,u}(\tilde{w})| &= |n_i - n_j| \cdot |\ln w - \ln \tilde{w}| \\ &\le \Delta \nu_w < \varepsilon_w. \end{aligned}$$

We now derive some consequences of the above induction hypothesis that will be helpful in carrying out the induction.

Consequence V.4. *If $|L(u)| \ge 1$, then $|Z_G(w)| > 0$.*

Proof: Note that $Z_G(w) = \sum_{i \in L(u)} Z^{(i)}_{G,u}(w)$. From item 4, we see that the angle between the complex numbers $Z^{(i)}_{G,u}(w)$ and $Z^{(j)}_{G,u}(w)$, for all $i, j \in L(u)$, is at most $d\varepsilon_I$. Applying Lemma II.7 we then have

$$\begin{aligned} \Big|\sum_{i \in L(u)} Z^{(i)}_{G,u}(w)\Big| &\ge |L(u)| \cos\frac{d\varepsilon_I}{2} \cdot \min_{i \in \Gamma_u}\left|Z^{(i)}_{G,u}(w)\right| \\ &\ge 0.9 \min_{i \in \Gamma_u}\left|Z^{(i)}_{G,u}(w)\right|, \end{aligned}$$

when $|L(u)| \ge 1$ and $d\varepsilon_I \le 0.01$. This last quantity is positive from item 1. ∎

Consequence V.5. *For all $\varepsilon_R, \varepsilon_I, \varepsilon_w$ small enough such that $\varepsilon_I \le \varepsilon_R$ and $\varepsilon_w \le 0.01\varepsilon_I$, the pseudo-probabilities approximate the real probabilities in the following sense: for any $j \in L(u)$,*

$$\begin{aligned} \left|\Im \ln \frac{\mathcal{P}_{G,w}[c(u)=j]}{\mathcal{P}_{G,\tilde{w}}[c(u)=j]}\right| &= |\Im \ln \mathcal{P}_{G,w}[c(u)=j]| \\ &\le d\varepsilon_I + 2\Delta\varepsilon_w; \\ \left|\Re \ln \frac{\mathcal{P}_{G,w}[c(u)=j]}{\mathcal{P}_{G,\tilde{w}}[c(u)=j]}\right| &\le d\varepsilon_R + d\varepsilon_I + 2\Delta\varepsilon_w, \end{aligned}$$

where d is the number of unpinned neighbors of u in G.

Proof: Using items 2 to 4 of the induction hypothesis, there exist complex numbers ξ_i (for all $i \in \Gamma_u$) satisfying $|\Re\xi_i| \le d\varepsilon_R + \varepsilon_w$ and $|\Im\xi_i| \le d\varepsilon_I + \varepsilon_w$ such that

$$\begin{aligned} \frac{\mathcal{P}_{G,\tilde{w}}[c(u)=j]}{\mathcal{P}_{G,w}[c(u)=j]} &= \mathcal{P}_{G,\tilde{w}}[c(u)=j] \sum_{i \in L(u)} \frac{Z^{(i)}_{G,u}(w)}{Z^{(j)}_{G,u}(w)} \\ &= \mathcal{P}_{G,\tilde{w}}[c(u)=j] \sum_{i \in L(u)} \frac{Z^{(i)}_{G,u}(\tilde{w})}{Z^{(j)}_{G,u}(\tilde{w})} e^{\xi_i}. \end{aligned} \tag{42}$$

Now, note that $\sum_{i \in L(u)} \frac{Z^{(i)}_{G,u}(\tilde{w})}{Z^{(j)}_{G,u}(\tilde{w})} = \frac{1}{\mathcal{P}_{G,\tilde{w}}[c(u)=j]}$, so that the sum above is a convex combination of the $\exp(\xi_i)$. From the bounds on the real and imaginary parts of the ξ_i quoted above, by a calculation similar to that in eq. (11), we also have (when $\varepsilon_I, \varepsilon_w \le 0.01/\Delta$)

$$\Re e^{\xi_i} \in (e^{-d\varepsilon_R - \varepsilon_w} - (d\varepsilon_I + \varepsilon_w)^2,\ e^{d\varepsilon_R + \varepsilon_w}), \text{ and}$$
$$|\arg e^{\xi_i}| \le d\varepsilon_I + \varepsilon_w.$$

The above will therefore be true also for any convex combination of the e^{ξ_i}, in particular the one in eq. (42). We therefore have, for $C := \frac{\mathcal{P}_{G,\tilde{w}}[c(u)=j]}{\mathcal{P}_{G,w}[c(u)=j]}$,

$$\Re C \in \left(e^{-d\varepsilon_R - \varepsilon_w} - (d\varepsilon_I + \varepsilon_w)^2,\ e^{d\varepsilon_R + \varepsilon_w}\right),$$
$$|\arg C| \le d\varepsilon_I + \varepsilon_w.$$

Now recall that for $|\theta| \le \pi/4$, we have $-\theta^2 \le \ln\cos\theta \le -\theta^2/2$. Thus, using the values of $\varepsilon_w, \varepsilon_I$ and ε_R, we have

$$|\Re \ln C| \le d\varepsilon_R + d\varepsilon_I + 2\Delta\varepsilon_w, \text{ and}$$
$$|\Im \ln C| \le d\varepsilon_I + \varepsilon_w.$$

■

As before we define $a^{(i)}_{G,u}(w) = \ln \mathcal{P}_{G,w}[c(u) = i]$, and recall the definition of the function $f_\gamma(x) := -\ln(1 - \gamma e^x)$.

Consequence V.6. *There exists a positive constant $\eta \in [0.9, 1)$ so that the following is true. Let d be the number of unpinned neighbors of u. Assume further that the vertex u is nice in G. Then, for any colors $i, j \in L(u)$, there exist a real constant $c = c_{G,u,i} \in [0, \frac{1}{d+\eta}]$ such that*

$$\Big|\Re f_\gamma(a^{(i)}_{G,u}(w)) - f_{\tilde{\gamma}}(a^{(i)}_{G,u}(\tilde{w})) - c \cdot \Re\Big(a^{(i)}_{G,u}(w) - a^{(i)}_{G,u}(\tilde{w})\Big)\Big| \le \varepsilon_I + \varepsilon_w; \quad (43)$$

$$\Big|\Im f_\gamma(a^{(i)}_{G,u}(w)) - \Im f_\gamma(a^{(j)}_{G,u}(w))\Big| \le \frac{1}{d+\eta} \cdot (d\varepsilon_I + 4\Delta\varepsilon_w) + 2\varepsilon_w. \quad (44)$$

Proof: Since u is nice in G, the bound $\mathcal{P}_{G,\tilde{w}}[c(u) = k] \le \frac{1}{d+2}$ (for any $k \in L(u)$) applies. Combining them with Consequence V.5 we see that $a^{(i)}_{G,u}(w), a^{(i)}_{G,u}(\tilde{w}), a^{(j)}_{G,u}(w), a^{(j)}_{G,u}(\tilde{w})$ lie in a domain D as described in Lemma II.6, with the parameters ζ and τ in that lemma chosen as

$$\zeta = \ln(d + 2) - d\varepsilon_R - d\varepsilon_I - 2\Delta\varepsilon_w, \text{ and}$$
$$\tau = d\varepsilon_I + 2\Delta\varepsilon_w.$$

Here, for the bound on ζ, we use the fact that for $k \in L(u)$, $\mathcal{P}_{G,\tilde{w}}[c(u) = k] \le \frac{1}{d+2}$, since u is nice in G. As in the proof of Consequence IV.5, we use the values of $\varepsilon_w, \varepsilon_I, \varepsilon_R$ to verify that the condition $\tau < 1/2$ and $\tau^2 + e^{-\zeta} < 1$ are satisfied, so that item 1 of Lemma II.6 applies (with the parameter κ therein set to $\tilde{\gamma}$) and further that ρ_R and ρ_I as set there satisfy $\rho_R \le \frac{1}{d+\eta}$ and $\rho_I < 3\varepsilon_I$, with $\eta = 0.94$. Using Lemma II.5 followed by the bound on ε_w, we then have

$$\begin{aligned}\Big|\Re f_{\tilde{\gamma}}(a^{(i)}_{G,u}(w)) - f_{\tilde{\gamma}}(a^{(i)}_{G,u}(\tilde{w})) \\ - c \cdot \Re\Big(a^{(i)}_{G,u}(w) - a^{(i)}_{G,u}(\tilde{w})\Big)\Big| &\le 3\varepsilon_I(d\varepsilon_I + 2\Delta\varepsilon_w) \\ &\le 4d\varepsilon_I^2 \le \varepsilon_I, \quad (45)\end{aligned}$$

for an appropriate positive $c \le 1/(d + \eta)$. This is almost eq. (43), whose difference will be handled later.

Similarly, applying Lemma II.5 to the imaginary part we have

$$\begin{aligned}&\Big|\Im\Big(f_{\tilde{\gamma}}(a^{(i)}_{G,u}(w)) - f_{\tilde{\gamma}}(a^{(j)}_{G,u}(w))\Big)\Big| \\ &\le \rho_R \cdot \max\Big\{\Big|\Im\Big(a^{(i)}_{G,u}(w) - a^{(j)}_{G,u}(w)\Big)\Big|, \\ &\qquad \Big|\Im a^{(i)}_{G,u}(w)\Big|, \Big|\Im a^{(j)}_{G,u}(w)\Big|\Big\}, \quad (46)\end{aligned}$$

where, as noted above, $\rho_R \le \frac{1}{d+\eta}$. Now, note that the first term in the above maximum is less than $d\varepsilon_I + \varepsilon_w$ by items 2 and 4 of the induction hypothesis, while the other two are at most $d\varepsilon_I + 2\Delta\varepsilon_w$ from item 2 of Consequence V.5.

Finally, we use item 2 of Lemma II.6 with the parameter κ' therein set to γ. To this end, we note that $|\gamma - \tilde{\gamma}| \le \varepsilon_w$, and that with the fixed values of $\varepsilon_w, \varepsilon_R$, and ε_I, the condition $(1 + \varepsilon_w) < e^\zeta$ is satisfied, so that the item applies. Using the item, we then see that for any $z \in D$,

$$|f_\gamma(z) - f_{\tilde{\gamma}}(z)| \le \varepsilon_w.$$

Thus, the quantities $|\Re f_\gamma(a^{(i)}_{G,u}(w)) - \Re f_{\tilde{\gamma}}(a^{(i)}_{G,u}(w))|$, $|\Im f_\gamma(a^{(i)}_{G,u}(w)) - \Im f_{\tilde{\gamma}}(a^{(i)}_{G,u}(w))|$, $|\Im f_\gamma(a^{(j)}_{G,u}(w)) - \Im f_{\tilde{\gamma}}(a^{(j)}_{G,u}(w))|$, and $|\Im f_\gamma(a^{(j)}_{G,u}(w)) - \Im f_{\tilde{\gamma}}(a^{(j)}_{G,u}(w))|$ are all at most ε_w. The desired bounds now follow from the triangle inequality and the bounds in eqs. (45) and (46). ■

Inductive proof of Lemma V.3

We are now ready to see the inductive proof of Lemma V.3; recall that the base case was already established following the statement of the lemma. Let G be any graph which satisfies Condition 1 and had at least two unpinned vertices (the base case when $|G| = 1$ was already handled above). We first prove induction item 1 for any vertex u in G. Consider the graph G' obtained from G by pinning vertex u to color i. Note that by the definition of the pinning operation, $Z^i_{G,u}(w) = Z_{G'}(w)$. Further, the graph G' also satisfies Condition 1, and has one fewer unpinned vertex than G. Thus, from Consequence V.4 of the induction hypothesis applied to G', we have that $\left|Z^{(i)}_{G,u}(w)\right| = |Z_{G'}(w)| > 0$.

We now consider item 2. When all neighbors of u in G are pinned, the fact that all pinned vertices have degree at most one implies that G can be decomposed into two disjoint components G_1 and G_2, where G_1 consists of u and its pinned neighbors, while G_2 has one fewer unpinned vertex than G. Let n_k be the number of neighbors of u pinned to color k. Now, since G_1 and G_2 are disjoint components, we have $Z^{(k)}_{G,u}(x) = x^{n_k} Z_{G_2}(x)$ for all $k \in L(u)$ and all $x \in \mathbb{C}$. Further, from Consequence V.4 of the induction hypothesis applied to G_2, we also have that $Z_{G_2}(w)$ and $Z_{G_2}(\tilde{w})$ are both non-zero. It therefore follows that

$$|\ln R^{(i,j)}_{G,u}(w) - \ln R^{(i,j)}_{G,u}(\tilde{w})| = |n_i - n_j| \cdot |\ln w - \ln \tilde{w}| \\ \leq \Delta\nu_w < \varepsilon_w.$$

We now consider items 3 and 4. Recall that by Lemma II.4, we have

$$R^{(i,j)}_{G,u}(w) = \prod_{k=1}^{\deg_G(u)} \frac{\left(1 - \gamma \mathcal{P}_{G_k^{(i,j)},w}[c(v_k) = i]\right)}{\left(1 - \gamma \mathcal{P}_{G_k^{(i,j)},w}[c(v_k) = j]\right)}.$$

As before, for simplicity we write $G_k := G_k^{(i,j)}$. Note that each G_k has exactly one fewer unpinned vertex than G, so that the induction hypothesis applies to each G_k. Without loss of generality, we relabel the unpinned neighbors of u as $v_1, v_2, \ldots, v_d$. Let n_k be the number of neighbors of u pinned to color k. Recalling that $1 - \gamma = w$, we can then simplify the above recurrence to

$$R^{(i,j)}_{G,u}(w) = w^{n_i - n_j} \prod_{k=1}^{d} \frac{\left(1 - \gamma \mathcal{P}_{G_k^{(i,j)},w}[c(v_k) = i]\right)}{\left(1 - \gamma \mathcal{P}_{G_k^{(i,j)},w}[c(v_k) = j]\right)}.$$

Now, as before, for $s \in L(v_k)$ we define $a^{(s)}_{G_k,v_k}(w) := \ln \mathcal{P}_{G_k,w}[c(v_k) = s]$. From the above recurrence, we then have,

$$-\ln R^{(i,j)}_{G,u}(w) = (n_i - n_j)\ln w \\ + \sum_{k=1}^{d} \left(f_\gamma\left(a^{(i)}_{G_k,v_k}(w)\right) - f_\gamma\left(a^{(j)}_{G_k,v_k}(w)\right) \right). \quad (47)$$

Note that the same recurrence also applies when w is replaced by $\tilde{w}$ (and hence γ by $\tilde{\gamma}$):

$$-\ln R^{(i,j)}_{G,u}(\tilde{w}) = (n_i - n_j)\ln \tilde{w} \\ + \sum_{k=1}^{d} \left(f_{\tilde{\gamma}}\left(a^{(i)}_{G_k,v_k}(\tilde{w})\right) - f_{\tilde{\gamma}}\left(a^{(j)}_{G_k,v_k}(\tilde{w})\right) \right).$$

(Recall that since $\Re w, \tilde{w} > 0$, $\ln w$ and $\ln \tilde{w}$ are well defined).

Using item 2 of Observation V.2, $|n_i - n_j| \leq \Delta$, and the fact that $\Delta\nu_w \leq \varepsilon_w$, we have

$$|n_i - n_j| \cdot |\ln w - \ln \tilde{w}| \leq \varepsilon_w.$$

Applying the triangle inequality to the real part of the difference of the two recurrences, we therefore get

$$\frac{1}{d}\left|\Re \ln R^{(i,j)}_{G,u}(w) - \ln R^{(i,j)}_{G,u}(\tilde{w})\right| \\ \leq \varepsilon_w + \max_{1 \leq k \leq d} \left\{ \left| \left(\Re f_\gamma\left(a^{(i)}_{G_k,v_k}(w)\right) - f_{\tilde{\gamma}}\left(a^{(i)}_{G_k,v_k}(\tilde{w})\right) \right) \right.\right. \\ \left.\left. - \left(\Re f_\gamma\left(a^{(j)}_{G_k,v_k}(w)\right) - f_{\tilde{\gamma}}\left(a^{(j)}_{G_k,v_k}(\tilde{w})\right) \right) \right| \right\}. \quad (48)$$

In what follows, we let v_k be the vertex that maximizes the above expression, and d_k be the number of unpinned neighbors of v_k in G_k. Before proceeding with the analysis, we note that the graphs G_k satisfy Condition 1, and further that v_k is nice in G_k (the latter fact follows from Lemma III.2 and the fact that G has Condition 1). Thus, the preconditions of Consequence V.6 applies to the vertex v_k in graph G_k. We now proceed with the analysis.

We begin by noting that

$$\left| \left(\Re f_\gamma\left(a^{(i)}_{G_k,v_k}(w)\right) - f_{\tilde{\gamma}}\left(a^{(i)}_{G_k,v_k}(\tilde{w})\right) \right) \right. \\ \left. - \left(\Re f_\gamma\left(a^{(j)}_{G_k,v_k}(w)\right) - f_{\tilde{\gamma}}\left(a^{(j)}_{G_k,v_k}(\tilde{w})\right) \right) \right| \\ \leq \max_{i',j' \in L(v_k)} \left| \left(\Re f_\gamma\left(a^{(i')}_{G_k,v_k}(w)\right) - f_{\tilde{\gamma}}\left(a^{(i')}_{G_k,v_k}(\tilde{w})\right) \right) \right. \\ \left. - \left(\Re f_\gamma\left(a^{(j')}_{G_k,v_k}(w)\right) - f_{\tilde{\gamma}}\left(a^{(j')}_{G_k,v_k}(\tilde{w})\right) \right) \right|.$$

On the other hand, for any color $s \in L(v_k)$, Consequence V.6 of the induction hypothesis instantiated on G_k and applied to v_k and s shows that there exists a $C_s = C_{s,v_k,G_k} \in [0, 1/(d_k + \eta)]$ such that

$$\left| \Re f_\gamma\left(a^{(s)}_{G_k,v_k}(w)\right) - f_{\tilde{\gamma}}\left(a^{(s)}_{G_k,v_k}(\tilde{w})\right) \right. \\ \left. - C_s(\Re a^{(s)}_{G_k,v_k}(w) - a^{(s)}_{G_k,v_k}(\tilde{w})) \right| \leq \varepsilon_I + \varepsilon_w.$$

Substituting this in the previous display shows that

$$\begin{aligned}
&\Big|\Big(\Re f_\gamma\Big(a^{(i)}_{G_k,v_k}(w)\Big) - f_{\tilde\gamma}\Big(a^{(i)}_{G_k,v_k}(\tilde w)\Big)\Big)\\
&\quad -\Big(\Re f_\gamma\Big(a^{(j)}_{G_k,v_k}(w)\Big) - f_{\tilde\gamma}\Big(a^{(j)}_{G_k,v_k}(\tilde w)\Big)\Big)\Big|\\
&\le \max_{i',j'\in L(v_k)}\Big|C_{i'}(\Re a^{(i')}_{G_k,v_k}(w) - a^{(i')}_{G_k,v_k}(\tilde w))\\
&\quad - C_{j'}(\Re a^{(j')}_{G_k,v_k}(w) - a^{(j')}_{G_k,v_k}(\tilde w))\Big| + 2\varepsilon_I + 2\varepsilon_w\\
&= 2\varepsilon_I + 2\varepsilon_w + \max_{i',j'\in L(v_k)}\left|C_{i'}\Re\xi_{i'} - C_{j'}\Re\xi_{j'}\right|,\\
&= 2\varepsilon_I + 2\varepsilon_w + C_s\Re\xi_s - C_t\Re\xi_t, \qquad (49)
\end{aligned}$$

where $\xi_l := a^{(l)}_{G_k,v_k}(w) - a^{(l)}_{G_k,v_k}(\tilde w)$ for $l \in \Gamma_{G_k,v_k}$, and s and t are given by

$$s := \arg\max_{i'\in L(v_k)} C_{i'}\Re\xi_{i'} \quad \text{and} \quad t := \arg\min_{i'\in L(v_k)} C_{i'}\Re\xi_{i'}.$$

We now have the following two cases:

Case 1: $(\Re\xi_s)\cdot(\Re\xi_t) \le 0$**.** Recall that C_s, C_t are non-negative and lie in $[0, 1/(d_k+\eta)]$. Thus, in this case, we must have $\Re\xi_s \ge 0$ and $\Re\xi_t \le 0$, so that

$$\begin{aligned}
C_s\Re\xi_s - C_t\Re\xi_t &= C_s\Re\xi_s + C_t\left|\Re\xi_t\right|\\
&\le \frac{\Re\xi_s + |\Re\xi_t|}{d_k+\eta} = \frac{|\Re\xi_s - \Re\xi_t|}{d_k+\eta}. \qquad (50)
\end{aligned}$$

Now, note that

$$\begin{aligned}
&\Re\xi_s - \Re\xi_t\\
&= \Re\ln\frac{\mathcal{P}_{G_k,w}[c(v_k)=s]}{\mathcal{P}_{G_k,\tilde w}[c(v_k)=s]} - \Re\ln\frac{\mathcal{P}_{G_k,w}[c(v_k)=t]}{\mathcal{P}_{G_k,\tilde w}[c(v_k)=t]}\\
&= \Re\ln\frac{\mathcal{P}_{G_k,w}[c(v_k)=s]}{\mathcal{P}_{G_k,w}[c(v_k)=t]} - \Re\ln\frac{\mathcal{P}_{G_k,\tilde w}[c(v_k)=s]}{\mathcal{P}_{G_k,\tilde w}[c(v_k)=t]}\\
&= \Re\ln R^{(s,t)}_{G_k,v_k}(w) - \ln R^{(s,t)}_{G_k,v_k}(\tilde w).
\end{aligned}$$

Note that all the logarithms in the above are well defined from Consequence V.5 of the induction hypothesis applied to G_k and v_k. Further, from items 2 and 3 of the induction hypothesis, the last term is at most $d_k\varepsilon_R + \varepsilon_w$ in absolute value. Substituting this in eq. (50), we get

$$C_s\Re\xi_s - C_t\Re\xi_t \le \frac{d_k}{d_k+\eta}\varepsilon_R + \varepsilon_w. \qquad (51)$$

This concludes the analysis of Case 1.

Case 2: $\Re\xi_{i'}$ **for** $i' \in L(v_k)$ **all have the same sign.** Suppose first that $\Re\xi_{i'} \ge 0$ for all $i' \in L(v_k)$. Then, we have

$$0 \le C_s\Re\xi_s - C_t\Re\xi_t \le \frac{\Re\xi_s}{d_k+\eta} \le \frac{d_k\cdot\varepsilon_R}{d_k+\eta} + \varepsilon_I + 4\Delta\varepsilon_w, \qquad (52)$$

where the last inequality follows from the second inequality in Consequence V.5 of the induction hypothesis applied to G_k at vertex v_k with color s, which states that $|\Re\xi_s| \le d_k(\varepsilon_R + \varepsilon_I) + 4\Delta\varepsilon_w$. Similarly, when $\Re\xi_{i'} \le 0$ for all $i' \in \Gamma_{G_k,v_k}$, we have

$$\begin{aligned}
0 \le C_s\Re\xi_s - C_t\Re\xi_t &= C_t|\Re\xi_t| - C_s|\Re\xi_s|\\
&\le \frac{1}{d_k+\eta}\left|\Re\xi_t\right|\\
&\le \frac{d_k}{d_k+\eta}\varepsilon_R + \varepsilon_I + 4\Delta\varepsilon_w, \qquad (53)
\end{aligned}$$

where the last inequality follows from the second inequality in Consequence V.5 of the induction hypothesis applied to G_k at vertex v_k with color t, which states that $|\Re\xi_t| \le d_k(\varepsilon_R + \varepsilon_I) + 4\Delta\varepsilon_w$. This concludes the analysis of Case 2.

Now, substituting eqs. (51) to (53) into eq. (49), we get

$$\begin{aligned}
&\Big|\Big(\Re f_\gamma\Big(a^{(i)}_{G_k,v_k}(w)\Big) - f_{\tilde\gamma}\Big(a^{(i)}_{G_k,v_k}(\tilde w)\Big)\Big)\\
&\quad -\Big(\Re f_\gamma\Big(a^{(j)}_{G_k,v_k}(w)\Big) - f_{\tilde\gamma}\Big(a^{(j)}_{G_k,v_k}(\tilde w)\Big)\Big)\Big|\\
&\qquad \le \frac{d_k}{d_k+\eta}\varepsilon_R + 3\varepsilon_I + 5\Delta\varepsilon_w. \qquad (54)
\end{aligned}$$

Substituting eq. (54) into eq. (48), we get

$$\begin{aligned}
&\frac{1}{d}\left|\Re\ln R^{(i,j)}_{G,u}(w) - \ln R^{(i,j)}_{G,u}(\tilde w)\right|\\
&\qquad \le \frac{d_k}{d_k+\eta}\varepsilon_R + 3\varepsilon_I + 7\Delta\varepsilon_w < \varepsilon_R, \qquad (55)
\end{aligned}$$

where the last inequality holds since $\eta\varepsilon_R > (\Delta+1)(3\varepsilon_I + 7\Delta\varepsilon_w)$ (recalling that $0 \le d_k \le \Delta$ and $\eta \in [0.9, 1)$). This verifies item 3 of the induction hypothesis.

Finally, to prove item 4, we consider the imaginary part of eq. (47). We first note that

$$|n_i - n_j|\cdot|\Im\ln w| \le \Delta\left|\ln w - \ln\tilde w\right| \le \Delta\nu_w \le \varepsilon_w.$$

We then have

$$\begin{aligned}
&\frac{1}{d}\left|\Im\ln R^{(i,j)}_{G,u}(w)\right| \le \varepsilon_w +\\
&\max_{1\le k\le d}\left|\Im f_\gamma\Big(a^{(i)}_{G_k,v_k}(w)\Big) - \Im f_\gamma\Big(a^{(j)}_{G_k,v_k}(w)\Big)\right|. \qquad (56)
\end{aligned}$$

Again, let v_k be the vertex that maximizes the above expression, and d_k be the number of unpinned neighbors of v_k in G_k. Applying eq. (44) of Consequence V.6 of the induction hypothesis to the graph G_k at vertex v_k with colors $i, j \in L(v_k)$ gives

$$\begin{aligned}
&\left|\Im f_\gamma\Big(a^{(i)}_{G_k,v_k}(w)\Big) - \Im f_\gamma\Big(a^{(j)}_{G_k,v_k}(w)\Big)\right|\\
&\qquad \le \frac{d_k}{d_k+\eta}\varepsilon_I + 6\Delta\varepsilon_w. \qquad (57)
\end{aligned}$$

Substituting eq. (57) into eq. (56) we then have

$$\frac{1}{d}\left|\Im \ln R^{(i,j)}_{G,u}(w)\right| \leq \frac{d_k}{d_k+\eta}\varepsilon_I + 8\Delta\varepsilon_w < \varepsilon_I,$$

where the last inequality holds since $\eta\varepsilon_I > 8(\Delta+1)\Delta\varepsilon_w$ (recalling that $0 \leq d_k \leq \Delta$ and $\eta \in [0.9, 1)$). This proves item 4, and also completes the inductive proof of Lemma V.3. □

We now use Lemma V.3 to prove Theorem V.1.

Proof of Theorem V.1: Let G be any graph of maximum degree Δ satisfying Condition 1. If G has no unpinned vertices, then $Z_G(w) = 1$ and there is nothing to prove. Otherwise, let u be an unpinned vertex in G. By Consequence V.4 of the induction hypothesis (which we proved in Lemma V.3), we then have $Z_w(G) \neq 0$ for w as in the statement of the theorem. ■

The proof of Theorem I.4 is now immediate.

Proof of Theorem I.4: Let the quantity $\nu_w = \nu_w(\Delta)$ be as in the statements of Theorems IV.1 and V.1. Fix the maximum degree Δ, and suppose that w satisfies

$$-\nu_w^2/8 \leq \Re w \leq 1 + \nu_w^2/8 \text{ and } |\Im w| \leq \nu_w^2/8. \quad (58)$$

Let G be a graph of maximum degree Δ satisfying Condition 1. When w satisfying eq. (58) is such that $\Re w \leq \nu_w/2$, we have $|w| \leq \nu_w$, so that $Z_G(w) \neq 0$ by Theorem IV.1, while when such a w satisfies $\Re w \geq \nu_w/2$, we have $Z_G(w) \neq 0$ from Theorem V.1. It therefore follows that $Z_G(w) \neq 0$ for all w satisfying eq. (58), and thus the quantity τ_Δ in the statement of Theorem I.4 can be taken to be $\nu_w^2/8$. ■

We conclude with a brief discussion of the dependence of τ_Δ on Δ. We saw above that τ_Δ can be taken to be $\nu_w(\Delta)^2/8$, so it is sufficient to consider the dependence of $\nu_w = \nu_w(\Delta)$ on Δ. Let $c = 10^{-6}$. As stated in the discussion following eq. (9), ν_w can be chosen to be $0.2c/(2^\Delta\Delta^7)$ for the case of general list colorings, or $c/(300\Delta^8)$ with the assumption of uniformly large list sizes (which, we recall from Remark 5, is satisfied in the case of uniform q-colorings). We have not tried to optimize these bounds, and it is conceivable that a more careful accounting of constants in our proofs can improve the value of the constant c by a few orders of magnitude.

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