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Singularities of Complex-Valued Solutions to Linear Parabolic Equations

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We construct examples of complex-valued singular solutions to linear, uniformly parabolic equations with complex coefficients in dimension $n \geq 2$, which are exactly as irregular as parabolic energy estimates allow.

1 Introduction

In this paper, we consider linear uniformly parabolic equations of the form

$$u_t - \text{div}(A(x, t)\nabla u) = 0. \tag{1}$$

Here $u : \mathbb{R}^{n+1} \to \mathbb{C}$, and the coefficients are bounded measurable, complex-valued functions satisfying

$$\text{Re}(A_{kl}(x, t)p_k\overline{p}_l) \geq \lambda|p|^2, \quad |A(x, t)\, p|^2 \leq \Lambda^2|p|^2 \tag{2}$$

for some constants $\lambda, \Lambda > 0$, and for all $(x, t) \in \mathbb{R}^{n+1}$ and $p \in \mathbb{C}^n$. By a solution, we mean that $u \in L^2_{loc,\,t}(H^1_{loc,\,x})$ solves (1) in the sense of distributions. We note that after writing

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u and A in terms of their real and imaginary parts as

$$u = v + iw, \quad A(x, t) = B(x, t) + iC(x, t),$$

Equation (1) can be viewed as the system of (real) equations

$$\begin{aligned} \partial_t v - \operatorname{div}(B(x, t)\nabla v) - \operatorname{div}(-C(x, t)\nabla w) = 0 \\ \partial_t w - \operatorname{div}(C(x, t)\nabla v) - \operatorname{div}(B(x, t)\nabla w) = 0. \end{aligned} \tag{3}$$

To motivate our results, we first discuss the elliptic case

$$\operatorname{div}(A(x)\nabla u) = 0, \tag{4}$$

where $u : \mathbb{R}^n \to \mathbb{C}$. Solutions to (4) are C^α when $n = 2$ by work of Morrey (see [19], Ch. 5.4). (Morrey in fact considers more general elliptic systems, where the solution can take values in $\mathbb{R}^m$ for any $m \geq 1$.) Real-valued solutions to (4) are C^α by fundamental work of De Giorgi [1] and Nash [20]. There are classical counterexamples to the continuity of solutions to elliptic systems in dimension $n \geq 3$ (see [2], [8], [12]). Discontinuous solutions to (4) were first constructed in dimension $n \geq 5$ [13] and later in dimension $n \geq 3$ [3]. In general, the best regularity we have for (4) is $u \in W^{1,2+\delta}_{loc}$ for some $\delta(n, \lambda, \Lambda) > 0$, which is only slightly better than the energy class of the solutions (see [6], Ch. 5 and the references therein, in particular the higher-integrability results of Gehring [5], Meyers [14], and Meyers-Elcrat [15]). In fact, for each $\gamma > 2$ there are solutions to (4) that are not in $W^{1,\gamma}_{loc}$ (see [3]).

Interestingly, the parabolic problem (1) has resisted a similar understanding. Real-valued solutions to (1) are C^α by Nash's theorem [20]. In general, we have the higher-integrability results $\nabla u \in L^{2+\delta}_{loc}$ and $u \in L^\infty_{loc,t}(L^{2+\delta}_{loc,x})$ for some $\delta(n, \lambda, \Lambda) > 0$ (see [21], [25]). There are also examples of discontinuity from smooth data for (1) when $n \geq 3$ (see [4] and [24], [23] for more general parabolic systems). However, all of these examples are in $L^\infty_{loc,t}(W^{1,2+\delta}_{loc,x})$ with $\delta > 0$ and are thus significantly more regular than the higher-integrability results predict. When $n = 2$ the known results do not imply continuity of solutions (unlike the elliptic case), which remained open for some time (see e.g., [9], [22]–[24]). We recently settled this problem with a counterexample [16]. Still, the example in [16] is barely irregular enough to develop a discontinuity (it is in $L^\infty_{loc,t}(L^p_{loc,x})$ for p large), so the regularity gap between theory and examples remained large.

The purpose of this paper is to complete the picture for (1) by constructing solutions in dimension $n \geq 2$ that are exactly as irregular as the parabolic higher-integrability results allow (see Theorem 2.2). We also prove some Liouville theorems that explain why previous approaches only produced "elliptic" discontinuities (see Theorems 2.4 and 2.5). Our results connect the regularity problem for (1) in $\mathbb{R}^{n+1}$, in parabolic geometry, to the regularity problem for the elliptic Equation (4) in $\mathbb{R}^{n+2}$. We discuss this connection further in the next section.

The paper is organized as follows. In Section 2, we give precise statements of our main results, Theorems 2.2, 2.4, and 2.5. In Section 3, we prove Theorem 2.2. In Section 4, we prove the Liouville Theorems 2.4 and 2.5. Finally, in Section 5, we discuss a few open questions motivated by this work.

2 Results

In this section, we state our results. We will deal with "spiraling" self-similar solutions to (1) of the form

$$u(x,t) = (-t)^{-\frac{\mu}{2}} e^{-\frac{i}{2}\log(-t)} w\left(\frac{x}{(-t)^{1/2}}\right). \tag{5}$$

Remark 2.1. Motivation for this ansatz (in the elliptic case) can be found in [13], Ch. 10.6.1, where the approach is to consider equations with constant complex coefficients in a thin cone and then flatten the boundary.

These are invariant under the rescalings $u \to \lambda^{\mu} e^{i\log\lambda} u(\lambda x, \lambda^2 t)$. We obtain a solution to (1) on $\mathbb{R}^n \times (-\infty, 0)$ with coefficients $A(x/(-t)^{1/2})$ if w solves the elliptic equation

$$\operatorname{div}(A(x)\nabla w) = \frac{1}{2}(iw + \mu w + x\cdot\nabla w) \tag{6}$$

on $\mathbb{R}^n$, and A satisfies (2) for some $\lambda, \Lambda > 0$. Furthermore, the solution defined by (5) is smooth up to $t = 0$ away from $x = 0$ and develops a "spiraling $-\mu$-homogeneous" discontinuity at $t = 0$ provided $\mu \geq 0$ and

$$w = |x|^{-\mu} g(x/|x|) e^{-i\log|x|}(1 + \mathcal{E}(|x|^{-2})) \text{ on } \mathbb{R}^n \backslash B_1. \tag{7}$$

Here, $g \in C^{\infty}(\mathbb{S}^{n-1})$ and $\mathcal{E}$ is a smooth function with $\mathcal{E}(0) = 0$. We can extend the solution to positive times, for example, by solving the heat equation with initial data $u(x,0) := |x|^{-\mu} g(x/|x|) e^{-i\log|x|}$, provided $\mu < n$.

Our 1st result is:

Theorem 2.2. For any $n \geq 2$ and $0 \leq 2\mu < n$, there exists a nontrivial solution w to a uniformly elliptic equation of the form (6) on $\mathbb{R}^n$, such that w satisfies (7).

By taking μ arbitrarily close to $\frac{n}{2}$, we obtain as a consequence:

Corollary 2.3. For any $n \geq 2$ and $\delta > 0$, there exists a solution u to a uniformly parabolic equation of the form (1) on $\mathbb{R}^{n+1}$ such that u satisfies

$$\lim_{t \to 0^-} \|u\|_{L_x^{2+\delta}(B_1 \times \{-t\})} = \infty, \quad \lim_{t \to 0^-} \|\nabla u\|_{L^{2+\delta}(B_1 \times (-1,-t))} = \infty.$$

(The ellipticity ratio λ/Λ degenerates as $\delta \to 0$, in accordance with the higher-integrability results.) We conclude, as in the elliptic case, that solutions to parabolic systems are only slightly better than their energy class.

Our remaining results are Liouville theorems for (6). It is natural to ask whether one can construct solutions that decay any faster than we managed. Our 1st Liouville theorem shows this is not possible:

Theorem 2.4. Assume that $w \in H^1_{loc}(\mathbb{R}^n)$ solves (6), with $|w| = O(|x|^{-\mu})$ and $2\mu \geq n$. Then $w \equiv 0$.

There are nontrivial $-\mu$-homogeneous solutions to elliptic systems of the form $\mathrm{div}(A(x)\nabla u) = 0$ in $\mathbb{R}^n$ provided $2\mu < n-2$, and there is a Liouville theorem for $-\mu$-homogeneous solutions on $\mathbb{R}^n \backslash \{0\}$ in the equality case (see [17]). Thus, Theorems 2.2 and 2.4 mirror the elliptic results in dimension $n+2$. This agrees with the observation that the parabolic energy $L_t^\infty(L_x^2) + L_t^2(H_x^1)$ in $\mathbb{R}^{n+1}$ and the elliptic energy H^1 in $\mathbb{R}^{n+2}$ are invariant under the matching rescalings

$$u \to \lambda^{n/2} u(\lambda x, \lambda^2 t), \quad \text{resp.} \quad u \to \lambda^{n/2} u(\lambda x).$$

Theorem 2.4 is a consequence of parabolic energy estimates. We can extend it to the "elliptic regime" $2\mu \geq n-2$ when w has the monotonicity property

$$(2\mu + x \cdot \nabla)|w|^2 \geq 0. \tag{8}$$

Theorem 2.5. Assume that $w \in H^1_{loc}(\mathbb{R}^n)$ solves (6), with $|w| = O(|x|^{-\mu})$ and $2\mu \geq n-2$. If in addition w satisfies (8), then $w \equiv 0$.

It is straightforward to check that previous examples ([4] and [24], [23] for more general systems) satisfy condition (8), which explains why they have "elliptic" discontinuities (i.e., $n \geq 3$ and $2\mu < n-2$).

3 Proof of Theorem 2.2

In this section, we prove Theorem 2.2. We exploit the useful observation from [3] that if $\mathrm{Im}(A)$ is symmetric, then the ellipticity condition (2) is satisfied provided $\mathrm{Re}(A)$ is uniformly positive definite, and $|A|$ is bounded.

Remark 3.1. Heuristically, this structure allows strong coupling between equations when we view (1) as the system (3). The example in [16] has skew-symmetric imaginary coefficients (which corresponds to the symmetry of the system coefficients). In that case, it is important to estimate the size of $\mathrm{Im}(A)$ because it affects the ellipticity condition.

3.1 Reduction to system of ODEs

We first reduce (6) to an ODE system. Let $r = |x|$ and let $\nu = r^{-1}x$ be the radial unit vector. We search for solutions of the form

$$w = \varphi(r)g(\nu)e^{-i\log r}, \tag{9}$$

where g is a smooth function on $\mathbb{S}^{n-1}$. For our calculations, it will be convenient to use the gradient and Laplace operators $\nabla_{\mathbb{S}^{n-1}}$ and $\Delta_{\mathbb{S}^{n-1}}$ on the sphere. When we view g as a zero-homogeneous function on $\mathbb{R}^n\backslash\{0\}$ (i.e., $g(x) = g(\nu)$ for $x \in \mathbb{R}^n\backslash\{0\}$), the spherical operators and the corresponding operators ∇ and Δ on $\mathbb{R}^n\backslash\{0\}$ are related by

$$\nabla g(x) = r^{-1}\nabla_{\mathbb{S}^{n-1}} g(\nu), \quad \Delta g(x) = \mathrm{div}(\nabla g)(x) = r^{-2}\Delta_{\mathbb{S}^{n-1}} g(\nu). \tag{10}$$

We note that the vector $\nabla_{\mathbb{S}^{n-1}} g(\nu)$ is orthogonal to ν.

Using these relations, we first compute

$$\nabla w = g e^{-i\log r}(\varphi'(r) - ir^{-1}\varphi)\nu + \varphi(r)e^{-i\log r} r^{-1}\nabla_{\mathbb{S}^{n-1}} g. \tag{11}$$

Now let

$$B = f(r)\nu \otimes \nu + h(r)(I - \nu \otimes \nu).$$

Since $B\,\nu = f(r)\,\nu$ and $B\,\tau = h(r)\,\tau$ for τ orthogonal to ν, it follows that:

$$B\nabla w = ge^{-i\log r}r^{n-1}(f\varphi' - ir^{-1}f\varphi)\frac{\nu}{r^{n-1}} + h\varphi e^{-i\log r}r^{-1}\nabla_{\mathbb{S}^{n-1}}g.$$

We will choose φ such that φ' and $r^{-1}\varphi$ are bounded. Taking the divergence of the above identity and using that $\frac{\nu}{r^{n-1}}$ is divergence-free away from the origin, that $\nabla_{\mathbb{S}^{n-1}}g$ is orthogonal to ν, and the relations (10), we arrive at

$$\begin{aligned}
&\mathrm{div}(B\nabla w)\\
&= \frac{g}{r^{n-1}}\left[e^{-i\log r}r^{n-1}(f\varphi' - ir^{-1}f\varphi)\right]' + \left(h\varphi e^{-i\log r}\right)\mathrm{div}(r^{-1}\nabla_{\mathbb{S}^{n-1}}g)\\
&= ge^{-i\log r}\left[\frac{(r^{n-1}f\varphi')'}{r^{n-1}} - \left(f - \frac{\Delta_{\mathbb{S}^{n-1}}g}{g}h\right)\frac{\varphi}{r^2} - i\left(\frac{(r^{n-2}f\varphi)'}{r^{n-1}} + \frac{f\varphi'}{r}\right)\right].
\end{aligned}$$

Let g be an eigenfunction of $\Delta_{\mathbb{S}^{n-1}}$ with eigenvalue $-\lambda_g < 0$. Then the previous expression becomes

$$\mathrm{div}(B\nabla w) = ge^{-i\log r}\left[\frac{(r^{n-1}f\varphi')'}{r^{n-1}} - (f + \lambda_g h)\frac{\varphi}{r^2} - i\frac{(r^{n-2}f\varphi^2)'}{r^{n-1}\varphi}\right].$$

Thus, if we take coefficients

$$A = \alpha I + i(\beta(r)\nu\otimes\nu + \gamma(r)(I - \nu\otimes\nu)) \tag{12}$$

with $\alpha > 0$ constant, we obtain

$$\begin{aligned}
\mathrm{div}(A\nabla w) = ge^{-i\log r}\Bigg[&\alpha\left(\frac{(r^{n-1}\varphi')'}{r^{n-1}} - (1+\lambda_g)\frac{\varphi}{r^2}\right) + \frac{(r^{n-2}\beta\varphi^2)'}{r^{n-1}\varphi}\\
&+ i\left(\frac{(r^{n-1}\beta\varphi')'}{r^{n-1}} - (\beta + \lambda_g\gamma)\frac{\varphi}{r^2} - \alpha\frac{(r^{n-2}\varphi^2)'}{r^{n-1}\varphi}\right)\Bigg].
\end{aligned}$$

Since

$$iw + \mu w + x \cdot \nabla w = ge^{-i\log r}(\mu\varphi + r\varphi'),$$

the Equation (6) becomes the ODE system

$$\begin{cases} \frac{(r^{n-2}\beta\varphi^2)'}{r^{n-1}\varphi} = \frac{1}{2}(\mu\varphi + r\varphi') + (1+\lambda_g)\alpha\frac{\varphi}{r^2} - \alpha\frac{(r^{n-1}\varphi')'}{r^{n-1}}, \\ \lambda_g\gamma\frac{\varphi}{r^2} = -\alpha\frac{(r^{n-2}\varphi^2)'}{r^{n-1}\varphi} + \frac{(r^{n-1}\beta\varphi')'}{r^{n-1}} - \beta\frac{\varphi}{r^2}. \end{cases} \tag{13}$$

Below we will fix an eigenfunction g of $\Delta_{\mathbb{S}^{n-1}}$ and fix φ and α depending on μ, such that $\varphi \sim r^{-\mu}$ for r large and $\alpha > 0$. Then the 1st equation determines β and the 2nd one γ. By the remark at the beginning of the section, the point is to make choices such that β and γ are bounded.

3.2 Solving the ODE system

To begin, we fix g to be any linear function restricted to the sphere so that

$$\lambda_g = n - 1.$$

Integrating the 1st equation in (13), we obtain

$$\begin{aligned} \beta = {} & \frac{1}{4}\left(r^2 + \frac{2\mu - n}{r^{n-2}\varphi^2}\int_0^r \varphi^2(s)s^{n-1}\,\mathrm{d}s\right) \\ & + \frac{n\alpha}{r^{n-2}\varphi^2}\int_0^r \varphi^2(s)s^{n-3}\,\mathrm{d}s \\ & + \frac{\alpha}{r^{n-2}\varphi^2}\int_0^r \varphi'^2(s)s^{n-1}\,\mathrm{d}s - \alpha\frac{r\varphi'}{\varphi}. \end{aligned} \tag{14}$$

Remark 3.2. It follows easily that if $2\mu \geq n$ and $\varphi = O(r^{-\mu})$, then β is unbounded (compare to Theorem 2.4).

We define

$$\varphi(r) = \begin{cases} r, \quad 0 \leq r < 3/4 \\ r^{-\mu} + C_\mu r^{-\mu-2}, \quad r > 1 \\ \text{positive and smooth}, \quad 1/2 < r < 3/2, \end{cases} \tag{15}$$

where $C_\mu \geq 0$ will be chosen later.

Remark 3.3. By Theorem 2.5, it will be necessary to take $C_\mu > 0$ when $2\mu \geq n-2$ (and in particular, to generate discontinuities in the case $n = 2$).

For $r < 3/4$, it is easy to check that β and γ are of the form $c_1(n, \alpha) + c_2(n, \mu)r^2$ (with c_i linear in α and μ) so we only need to analyze the solutions for r large. We divide into three cases.

Case 1: $2\mu < n - 2$. We take $C_\mu = 0$ and $\alpha = 1$. It is easy to check that β and γ have the form $c_1 + c_2 r^{2-n+2\mu}$ for $r > 1$, which is bounded.

Case 2: $n - 2 < 2\mu < n$. Now the quantities

$$D := \int_0^\infty (\varphi^2 - s^{-2\mu})s^{n-1}\,ds, \quad E := \int_0^\infty \varphi^2 s^{n-3}\,ds, \quad F := \int_0^\infty \varphi'^2 s^{n-1}\,ds$$

are bounded, for any fixed $C_\mu \geq 0$. The solution (14) becomes

$$\beta = \left(-\frac{n-2\mu}{4}D + \alpha(nE + F)\right) r^{2\mu-n+2} + \mathcal{R}(1).$$

Here and below, $\mathcal{R}(1)$ denotes any smooth function on $(1, \infty)$ whose j^{th} derivative is $O(r^{-j})$ as $r \to \infty$ for each $j \geq 0$. Using the definition of φ, we estimate

$$\begin{aligned} D &\geq -\int_0^1 s^{n-1-2\mu}\,ds + 2C_\mu \int_1^\infty s^{n-3-2\mu}\,ds \\ &= -\frac{1}{n-2\mu} + \frac{2C_\mu}{2\mu - n + 2}. \end{aligned}$$

We conclude that

$$-\frac{n-2\mu}{4}D \leq \frac{1}{4} - \frac{n-2\mu}{2(2\mu - n + 2)}C_\mu < 0$$

provided we choose C_μ large. We may then choose $\alpha > 0$ small so that

$$-\frac{n-2\mu}{4}D + \alpha(nE + F) = 0,$$

hence

$$\beta = \mathcal{R}(1).$$

Solving the 2nd equation in (13) for γ gives

$$\gamma = \mathcal{R}(1),$$

which completes this case.

Case 3: $2\mu = n - 2$. This case is similar to the case $2\mu > n - 2$, except to leading order β grows logarithmically. Computing (14) gives

$$\beta = \left(-C_\mu + \alpha\left(n + \frac{1}{4}(n-2)^2\right)\right)\log r + \mathcal{R}(1).$$

Choosing C_μ and α to satisfy the relation

$$C_\mu = \left(n + \frac{1}{4}(n-2)^2\right)\alpha$$

we arrive at the same conclusion as in Case 2, completing the construction.

3.3 Proof of Theorem 2.2

For $0 \le 2\mu < n$, by taking w and A as constructed above, we obtain a nontrivial solution to the Equation (6) on $\mathbb{R}^n$, such that A has the desired ellipticity properties and w has the desired asymptotics. This proves Theorem 2.2. More precisely:

Proof of Theorem 2.2 For $0 \le 2\mu < n$, take φ, g, α, β, γ as constructed above. Then the function

$$w = \varphi(r)g(\nu)e^{-i\log r}$$

solves the Equation (6) in $\mathbb{R}^n$ with bounded coefficients

$$A = \alpha I + i(\beta(r)\nu \otimes \nu + \gamma(r)(I - \nu \otimes \nu)).$$

By the choice (15) of φ, the function w has the asymptotics (7). Since $\alpha > 0$ is constant and $\mathrm{Im}(A)$ is symmetric, the coefficients satisfy the ellipticity condition (2), completing the proof. ■

Remark 3.4. In our construction, w is Lipschitz but no better at 0 and smooth but not analytic away from 0. This is a consequence of choices we made for computational convenience. It is not hard to modify the construction so that w is analytic on all of $\mathbb{R}^n$, for example, by taking $w = \varphi(r)g(\nu)e^{-\frac{i}{2}\log(1+r^2)}$ with g as above and

$$\varphi = r\left((1+r^2)^{-\frac{\mu+1}{2}} + C_\mu(1+r^2)^{-\frac{\mu+3}{2}}\right).$$

The coefficients $A(x)$ also become analytic on all of $\mathbb{R}^n$ with these modifications.

4 Liouville Theorems

In this section, we prove the Liouville theorems Theorem 2.4 and Theorem 2.5.

4.1 Proof of Theorem 2.4

Theorem 2.4 says that if $w \in H^1_{loc}(\mathbb{R}^n)$ solves the uniformly elliptic Equation (6) on $\mathbb{R}^n$, namely

$$\mathrm{div}(A(x)\nabla w) = \frac{1}{2}(iw + \mu w + x \cdot \nabla w),$$

and $|w| = O(|x|^{-\mu})$ with $2\mu \geq n$, then $w \equiv 0$. We prove it here.

Proof. of Theorem 2.4 Let $\psi \in C_0^\infty(\mathbb{R}^n)$ be real-valued. Multiplying (6) by $\overline{w}\psi^2$, we obtain

$$2\mathrm{Re}\left(\mathrm{div}(A\nabla w)\overline{w}\psi^2\right) = \frac{1}{2}(2\mu|w|^2 + x \cdot \nabla|w|^2)\psi^2. \tag{16}$$

Integrating by parts and using the ellipticity condition (2), we get

$$\begin{aligned}\int_{\mathbb{R}^n} &(-\lambda|\nabla w|^2\psi^2 + C(\lambda, \Lambda)|w|^2|\nabla\psi|^2)\,\mathrm{d}x \\ &\geq \frac{2\mu - n}{2}\int_{\mathbb{R}^n}|w|^2\psi^2\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^n}|w|^2 x \cdot \nabla(\psi^2)\,\mathrm{d}x.\end{aligned} \tag{17}$$

Since $2\mu \geq n$, the 1st term on the right side is non-negative.

We now fix our choice of ψ. Let ψ_1 be a smooth, radially decreasing function supported in B_2 with $\psi_1 \equiv 1$ in B_1, and let $\psi_R := \psi_1(R^{-1}x)$. Take $\psi = \psi_R$. Then the 2nd term on the right side of (17) is non-negative, so the right side is non-negative. Using that $|w|^2|\nabla\psi|^2 = O(R^{-2\mu-2})$ in $B_{2R}\backslash B_R$, we conclude that

$$\int_{B_R}|\nabla w|^2\,\mathrm{d}x = O(R^{n-2\mu-2}) = O(R^{-2}).$$

By taking $R \to \infty$, we conclude that w is constant, and by the Equation (6) this constant is zero. ■

4.2 Proof of Theorem 2.5

Theorem 2.5 says that if $w \in H^1_{loc}(\mathbb{R}^n)$ solves (6) on $\mathbb{R}^n$, with $|w| = O(|x|^{-\mu})$ and $2\mu \geq n-2$, and in addition w satisfies the monotonicity property (8):

$$(2\mu + x \cdot \nabla)|w|^2 \geq 0,$$

then $w \equiv 0$. We prove it here.

Proof. of Theorem 2.5 We start again with the identity (16). By (8) the right side of (16) is non-negative. Integrating by parts gives the Caccioppoli inequality

$$\int_{\mathbb{R}^n} |\nabla w|^2 \psi^2 \, \mathrm{d}x \le C(\lambda, \Lambda) \int_{\mathbb{R}^n} |w|^2 |\nabla \psi|^2 \, \mathrm{d}x.$$

Choosing ψ as before, we recover the inequality

$$\int_{B_R} |\nabla w|^2 \, \mathrm{d}x = O(R^{n-2\mu-2}),$$

which proves the theorem when $2\mu > n-2$. In the critical case $2\mu = n-2$, use instead

$$\psi = \begin{cases} 1 \text{ in} B_1, \\ 1 - \log(r)/\log(R) \text{ in} B_R \backslash B_1, \\ 0 \text{ in} \mathbb{R}^n \backslash B_R \end{cases}$$

to obtain

$$\int_{B_{\sqrt{R}}} |\nabla w|^2 \, \mathrm{d}x = O\left(\frac{1}{\log R}\right).$$

Again, taking $R \to \infty$ we conclude that w is constant, and by the Equation (6) this constant is zero. ■

5 Some Questions

Our results motivate several questions about the regularity of solutions to parabolic systems. First, the coefficients in our examples allow for strong coupling between equations. It is natural to ask if there are structure conditions on the coefficients that give positive regularity results, and our 1st two questions address this issue. Second, our examples exhibit blowup at a single point. The 3rd question below concerns the possibility of constructing solutions with larger singular sets. Finally, parabolic systems with quasilinear structure arise naturally, and it would be interesting to construct singular solutions to systems of that type. Our last question addresses this problem.

1. Our examples have coefficients with symmetric imaginary part. Similar constructions might be possible with skew-symmetric imaginary coefficients,

using techniques from [16]. In this setting, the imaginary coefficients play a role in ellipticity.

2. For elliptic systems, there is a sharp condition on the spectrum of the coefficients that guarantees continuity of solutions [11]. Sufficient conditions are known in the parabolic case ([10], [11]). It would be interesting to investigate how closely our counterexamples match these conditions. Similarly, it would be interesting to optimize in our examples the dependence of the higher-integrability exponent δ on the ellipticity ratio λ/Λ.
3. Solutions to parabolic systems in dimension $n \geq 3$ can be discontinuous on very large sets [22]. It is natural to ask how large the discontinuity set can be when $n = 2$. Known results imply spatial continuity at almost every time, which is false when $n \geq 3$ by elliptic examples.
4. Parabolic systems with the quasilinear structure

$$u_t - \text{div}(A(u)\nabla u) = 0 \tag{18}$$

have a well-developed partial regularity theory and are important in applications [7]. Here, the coefficients depend smoothly on u. Constructing solutions to (18) becomes easier when $u : \mathbb{R}^n \to \mathbb{R}^m$ for m large because there is more room to "disperse u."; [16] contains examples of discontinuity formation for (18) when $n = 2$, $m = 4$. One can improve to $n = 2$, $m = 3$ using similar techniques [18]. Continuity for solutions to (18) in the case $n = m = 2$ (in particular, the $\mathbb{C}$-valued scalar case) remains open. It seems possible in view of Theorem 2.5 that the restrictive geometry of the target could play in favor of regularity (see the discussion in [18]).

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