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Optimal error estimate of elliptic problems with Dirac sources for discontinuous and enriched Galerkin methods

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ABSTRACT

We present an optimal a priori error estimates of the elliptic problems with Dirac sources away from the singular point using discontinuous and enriched Galerkin finite element methods. It is widely shown that the finite element solutions for elliptic problems with Dirac source terms converge sub-optimally in classical norms on uniform meshes. However, here we employ inductive estimates and L^2 norm to obtain the optimal order by excluding the small ball regions with the singularities for both two and three dimensional domains. Numerical examples are presented to substantiate our theoretical results.

1. Introduction

Optimal numerical error estimations of elliptic problems with Dirac sources are crucial to improve the efficiency and robustness of realistic simulations. In this paper, we consider the following system,

$$\begin{cases} -\nabla \cdot (\beta \nabla u) &= \displaystyle\sum_{i=1}^{N} \pi \delta_{\mathbf{x}_n} \quad \text{in } \Omega, \\ \qquad\qquad u &= 0 \qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where a bounded, open, and convex domain is denoted by $\Omega \subset \mathbb{R}^d (d \in 2, 3)$ and $\delta_{\mathbf{x}_n}$ is the Dirac measure at $\mathbf{x}_n \in \Omega$, where $n = 1, \cdots, N$ ($N > 0$ is a number of Dirac sources). Optimal error estimates for these systems (1) will be brought to bear on problems in different area including electrodynamics [19], control problems [7], and especially subsurface ground flow, in which flow injections and productions are concentrated in very small regions as compared to the total area of interest [10,26,28] (see, Fig. 1). For the latter problems, $u : \Omega \to \mathbb{R}$ is a solution which often refers to the flow pressure in subsurface, and β is a given positive function or material property which is sufficiently smooth.

The accuracy of finite element approximations of the above systems have difficulties to obtain optimal order of convergence for errors since the solution (u) of (1) is not a $H^1(\Omega)$ function. In the past years, various approaches were employed

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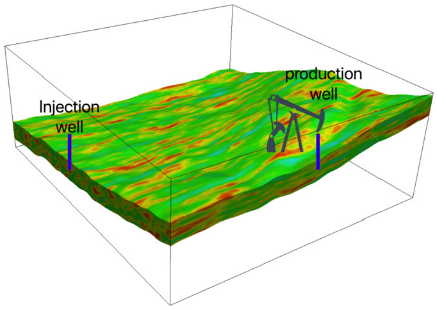


Fig. 1. An example of a large reservoir with injection and production wells.

to obtain the optimal convergence rate with these singularities which includes changing the mesh or right-hand side, using weighted Sobolev spaces, or using mappings [1,3,6,11,15,32,33,35,37]. For conforming continuous Galerkin (CG) finite element methods, Casas [6] showed that the $L^2(\Omega)$ convergence error is of order $h^{2-\frac{d}{2}}$ regardless of the polynomial order in the approximation. Recently, Kopple-Wohlmuth [21] obtained the sharp L^2-convergence estimate on compact subsets in $\Omega \setminus \{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$ for conforming CG finite element methods. The result shows that the L^2 error is of order h^{l+1} for $l \geq 2$ and $h^2|\log h|$ for $l = 1$, where l denotes the polynomial order in the approximation. For the non-conforming discontinuous Galerkin (DG) approximations, Houston-Wihler [18] obtained the sharp L^2-convergence estimate on two dimensional domain Ω.

In this paper, we obtain L^2-convergence estimate on compact subsets in $\Omega \setminus \{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$ for the non-conforming discontinuous Galerkin (DG) and enriched Galerkin (EG) approximations. Our estimate is obtained for both two and three dimensional domains.

For subsurface flow simulations, it is well known that employing classical continuous Galerkin conforming finite element methods (CG) for (1) and coupling with a transport equation yields non-physical numerical oscillations due to lack of the local conservation on the existing mesh. Thus, it is important to choose a physics-preserving numerical approximation which provides local mass conservation to avoid spurious numerical oscillations [20,36]. One of the most popular and successful methods in terms of the local flux conservation is discontinuous Galerkin (DG) [2,40]. DG provides robust results with general partial differential equations with highly varying material properties [29–31,39,41]. However, DG requires large number of degrees of freedom and optimal linear solver for higher order approximations are still ongoing study.

Recently studied enriched Galerkin (EG) approximations also provide locally and globally conservative fluxes [5,24,38]. EG, which enriches CG with piecewise constant functions, has the same bilinear forms as the interior penalty DG schemes. The main advantage of EG is that EG has substantially fewer degrees of freedom in comparison with DG and a fast effective solver for elliptic/parabolic problems [24]. Moreover, EG has been successfully employed to realistic multi scale and multi physics applications [8,14,22,23,25,27] with dynamic mesh adaptivity.

Therefore, we focus on optimal error estimates of (1) for both non-conforming DG and EG finite element methods in this paper. In Section 2, we introduce the general notations and each finite element spaces which we will use throughout the paper. The main theorem of this paper is stated with the proof of the corresponding lemmas in Section 3. In Section 4, the main theorem is proved, and remaining lemma related to L^1 norm is proved in Section 5. Several numerical simulations are illustrated in Section 6. Moreover, supplementary proofs for some lemmas are given in the appendix section.

2. Preliminaries

In this section, we briefly discuss the preliminary lemmas and notations. For the simplicity, we only consider a single Dirac source which is denoted by $x_0 \in \Omega$, but extension to multiple Dirac sources are trivial since the problem is linear. Then the weak solution $u \in W_0^{1,p}(\Omega)$ of (1) is given by

$$\int_{\Omega} \beta \nabla u \cdot \nabla v dx = v(x_0), \qquad \forall\, v \in C_0(\Omega), \tag{2}$$

where

$$W_0^{1,p}(\Omega) := \Big\{ v \in L^p(\Omega) : \nabla v \in L^p(\Omega) \text{ and } v|_{\partial\Omega} = 0 \Big\},$$

with a fixed $p \in \left(1, \frac{d}{d-1}\right)$. Here we assume β is a sufficiently smooth positive function. The existence of the Green's function and its point-wise estimates can be found in [12,13,17], and we recall the point-wise estimates of the Green's function in the following lemma.

Lemma 2.1. *Let $u \in L^2(\Omega)$ be the weak solution of (1) and assume that $\beta \in C^{k+1}(\Omega)$. Then, u belongs to $C^k(\Omega \setminus \{x_0\})$ and satisfies the following bounds.*

1. *For $d = 2$,*

$$|u(x)| \le C \log|x - x_0| \quad \text{and} \quad |\partial_x^\alpha u(x)| \le \frac{C_k}{|x - x_0|} \quad \forall x \in \Omega \setminus \{x_0\},$$

for any multi-index α with $1 \le |\alpha| \le k$.

2. *For $d = 3$,*

$$|u(x)| \le \frac{C}{|x - x_0|} \quad \text{and} \quad |\partial_x^\alpha u(x)| \le \frac{C_k}{|x - x_0|^{1+|\alpha|}} \quad \forall x \in \Omega \setminus \{x_0\},$$

for any multi-index α with $1 \le |\alpha| \le k$. Here the constants C and C_k are determined by β and k.

The fundamental solution u is smooth enough away from the singular point x_0, which can be seen from the following classical elliptic regularity result [16].

Lemma 2.2. *Let $r_2 > r_1 > 0$ and assume that $\beta \in C^{k+1}(B(0, r_2))$ for some $k \in \mathbb{N}$. Then, for any $w \in H^2(B(0, r_2))$ satisfying*

$$\int_{B(0,r_2)} \beta \nabla w \cdot \nabla \phi \, dx = 0 \quad \forall\, \phi \in H_0^1(B(0, r_2)),$$

there exists a constant $C = C(d, k, r_1, r_2, \|\beta\|_{C^{k+1}(B(0,r_2))}) > 0$ such that

$$\|w\|_{H^{k+1}(B(0,r_1))} \le C\|w\|_{H^2(B(0,r_2))}.$$

2.1. Finite element approximations

Let $\mathcal{T}_h$ be the shape-regular (in the sense of Ciarlet) and disjoint triangular or quadrilateral elements by a family of partitions of Ω into d-simplices T (triangles/squares in $d = 2$ or tetrahedra/cubes in $d = 3$). We denote by h_T the diameter of T and we set $h = \max_{T \in \mathcal{T}_h} h_T$. Also we denote by $\mathcal{E}$ the set of all edges and by $\mathcal{E}^I$ and $\mathcal{E}^\partial$ the collection of all interior and boundary edges, respectively. In the following notation, we assume edges for two dimension but the results hold analogously for faces in three dimensional case. The space $H^s(\mathcal{T}_h)$ $(s \in \mathbb{R})$ is the set of element-wise H^s functions on $\mathcal{T}_h$, and $L^2(\mathcal{E})$ refers to the set of functions whose traces on the elements of $\mathcal{E}$ are square integrable. Let $\mathbb{Q}_l(T)$ denote the space of polynomials of partial degree at most l. Throughout the paper, we use the standard notation for Sobolev spaces and their norms. For example, let $E \subseteq \Omega$, then $\|\cdot\|_{1,E}$ and $|\cdot|_{1,E}$ denote the $H^1(E)$ norm and seminorm, respectively. For simplicity, we eliminate the subscripts on the norms if $E = \Omega$. For any $e \in \mathcal{E}^I$, let T^+ and T^- be two neighboring elements such that

$$e = \partial T^+ \cap \partial T^-$$

and we denote by h_e the length of the edge e. Let $\mathbf{n}^+$ and $\mathbf{n}^-$ be the outward normal unit vectors to ∂T^+ and ∂T^-, respectively ($\mathbf{n}^\pm := \mathbf{n}_{|T^\pm}$). For any given function ξ and vector function $\boldsymbol{\xi}$, defined on the triangulation $\mathcal{T}_h$, we denote $\xi^\pm$ and $\boldsymbol{\xi}^\pm$ by the restrictions of ξ and $\boldsymbol{\xi}$ to $T^\pm$, respectively.

Next, we define the average $\{\cdot\}$ as follows: for $\zeta \in L^2(\mathcal{T}_h)$ and $\boldsymbol{\tau} \in L^2(\mathcal{T}_h)^d$,

$$\{\zeta\} := \frac{1}{2}(\zeta^+ + \zeta^-) \quad \text{and} \quad \{\boldsymbol{\tau}\} := \frac{1}{2}(\boldsymbol{\tau}^+ + \boldsymbol{\tau}^-) \quad \text{on } e \in \mathcal{E}^I. \tag{3}$$

On the other hand, for $e \in \mathcal{E}^\partial$, we set $\{\zeta\} := \zeta$ and $\{\boldsymbol{\tau}\} := \boldsymbol{\tau}$. The jump across the interior edge will be defined as

$$[\zeta] = \zeta^+\mathbf{n}^+ + \zeta^-\mathbf{n}^- \quad \text{and} \quad [\boldsymbol{\tau}] = \boldsymbol{\tau}^+ \cdot \mathbf{n}^+ + \boldsymbol{\tau}^- \cdot \mathbf{n}^- \quad \text{on } e \in \mathcal{E}^I.$$

For $e \in \mathcal{E}^\partial$, we let $[\zeta] := \zeta\mathbf{n}$ and $[\boldsymbol{\tau}] := \boldsymbol{\tau} \cdot \mathbf{n}$.

We introduce the space of piecewise discontinuous polynomials of degree l by

$$M^l(\mathcal{T}_h) := \left\{ \psi \in L^2(\Omega) |\ \psi_{|T} \in \mathbb{Q}_l(T),\ \forall T \in \mathcal{T}_h \right\}, \tag{4}$$

and let $M_0^l(\mathcal{T}_h)$ be the subspace of $M^l(\mathcal{T}_h)$ consisting of continuous piecewise polynomials

$$M_0^l(\mathcal{T}_h) := M^l(\mathcal{T}_h) \cap C(\Omega).$$

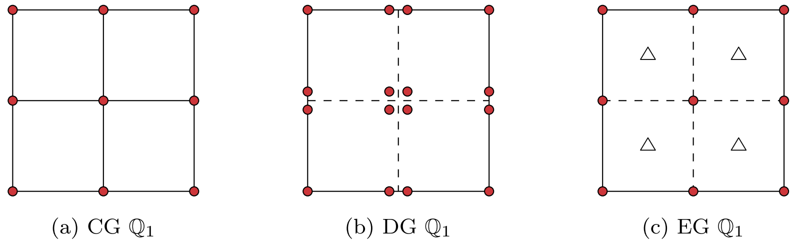


Fig. 2. A sketch of the degrees of freedom for (a) CG, (b) DG, and (c) EG in a two-dimensional Cartesian grid ($\mathbb{Q}$) with $l = 1$. Four circles ($\circ$) are the degrees of freedom for continuous Galerkin ($M_0^l(\mathcal{T}_h)$) and ($\triangle$) is the discontinuous constant ($M^0(\mathcal{T}_h)$).

Here, $\mathbb{Q}_l(T)$ denotes either the space of all polynomials of total degree at most l on T, when T is a triangle, or the space of all polynomials of degree at most l in each coordinate direction, when T is a quadrilateral. Then the finite element space for the discontinuous Galerkin method is defined as

$$V_{h,l}^{\mathrm{DG}}(\mathcal{T}_h) = M^l(T_h), \tag{5}$$

and on the other hand, the enriched Galerkin finite element space is defined as

$$V_{h,l}^{\mathrm{EG}}(\mathcal{T}_h) := M_0^l(\mathcal{T}_h) + M^0(\mathcal{T}_h), \tag{6}$$

where $l \geq 1$ [5,24,38]. Fig. 2 illustrates the different degrees of freedom for CG, DG, and EG methods on a two dimensional Cartesian grid ($\mathbb{Q}$) with a polynomial order $l = 1$.

For simplicity, we define the uniform notation

$$V_{h,l}(\mathcal{T}_h) := V_{h,l}^{\mathrm{DG}}(\mathcal{T}_h) \text{ or } V_{h,l}^{\mathrm{EG}}(\mathcal{T}_h), \tag{7}$$

which indicates either non-conforming DG or EG spaces throughout the paper. Moreover, for any subspace $D \subset \Omega$, we consider the discontinuous Sobolev space

$$H_h^l(D) := \{v \;:\; v \in H^l(T \cap D) \text{ for each } T \in \mathcal{T}_h \text{ and } \|v\|_{H_h^l(D)} < \infty\}, \tag{8}$$

equipped with the norm

$$\|v\|^2_{H_h^l(D)} := \sum_{i=0}^{l} |v|^2_{H_h^i(D)}, \quad |v|^2_{H_h^i(D)} := \sum_{T \in \mathcal{T}_h} |v|^2_{H^i(T\cap D)}. \tag{9}$$

We also set a locally defined norm $|||\cdot|||_{H_h^1(D)}$ by

$$|||v|||_{H_h^1(D)} = \|v\|_{H_h^1(D)} + \left(\sum_{e\in\mathcal{E}} h^{-1} \int_{e\cap D} |[v]|^2 ds\right)^{1/2} + \left(\sum_{e\in\mathcal{E}} h \int_{e\cap D} \sum_{i=1}^{d} \left|\left\{\frac{\partial v}{\partial x_i}\right\}\right|^2 ds\right)^{1/2}. \tag{10}$$

Finally, the approximate solution of u in (1) is defined as $u_h \in V_{h,l}(\mathcal{T}_h)$ such that

$$a(u_h, v) = v(x_0), \quad \forall v \in V_{h,l}(\mathcal{T}_h), \tag{11}$$

where

$$a(w,v) = \sum_{T\in\mathcal{T}_h} \int_T \beta \nabla w \cdot \nabla v \, dx - \sum_{e\in\mathcal{E}} \int_e \beta \{\nabla w\} \cdot [v] \, ds - \sum_{e\in\mathcal{E}} \int_e \beta \{\nabla v\} \cdot [w] \, ds + \frac{\gamma}{h} \sum_{e\in\mathcal{E}} \int_e [v][w] \, ds, \tag{12}$$

where $\gamma > 0$ is a sufficiently large positive constant.

Before we state and prove the main theorem of this paper in the next section, we note that the general convergence rate of the error estimation for (11)-(12) is not optimal as discussed in the introduction. For example, it was proved in [18] that the error for (11)-(12), is bounded by

$$\|u - u_h\|_{L^2(\Omega)} \leq Ch, \tag{13}$$

where C is a positive constant independent of h for the DG approximate solution u_h in a two dimensional space. We note that the convergence order is h since the error was measured on the entire domain including the singularities in [18]. In this paper, we will recover the optimal convergence rate on domains which excludes a small region near the singularity. For this aim, we extend the above estimate (13) to the three dimensional case for both DG and EG approximations for our further use in the following theorem.

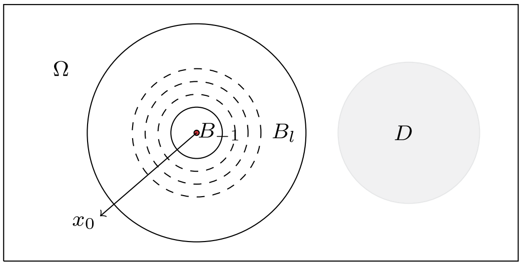


Fig. 3. A figure illustrating a series of balls $\mathbf{B}_{-1} \subset \mathbf{B}_0 \subset \cdots \subset \mathbf{B}_l$, a singularity x_0, and a subdomain D.

Theorem 2.3. *Let u be the weak solution of* (1) *and let u_h be the approximate solution solving* (11) *in $d \in \{2, 3\}$. Then, there exists a constant $C > 0$ independent of $h \in (0, 1)$ such that*

$$\|u - u_h\|_{L^2(\Omega)} \le Ch^{2-\frac{d}{2}}. \tag{14}$$

For the exposition of the paper, we defer the proof of this result to the appendix.

3. Main theorem and outline of the proof

The main goal of this paper is to prove the following theorem.

Theorem 3.1. *Let $u \in W_0^{1,p}(\Omega)$ be the weak solution of* (2) *and $u_h \in V_{h,l}(\mathcal{T}_h)$ be the approximated solution solving* (11)*. For any compact set $D \subset\subset \Omega \setminus \{x_0\}$, we have*

$$\|u - u_h\|_{L^2(D)} = O(h^{l+1}). \tag{15}$$

For the proof, we extend the idea in [21], which was applied for conforming CG methods to non-conforming DG and EG methods.

To begin with, we choose a series of balls $\mathbf{B}_{-1} \subset \mathbf{B}_0 \subset \cdots \subset \mathbf{B}_l$ such that $D \subset \Omega \setminus \mathbf{B}_l$ and $x_0 \in \mathbf{B}_{-1}$ holds and

$$\mathrm{dist}(\partial \mathbf{B}_j, \partial \mathbf{B}_{j-1}) > 3\delta \quad \text{for} \quad j = 0, 1, \cdots, l, \tag{16}$$

where $\delta > 0$ is a small fixed value (Fig. 3). Our aim is to find an inductive estimate on $\|u - u_h\|_{L^2(\Omega\setminus\mathbf{B}_j)}$ for $j \in \{0, 1, \cdots, l\}$. For this reason, we choose an auxiliary set $\tilde{\mathbf{B}}_j \subset \Omega$ for $j \in \{0, 1, \cdots, l\}$ such that $\mathbf{B}_{j-1} \subsetneq \tilde{\mathbf{B}}_j \subsetneq \mathbf{B}_j$ satisfying

$$\mathrm{dist}(\partial \mathbf{B}_{j-1}, \partial \tilde{\mathbf{B}}_j) > \delta \quad \text{and} \quad \mathrm{dist}(\partial \tilde{\mathbf{B}}_j, \partial \mathbf{B}_j) > \delta. \tag{17}$$

As the first step of the proof, we will prove

$$\begin{aligned} &\|u - u_h\|_{L^2(\Omega\setminus\mathbf{B}_j)} \\ &\quad \le Ch^l \left(\|\nabla(u - u_h)\|_{L^1(\tilde{\mathbf{B}}_j)} + \sum_{e\in\mathcal{E}} \int_{e\cap\tilde{\mathbf{B}}_j} \Big|[u - u_h]\Big| ds \right) \\ &\qquad + Ch \left(\|\nabla(u - u_h)\|^2_{L^2(\Omega\setminus\tilde{\mathbf{B}}_j)} + \frac{1}{h} \sum_{e\in\mathcal{E}\cap(\Omega\setminus\tilde{\mathbf{B}}_j)} \int_e |[u - u_h]|^2 \, ds \right)^{\frac{1}{2}} \end{aligned} \tag{18}$$

in Lemma 4.1 for each $j \in \{0, 1, 2, \cdots, l\}$. Next, based on the idea for local error estimate from [9], we estimate the second term in right hand side of (18) by

$$\|u - u_h\|_{L^2(\Omega\setminus\mathbf{B}_j)} \le Ch^l \left(\|\nabla(u - u_h)\|_{L^1(\tilde{\mathbf{B}}_j)} + \sum_{e\in\mathcal{E}} \int_{e\cap\tilde{\mathbf{B}}_j} \Big|[u - u_h]\Big| ds \right) + Ch^{l+1} + Ch\|u - u_h\|_{L^2(\Omega\setminus\mathbf{B}_{j-1})}. \tag{19}$$

The above estimate is stated and proved in Lemma 4.3.

Finally, we estimate the L^1 norms on the right hand side in (19) by

$$\|u-u_h\|_{L^2(\Omega\backslash\mathbf{B}_j)} \le \begin{cases} Ch^{l+1}+Ch\|u-u_h\|_{L^2(\Omega\backslash\mathbf{B}_{j-1})} & \text{if } l\ge 2 \\ Ch^2|\log h|+Ch\|u-u_h\|_{L^2(\Omega\backslash\mathbf{B}_{j-1})} & \text{if } l=1, \end{cases} \tag{20}$$

which give us the inductive estimate. The proof of estimate (20) will be presented in Lemma 4.5. This inductive estimates give the proof of the main theorem as presented in Section 4.1.

3.1. Preliminary lemmas

Before we provide the proofs for Lemmas 4.1, 4.3, 4.5 and the main Theorem 3.1, we first present a few preliminary results in this section. We begin with the trace embedding theorem

$$\|f\|_{L^2(\partial T)} \le Ch^{-\frac{1}{2}}\|f\|_{L^2(T)}+Ch^{\frac{1}{2}}\|\nabla f\|_{L^2(T)}, \tag{21}$$

for $T\in T_h$ and $f\in H^1(T)$, where the constant $C>0$ is independent of the discretization parameter h.

In addition, we employ the Scott-Zhang type interpolation operator [34]

$$S_h : W^{1,p}(\Omega)\to V_{h,l}(\mathcal{T}_h) \tag{22}$$

with $p\in[1,\infty]$, which is known to satisfy the following estimates.

Lemma 3.2. *Consider two compact sets $T_1\subset\subset T_2\subset\subset\overline{\Omega}$.*

1. *We have*
$$\|z-S_hz\|_{W^{m,p}(T_1)} \le Ch^{n-m}|z|_{W^{n,p}(T_2)} \quad \forall\, z\in W^{n,p}(\Omega), \tag{23}$$
where $0\le m\le n$, and $1\le n\le l+1$.
2. *We have*
$$\|S_hz\|_{W^{n,p}(T_1)} \le C\|z\|_{W^{n,p}(T_2)}, \tag{24}$$
where $0\le n\le l+1$.

In the following lemma, we recall the local error estimate from [9].

Lemma 3.3. *Consider three compact sets $T_0\subsetneq T_1\subsetneq T_2\subsetneq\overline{\Omega}$ and assume that $v\in H^1(T_2)$ and $v_h\in V_{h,l}(T_h)$ satisfy*

$$a(v-v_h,\phi)=0, \quad \forall\phi\in V_{h,l}(\mathcal{T}_h) \text{ such that supp } \phi\subset T_2. \tag{25}$$

Then, there exists a constant $C>0$ which is independent of $h>0$ such that

$$|||v-v_h|||_{H_h^1(T_0)} \le C\inf_{\chi\in V_{h,l}(T_h)}|||v-\chi|||_{H_h^1(T_1)}+C\|v-v_h\|_{L^2(T_1)}, \tag{26}$$

where $|||\cdot|||$ is the norm defined in (10).

Proof 3.4. This lemma is a slight modification of the Lemma 4.1 in [9], which is stated for globally defined solution of Ω instead of the local one stated as in (25). Since the extension of the proof in [9] to the above local problem (25) is straightforward as the proof makes use of only test functions defined locally near the support of the inequality (26), we omit the detailed proof and refer to [9].

Let $r:=|x-x_0|$ be the distance from the singularity point x_0 to a point $x\in\Omega$. Then the following lemma gives an interpolation estimate in the weighted norms.

Lemma 3.5. *Choose any $\alpha\in\mathbb{R}$. Then, for $z\in W^{l+1,2}(\Omega\setminus B(x_0,4h))$ and $a\in\{1,\cdots,l\}$, we have*

$$\|r^{\alpha}\nabla(z-S_hz)\|_{L^2(\Omega\backslash B(x_0,8h))}+\left(h\sum_{e\in\mathcal{E}}\int_{e\cap\Omega\backslash B(x_0,8h)} r^{2\alpha}|\{\nabla(z-S_hz)\}|^2ds\right)^{1/2}$$
$$\le Ch^a\sum_{|\xi|=a+1}\|r^{\alpha}\partial_x^{\xi}z\|_{L^2(\Omega\backslash B(x_0,4h))}. \tag{27}$$

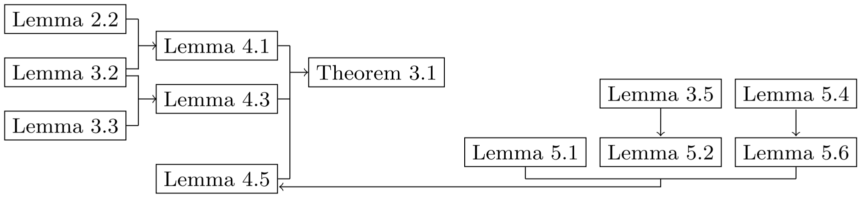


Fig. 4. First, the estimate (18) of Lemma 4.1 is proved by employing a decomposition argument and the interpolation estimate of Lemma 3.2. The local error estimate of Lemma 4.3 is a direct consequence of Lemma 3.3 and Lemma 3.2. To obtain the L^1 estimate in Lemma 4.5 for the right hand side of (19), we use the Holder's inequality with the weighted interpolation estimate of Lemma 3.5.

The proof of this lemma is obtained by using the rescaling argument [21, Lemma 3.4] and it is given in the Appendix B. Before we end this section, we provide a flowchart for the Theorems and Lemmas in this paper to help the readers in Fig. 4.

4. The sharp error estimate

In this section, we establish the proofs of three Lemmas 4.1, 4.3, and 4.5, and finally conclude the main Theorem 3.1. As in (16) and (17), we choose a series of balls $\mathbf{B}_{-1} \subset \mathbf{B}_0 \subset \cdots \subset \mathbf{B}_l$, where $\mathbf{B}_l$ satisfies $D \subset \Omega \setminus \mathbf{B}_l$, and $\tilde{\mathbf{B}}_j$ such that $\mathbf{B}_{j-1} \subsetneq \tilde{\mathbf{B}}_j \subsetneq \mathbf{B}_j$ for $j \in \{0, 1, \cdots, l\}$. Since

$$\int_D |u - u_h|^2 dx \le \int_{\Omega \setminus \mathbf{B}_l} |u - u_h|^2 dx, \tag{28}$$

it suffices to estimate the right hand side for our aim. Thus, we focus to find an inductive estimate on $\|u - u_h\|_{L^2(\Omega \setminus \mathbf{B}_j)}$ for $j \in \{0, 1, \cdots, l\}$.

Lemma 4.1. *For each $j \in \{0, 1, 2, \cdots, l\}$, we have the estimate*

$$\begin{aligned}\|u - u_h\|_{L^2(\Omega\setminus\mathbf{B}_j)} &\le Ch^l \left(\|\nabla(u - u_h)\|_{L^1(\tilde{\mathbf{B}}_j)} + \sum_{e\in\mathcal{E}} \int_{e\cap\tilde{\mathbf{B}}_j} \Big|[u - u_h]\Big| dx \right) \\ &+ Ch \left(\|\nabla(u - u_h)\|^2_{L^2(\Omega\setminus\tilde{\mathbf{B}}_j)} + \frac{1}{h} \sum_{e\in\mathcal{E}} \int_{e\cap(\Omega\setminus\tilde{\mathbf{B}}_j)} |[u - u_h]|^2 ds \right)^{\frac{1}{2}}.\end{aligned} \tag{29}$$

Proof 4.2. We begin with choosing $w_h \in H^1_0(\Omega)$ such that

$$\begin{cases} -\nabla \cdot (\beta \nabla w_h) &= (u - u_h) 1_{\Omega\setminus\mathbf{B}_j} \quad \text{in } \Omega, \\ \quad w_h &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{30}$$

Since Ω is convex and $(u - u_h)1_{\Omega\setminus\mathbf{B}_j} \in L^2(\Omega)$, we have $w_h \in H^2(\Omega)$. Multiplying $(u - u_h)$ on both sides of (30) and integrating over Ω, we obtain

$$\int_{\Omega\setminus\mathbf{B}_j} |u - u_h|^2 dx = \sum_{T\in\mathcal{T}_h} \int_T (u - u_h)(-\nabla \cdot (\beta \nabla w_h)) dx. \tag{31}$$

Using integration by parts, we obtain

$$\begin{aligned}&\int_{\Omega\setminus\mathbf{B}_j} |u - u_h|^2 dx \\ &= \sum_{T\in\mathcal{T}_h} \int_T \beta \nabla(u - u_h) \nabla w_h dx - \sum_{T\in\mathcal{T}_h} \int_{\partial T} \beta (u - u_h) \frac{\partial w_h}{\partial n} ds \\ &= \sum_{T\in\mathcal{T}_h} \int_T \beta \nabla(u - u_h) \nabla w_h dx - \sum_{e\in\mathcal{E}} \int_e \beta [u - u_h] \left\{ \frac{\partial w_h}{\partial n} \right\} ds,\end{aligned} \tag{32}$$

where we used the fact that $w_h \in H^2(\Omega)$ in the second equality. Now we recall that $a(u-u_h,\phi)=0$ for any $\phi \in V_{h,l}(\mathcal{T}_h)$, which implies

$$\begin{aligned}0 &= \sum_{T\in\mathcal{T}_h}\int_T \beta\nabla(u-u_h)\nabla\phi\, dx \\ &- \sum_{e\in\mathcal{E}}\int_e \beta[u-u_h]\left\{\frac{\partial\phi}{\partial n}\right\}ds - \sum_{e\in\mathcal{E}}\int_e \beta\left\{\frac{\partial}{\partial n}(u-u_h)\right\}[\phi]ds + \sum_{e\in\mathcal{E}}\frac{\gamma}{h}\int_e \beta[u-u_h][\phi]ds.\end{aligned} \tag{33}$$

Note that the last two terms are zeros in (33) provided $\phi \in H^2 \cap V_{h,l}$. Therefore, by choosing $\phi = P_h w_h \in H^2(\Omega)$, we get

$$0 = \sum_{T\in\mathcal{T}_h}\int_T \beta\nabla(u-u_h)\nabla P_h w_h dx - \sum_{e\in\mathcal{E}}\int_e \beta[u-u_h]\left\{\frac{\partial}{\partial n}(P_h w_h)\right\}ds. \tag{34}$$

Combining the above result (34) with (32), we obtain

$$\begin{aligned}\int_{\Omega\setminus\mathbf{B}_j}|u-u_h|^2 dx &= \sum_{T\in\mathcal{T}_h}\int_T \beta\nabla(u-u_h)\nabla(w_h-P_h w_h)dx - \sum_{e\in\mathcal{E}}\int_e \beta[u-u_h]\left\{\frac{\partial}{\partial n}(w_h-P_h w_h)\right\}ds \\ &= I_1 - I_2.\end{aligned} \tag{35}$$

From now, we estimate I_1 and I_2, separately. First, we split I_1 as

$$I_1 = \int_{\tilde{\mathbf{B}}_j}\beta\nabla(u-u_h)\nabla(w_h-P_h w_h)dx + \int_{\Omega\setminus\tilde{\mathbf{B}}_j}\beta\nabla(u-u_h)\nabla(w_h-P_h w_h)dx, \tag{36}$$

and apply Holder's inequality to get

$$\begin{aligned}|I_1| &\le \|\beta\|_{L^\infty}\|\nabla(u-u_h)\|_{L^1(\tilde{\mathbf{B}}_j)}\|\nabla(w_h-P_h w_h)\|_{L^\infty(\tilde{\mathbf{B}}_j)} \\ &\quad + \|\beta\|_{L^\infty}\|\nabla(u-u_h)\|_{L^2(\Omega\setminus\tilde{\mathbf{B}}_j)}\|\nabla(w_h-P_h w_h)\|_{L^2(\Omega\setminus\tilde{\mathbf{B}}_j)}.\end{aligned} \tag{37}$$

Using (23) in Lemma 3.2 and Lemma 2.2 with (30) on $\tilde{\mathbf{B}}_j \subset \mathbf{B}_j$, we obtain

$$\begin{aligned}\|\nabla(w_h-P_h w_h)\|_{L^\infty(\tilde{\mathbf{B}}_j)} &\le h^l\|w_h\|_{W^{l+1,\infty}(\mathbf{B}_j^*)} \\ &\le Ch^l\|w_h\|_{H^2(\mathbf{B}_j)} \\ &\le Ch^l\|u-u_h\|_{L^2(\Omega\setminus\mathbf{B}_j)},\end{aligned} \tag{38}$$

where we also used the $L^2 \to H^2$ estimate of (30). Here $\mathbf{B}_j^*$ is suitably chosen so that $\tilde{\mathbf{B}}_j \subset \mathbf{B}_j^* \subset \mathbf{B}_j$ with $\mathrm{dist}(\partial\mathbf{B}_j^*, \partial\tilde{\mathbf{B}}_j) > \delta/3$ and $\mathrm{dist}(\partial\mathbf{B}_j^*, \partial\mathbf{B}_j) > \delta/3$. On the other hand, by Lemma 3.2, we get

$$\begin{aligned}h^{-1}\|\nabla(w_h-P_h w_h)\|_{L^2(\Omega\setminus\tilde{\mathbf{B}}_j)} + \|\nabla^2(w_h-P_h w_h)\|_{L^2(\Omega\setminus\tilde{\mathbf{B}}_j)} &\le C\|w_h\|_{H^2(\Omega)} \\ &\le C\|u-u_h\|_{L^2(\Omega\setminus\mathbf{B}_j)}.\end{aligned} \tag{39}$$

By inserting the above two estimates (38)-(39) into (37), we obtain

$$|I_1| \le C\Big(h^l\|\nabla(u-u_h)\|_{L^1(\tilde{\mathbf{B}}_j)} + Ch\|\nabla(u-u_h)\|_{L^2(\Omega\setminus\tilde{\mathbf{B}}_j)}\Big)\|u-u_h\|_{L^2(\Omega\setminus\mathbf{B}_j)}. \tag{40}$$

Next, to estimate I_2 in (35), we divide it into two terms as

$$I_2 = \sum_{e\in\mathcal{E}}\int_{e\cap\tilde{\mathbf{B}}_j}\beta[u-u_h]\left\{\frac{\partial}{\partial n}(w_h-P_h w_h)\right\}ds + \sum_{e\in\mathcal{E}}\int_{e\cap(\Omega\setminus\tilde{\mathbf{B}}_j)}\beta[u-u_h]\left\{\frac{\partial}{\partial n}(w_h-P_h w_h)\right\}ds. \tag{41}$$

The first term in (41) is bounded by

$$\left|\sum_{e\in\mathcal{E}}\int_{e\cap\tilde{\mathbf{B}}_j}\beta[u-u_h]\left\{\frac{\partial}{\partial n}(w_h-P_h w_h)\right\}ds\right|$$

$$\leq \|\beta\|_{L^\infty} \left(\sum_{e\in\mathcal{E}} \int_{e\cap\tilde{\mathbf{B}}_j} \left|[u-u_h]\right| ds \right) \|\nabla(w_h - P_h w_h)\|_{L^\infty(\tilde{\mathbf{B}}_j)} \tag{42}$$

$$\leq Ch^l \left(\sum_{e\in\mathcal{E}} \int_{e\cap\tilde{\mathbf{B}}_j} \left|[u-u_h]\right| ds \right) \|u-u_h\|_{L^2(\Omega\setminus\mathbf{B}_j)}$$

using (38). Next, we apply Holder's inequality to estimate the second term in (41) by

$$\left| \sum_{e\in\mathcal{E}} \int_{e\cap(\Omega\setminus\tilde{\mathbf{B}}_j)} \beta[u-u_h] \left\{ \frac{\partial}{\partial n}(w_h - P_h w_h) \right\} ds \right|$$

$$\leq \|\beta\|_{L^\infty} \left(\sum_{e\in\mathcal{E}} \int_{e\cap\Omega\setminus\tilde{\mathbf{B}}_j} \left|[u-u_h]\right|^2 ds \right)^{\frac{1}{2}} \left(\sum_{e\in\mathcal{E}} \int_{e\cap\Omega\setminus\tilde{\mathbf{B}}_j} \left|\left\{ \frac{\partial}{\partial n}(w_h - P_h w) \right\}\right|^2 ds \right)^{\frac{1}{2}} \tag{43}$$

$$\leq \|\beta\|_{L^\infty} \left(\sum_{e\in\mathcal{E}} \int_{e\cap\Omega\setminus\tilde{\mathbf{B}}_j} \left|[u-u_h]\right|^2 ds \right)^{\frac{1}{2}} \left(h^{-\frac{1}{2}} \|\nabla(w_h - P_h w_h)\|_{L^2(\Omega\setminus\tilde{\mathbf{B}}_j)} + h^{\frac{1}{2}} \|\nabla^2(w_h - P_h w_h)\|_{L^2(\Omega\setminus\tilde{\mathbf{B}}_j)} \right).$$

Here we used the trace inequality (21) for the second inequality. The above estimate is again bounded by (39) and we obtain

$$\left| \sum_{e\in\mathcal{E}} \int_{e\cap(\Omega\setminus\tilde{\mathbf{B}}_j)} \beta[u-u_h] \left\{ \frac{\partial}{\partial n}(w_h - P_h w_h) \right\} ds \right|$$
$$\leq C\|\beta\|_{L^\infty} h \left(\frac{1}{h} \sum_{e\in\mathcal{E}} \int_{e\cap\Omega\setminus\tilde{\mathbf{B}}_j} \left|[u-u_h]\right|^2 ds \right)^{\frac{1}{2}} \|u-u_h\|_{L^2(\Omega\setminus\mathbf{B}_j)}. \tag{44}$$

Therefore, the estimate of I_2 is given by (42) and (44). Finally, the desired estimate (29) is obtained by combining both estimates for I_1 and I_2.

In order to get the desired inductive inequality (20), the next lemma provides the estimate of the last terms in (29) Lemma 4.1 by employing the local estimates of Lemma 3.3.

Lemma 4.3. *For each $j \in \{0, 1, \cdots, l\}$, we have*

$$\|u-u_h\|_{H^1(\Omega\setminus\tilde{\mathbf{B}}_j)} + \left(\frac{1}{h} \sum_{e\in\mathcal{E}} \int_{e\cap\Omega\setminus\tilde{\mathbf{B}}_j} \left|[u-u_h]\right|^2 ds \right)^{\frac{1}{2}}$$
$$\leq Ch^l + C\|u-u_h\|_{L^2(\Omega\setminus\mathbf{B}_{j-1})}. \tag{45}$$

Proof 4.4. Since $u \in H^2(\Omega \setminus \mathbf{B}_{j-1})$, it satisfies

$$a(u, v) = v(x_0) \quad \text{for all } v \in V_{h,l}(\mathcal{T}_h) \quad \text{such that supp } v \subset \Omega \setminus \mathbf{B}_{j-1}, \tag{46}$$

which yields $a(u-u_h, v) = 0$. Now we apply Lemma 3.3 to find

$$
\begin{aligned}
&\|u-u_h\|_{H^1(\Omega\backslash\tilde{\mathbf{B}}_j)} + \left(\frac{1}{h}\sum_{e\in\mathcal{E}}\int_{e\cap\Omega\backslash\tilde{\mathbf{B}}_j} |[u-u_h]|^2 ds\right)^{\frac{1}{2}} \\
&\quad \le C \inf_{\chi\in S^h(\Omega)} |||u-\chi|||_{H_h^1(\Omega\backslash\mathbf{B}_{j-1})} + C\|u-u_h\|_{L^2(\Omega\backslash\mathbf{B}_{j-1})} \qquad (47)\\
&\quad \le Ch^l + C\|u-u_h\|_{L^2(\Omega\backslash\mathbf{B}_{j-1})},
\end{aligned}
$$

where we used the trace inequality (21) and (23) for the second inequality. This concludes the proof.

Next, to bound the first term of the right hand side in (29), we shall prove the following estimate.

Lemma 4.5. *We have*

$$
\left(\int_{\tilde{\mathbf{B}}_j} |\nabla(u-u_h)|dx + \sum_{e\in\mathcal{E}}\int_{e\cap\tilde{\mathbf{B}}_j} \Big|[u-u_h]\Big|ds\right) \le \begin{cases} Ch & \text{if } l\ge 2,\\ Ch|\log h| & \text{if } l=1.\end{cases} \qquad (48)
$$

The proof of this Lemma is postponed to the next Section 5 due to its complexity.

4.1. The proof for the main Theorem 3.1

With the result of Lemmas 4.1, 4.3, and 4.5, finally we conclude the proof of the main result.

Proof 4.6 *(Proof of Theorem 3.1).* By inserting the estimates (45) and (48) into (29), for each $j\in\{0,1,\cdots,l\}$, we obtain the following inductive estimate (stated in (20))

$$
\|u-u_h\|_{L^2(\Omega\backslash\mathbf{B}_j)} \le \begin{cases} Ch^{l+1} + Ch\|u-u_h\|_{L^2(\Omega\backslash\mathbf{B}_{j-1})} & \text{if } l\ge 2,\\ Ch^2|\log h| + Ch\|u-u_h\|_{L^2(\Omega\backslash\mathbf{B}_{j-1})} & \text{if } l=1.\end{cases} \qquad (49)
$$

Using this estimate for $j=0,1,\cdots,l$ recursively and Theorem 2.3, we get the desired estimate

$$
\|u-u_h\|_{L^2(\Omega\backslash\mathbf{B}_l)} \le \begin{cases} Ch^{l+1} & \text{if } l\ge 2\\ Ch^2|\log h| & \text{if } l=1.\end{cases} \qquad (50)
$$

The proof is finished.

5. L^1 estimate near the singularity

The aim of this section is to prove Lemma 4.5. For this, we first find weighted L^2 estimates of $|u-u_h|$ near the singularity and then use it to bound the L^1 norms in (48) with Holder's inequality. For the weighted estimate, we need a weighted version of the local error estimate of Lemma 3.3, which is achieved in the following lemma. In order to simplify the notation, we shall use B_a and r to denote $B(x_0,a)$ and $|x-x_0|$, respectively, for each $a>0$.

Lemma 5.1. *For any $\alpha\in\mathbb{R}$, there exists a constant $C>0$ depending on α such that*

$$
\begin{aligned}
&\|r^\alpha\nabla(u-u_h)\|_{L^2(\Omega\backslash B_{8h})} + \left(\frac{1}{h}\sum_{e\in\mathcal{E}}\int_{e\cap(\Omega\backslash B_{8h})} r^{2\alpha}[u-u_h]^2 ds\right)^{\frac{1}{2}} \\
&\qquad + \left(h\sum_{e\in\mathcal{E}}\int_{e\cap(\Omega\backslash B_{8h})} r^{2\alpha}|\{\nabla(u-u_h)\}|^2 ds\right)^{1/2} \le C\|r^\alpha\nabla(u-S_hu)\|_{L^2(\Omega\backslash B_{4h})} \\
&\qquad + \left(h\sum_{e\in\mathcal{E}}\int_{e\cap(\Omega\backslash B_{4h})} r^{2\alpha}|\{\nabla(u-S_hu)\}|^2 ds\right)^{1/2} + C\|r^{\alpha-1}(u-u_h)\|_{L^2(\Omega\backslash B_{4h})}, \qquad (51)
\end{aligned}
$$

where $r:=|x-x_0|$.

The proof of the above result is given in the appendix.

In proving Lemma 4.5, we shall need to estimate the right hand side of (51). To estimate the first two terms, we prove the weighted interpolation estimate of u in the following lemma.

Lemma 5.2. *Choose $\alpha \in \mathbb{R}$ such that $\alpha + 1 - d/2 \le l$. Then there exists a constant $C > 1$ depending on α such that*

$$\|r^\alpha \nabla(u - S_h u)\|_{L^2(\Omega\setminus B_{4h})} + \left(h \sum_{e \in \mathcal{E}} \int_{e\cap(\Omega\setminus B_{4h})} r^{2\alpha} |\{\nabla(u - S_h u)\}|^2 ds \right)^{1/2} \le \begin{cases} Ch^{\alpha+1-d/2} & \text{if } \alpha + 1 - d/2 < l, \\ C|\log h| & \text{if } \alpha + 1 - d/2 = l. \end{cases} \tag{52}$$

Proof 5.3. By Lemma 3.5, we have

$$\|r^\alpha \nabla(u - S_h u)\|_{L^2(\Omega\setminus B_{4h})} + \left(h \sum_{e \in \mathcal{E}} \int_{e\cap(\Omega\setminus B_{4h})} r^{2\alpha} |\{\nabla(u - S_h u)\}|^2 ds \right)^{1/2} \le Ch^l \sum_{|\xi|=l+1} \|r^\alpha \partial_x^\xi u\|_{L^2(\Omega\setminus B_{2h})}. \tag{53}$$

We recall from lemma (2.1) that $|\partial_x^\xi u(x)| \le C|x - x_0|^{1-d-l}$ if $|\xi| = l + 1$. Therefore,

$$\|r^\alpha \partial_x^\xi u\|_{L^2(\Omega\setminus B_{2h})} \le \begin{cases} Ch^{\alpha+1-d/2} & \text{if } \alpha + 1 - d/2 < l, \\ C|\log h| & \text{if } \alpha + 1 - d/2 = l. \end{cases} \tag{54}$$

Finally, we insert the above inequality into (53) and get the desired result.

Next, we estimate the last term $\|r^{\alpha-1}(u - u_h)\|_{L^2(\Omega\setminus B_{4h})}$ in (51). Here, we need to consider both cases such that $(\alpha - 1)$ is negative and positive. The estimate for the negative case will be first shown in the next lemma, and based on it, we will treat the positive case.

Lemma 5.4. *Let $\alpha \in (0, 1)$ for $d = 2$ and $\alpha \in (0, 1/2)$ for $d = 3$. Then we have the following estimate*

$$\int_\Omega r^{-2\alpha} |u - u_h|^2 dx \le Ch^{-2\alpha+4-d}. \tag{55}$$

Proof 5.5. We recall from [21, (3.7)] that there exists a constant $C = C_\alpha > 0$ satisfying

$$\|r^{-\alpha} f\|_{L^2(B_s)} \le C\|r^{-\alpha+1}\nabla f\|_{L^2(B_{2s})} + Cs^{-\alpha}\|f\|_{L^2(B_{2s})}, \tag{56}$$

for any $s > 0$ and $f \in L^2(B_{2s})$ such that $r^{-\alpha+1}\nabla f \in L^2(B_{2s})$. We use the triangle inequality and (56) in order to find

$$\begin{aligned} \|r^{-\alpha}(u - u_h)\|_{L^2(\Omega)} &\le \|r^{-\alpha}(u - u_h)\|_{L^2(B_{4h})} + \|r^{-\alpha}(u - u_h)\|_{L^2(\Omega\setminus B_{4h})} \\ &\le C\left(\|r^{-\alpha+1}\nabla(u - u_h)\|_{L^2(B_{8h})} + h^{-\alpha}\|u - u_h\|_{L^2(B_{8h})}\right) + Ch^{-\alpha}\|u - u_h\|_{L^2(\Omega\setminus B_{4h})}. \end{aligned} \tag{57}$$

Next, by applying the L^2 estimate (14) in the right hand side of (57), we obtain

$$\|r^{-\alpha}(u - u_h)\|_{L^2(\Omega)} \le C\|r^{-\alpha+1}\nabla(u - u_h)\|_{L^2(B_{8h})} + Ch^{-\alpha+2-d/2}. \tag{58}$$

To estimate the first term on the right hand side in (58), we employ the triangle inequality to get

$$\|r^{-\alpha+1}\nabla(u - u_h)\|_{L^2(B_{8h})} \le \|r^{-\alpha+1}\nabla u\|_{L^2(B_{8h})} + \|r^{-\alpha+1}\nabla(S_h u)\|_{L^2(B_{8h})} + \|r^{-\alpha+1}\nabla(S_h u - u_h)\|_{L^2(B_{8h})}. \tag{59}$$

We note that the first term in the right hand side of (59) is bounded by

$$\|r^{-\alpha+1}\nabla u\|_{L^2(B_{8h})} \le Ch^{-\alpha+2-d/2} \tag{60}$$

by using Lemma 2.1. For the second right hand side term, we use the inverse inequality (24), and Lemma 2.1 to get

$$\begin{aligned}\|r^{-\alpha+1}\nabla(S_h u)\|_{L^2(B_{8h})} &\le Ch^{1-\alpha}h^{-d/2}\|\nabla(S_h u)\|_{L^1(B_{8h})}\\ &\le Ch^{-\alpha+1-d/2}\|\nabla u\|_{L^1(B_{10h})} \le Ch^{2-d/2-\alpha}.\end{aligned} \tag{61}$$

For the last term of (59), we use the inverse inequality and (14) to deduce

$$\begin{aligned}\|r^{-\alpha+1}\nabla(S_h u-u_h)\|_{L^2(B_{8h})} &= h^{-\alpha+1}\|\nabla(S_h u - S_h u_h)\|_{L^2(B_{8h})}\\ &\le Ch^{-\alpha+1}h^{-1}\|S_h u - S_h u_h\|_{L^2(B_{8h})}\\ &\le Ch^{-\alpha+2-d/2}.\end{aligned} \tag{62}$$

By summing up the above three estimates (60)-(62), we find that (59) is bounded by $Ch^{-\alpha+1}$. Thus, we get

$$\|r^{-\alpha}(u-u_h)\|_{L^2(\Omega)} \le Ch^{-\alpha+2-d/2} \tag{63}$$

from (58) and the proof is done.

Next we obtain the error estimate for the positive order case. The main idea is to split Ω in a dyadic way according to distance from x_0, and estimate the L^2 norm of $u-u_h$ on each annulus using a duality argument.

Lemma 5.6. *Let $d \in \{2,3\}$ and choose any $\alpha \in [0,1)$. Then we have*

$$\int_\Omega r^{2\alpha}|u-u_h|^2 dx \le Ch^{2\alpha+4-d}. \tag{64}$$

Proof 5.7. We choose the smallest $M \in \mathbb{N}$ such that

$$\Omega \subset B(x_0,h) \cup \bigcup_{k=1}^{M} B(x_0,2^k h) \setminus B(x_0,2^{k-1}h), \tag{65}$$

and set $S_k = B(x_0,2^k h)\setminus B(x_0,2^{k-1}h)$ for each $k \in \mathbb{N}$. Then, we get

$$\begin{aligned}\int_\Omega r^{2\alpha}|u-u_h|^2dx &\le \int_{B(x_0,h)} r^{2\alpha}|u-u_h|^2dx + \sum_{k=1}^{M}\int_{S_k} r^{2\alpha}|u-u_h|^2dx\\ &\le Ch^{2\alpha+4-d} + C\sum_{k=1}^{M}(2^k h)^{2\alpha}\int_{S_k}|u-u_h|^2dx\end{aligned} \tag{66}$$

and we only need to estimate $\|u-u_h\|_{L^2(S_k)}$. For this, we consider $w \in H^2(\Omega)$ such that

$$\begin{cases} -\nabla\cdot(\beta\nabla w) &= (u-u_h)1_{S_k} \quad \text{in } \Omega,\\ \qquad\quad w &= 0 \qquad\qquad\quad\ \ \text{on } \partial\Omega.\end{cases} \tag{67}$$

Since Ω is convex, we have $w \in H^2(\Omega)$. Next, we multiply both sides of (67) by $(u-u_h)$ and integrate by parts to obtain

$$\begin{aligned}\int_{S_k}|u-u_h|^2dx &= \int_\Omega (u-u_h)(-\nabla\cdot(\beta\nabla w))dx\\ &= \sum_{T\in\mathcal{T}_h}\left(\int_T \beta\nabla(u-u_h)\nabla w\,dx - \int_{\partial T}\beta(u-u_h)\frac{\partial w}{\partial n}ds\right)\\ &= \sum_{T\in\mathcal{T}_h}\left(\int_T \beta\nabla(u-u_h)\nabla w\,dx - \int_{\partial T}\beta[u-u_h]\left\{\frac{\partial w}{\partial n}\right\}ds\right),\end{aligned} \tag{68}$$

where we used the fact that $w \in H^2(\Omega)$ in the last equality. In addition, since it holds that $a(u-u_h, S_h w)=0$ for $S_h w \in V_{h,I}(\mathcal{T}_h)\cap H^2(\Omega)$, we have

$$\sum_{T\in\mathcal{T}_h}\int_T \beta\nabla(u-u_h)\cdot\nabla(S_h w)dx - \sum_{e\in\mathcal{E}}\int_e \beta[u-u_h]\left\{\frac{\partial(S_h w)}{\partial n}\right\}ds = 0. \tag{69}$$

Using (69) and (68), we find

$$\begin{aligned}\int_{S_k} |u-u_h|^2 dx = &\underbrace{\sum_{T\in\mathcal{T}_h}\int_{T\cap(B_{4h})^c} \beta\nabla(u-u_h)\nabla(w-S_h w)dx}_{I_1} - \underbrace{\sum_{e\in\mathcal{E}}\int_{e\cap(B_{4h})^c} \beta[u-u_h]\left\{\frac{\partial}{\partial n}(w-S_h w)\right\}ds}_{I_2} \\ &+ \underbrace{\sum_{T\in\mathcal{T}_h}\int_{T\cap(B_{4h})} \beta\nabla(u-u_h)\nabla(w-S_h w)dx - \sum_{e\in\mathcal{E}}\int_{e\cap(B_{4h})} \beta[u-u_h]\left\{\frac{\partial}{\partial n}(w-S_h w)\right\}ds.}_{I_3}\end{aligned} \tag{70}$$

Now, we shall provide the estimates for I_1, I_2, and I_3 by a constant multiple of $h^{2-\frac{d}{2}}2^{-kq}\|u-u_h\|_{L^2(S_k)}$, where we fix any $q\in(\alpha,1)$ for $d=2$ and $q\in(\alpha,\min(\alpha+1/2,1))$ for $d=3$.

Estimate of I_1. We apply Holder's inequality for I_1 to get

$$\begin{aligned} I_1 &= \int_{\Omega\setminus B_{4h}} \beta r^q\nabla(u-u_h)r^{-q}\nabla(w-S_h w)dx \\ &\le C\|r^q\nabla(u-u_h)\|_{L^2(\Omega\setminus B_{4h})}\cdot\|r^{-q}\nabla(w-S_h w)\|_{L^2(\Omega\setminus B_{4h})}.\end{aligned} \tag{71}$$

To estimate the right hand side, we apply Lemmas 5.1, 5.2, and 5.4 to derive

$$\begin{aligned} &\|r^q\nabla(u-u_h)\|_{L^2(\Omega\setminus B_{4h})} \\ &\quad\le \|r^q\nabla(u-S_h u)\|_{L^2(\Omega\setminus B_{3h})} + \left(h\sum_{e\in\mathcal{E}}\int_{e\cap\Omega\setminus B_{3h}} r^{2q}|\{u-S_h u\}|^2 ds\right)^{\frac{1}{2}} + \|r^{q-1}(u-u_h)\|_{L^2(\Omega\setminus B_{3h})} \\ &\quad\le Ch^{q+1-d/2},\end{aligned} \tag{72}$$

where it was noted that $q-1\in(-1,0)$ for $d=2$ and $q-1\in(-1/2,0)$ for $d=3$ in using Lemma 5.4. In addition, for the last term in (71), we first obtain

$$\begin{aligned} &[ll]\|r^{-q}\nabla(w-S_h w)\|_{L^2(\Omega\setminus B_{4h})} \\ &\quad\le \|r^{-q}\nabla(w-S_h w)\|_{L^2(|x-x_0|\le 2^{k-2}h))} + \|r^{-q}\nabla(w-S_h w)\|_{L^2(|x-x_0|>2^{k-2}h)}.\end{aligned} \tag{73}$$

Using Lemma 2.2 with equation (67) of w in $B(x_0,2^{k-1}h)$, we obtain

$$\begin{aligned}\|\nabla(w-S_h w)\|_{L^\infty(|x-x_0|\le 2^{k-2}h)} &\le h\|\nabla^2 w\|_{L^\infty(B_{2^{k-3/2}h})} \\ &\le h(2^k h)^{-\frac{d}{2}}\|\nabla^2 w\|_{L^2(B_{2^{k-1}h})} \\ &\le h(2^k h)^{-\frac{d}{2}}\|u-u_h\|_{L^2(S_k)}.\end{aligned} \tag{74}$$

By the above estimate (74), we get

$$\begin{aligned} &\|r^{-q}\nabla(w-S_h w)\|_{L^2(|x-x_0|\le 2^{k-2}h))} \\ &\quad\le \|r^{-q}\|_{L^2(|x-x_0|\le 2^{k-2}h)}\|\nabla(w-S_h w)\|_{L^\infty(|x-x_0|\le 2^{k-2}h)} \\ &\quad\le Ch(2^k h)^{-q}\|u-u_h\|_{L^2(S_k)}.\end{aligned} \tag{75}$$

On the other hand, by using Lemma 3.2 we deduce

$$\begin{aligned}\|r^{-q}\nabla(w-S_h w)\|_{L^2(|x-x_0|>2^{k-2}h)} &\le C(2^k h)^{-q}\|\nabla(w-S_h w)\|_{L^2(|x-x_0|>2^{k-2}h)} \\ &\le Ch(2^k h)^{-q}\|\nabla^2 w\|_{L^2(|x-x_0|>2^{k-3}h)} \\ &\le Ch(2^k h)^{-q}\|u-u_h\|_{L^2(S_k)}.\end{aligned} \tag{76}$$

Combining the above estimates with (73), we arrive at the following estimate

$$\|r^{-q}\nabla(w-S_h w)\|_{L^2(\Omega)} \le Ch(2^k h)^{-q}\|u-u_h\|_{L^2(S_k)}. \tag{77}$$

We merge this estimate with (72) and (71) to get

$$I_1 \le h^{2-\frac{d}{2}}(2^k)^{-q}\|u-u_h\|_{L^2(S_k)}. \tag{78}$$

Estimate of I_2. By Holder's inequality, we first get

$$\begin{aligned}&\sum_{e\in\mathcal{E}}\int_{e\cap(B_{4h})^c}\beta[u-u_h]\left\{\frac{\partial}{\partial n}(w-S_hw)\right\}ds\\&\le\|\beta\|_{L^\infty}\left(\sum_{e\in\mathcal{E}}\int_{e\cap(B_{4h})^c}r^{2q}[u-u_h]^2ds\right)^{\frac12}\left(\sum_{e\in\mathcal{E}}\int_{e\cap(B_{4h})^c}r^{-2q}\left|\left\{\frac{\partial}{\partial n}(w-S_hw)\right\}\right|^2ds\right)^{\frac12}.\end{aligned}\tag{79}$$

To estimate the first term on the right hand side in (79), we apply Lemma 4.3, Lemma 5.1 and Lemma 5.2 to obtain

$$\begin{aligned}&\left(\sum_{e\in\mathcal{E}}\int_{e\cap(B_{4h})^c}r^{2q}[u-u_h]^2ds\right)^{\frac12}\\&\le Ch^{\frac12}h^qh^{1-\frac{d}{2}}+C\sqrt{h}\left(\sum_{T\in\mathcal{T}_h}\int_{T\cap(B_{2h})^c}r^{2q-2}|u-u_h|^2dx\right)^{\frac12}+Ch^{1/2}\|u-u_h\|_{L^2(\Omega\setminus B_1)}\\&\le Ch^{\frac12}h^qh^{1-\frac{d}{2}}+Ch^{1/2}\|u-u_h\|_{L^2(\Omega\setminus B_1)}.\end{aligned}\tag{80}$$

Here, Lemma 5.4 is used in the second inequality. Next, to estimate the last term of (79) we split it by the triangle inequality as

$$\begin{aligned}&\left(\sum_{e\in\mathcal{E}}\int_{e\cap(B_{4h})^c}r^{-2q}\left|\left\{\frac{\partial}{\partial n}(w-S_hw)\right\}\right|^2ds\right)^{\frac12}\\&\le\left(\sum_{e\in\mathcal{E}}\int_{e\cap(B_{4h})^c\cap B_{2^kh}}r^{-2q}\left|\left\{\frac{\partial}{\partial n}(w-S_hw)\right\}\right|^2ds\right)^{\frac12}+\left(\sum_{e\in\mathcal{E}}\int_{e\cap(B_{2^kh})^c}r^{-2q}\left|\left\{\frac{\partial}{\partial n}(w-S_hw)\right\}\right|^2ds\right)^{\frac12}.\end{aligned}\tag{81}$$

For the first integration conducted on $(B_{4h})^c\cap B_{2^kh}$, we apply (74) to get

$$\begin{aligned}&\left(\sum_{e\in\mathcal{E}}\int_{e\cap(B_{4h})^c\cap B_{2^kh}}r^{-2q}\left|\left\{\frac{\partial}{\partial n}(w-S_hw)\right\}\right|^2ds\right)^{\frac12}\\&\le C(2^kh)^{-\frac{d}{2}}\|u-u_h\|_{L^2(S_k)}\left(\sum_{e\in\mathcal{E}}\int_{e\cap(B_{4h})^c\cap B_{2^kh}}r^{-2q}ds\right)^{\frac12}\\&\le C(2^kh)^{-\frac{d}{2}}\|u-u_h\|_{L^2(S_k)}\left(h^{d-1-2\beta}2^{k(d-2q)}\right)^{\frac12}\\&=Ch^{\frac12-q}2^{-qk}\|u-u_h\|_{L^2(S_k)},\end{aligned}\tag{82}$$

where we used

$$\begin{aligned}&\left(\sum_{e\in\mathcal{E}}\int_{e\cap(B_{4h})^c\cap B_{2^kh}}r^{-2q}ds\right)\\&\qquad\le\sum_{j=1}^{k}\left(\sum_{e\in\mathcal{E}}\int_{e\cap(B_{2^jh}\setminus B_{2^{j-1}h})}r^{-2q}ds\right)\end{aligned}$$

$$
\begin{aligned}
&\leq C\sum_{j=1}^{k}(2^j h)^{-2q}h^{d-1}|\{e\in\mathcal{E}^1 \mid e\cap(B_{2^j h}\setminus B_{2^{j-1}h})\neq\phi\}| \\
&\leq C\sum_{j=1}^{k}(2^j h)^{-2q}h^{d-1}\cdot 2^{jd} \\
&= h^{d-1-2q}\sum_{j=1}^{k}2^{j(d-2q)}\leq Ch^{d-1-2q}2^{k(d-2\beta)}.
\end{aligned}
\tag{83}
$$

To estimate the last term of (81), we use the trace inequality (21) to deduce

$$
\begin{aligned}
&\left(\sum_{e\in\mathcal{E}_{e\cap(B_{2^k h})^c}}\int r^{-2q}\left|\left\{\frac{\partial}{\partial n}(w-S_h w)\right\}\right|^2 ds\right)^{\frac{1}{2}} \\
&\leq (2^k h)^{-q}\left(\sum_{e\in\mathcal{E}_{e\cap(B_{2^k h})^c}}\int \left|\left\{\frac{\partial}{\partial n}(w-S_h w)\right\}\right|^2 ds\right)^{\frac{1}{2}} \\
&\leq C(2^k h)^{-q}\left(\sum_{K\in\mathcal{T}_h}h^{-1}\int_K|\nabla(w-S_h w)|^2dx+h\int_K|\nabla^2(w-S_h w)|^2dx\right)^{\frac{1}{2}} \\
&\leq C(2^k h)^{-q}h^{\frac{1}{2}}\|w\|_{H^2(\Omega)} \\
&\leq C(2^k h)^{-q}h^{\frac{1}{2}}\|u-u_h\|_{L^2(S_k)}.
\end{aligned}
\tag{84}
$$

Next, we combine the above with (82), and get

$$
\left(\sum_{e\in\mathcal{E}_{e\cap(B_{4h})^c}}\int r^{-2q}\left|\left\{\frac{\partial}{\partial n}(w-S_h w)\right\}\right|^2 ds\right)^{\frac{1}{2}}\leq Ch^{\frac{1}{2}-q}2^{-qk}\|u-u_h\|_{L^2(S_k)}.
$$

By inserting this estimate and (80) into (79), we arrive at the following estimate

$$
I_2=\sum_{e\in\mathcal{E}_{e\cap(B_{4h})^c}}\int \beta[u-u_h]\left\{\frac{\partial}{\partial n}(w-S_h w)\right\}ds\leq Ch^{2-\frac{d}{2}}2^{-kq}\|u-u_h\|_{L^2(S_k)}.
\tag{85}
$$

Thus, we obtain

$$
I_1+I_2\leq Ch^{2-\frac{d}{2}}(2^k)^{-q}\|u-u_h\|_{L^2(S_k)}.
\tag{86}
$$

Estimate of I_3. We use Holder's inequality and the triangle inequality to deduce

$$
\begin{aligned}
&\left|\int_{T\cap(B_{4h})}\beta\nabla(u-u_h)\cdot\nabla(w-S_h w)dx\right| \\
&\leq C\|\nabla(w-S_h w)\|_{L^\infty(B_{4h})}\|\nabla(u-u_h)\|_{L^1(B_{4h})} \\
&\leq C\|\nabla(w-S_h w)\|_{L^\infty(B_{4h})}\left(\|\nabla(u-S_h u)\|_{L^1(B_{4h})}+\|\nabla(S_h u-u_h)\|_{L^1(B_{4h})}\right).
\end{aligned}
\tag{87}
$$

We note that by Lemma 2.1 and Lemma 3.2, we have

$$
\|\nabla(u-S_h u)\|_{L^1(B_{4h})}\leq\|\nabla u\|_{L^1(B_{4h})}+\|\nabla S_h u\|_{L^1(B_{4h})}\leq Ch.
\tag{88}
$$

On the other hand, since $S_h u-u_h\in V_{h,l}(\mathcal{T}_h)$, we get

$$\begin{aligned}\|\nabla(S_h u-u_h)\|_{L^1(B_{4h})} &\le Ch^{-1}\|S_h u-u_h\|_{L^1(B_{8h})}\\ &\le Ch^{-1}h^{d/2}\|S_h u-u_h\|_{L^2(B_{8h})}\\ &\le Ch^{-1}h^{d/2}(\|S_h u\|_{L^2(B_{8h})}+\|u\|_{L^2(B_{8h})}+\|u-u_h\|_{L^2(B_{8h})})\\ &\le Ch.\end{aligned} \tag{89}$$

By combining the above two estimates with (74), we have

$$\left|\int_{T\cap(B_{4h})}\beta\nabla(u-u_h)\cdot\nabla(w-S_h w)dx\right|\le Ch^{2-\frac{d}{2}}2^{-k\frac{d}{2}}\|u-u_h\|_{L^2(S_k)}. \tag{90}$$

Next, using (74) again and the fact that $u\in C(\Omega\setminus\{x_0\})$, we obtain

$$\begin{aligned}&\left|\sum_{e\in\mathcal{E}}\int_{e\cap(B_{4h})}\beta[u-u_h]\left\{\frac{\partial}{\partial n}(w-S_h w)\right\}ds\right|\\ &\le Ch^{1-\frac{d}{2}}2^{-\frac{k}{2}d}\|u-u_h\|_{L^2(S_k)}\sum_{e\in\mathcal{E}}\int_{e\cap(B_{4h})}|[S_h u-u_h]|ds\\ &\le Ch^{1/2}2^{-\frac{k}{2}d}\|u-u_h\|_{L^2(S_k)}\left(\sum_{e\in\mathcal{E}}\int_{e\cap(B_{4h})}|[S_h u-u_h]^2ds\right)^{1/2}\\ &\le Ch^{1/2}2^{-\frac{k}{2}d}\|u-u_h\|_{L^2(S_k)}\left(h\int_{B_{4h}}|\nabla(S_h u-u_h)|^2dx+h^{-1}\int_{B_{4h}}|S_h u-u_h|^2dx\right)^{1/2}\\ &\le Ch^{2-\frac{d}{2}}2^{-\frac{k}{2}d}\|u-u_h\|_{L^2(S_k)},\end{aligned} \tag{91}$$

where we used the trace embedding (21) in the third inequality and (89) in the fourth inequality. Finally, we add the above estimates for I_1, I_2, and I_3 to get the following estimate

$$\|u-u_h\|_{L^2(S_k)}\le Ch^{2-\frac{d}{2}}(2^k)^{-q}\quad\text{for } k\in\mathbb{N}. \tag{92}$$

Inserting this estimate into (66) and reminding that $q>\alpha$, we obtain the estimate (64) immediately. The proof is done.

5.1. Proof of Lemma 4.5

Now we are ready to prove Lemma 4.5 which provides the sharp estimates of the L^1 norms in (29).

Proof 5.8 *(Proof of Lemma 4.5).* For case $l=1$, we choose $\alpha=l+d/2-l$, and for case $l>1$, we choose $\alpha\in(1/2,1)$ satisfying $\alpha<l+d/2-2$. We apply Lemma 5.1, Lemma 5.2 and Lemma 5.6 to deduce

$$\begin{aligned}&\int_{\Omega\setminus B_{8h}}r^{2\alpha+2}|\nabla(u-u_h)|^2dx+\frac{1}{h}\sum_{e\in\mathcal{E}}\int_{e\cap(\Omega\setminus B_{8h})}r^{2\alpha+2}\Big|[u-u_h]\Big|^2ds\\ &\le\int_{\Omega\setminus B_{4h}}r^{2\alpha+2}|\nabla(u-S_h u)|^2dx\\ &+\left(h\sum_{e\in\mathcal{E}}\int_{e\cap\Omega\setminus B_{4h}}r^{2\alpha+2}|\{\nabla(u-S_h u)\}|^2ds\right)+\int_{\Omega\setminus B_{4h}}r^{2\alpha}\Big|(u-u_h)\Big|^2dx\\ &\le\begin{cases}Ch^{2\alpha+2}h^{2-d} & \text{if } l\ge 2\\ C|\log h|h^2 & \text{if } l=1.\end{cases}\end{aligned} \tag{93}$$

Using (93) with Holder's inequality, we obtain

$$\begin{aligned}
&\int_{\Omega\setminus B_{8h}} |\nabla(u-u_h)|dx \\
&\quad \le \left(\int_{\Omega\setminus B_{8h}} r^{-2\alpha-2}dx\right)^{\frac{1}{2}} \left(\int_{\Omega\setminus B_{8h}} r^{2\alpha+2}|\nabla(u-u_h)|^2 dx\right)^{\frac{1}{2}} \\
&\quad \le \begin{cases} C(h^{-2\alpha-2+d})^{1/2}\cdot h^{\alpha+2-d/2} = Ch & \text{if } l\ge 2 \\ C|\log h|h & \text{if } l=1. \end{cases}
\end{aligned} \tag{94}$$

Similarly, using (93) we find

$$\begin{aligned}
&\sum_{e\in\mathcal{E}} \int_{e\cap(\Omega\setminus B_{8h})} \Big|[u-u_h]\Big|dx \\
&\quad \le \left(\sum_{e\in\mathcal{E}\cap(\Omega\setminus B_{8h})} r^{-2\alpha-2}|e|\right)^{\frac{1}{2}} \left(\sum_{e\in\mathcal{E}} r^{2\alpha+2} \int_{e\cap(\Omega\setminus B_{8h})} \Big|[u-u_h]\Big|^2 ds\right)^{\frac{1}{2}} \\
&\quad \le \begin{cases} C\sqrt{h}\cdot h^{\alpha+2-\frac{d}{2}} \left(\sum_{e\in\mathcal{E}\cap(\Omega\setminus B_{8h})} r^{-2\alpha-2}|e|\right)^{\frac{1}{2}} & \text{if } l\ge 2 \\ C\sqrt{h}|\log h|^{1/2} h \left(\sum_{e\in\mathcal{E}\cap(\Omega\setminus B_{8h})} r^{-2\alpha-2}|e|\right)^{\frac{1}{2}} & \text{if } l=1. \end{cases}
\end{aligned} \tag{95}$$

It is easy to show that

$$\begin{aligned}
\sum_{e\in\mathcal{E}\cap(\Omega\setminus B_{8h})} r^{-2\alpha-2}|e| &\le C\sum_{j=1}^{\infty}(hj)^{-2\alpha-2} j^{(d-1)}\cdot h^{(d-1)} \\
&\le \begin{cases} Ch^{-2\alpha-2}h^{(d-1)} & \text{if } l\ge 2 \\ Ch^{-1}|\log h| & \text{if } l=1. \end{cases}
\end{aligned} \tag{96}$$

By combining the above estimate and (95), we finally get

$$\sum_{e\in\mathcal{E}} \int_{e\cap(\Omega\setminus B_{8h})} \Big|[u-u_h]\Big|dx \le \begin{cases} Ch & \text{if } l\ge 2 \\ Ch|\log h| & \text{if } l=1. \end{cases} \tag{97}$$

Thus, by (94)-(97), we have the desired estimate (48).

6. Numerical results

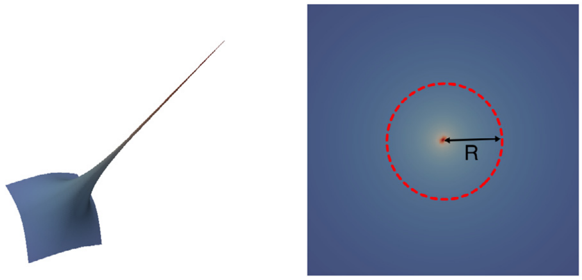
This section verifies our theoretical results by demonstrating several numerical examples including single and multiple Dirac sources in two and three dimensional domains. All the following CG, DG, and EG numerical schemes are coded by the authors using the open-source finite element package deal.II [4]. The current solver for the CG, DG, and EG system is GMRES with applying Algebraic Multigrid(AMG) block diagonal preconditioner for EG [24] and SSOR preconditioner for CG and DG.

6.1. Example 1. A single Dirac source term

First, we illustrate the convergence rate of the errors of system (1) with a Dirac source term for each different spatial discretization, CG, DG and EG. Here, the exact solution is given by

$$u(\mathbf{x}) = -0.5\pi\ \ln\sqrt{(x-x_1)^2+(y-y_1)^2},\ \mathbf{x}=(x,y)\in\Omega, \tag{98}$$

in the domain $\Omega=(0,1)^2$, where $(x_1,y_1)=(0.5,0.5)$ is the position of the Dirac source. See Fig. 5 for the detailed setup. A Dirichlet boundary condition is applied. Six computation cycles on uniform meshes were computed where the mesh size h is divided by two for each cycle. Also, two different polynomial orders for the approximation, linear and quadratic ($l=1$ and $l=2$), are tested. The maximum numbers of degrees of freedom with the finest mesh size (the last cycle) for the quadratic case ($l=2$) are 66049, 147456, and 82433, for CG, DG, and EG, respectively. Here, we set the penalty coefficient as $\gamma=100$ for $l=1$ and $\gamma=1000$ for $l=2$ for DG and EG.



(a) (b)

Fig. 5. Example 1. Setup: a) given exact solution with a Dirac source term. b) definition of the ball B (dotted) with radius R for error computations.

The behavior of the $\|u - u_h\|_{L^2(\Omega)\setminus B}$ errors for the approximated solution versus the mesh size h are depicted in Figs. 6 and 7 for $l = 1$ and $l = 2$, respectively. For each order, different radii $R = 0, 0.05, 0.1$ and 0.2 are tested. We do not observe the optimal error convergence rate for the case $R = 0$, but the optimal order of convergences as discussed in Theorem 3.1 are observed for the other cases as expected. In addition, we note that the optimal convergence is recovered provided $R \gtrapprox h$ (Fig. 6b).

Note that CG, DG, and EG have the same optimal error convergence rates for the given system. However, DG and EG become crucial when the system (1) is coupled with transport system since DG and EG fluxes are locally conservative. It is well known that DG and EG will perform better than CG if the coefficient β is highly heterogeneous. Readers are referred to [24,36] for more discussion.

6.2. Example 2. Multiple Dirac source terms

Next, we consider multiple Dirac source terms with the given exact solution

$$u(\mathbf{x}) = -0.5\pi \ \ln\sqrt{(x - x_n)^2 + (y - y_n)^2}, \ n = 1, 2, 3, \ \mathbf{x} = (x, y) \in \Omega = (0, 1)^2. \tag{99}$$

The Dirac sources ($\delta_{\mathbf{x}_n}$) are positioned on three different points $\delta_{\mathbf{x}_1} = (x_1, y_1) = (0.25, 0.75)$, $\delta_{\mathbf{x}_2} = (x_2, y_2) = (0.25, 0.25)$, and $\delta_{\mathbf{x}_3} = (x_3, y_3) = (0.75, 0.5)$ as shown in Fig. 8. We also define $R_n (n = 1, 2, 3)$ for each Dirac source terms. See Fig. 8b. A Dirichlet boundary condition is applied for the system.

Six computations on uniform meshes were computed where the mesh size h is divided by two for each cycle. Two different orders, $l = 1$ and $l = 2$, are tested and the maximum number of degrees of freedom with the finest mesh size for the case $l = 2$ are 66049, 147456, and 82433, for CG, DG, and EG, respectively. The penalty coefficients are $\gamma = 100$ for $l = 1$ and $\gamma = 1000$ for $l = 2$. The behavior of the $\|u - u_h\|_{L^2(\Omega)\setminus B_1 \cup B_2 \cup B_3}$ errors for the approximated solution versus the mesh size h are depicted in Figs. 9 and 10 for $l = 1$ and $l = 2$, respectively. Different radii $R_n = 0, 0.05, 0.1$ and 0.2 are tested for $l = 1, 2$. The optimal order of convergences as discussed in Theorem 3.1 are observed if $R \gtrapprox h$.

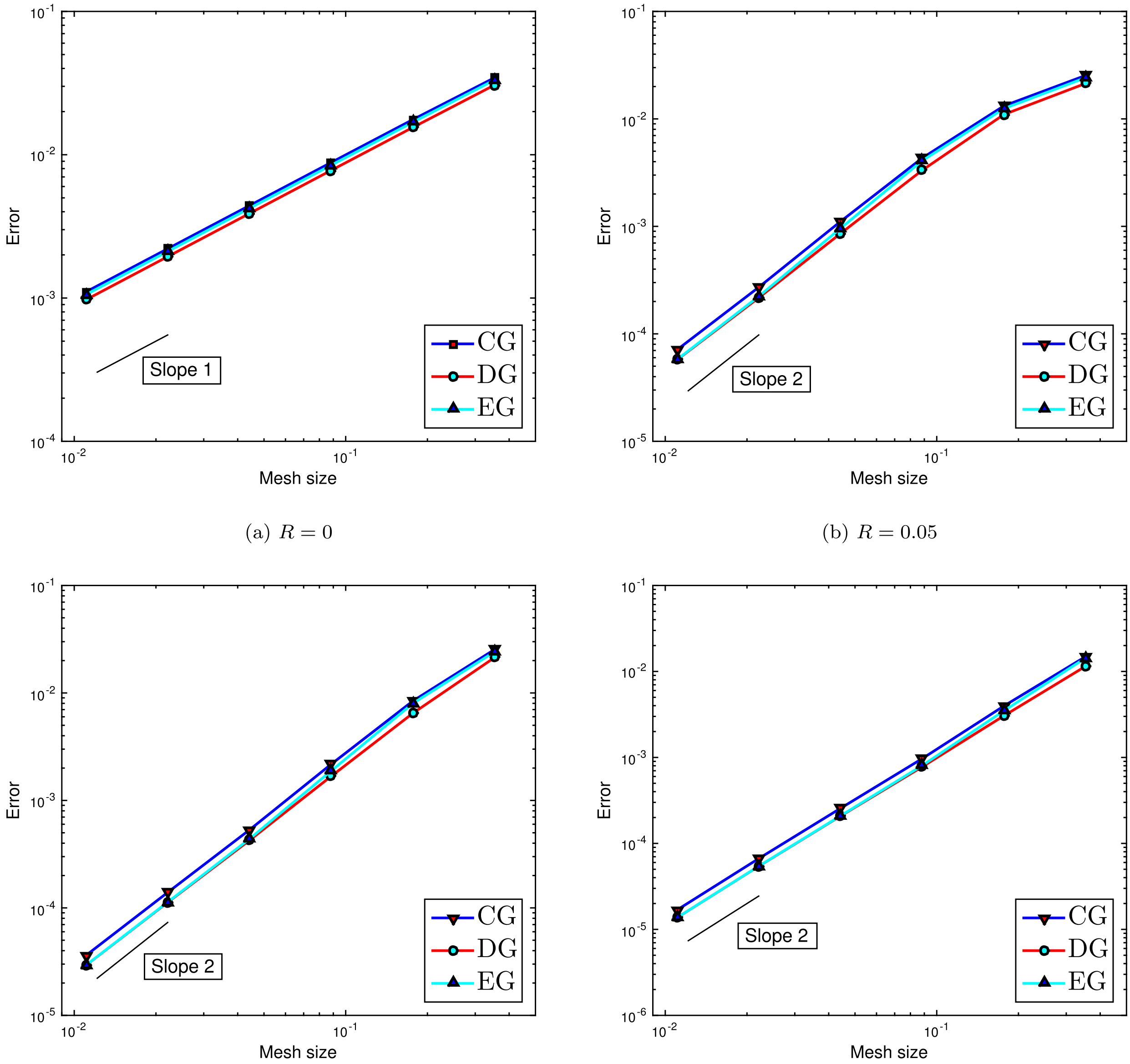
6.3. Example 3. A single Dirac source term in a three dimensional domain

In this final example, we extend the Example 1 to a three dimensional domain $\Omega = (0, 1)^3$, where the exact solution is given as

$$u(\mathbf{x}) = \frac{0.5}{\sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2}}, \ \mathbf{x} = (x, y, z) \in \Omega. \tag{100}$$

The Dirac source is positioned on $(x_1, y_1, z_1) = (0.5, 0.5, 0.5)$. Total six computations on uniform meshes were computed where the mesh size h is divided by two for each cycle. Two different orders, $l = 1$ and $l = 2$, are tested and the maximum number of degrees of freedom with the finest mesh size for the case $l = 2$ are 2146689, 7077888, and 2408833, for CG, DG, and EG, respectively. Here, we note the significant difference in the number of degrees of freedom in this three dimensional case. This example is computed by eight Intel(R) Xeon(R) CPU E5-2680 v3 @ 2.50 GHz processors by employing MPI parallel computing. The mesh sizes are $h = 0.8660, 0.4330, 0.2165, 0.1083, 0.0541$, and 0.027, for the six cycles, respectively. Here, we set the penalty coefficient as $\gamma = 100$ for $l = 1$ and $\gamma = 1000$ for $l = 2$.

As done in the previous example, the behavior of the $\|u - u_h\|_{L^2(\Omega\setminus B)}$ errors for the approximated solution versus the mesh size h are depicted in Figs. 11 and 12 for $l = 1$ and $l = 2$, respectively. For each order, different radii $R = 0, 0.05, 0.1$ and 0.2 are tested. We note that the suboptimal error convergence rate for the case $R = 0$ in three dimensional is smaller than the two dimensional setup as shown in Theorem 2.3. The optimal order of convergences as discussed in Theorem 3.1 are observed provided $h \lessapprox R$.



(a) $R = 0$ (b) $R = 0.05$ (c) $R = 0.1$ (d) $R = 0.2$

Fig. 6. Example 1. Convergence of the errors with order $l = 1$. Given mesh sizes are $h = 0.3536, 0.1768, 0.0884, 0.0442, 0.0221$, and 0.011. We do not obtain the expected optimal convergence rate if $h \gtrapprox R$ as shown in (a) and (b).

Acknowledgements

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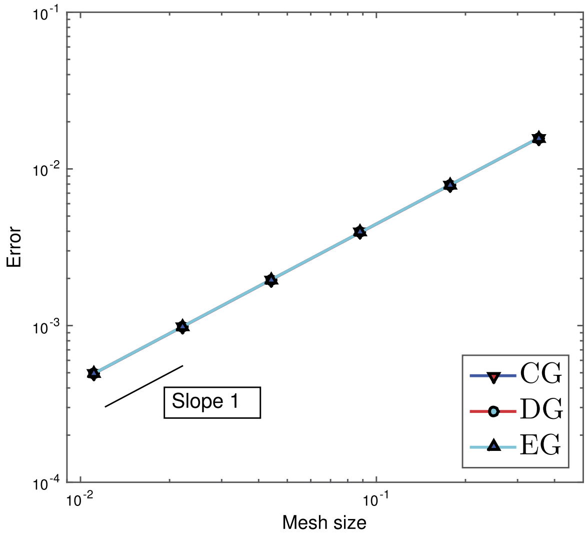
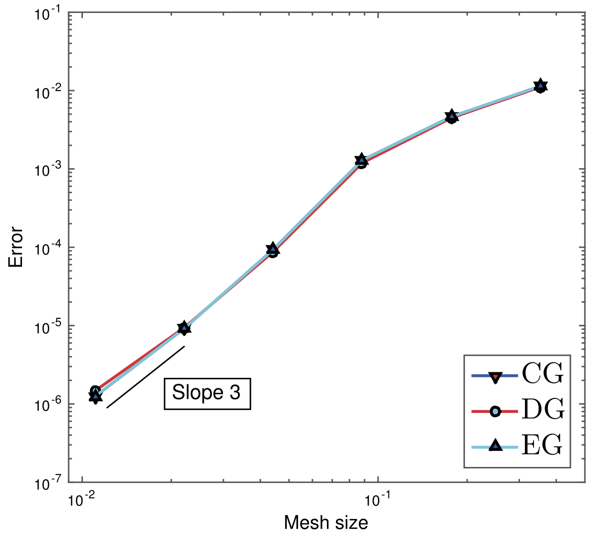
Appendix A. Global L^2 estimates

In this appendix, we provide the proof of Theorem 2.3 which considers the global L^2 estimate for the Dirac source problem. We are based on the idea presented in [21] where they used an approximated delta function, which is described as follows. Choose a unique triangle $T_h \in \mathcal{T}_h$ such that $x_0 \in T_h$. We find $\delta_{T_h} \in \mathbb{Q}^l(T_h)$ such that

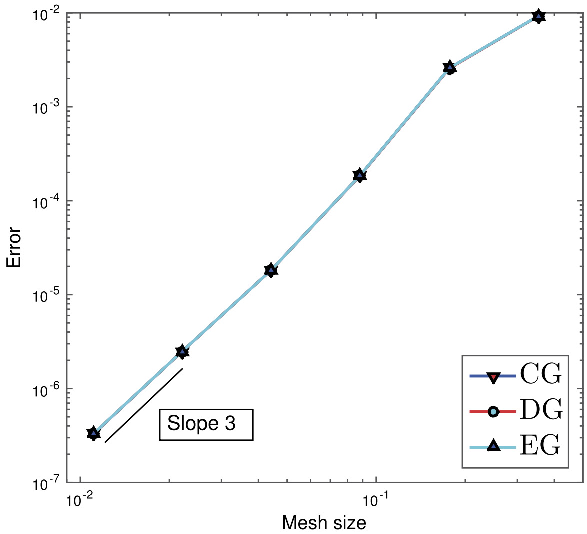
$$\int_{T_h} \delta_{T_h}(x) q(x) dx = q(x_0) \quad \forall q \in \mathbb{Q}^l(T_h). \tag{A.1}$$



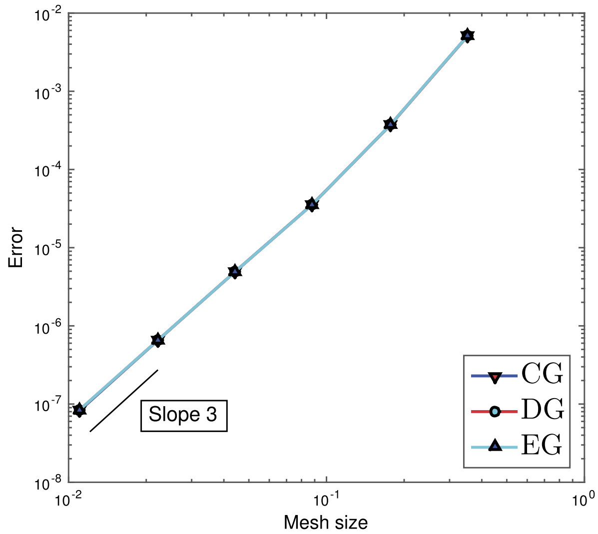
(a) $R = 0$



(b) $R = 0.05$

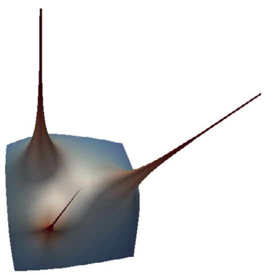


(c) $R = 0.1$

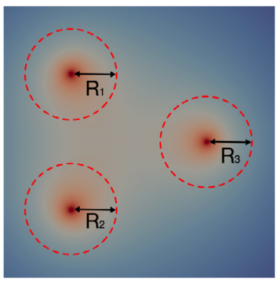


(d) $R = 0.2$

Fig. 7. Example 1. Convergence of the errors with order $l = 2$.

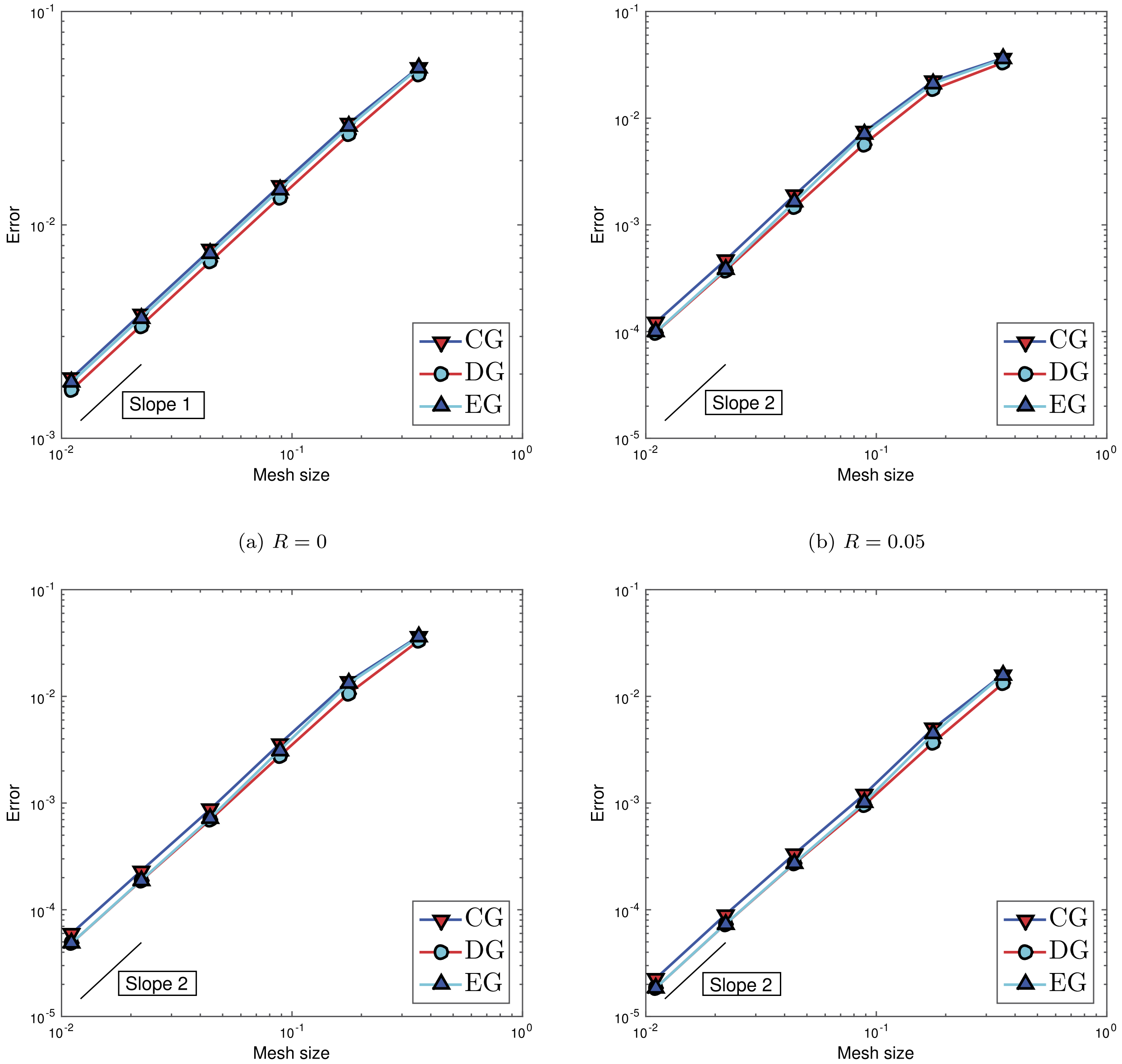


(a)



(b)

Fig. 8. Example 2. Setup: a) given exact solution with multiple Dirac source terms. b) definition of R_n for each $\delta_{\mathbf{x}_n}$.



(a) $R = 0$ (b) $R = 0.05$

(c) $R = 0.1$ (d) $R = 0.2$

Fig. 9. Example 2. Results with order 1.

Next, let $\delta_h \in V_{h,l}(\mathcal{T}_h)$ be

$$\delta_h(x) = \begin{cases} \delta_{T_h}(x) & x \in T_h, \\ 0 & x \notin T_h, \end{cases} \tag{A.2}$$

then we have

$$\int\limits_{\Omega} \delta_h(x) q(x) dx = q(x_0), \quad \forall\, q \in V_{h,l}(\mathcal{T}_h). \tag{A.3}$$

Now, we consider the weak solution $U^h \in H_0^1(\Omega)$ of the problem

$$\begin{cases} -\Delta U^h &= \delta_h \quad \text{in } \Omega, \\ \quad U^h &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{A.4}$$

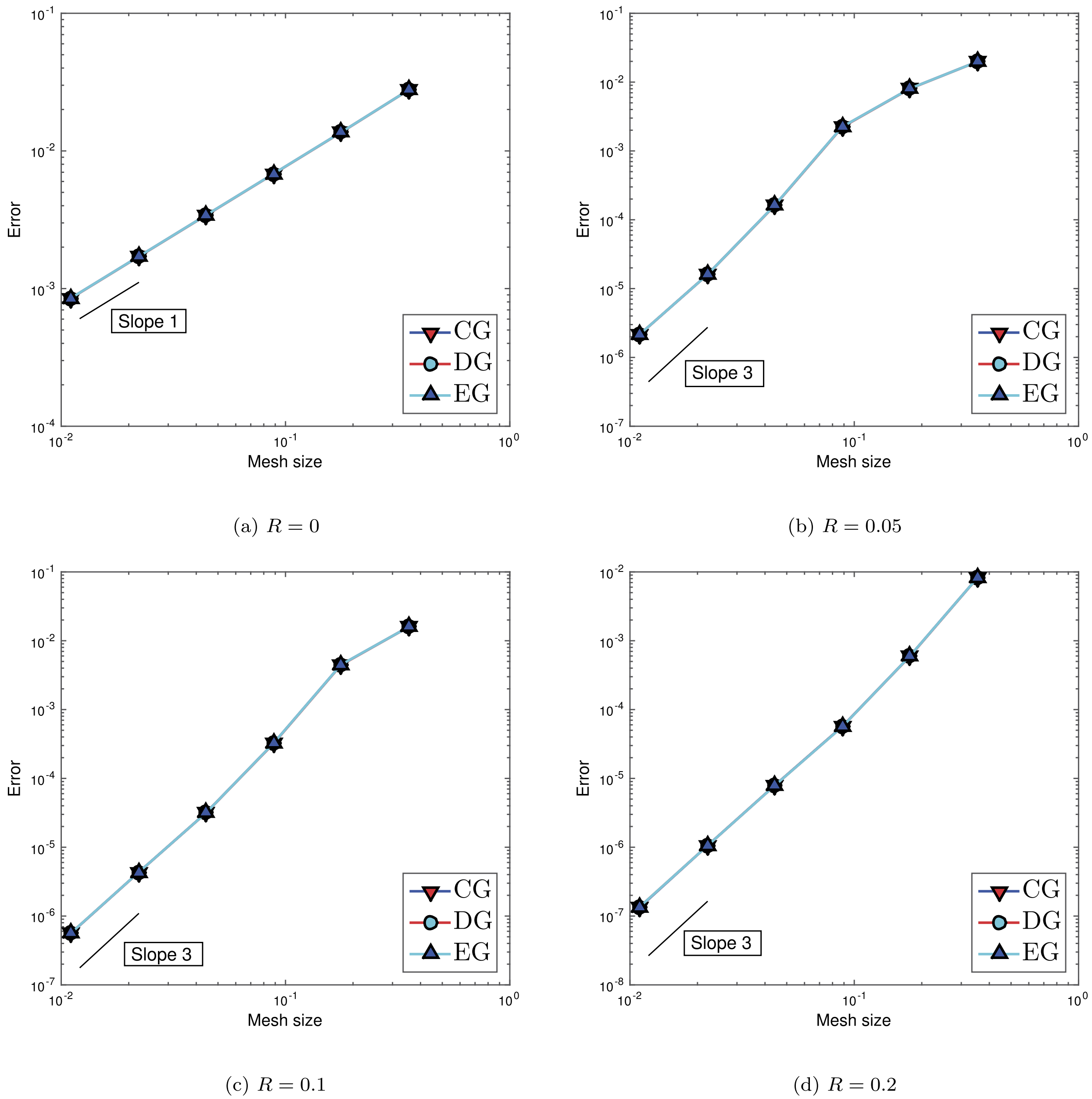


Fig. 10. Example 2. Results with order 2.

and derive the following estimate.

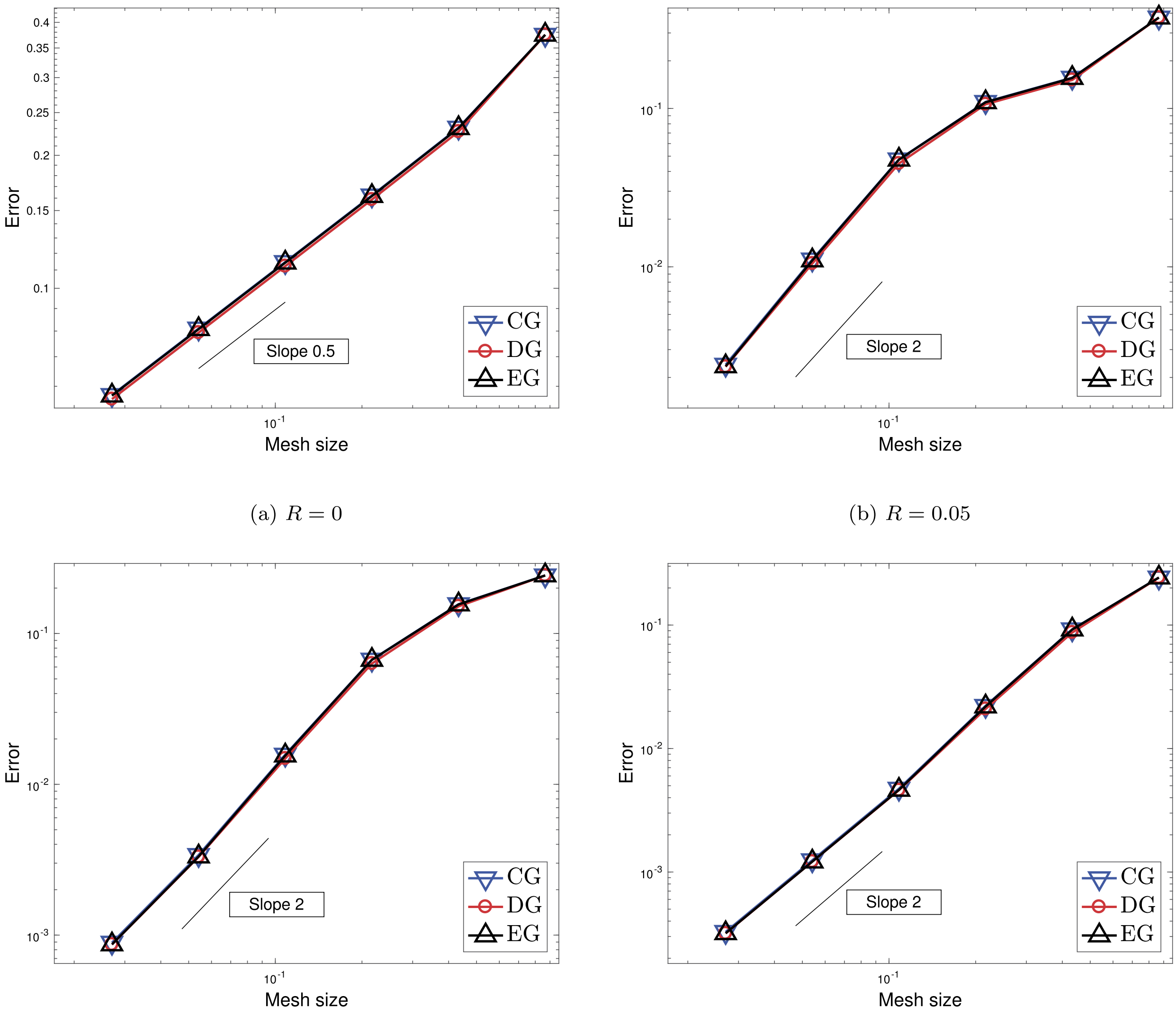
Lemma A.1. *We have*

$$\|u - U^h\|_{L^2(\Omega)} \leq Ch^{2-\frac{d}{2}}. \tag{A.5}$$

Proof A.2. First, we find $w_h \in H^2(\Omega)$ such that

$$\begin{cases} -\Delta w_h &= u - U^h \qquad \text{in } \Omega, \\ \quad w_h &= 0 \qquad\qquad \text{on } \partial\Omega. \end{cases} \tag{A.6}$$

Then, we obtain



(a) $R = 0$ (b) $R = 0.05$

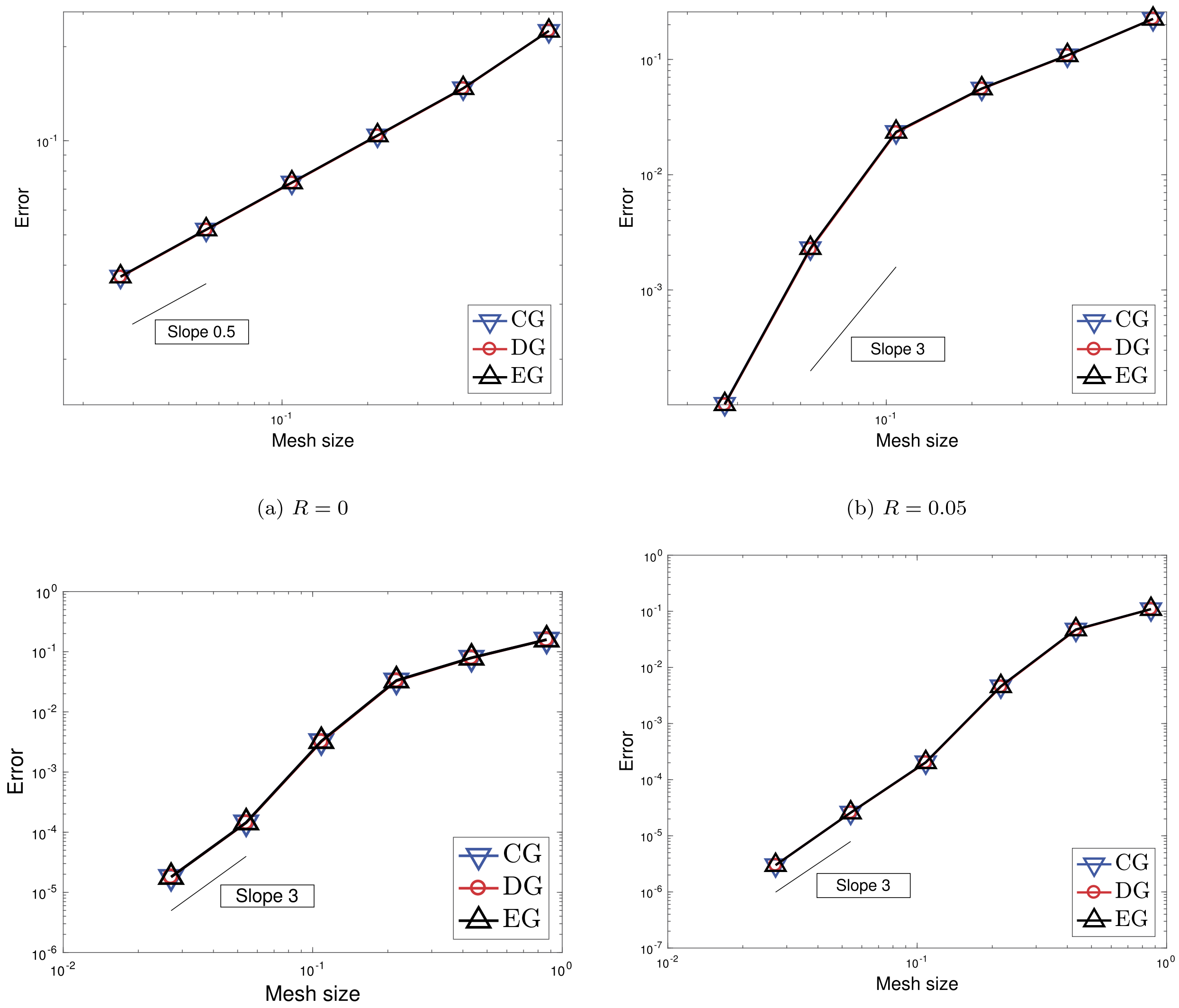
(c) $R = 0.1$ (d) $R = 0.2$

Fig. 11. Example 3. Results with order 1 in a three dimensional domain.

$$\begin{aligned}\int_\Omega (u - U^h)^2 dx &= \int_\Omega (u - U^h)(-\Delta w_h) dx \\ &= \int_\Omega \nabla(u - U^h)\nabla w_h dx \\ &= w_h(x_0) - \frac{1}{|T_h|}\int_{T_h} w_h(x) dx.\end{aligned} \tag{A.7}$$

If $d = 2$, the embedding $H^1(\Omega) \to L^\infty(\Omega)$ holds, and we get

$$\begin{aligned}\left| w_h(x_0) - \frac{1}{|T_h|}\int_{T_h} w_h(x) dx \right| &\le Ch\|\nabla w_h\|_{L^\infty(T_h)} \\ &\le Ch\|\nabla^2 w_h\|_{L^2(\Omega)}\end{aligned} \tag{A.8}$$



(a) $R = 0$ (b) $R = 0.05$

(c) $R = 0.1$ (d) $R = 0.2$

Fig. 12. Example 3. Results with order 2 in a three dimensional domain.

$$\leq Ch\|u - U^h\|_{L^2(\Omega)}.$$

If $d = 3$, thanks to the embedding $H^2(\Omega) \to C^\alpha(\Omega)$ with $\alpha = 2 - 3/2 = 1/2$, and we have

$$\begin{aligned}\left| w_h(x_0) - \frac{1}{|T_h|}\int_{T_h} w_h(x)dx \right| &\leq Ch^{1/2}\|w_h\|_{C^{1/2}(T_h)} \\ &\leq Ch^{1/2}\|\nabla^2 w_h\|_{L^2(\Omega)} \\ &\leq Ch^{1/2}\|u - U^h\|_{L^2(\Omega)}.\end{aligned} \tag{A.9}$$

The proof is finished.

Now we are ready to prove the main theorem of this section.

Theorem A.3. *Let $u \in L^2(\Omega)$ be the weak solution of (1) and let $u_h \in V_{h,l}(\mathcal{T}_h)$ be the DG or EG approximated solution given by (11). Then, we have*

$$\|u-u_h\|_{L^2} \le Ch^{2-\frac{d}{2}}. \tag{A.10}$$

Proof A.4. Let $f_h \in V_{h,l}(\mathcal{T}_h)$ be the DG or EG solution of (A.4). Then, for $\phi \in V_{h,l}(\mathcal{T}_h)$, we have

$$a(f_h,\phi) = \int_\Omega f_h(x)\phi(x)dx = \phi(x_0), \tag{A.11}$$

which implies that $f_h = u_h$. Since $\delta_h \in L^2(\Omega)$, we may apply the already known results to get

$$\|U^h-u_h\|_{L^2} \le Ch^2\|U^h\|_{H^2(\Omega)} \le h^2\|\delta_{T_h}\|_{L^2(\Omega)} \le Ch^{2-\frac{d}{2}}. \tag{A.12}$$

For $q \in L^2(\Omega)$, we denote by $P_k q \in V_{h,l}(\mathcal{T}_h)$ the projection of q into the subspace $V_{h,l}(\mathcal{T}_h)$ with respect to the $L^2(\Omega)$ inner product. Here the last inequality follows by

$$\begin{aligned}\|\delta_{T_h}\|_{L^2(T_h)} &= \sup_{\|q\|_{L^2(T_h)}\le 1} \int_{T_h} \delta_{T_h} q(x)dx \\ &= \sup_{\|q\|_{L^2}\le 1} \int_{T_h} \delta_{T_h} P_k q(x)dx \\ &= \sup_{\|q\|_{L^2}\le 1} (P_k q)(x_0) \\ &\le \sup_{\|q\|_{L^2}\le 1} h^{-\frac{d}{2}}\|P_k q\|_{L^2} \le Ch^{-\frac{d}{2}}.\end{aligned} \tag{A.13}$$

Finally, we get

$$\|u-u_h\|_{L^2} \le \|u-U^h\|_{L^2} + \|U^h-u_h\|_{L^2} \le Ch^{2-\frac{d}{2}}. \tag{A.14}$$

The proof is finished.

Appendix B. Weighted estimates

In this section, we prove Lemma 3.5 and Lemma 5.1.

Proof B.1 *(Proof of Lemma 3.5).* For the simplicity of the proof, we assume $x_0 = 0$ by using a change of coordinate $x \to x + x_0$. Recall that the shape-regular triangulation $\mathcal{T}_h$ is assumed and the interpolation operator S_h defined in (22). Let $u \in W^{l+1,2}(\Omega \setminus B(x_0,4h))$ and we fix any $j \in \mathbb{N}$ with $3 \le j \le C|\log h|$ with a large fixed value $C > 1$. Moreover, we set

$$\mathbf{D} = B(0,2)\setminus B(0,1), \qquad \widetilde{\mathbf{D}} = B(0,4)\setminus B(0,1/2), \tag{B.1}$$

and define

- $\mathbf{T}_j := \left\{(2^j h)^{-1} T \;\mid\; T \in \mathcal{T}_h\right\}$,
- $\mathbf{V}_{j,l} := \left\{y \mapsto \psi\left((2^j h)\, y\right) \;\mid\; \psi \in V_{h,l}(\mathcal{T}_h)\right\}$,
- $\mathbf{E}_j := \left\{(2^j h)^{-1} e \;\mid\; e \in \mathcal{E}\right\}$.

Also, for $\lambda > 0$, we define $\lambda T := \{\lambda x \mid x \in T\}$. Let $v(x) = u(2^j h\cdot x)$ and $S_h^j v(x) = S_h u(2^j h \cdot x)$ be functions defined on $\widetilde{\mathbf{D}}$. Then $S_h^j v \in \mathbf{V}_{j,l}$ equipped with shape-regular triangulation $\mathbf{T}_j$ whose mesh size is comparable to 2^{-j}.

Therefore, by (21) and Lemma 3.2 we obtain

$$\begin{aligned}&\left(\sum_{e\in \mathbf{E}_{j_{e\cap \mathbf{D}}}} \int |\{\nabla(v-S_h^j v)\}|^2 ds\right)^{1/2} \\ &\le \left(\sum_{T\in \mathbf{T}_{j_{T\cap\widetilde{\mathbf{D}}}}} \int 2^{-j}|\nabla^2(v-S_h^j v)|^2 + 2^j|\nabla(v-S_h^j v)|^2 dx\right)^{1/2} \\ &\le C(2^{-j})^{a-1/2} \sum_{|\xi|=a+1} \|D^\xi v\|_{L^2(\widetilde{\mathbf{D}})}\end{aligned} \tag{B.2}$$

for $a \in \{1, 2, \cdots, l\}$. Let $D_j = B(0, 2^{j+1}h) \setminus B(0, 2^j h)$ for each $j \in \mathbb{N}$ with $j \le J := C|\log h|$ with a suitable constant $C > 1$ so that $\Omega \setminus B(0, 8h) \subset \cup_{j=3}^{J} D_j$. We also let $\tilde{D}_j = B(0, 2^{j+2}h) \setminus B(0, 2^{j-1}h)$. By change of variables, we derive

$$\begin{aligned}
&\sum_{e\in \mathbf{E}_j} \int_{e\cap\widetilde{\mathbf{D}}} |\{\nabla(v - S_h^j v)\}|^2 ds \\
&\quad = (2^j h)^2 (2^j h)^{-(d-1)} \sum_{e\in\mathcal{E}} \int_{e\cap\tilde{D}_j} |\{\nabla(u - S_h u)\}|^2 ds.
\end{aligned} \tag{B.3}$$

Moreover,

$$\sum_{|\xi|=a+1} \|D^\xi v\|_{L^2(\widetilde{\mathbf{D}})} = (2^j h)^{a+1} (2^j h)^{-\frac{d}{2}} \sum_{|\xi|=a+1} \|D^\xi u\|_{L^2(\tilde{D}_j)}. \tag{B.4}$$

By inserting the above estimates in (B.2), we obtain

$$\begin{aligned}
&(2^j h)^{3/2-d/2} \left(\sum_{e\in\mathcal{E}} \int_{e\cap D_j} |\{\nabla(u - S_h u)\}|^2 ds \right)^{1/2} \\
&\le C (2^{-j})^{a-1/2} (2^h h)^{j+1-d/2} \sum_{|\xi|=a+1} \|D^\xi u\|_{L^2(\tilde{D}_j)}.
\end{aligned} \tag{B.5}$$

Thus, we have

$$\left(h \sum_{e\in\mathcal{E}} \int_{e\cap D_j} |\{\nabla(u - S_h u)\}|^2 ds \right)^{1/2} \le C h^k \sum_{|\xi|=k+1} \|D^\xi u\|_{L^2(\tilde{D}_j)}. \tag{B.6}$$

Since $2^{j+2}h \le |x| \le 2^{j-1}h$ on the domain $\tilde{D}_j$, we get

$$\left(h \sum_{e\in\mathcal{E}} \int_{e\cap D_j} r^{2\alpha} |\{\nabla(u - S_h u)\}|^2 ds \right)^{1/2} \le C h^k \sum_{|\xi|=k+1} \|r^\alpha D^\xi u\|_{L^2(\tilde{D}_j)}. \tag{B.7}$$

By summing the square of this inequality for $j \ge 3$, we obtain the desired weighted estimate.

Finally, we prove the Lemma 5.1.

Proof B.2 *(Proof of Lemma 5.1).* For the proof, we only need to verify that

$$\begin{aligned}
&\|r^\alpha \nabla(u - u_h)\|_{L^2(B_1\setminus B_{8h})} + \left(\frac{1}{h} \sum_{e\in\mathcal{E}} \int_{e\cap(B_1\setminus B_{8h})} r^{2\alpha} [u - u_h]^2 ds \right)^{\frac{1}{2}} \\
&\quad + \left(h \sum_{e\in\mathcal{E}} \int_{e\cap(B_1\setminus B_{8h})} r^{2\alpha} |\{\nabla(u - u_h)\}|^2 ds \right)^{1/2} \le C \|r^\alpha \nabla(u - S_h u)\|_{L^2(B_2\setminus B_{4h})} \\
&\quad + \left(h \sum_{e\in\mathcal{E}} \int_{e\cap(B_2\setminus B_{4h})} r^{2\alpha} |\{\nabla(u - S_h u)\}|^2 ds \right)^{1/2} + C \|r^{\alpha-1}(u - u_h)\|_{L^2(B_2\setminus B_{4h})},
\end{aligned} \tag{B.8}$$

since the estimate for the domain $\Omega \setminus B_1$ can be derived easily by the fact that $1/2 \le r = |x - x_0| \le L$ on the domain $\Omega \setminus B_{1/2}$, where L is the diameter of Ω.

To prove the above estimate, we shall combine a dyadic decomposition with Lemma 3.3 to obtain this weighted version. As we did in the previous proof, we set $x_0 = 0$ for the simplicity. Let $D_j = B(x_0, 2^{j+1}h) \setminus B(x_0, 2^j h)$ for each $j \in \mathbb{N}$ and let $\tilde{D}_j = B(0, 2^{j+2}h) \setminus B(0, 2^{j-1}h)$. Here we also use the same notations $\mathbf{T}_j$, $\mathbf{V}_{j,l}$, $\mathbf{E}_j$ as defined in the previous proof and we set

$$\mathbf{D} = B(0,2) \setminus B(0,1) \quad \text{and} \quad \widetilde{\mathbf{D}} = B(0,4) \setminus B(0,1/2). \tag{B.9}$$

First, we take a minimal value $J \in \mathbb{N}$ such that

$$\Omega \setminus B(x_0, 8h) \subset \cup_{j=3}^{J} D_j. \tag{B.10}$$

For fixed $j \geq 3$, we let

$$v(x) = u(2^j h \cdot x) \quad \text{and} \quad v_h(x) = u_h(2^j h \cdot x) \tag{B.11}$$

be functions defined on $\widetilde{\mathbf{D}}$. Then v satisfies $-\Delta v = 0$ on $\widetilde{\mathbf{D}}$ and $v_h \in \mathbf{V}_{j,l}$ is the approximation solution of v on $\widetilde{\mathbf{D}}$ with mesh length comparable to 2^{-j}. Then, we deduce from Lemma 3.3 that

$$\begin{aligned}
&\|\nabla(v - v_h)\|_{L^2(\mathbf{D})} + \left(2^j \sum_{e \in \mathbf{E}_{j e \cap \mathbf{D}}} \int |[v - v_h]|^2 ds\right)^{1/2} + \left(2^{-j} \sum_{e \in \mathbf{E}_{j e \cap \mathbf{D}}} \int |\{\nabla(v - v_h)\}|^2 ds\right)^{1/2} \\
&\leq C\|\nabla(v - S_h^j v)\|_{L^2(\widetilde{\mathbf{D}})} + C\left(2^{-j} \sum_{e \in \mathbf{E}_{j_{e \cap \widetilde{\mathbf{D}}}}} \int |\left\{\nabla(v - S_h^j v)\right\}|^2 ds\right)^{1/2} + C\|v - v_h\|_{L^2(\widetilde{\mathbf{D}})},
\end{aligned} \tag{B.12}$$

where $S_h^j v(x) = S_h u(2^j h \cdot x)$. Next, we write this inequality in terms of u and u_h. By a change of variables,

$$\|v - v_h\|_{L^2(\widetilde{\mathbf{D}})} = (2^j h)^{-\frac{d}{2}} \|u - u_h\|_{L^2(\tilde{D}_j)}$$

and

$$\|\nabla(v - v_h)\|_{L^2(\mathbf{D})} = (2^j h)^{-\frac{d-2}{2}} \|\nabla(u - u_h)\|_{L^2(D_j)}.$$

Thus we get,

$$\begin{aligned}
&\left(2^j \sum_{e \in \mathbf{E}_{j e \cap \mathbf{D}}} \int |[v - v_h]|^2(x) ds\right)^{1/2} \\
&= \left(2^j \sum_{e \in \mathbf{E}_{j e \cap \mathbf{D}}} \int |[u - u_h]|^2(2^j h \cdot x) ds\right)^{1/2} \\
&= (2^j h)^{-\frac{(d-1)}{2}} \left(2^j \sum_{e \in \mathcal{E}_{e \cap D_j}} \int |[u - u_h]|^2(x) ds\right)^{1/2},
\end{aligned}$$

and

$$\begin{aligned}
&\left(2^{-j} \sum_{e \in \mathbf{E}_{j e \cap \mathbf{D}}} \int |\{\nabla(v - v_h)\}|^2(x) ds\right)^{1/2} \\
&= \left(2^{-j}(2^j h)^2 \sum_{e \in \mathbf{E}_{j e \cap \mathbf{D}}} \int |\{\nabla(u - u_h)\}|^2(2^j h \cdot x) ds\right)^{1/2} \\
&= 2^{-j/2}(2^j h)(2^j h)^{-\frac{(d-1)}{2}} \left(\sum_{e \in \mathcal{E}_{e \cap D_j}} \int |\{\nabla(u - u_h)\}|^2(x) ds\right)^{1/2}.
\end{aligned}$$

Similarly, for the terms in the right hand side of (B.12), we replace the terms in (B.12) with u and multiply by $(2^j h)^{\frac{d-2}{2}}$ to obtain

$$\|\nabla(u-u_h)\|_{L^2(D_j)} + \left(\frac{1}{h}\sum_{e\in\mathcal{E}}\int_{e\cap D_j}[u-u_h]^2(x)ds\right)^{1/2} + \left(h\sum_{e\in\mathcal{E}}\int_{e\cap D_j}|\{\nabla(u-u_h)\}|^2(x)ds\right)^{1/2}$$

$$\le C\|\nabla(u-S_hu)\|_{L^2(\tilde{D}_j)} + C\left(\frac{1}{h}\sum_{e\in\mathcal{E}}\int_{e\cap\tilde{D}_j}|[u-S_hu]|^2(x)ds\right)^{1/2} \tag{B.13}$$

$$+C\left(h\sum_{e\in\mathcal{E}}\int_{e\cap\tilde{D}_j}|\{\nabla(u-S_hu)\}|^2(x)ds\right)^{1/2} + C(2^jh)^{-1}\|u-u_h\|_{L^2(\tilde{D}_j)}.$$

Having in mind that $r=|x|$ is comparable to 2^jh on D_j and $\tilde{D}_j$, we sum the squares of the above inequalities for $j\in\{3,4,\cdots,J\}$ to get

$$\|r^\alpha\nabla(u-u_h)\|_{L^2(\Omega\setminus B_{8h})} + \left(\frac{1}{h}\sum_{e\in\mathcal{E}}\int_{e\cap(\Omega\setminus B_{8h})}r^{2\alpha}[u-u_h]^2(x)ds\right)^{1/2}$$

$$+\left(h\sum_{e\in\mathcal{E}}\int_{e\cap(\Omega\setminus B_{8h})}r^{2\alpha}|\{\nabla(u-u_h)\}|^2(x)ds\right)^{1/2}$$

$$\le C\|r^\alpha\nabla(u-S_hu)\|_{L^2(\Omega\setminus B_{4h})} + C\left(\frac{1}{h}\sum_{e\in\mathcal{E}}\int_{e\cap(\Omega\setminus B_{4h})}r^{2\alpha}[u-S_hu]^2(x)ds\right)^{1/2} \tag{B.14}$$

$$+C\left(h\sum_{e\in\mathcal{E}}\int_{e\cap(\Omega\setminus B_{4h})}r^{2\alpha}|\{\nabla(u-S_hu)\}|^2(x)ds\right)^{1/2} + C\|r^{\alpha-1}(u-u_h)\|_{L^2(\Omega\setminus B_{4h})}.$$

This is the desired estimate and the proof is done.

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